# Lecture Notes

Course:

Quantum Cohomology of $G/P$

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## Lecture 1 February 7 1997

**Course Outline:**

let $G$: semisimple algebraic group over $\mathbb{C}$

$B \subset G$: a Borel

$P \supset B$: a parabolic

$K \subset G$: maximal compact

$T = K \cap B$: maximal torus in $K$

$W = N_K(T)/T$: Weyl group

Then $G/B = K/T$

and $W$ acts on the de Rham cohomology space $H^*(K/T)$.

Moreover, since $K/T$ maps to the classifying space $B_T$, we have a morphism $H^*(B_T) \to H^*(K/T)$ of algebras.

The map $G/B \to G/P$ : $gB \mapsto gP$ gives inclusion

$$H^*(G/P) \hookrightarrow H^*(G/B) = H^*(K/T).$$

$G/P$ is a smooth projective variety.

The de Rham cohomology $H^*(G/p)$ can be used to answer the following question: suppose that three subvarities $X_1$, $X_2$ & $X_3$ of $G/p$ are in general position, and that $\Sigma_{k=1}^3 \dim X_k = \dim G/p$. What is the number of points in the intersection $X_1 \cap X_2 \cap X_3$ ?

The quantum cohomology $\mathfrak{z}H^*(G/p)$ answers a more general question: what is the number of holomorphic maps $\phi: \mathbb{P}^1 \to G/p$ with a fixed degree such that

$$\phi(0) \in X_1$$
$$\phi(1) \in X_2$$
$$\phi(\infty) \in X_3 \quad ?$$

Some features of $\mathfrak{z}H^*(G/p)$:

- There is no natural homomorphism $\mathfrak{z}H^*(G/p) \to \mathfrak{z}H^*(G/B)$;
- If $P \subset Q \subset G$ is a filtration, $\exists$ sth. similar to
  $$gr\, H^*(G/p) = H^*(G/Q) \otimes H^*(Q/P);$$
- $W$ does not act on $\mathfrak{z}H^*(G/B)$.



So take equivariant cohomology $H^T(G/p)$, where $T$ acts on $G/p$ from the left by left translations.

Can define " T-equivariant quantum cohomology $\mathfrak{z}H^T(G/p$

Then

- $W$ acts on $\mathfrak{z}H^T(G/p)$;
- The affine Weyl group $W_{af}$ acts on $\mathfrak{z}H^T(G/p)_{(\mathfrak{z}^{-1})}$

(the parameter $\mathfrak{z}$ inverted).

- Have creation and annihilation operators
- Have Schubert basis for $\mathfrak{z}H^T(G/p)$
- Have "stable" Bruhat order on $W_{af}$;
- Formula for multiplication by $H^2$.
- Special for symmetric spaces of the form $G/p$;
- Borel presentation;
- Pieri formula
-

## Geometrical models — the variety $Y$.

For each parabolic, have $y_P \in Y$ and

$$Y_P^{\pm} := \{ y \in Y : \lim_{t\to\infty} t \cdot y = y_P \}$$

$$Y = \coprod_P Y_P^{\pm} \qquad \text{over } \mathbb{C}$$

and

$$\mathcal{O}(Y_P^+) \cong qH^*(G/P) \qquad \text{over } \mathbb{Z}$$

$$\mathcal{O}(Y_P^-) \cong H_*(\Omega(K \cap P))$$

where $\Omega(K \cap P)$ is the group of loops in $K \cap P$.

Moreover, $Y_P^- \cong \mathbb{C}^n$ for some $n$, and

$$Y = \overline{Y_G^-} = \overline{Y_B^+}$$

$$\Rightarrow \quad \begin{array}{ccccc} \mathcal{O}(Y_G^-) & \longrightarrow & \mathcal{O}(Y_G^- \cap Y_B^+) & \longleftarrow & \mathcal{O}(Y_B^+) \\ \wr\| & & \wr\| & & \| \\ H_*(\Omega K) & & qH^*(G/B)_{(q)} & & qH^*(G/B) \end{array}$$

Will express the Schubert basis elements as matrix entries of some representations. The variety $Y$ lies in $G^\vee/B^\vee$, where $G^\vee$ is the Langland dual of $G$.

End of Lecture 1

# Lecture 2 February 11, 1997

## Kac-Moody root datum

**Definition**: A generalized Cartan matrix is a matrix $A = (a_{ij})_{i,j\in I}$ with integer entries for some finite set $I$ such that

(1) $a_{ii} = 2 \qquad \forall\, i \in I$

(2) $a_{ij} \leq 0 \qquad \forall\, i \neq j$

(3) $a_{ij} = 0 \iff a_{ji} = 0$

A Kac-Moody root datum consists of

- a generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$
- two finitely generated free $\mathbb{Z}$-modules $h_{\mathbb{Z}}^{\vee}$ and $h_{\mathbb{Z}}$ end with a perfect pairing $\langle \quad \rangle$ between them
- two maps $I \longrightarrow h_{\mathbb{Z}}^{\vee} : \quad i \mapsto \alpha_i$

