An introduction to universal central extensions of Lie superalgebras

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Introduction

Central extensions play an important role in the theory of Lie algebras, and it is therefore not surprising that there are many results on central extensions of various classes of Lie algebras, too many for any meaningful survey. Recently, several authors have considered central extensions of Lie superalgebras. Scheunert and Zhang [**24**] develop a theory of central extensions of arbitrary Lie superalgebras over a field, while others have looked at central extensions of specific classes of Lie superalgebras: Iohara and Koga [**14**] (basic classical Lie superalgebras extended by a commutative associative algebra), Mikhalev and Pinchuk [**18**] (Lie superalgebras $\mathfrak{sl}(m,n;A)$ for A an associative algebra) and Duff [**9**, Ch.V] (orthosymplectic Lie superalgebras over commutative superrings). See also 1.17 for more hints to the literature.

The aim of this paper is to provide an introduction to the theory of universal central extensions of Lie superalgebras, with emphasis on the general part of the theory, rather than on specific examples. We have tried to state the theory in what seemed to us the appropriate generality, in particular since this could be done without any extra "cost". Thus, we consider Lie superalgebras over a commutative superring, i.e., allow scalars of even or odd parity. In a theory where vectors may have even or odd parity, it seems to be more natural to allow scalars of both parity. Another instance is the definition of a universal central extension itself, which some authors require to be a perfect although this follows from the universal mapping property.

In §1 we first present all the necessary definitions, 1.1 – 1.3. We then develop the general theory of universal extensions. In particular, we give two characterizations of universal central extensions in Th. 1.8 (being simply connected respectively centrally closed) and show that a Lie superalgebra has a universal central extension if and only if it is perfect (Th. 1.8 and Th. 1.14). That perfectness is sufficient for the existence of a universal central extension is a result whose proof we have postponed as long as possible, so that the reader can see how far the theory can be developed by just using the universal property. We prove the existence of a universal central extension in Th. 1.14 by providing a concrete model. Our model is the super version of the universal central extension of a perfect Lie

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algebra, due to van der Kallen [**28**]. This construction provides a Lie superalgebra $\mathfrak{uce}(L)$ for *any* Lie superalgebra L, and the assignment $L \to \mathfrak{uce}(L)$ is a covariant functor $\mathfrak{uce}$ on the category of Lie superalgebras. Other models are discussed in 1.17.

The emphasis on the functor $\mathfrak{uce}$ is one of the novelties of our presentation. It makes it easy to see that automorphisms and derivations lift to central extensions. This is shown in §2. As an application we revisit a theme of Benkart and Moody [**2**]: what is the central extension of a semidirect product $L = K \rtimes M$ of two Lie (super)algebras K and M? In Th. 2.7 we describe $\mathfrak{uce}(K \rtimes M)$ and show that $\mathfrak{uce}(K \rtimes M) = \mathfrak{uce}(K) \rtimes \mathfrak{uce}(M)$ if and only if $\mathfrak{uce}(M)$ operates trivially on the second homology $\mathrm{H}_2(K)$ of K. This result clarifies [**2**, Th. 3.8].

Most of the results in the paper are known for Lie algebras, although some of the published proofs are different. Some hints to the literature are provided in the notes 1.17 and 2.8 at the end of §1 and §2 respectively. These notes should not be considered as a complete overview of the theory of central extensions.

1. Universal central extensions of Lie superalgebras: Some general results

1.1. Terminology and notation. In this subsection we review some terminology and establish the notation used in the paper. We mainly follow [**8**, Ch. 1], [**16**, Ch. 1] and [**17**, Ch. 3], although with some modifications, for example in our definition of a commutative $\mathbb{Z}_2$-graded ring, see (1) below.

We write $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ and use its standard field structure. We put $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$. Most objects will be $\mathbb{Z}_2$-graded, say $M = M_{\bar{0}} \oplus M_{\bar{1}}$. In this case, elements in $M_{\bar{0}} \cup M_{\bar{1}}$ are called *homogenous*. For a homogenous $m \in M_{\bar{\varepsilon}}$, $\bar{\varepsilon} \in \mathbb{Z}_2$, its *degree* is denoted by $|m| = \bar{\varepsilon} \in \mathbb{Z}_2$. We adopt the convention that whenever the degree function occurs in a formula, the corresponding elements are supposed to be homogeneous.

An arbitrary (not necessarily associative) ring S is called $\mathbb{Z}_2$*-graded* if $S = S_{\bar{0}} \oplus S_{\bar{1}}$ as abelian group and $S_{\bar{\alpha}} S_{\bar{\beta}} \subset S_{\bar{\alpha}+\bar{\beta}}$ for $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}_2$. A $\mathbb{Z}_2$-graded ring is called *unital* if there exists $1 \in S_{\bar{0}}$ such that $1s = s$ for all $s \in S$, and it is called *commutative* (sometimes also called *supercommutative*) if

$$st = (-1)^{|s||t|} ts \quad \text{for } s, t \in S \text{ and } \quad s_{\bar{1}}^2 = 0 \quad \text{for } s_{\bar{1}} \in S_{\bar{1}}. \tag{1}$$

Obviously the second condition in (1) is not necessary if S does not have 2-torsion. Note the difference between a commutative ring and a commutative $\mathbb{Z}_2$-graded ring. Also compare the definition of a commutative $\mathbb{Z}_2$-graded ring with that of a Lie superalgebra in 1.2. We will call S a *base superring* if S is a commutative unital associative $\mathbb{Z}_2$-graded ring. Unless specified otherwise, S will always denote such a base superring and all structures considered here will be defined over S in a sense to be explained in the following.

A S*-supermodule* is a left module M over (the associative ring) S whose underlying abelian group is $\mathbb{Z}_2$-graded, i.e., $M = M_{\bar{0}} \oplus M_{\bar{1}}$, such that $S_{\bar{\alpha}} M_{\bar{\beta}} \subset M_{\bar{\alpha}+\bar{\beta}}$ for $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}_2$. It is convenient to consider S-supermodules also as S-bimodules by defining the right action as

$$ms = (-1)^{|s||m|} sm \tag{2}$$

for $s \in S$ and $m \in M$. Alternatively, one can define S-supermodules as S-bimodules satisfying (2).

Let M and N be two S-supermodules, and let $\bar{\alpha} \in \mathbb{Z}_2$. A *homomorphism of degree* $\bar{\alpha}$ from M to N is a map $f\colon M \to N$ satisfying

(i) $f(M_{\bar{\beta}}) \subset N_{\bar{\alpha}+\bar{\beta}}$ for all $\bar{\beta} \in \mathbb{Z}_2$,

(ii) f is additive and $f(ms) = f(m)s$ for $m \in M$ and $s \in S$.

Note that then $sf(m) = (-1)^{|s||f|} f(sm)$. We denote by $\mathrm{Hom}_S(M,N)_{\bar{\alpha}}$ the abelian group of homomorphisms of degree $\bar{\alpha}$ and put $\mathrm{Hom}_S(M,N) = \mathrm{Hom}_S(M,N)_{\bar{0}} \oplus \mathrm{Hom}_S(M,N)_{\bar{1}}$. This becomes a S-supermodule by defining $(sf)(m) = sf(m)$.

Let M be a S-supermodule. A *submodule* of M is a submodule N of the S-module M which respects the $\mathbb{Z}_2$-grading, i.e., $N = (N \cap M_{\bar{0}}) \oplus (N \cap M_{\bar{1}})$. In particular, N is a S-supermodule. The quotient of M by a submodule is again a S-supermodule with respect to the canonical S-module structure and $\mathbb{Z}_2$-grading. The tensor product of two S-supermodules M and N in the category of S-bimodules is a S-supermodule ([**5**, II, §11.5]). We will often write $\otimes$ for $\otimes_S$ if S is clear from the context. Recall that $ms \otimes n = m \otimes sn$ for $m \in M$, $s \in S$ and that the S-action on $M \otimes N$ is given by $s.(m \otimes n) = (sm) \otimes n$ and $(m \otimes n).s = m \otimes (ns)$.

Given a third S-supermodule P, a *S-bilinear map of degree* $\bar{\gamma}$ is a map $b\colon M \times N \to P$ satisfying

(i) $b(M_{\bar{\alpha}}, N_{\bar{\beta}}) \subset P_{\bar{\alpha}+\bar{\beta}+\bar{\gamma}}$ for all $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}_2$,

(ii) b is additive in each argument,

(iii) $b(ms,n) = b(m,sn)$ and $b(m,ns) = b(m,n)s$ for all $m \in M$, $n \in N$ and $s \in S$.

We note that then also $sb(m,n) = (-1)^{|s||b|} b(sm,n)$ holds. The tensor product $M \otimes_S N$ has the *universal property* that there is an isomorphism between S-bilinear maps of degree $\bar{\gamma}$ and $\mathrm{Hom}_S(M \otimes_S N, P)_{\bar{\gamma}}$ by mapping b to the homomorphism $m \otimes n \mapsto b(m,n)$.

A *S-superalgebra*, also called a *superalgebra over* S, is a S-supermodule $A = A_{\bar{0}} \oplus A_{\bar{1}}$ together with a S-bilinear map $A \times A \to A$, called *product*, of degree $\bar{0}$. In particular A is a $\mathbb{Z}_2$-graded ring. A *S-superextension* is an associative, unital and commutative S-superalgebra. In particular, a S-superextension can serve as a new base superring.

