Koszul Duality
in Field Theory and Holography

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Based on 1911.xxxxx + works in progress
w/ K. Costello

Motivation

i.e. what the heck is Koszul duality and why should I listen?

- We are learning a lot about the physics of boundary conditions and defects.

- Sometimes these defects support local operators that we can describe precisely and are algebraically interesting.

- D-branes couple to closed string theory as defects: applications include AdS/CFT. Today we will play with AdS3.

- We seek a more fundamental, mathematical perspective on algebras of (defect) local operators and how they couple to bulk physics.

Koszul duality relates

1.) an algebra of local operators in a bulk theory

$$\mathscr{A}$$

2.) a ***universal defect*** algebra

$$\mathscr{A}^!$$

the most general algebra to which it can couple in an anomaly-free way

That is too coarse (of course!)
We will improve on this statement, adding all the requisite caveats, soon.

[Quillen, Sullivan, Priddy, Bernstein-Gelfand-Gelfand, Ginzburg-Kapranov, Beilinson-Ginzburg-Soergel, Goresky-Kottwitz-MacPherson, Costello-Gwilliam, Ayala-Francis, many, many others….]

Warm up:

topological line defects in QFT

- A Euclidean QFT on $\mathbb{R} \times \mathbb{R}^n$

- Assume the theory is topological in $\mathbb{R}$

- Q: *What is the most general topological line operator to which we can couple our system?*

- Say line described by some QM system with operator algebra $\mathscr{B}$ coupled to the bulk theory with algebra $\mathscr{A}$

- A: Any coupling of $\mathscr{B}$ to $\mathscr{A}$ is the same as giving an algebra homomorphism Hom$(\mathscr{A}^!, \mathscr{B})$

E.g: Chern-Simons w/ gauge algebra $\mathfrak{g}$

- Topological line implies local ops have dimension 0. Write down the most general dimensionless coupling:
- $\int_{\mathbb{R}^3} CS(A) + \int_{\mathbb{R}} \mathscr{L}(\phi) + \text{PExp}\left(\int_{\mathbb{R}} A_t^a \rho_a(\phi)\right), \quad \rho_a \in \mathscr{B}$
- Insert $\delta A^a = dc^a + f_{bc}^a c^b A^c$ into PExp
- $\sum_{n \geq 1} \sum_{i=1}^{n} \int_{t_1 \leq \ldots t_n} A_t^{a_1}(t_1)\rho_{a_1}(t_1)\ldots\left(dc^{a_i} + f_{bc}^{a_i} c^b(t_i) A_t^c(t_i)\right)\ldots A_t^{a_n}(t_n)$
- Int by parts: dc can pick up bdy terms at t_i, t_{i+1}. Collision: $\rho_{a_i}, \rho_{a_{i+1}}$ multiplied with algebra
- Cancellation of the gauge variation: $\rho_b \rho_c - \rho_c \rho_b = f_{bc}^a \rho_a$
- ρ_a **satisfy the relations of the universal enveloping algebra:** $U(\mathfrak{g}) = \mathscr{A}^!$

Mathematical Perspective:

CS has no gauge-inv't local ops of ghost # 0 but $\mathscr{A}$ nontrivial

$$Q_{BRST} c^a = \frac{1}{2} f^a_{bc} c^b c^c$$

Abelian case: $f^a_{bc} = 0$ hence $\mathscr{A} = \wedge^* \mathfrak{g}^*$ $\leftarrow$

ghost # > 0

(some standard algebraic results)

$\rightarrow \mathscr{A}^! = \textbf{Sym}^* \mathfrak{g}$

ghost # 0

Can apply deformations to both sides:

- *non-Abelian: add a differential to* $\mathscr{A}$*:* $d = \frac{1}{2} f^i_{jk} c^j c^k \partial_{c_i}$
- $d^2 = 0 \leftrightarrow f^i_{jk}$ *are structure constants*
- *Correspondingly,* $\mathscr{A}^!$ *becomes non-commutative:*
- $\left[\rho_i, \rho_j\right] = f^k_{ij} \rho_k$
- *Differential graded algebras (w/ coh. grading):*

$\mathscr{A} = (\wedge^* \mathfrak{g}^*, d) \leftrightarrow \mathscr{A}^! = (U(\mathfrak{g}), 0)$

