Foundations for Category Theory

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Disclaimer: I am by no means a set theorist or any kind of expert. This is the record of my search for peace of mind with respect to foundations in category theory. Perhaps my experience will save another graduate student some frustration. There is a large literature on this subject, of which I am largely ignorant. The reader might find [Isb66], [Kru65], [ML71], [Mak96] and [Son62] useful.

Contents

1 Introduction

One reason for the increase in importance of mathematical logic was the discovery of paradoxes in naive set theory. There are many different axiomatic theories that can serve as a foundation for set theory, some more obscure than others. The most popular form of axiomatic set theory is Zermelo-Frankel (ZF) together with the Axiom of Choice (ZFC), with most modern research in set theory on questions of independence and consistency carried out with respect to ZF. This form of set theory is sufficient for many parts of mathematics, but in category theory (and therefore fields like algebraic geometry where categorical techniques are prevalent) this is not enough, because we need to talk about structures like the "category of all sets" which have no place in ZFC.

In many of the standard english references on category theory, little attention is paid to set-theoretic problems. Notable exceptions are Schubert's excellent book [Sch72] and also [AHS90]. In the author's opinion, among all references on category theory [Sch72] comes the closest to a complete, careful, *working* foundation for category theory. The available foundations that the author is aware of are:

(a) An alternative version of set theory called NBG (due to von Neumann, Robinson, Bernays and Gödel) which introduces *classes* to play the role of sets which are "too big" to exist in ZF. A good reference is Chapter 4 of [Men97]. The author learnt category theory from [Mit65], so began life believing in NBG. But if you're careful you soon observe a whole raft of things that need to be added to make NBG work, and in the end you might as well learn about universes.

(b) Extend ZFC by adding a new axiom describing *grothendieck universes*. Intuitively speaking, you fix a grothendieck universe $\mathscr{U}$ and call elements of $\mathscr{U}$ *sets*, while calling subsets of $\mathscr{U}$ *classes*. We describe this approach in detail in Section 4. This is the approach preferred by the author, and seems to be the only serious foundation available for modern research involving categories.

(c) The first two options are conversative, in that they seek to extend set theory by as little as possible to make things work. More exotically, we can introduce categories as foundational objects. This approach focuses on topoi as the fundamental logical objects (as well as the connection with the more familiar world of naive set theory). While such a foundation shows promise, it is not without its own problems [Hel03] and is probably not ready for "daily use".

We begin this note by reminding the reader briefly about first order theories. For this and the detailed treatment of NBG the careful reader should consult [Men97]. At some point we must admit that mathematics is a game played with pencils and paper, and can never aspire to be "perfect". Not least of all because every mathematician performs their work using an imperfect organic processing device, and it is only this device which attaches meaning to the symbols on a page. Our foundation of mathematics will make reference to concepts like "collection","finite","countable" and "sequence" which will not be formalised. There will always be some level of ambiguity and imperfection in our foundations: the aim is to reduce this to a minimum and then hopefully never worry about it again. To this end, formal theories are described using a mathematically weak portion of the English language.

2 First order theories

Underlying Philosophy 1. Mathematics consists of formal theories and interpretations of these theories. The definition of a formal theory refers to the simplest innate capabilities of the human mind, corresponding loosely to mental manipulation of discrete objects:

- The ability to conceive of individual objects
- The ability to group objects together into collections
- The ability to distinguish those objects of a collection which satisfy some condition

A formal theory is a conservative mental structure dealing with sequences of symbols and the relations between these sequences (symbols being objects, sequences and relations both being types of collections).

Definition 1. A *formal theory* $\mathcal{S}$ is defined by the following information:

1. A countable collection of symbols. A finite sequence of symbols of $\mathcal{S}$ is called an *expression* of $\mathcal{S}$.
2. There is a collection of expressions of $\mathcal{S}$ called the *well formed formulas* (wfs) of $\mathcal{S}$. There is usually an effective procedure to determine whether a given expression is a wf.
3. There is a collection of wfs called the *axioms* of $\mathcal{S}$. Most often, one can effectively decide whether a given wf is an axiom; in such a case, $\mathcal{S}$ is called an *axiomatic* theory.
4. There is a finite collection $R_1, \ldots, R_n$ of relations among the wfs, called *rules of inference*. For each R_i there is a unique positive integer j such that, for every collection of j wfs and each wf $\mathscr{B}$, one can effectively decide whether the given j wfs are in relation R_i to $\mathscr{B}$, and if so, $\mathscr{B}$ is said to *follow from* or to be a *direct consequence* of the given wfs by virtue of R_i.