Let T be a S-superextension, and let M be a S-supermodule. Then $T \otimes_S M$ has a canonical left T-module structure which can be used to make $T \otimes_S M$ a T-supermodule. If A is a S-superalgebra then $T \otimes_S A$ becomes a T-superalgebra, called the *base superring extension* by

$$(t \otimes a)(t' \otimes a') = (-1)^{|a||t'|} tt' \otimes aa'. \tag{3}$$

If A is a commutative S-superalgebra or a S-superextension, then $T \otimes_S A$ is a commutative T-superalgebra respectively a T-superextension. An example of a $\mathbb{Z}$-superextension is the algebra of dual numbers $\mathbb{Z}[\varepsilon] = \mathbb{Z} \oplus \mathbb{Z}\varepsilon$ where ε is a homogenous element satisfying $\varepsilon^2 = 0$. It gives rise to the *S-superalgebra of dual numbers* $S[\varepsilon] = S \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon]$. We have

$$S[\varepsilon]_{\bar{0}} = \begin{cases} S_{\bar{0}} \oplus S_{\bar{0}}\varepsilon & \text{if } |\varepsilon| = \bar{0}, \\ S_{\bar{0}} \oplus S_{\bar{1}}\varepsilon & \text{if } |\varepsilon| = \bar{1}, \end{cases} \quad \text{and} \quad S[\varepsilon]_{\bar{1}} = \begin{cases} S_{\bar{1}} \oplus S_{\bar{1}}\varepsilon & \text{if } |\varepsilon| = \bar{0}, \\ S_{\bar{1}} \oplus S_{\bar{0}}\varepsilon & \text{if } |\varepsilon| = \bar{1}. \end{cases} \tag{4}$$

A related example is the *Grassmann algebra* over S, defined as $G_S = S \otimes_{\mathbb{Z}} G_{\mathbb{Z}}$ where $G_{\mathbb{Z}} = \mathbb{Z}[\xi_i; i \in I]$ is the exterior algebra over $\mathbb{Z}$ in a finite or countable number of odd generators ξ_i satisfying $\xi_i^2 = 0$ and $\xi_i\xi_j + \xi_j\xi_i = 0$ for $i,j \in I$.

1.2. Lie superalgebras. A S-superalgebra L with product $[..]$ is a *Lie S-superalgebra* if for all $x,y,z \in L$

$$[xy] = -(-1)^{|x||y|}[yx], \tag{1}$$

$$[x[yz]] = [[xy]z] + (-1)^{|x||y|}[y[xz]] \quad \text{and} \tag{2}$$
$$[w_{\bar 0} w_{\bar 0}] = 0 \quad \text{for all } w_{\bar 0} \in L_{\bar 0}\,. \tag{3}$$

Note that (3) is not needed if $\frac{1}{2} \in S$. Also, under the presence of (1) it is easily seen that (2) is equivalent to the more symmetrical identity

$$(-1)^{|x||z|}[x[yz]] + (-1)^{|y||x|}[y[zx]] + (-1)^{|z||y|}[z[xy]] = 0\,. \tag{4}$$

For ease of reading we will sometimes denote the product of a Lie superalgebra by $[x, y]$ instead of $[xy]$.

Lie superalgebras over fields have been extensively studied, and the reader is referred to the basic references [**15**] and [**23**]. Lie superalgebras over superextensions naturally arise in the setting of root graded Lie superalgebras, see for example [**12**]. Any associative superalgebra A becomes a Lie superalgebra with respect to the new product given by the commutator $[a, b] = ab-(-1)^{|a||b|}ba$. For example, for any S-supermodule M, $\mathrm{End}_S(M) = \mathrm{Hom}_S(M, M)$ is an associative S-superalgebra with respect to composition and hence becomes a Lie superalgebra over S with respect to the commutator.

Let L be a Lie superalgebra over S. We note that then $L_{\bar 0}$ is a Lie algebra over $S_{\bar 0}$ and $L_{\bar 1}$ is a module for $L_{\bar 0}$. The *centre of* L is defined as $Z(L) = \{z \in L : [zx] = 0 \text{ for all } x \in L\}$. The *derived algebra* $[LL]$ is the S-span (equivalently, the $\mathbb{Z}$-span) of all products $[xy]$, $x, y \in L$. One calls L *perfect* if $L = [LL]$. A *subalgebra* (respectively an *ideal*) of L is a S-submodule I satisfying $[I, I] \subset I)$ (respectively $[L, I] \subset I$). Both $Z(L)$ and $[LL]$ are ideals of L. If I is an ideal of L, the quotient L/I inherits a canonical Lie superalgebra structure such that the natural projection map becomes a homomorphism, as defined below. If T is a S-superextension, the base superring extension $T \otimes_S L$, cf. 1.1.3, is a Lie superalgebra over T. For example, this applies to the Grassmann algebra G_S and yields that $G_S \otimes_S L$ is a Lie superalgebra over G_S, in particular the so-called *Grassmann envelope* $G(L) := (G_S \otimes_S L)_{\bar 0} = (G_{\bar 0} \otimes L_{\bar 0}) \oplus (G_{\bar 1} \otimes L_{\bar 1})$ is a Lie algebra over $S_{\bar 0}$. It is easily seen that this characterizes Lie superalgebras: a superalgebra L is a Lie superalgebra if and only if its Grassmann envelope $G(L)$ is an ordinary Lie algebra.

If L' is another Lie superalgebra over S, a *homomorphism* from L to L' is a map $f \in \mathrm{Hom}_S(L, L')_{\bar 0}$ satisfying $f[x, y] = [f(x), f(y)]$ for all $x, y \in L$. We point out that, by definition, homomorphism are always even maps. We denote by $\mathbf{Lie}_S$ the category of all Lie superalgebras over S with homomorphisms as just defined. The notions of *epimorphisms, isomorphisms* and *automorphisms* have the obvious meaning. We denote by $\mathrm{Aut}(L)$ the group of automorphisms of L. For a homogenous $d \in \mathrm{End}_S L$ the following two conditions are equivalent:

(i) $d([x, y]) = [d(x), y] + (-1)^{|d||x|}[x, d(y)]$ for $x, y \in L$,

(ii) $\mathrm{Id} + \varepsilon d$ is an automorphism of the base superring extension $S[\varepsilon] \otimes_S L$ where $S[\varepsilon]$ is the S-superalgebra of dual numbers with $|\varepsilon| = |d|$, cf. 1.1.4.

We denote by $(\mathrm{Der}_S L)_{\bar\alpha} \subset \mathrm{End}_S L$, $\bar\alpha \in \mathbb{Z}_2$, the set of such maps and put $\mathrm{Der}_S L = (\mathrm{Der}_S L)_{\bar 0} \oplus (\mathrm{Der}_S L)_{\bar 1}$. Elements of $\mathrm{Der}_S L$ will be called *derivations*. It is easily seen that $\mathrm{Der}_S L$ is a subalgebra of the Lie superalgebra $\mathrm{End}_S(M)$. Each $\mathrm{ad}\, x\colon L \to L : y \mapsto [xy]$ is a derivation, a so-called *inner derivation*, and $\mathrm{IDerL} = \{\mathrm{ad}\, x : x \in L\}$ is an ideal of $\mathrm{Der}_S L$.

For a Lie superalgebra M and a homomorphism $g\colon M \to \mathrm{Der}_S L$ one defines the *semidirect product* $L \rtimes M$ as usual: its underlying S-supermodule is $L \oplus M$ and its product is determined by $[m, x] = g(m)(x)$ for $m \in M, x \in L$ and the requirement that L and M are subalgebras. If $g = 0$ we call $L \rtimes M = L \times M$ a *direct product*.

1.3. Extensions of Lie superalgebras. An *extension* of a Lie superalgebra L is a short exact sequence in the category $\mathbf{Lie}_S$:

$$0 \longrightarrow I \xrightarrow{e} K \xrightarrow{f} L \longrightarrow 0\,. \tag{1}$$

Since $e\colon I \to e(I) = \operatorname{Ker} f$ is an isomorphism we will usually identify I and $e(I)$. An extension of L is then the same as an epimorphism $f\colon K \to L$. A *homomorphism* from an extension $f\colon K \to L$ to another extension $f'\colon K' \to L$ is a Lie superalgebra homomorphism $g\colon K \to K'$ satisfying $f = f' \circ g$; in other words, we have a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\quad g \quad} & K' \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle f'} & \\ & L & \end{array} \tag{2}$$

and hence in particular,

$$\operatorname{Ker} g \subset g^{-1}(\operatorname{Ker} f') = \operatorname{Ker} f \quad \text{and} \quad K' = g(K) + \operatorname{Ker} f'. \tag{3}$$

An extension (1) is called *split* if there exists a Lie superalgebra homomorphism $s\colon L \to K$, called *splitting homomorphism*, such that $f \circ s = \mathrm{Id}_L$. In this case, $K = I \oplus s(L)$ where $I = \operatorname{Ker} f$ and $s : L \to s(L)$ is an isomorphism with inverse $s^{-1} = f|s(L)$. Moreover, $K = I \rtimes s(L)$ is a semidirect product. Conversely, any semidirect product $K = I \rtimes L$ gives rise to a canonical split extension $0 \to I \to K \to L \to 0$, where $I \to K : i \mapsto i \oplus 0$ and $K \to L : i \oplus l \mapsto l$. In this way, semidirect products and split exact sequences correspond to each other. We point out that in general an extension need not be split; there need not even exist a S-supermodule homomorphism $s\colon L \to K$ such that $f \circ s = \mathrm{Id}_L$. We will say the extension (1) *splits uniquely* if there exists a unique $s\colon L \to K$ with $f \circ s = \mathrm{Id}_L$.

A *central extension of* L is an extension (1) such that $\operatorname{Ker} f \subset Z(K)$. We note that for a split central extension (1) with splitting homomorphism s the Lie superalgebra K is a direct product $K = \operatorname{Ker} f \times s(L)$. A central extension K of L is called a *covering* if K is perfect. In that case, also L is perfect.

