Representability of cohomology theories

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Cohomology

Space = pointed simplicial (or cell) complex.

Definition

A reduced cohomology theory H is:

$$\text{Topology} \to \text{Algebra}$$

$$\text{space } X \mapsto H^n(X) \text{ abelian group}, \quad n \in \mathbb{Z},$$

$$\text{map } X \xrightarrow{f} Y \mapsto H^n(X) \xleftarrow{H^n(f)} H^n(Y) \text{ homomorphism},$$

$$\text{homotopic } f \simeq g\colon X \to Y \mapsto H^n(f) = H^n(g) \text{ equal},$$

$$\text{base point union } \coprod_{i \in I} X_i \mapsto \prod_{i \in I} H^n(X_i) \text{ product},$$

$$\text{cofiber sequence} \qquad \text{exact sequence}$$

$$X \to Y \to C_f \to \Sigma X \mapsto H^n(X) \leftarrow H^n(Y) \leftarrow H^n(C_f) \leftarrow H^n(\Sigma X),$$

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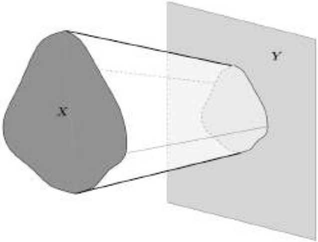

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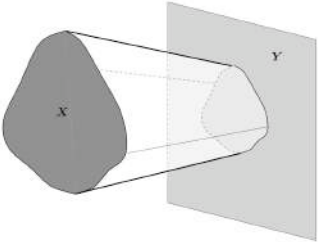
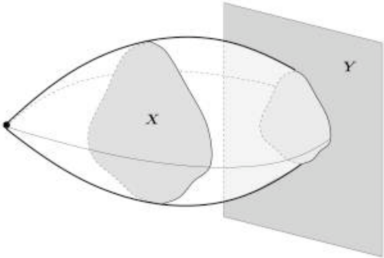


mapping cone

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{q} \Sigma X$$

inclusion



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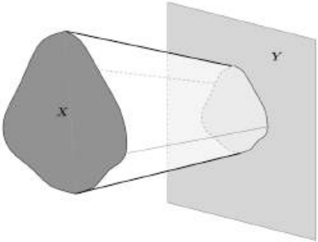

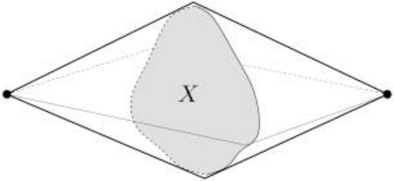

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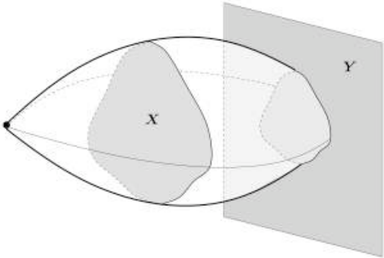
$$X \xrightarrow{f} Y \xrightarrow[\text{inclusion}]{i} C_f \xrightarrow[\text{collapse}]{q} \Sigma X \quad \text{suspension}$$



Examples of cohomologies

Example

- *Singular cohomology* $H^*(X,\mathbb{Z})$, *defined on all spaces,*

$$H^n(discrete) = 0, \ \ for\ n \neq 0,$$
$$H^0(X,\mathbb{Z}) = pointed\ maps\ X \to \mathbb{Z}.$$

▸ course

- *Complex K-theory* $K^*(X)$, X *compact,*

$$K^0(X) = stable\ isomorphism\ classes\ of\ \mathbb{C}\text{-}vector\ bundles/X,$$
$$K^n(X) \cong K^{n+2}(X),\ \ n \in \mathbb{Z},\ \ Bott\ periodicity,$$
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A *spectrum* $E = \{E_0, E_1, \ldots, E_n, \ldots\}$ *is a sequence of spaces together with* *bonding maps*,

$$\Sigma E_n \longrightarrow E_{n+1}, \quad n \geq 0.$$

An *Ω-spectrum* *is a spectrum E where the adjoints of the bonding maps $E_n \to \Omega E_{n+1}$ are homotopy equivalences.*

Example

Any space X defines a spectrum $\Sigma^\infty X = \{X, \Sigma X, \ldots, \Sigma^n X, \ldots\}$, which is not an Ω-spectrum.