A *proof* in $\mathcal{S}$ is a sequence $\mathscr{B}_1, \ldots, \mathscr{B}_k$ of wfs such that, for each i, either $\mathscr{B}_i$ is an axiom of $\mathcal{S}$ or $\mathscr{B}_i$ is a direct consequence of some of the preceeding wfs in the sequence by virtue of one of the rules of inference of $\mathcal{S}$.

A *theorem* of $\mathcal{S}$ is a wf $\mathscr{B}$ of $\mathcal{S}$ such that $\mathscr{B}$ is the last wf of some proof in $\mathcal{S}$. Such a proof is called a *proof of $\mathscr{B}$ in $\mathcal{S}$*. A wf $\mathscr{C}$ is said to be a *consequence* in $\mathcal{S}$ of a collection Γ of wfs if and only if there is a sequence $\mathscr{B}_1, \ldots, \mathscr{B}_k$ of wfs such that $\mathscr{C}$ is $\mathscr{B}_k$ and, for each i, either $\mathscr{B}_i$ is an axiom or $\mathscr{B}_i$ is in Γ, or $\mathscr{B}_i$ is a direct consequence by some rule of inference of some of the preceeding wfs in the sequence. Such a sequence is called a *proof* (or *deduction*) *of* $\mathscr{C}$ *from* Γ. The members of Γ are called the *hypotheses* or *premises* of the proof.

Definition 2. A *first order theory* $\mathcal{L}$ is a formal theory of a certain type, as described below.

1. The symbols of $\mathcal{L}$ are given by:
   (a) The propositional connectives $\neg$ and $\Rightarrow$, and the universal quantifier symbol $\forall$.
   (b) Punctuation marks: the left parenthesis (the right parenthesis) and the comma.
   (c) Denumerably many individual variables $x_1, x_2, \ldots$.
   (d) A finite or denumerable, possibly empty, collection of function letters. Associated with a function letter f is a positive integer n indicating the number of arguments.
   (e) A finite or denumerable, possibly empty, collection of individual constants
   (f) A non-empty collection of predicate letters. Associated with a predicate letter A is a positive integer n indicating the number of arguments.

2. To define the wfs of $\mathcal{L}$ we first give two collections of expressions of $\mathcal{L}$: the *terms* and the *atomic formulas*. The terms are defined as follows:
   (a) Variables and individual constants are terms.
   (b) If f is a function letter with n arguments and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.
   (c) An expression is a term only if it can be shown to be a term on the basis of conditions 1 and 2.

   An expression is an *atomic formula* if it is $A(t_1, \ldots, t_n)$ where A is a predicate letter with n arguments and $t_1, \ldots, t_n$ are terms. The well-formed formulas of $\mathcal{L}$ are defined as follows:
   (a) Every atomic formula is a wf.
   (b) If $\mathscr{B}$ and $\mathscr{C}$ are wfs and y is a variable then $(\neg\mathscr{B}), (\mathscr{B} \Rightarrow \mathscr{C})$ and $((\forall y)\mathscr{B})$ are wfs.
   (c) An expression is a wf only if it can be shown to be a wf on the basis of conditions 1 and 2.

   In $((\forall y)\mathscr{B})$ the wf $\mathscr{B}$ is called the *scope* of the quantifier $(\forall y)$. An occurrence of a variable x is said to be *bound* in a wf $\mathscr{B}$ if either it is the occurrence of x in a quantifier $(\forall x)$ in $\mathscr{B}$ or it lies within the scope of a quantifier $(\forall x)$ in $\mathscr{B}$. Otherwise the occurrence is said to be *free* in $\mathscr{B}$. A variable is said to be *free* (*bound*) in a wf $\mathscr{B}$ if it has a free (bound) occurrence in $\mathscr{B}$.

   If $\mathscr{B}$ is a wf and t is a term, then t is said to be *free* for the variable x_i in $\mathscr{B}$ if no free ocurrence of x_i lies in $\mathscr{B}$ within the scope of any quantifier $(\forall x_j)$, where x_j is a variable in t. That is, if t is substituted for all free occurrences of x_i in $\mathscr{B}$, no occurrence of a variable in t becomes a bound occurrence in $\mathscr{B}$.