A central extension $\mathfrak{u}\colon \mathfrak{L} \to L$ is called a *universal central extension* if there exists a unique homomorphism from $\mathfrak{u}\colon \mathfrak{L} \to L$ to any other central extension $f\colon K \to L$ of L. It is obvious from the universal property that

$$\textit{two universal central extensions of } L \textit{ are isomorphic as extensions,} \tag{4}$$

and hence in particular their underlying Lie superalgebras are isomorphic. We will prove two characterizations of universal central extensions in Theorem 1.8, after we have established some auxiliary results of independent interest. In particular, we will see that L has a universal central extension if and only if L is perfect (1.8 and 1.14).

1.4. Lemma (central trick). *Let $f\colon K \to L$ be a central extension.*

(a) *If $f(x) = f(x')$ and $f(y) = f(y')$ then $[x, y] = [x', y']$.*

(b) *If g and g' are homomorphisms from some Lie superalgebra P to K such that $f \circ g = f \circ g'$, then $g|[P, P] = g'|[P, P]$. In particular, there exists at most one homomorphism from a covering $P \to L$ to the central extension $f\colon K \to L$.*

Proof. In (a) we have $x' = x + z$ and $y' = y + z'$ for some $z, z' \in \operatorname{Ker} f \subset Z(K)$. This immediately gives the claim. For the proof of (b) we use (a): $g([x, y]) = [g(x), g(y)] = [g'(x), g'(y)] = g'([x, y])$.

1.5. Lemma. *Let $f\colon K \to L$ be a central extension of a perfect L.*

(a) *$K = [K, K] + \operatorname{Ker} f$, and $f\colon [K, K] \to L$ is a covering.*

(b) *$Z(K) = f^{-1}(Z(L))$ and $f(Z(K)) = Z(L)$.*

(c) *If $g\colon L \to M$ is a central extension, then so is $g \circ f\colon K \xrightarrow{f} L \xrightarrow{g} M$.*

(d) *If $f'\colon K' \to L$ is a covering and $g\colon K \to K'$ a homomorphism from the extension $f\colon K \to L$ to the extension $f'\colon K' \to L$, cf.* 1.3.2,

$$\begin{array}{ccc} K & \xrightarrow{\quad g \quad} & K' \\ & {\scriptstyle f}\searrow \qquad \swarrow{\scriptstyle f'} & \\ & L & \end{array}$$

then $g\colon K \to K'$ is a central extension. In particular, g is surjective.

Proof. (a) We have $f([K, K]) = L$, so $f|[K, K]$ is a central extension of L. Moreover, $f([K, K]) = L$ also implies that $K = [K, K] + \operatorname{Ker} f$, from which it then easily follows that $[K, K]$ is perfect.

(b) Let $z \in K$. Then $z \in f^{-1}(Z(L)) \Leftrightarrow [z, K] \subset \operatorname{Ker} f$. In particular, $Z(K) \subset f^{-1}(Z(L))$. To prove the other inclusion, let $z \in f^{-1}(Z(L))$. Then $[z, K] \subset Z(K)$, and so $[z, K] = [z\,,\, [K, K] + \operatorname{Ker} f] = [z\,,\, [K, K]] = [[z, K], K] + [K, [z, K]] = 0$, i.e., $z \in Z(K)$. The second formula in (b) is then immediate using surjectivity of f.

(c) follows from $\operatorname{Ker}(g \circ f) = f^{-1}(\operatorname{Ker} g) \subset f^{-1}(Z(L)) = Z(K)$.

(d) By 1.3.3 we have $K' = [K', K'] = [g(K), g(K)] = g([K, K])$, and $\operatorname{Ker} g \subset \operatorname{Ker} f$ is central.

1.6. Corollary. *Let L be an arbitrary Lie superalgebra. If $L/Z(L)$ is perfect, then $Z(L/Z(L)) = 0$.*

Proof. We apply the second formula of Lemma 1.5(b) to the central extension $f\colon L \to L/Z(L)$, where f is the canonical map.

In particular for a perfect L the corollary says that $L/Z(L)$ is the "smallest" central quotient, sometimes informally referred to as the *bottom algebra*.

1.7. Pullback Lemma. *Let $f\colon L \to M$ be a homomorphism of Lie superalgebras, and suppose $g\colon N \to M$ is a central extension. Then $P = \{(l, n) \in L \times N : f(l) = g(n)\}$ is a Lie superalgebra (a subalgebra of the direct product $L \times N$), and $\mathrm{pr}_1\colon P \to L : (l, n) \mapsto l$ is a central extension. The extension $\mathrm{pr}_1\colon P \to L$ splits (uniquely) if and only if there exists a (unique) Lie superalgebra homomorphism $h\colon L \to N$ such that $g \circ h = f$.*

It is useful to visualize the situation of the lemma by the following commutative diagram, where pr_2 is the canonical projection map

$$\begin{array}{ccc} P & \xrightarrow{\quad \mathrm{pr}_2 \quad} & N \\ {\scriptstyle \mathrm{pr}_1}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\ L & \xrightarrow{\quad f \quad} & M \end{array} \tag{1}$$

Proof. It is easily seen that P is a Lie superalgebra and that $\mathrm{pr}_1\colon P \to L$ is a central extension. A map $s\colon L \to P$ splits the extension pr_1 if and only if there exists a Lie superalgebra homomorphism $h\colon L \to N$ such that $s(l) = (l, h(l)) \in P$ for all $l \in L$, equivalently, $g \circ h = f$. Uniqueness of s is clearly equivalent to uniqueness of h.

1.8. Theorem (Characterization and properties of universal central extensions). *For a Lie superalgebra L the following are equivalent:*

(i) *L is* simply connected, *i.e., every central extension $L' \to L$ splits uniquely;*

(ii) *L is* centrally closed, *i.e.,* Id: $L \to L$ *is a universal central extension.*

If $\mathfrak{u}\colon L \to M$ is a central extension, then (i) *and* (ii) *are also equivalent to*

(iii) *$\mathfrak{u}\colon L \to M$ is a universal central extension of M.*

In this case,

(a) *both L and M are perfect, and*

(b) $Z(L) = \mathfrak{u}^{-1}(Z(M))$, $\mathfrak{u}(Z(L)) = Z(M)$.

Proof. (i) $\Leftrightarrow$ (ii): (i) holds if and only if for every central extension $f\colon L' \to L$ there exists a unique homomorphism $g\colon L \to L'$ such that $f \circ g = \mathrm{Id}_L$. By definition of a universal central extension, this is equivalent to (ii).

(i) $\Rightarrow$ (iii): Let $g\colon N \to M$ be a central extension, and let $\mathrm{pr}_1\colon P \to L$ be the central extension constructed in Lemma 1.7. By assumption, pr_1 splits uniquely. Hence again by 1.7, there exists a unique homomorphism $h\colon L \to N$ such that $g \circ h = \mathfrak{u}$.

Suppose (iii) holds. We will first show (a). By Lemma 1.5 we know that $\mathfrak{u}\colon [L, L] \to M$ is a covering. Hence, by the universal property of $\mathfrak{u}$ there exists a unique homomorphism $f\colon L \to [L, L]$ such that $\mathfrak{u} \circ f = \mathfrak{u}$. Now let $\iota\colon [L, L] \to L$ be the canonical injection. Then $\iota \circ f\colon L \to L$ is a homomorphism with $\mathfrak{u} \circ (\iota \circ f) = \mathfrak{u}$. The universal property of $\mathfrak{u}$, applied to the central extension $\mathfrak{u}$, then shows that $\iota \circ f = \mathrm{Id}_L$. But then $L = \iota(f(L)) \subset [L, L]$, and so $L = [L, L]$. By surjectivity of $\mathfrak{u}$, also M is perfect.

We can now easily prove (iii) $\Rightarrow$ (i). Indeed, if $f'\colon L' \to L$ is a central extension, Lemma 1.5(c) implies that $\mathfrak{u} \circ f'$ is a central extension of M. Since $\mathfrak{u}$ is a universal central extension, there exists a unique homomorphism $g\colon L \to L'$ such that $\mathfrak{u} = \mathfrak{u} \circ f' \circ g$. By Lemma 1.4(b), this implies $f' \circ g = \mathrm{Id}_L$.

The assertion (b) is a special case of Lemma 1.5(b).

1.9. Corollary. *Let $f\colon K \to L$ and $g\colon L \to M$ be central extensions. Then $g \circ f\colon K \to M$ is a universal central extension if and only if $f\colon K \to L$ is a universal central extension.*

Proof. The conditions (i) and (ii) of Th. 1.8 are independent of the maps f or $g \circ f$ and involve only the Lie superalgebra K. Hence, if $g \circ f$ is a central extension, $g \circ f$ is universal if and only if f is so. However, if f is universal, K is perfect by 1.8(a) and so $g \circ f$ is indeed a central extension by Lemma 1.5(c).

1.10. Corollary. *Let L and L' be perfect Lie superalgebras with universal central extensions $\mathfrak{u}\colon \mathfrak{L} \to L$ and $\mathfrak{u}'\colon \mathfrak{L}' \to L'$ respectively. Then*

$$L/Z(L) \cong L'/Z(L') \quad \Leftrightarrow \quad \mathfrak{L} \cong \mathfrak{L}'. \tag{1}$$

If the equivalent conditions in (1) are fulfilled, one calls L and L' *isogenous.* It can sometimes be easier to describe isogeny classes of a category of Lie superalgebras, rather than isomorphism classes. For example, this concept has been very useful in the early stages of the theory of root graded Lie algebras ([**1**, **4**, **21**]).

Proof. We claim that in the diagram

$$\begin{array}{ccccc}
\mathfrak{L} & \xrightarrow{\ \mathfrak{u}\ } & L & \xrightarrow{\ \pi\ } & L/Z(L) \\
\Big\downarrow{\scriptstyle \Phi} & & & & \Big\downarrow{\scriptstyle \varphi} \\
\mathfrak{L}' & \xrightarrow{\ \mathfrak{u}'\ } & L' & \xrightarrow{\ \pi'\ } & L'/Z(L')
\end{array}$$

Φ exists and is an isomorphism if and only φ exists and is an isomorphism.