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Spectra and cohomology

Theorem (G. W. Whitehead'62)

A spectrum E represents a cohomology theory H defined on compact spaces by

$$H^n(X) = \lim_{k\to\infty}[\Sigma^{k-n}X, E_k], \quad n \in \mathbb{Z},$$

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If E is an Ω-spectrum then E represents a cohomology theory H defined on all spaces by the formula above.

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Theorem (E. H. Brown'63)

Any cohomology theory defined on all spaces is represented by a spectrum.

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The following corollary can be applied to complex K-theory.

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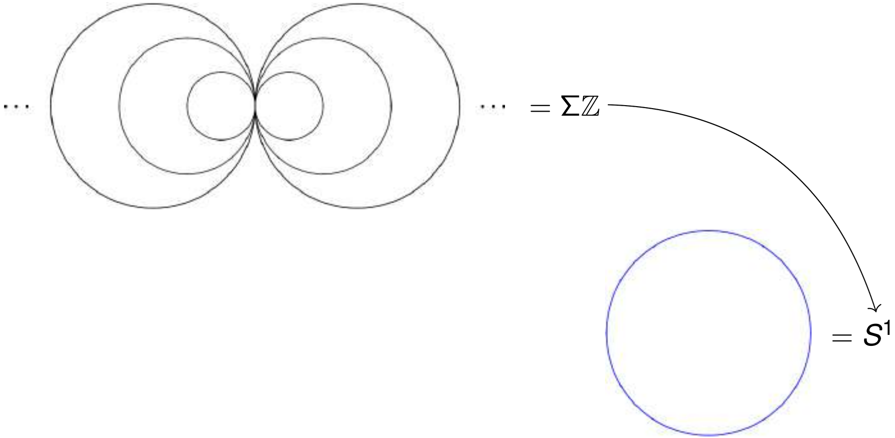

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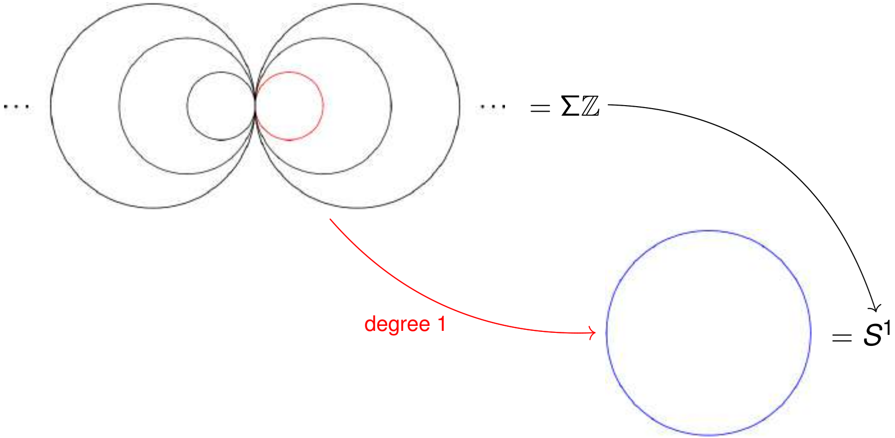

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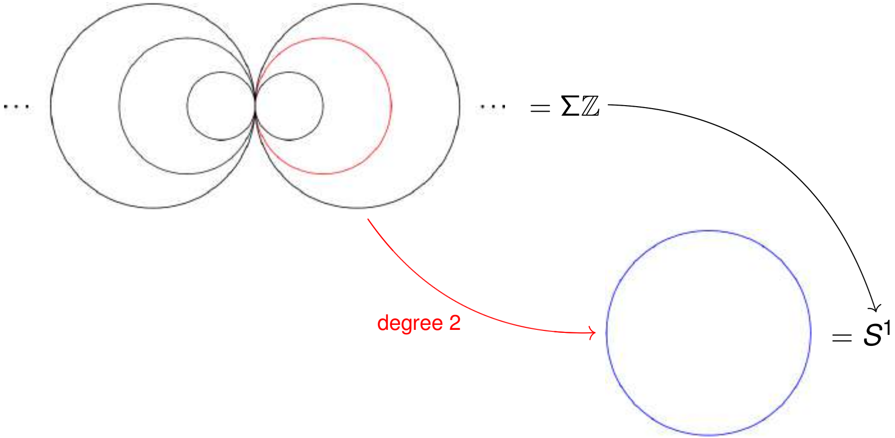