3. We only require that the axioms of $\mathcal{L}$ include the following *logical axioms* for wfs $\mathscr{B}, \mathscr{C}$ and $\mathscr{D}$ of $\mathcal{L}$
   (A1) $\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{B})$
   (A2) $(\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{D})) \Rightarrow ((\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{D}))$
   (A3) $(\neg\mathscr{C} \Rightarrow \neg\mathscr{B}) \Rightarrow ((\neg\mathscr{C} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{C})$
   (A4) $(\forall x_i)\mathscr{B}(x_i) \Rightarrow \mathscr{B}(t)$ if $\mathscr{B}(x_i)$ is a wf of $\mathcal{L}$ and t is a term of $\mathcal{L}$ that is free for x_i in $\mathscr{B}(x_i)$. Note here that t may be identical with x_i so that all wfs $(\forall x_i)\mathscr{B} \Rightarrow \mathscr{B}$ are axioms by virtue of (A4).
   (A5) $(\forall x_i)(\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\mathscr{B} \Rightarrow (\forall x_i)\mathscr{C})$ if $\mathscr{B}$ contains no free occurrences of x_i.

4. There are precisely two rules of inference:

   (a) Modus ponens: $\mathscr{C}$ follows from $\mathscr{B}$ and $\mathscr{B} \Rightarrow \mathscr{C}$.

   (b) Generalisation: $(\forall x_i)\mathscr{B}$ follows from $\mathscr{B}$.

Underlying Philosophy 2. A first other theory is meant to describe formal manipulations of imaginary symbols. Of course, in practice (for example in the first other theory of sets) we say that the variables "stand for" sets, and the wfs of the theory "say things" about sets. So a formal theory of sets is an attempt to be as unbiased as possible about what is and what is not true about sets.

3 ZFC

Before we study grothendieck universes, let us first agree on what we mean by ZFC. The first order theory ZFC has two predicate letters A, B but no function letter, or individual constants. Traditionally the variables are given by uppercase letters $X_1, X_2, \ldots$ (As usual, we shall use X, Y, Z to represent arbitrary variables). We shall abbreviate $A(X, Y)$ by $X \in Y$ and $B(X, Y)$ by $X = Y$. Intuitively $\in$ is thought of as the membership relation and the values of the variables are to be thought of as sets (in ZFC we have no concept of "class"). The proper axioms are as follows (there are an infinite number of axioms since an axiom scheme is used):

Axiom of Extensionality Two sets are the same if and only if they have the same elements

$$\forall A, \forall B : A = B \Leftrightarrow (\forall C : C \in A \Leftrightarrow C \in B)$$

Axiom of Empty Set There is a set with no elements. By the previous axiom, it must be unique and we denote it $\emptyset$

$$\exists \emptyset, \forall A : \neg(A \in \emptyset)$$

Axiom of Pairing If x, y are sets, then there exists a set containing x, y as its only elements, which we denote $\{x, y\}$. Therefore given any set x there is a set $\{x\} = \{x, x\}$ containing just the set x

$$\forall A, \forall B, \exists C, \forall D : D \in C \Leftrightarrow (D = A \vee D = B)$$

Axiom of Union For any set x, there is a set y such that the elements of y are precisely the elements of the elements of x

$$\forall A, \exists B, \forall C : C \in B \Leftrightarrow (\exists D : C \in D \wedge D \in A)$$

Axiom of Infinity There exists a set x such that $\emptyset$ is in x and whenever y is in x, so is $y \cup \{y\}$

$$\exists \mathbb{N} : \emptyset \in \mathbb{N} \wedge (\forall A : A \in \mathbb{N} \Rightarrow A \cup \{A\} \in \mathbb{N})$$

Axiom of Power Set Every set has a power set. That is, for any set x there exists a set y, such that the elements of y are precisely the subsets of x.

$$\forall A, \exists \mathcal{P}A, \forall B : B \in \mathcal{P}A \Leftrightarrow (\forall C : C \in B \Rightarrow C \in A)$$

Axiom of Comprehension Given any set and any wf $\mathscr{B}(x)$ with x free, there is a subset of the original set containing precisely those elements x for which $\mathscr{B}(x)$ holds (this is an axiom schema)

$$\forall A, \exists B, \forall C : C \in B \Leftrightarrow C \in A \wedge \mathscr{B}(C)$$

Here we make the technical assumption that the variables A, B, C do not occur in $\mathscr{B}$.