By Cor. 1.9, both $\pi \circ \mathfrak{u}\colon \mathfrak{L} \to L/Z(L)$ and $\pi' \circ \mathfrak{u}'\colon \mathfrak{L}' \to L'/Z(L')$ are universal central extensions. Hence, if $\varphi\colon L/Z(L) \to L'/Z(L')$ is an isomorphism, their universal central extensions are isomorphic too (1.3.4). Conversely, suppose that $\Phi\colon \mathfrak{L} \to \mathfrak{L}'$ is an isomorphism. Since $L/Z(L)$ is centreless by Cor. 1.6, it follows from Lemma 1.5(b) that $Z(\mathfrak{L}) = \operatorname{Ker}(\pi \circ \mathfrak{u})$ and, analogously, $Z(\mathfrak{L}') = \operatorname{Ker}(\pi' \circ \mathfrak{u}')$. Therefore $\operatorname{Ker}(\pi' \circ \mathfrak{u}' \circ \Phi) = \Phi^{-1}(\operatorname{Ker}(\pi' \circ \mathfrak{u}') = \Phi^{-1}(Z(\mathfrak{L}')) = Z(\mathfrak{L}) = \operatorname{Ker}(\pi \circ \mathfrak{u})$. Since both $\pi \circ \mathfrak{u}$ and $\pi' \circ \mathfrak{u}' \circ \Phi$ are surjective, φ exists and is an isomorphism.

1.11. The functor $\mathfrak{uce}$. Let L be a Lie superalgebra over S. We denote by $\mathcal{B} = \mathcal{B}_L$ the S-submodule of the S-supermodule $L \otimes_S L$ spanned by all elements of type $(x, y, z \in L)$

$$\begin{gathered} x \otimes y + (-1)^{|x||y|} y \otimes x, \\ (-1)^{|x||z|} x \otimes [yz] + (-1)^{|y||x|} y \otimes [zx] + (-1)^{|z||y|} z \otimes [xy] \text{ and} \\ w_{\bar 0} \otimes w_{\bar 0} \text{ for } w_{\bar 0} \in L_{\bar 0}, \end{gathered}$$

and put

$$\mathfrak{uce}(L) = (L \otimes_S L)/\mathcal{B} \quad \text{and} \quad \langle x, y \rangle = x \otimes y + \mathcal{B} \in \mathfrak{uce}(L).$$

By construction, the following identities then hold in $\mathfrak{uce}(L)$, where $x, y, z \in L$:

$$\langle x, y \rangle = -(-1)^{|x||y|} \langle y, x \rangle, \tag{1}$$

$$\langle x, [y, z] \rangle = \langle [x, y], z \rangle + (-1)^{|x||y|} \langle y, [x, z] \rangle, \tag{2}$$

$$\langle w_{\bar 0}, w_{\bar 0} \rangle = 0 \quad \text{for } w_0 \in L_{\bar 0}. \tag{3}$$

The Lie product on L gives rise to a S-linear map $L \otimes L \to L : x \otimes y \mapsto [xy]$ of degree $\bar 0$ which vanishes on $\mathcal{B}$ and hence descends to a S-linear map of degree $\bar 0$

$$\mathfrak{u} = \mathfrak{u}_L\colon \mathfrak{uce}(L) \to L : \langle x, y \rangle \mapsto [xy]. \tag{4}$$

Note that

$$\operatorname{Ker} \mathfrak{u} = \{ \textstyle\sum_i \langle x_i, y_i \rangle \in \mathfrak{uce}(L) : \sum_i [x_i, y_i] = 0 \} = \mathrm{H}_2(L), \tag{5}$$

the second homology group of L with trivial coefficients. The supermodule $\mathfrak{uce}(L)$ becomes a S-superalgebra with respect to the product

$$[l_1\, l_2] = \langle \mathfrak{u}(l_1), \mathfrak{u}(l_2) \rangle, \tag{6}$$

where $l_i \in \mathfrak{uce}(L)$. Hence $\mathfrak{u}$ is a homomorphism. In particular, we have

$$[\langle x, y \rangle\, ,\, \langle x', y' \rangle] = \langle [x, y], [x', y'] \rangle \qquad \text{for } x, y, x', y' \in L. \tag{7}$$

We claim that $\mathfrak{uce}(L)$ *is a Lie superalgebra over* S. Indeed, since the defining identities 1.2.1 and 1.2.2 of a Lie superalgebra are linear in each argument, it is sufficient to verify them for elements of the spanning set $\{\langle x, y \rangle : x, y \in L\}$. For these they follow from (1), (2) and (7). The remaining identity 1.2.3 follows from (3). It is now immediate from (6) that

$$\mathfrak{u}\colon \mathfrak{uce}(L) \to [LL] \quad \textit{is a central extension of } [L, L]. \tag{8}$$

Let $f\colon L \to M$ be a homomorphism of Lie superalgebras over S. Let $\mathcal{B}_M \subset M \otimes_S M$ be defined analogously to $\mathcal{B}_L \subset L \otimes_S L$. The canonical map $f \otimes_S f$ maps $\mathcal{B}_L$ into $\mathcal{B}_M$, hence induces a S-linear map

$$\mathfrak{uce}(f)\colon \mathfrak{uce}(L) \to \mathfrak{uce}(M) : \langle x, y\rangle \mapsto \langle f(x), f(y)\rangle\,. \tag{9}$$

We note that by construction the diagram

$$\begin{array}{ccc} \mathfrak{uce}(L) & \xrightarrow{\mathfrak{uce}(f)} & \mathfrak{uce}(M) \\ {\scriptstyle \mathfrak{u}_L}\downarrow & & \downarrow{\scriptstyle \mathfrak{u}_M} \\ L & \xrightarrow{f} & M \end{array} \tag{10}$$

commutes. To check that $\mathfrak{uce}(f)$ is a homomorphism, it suffices to show that

$$\mathfrak{uce}(f)([\langle x, y\rangle\,,\, \langle x', y'\rangle]) = [\,\mathfrak{uce}(f)(\langle x, y\rangle)\,,\, \mathfrak{uce}(f)(\langle x', y'\rangle)\,]$$

for $x, y, x', y' \in L$. But this is immediate from (7) and (9). It is now easily verified that

$\mathfrak{uce}$: $\mathbf{Lie}_S \to \mathbf{Lie}_S$ *is a covariant functor.*

Since $\mathfrak{uce}$ is a covariant functor, an automorphism f of L gives rise to the automorphism $\mathfrak{uce}(f)$ of $\mathfrak{uce}(L)$. The commutativity of the diagram (10) implies that $\mathfrak{uce}(f)$ leaves $\mathrm{H}_2(L)$ invariant. Thus, we obtain a group homomorphism

$$\mathrm{Aut}(L) \longrightarrow \{g \in \mathrm{Aut}(\mathfrak{uce}(L)) : g(\mathrm{H}_2(L)) = \mathrm{H}_2(L)\} : f \mapsto \mathfrak{uce}(f), \tag{11}$$

see also 2.2 and 2.3. The following lemma shows that the functor $\mathfrak{uce}$ is natural in the sense that it commutes with base superring extensions.

1.12. Lemma. *Let L be a Lie superalgebra over S and let T be a superextension of S. Then*

$$\mathfrak{uce}(T \otimes_S L) \cong T \otimes_S \mathfrak{uce}(L) \tag{1}$$

where $T \otimes_S L$ and $T \otimes_S \mathfrak{uce}(L)$ are the base superring extensions as defined in 1.1.3. *If T is flat over S, e.g. a Grassmann algebra, then*

$$\mathrm{H}_2(T \otimes_S L) \cong T \otimes_S \mathrm{H}_2(L). \tag{2}$$

Proof. By construction, we have the exact sequence of Lie superalgebras over S,

$$0 \to \mathcal{B}_L \to L \otimes_S L \to \mathfrak{uce}(L) \to 0\,.$$

Since $T \otimes_S -$ is a right exact functor, this yields the exact sequence

$$T \otimes_S \mathcal{B}_L \to T \otimes_S (L \otimes_S L) \to T \otimes_S \mathfrak{uce}(L) \to 0\,. \tag{3}$$

We now note that

$$\begin{gathered} T \otimes_S (L \otimes_S L) \to (T \otimes_S L) \otimes_T (T \otimes_S L) \\ t \otimes_S (x \otimes_S y) \mapsto (t \otimes_S x) \otimes_T (1 \otimes_S y) \end{gathered}$$

is an isomorphism of T-supermodules, see [**5**, II, §5.1 Prop. 3], which maps $T \otimes_S \mathcal{B}_L$ onto $\mathcal{B}_{T\otimes_S L}$. This, together with (3), implies (1). The isomorphism (2) follows from the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T \otimes_S \mathrm{H}_2(L) & \xrightarrow{\iota_T} & T \otimes_S \mathfrak{uce}(L) & \xrightarrow{\mathfrak{u}_T} & T \otimes_S [L, L] & \longrightarrow & 0 \\
 & & & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{H}_2(T \otimes_S L) & \longrightarrow & \mathfrak{uce}(T \otimes_S L) & \xrightarrow{\mathfrak{u}_{T\otimes_S L}} & [T \otimes_S L, T \otimes_S L] & \longrightarrow & 0
\end{array}$$

with exact rows. The top row is obtained by tensoring the exact sequence

$$0 \longrightarrow \mathrm{H}_2(L) \xrightarrow{\iota} \mathfrak{uce}(L) \xrightarrow{\mathfrak{u}} [L, L] \longrightarrow 0 \tag{4}$$

with T and using that $T \otimes_S -$ is exact by flatness of T. The bottom row is (4) for L replaced by $T \otimes_S L$. The horizontal maps are the canonical ones.