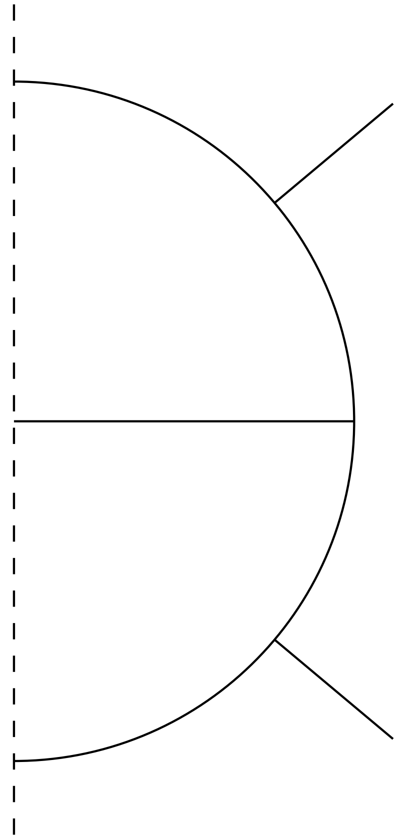
Quantum corrections

- E.g: 4d Chern-Simons
  [Costello-Witten-Yamazaki ’17]

- $PExp \int t_{a,k} \partial_z^k A^a, \quad t_{a,k} \in \mathscr{B}$

Classically: $\left[t_{a,k}, t_{b,l}\right] = f_{ab}^c t_{c,k+l}$

A 2-loop anomaly



The diagram modifies the commutation relations so that

$$t_{a,k} \in Y(\mathfrak{g})$$

We learn that the algebra of local operators of 4d CS is the Koszul dual of the Yangian

A general argument:

Before coupling: $\mathscr{A} \otimes \mathscr{B}$ is algebra of operators on line

A deformation takes generic form:

1st order

$$\textbf{TrPExp} \int_{\mathbb{R}_t} \mathcal{O}^{(0)} \qquad \text{satisfying} \qquad Q_{BRST}\mathcal{O}^{(0)} = \partial_t \mathcal{O}^{(1)} + \left[\mathcal{O}^{(1)}, \mathcal{O}^{(0)}\right]$$

(sol'n of quantum master equation)

Assume bulk/defect theories have $\hat{Q}$ s.t. $\left[Q_{BRST}, \hat{Q}\right] = \partial_t$ $\qquad \rightarrow \mathcal{O}^{(0)} = \hat{Q}\mathcal{O}^{(1)}$

so that now BRST invariance holds $\leftrightarrow Q_{BRST}\mathcal{O}^{(1)} + \frac{1}{2}\left[\mathcal{O}^{(1)}, \mathcal{O}^{(1)}\right] = 0$

Maurer-Cartan equation!

Known equivalence: **MC**$(\mathscr{A} \otimes \mathscr{B}) \simeq$ **Hom**$(\mathscr{A}^!, \mathscr{B})$ [Lurie, Costello-Gwilliam,…]

Two ways of writing space of ways to couple the theories along the line!

Koszul Duality & Boundary Conditions

[Beem-Ben-Zvi-Bullimore-Dimofte-Neitzke, Dimofte's String-Math 2017 talk, …]

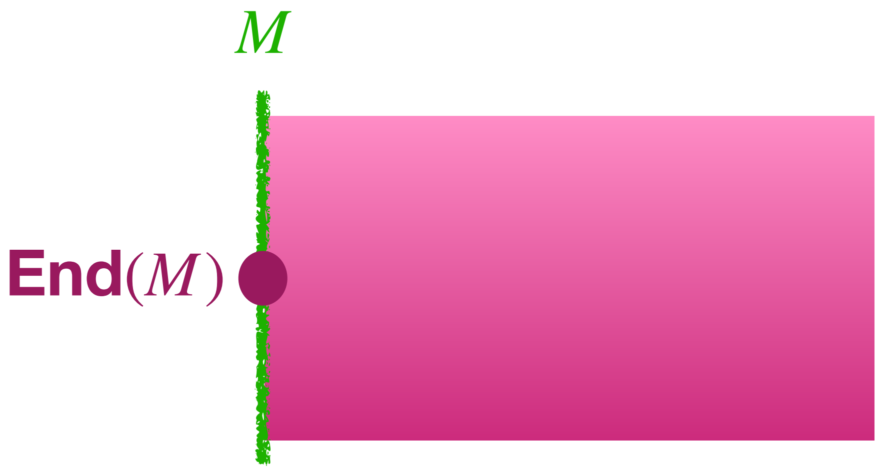
- Continue w/ associative algebras (i.e. QM)

- $h : \mathscr{A} \to \mathbb{C}$,
  $h(ab) = h(a)h(b),\ h(\mathrm{d}a) = 0$
  augmentation

- $\mathscr{A}^! = \mathrm{Ext}^*_{\mathscr{A}}(\mathbb{C}_h, \mathbb{C}_h)$, symmetries of the trivial $\mathscr{A}$-module