Axiom of Replacement Given any set and any mapping, formally defined as a wf $\mathscr{B}(x,y)$ with x,y free such that $\mathscr{B}(x,y_1)$ and $\mathscr{B}(x,y_2)$ implies $y_1=y_2$, there is a set containing precisely the images of the original set's elements (this is an axiom schema)

$$(\forall X, \exists! Y : \mathscr{B}(X,Y)) \Rightarrow \forall A, \exists B, \forall C : C \in B \Leftrightarrow \exists D : D \in A \wedge \mathscr{B}(D,C)$$

Axiom of Foundation A *foundation member* of a set x is $y \in x$ such that $y \cap x$ is empty. Every nonempty set has a foundation member.

$$\exists A(A \in B) \Rightarrow \exists A(A \in B \wedge \neg \exists C(C \in A \wedge C \in B))$$

Axiom of Choice Given any set of mutually disjoint nonempty sets, there exists at least one set that contains exactly one element in common with each of the nonempty sets.

4 Grothendieck Universes

Whatever foundation we use for category theory, it must somehow provide us with a notion of "big sets". In Grothendieck's approach, one fixes a particular set $\mathfrak{U}$ (called the *universe*) and thinks of elements of $\mathfrak{U}$ as "normal sets", subsets of $\mathfrak{U}$ as "classes", and all other sets as "unimaginably massive". The original (and to the author's knowledge, only) complete reference for grothendieck universes is SGA4, although the definition is given in [Sch72] and various other places. The presentation in this section closely follows SGA4, up to the definition of the "conglomerate convention".

Definition 3. A *grothendieck universe* (or just a *universe*) is a nonempty set $\mathscr{U}$ with the following properties:

U1. If $x \in \mathscr{U}$ and $y \in x$ then $y \in \mathscr{U}$ (that is, if $x \in \mathscr{U}$ then $x \subseteq \mathscr{U}$).

U2. If $x, y \in \mathscr{U}$ then $\{x,y\} \in \mathscr{U}$.

U3. If $x \in \mathscr{U}$, then $\mathcal{P}(x) \in \mathscr{U}$.

U4. If $I \in \mathscr{U}$ and $\{x_i\}_{i \in I}$ is a family of elements of $\mathscr{U}$, then the union $\bigcup_{i\in I} x_i$ belongs to $\mathscr{U}$.

Remark 1. From these axioms, one deduces the following properties

- If $x \in \mathscr{U}$ then $\{x\} \in \mathscr{U}$.
- if x is a subset of $y \in \mathscr{U}$, then $x \in \mathscr{U}$ (in particular $\emptyset \in \mathscr{U}$).
- If $x, y \in \mathscr{U}$ then the Kuratowski ordered pair $(x,y) = \{x, \{x,y\}\}$ is an element of $\mathscr{U}$.
- If $x, y \in \mathscr{U}$ then $x \cup y$ and $x \times y$ are elements of $\mathscr{U}$.
- If $x, y \in \mathscr{U}$ then the set of all functions $x \longrightarrow y$ belongs to $\mathscr{U}$.
- If $I \in \mathscr{U}$ and $\{x_i\}_{i\in I}$ is a family of elements of $\mathscr{U}$, then the sets $\prod_{i\in I} x_i, \coprod_{i\in I} x_i$ both belong to $\mathscr{U}$. If I is nonempty then $\bigcap_{i\in I} x_i \in \mathscr{U}$.
- If $x \in \mathscr{U}$ then $x \cup \{x\} \in \mathscr{U}$, and therefore $\mathbb{N} \subseteq \mathscr{U}$ (you can't say in general that $\mathbb{N} \in \mathscr{U}$). Therefore any finite union, product and disjoint union of elements of $\mathscr{U}$ belongs to $\mathscr{U}$. In particular every finite subset of $\mathscr{U}$ belongs to $\mathscr{U}$.
- If $x \in \mathscr{U}$ then the union set of x belongs to $\mathscr{U}$.

