The Green-Wald conjecture and its aftermath

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LABEX LIO
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Averaging/smoothing the metric and calculating the curvature tensors do not commute

$$
\begin{array}{ccccccccccccc}
g^n_{ab} & \longrightarrow & \langle\rangle & \longrightarrow & g^{n-1}_{ab} & \longrightarrow & \langle\rangle & \longrightarrow & g^1_{ab} & \longrightarrow & \langle\rangle & \longrightarrow & g^0_{ab} \\
\downarrow & & & & \downarrow & & & & \downarrow & & & & \downarrow \\
G^n_{ab} = \kappa T_{ab} & \longrightarrow & \langle\rangle & \longrightarrow & \langle G_{ab}(g^n_{ab})\rangle \neq G_{ab}(g^{n-1}_{ab}) & \longrightarrow & \langle\rangle & \longrightarrow & \ldots & \longrightarrow & \langle\rangle & \longrightarrow & \langle G_{ab}(g^n_{ab})\rangle \neq G_{ab}(g^0_{ab})
\end{array}
$$

which leads to:

$$
\boxed{G_{\mu\nu}(\langle g_{\mu\nu}\rangle_{\mathcal{D}}) = 8\pi\langle T_{\mu\nu}\rangle_{\mathcal{D}} + t_{\mu\nu}}
$$

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- Response to rebuttal, by Green and Wald: ‘Comments on Backreaction’ (arXiv: 1506.06452[gr-qc])
- Further examples and applications

From Green and Wald, 2011:

But if one cannot neglect nonlinear terms in Einstein's equation on small scales, how can one justify neglecting them on large (i.e., 100 Mpc or larger) scales?

(...) Indeed, it is far from obvious, a priori, that nonlinearities associated with small-scale inhomogeneities could not produce important effects on the large-scale dynamics of the FLRW model itself (...)

Nevertheless, the situation (...) referring to ΛCDM assumptions *(...) is quite unsatisfactory from the perspective of having a mathematically consistent theory wherein the assumptions and approximations are justified in a systematic manner.*

In particular, nonlinear effects play an essential role in Newtonian dynamics (...)

Assumptions

We fix the background manifold and choose the gauge in which:

- For all $\lambda > 0$ the metric $g_{ab}(\lambda, x)$ satisfies:

$$G_{ab}(g(\lambda, x)) + \Lambda g_{ab}(\lambda, x) = 8\pi T_{ab}(\lambda),$$

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- There exists a smooth tensor field μ_{abcdef} on M such that:

$$\underset{\lambda \searrow 0}{\text{w–lim}} (\nabla_a h_{cd}(\lambda, x) \nabla_b h_{ef}(\lambda, x)) = \mu_{abcdef}.$$

Weak limit, Green and Wald equations

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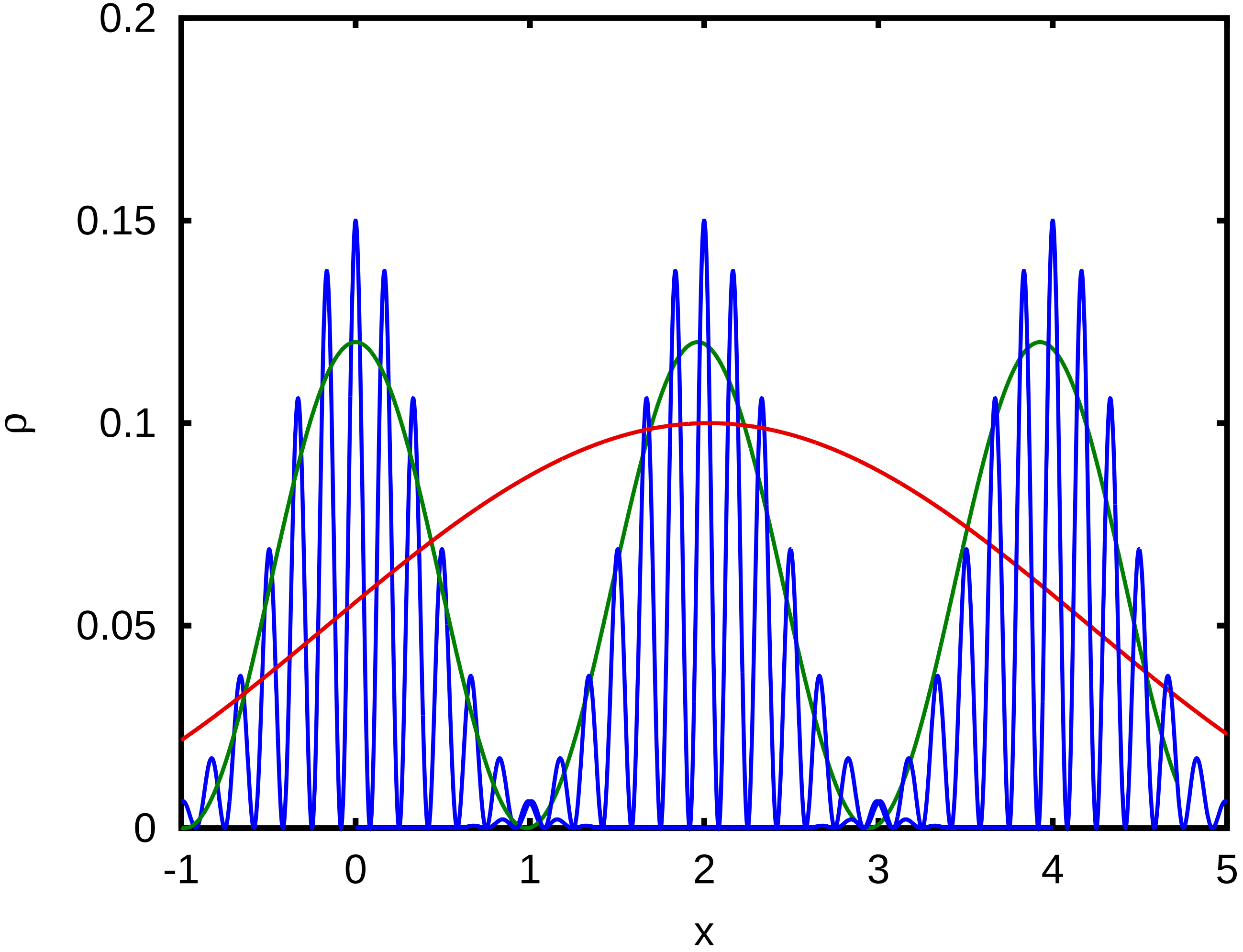
Green and Wald formalism: key points