- 2d: category of boundary conditions in a TFT, rep category of an algebra

- Any rep M: alg of local ops on bdy: $\mathrm{Ext}^*_{\mathscr{A}}(M, M)$



$\mathscr{H}_{B,B^!} =$ **Hom**$(B, B!) \simeq \mathbb{C}$

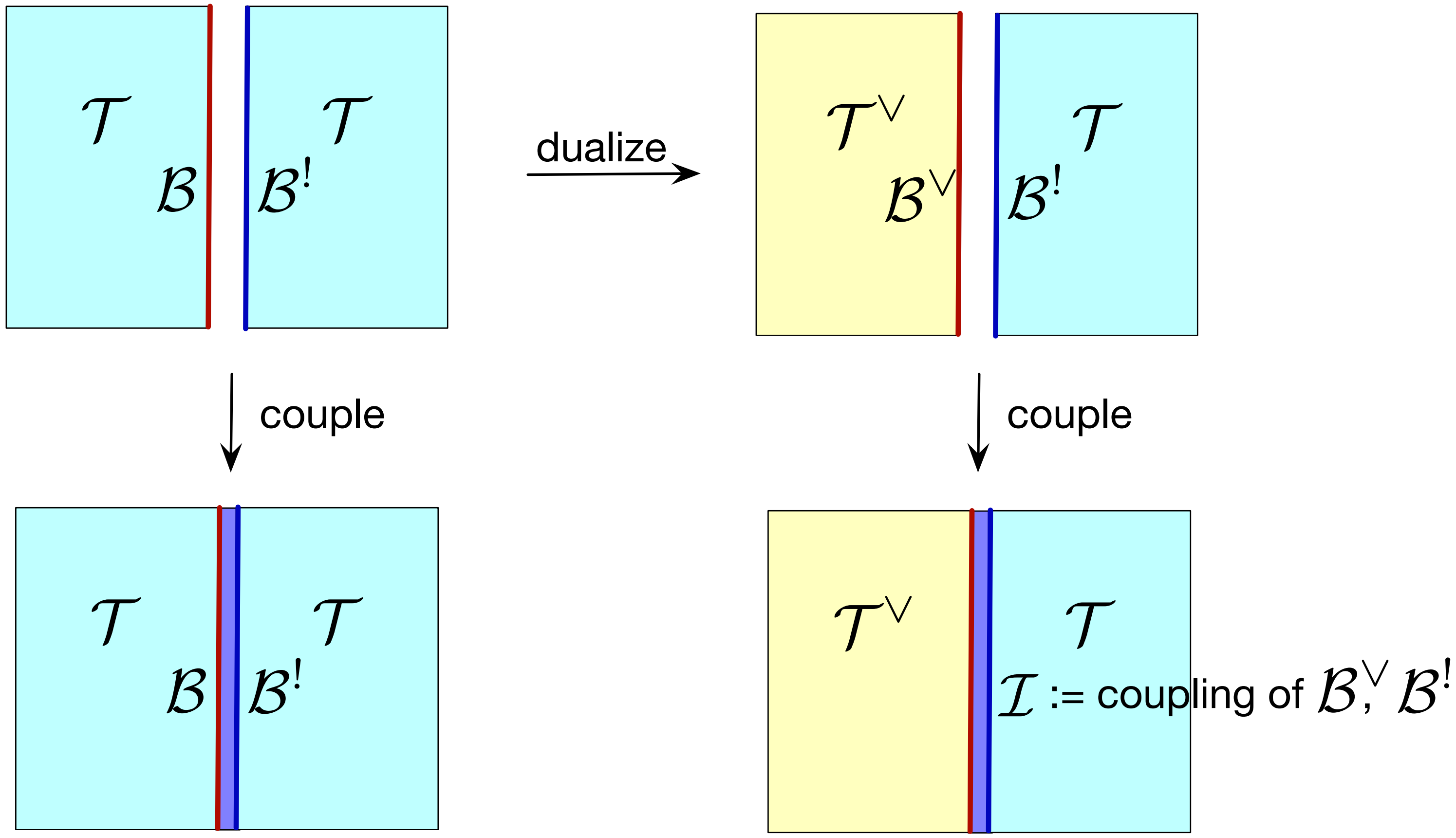
E.g. Neumann/Dirichlet!

Higher Dimensions?

Soon I'll *define* Koszul duality for chiral/vertex algebras from physics

- 2d CFTs (holomorphic half)
- Twist of 4d $\mathcal{N}=2$ theories [Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees]
- Ω-deformations [Nekrasov, Yagi, Gaiotto-Oh,…]
- Holomorphic/topological twists of 3d $\mathcal{N}=2$ [Aganagic-Costello-McNamara-Vafa]
- $\vdots$

Boundary conditions supporting Koszul dual algebras cut and glue the space, useful for duality interfaces



[Dimofte-Gaiotto-NMP]

(see also [Bullimore-Dimofte-Gaiotto-Hilburn])

Koszul duality for chiral algebras: a physical prescription

- A Euclidean QFT on $\mathbb{C} \times \mathbb{R}^n$
- Assume the theory is holomorphic along $\mathbb{C}$
- Define: $\mathscr{A}^!$ is the universal vertex algebra that can be coupled to the theory as algebra of operators of a defect wrapping $\mathbb{C}$
- Any coupling of $\mathscr{B}$ to $\mathscr{A}$ is the same as giving an algebra homomorphism Hom$(\mathscr{A}^!, \mathscr{B})$