We can therefore perform all the usual operations of the theory of sets beginning with the elements of a universe without the final result escaping our universe. To be more precise, the axioms of ZFC strictly limit the ways you can produce new sets from known ones. Looking at the axioms, only the Axiom of Replacement can produce a set outside our universe (beginning with sets inside the universe), although one could argue that the Axiom of Infinity also "produces" the set $\mathbb{N}$, which may not belong to $\mathscr{U}$. To get around the latter difficulty, we add the following axiom to ZFC (write Unv(X) to mean that the set X is a universe)

UA. Every set is contained in some universe

$$\forall A \exists \mathscr{U} : Unv(\mathscr{U}) \wedge A \in \mathscr{U}$$

Definition 4. The first order theory ZFCU is the first order theory ZFC together with the axiom UA. We also refer to this as ZFC *with universes*. An *infinite universe* is a universe $\mathscr{U}$ with $\mathbb{N} \in \mathscr{U}$. By UA there is at least one infinite universe. Throughout our notes, all universes will be infinite.

Remark 2. Let $\mathscr{U}$ be a grothendieck universe. Then by our convention $\mathscr{U}$ contains $\mathbb{N}$, and therefore also $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ and all structures built from these using the theory of sets.

Definition 5. Let $\mathscr{U}$ be a grothendieck universe. An *$\mathscr{U}$-set* is an element of $\mathscr{U}$. An *$\mathscr{U}$-group* (resp. *$\mathscr{U}$-abelian group*) is a group (resp. abelian group) in the usual sense whose underlying set belongs to $\mathscr{U}$. In this way we define $\mathscr{U}$-rings, $\mathscr{U}$-algebras, $\mathscr{U}$-topological spaces, etc. An *$\mathscr{U}$-class* is a subset of $\mathscr{U}$. Therefore we have the set $\mathcal{P}(\mathscr{U})$ of all $\mathscr{U}$-classes. In our notes we will usually fix a grothendieck universe $\mathscr{U}$ and then use the following notation convention, called the *conglomerate convention for $\mathscr{U}$* (CC$\mathscr{U}$):

CC$\mathscr{U}$	*ZFC*
set	$\mathscr{U}$-set
class	$\mathscr{U}$-class
conglomerate	set
group	$\mathscr{U}$-group
abelian group	$\mathscr{U}$-abelian group
ring	$\mathscr{U}$-ring
topological space	$\mathscr{U}$-topological space
$\vdots$	$\vdots$

While this convention is in force we use it exclusively: that is, we *never* refer to conglomerates as "sets" in the usual fashion of ZFC. In this notation we have the class $\mathscr{U}$ of all sets, and the conglomerate $\mathcal{P}(\mathscr{U})$ of all classes, but there is no "collection" of all conglomerates.

- Any set is a class, and any class is a conglomerate.
- Call a class *proper* if it is not a set. Then a class is proper if and only if it is not contained in any other class (which agrees with the notion of proper class in NBG). If X, Y are classes then so are $X \cup Y, X \times Y$ and $X \cap Y$. If I is a set then any I-indexed union, product or intersection of classes is a class.
- By the Axiom of Comprehension, we can form the conglomerate of *all classes* with a certain property.
- By the Axiom of Choice, given a conglomerate of disjoint nonempty classes, we can produce a class which contains precisely one element from each of the classes. Of course the usual Axiom of Choice for sets applies: given a set of disjoint nonempty sets, there is a set containing precisely one elements from each of the sets.

Remark 3. Apparently UA is equivalent to the existence of inaccessible cardinals, and is therefore logically independent of ZFC. For details see [Bou72].

Now that we have described in detail our logical foundation, it remains to give the definition of a category. In Section 4.1 we give our approach to the definition of a category and show how this framework solves many of the set-theoretic issues that come up in the "daily life" of someone doing algebraic geometry. In Section 4.2 we outline the approach taken in SGA4, and explain why we have chosen to do things differently.

4.1 Categories

Return for a moment to the notation of ZFC. Firstly we define a category just as an algebraic object similar to a group or a ring (AC,Definition 1), with no mention of universes. We only need to introduce fancy set-theoretic ideas to satisfy our desire to talk about the category of "all sets", "all groups" or "all topological spaces". It is the choice of a grothendieck universe $\mathfrak{U}$ which furnishes us with such concepts, by allowing us to form the $\mathfrak{U}$-classes of all $\mathfrak{U}$-sets, $\mathfrak{U}$-groups and $\mathfrak{U}$-topological spaces.