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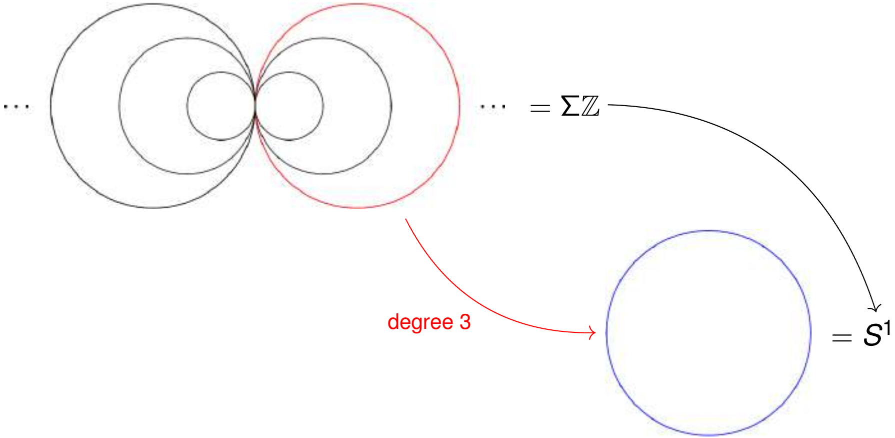

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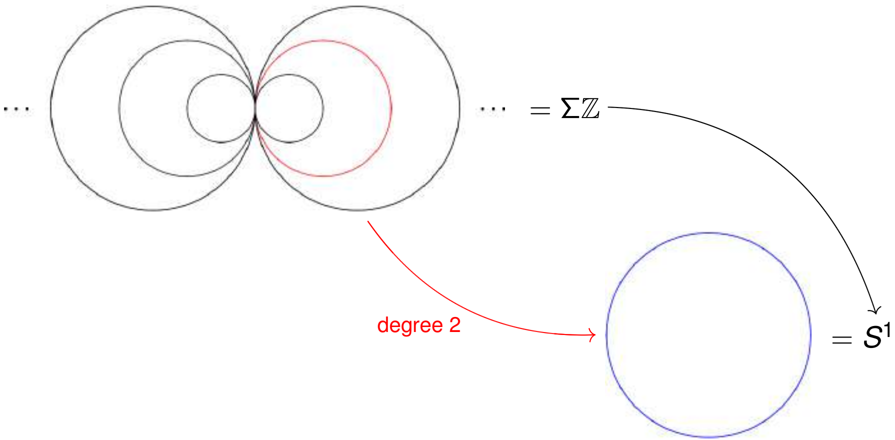

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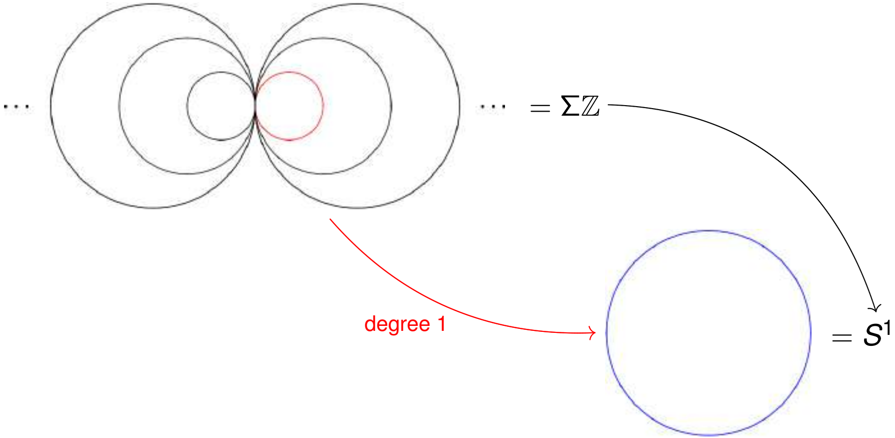