We want to compute the Koszul dual of an algebra in a theory of interest

- We have done this at the classical level, e.g. in our Chern-Simons example
- As alluded to, quantum corrections will modify the algebra
- Order-by-order: Feynman diagrams in coupled defect/bulk system
- **A field theory warm-up w/ holomorphic Chern-Simons** on $\mathbb{C}^3$ [Costello-NMP]

$$\mathbb{C}_z \times \mathbb{C}^2_{w_1,w_2} \qquad A = A_{\bar z} d\bar z + A_{\bar w_1} d\bar w_1 + A_{\bar w_2} d\bar w_2 \qquad \int dz dw_1 dw_2 CS(A)$$

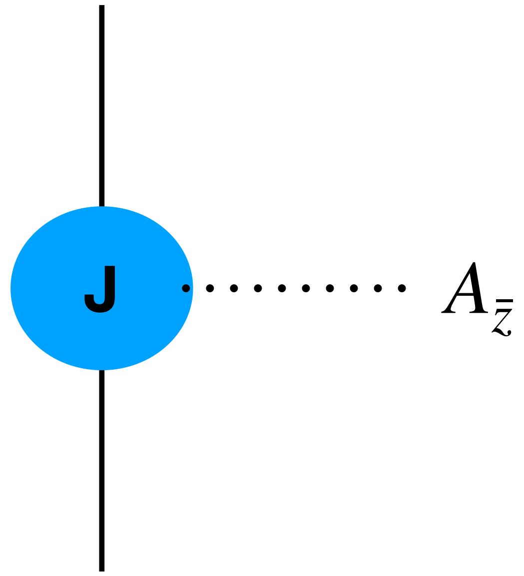
Introduce a defect at $w_1 = w_2 = 0$ with operator algebra

$$\mathscr{B}$$

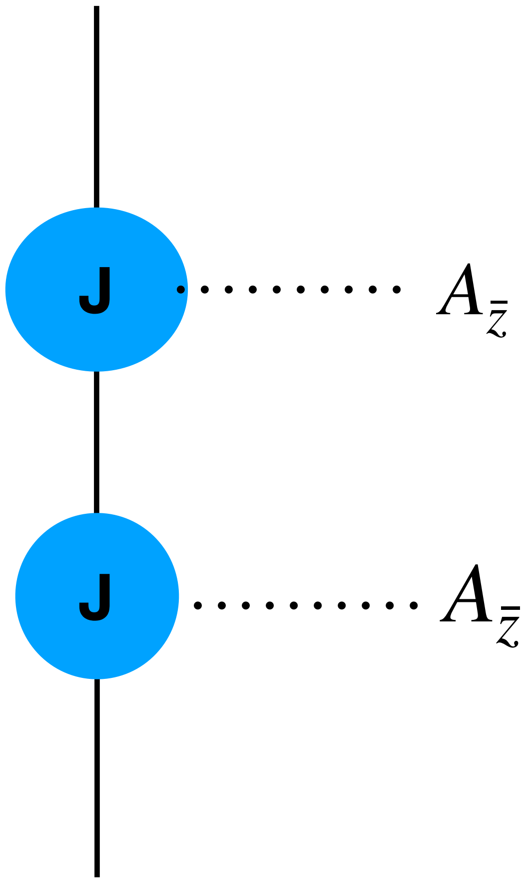
- Write down BRST invariant couplings with $J \in \mathscr{B}$

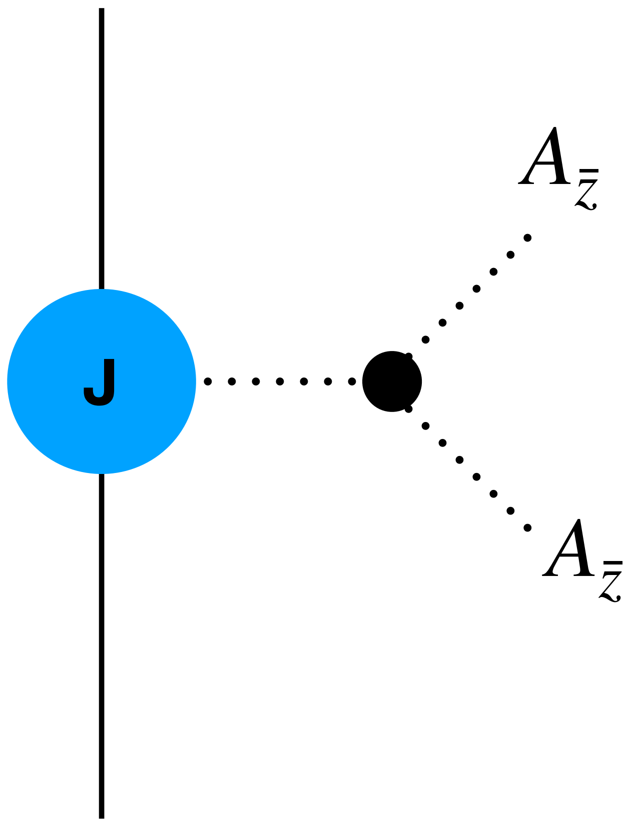
$$\sum_{k_1,k_2\geq 0} \int \frac{1}{k_1!k_2!} \partial^{k_1}_{w_1} \partial^{k_2}_{w_2} A^a_{\bar z} J_a[k_1,k_2]$$