An $\mathfrak{U}$-category (AC,Definition 6) is a category whose objects and morphisms form $\mathfrak{U}$-classes, and whose morphisms sets are $\mathfrak{U}$-sets. The categories of all $\mathfrak{U}$-sets, $\mathfrak{U}$-groups, $\mathfrak{U}$-topological spaces etc. are all $\mathfrak{U}$-categories. In fact, throughout our notes we not only fix the universe $\mathfrak{U}$ and adopt the CC$\mathfrak{U}$ as described above, we also agree that all categories are $\mathfrak{U}$-categories unless otherwise specified. This is the meaning that the word "category" has throughout the development of (AC,Section 1), and in most of our other notes. One should see Section 4.2 for a discussion of a way in which one naturally encounters categories which are not $\mathfrak{U}$-categories, and how to avoid this problem.

Let us now describe how ZFCU solves set-theoretic problems occurring naturally in algebraic geometry and category theory. Some examples are:

- Let $\mathcal{C}$ be a small category, $\mathcal{D}$ any category with binary products. Then the category of all covariant functors $\mathcal{C} \longrightarrow \mathcal{D}$ has binary products. Given functors F, G the construction works by *choosing*, for every $C \in \mathcal{C}$, a product $F(C) \times G(C)$ in $\mathcal{D}$. This requires the application of a powerful axiom of choice, not even available in standard NBG. In ZFCU there is no problem, since we have AC for conglomerates. For details see Lemma 1 below.

- Here are some other examples where AC for conglomerates is needed:

  (i) In proving an abelian category with enough projectives has a simultaneous assignment of projective resolutions to every object (even if we have available a canonical projective generator, if there are no canonical coproducts one must still simultaneously choose coproducts out of which to project).

  (ii) In proving basic facts about equivalences of categories.

  (iii) In describing "taking limits" as a functor on diagrams, which leads to the easy proof that limits preserve limits.

- There are situations where we need collections larger than the classes provided by NBG. For example, given a ringed space $(X, \mathcal{O}_X)$ an important invariant is the *Picard group* of all isomorphism classes of invertible sheaves on X under the tensor product. The class of all invertible sheaves on X is proper, so within NBG it is impossible to even talk about the "collection" of equivalence classes, since each equivalence class will be a proper class which can therefore not belong to any other class. In ZFCU there is no problem, since we have conglomerates.

Fix a universe $\mathscr{U}$ and adopt the conglomerate convention, so in particular all categories are $\mathfrak{U}$-categories. To be perfectly clear, we elaborate on the first point above.

Lemma 1. *Let $\mathcal{C}$ be a category with binary products, and denote the object and morphism classes by O, M respectively. Then there exists a function $f : O \times O \longrightarrow M \times M$ which maps a pair of objects (A, C) to a product (u, v).*

Proof. When we say the tuple (u, v) is a product, we mean the morphisms $u : D \longrightarrow A, u : D \longrightarrow C$ are a product in $\mathcal{C}$. Let Q be the conglomerate of pairs (u, v) in $M \times M$ which are a product in $\mathcal{C}$. In the usual way (using the Axiom of Replacement) we can define a function $g : O \times O \longrightarrow \mathcal{P}(Q)$ which maps a pair (C, D) to the conglomerate of all pairs (u, v) consisting of morphisms $u : D \longrightarrow A, u : D \longrightarrow C$ which are a product. The conglomerate $Im(g)$ is nonempty and all its elements are disjoint and nonempty, so we can apply the Axiom of Choice to produce a function $c : Im(g) \longrightarrow M \times M$ choosing a particular product. The composite $f = cg$ is the required function. □

Remark 4. To summarise, throughout our notes we work in the first order theory ZFCU and fix a universe $\mathscr{U}$ containing $\mathbb{Z}$. With respect to this universe we adopt the conglomerate notation convention, so that *set, class* and *conglomerate* have specific meaning. All categories are $\mathfrak{U}$-categories unless specified otherwise. If we wish to adopt the notation of ZFC ("calling a set a set") we will say so explicitly.

4.2 Categories in SGA4

Throughout this section we work in notation of ZFC. That is, the conglomerate convention is not in force and a category is an algebraic object as defined in (AC,Definition 1). In SGA4 grothendieck universes are defined as above, but the development of category theory proceeds along different lines. To avoid confusing notation, the categories called "$\mathfrak{U}$-categories" in SGA4 will here be referred to as "$\mathfrak{U}$-gategories". Here are the relevant definitions from SGA4. As before, all universes are assumed to be infinite.