  $I \longrightarrow h_{\mathbb{Z}} : \quad i \longmapsto \check{\alpha}_i$

  such that $\langle \alpha_j, \check{\alpha}_i \rangle = a_{ij}$ (backwards)

$\underline{\text{rk}}$

- Can form direct sums of root data
- Can form the "dual" root data:

$$(A,\ h_{\mathbb{Z}}^{\vee},\ h_{\mathbb{Z}}) \longrightarrow (A^{t},\ h_{\mathbb{Z}},\ h_{\mathbb{Z}}^{\vee})$$

$$\alpha_i \longleftrightarrow \alpha_i^{\vee}$$

ition

Simple roots: $\Pi = \{\alpha_i\}_{i\in I} \subset h_{\mathbb{Z}}^{\vee}$

simple coroots: $\Pi^{\vee} = \{\alpha_i^{\vee}\}_{i\in I} \subset h_{\mathbb{Z}}$

weight lattice: $h_{\mathbb{Z}}^{\vee}$

coweight lattice: $h_{\mathbb{Z}}$

root lattice: $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$

co-root lattice: $Q^{\vee} = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^{\vee}$

re

$Q \longrightarrow h_{\mathbb{Z}}^{\vee}$ isom. $\Leftrightarrow$ of adjoint type

$Q^{\vee} \longrightarrow h_{\mathbb{Z}}$ isom $\Leftrightarrow$ of simply connected type.

$\underline{k}$ In the classical case, root datum comes from connected reductive algebraic groups over $\mathbb{C}$.

**Definition**: We say that $A = (a_{ij})_{i,j\in I}$ is symmetrizable if

$$A = (\text{diagonal})\cdot(\text{symmetric})$$

**Assumption**: Will assume that $A$ is symmetrizable.

The numbers $m_{ij}$: Define, for $i\neq j$, $i,j\in I$

$$m_{ij} = \begin{cases} 2 & \text{if } a_{ij}a_{ji} = 0 \\ 3 & \text{if } a_{ij}a_{ji} = 1 \\ 4 & \text{if } a_{ij}a_{ji} = 2 \\ 6 & \text{if } a_{ij}a_{ji} = 3 \\ \infty & \text{if } a_{ij}a_{ji} \geq 4 \end{cases}$$

The Weyl group $W$ is the group with generators $r_i$, $i\in I$ with relations

$$r_i^2 = 1 \qquad i\in I$$

$$(r_i r_j)^{m_{ij}} = 1 \qquad i,j\in I,\ i\neq j$$

The $r_i$'s are called the simple reflections.

ation Notation:

- $w = r_{i_1} r_{i_2} \cdots r_{i_n}$ [red] means that this is an reduced expression, i.e., $n$ is the minimum number such that $w$ is a product of $n$ simple reflections. Also write $n = \ell(w)$.

- If $W$ is finite, use $w_0$ to denote the longest element.

- From now on, write $h_{\mathbb{Z}}^{\vee} = h_{\mathbb{Z}}^{*}$. using $<\ \ >$.

tions
ons of W on $h_{\mathbb{Z}}^{*}$, $h_{\mathbb{Z}}$, $Q$, $Q^{\vee}$, $S = S(h_{\mathbb{Z}}^{*})$

- $W$ acts on $h_{\mathbb{Z}}^{*}$ by

$$r_i \lambda = \lambda - <\lambda, \alpha_i^{\vee}> \alpha_i$$

$W(Q) \subset Q$.

- $W$ acts on $h_{\mathbb{Z}}$ by

$$r_i h = h - <\alpha_i, h> \alpha_i^{\vee}$$

$W(Q^{\vee}) \subset Q^{\vee}$

- The $W$ actions on $h_{\mathbb{Z}}^{*}$ and on $h_{\mathbb{Z}}$ preserve the pairing

- The $W$ actions on $Q$ and on $Q^{\vee}$ are faithful

- $W$ acts on $S = S(h_{\mathbb{Z}}^{*})$, the symmetric algebra of $h_{\mathbb{Z}}^{*}$ (v the action on $h_{\mathbb{Z}}^{*}$). $W$ acts by algebra automorphisms

The action of $W$ on $S$ will be denoted by

$$s \xmapsto{w} w(s)\ w \cdot s$$

Nil-Hecke ring $\underline{A}$

Definition: The Nil-Hecke ring $\underline{A}$ associated to the root datum $(A=(a_{ij})_{i,j\in I},\ h_{\mathbb{Z}}^*,\ h_{\mathbb{Z}},\ \{\alpha_i\}_{i\in I},\ \{\alpha_i^\vee\}_{i\in I})$ is the associated ring with 1 with generators

$$\hat{\lambda},\quad A_i, \qquad \hat{\lambda}\in h_{\mathbb{Z}}^*,\quad i\in I$$

and relations:

$$\hat{\lambda}+\hat{\mu} = \widehat{\lambda+\mu}$$
$$\hat{\lambda}\hat{\mu} = \hat{\mu}\hat{\lambda} \qquad \lambda,\mu\in h_{\mathbb{Z}}^*$$
$$A_i\hat{\lambda} = \widehat{r_i\lambda}\,A_i + \langle\lambda,\alpha_i^\vee\rangle 1 \qquad \lambda\in h_{\mathbb{Z}}^*,\ i\in I$$
$$A_iA_i = 0 \qquad i\in I$$
$$\underbrace{A_iA_jA_i\cdots}_{m_{ij}} = \underbrace{A_jA_iA_j\cdots}_{m_{ij}} \qquad (i\neq j,\ i,j\in I)$$