1.13. Proposition. *Let $f\colon L \to M$ be a homomorphism of Lie superalgebras and suppose $g\colon M' \to M$ is a central extension. Then there exists a homomorphism $\mathfrak{f}\colon \mathfrak{uce}(L) \to M'$ satisfying $g \circ \mathfrak{f} = f \circ \mathfrak{u}$, i.e., the following diagram commutes:*

$$\begin{array}{ccc}
\mathfrak{uce}(L) & \xrightarrow{\ \mathfrak{f}\ } & M' \\
\mathfrak{u} \downarrow & & \downarrow g \\
L & \xrightarrow{\ f\ } & M
\end{array} \tag{1}$$

The map $\mathfrak{f}$ is uniquely determined on the derived algebra $[\mathfrak{uce}(L), \mathfrak{uce}(L)]$ by the commutativity of (1).

Proof. We choose a section $s\colon M \to M'$ of g in the category of $\mathbb{Z}_2$-graded sets, i.e., a map s such that $g \circ s = \mathrm{Id}_M$ and $s(M_{\bar\varepsilon}) \subset M'_{\bar\varepsilon}$ for $\bar\varepsilon \in \mathbb{Z}_2$. While s may not be linear, we at least have $as(m) - s(am) \in Z(M')$, $s(ma) - s(m)a \in Z(M')$ and $s(m+n) - s(m) - s(n) \in Z(M')$ for all homogeneous $a \in S$ and $m, n \in M$. This is enough to ensure that the map $L \times L \to M' : l_1 \times l_2 \mapsto [(s \circ f)(l_1), (s \circ f)(l_2)]$ is S-bilinear of degree $\bar 0$. Using the universal property of the tensor product, we thus obtain a S-linear map

$$L \otimes L \to M' : l_1 \otimes l_2 \mapsto [s(f(l_1)), s(f(l_2))].$$

Using that M' is a Lie superalgebra and $s[m, n] - [s(m), s(n)] \in Z(M')$, one verifies that this map annihilates $\mathcal{B}_L$. Thus we obtain a S-linear map

$$\mathfrak{f}\colon \mathfrak{uce}(L) \to M : \langle l_1, l_2\rangle \mapsto [s(f(l_1)), s(f(l_2))] \tag{2}$$

of degree $\bar 0$ which turns out to be a superalgebra homomorphism:

$$\begin{aligned}
\mathfrak{f}[\langle l_1, l_2\rangle, \langle l_3, l_4\rangle] = \mathfrak{f}\langle [l_1, l_2], [l_3, l_4]\rangle &= [\, s[f(l_1), f(l_2)]\, ,\, s[f(l_3), f(l_4)]\,] \\
&= [\, [s(f(l_1)), s(f(l_2))]\, ,\, [s(f(l_3)), s(f(l_4))]\,] \\
&= [\, \mathfrak{f}(\langle l_1, l_2\rangle)\, ,\, \mathfrak{f}(\langle l_3, l_4\rangle)\,].
\end{aligned}$$

Moreover, $(g \circ \mathfrak{f})\langle l_1, l_2\rangle = g[s(f(l_1)), s(f(l_2))] = [g(s(f(l_1))), g(s(f(l_2)))] = [f(l_1), f(l_2)] = (f \circ \mathfrak{u})\langle l_1, l_2\rangle$ implies $g \circ \mathfrak{f} = f \circ \mathfrak{u}$. The uniqueness assertion follows from 1.4(b).

In the following theorem we summarize some of the results obtained so far.

1.14. Theorem. *Let L be a perfect Lie superalgebra over S. Then*

$$0 \longrightarrow \mathrm{H}_2(L) \longrightarrow \mathfrak{uce}(L) \xrightarrow{\mathfrak{u}} L \longrightarrow 0 \tag{1}$$

is a universal central extension of L. Moreover, the following holds:

(a) *If L is centreless, then $\mathrm{H}_2(L) = Z(\mathfrak{uce}(L))$.*

(b) *If T is a superextension of S, then $T \otimes_S \mathfrak{uce}(L)$ is a universal central extension of the Lie T-superalgebra $T \otimes_S L$. Moreover, $\mathrm{H}_2(T \otimes_S L) \cong T \otimes_S \mathrm{H}_2(L)$ in case T is a flat S-module.*

Proof. If L is perfect then so is $\mathfrak{uce}(L)$. In any diagram 1.13.1 the map $\mathfrak{f}$ is then uniquely determined. Applying this to the special case $f = \mathrm{Id}$, we see that (1) is a universal central extension. If L is centreless, $\mathrm{H}_2(L) = Z(\mathfrak{uce}(L))$ by 1.5(b). The statement in (b) follows from 1.12.

1.15. Gradings. Let L be a Lie superalgebra over a base superring S, and let Γ be an abelian group written additively. A *Γ-grading of* L is a family $(L_\gamma : \gamma \in \Gamma)$ of S-submodules $L_\gamma \subset L$ such that

$$L = \bigoplus_{\gamma \in \Gamma} L_\gamma \quad \text{and} \quad [L_\gamma\,,\, L_\delta] \subset L_{\gamma+\delta} \text{ for all } \gamma, \delta \in \Gamma. \tag{1}$$

Recall that S-submodules respect the $\mathbb{Z}_2$-grading of $L = L_{\bar 0} \oplus L_{\bar 1}$, hence $L_\gamma = (L_\gamma \cap L_{\bar 0}) \oplus (L_\gamma \cap L_{\bar 1})$ for $\gamma \in \Gamma$. A *homomorphism* from a Γ-graded L to another Γ-graded Lie superalgebra L' is a homomorphism $f\colon L \to L'$ respecting the Γ-grading, i.e., $f(L_\gamma) \subset L'_\gamma$ for all $\gamma \in \Gamma$.

Let L be a Γ-graded Lie superalgebra. A *graded submodule* of L is a submodule M respecting the Γ-grading. For example, both $Z(L)$ and $[L, L]$ are graded ideals of L. Following [**30**], a covering $f\colon K \to L$ is called a *Γ-cover* if K is a Γ-graded Lie superalgebra and f is a homomorphism of Γ-graded superalgebras. It follows from the result below that the universal central extension of a perfect Γ-graded L is a Γ-cover, but this is in general not so for an arbitrary covering of L.

1.16. Proposition. *Let $L = \bigoplus_{\gamma\in\Gamma} L_\gamma$ be a Γ-graded Lie superalgebra. Then $\mathfrak{uce}(L)$ is also Γ-graded,*

$$\mathfrak{uce}(L) = \bigoplus_{\gamma \in \Gamma} \mathfrak{uce}(L)_\gamma, \quad \textit{where} \quad \mathfrak{uce}(L)_\gamma = \sum_{\delta \in \Gamma} \langle L_\delta, L_{\gamma-\delta} \rangle\,, \tag{1}$$

the canonical map $\mathfrak{u}\colon \mathfrak{uce}(L) \to L$ is a homomorphism of Γ-graded Lie superalgebras and hence a Γ-cover if L is perfect. Moreover, $\mathrm{H}_2(L)$ is a graded submodule.

If $f\colon K \to L$ is a covering, and hence $K = \mathfrak{uce}(L)/C$ for some central ideal C, then f is a Γ-cover if and only if C is a Γ-graded submodule. In this case, we have

$$K_0 = \sum_{0 \neq \gamma \in \Gamma} [K_\gamma, K_{-\gamma}] \quad \Longleftrightarrow \quad L_0 = \sum_{0 \neq \gamma \in \Gamma} [L_\gamma, L_{-\gamma}]\,. \tag{2}$$

We note that the condition in (2) is one of the defining properties of root graded Lie algebras ([**4**]).

Proof. The submodule $\mathcal{B}_L$ (cf. 1.11) is a graded submodule of the Γ-graded S-supermodule $L \otimes_S L$, hence $\mathfrak{uce}(L) = (L \otimes_S L)/\mathcal{B}_L$ is Γ-graded too, i.e., we have the decomposition $\mathfrak{uce}(L) = \bigoplus_{\gamma \in \Gamma} \mathfrak{uce}(L)_\gamma$ with the description of $\mathfrak{uce}(L)_\gamma$ as stated in (1). By definition, $\mathfrak{u}(\mathfrak{uce}(L)_\gamma) \subset L_\gamma$ which, by 1.11.6, implies that $\mathfrak{uce}(L)$ is Γ-graded. The statements concerning $\mathfrak{u}$, $\mathrm{H}_2(L)$ and the characterization of Γ-coverings are then easily seen. In (2) the implication $\Rightarrow$ is immediate from $f(K_\gamma) = L_\gamma$ for all $\gamma \in \Gamma$. Conversely, if the right side of (2) holds, then $K_0 = [K_0, K_0] + A$ where $A = \sum_{0 \neq \gamma} [K_\gamma, K_{-\gamma}]$ satisfies $f(A) = L_0$, whence $K_0 \subset A + \operatorname{Ker} f$. Since $\operatorname{Ker} f$ is central, we have $[K_0, K_0] \subset [A, A]$, and it suffices to show that A is a subalgebra which is immediate from the Jacobi identity: $[K_0, [K_\gamma, K_{-\gamma}]] = [[K_0, K_\gamma], K_{-\gamma}] + [K_\gamma, [K_0, K_{-\gamma}]] \subset [K_\gamma, K_{-\gamma}]$.

1.17. Notes. Our terminology follows the one for Lie algebras, however with some exceptions. For example, a split extension is called inessential in [**6**, I, §1.7 and 1.8]. Also, some authors, e.g. Garland [**13**] or Moody-Pianzola [**19**], require of a universal central extension $\mathfrak{u}\colon \mathfrak{L} \to L$ that $\mathfrak{L}$ be perfect, in addition to the universal mapping property. As we have seen in 1.8, this is however not necessary.