Definition 6. Let $\mathfrak{U}$ be a universe. A set X is $\mathfrak{U}$-small if there is some $\mathfrak{U}$-set Y and a bijection $X \cong Y$. A group, ring, topological space etc. is $\mathfrak{U}$-small if its underlying set is $\mathfrak{U}$-small. A category is $\mathfrak{U}$-small if its object and morphisms sets O, M are $\mathfrak{U}$-small.

Definition 7. Let $\mathfrak{U}$ be a universe. A category $\mathcal{C}$ is a $\mathfrak{U}$-gategory if for every pair of objects X, Y the set $Hom_{\mathcal{C}}(X, Y)$ is $\mathfrak{U}$-small.

Remark 5. Observe that the set of objects is not required to be a $\mathfrak{U}$-class, or even to be bijective to a $\mathfrak{U}$-class. This is not so serious, since the set-theoretic arguments occurring in category theory rarely rely on smallness conditions on the class of objects. The important increase in generality is that instead of requiring the morphism sets to *be* $\mathfrak{U}$-sets, they only have to be bijective to a $\mathfrak{U}$-set. This might seem innocuous (after all, one of the basic lessons of category theory is that we shouldn't distinguish isomorphic objects) but it introduces irritating technical problems.

Remark 6. Given categories $\mathcal{C}, \mathcal{D}$ let $[\mathcal{C}, \mathcal{D}]$ denote the category of covariant functors $\mathcal{C} \longrightarrow \mathcal{D}$. Then

(i) If $\mathcal{C}, \mathcal{D}$ belong to the universe $\mathfrak{U}$ (resp. are $\mathfrak{U}$-small) then $[\mathcal{C}, \mathcal{D}]$ is an element of the universe (resp. is $\mathfrak{U}$-small).

(ii) If $\mathcal{C}$ is $\mathfrak{U}$-small and $\mathcal{D}$ is a $\mathfrak{U}$-gategory, then $[\mathcal{C}, \mathcal{D}]$ is a $\mathfrak{U}$-gategory.

Remark 7. Here is one way in which $\mathfrak{U}$-gategories arise naturally. Given an abelian category $\mathcal{A}$ we can form the derived category $D(\mathcal{A})$. One key technical point in this definition is the formation of a "category of fractions". Let $\mathcal{C}$ be a $\mathfrak{U}$-category and Σ a set of morphisms to "invert". We form a category $\mathcal{C}[\Sigma^{-1}]$ in which the morphisms of Σ become isomorphisms. For bad Σ the morphism sets of $\mathcal{C}[\Sigma^{-1}]$ are not even necessarily $\mathfrak{U}$-small. With enough hypothesis, we can show that the morphism sets of $\mathcal{C}[\Sigma^{-1}]$ are $\mathfrak{U}$-small, *but they are not necessarily $\mathfrak{U}$-sets*. So even if we start with $\mathfrak{U}$-category, we can end up with a $\mathfrak{U}$-gategory.

The philosophy expressed in SGA4 is that one should replace $\mathfrak{U}$-categories by $\mathfrak{U}$-gategories throughout. This has the advantage that our "categories" are closed under constructions like categories of fractions. It also has a serious technical disadvantage. Let $\mathfrak{U}$-**Sets** be the $\mathfrak{U}$-category of all $\mathfrak{U}$-sets. If $\mathcal{C}$ is a $\mathfrak{U}$-category then every object X gives rise to a covariant functor $Hom_{\mathcal{C}}(X, -)$:

$\mathcal{C} \longrightarrow \mathfrak{U}$-**Sets** which is widely used. But if $\mathcal{C}$ is a $\mathfrak{U}$-gategory with objects X, Y then $Hom_{\mathcal{C}}(X, Y)$ does not necessarily belong to $\mathfrak{U}$-**Sets**, so we cannot define $Hom_{\mathcal{C}}(X, -)$ in the naive way. Instead we have to *introduce a new axiom* UB to ZFCU and and replace the set $Hom_{\mathcal{C}}(X, Y)$ by a certain $\mathfrak{U}$-set (see SGA4.I Construction-définition 1.3 for details).

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