Towards a Solution of Large N Double-Scaled SYK

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Based on work with M. Berkooz, M. Isachenkov, and G. Torrents, arXiv:1811.02584.

The Sachdev-Ye-Kitaev (SYK) model

SYK is a quantum mechanical model ($0+1$ dimensions) involving N Majorana fermion ψ_i, $i = 1, \cdots, N$, with random all-to-all interactons

$$H = i^{p/2} \sum_{1 \leq i_1 < \cdots < i_p \leq N} J_{i_1 \cdots i_p} \psi_{i_1} \cdots \psi_{i_p}$$

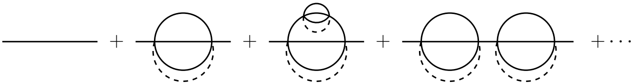
The fermions satisfy the algebra

$$\{\psi_i, \psi_j\} = 2\delta_{ij}$$

Interesting because can do calculations at large N and strong coupling, and find that it is maximally chaotic (chaos bounded [Maldacena, Shenker, and Stanford, 2016]). Features of holographic theories with semi-classical duals, so can study holography.

The SYK model

Usually studied using summation of Feynman diagrams, leading to Schwinger-Dyson equations. In the IR, a conformal ansatz is plugged in and solved ([Polchinski and Rosenhaus, 2016], [Maldacena and Stanford, 2016]).



$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega), \qquad \Sigma(\tau) = J^2 G(\tau)^{p-1}$$

Goal: We will take a combinatorial approach, allowing to do exact computations (at all energy scales).

Double-Scaled SYK

Usually p is held fixed (independent of N) and $N \to \infty$.

Double-scaled SYK $=$ we take p (even) to scale as $\sqrt{N}$:

$$N \to \infty, \qquad \lambda = \frac{2p^2}{N} = \text{fixed} \tag{1}$$

[Erdős and Schröder, 2014], [Cotler et al., 2017], [Berkooz, Narayan, and Simon, 2018]
Will be natural to denote

$$q \equiv e^{-\lambda}$$

The more standard SYK $\leftrightarrow$ $q \to 1$

The J's are independent and Gaussian (actually enough to assume they are independent, have zero mean, unifromly bounded moments) with

$$\langle J^2_{i_1 \cdots i_p} \rangle_J = \binom{N}{p}^{-1}$$

In the double-scaling (1), this differs by a factor of λ from usual convention [Maldacena and Stanford, 2016].

Chord diagrams

Chord diagrams for the partition function

Consider moments from which can get immediately the (averaged) partition function

$$\langle \operatorname{tr} H^k \rangle_J$$

Denote $\{i_1, \cdots, i_p\} \leftrightarrow I$, so

$$H = i^{p/2} \sum_I J_I \cdot \psi_I$$

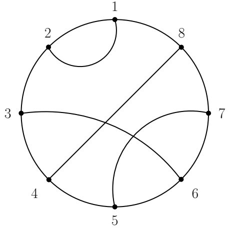
where $\psi_I = \psi_{i_1} \cdots \psi_{i_p}$.

$$\langle \operatorname{tr} H^k \rangle_J = i^{kp/2} \sum_{I_1, \cdots, I_k} \underbrace{\langle J_{I_1} \cdots J_{I_k} \rangle_J}\, \operatorname{tr} \psi_{I_1} \cdots \psi_{I_k}.$$

By Wick's theorem, the I_j come in pairs.

Chord diagrams for the partition function

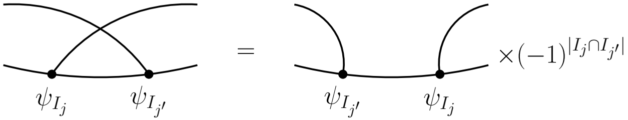
Wick's theorem $\rightarrow$ sum over pairings $\Leftrightarrow$ sum over chord diagrams (circular since trace). Each node $\leftrightarrow$ H insertion.



For each chord diagram left with

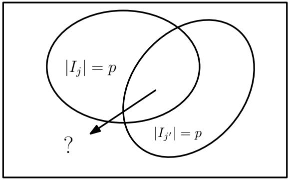
$$\binom{N}{p}^{-k/2} i^{kp/2} \sum_{I_1,\cdots,I_{k/2}} \operatorname{tr} \psi_{I_1} \cdots \psi_{I_1} \cdots$$

Now commute nodes to bring all pairs to be neighboring:



Chord diagrams for the partition function

The $\binom{N}{p}^{-k/2}$ factor (number of terms in the sum) turns counting to probabilities.