It seems to be customary to attribute the theory of universal central extensions to Garland's paper [**13**], although [**28**, §1] by van der Kallen is an earlier and more general reference. The setting of [**28**] is Lie algebras over rings. There the reader will find the important central trick (1.4) which goes back to the pioneering work of Steinberg on central extensions of algebraic groups [**25**, **26**, **27**]. Moreover, the Lie algebra version of our model of a universal central extension (Th. 1.14) and parts of Lemma 1.5 and Th. 1.8 are already given in Prop. 1.3 of [**28**] in the setting of Lie algebras. Van der Kallen's construction was later generalized by Ellis in [**10**, **11**] who introduced a so-called non-abelian tensor product $L \otimes_{\mathrm{Lie}} M$ of two Lie algebras L and M. It satisfies $L \otimes_{\mathrm{Lie}} L = \mathfrak{uce}(L)$.

Garland studies universal central extensions of Lie algebras over fields in §1 of [**13**]. In particular, he constructs a model of a universal central extension of a perfect Lie algebra L, using (in our notation) the "universal" 2-cocycle $L \times L \to \mathfrak{uce}(L) : (x, y) \mapsto \langle x, y\rangle$. His model is different from the one in [**28**] (but they are of course isomorphic, cf. 1.3.4). He also proves that a covering of L is universal if and only if it is simply connected (cf. Th. 1.8).

Yet another model of a universal central extension (a quotient of the derived algebra of a free Lie algebra mapping onto L) is given in §7.9 of Weibel's book [**29**]. This model is the direct analogue of the standard construction of a universal central extension of a perfect group. Weibel's theory works for Lie algebras over rings.

Central extensions in the category of certain topological Lie algebras, such as Fréchet Lie algebras, are studied in the recent preprint [**20**] of Neeb. In particular, Neeb introduces a topological version of van der Kallen's model.

As for Lie algebras, a central extension of a Lie S-superalgebra L with kernel C can be constructed on $L \oplus C$ by using a (suitably defined) 2-cocycle $\tau\colon L \times L \to C$: the product on $L \oplus C$ is given by $[x_1 \oplus c_1\,,\, x_2 \oplus c_2]_{L\oplus C} = [x_1, x_2]_L \oplus \tau(x_1, x_2)$. It is proven in §IV of the paper [**24**] by Scheunert and Zhang that if S is a field, this sets up a bijection between the isomorphism classes of central extensions of L in the sense of 1.3.2 and the cohomology group $\mathrm{H}^2(L, C)$ of L with values in the trivial L-module C. In the setting of Lie algebras this is classical result, see for example [**7**, Exp. 5] or [**29**, 7.6] where this is proven for Lie algebras over rings and extensions 1.3.1 with an abelian I. The paper [**24**] also gives the super version of Garland's model of a universal central extension and shows that a covering of a Lie superalgebra is universal if and only if it is simply connected. It should be mentioned that [**24**] considers ε-Lie algebras, sometimes also called colour or color Lie algebras, which are generalizations of Lie superalgebras.

2. Lifting automorphisms and derivations

2.1. Notation. Let $f\colon L' \to L$ be a covering. The commutative diagram 1.11.10 then becomes the following diagram where we abbreviated $\mathfrak{L}' = \mathfrak{uce}(L')$, $\mathfrak{f} = \mathfrak{uce}(f)$, $\mathfrak{L} = \mathfrak{uce}(L)$, $\mathfrak{u}' = \mathfrak{u}_{L'}$ and $\mathfrak{u} = \mathfrak{u}_L$:

$$\begin{array}{ccc} \mathfrak{L}' & \xrightarrow{\mathfrak{f}} & \mathfrak{L} \\ {\scriptstyle \mathfrak{u}'}\downarrow & & \downarrow{\scriptstyle \mathfrak{u}} \\ L' & \xrightarrow{f} & L \end{array} \tag{1}$$

Since both $\mathfrak{u}'$ and f are central extensions, we conclude from Corollary 1.9 that $f \circ \mathfrak{u}'\colon \mathfrak{L}' \to L$ is a universal central extension of L. Moreover, $\mathfrak{f}$ is a homomorphism from this universal central extension to the universal central extension $\mathfrak{u}\colon \mathfrak{uce}(L) \to L$. Therefore, by 1.3.4, $\mathfrak{f}$ *is an isomorphism,* and we obtain a covering $\mathfrak{u}' \circ \mathfrak{f}^{-1}\colon \mathfrak{L} \to L'$ with kernel

$$C := \operatorname{Ker}(\mathfrak{u}' \circ \mathfrak{f}^{-1}) = \mathfrak{f}(\operatorname{Ker} \mathfrak{u}') = \mathfrak{f}(\mathrm{H}_2(L')). \tag{2}$$

2.2. Theorem (lifting of automorphisms). *We use the setting and notation of* 2.1. *In particular, $f\colon L' \to L$ is a covering.*

(a) *Let $h \in \operatorname{Aut}(L)$. Then there exists $h' \in \operatorname{Aut}(L')$ such that the diagram*

$$\begin{array}{ccc} L' & \xrightarrow{f} & L \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ L' & \xrightarrow{f} & L \end{array} \tag{1}$$

commutes if and only if the automorphism $\mathfrak{uce}(h)$ of $\mathfrak{L}$, cf. 1.11.11, *satisfies $\mathfrak{uce}(h)(C) = C$. In this case, h' is uniquely determined by* (1) *and $h'(\operatorname{Ker} f) = \operatorname{Ker} f$.*

(b) *With the notation of* (a), *the map $h \mapsto h'$ is a group isomorphism*

$$\{h \in \operatorname{Aut}(L) : \mathfrak{uce}(h)(C) = C\} \longrightarrow \{g \in \operatorname{Aut}(L') : g(\operatorname{Ker} f) = \operatorname{Ker} f\}. \tag{2}$$

Proof. (a) If h' exists, it is a homomorphism from the covering $h \circ f$ to the covering f and therefore by 1.4(b) uniquely determined by the commutativity of (1). Applying the $\mathfrak{uce}$-functor to (1), yields the commutative diagram

$$\begin{array}{ccc} \mathfrak{L}' & \xrightarrow{\mathfrak{f}} & \mathfrak{L} \\ {\scriptstyle \mathfrak{uce}(h')}\downarrow & & \downarrow{\scriptstyle \mathfrak{uce}(h)} \\ \mathfrak{L}' & \xrightarrow{\mathfrak{f}} & \mathfrak{L} \end{array} \tag{3}$$

whence, by 2.1.2 and 1.11.11, $\mathfrak{uce}(h)(C) = (\mathfrak{uce}(h) \circ \mathfrak{f})(\mathrm{H}_2(L')) = (\mathfrak{f} \circ \mathfrak{uce}(h'))(\mathrm{H}_2(L')) = \mathfrak{f}(\mathrm{H}_2(L') = C$. For the proof of the other direction, note that $\mathfrak{u} = f \circ \mathfrak{u}' \circ \mathfrak{f}^{-1}$ by 2.1.1. Hence, the commutative diagram 1.11.10 becomes

$$\begin{array}{ccccc} \mathfrak{L} & \xrightarrow{\mathfrak{u}' \circ \mathfrak{f}^{-1}} & L' & \xrightarrow{f} & L \\ \Big\downarrow{\scriptstyle \mathfrak{uce}(h)} & & \Big\downarrow & & \Big\downarrow{\scriptstyle h} \\ \mathfrak{L} & \xrightarrow{\mathfrak{u}' \circ \mathfrak{f}^{-1}} & L' & \xrightarrow{f} & L \end{array} \tag{4}$$

If $\mathfrak{uce}(h)(C) = C$, the kernel of the epimorphism $\mathfrak{u}' \circ \mathfrak{f}^{-1} \circ \mathfrak{uce}(h)$ is C. By 2.1.2 we therefore obtain an automorphism h': $L' \to L'$ such that (1) = right half of (4) commutes. Commutativity of (1) then implies that $h'(\operatorname{Ker} f) = \operatorname{Ker} f$.

(b) By (a), the map is well-defined. It is a group monomorphism by uniqueness in (a). Any automorphism g of L' with $g(\operatorname{Ker} f) = \operatorname{Ker} f$ descends to an automorphism h: $L \to L$ such that $f \circ g = h \circ f$. Hence, by (a), $g = h'$ and $\mathfrak{uce}(h)(C) = C$.

2.3. Corollary. *If L is perfect, the map*

$$\operatorname{Aut}(L) \to \{g \in \operatorname{Aut}(\mathfrak{uce}(L)) : g(\mathrm{H}_2(L)) = \mathrm{H}_2(L)\} : f \mapsto \mathfrak{uce}(f) \tag{1}$$

is a group isomorphism. In particular, $\operatorname{Aut}(L) \cong \operatorname{Aut}(\mathfrak{uce}(L))$ *if L is centreless.*

Proof. We apply Th. 2.2 to the covering $\mathfrak{u}$: $\mathfrak{uce}(L) \to L$. In this case $C = 0$ so that (1) follows from 2.2.2. If L is centreless, $\mathrm{H}_2(L) = Z(\mathfrak{uce}(L))$ by 1.5(b). Since every automorphism leaves the centre invariant, the second claim is a special case of (1).