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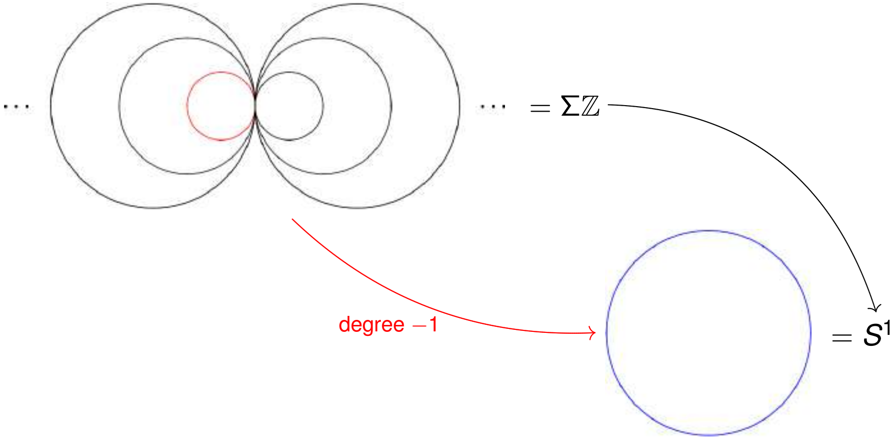

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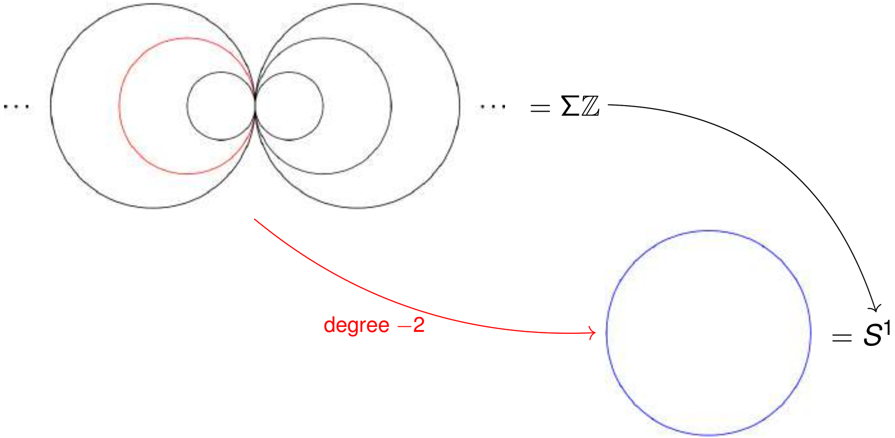

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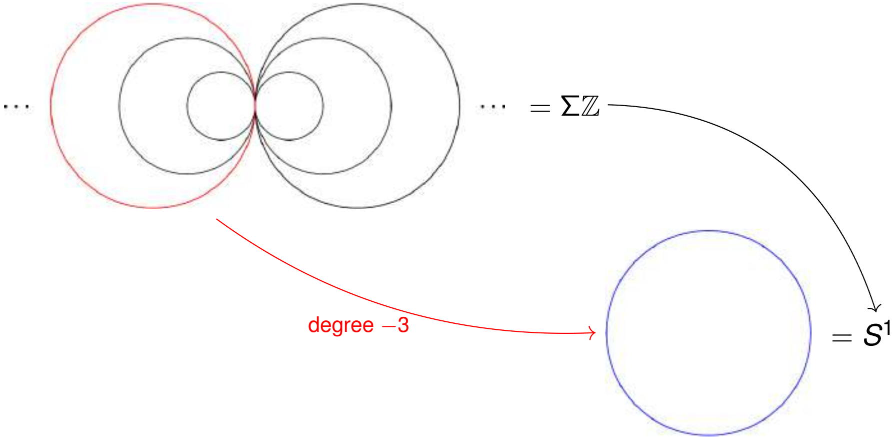