For $p \ll N$ can do this by choosing independently the p points of $I_{j'}$ (p trials, in intersection with probability p/N).

$$|I_j \cap I_{j'}| \sim Pois\left(\frac{p^2}{N}\right)$$

Since $p^2/N \sim O(1)$, the different intersections are independent.

Chord diagrams for the partition function

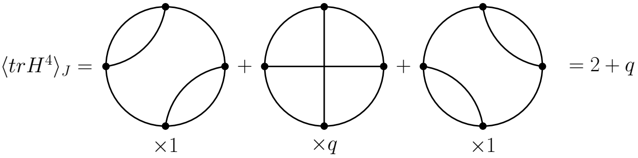
Each intersection then gives $(n = |I_j \cap I_{j'}|)$

$$\sum_{n=0}^{\infty} \left(\frac{(p^2/N)^n}{n!} e^{-p^2/N} \right) (-1)^n = e^{-\lambda} = q$$

Using $i^{kp/2} \operatorname{tr} \psi_{I_1} \psi_{I_1} \psi_{I_2} \psi_{I_2} \cdots = 1$ we get

$$\langle \operatorname{tr} H^k \rangle_J = \sum_{\text{Chord diagrams}} q^{\#\text{ intersections}}$$

For example



Operators

Similarly to the Hamiltonian, consider random operators with different $p_A \sim \sqrt{N}$

$$M_A = i^{p_A/2} \sum_{1 \leq i_1 < \cdots < i_{p_A} \leq N} J^{(A)}_{i_1 \cdots i_{p_A}} \psi_{i_1} \cdots \psi_{i_{p_A}}$$

A - flavor. The J's are again random, independent, with zero mean and

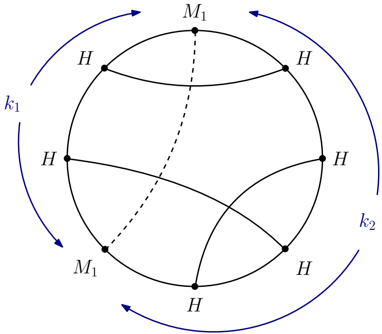
$$\langle J^{(A)}_{i_1 \cdots i_{p_A}} J^{(B)}_{j_1 \cdots j_{p_B}} \rangle_J = \binom{N}{p_A}^{-1} \delta^{AB} \delta_{i_1, j_1} \delta_{i_2, j_2} \cdots$$

(and independent of the Hamiltonian couplings).

Correlation function moments

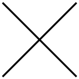
$$\langle \mathrm{tr}\, H^{k_1} M_1 H^{k_2} M_1 \cdots \rangle_J$$

From the averaging, the Hamiltonian insertions are paired, the M_1 insertions are paired.

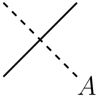


Correlation function moments

The only difference is the probability distribution of the number of sites in the intersection. For sets of size p, p_A the intersection is distributed $Pois\left(\frac{pp_A}{N}\right)$. So



$$= q = e^{-2p^2/N}$$



$$= \tilde{q}_A = e^{-2pp_A/N}$$

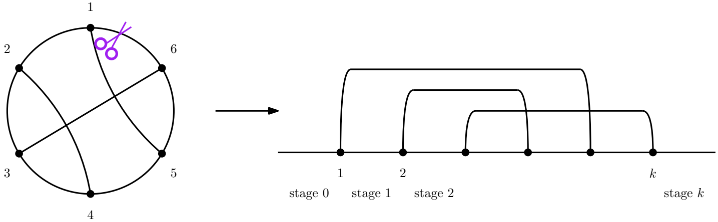
$$\langle \mathrm{tr}\, H^{k_1} M_1 H^{k_2} M_1 \cdots \rangle_J =$$

$$= \sum_{\text{Chord diagrams}} q^{\#\ H\text{-}H \text{ intersections}} \prod_A \tilde{q}_A^{\#\ H\text{-}M_A \text{ intersections}}$$

Effective Hilbert space and analytic evaluation

Partition function

Want to evaluate the sum over chord diagrams [Berkooz, Narayan, and Simon, 2018]. Cut open the chord diagrams at an arbitrary point.