2.4. Lifting derivations to $\mathfrak{uce}(L)$. In this subsection we will describe the analogue of 1.11.11 for derivations. Thus let L be a Lie superalgebra over S and let $d \in \operatorname{Der}_S L$ be a derivation of L. The S-linear map $L \otimes_S L \to L \otimes_S L : x \otimes y \mapsto d(x) \otimes y + (-1)^{|d||x|} x \otimes d(y)$ leaves $\mathcal{B}_L \subset L \otimes L$ invariant and hence induces a S-linear map

$$\mathfrak{uce}(d)\colon \mathfrak{uce}(L) \to \mathfrak{uce}(L) : \langle x, y\rangle \mapsto \langle d(x), y\rangle + (-1)^{|d||x|}\langle x, d(y)\rangle \tag{1}$$

rendering the following diagram commutative

$$\begin{array}{ccc} \mathfrak{uce}(L) & \xrightarrow{\mathfrak{uce}(d)} & \mathfrak{uce}(L) \\ \Big\downarrow{\scriptstyle \mathfrak{u}} & & \Big\downarrow{\scriptstyle \mathfrak{u}} \\ L & \xrightarrow{d} & L \end{array}$$

In particular $\mathfrak{uce}(d)$ leaves $\operatorname{Ker} \mathfrak{u} = \mathrm{H}_2(L)$ invariant. A straightforward verification also shows that $\mathfrak{uce}(d)$ is a derivation of $\mathfrak{uce}(L)$ and that

$$\mathfrak{uce}\colon \operatorname{Der}_S L \to \{e \in \operatorname{Der}(\mathfrak{uce}(L)) : e(\mathrm{H}_2(L)) \subset \mathrm{H}_2(L)\} : d \mapsto \mathfrak{uce}(d) \tag{2}$$

is a Lie superalgebra homomorphism. Its kernel is contained in the subalgebra of those derivations vanishing on $[L, L]$. It is also easily seen that $\mathfrak{uce}(\operatorname{ad}_L[x, y]) = \operatorname{ad}_{\mathfrak{L}}\langle x, y\rangle$ whence

$$\mathfrak{uce}(\operatorname{ad}_L \mathfrak{u}(z)) = \operatorname{ad}_{\mathfrak{L}} z \quad \text{for } z \in \mathfrak{L}, \text{ and} \quad \mathfrak{uce}(\operatorname{ad}_L[L, L]) = \operatorname{IDer}(\mathfrak{uce}(L)). \tag{3}$$

Functoriality of $\mathfrak{uce}$ for derivations is expressed in the following lemma.

2.5. Lemma *Let $f\colon K \to L$ be a homomorphism of Lie S-superalgebras, and let $d_K \in \operatorname{Der}_S K$ and $d_L \in \operatorname{Der}_S L$ be related by f in the sense that $f \circ d_K = d_L \circ f$, e.g., $d_K = \operatorname{ad} x$ and $d_L = \operatorname{ad} f(x)$. Then, with the definitions* 1.11.9 *and* 2.4.1, *we have*

$$\mathfrak{uce}(f) \circ \mathfrak{uce}(d_K) = \mathfrak{uce}(d_L) \circ \mathfrak{uce}(f)\,. \tag{1}$$

Proof. Since f has degree $\bar{0}$, we can assume that d_K and d_L are homogenous of the same degree. It suffices to establish (1) when evaluated on $\langle k, k'\rangle \in \mathfrak{uce}(K)$ where $k, k' \in K$ are homogenous. We have

$$\begin{aligned}
\Big(\mathfrak{uce}(f) \circ \mathfrak{uce}(d_K)\Big)(\langle k, k'\rangle) &= \mathfrak{uce}(f)\Big(\langle d_K(k), k'\rangle + (-1)^{|d_K||k|}\langle k, d_K(k')\rangle\Big) \\
&= \langle f(d_K(k)), f(k')\rangle + (-1)^{|d_K||k|}\langle f(k), f(d_K(k'))\rangle \\
&= \langle d_L(f(k)), f(k')\rangle + (-1)^{|d_L||f(k)|}\langle f(k), d_L(f(k'))\rangle \\
&= \mathfrak{uce}(d_L)\Big(\langle f(k), f(k')\rangle\Big) = \Big(\mathfrak{uce}(d_L) \circ \mathfrak{uce}(f)\Big)(\langle k, k'\rangle)
\end{aligned}$$

We now have the analogous result to Th. 2.2 and Cor. 2.3.

2.6. Theorem (lifting of derivations). *Let $f\colon L' \to L$ be a covering. As in* 2.1.2 *we denote $C = \mathfrak{uce}(f)(\mathrm{H}_2(L')) \subset \mathrm{H}_2(L)$.*

(a) *A derivation d of L lifts to a derivation d' of L' satisfying $d' \circ f = f \circ d$ if and only if the derivation $\mathfrak{uce}(d)$ of $\mathfrak{uce}(L)$ satisfies $\mathfrak{uce}(d)(C) \subset C$. In this case, d' is uniquely determined and leaves $\operatorname{Ker} f$ invariant. In particular, any inner derivation $\operatorname{ad} x$, $x \in L$, lifts uniquely to the inner derivation $\operatorname{ad} x'$ where $x' \in L'$ satisfies $f(x') = x$.*

(b) *Using the notation of* (a), *the map*

$$\{d \in \operatorname{Der}_S L : \mathfrak{uce}(d)(C) \subset C\} \;\to\; \{e \in \operatorname{Der}_S L' : e(\operatorname{Ker} f) \subset \operatorname{Ker} f\} : d \mapsto d'$$

is an isomorphism of Lie superalgebras mapping $\operatorname{IDer} L$ onto $\operatorname{IDer} L'$.

(c) *In particular, for the covering $\mathfrak{u}\colon \mathfrak{uce}(L) \to L$ we obtain that the map*

$$\mathfrak{uce}\colon \operatorname{Der}_S L \to \{e \in \operatorname{Der}_S \mathfrak{uce}(L) : e(\mathrm{H}_2(L)) \subset \mathrm{H}_2(L)\}$$

of 2.4.2 *is an isomorphism preserving inner derivations. If L is centreless, we even have $\operatorname{Der}_S L \cong \operatorname{Der}_S \mathfrak{uce}(L)$.*

Proof. With the exception of the statements concerning inner derivations, a proof of this result can be given along the lines of the proof of the corresponding statements on automorphisms in 2.2 and 2.2.1. Alternatively, one can use that d is a derivation if and only if $\mathrm{Id} + \varepsilon d$ is an automorphism of the base superring extension $S[\varepsilon] \otimes_S L$, cf. 1.2. The claims on inner derivations are easily checked, cf. 2.4.3. Details will be left to the reader.

2.7. Theorem. *Let $0 \longrightarrow K \xrightarrow{f} L \underset{s}{\overset{g}{\rightleftarrows}} M \longrightarrow 0$ be a split exact sequence of perfect Lie superalgebras. We abbreviate the notations introduced in* 1.11 *as follows*

$$\begin{aligned}
\mathfrak{K} &= \mathfrak{uce}(K), & \mathfrak{L} &= \mathfrak{uce}(L), & \mathfrak{M} &= \mathfrak{uce}(M), \\
\varphi &= \mathfrak{uce}(f), & \gamma &= \mathfrak{uce}(g), & \sigma &= \mathfrak{uce}(s),
\end{aligned}$$

and thus have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & \mathfrak{K} & \xrightarrow{\ \varphi\ } & \mathfrak{L} & \underset{\sigma}{\overset{\gamma}{\rightleftarrows}} & \mathfrak{M} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathfrak{u}_K} & & \downarrow{\scriptstyle \mathfrak{u}_L} & & \downarrow{\scriptstyle \mathfrak{u}_M} & & \\
0 & \longrightarrow & K & \xrightarrow{\ f\ } & L & \underset{s}{\overset{g}{\rightleftarrows}} & M & \longrightarrow & 0\,.
\end{array}$$

For $m \in \mathfrak{M}$ define $h(m) = \mathfrak{uce}\big(\operatorname{ad} s(\mathfrak{u}_M(m))\,|K\,\big) \in \operatorname{Der}_S \mathfrak{K}$, cf. 2.4, *and let $\mathfrak{K} \rtimes \mathfrak{M}$ be the semidirect product corresponding to the homomorphism $h\colon \mathfrak{M} \to \operatorname{Der}_S \mathfrak{K}$.*

(a) *Then $\mathfrak{L}$ is a semidirect product,*

$$\mathfrak{L} = \varphi(\mathfrak{K}) \rtimes \sigma(\mathfrak{M}), \tag{1}$$

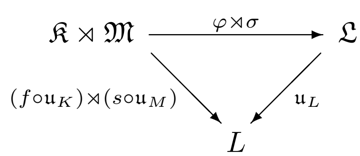
where $\varphi(\mathfrak{K}) \cong \mathfrak{K}/\operatorname{Ker}\varphi$ with

$$\operatorname{Ker}\varphi = h(\mathfrak{M})(\mathrm{H}_2(K)) \subset \mathrm{H}_2(K), \tag{2}$$
$$\sigma(\mathfrak{M}) \cong \mathfrak{M}, \quad \textit{and} \tag{3}$$
$$\mathrm{H}_2(L) = \varphi(\mathrm{H}_2(K)) \oplus \sigma(\mathrm{H}_2(M)). \tag{4}$$

Moreover, the maps

$$\varphi \rtimes \sigma : \mathfrak{K} \rtimes \mathfrak{M} \to \mathfrak{L} : k \oplus m \mapsto \varphi(k) \oplus \sigma(m) \quad \textit{and}$$
$$(f \circ \mathfrak{u}_K) \rtimes (s \circ \mathfrak{u}_M) : \mathfrak{K} \rtimes \mathfrak{M} \to L : k \oplus m \mapsto (f \circ \mathfrak{u}_K)(k) \oplus (s \circ \mathfrak{u}_M)(m)$$

are epimorphisms such that



(5)

commutes.

(b) *The following are equivalent:*

(i) *$(f \circ \mathfrak{u}_K) \rtimes (s \circ \mathfrak{u}_M) : \mathfrak{K} \rtimes \mathfrak{M} \to L$ is a central extension, and hence a covering;*
(ii) *$h(\mathfrak{M})(\mathrm{H}_2(K)) = 0$;*
(iii) *$\varphi \rtimes \sigma$ is an isomorphism, and hence $\mathfrak{K} \rtimes \mathfrak{M}$ is a universal central extension of L;*
(iv) *φ is injective.*

In particular, for a direct product $L = K \times M$ we have

$$\mathfrak{uce}(K \times M) \cong \mathfrak{uce}(K) \times \mathfrak{uce}(M) \tag{6}$$

Proof. In the proof we will interpret f and s as identifications, and therefore have $L = K \rtimes M$.