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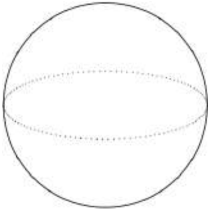
$\Sigma S^1 = S^2 =$  $= \mathbb{C}P^1 \xrightarrow{\text{inclusion}} \mathbb{C}P^\infty$

Stable homotopy

Recall that cohomology regards suspension as an invertible functor:

$$H^n(X) \cong H^{n+1}(\Sigma X).$$

Definition

The compact stable homotopy category $\mathbf{SH}^c$:

- *Objects:* (X, n), X *compact space*, $n \in \mathbb{Z}$,

$$(X, n) \sim \Sigma^n X.$$

- *Morphisms:* $\mathrm{Hom}((X, n), (Y, m)) = \lim_{k \to \infty} [\Sigma^{k+n} X, \Sigma^{k+m} Y]$.
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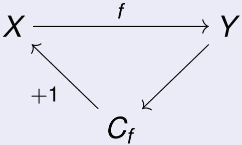

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Homotopy and derived categories

Example

Let R be a ring.

- *The homotopy category* $\mathbf{K}(R)$, *objects are complexes of R-modules,*

$$C = \cdots \to C_{n+1} \xrightarrow{d_C} C_n \xrightarrow{d_C} C_{n-1} \to \cdots, \quad d_C^2 = 0,$$

$$(\Sigma C)_n = C_{n-1}, \quad d_{\Sigma C} = -d_C,$$

morphisms are chain homotopy classes of maps, and exact triangles come from cofiber sequences of complexes.

- *The derived category* $\mathbf{D}(R) \subset \mathbf{K}(R)$ *is the full subcategory spanned by 'injective resolutions' of complexes.*
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A triangulated category $\mathbf{T}$ *is cocomplete if it has all coproducts,*

$$\coprod_{i \in I} X_i, \quad I \text{ a set.}$$

An object Y *in* $\mathbf{T}$ *is compact if any map*

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- $\mathbf{D}(\mathbf{A})$ *and* $\mathbf{K}(\mathbf{A})$ *are cocomplete.*
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- *The compact stable homotopy category* $\mathbf{SH}^c$ *admits a cocompletion* $\mathbf{SH}$ *called the full stable homotopy category, whose objects are spectra.*
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A cocomplete triangulated category $\mathbf{T}$ is compactly generated if any non-trivial object X in $\mathbf{T}$ admits a non-trivial map $Y \to X$ from a compact object Y in $\mathbf{T}^c$.

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This theorem applies to $\mathbf{SH}$, but it also applies to $\mathbf{D}(\mathbb{Z})$.

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Let R be a 'polynomial ring with a proper class of indeterminates', the subcategory of acyclic complexes in $\mathbf{K}(R)$ does not satisfy Brown representability.

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A homological criterion for Adams representability

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A right $\mathbf{T}^c$*-module* *A is an additive functor* $A\colon (\mathbf{T}^c)^{op} \to \mathbf{Ab}$. *The category* $\mathbf{Mod}(\mathbf{T}^c)$ *of right* $\mathbf{T}^c$*-modules is a Grothendieck abelian category with a set of small projective generators.*

Theorem (Neeman'97, Beligiannis'00)

If $\mathbf{T}$ *is compactly generated and* $H\colon (\mathbf{T}^c)^{op} \to \mathbf{Ab}$ *is a cohomology theory then:*

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For any ring R,

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Remark (Baer–Brune–Lenzing'82)

If R is a finite-dimensional hereditary algebra over an uncountable algebraically closed field, then $\mathbf{D}(R)$ satisfies the Adams representability theorem $\Leftrightarrow$ has finite representation type.

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$$\sup_{\substack{H\colon (\mathbf{D}(R)^c)^{op}\to\mathbf{Ab}\\ \text{cohomology}}} \operatorname{proj\,dim} H \geq \operatorname{pure\,proj\,dim} \mathbf{Mod}(R).$$

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If R is a finite-dimensional hereditary algebra over an uncountable algebraically closed field, then $\mathbf{D}(R)$ satisfies the Adams representability theorem $\Leftrightarrow$ has finite representation type.