represented by new bulk/defect vertex



Classical algebra



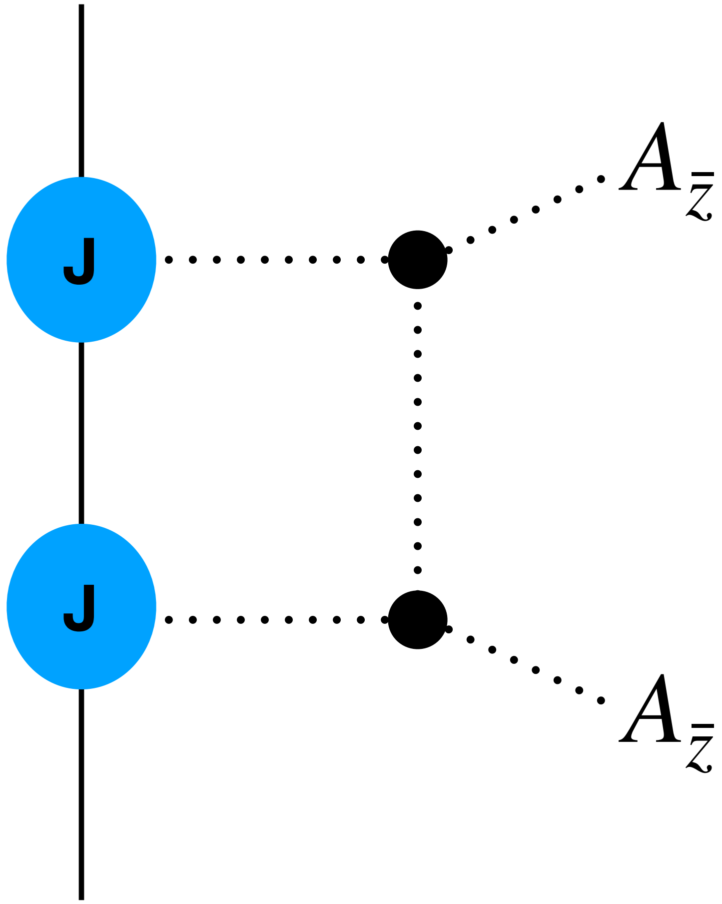


Compute gauge variation of the tree-level Feynman diagrams

Requiring that the two variations cancel each other gives:

$$J_b[l_1, l_2](0)J_c[m_1, m_2](z) \sim \frac{1}{z} f^a_{bc} J_a[l_1 + m_1, l_2 + m_2]$$

Quantum algebra



It turns out that this diagram has a gauge anomaly:

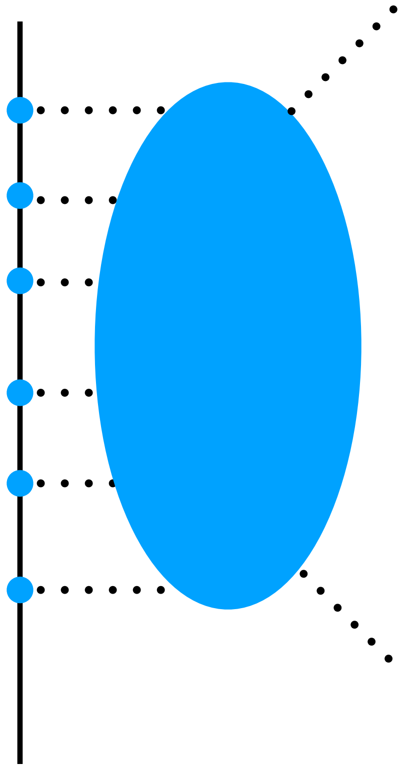
$$\hbar \int_{w_1=w_2=0} \epsilon_{ij}(\partial_{w_i} A^a_{\bar{z}})(\partial_{w_j} c^b) K^{fe} f^c_{ae} f^d_{bf} J_c J_d + \dots$$

Cancelled if the classical OPE gets corrected:

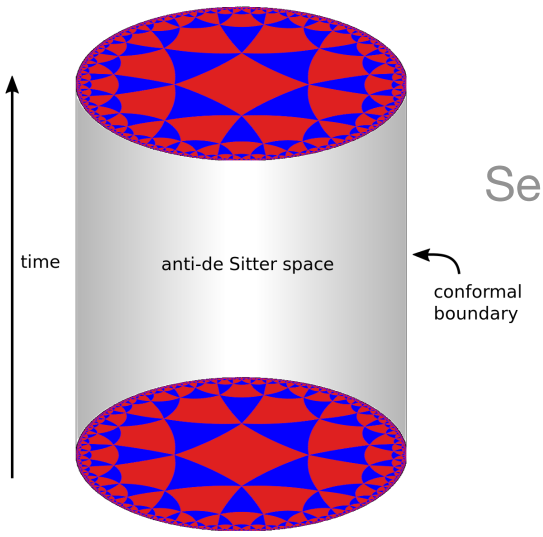
$$J_a[1,0](0) J_b[0,1](z) \sim \frac{1}{z} f^c_{ab} J_c[1,1] + \hbar \frac{1}{z} K^{fe} f^c_{ae} f^d_{bf} J_c[0,0] J_d[0,0]$$

Quantum algebra 2

In general, this theory has anomalous diagrams at each loop order w/ 2 external gluons



- k vertices, l loops
- cancel with an order $\hbar^l$ correction to OPE
- OPE of 2 operators J[r, s] expressed in terms of k operators or z-derivatives



Twisted Holography

[Costello-NMP]

See also [Costello, Costello-Li, **Costello-Gaiotto**, Ishtiaque-Moosavian-Zhao]

many analogous computations

$$AdS_3 \times S^3 \times T^4 \leftrightarrow Sym^N(T^4), \quad N \to \infty$$

- **Twisted holography:** restrict to subset of protected, cohomological observables
  - Toy model of holography, part of original duality
  - Goldilocks case: mathematically rigorous, physically rich

Goals:

1. Study twisted holography in a well-studied, physically central-but-simple context
2. Illustrate role of Koszul duality in holography in a concrete but familiar setting

Gravitational backreaction ↔ *Deformation* of Koszul duality

Twisted AdS3

Today: Focus mostly on gravitational side of duality

Conjecture: [Costello-Li] One can choose a localizing supercharge for type IIB on $\mathbb{R}^{10}$ such that the twisted theory is Kodaira-Spencer theory (B-model) $\mathbb{C}^5$

[BCOV]

Very similar arguments apply to $\mathbb{C}^3 \times T^4$

Twisted closed string sector

- Basic fields of KS: $PV^{i,j}(X) = \Omega^{(0,j)}(X, \bigwedge^{i} TX) \overset{\vee\, \Omega_X}{\simeq} \Omega^{5-i,j}(X)$

(i, j)=(1,1): Beltrami differential, deformations of complex structure

(i,j)=(k, 5-k-1): sourced by a D_{2k-1}-brane, RR (9-2k)-form

Compactification on T^4:

$$\mu \in \oplus_{i,j} PV^{i,j}(\mathbb{C}^3) \otimes \mathbb{C}[\eta_a] \xleftarrow[a = 1,\ldots,4]{dz_a \text{ on } T^4}$$

Twisted open string sector

N_1 D1 branes on $\mathbb{C}_z \subset \mathbb{C}^3$, N_5 D5 branes on $\mathbb{C}_z \times T^4$

$$F_1 = \sum F_1^{ab} \eta_a \eta_b \qquad F_5 = \sum F_5^{ab} \eta_a \eta_b \qquad F = F_1 + F_5 \rightarrow \langle F, F \rangle = N_1 N_5 =: N$$

Backreaction from Brane Sources

$$\mu \in PV^{1,1}(\mathbb{C}^3) \otimes \mathbb{C}[\eta_a] \simeq \Omega^{2,1}(\mathbb{C}^3) \otimes \mathbb{C}[\eta_a]$$

$$\bar{\partial}\mu = F^{ab}\eta_a\eta_b\delta_{\mathbb{C}}$$

Solve equation of motion (+ a constraint I suppressed):

$$\mu = F^{ab}\eta_a\eta_b \frac{\epsilon^{ij}\bar{w}_i d\bar{w}_j}{(w_1\bar{w}_1 + w_2\bar{w}_2)^2} dw_1 dw_2$$

Which functions are holomorphic in the deformed complex structure?

$$u_1 = w_1 z - F^{ab}\eta_a\eta_b \frac{\bar{w}_2}{|w|^2} \qquad u_2 = w_2 z + F^{ab}\eta_a\eta_b \frac{\bar{w}_1}{|w|^2}$$

Notice: $u_2 w_1 - u_1 w_2 = F^{ab}\eta_a\eta_b$ cf. deformed conifold in [Costello-Gaiotto]

$F^3 = 0$ in $H^*(T^4) \rightarrow$ **Finite order corrections to flat space computations!**