(a) Let $m \in M$, $k, k' \in K$. Then $\langle m\,,\,[k,k']\rangle = \langle [m,k]\,,\,k'\rangle + (-1)^{|m||k|}\langle k\,,\,[m,k']\rangle$ holds in $\mathfrak{L}$. This, together with perfectness of K, implies $\langle M,K\rangle \subset \langle K,K\rangle$, and hence $\mathfrak{L} = \langle K,K\rangle + \langle M,M\rangle$. We have $\langle K,K\rangle = \varphi(\mathfrak{K})$ and $\langle M,M\rangle = \sigma(\mathfrak{M})$ by definition of φ and σ respectively. Since $g \circ s = \mathrm{Id}_M$ we have $\gamma \circ \sigma = \mathrm{Id}_{\mathfrak{M}}$. Hence $\mathfrak{L} = \operatorname{Ker}\gamma \oplus \sigma(\mathfrak{M})$ is a semidirect product and σ is an isomorphism from $\mathfrak{M}$ to $\sigma(\mathfrak{M})$. Since clearly $\langle K,K\rangle \subset \operatorname{Ker}\gamma$ it now follows from $\mathfrak{L} = \langle K,K\rangle + \langle M,M\rangle$ that $\operatorname{Ker}\gamma = \langle K,K\rangle = \varphi(\mathfrak{K})$. This implies (1) and (3). We will postpone the proof of (2) and move on to (4). Because of (1), any element of $\mathfrak{L}$ has the form $\varphi(k) \oplus \sigma(m)$ for suitable $k \in \mathfrak{K}$ and $m \in \mathfrak{M}$. Such an element lies in $\mathrm{H}_2(L)$ if and only if $0 = \mathfrak{u}_L\varphi(k) = \mathfrak{u}_K(k)$ and $0 = \mathfrak{u}_L\sigma(m) = \mathfrak{u}_M(m)$ which implies (4).

Since both φ and σ are epimorphisms, $\varphi \rtimes \sigma$ will be a homomorphism (and hence an epimorphism) if, for $m \in \mathfrak{M}$ and $k \in \mathfrak{K}$,

$$\varphi(m.k) = [\sigma(m)\,,\,\varphi(k)] \tag{7}$$

where we abbreviated $m.k = h(m)(k)$. We abbreviate $l = s(\mathfrak{u}_M(m)) = \mathfrak{u}_M(m) = \mathfrak{u}_L(\sigma(m))$. It follows from 2.4.3 and 2.6(b) that the inner derivation $\operatorname{ad}_{\mathfrak{L}} \sigma(m)$ is the unique lift of $\operatorname{ad}_L l \in \operatorname{Der}_S L$:

$$\operatorname{ad}_{\mathfrak{L}} \sigma(m) = \mathfrak{uce}(\operatorname{ad}_L l)\,. \tag{8}$$

Note that the derivations $d_K = (\operatorname{ad}_L l)|K$ and $d_L = \operatorname{ad}_L l$ are related by f so that Lemma 2.5 applies. We can now prove (7):

$$\begin{aligned} \varphi(m.k) &= \mathfrak{uce}(f)\big(\mathfrak{uce}(\operatorname{ad}_L l|K).k\big) \ \ \text{(by definition of } \varphi \text{ and } h) \\ &= \mathfrak{uce}(\operatorname{ad}_L l)(\mathfrak{uce}(f)(k)) \ \ \text{(by Lemma 2.5)} \\ &= [\sigma(m)\,,\, \varphi(k)] \ \ \text{(by (8))}. \end{aligned}$$

We apply $\mathfrak{u}_L$ to (7) and obtain

$$\mathfrak{u}_K(m.k) = (\mathfrak{u}_L \circ \varphi)(m.k) = [\mathfrak{u}_L(\sigma(m)), \mathfrak{u}_L(\varphi(k))] = [\mathfrak{u}_M(m), \mathfrak{u}_K(k)].$$

This relation easily implies that also $\mathfrak{u}_K \rtimes \mathfrak{u}_K$ is an epimorphism. Commutativity of (5) holds by definition.

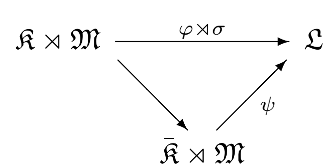
We now come to the proof of (2). Since $\varphi(\mathrm{H}_2(K)) \subset \mathrm{H}_2(L) by (\text{by}(4)) \subset Z(\mathfrak{L})$ we have

$$\mathfrak{M}.\mathrm{H}_2(K) \subset \operatorname{Ker} \varphi\,. \tag{9}$$

For the proof of the other inclusion we first note that

$$\operatorname{Ker} \varphi \subset \mathrm{H}_2(K) \tag{10}$$

since $0 = \mathfrak{u}_L(\varphi(\operatorname{Ker} \varphi)) = f(\mathfrak{u}_K(\operatorname{Ker} \varphi)) = \mathfrak{u}_K(\operatorname{Ker} \varphi)$. This together with (9) implies that $\mathfrak{M}.\mathrm{H}_2(K)$ is $h(\mathfrak{M})$ invariant. Hence the action of $\mathfrak{M}$ on $\mathfrak{K}$ descends to an action of $\mathfrak{M}$ on $\bar{\mathfrak{K}} = \mathfrak{K}/\mathfrak{M}.\mathrm{H}_2(K)$. Denote by $\bar\varphi\colon \bar{\mathfrak{K}} \to \varphi(\mathfrak{K})$ the canonical map induced by φ. Identifying $\operatorname{Ker} \varphi = \operatorname{Ker}(\varphi \rtimes \sigma)$ we obtain an induced epimorphism $\psi = \bar\varphi \rtimes \sigma$



with kernel $\operatorname{Ker} \varphi/\mathfrak{M}.\mathrm{H}_2(K)$. We claim that ψ is a central extension. Indeed, for arbitrary $k \in \mathfrak{K}$, $m \in \mathfrak{M}$ and $\bar x \in \operatorname{Ker} \varphi/\mathfrak{M}.\mathrm{H}_2(K)$ we have, with obvious notation, $[\bar k \oplus m, \bar x] = \overline{[k, x]} + \overline{m.x} = 0$ by (10). Since $\mathfrak{L}$ is centrally closed and $\bar{\mathfrak{K}} \rtimes \mathfrak{M}$ is perfect, ψ is an isomorphism, and hence $\operatorname{Ker} \varphi = \mathfrak{M}.\mathrm{H}_2(\mathfrak{K})$.

(b) The kernel of $(f \circ \mathfrak{u}_K) \rtimes (s \circ \mathfrak{u}_M) = \mathfrak{u}_K \rtimes \mathfrak{u}_M$ is $\mathrm{H}_2(K) \oplus \mathrm{H}_2(M)$. Here $\mathrm{H}_2(M)$ is always central in $\mathfrak{K} \rtimes \mathfrak{M}$ since $\mathrm{H}_2(M)$ is central in $\mathfrak{M}$ and $h(\mathrm{H}_2(M)) = 0$. Since $\mathrm{H}_2(K)$ is central in $\mathfrak{K}$ we have (i) $\Leftrightarrow$ (ii). The equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) are immediate from (2).

In the case of a direct product we have $h = 0$, so $\mathfrak{K} \rtimes \mathfrak{M} = \mathfrak{K} \times \mathfrak{M}$. Clearly (ii) is fulfilled, proving that $\varphi \times \sigma\colon \mathfrak{K} \times \mathfrak{M} \to \mathfrak{L}$ is an isomorphism. Thus (6) holds.

Remark. In general, the condition (b.ii) is not fulfilled. For example, let $K = \mathfrak{sl}_N(C_q)$, $N \geq 3$, where C_q is a quantum torus defined with respect to a $n \times n$-matrix, $n \geq 2$. If C_q has a nontrivial centre, it follows from the results in [**3**] that $\operatorname{Der} K$ operates nontrivially on $\mathrm{H}_2(K)$. Because of (b.iv) this also means that $\varphi\colon \mathfrak{K} \to \mathfrak{L}$ is in general not injective.

Since K is a perfect centreless Lie algebra, this example contradicts [**2**, Th. 3.8] which in the notation from above claims that the universal central extension of a semidirect product $K \rtimes \operatorname{Der} K$ is always isomorphic to $\mathfrak{uce}(K) \rtimes \mathfrak{uce}(\operatorname{Der} K)$. Note however that the main application of [**2**, Th. 3.8] in [**2**], the preceding result [**2**, Th. 3.7], remains valid since the condition (b.ii) can easily be verified in the setting of [**2**, Th. 3.7].

Formula (2) for the kernel of φ has been suggested by Georgia Benkart and Bob Moody.

2.8. Notes. That automorphisms lift uniquely to the universal central extension of a perfect Lie algebra, cf. 2.3, is already contained in [**28**, 3.1]. Using a different model of the universal central extension, this is also proven in Pianzola's recent preprint [**22**]. The corresponding result for derivations of Lie algebras, cf. 2.6, can be found in [**2**, Th. 2.2].

Let $0 \longrightarrow K \longrightarrow L \xrightarrow{g} M \longrightarrow 0$ be a not necessarily split exact sequence of Lie superalgebras, cf. 2.7. By functoriality we then obtain an epimorphism $\mathfrak{uce}(g)\colon \mathfrak{uce}(L) \longrightarrow \mathfrak{uce}(M)$. In the setting of Lie algebras the kernel of the map $\mathfrak{uce}(g)$ was determined by Ellis, see the remark after Prop. 9 of [**11**]: it is $(L \otimes_{\mathrm{Lie}} K) \rtimes (K \otimes_{\mathrm{Lie}} L)$ where $\otimes_{\mathrm{Lie}}$ denotes the non-abelian tensor product of Lie algebras.

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