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Let $\mathbf{T}$ be a cocomplete triangulated category. For any regular cardinal α there is a notion of α-compact object. These objects span a full triangulated subcategory $\mathbf{T}^\alpha \subset \mathbf{T}$. For $\alpha = \aleph_0$ this coincides with classical compactness.

If Y is in $\mathbf{T}^\alpha$ then any map

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Well generated categories

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A cocomplete triangulated category $\mathbf{T}$ *is* α*-compactly generated if any non-trivial object* X *in* $\mathbf{T}$ *admits a non-trivial map* $Y \to X$ *from an* α*-compact object* Y *in* $\mathbf{T}^\alpha$*. A cocomplete triangulated category is well generated if it is* α*-compactly generated for some regular cardinal* α*.*

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- *All compactly generated triangulated categories are well generated.*
- $\mathbf{D}(\text{sheaves}/M)$ *is well generated, actually* $\aleph_1$*-compactly generated.*
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Representability in well generated categories

Theorem (Brown representability, Neeman'01)

Any cohomology theory $H\colon \mathbf{T}^{op} \to \mathbf{Ab}$ on a well generated triangulated category $\mathbf{T}$ is representable.

Definition

A cohomology theory H for α-compact objects is an additive functor

$$H\colon (\mathbf{T}^{\alpha})^{op} \longrightarrow \mathbf{Ab}$$

taking exact triangles to exact sequences and

$$H(\coprod_{i\in I} X_i) \cong \prod_{i\in I} H(X_i), \quad \text{card } I < \alpha.$$

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What about α-Adams representability?

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Conjecture (Rosický'05, Neeman'09)

T *is a well generated triangulated category* $\Leftrightarrow$ *there exists a regular cardinal* α *such that the* α*-Adams representability theorem holds:*

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Proposition (The 'easy' part, Rosicky'09)

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Almost nothing is known about $\Rightarrow$ for $\alpha > \aleph_0$.

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A homological criterion for Adams representability

Definition

A right continuous $\mathbf{T}^\alpha$-module A is an additive functor $A\colon (\mathbf{T}^\alpha)^{op} \to \mathbf{Ab}$ with

$$A(\coprod_{i\in I} X_i) \cong \prod_{i\in I} A(X_i), \quad \text{card } I < \alpha.$$

The category $\mathbf{Mod}_\alpha(\mathbf{T}^\alpha)$ of right continuous $\mathbf{T}^\alpha$-modules is an abelian category but not Grothendienck.

Theorem (M-Raventós'09)

If $\mathbf{T}$ is α-compactly generated and $H\colon (\mathbf{T}^\alpha)^{op} \to \mathbf{Ab}$ is a cohomology theory then:

- proj dim $H \leq 2 \Rightarrow$ *H is representable.*
- proj dim $H \leq 1$ *for all H* $\Leftrightarrow$ *α-Adams representability theorem for $\mathbf{T}$.*

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α-purity is the relative homological algebra in **Mod**(R) obtained by regarding all R-modules with $< \alpha$ generators and relations as projectives.

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For any ring R,

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If R is a finite-dimensional wild hereditary k-algebra, card $k \geq \aleph_\omega$, then the α-Adams representability theorem is false for all $\alpha < \aleph_\omega$.

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If the conjecture is true, then for any ring R there exists a cardinal α such that for any R-module A there is a short exact sequence,

$$0 \to \bigoplus_{i \in I} P_i \longrightarrow \bigoplus_{j \in J} Q_j \longrightarrow A \to 0,$$

such that the modules P_i and Q_j have $< \alpha$ generators and relations.

Moreover, if B is an R-module with $< \alpha$ generators and relations, then

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Under the continuum hypothesis, the following theorem can be applied to $\mathbf{D}(\text{sheaves}/M)$.

Theorem ($\aleph_1$-Adams representability, M-Raventós'09)

Let $\mathbf{T}$ be an $\aleph_1$-generated triangulated category with $\operatorname{card} \mathbf{T}^{\aleph_1} \leq \aleph_1$. Then:

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This is the only available non-trivial result on α-Adams representability for $\alpha > \aleph_0$.