Recall each node is a Hamiltonian insertion, and between each two insertions there is a propagating state $\cdots HH\cdots = \cdots H|l\rangle\langle l|H\cdots$.

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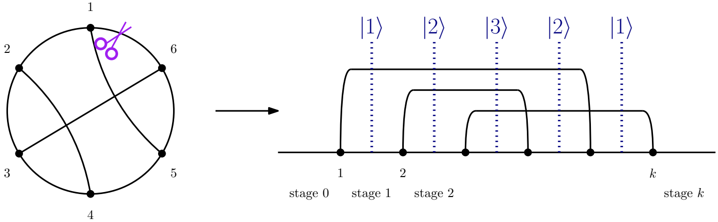
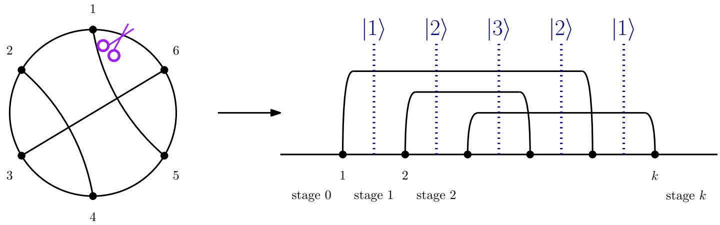


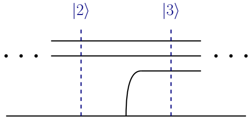
Recall each node is a Hamiltonian insertion, and between each two insertions there is a propagating state $\cdots HH\cdots = \cdots H|l\rangle\langle l|H\cdots$. Effective Hilbert space $\mathcal{H}$ with basis $|l\rangle$, number of chords $l = 0, 1, 2, \cdots$.

Partition function

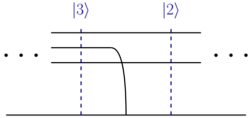


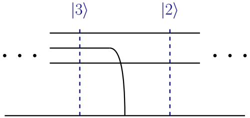
Node ↔ Hamiltonian insertion. 2 transitions only:

$|l\rangle \to |l+1\rangle$



$|l\rangle \to |l-1\rangle$





In the latter case can close either of the l chords. Crossings $\to$ $1+q+q^2+\cdots q^{l-1} = \frac{1-q^l}{1-q}$. Effective Hamiltonian

$$T = \begin{pmatrix} 0 & \frac{1-q}{1-q} & 0 & 0 & \cdots \\ 1 & 0 & \frac{1-q^2}{1-q} & 0 & \cdots \\ 0 & 1 & 0 & \frac{1-q^3}{1-q} & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Can simply diagonalize and get the energies

$$E(\theta) = \frac{2\cos(\theta)}{\sqrt{1-q}}, \qquad \theta \in [0, \pi).$$

Partition function is just

$$\langle \mathrm{tr}\, e^{-\beta H} \rangle_J = \int_0^\pi d\mu(\theta)\, e^{-\beta E(\theta)}$$

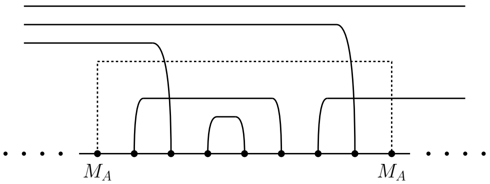
The measure is

$$d\mu(\theta) \equiv \frac{d\theta}{2\pi}(q;q)_\infty (e^{2i\theta};q)_\infty (e^{-2i\theta};q)_\infty, \text{ where } (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k).$$

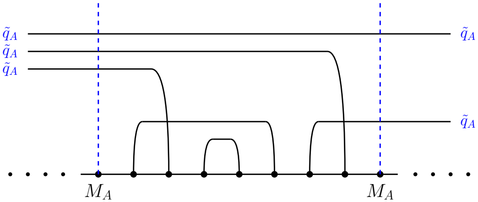
Correlation functions

Correlation functions

Consider a region enclosed by a contracted pair of M-nodes.



Time evolution over this region (before was T^k)? In the Hilbert space we keep only number of solid chords, can we do that? Yes!