Twisted supergravity on $AdS_3 \times S^3 \times T^4$ is

Kodaira-Spencer Theory on $\sim \mathbb{C}^{3|4}$

Twisted Holography & Koszul Duality

- Twisted dual CFT is related to 2 $bc\beta\gamma$ systems on T^4, BPS states in CFT

[Witten, Kapustin, Malikov-Schechtman-Vaintrob]

- Compute correlation functions/OPEs, compare with Witten diagrams of KS

[Costello-Gaiotto, Costello-NMP (to appear)]

Today: different route to same answers from Koszul duality, focus on KS theory

1. Enumerate single-particle gravitational states

Recover short psu(1,1|2) reps of the CFT [de Boer]

2. Form a Lie algebra of global symmetries of twisted SUGRA

$$\mathscr{V}_\infty := \left\{ Vect_0(\mathbb{C}^x \times \mathbb{C}^2) \oplus \left(\Pi\mathscr{O}(\mathbb{C}^x \times \mathbb{C}^2)\right)^2 \right\} \otimes H^*(T^4)$$

(+ central extension which we can compute)

Holomorphic divergence-free vector fields

μ

Fermionic holomorphic functions

α, γ

OPEs from Gravity, Pt 1: No Backreaction

Enumerating the states following [Costello-Gaiotto]

Short reps of $SU(1,1|2)$!

[de Boer]

- Solve e.o.m.
- Satisfy ``vacuum” b.c. except z=0
- All occur with multiplicity from $H^*(T^4)$

KS field	Statistics	$SU(2)_R$ Highest weight	$SO(2)$ Highest weight
$\alpha \sim \delta^{(l)}_{z=0} n^{-k} + \ldots$	Fermionic	k/2 $k \geq 0$	k/2 + 1
$\gamma \sim \delta^{(l)}_{z=0} n^{-k} + \ldots$	Fermionic	k/2 $k \geq 0$	k/2 + 1
$\mu_0 \sim d \log n dz n^{-k} \delta^{(l)}_{z=0} + \ldots$	Bosonic	k/2 $k > 0$	k/2
$\mu_2 \sim d \log n dw n^{-k-2} \delta^{(l)}_{z=0}$	Bosonic	k/2 $k \geq 0$	k/2 + 2

OPEs from Gravity, Pt 1: No Backreaction

Q: Which chiral algebra operators can we couple to as a defect?

These are our dual twisted CFT operators

$SO(2): \ 1 + \frac{m+n}{2}$

Example:

$$\frac{1}{m!n!}\int_{\mathbb{C}^{1|4}} G_1[m,n](\eta_a)\partial^m_{w_1}\partial^n_{w_2}\alpha(\eta_a)dzd^4\eta \qquad \& \qquad \frac{1}{m!n!}\int_{\mathbb{C}^{1|4}} G_2[m,n](\eta_a)\partial^m_{w_1}\partial^n_{w_2}\gamma(\eta_a)dzd^4\eta$$

$$G_{1,2}[m,n](\eta^a) = G^0_{1,2}[m,n] + G^1_{1,2;a}[m,n]\eta^a + \ldots$$

$G^0_1[1,0], G^0_1[0,1], G^0_2[1,0], G^0_2[0,1]$

$\mathcal{N} = 4$ SUSYs!

Run through the same (classical) argument as we did before!

Compute gauge variation & require cancellation

$$G_1[r,s](0,\eta^a)G_2[k,l](\tilde{\eta}^a) \sim -\frac{1}{z^2}J[k+r,s+l](0,\eta^a+\tilde{\eta}^a) + \frac{1}{z}(rl-ks)T[r+k-1,s+l-1](0,\eta^a+\tilde{\eta}^a)$$

couples to μ_0

couples to μ_2

Holography & a deformation of Koszul Duality

Momentarily return to holomorphic Chern-Simons theory (N B-branes)

- Include its coupling to Kodaira-Spencer

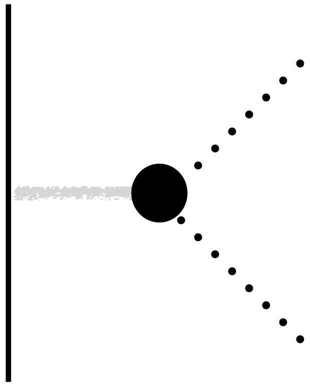
- **Branes backreact on the geometry** $\longleftrightarrow$ **Koszul duality gets deformed**

$S \to S + S_{BR}$

$$S_{BR} = \frac{1}{2}\int_{\mathbb{C}^3} A^a \mu_{BR} A^a dz dw_1 dw_2 \sim N \int_{\mathbb{C}^3} \frac{\epsilon^{ij}\bar{w}_i d\bar{w}_j}{(w_1\bar{w}_1 + w_2\bar{w}_2)^2} A^a \partial_z A^a dz dw_1 dw_2$$