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Representability of cohomology theories

Fernando Muro

Universidad de Sevilla
Departamento de Álgebra

Joint Mathematical Conference CSASC 2010
Prague, January 2010

Cofiber sequence of complexes

$$
\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Y \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
X_{n+1} & \xrightarrow{\quad f \quad} & Y_{n+1} \\
\downarrow{\scriptstyle d_X} & & \downarrow{\scriptstyle d_Y} \\
X_n & \xrightarrow{\quad f \quad} & Y_n \\
\downarrow{\scriptstyle d_X} & & \downarrow{\scriptstyle d_Y} \\
X_{n-1} & \xrightarrow{\quad f \quad} & Y_{n-1} \\
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\vdots & & \vdots
\end{array}
$$

Cofiber sequence of complexes

$$
\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \qquad & C_f \\
 & & & & \text{mapping cone} \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
X_{n+1} & \xrightarrow{f} & Y_{n+1} & & Y_{n+1} \oplus X_n \\
\downarrow{\scriptstyle d_X} & & \downarrow{\scriptstyle d_Y} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} d_Y & f \\ 0 & -d_X \end{smallmatrix}\right) = d_{C_f}} \\
X_n & \xrightarrow{f} & Y_n & & Y_n \oplus X_{n-1} \\
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X_{n-1} & \xrightarrow{f} & Y_{n-1} & & Y_{n-1} \oplus X_{n-2} \\
\downarrow & & \downarrow & & \downarrow \\
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\end{array}
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X_{n+1} & \xrightarrow{\ f\ } & Y_{n+1} & \xrightarrow{\text{inclusion}} & Y_{n+1} \oplus X_n \\
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\begin{array}{ccccccc}
X & \xrightarrow{\;f\;} & Y & \xrightarrow{\;i\;} & C_f & & \Sigma X \quad \text{suspension} \\
 & & & & \text{mapping cone} & & \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
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$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i} & C_f & \xrightarrow{q} & \Sigma X \quad \color{red}{\text{suspension}} \\
 & & & & \color{blue}{\text{mapping cone}} & & \\
\vdots & & \vdots & & \vdots & & \vdots \\
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\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

back

Crash course on singular cohomology

Compute the cohomology of spheres,

$$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1\};$$

$$S^0 = \{\pm 1\} = \bullet \quad \bullet\,, \quad H^0(S^0, \mathbb{Z}) = \text{pointed maps } S^0 \to \mathbb{Z} \cong \mathbb{Z};$$

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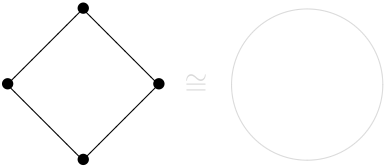
$\Sigma S^0 =$  $= S^1,$

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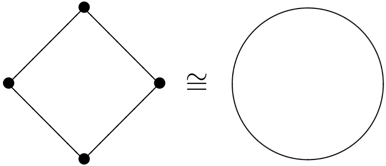

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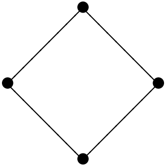
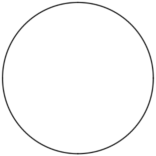
$\Sigma S^0 =$  $\cong$  $= S^1$,

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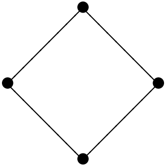
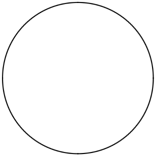
$\Sigma S^0 =$  $\cong$  $= S^1,$

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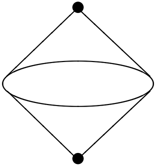
$\Sigma S^1 =$  $\cong$ $= S^2,$

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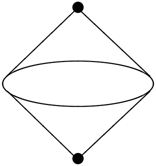
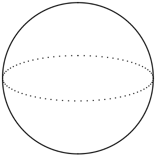
$\Sigma S^1 =$  $\cong$  $= S^2$,