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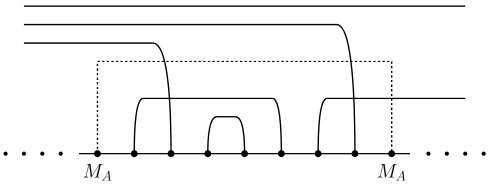

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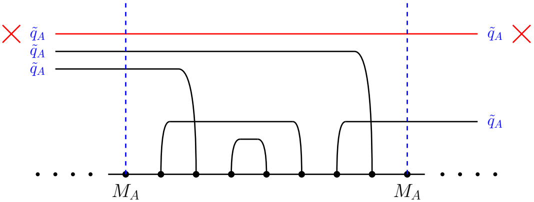
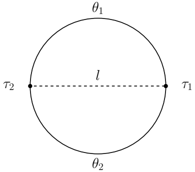
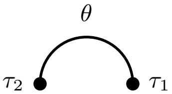


Diagrammatic rules

Similarly to [Mertens, Turiaci, and Verlinde, 2017], it is convenient to organize the results for correlation functions using non-perturbative diagrammatic rules. The diagrams arise naturally here; these are just chord diagrams.

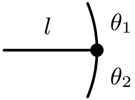


- Propagator



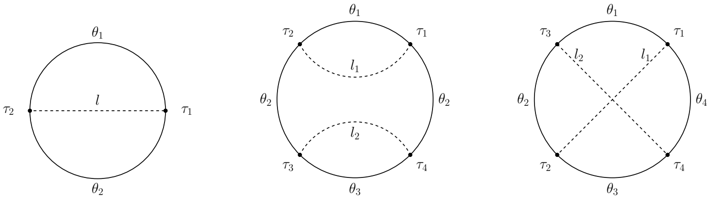
$$= e^{-\Delta\tau \cdot E(\theta)}$$

- Sum over energy eigenstates that propagate, or equivalently over θ, with measure $d\mu(\theta) = \frac{d\theta}{2\pi}(q, e^{\pm 2i\theta}; q)_\infty$.
- Vertex



$$= \gamma_l(\theta_1, \theta_2) = \sqrt{\frac{(\tilde{q}_A^2; q)_\infty}{(\tilde{q}_A e^{i(\pm\theta_1 \pm\theta_2)}; q)_\infty}}, \qquad \tilde{q}_A = q^{l_A}$$

Diagrammatic rules

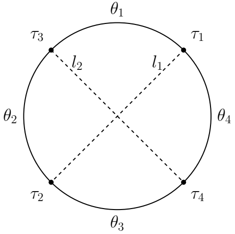


These rules give precisely the 2-point function and the first 4-point function $\langle M_1 M_1 M_2 M_2 \rangle$.

$$\langle M_1 M_2 M_1 M_2 \rangle = \int \prod_{j=1}^{4} d\mu(\theta_j) e^{-\sum \beta_j E(\theta_j)} \gamma_{l_1}(\theta_1, \theta_4) \gamma_{l_1}(\theta_2, \theta_3) \gamma_{l_2}(\theta_1, \theta_2) \gamma_{l_2}(\theta_3, \theta_4) \cdot R$$

So R is associated to the crossing of chords.

The R-matrix



The chord diagram is reminiscent of holography, representing the hyperbolic disc, boundary is exactly our QM system. The chords intersection is scattering in the bulk – the R-matrix.

For the Schwarzian, the R-matrix is the $6j$ symbol of $SU(1,1)$.

In double-scaled SYK, the R-matrix is the $6j$ symbol of the quantum group $U_q(su(1,1))$!

The spectrum also matches to this quantum group. Suggests that the theory can be **completely solved** by symmetry considerations.

More on the results

- In $q \to 1$ and low energies, these results reduce exactly to those of the Schwarzian. But the results above are at all energies and for any q.

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- Using saddle point for the 4-point function [Lam et al., 2018], calculated the Lyapunov exponent for small λ and low energies $T \ll \sqrt{\lambda}$

$$\lambda_L = 2\pi T - 4\pi\lambda^{-1/2}T^2 + \cdots$$

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$$\lambda_L = 2\pi T - 4\pi\lambda^{-1/2}T^2 + \cdots$$

- The analysis did not use the trace, so holds trivially also for pure states in agreement with [Kourkoulou and Maldacena, 2017].

Summary and future directions

Calculated exact correlation functions, including the 4-point function, in large N double-scaled SYK and saw an emerging quantum group.

- Solving the model by $U_q(su(1,1))$ symmetry considerations.
- Computing chaos for large λ and temperature.
- The leading order in N is basically completely solved. Suggests we can go to subleading in N.
- Bulk dual.
- Non-thermal mixed states.

Thank you!