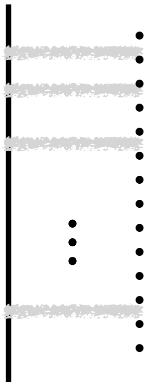
$\delta_{gauge} S_{BR} \neq 0$ on $\mathbb{C}_z$

New vertices:



Coupling to identity operator of chiral algebra

Deformation away from flat space



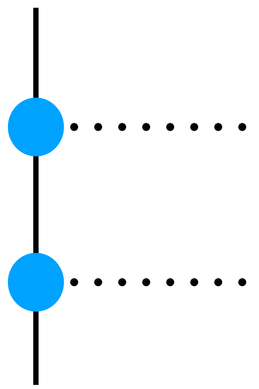
propagator in backreacted geometry

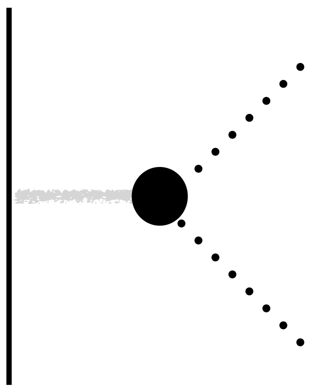
Universal chiral algebra couples to bulk gravitational theory

such that it has a non-zero anomaly that cancels that of S_{BR}

Holography & a deformation of Koszul Duality

To first order (tree-level), we need these two diagrams to cancel:





Here's the answer: $$J^a[0,0](0)J^b[0,0](z) \sim \frac{1}{z} f^{ab}_c J^c[0,0] + \delta^{ab} N \mathbf{Id} \frac{1}{z^2}$$

Algebra of the currents: Kac-Moody algebra at level-N, with the central extension!

$J^a[r,s]J^b[k,l]$ OPEs will also get deformed!

Again, can work order-by-order in perturbation theory.

Holographic dictionary: structure constants in the deformed Koszul dual algebra

↕

2-particle scattering in backreacted geometry

OPEs from Gravity, Pt 2: Backreaction's Back (alright!)

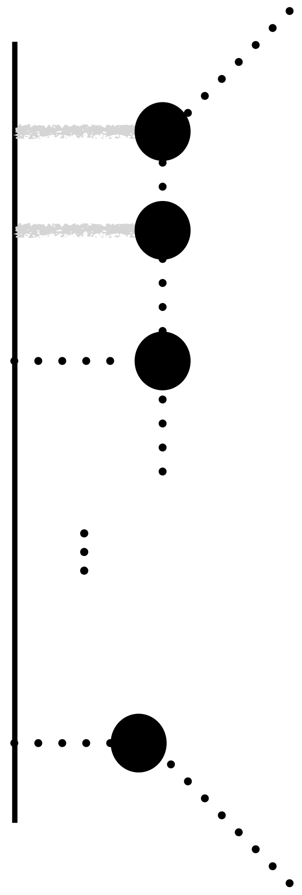
Very similar to what we just did! Formally, replace:

$$N \mapsto F^{ab}\eta_a\eta_b$$

Recover $\mathcal{N} = 4$ superconformal algebra, with central extensions, from OPEs

More generally: truncation saves us from summing up an infinite number of diagrams!

At most two `flux' legs!!



Leading order in 1/N: only 4 new diagrams.

Summary

- Koszul duality from defects: universal (topological, holomorphic,…) defects to which one can couple a bulk theory

- Koszul duality from boundaries: transverse boundary conditions support Koszul dual algebras. Re-glue theories.

- Can obtain defect algebra from bulk algebra order-by-order with Feynman diagrams by requiring that theories couple non-anomalously. Illustrations with (ordinary, 4d, holomorphic) Chern-Simons theories

- Useful in (twisted) holography, remembering open-closed coupled string theory. Requires deformation. We illustrated how to obtain algebra of boundary local operators from Koszul duality in $AdS_3 \times S^3 \times T^4 / Sym^N T^4$ dual pair.

More to come about… [Costello-NMP]

- More details about twisted CFT
- Match correlators & Witten diagrams in the twisted duality
- $T^4 \rightarrow K3$ (enumerated states visible in (large-N) elliptic genus)
- Black holes in AdS_3 via quotients
- Giant gravitons in AdS_3?

Even more we'd like to know…

- Use Koszul duality in other examples! Push computations to higher loops directly or prove what the algebra must be
- Koszul duality for… factorization algebras? Von Neumann algebras?
- Connections between spacetime/worldsheet twists
- Higher B-model loops, holomorphic anomaly… connect to stringy exclusion principle, etc.
- Full mathematical formulation of chiral algebra & deformed Koszul duality… proofs of this holographic correspondence?
- Embed other toy examples of holography into this framework? Connections with integrability? [Costello-NMP: WIP w/ 2d Yang-Mills]

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Thank you!