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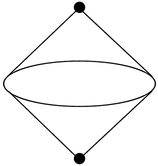
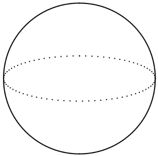
$\Sigma S^1 =$  $\cong$  $= S^2,$

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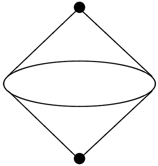
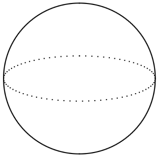
$\Sigma S^1 =$  $\cong$  $= S^2,$

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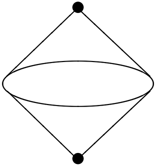
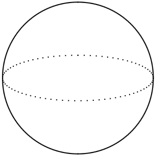
$\Sigma S^1 =$  $\cong$  $= S^2,$

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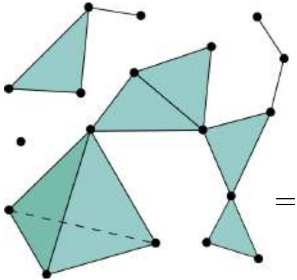
Let C_f be a simplicial complex,



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Crash course on singular cohomology

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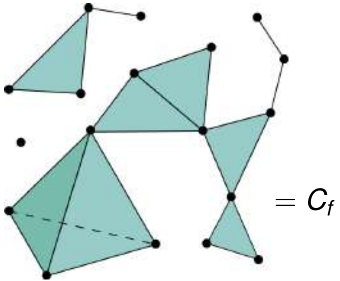
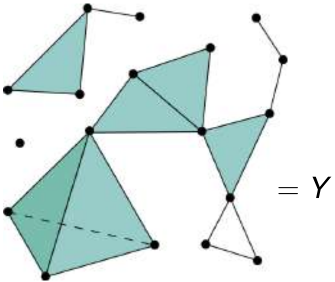


Any simplex is a cone over its boundary, therefore C_f can be obtained from Y,



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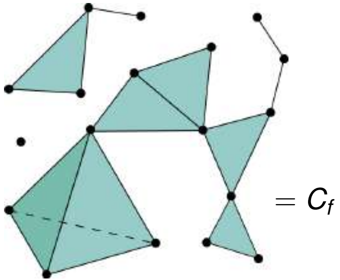

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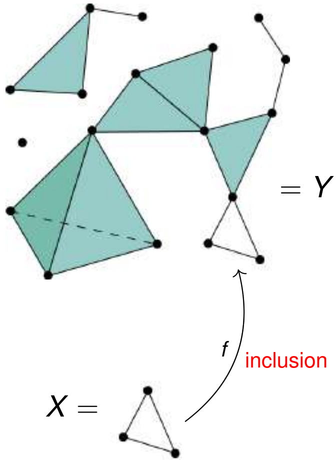

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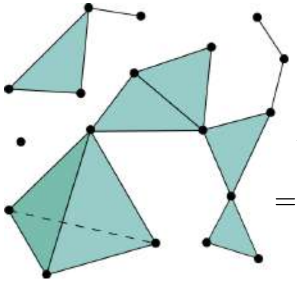

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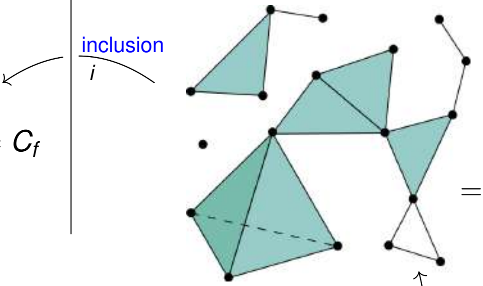

$= Y$

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$$S^1 = X \xrightarrow[\text{inclusion}]{f} Y \xrightarrow[\text{inclusion}]{i} C_f \xrightarrow{q} \Sigma X = S^2.$$

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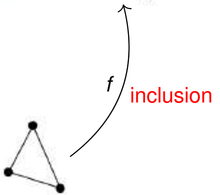

Crash course on singular cohomology

Given a cofiber sequence

$$S^n \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{q} S^{n+1};$$

$$H^m(Y,\mathbb{Z}) = H^m(C_f,\mathbb{Z}), \text{ if } m \neq n, n+1;$$

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