- physical metric has to be close to the background everywhere
- first and second metric derivatives converge weakly to its background values, products of derivatives NOT - **backreaction**
- for the backreaction to be non-zero, metric deviations have to behave like e.g. $h_{ab}(x, \lambda) = \lambda \sin(x/\lambda)$ hence their derivatives can not converge point-wise and have to blow-up at $\lambda = 0$
- one of the most important questions is: what is the physical meaning of Green and Wald formalism?

Green and Wald: applications

- General relativity
  - $\rightarrow$ confirming GW conjectures: Wainwright-Marshman space-times (trivial example), Einstein-Rosen waves coupled to a massless scalar field; **Szybka et al 2016**
  - $\rightarrow$ comparing GW formalism with scalar averaging; **Glod, Sikora 2016**
- Modified gravity
  - $\rightarrow$ contradictory results; **Saito 2012, Preston 2014**
- Mathematics
  - $\rightarrow$ weak limits of vacuum space-times; **Cecile 2017, Lott 2018**
- So far, no realistic cosmological applications...

Averaging: coarse-graining vs homogenization



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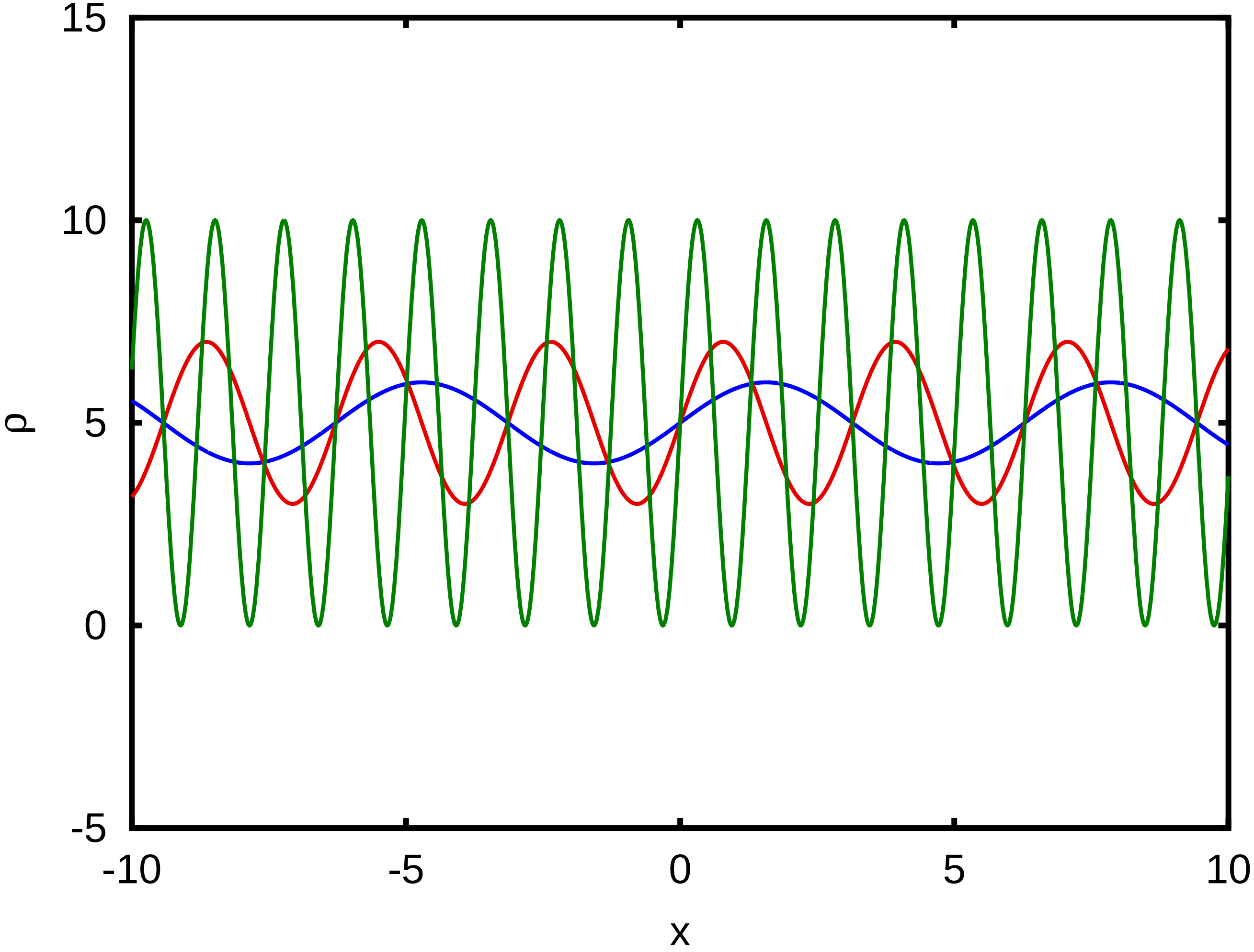


Two-scale asymptotic homogenization:

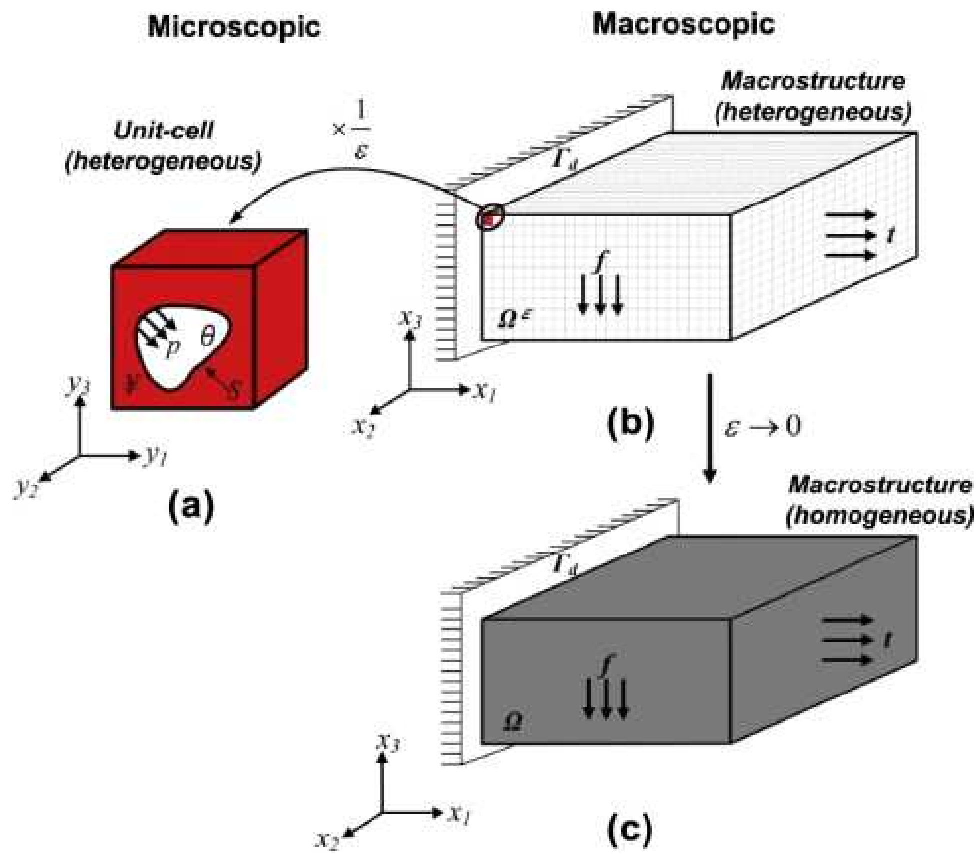
- We look for the solution to differential equation with rapidly oscillating coefficients e.g.:

$$\nabla \cdot \left(A \left(\frac{x}{\epsilon} \right) \nabla u_\epsilon \right) = f \ ,$$

  where ϵ is a ratio of two characteristic scales

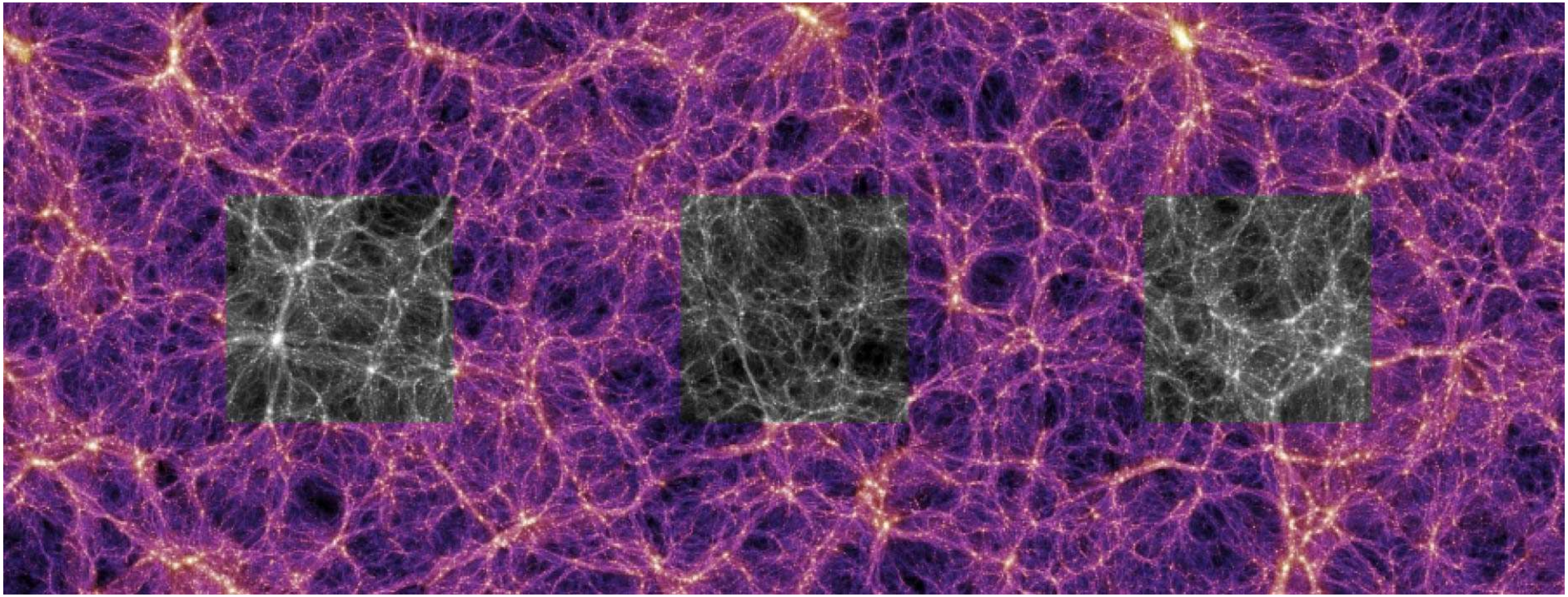
- The averaged equation reads: $\nabla \cdot (A^\star \nabla u) = f$
- both A and u converge weakly to their background values, but not their product - **backreaction**
- λ in Green and Wald plays the same role as ϵ - metric derivatives blow up at $\lambda = 0$

Representative elementary volume - big enough to contain all representative features of fluid, small compared to homogeneity scale



Nasution et al, 2014

The real universe has hierarchical structure - at each scale, up to homogeneity, one finds characteristic features of cosmic web



Clarkson 2010, Springel et al 2004

Conclusions:

- Green and Wald formalism extends the work by Isaacson and Burnett which was designed to examine the high frequency limit of gravitational radiation; density field however, unlike gravitational waves forms persistent, hierarchical structures
- representative elementary volume for cosmology would have to contain clusters, voids and filaments for which we would need to know the Einstein equations exactly
- even then, in cosmic web there is no clear scale separation between the microscale and the macroscale
- Green and Wald formalism, being a special case of two-scale asymptotic homogenization is not applicable to gravitational systems with hierarchical structures
- any features of backreaction (including backreaction being trace-free) based on Green and Wald formalism are unjustified