The grading on $\underline{A}$ is defined to be

$$\deg\hat{\lambda} = 2$$
$$\deg A_i = -2$$



For $w\in W$ and for any

$$w = r_{i_1}r_{i_2}\cdots r_{i_n}\ [\text{red}],$$

set

$$\underline{A_w} = \underline{A_{i_1}A_{i_2}\cdots A_{i_n}}$$
$$(A_{id} = 1)$$

Then it is clear that

① $A_w$ is independent of the reduced expression

② $$A_vA_w = \begin{cases} A_{vw} & \text{if } \ell(v)+\ell(w)=\ell(vw) \\ 0 & \text{otherwise.}\end{cases}$$

Clearly $S\subset\underline{A}$ as a ~~subal.~~ subring.

Proposition: $\{A_w : w\in W\}$ is an $S$-basis for $\underline{A}$

(Does this need a proof?)

**position**
**ition**: The map

$$\mathbb{Z}W \to \underline{A}: \quad r_i \mapsto 1-\hat{\alpha}_i A_i = A_i\hat{\alpha}_i - 1 \qquad i\in I$$

defines an injective ring homomorphism.

**f**: Only need to check $r_i^2=1$ $i\in I$ and $(r_i r_j)^{m_{ij}}=1$ for $i\neq j$.

Injectivity is clear (?)

**osition**
**sition**: The following defines an $\underline{A}$-module structure on $S$:

$$s'\cdot s = s's$$
$$A_i\cdot s = \frac{1}{\alpha_i}(s - r_i\cdot s)$$

**of**: The induced $r_i$ action on $S$ is $s\mapsto r_i\cdot s$, as the usuall one.

//

**nark**
**rk**: Suppose we need to check certain specified operators for $s\in S$ and $A_i$ on some space $M$ is an action. We first check $r_i = 1-\hat{\alpha}_i A_i = A_i\hat{\alpha}_i - 1$. Then this is how $r_i$ acts. If this gives a $W$-action, we are done.

**sition**: For $s\in S$, $i\in I$ and $w\in W$

$$ws = (w\cdot s)\, w$$
$$A_i s = r_i(s)\, A_i + A_i\cdot s$$

in $\underline{A}$. $\quad A_i s = s A_i + (A_i\cdot s)\, r_i$



**Proof**: Just check ~~need to~~ check that

$$r_i = 1-\hat{\alpha}_i A_i = A_i\hat{\alpha}_i - 1$$

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**The anti-automorphism $*$ on $\underline{A}$**

$$*(s) = s$$
$$*A_w = A_{w^{-1}}$$

To check that this is an anti-automorphism, need to check ~~of~~ only

$$\lambda A_i^* = A_i\, r_i\lambda + \langle\lambda, \alpha_i^\vee\rangle 1 \qquad \lambda\in \mathfrak{h}_{\mathbb{Z}}^*,\ i\in I$$

This is easy. Now since

$$r_i = 1-\hat{\alpha}_i A_i = A_i\hat{\alpha}_i - 1$$

we get

$$*r_i = \hat{\alpha}_i A_i - 1 = -r_i.$$

$$\boxed{\begin{aligned} A_i r_i &= -A_i \\ r_i A_i &= A_i \end{aligned}}$$

Consequently,

$$\underline{*w = (-1)^{\ell(w)}\, w^{-1}}$$

tions

<u>s of $\underline{A}$ on $M\otimes_S N$ and $Hom_S(M,N)$</u>

Assume that $M$ and $N$ are $\underline{A}$-module and are thus
odules. Form

$$M\otimes_S N = M\otimes N \,/\, \{sm\otimes n - m\otimes sn\}$$

$$Hom_S(M,N) = \{f: M\to N : f(sm) = s f(m)\}$$

want to define $\underline{A}$-module structures on $M\otimes_S N$ and $Hom_S(M,N)$.

<u>$M\otimes_S N$</u>:

$$s\cdot(m\otimes n) = sm\otimes n$$

$$A_i\cdot(m\otimes n) = A_i\cdot m\otimes n + r_i\cdot m\otimes A_i\cdot n$$
$$= m\otimes A_i\cdot n + A_i\cdot m\otimes r_i\cdot n$$

check that this is an action, we first need to show that the
ove operators are well-defined. The $s$-operator is clearly OK.
$s\in S$, $i\in I$, we have, by definition

$$A_i\cdot(sm\otimes n - m\otimes sn) = (A_i s)\cdot m\otimes n + (r_i s)\cdot m\otimes A_i\cdot n - A_i\cdot m\otimes sn - r_i\cdot m\otimes (A_i s)\cdot n$$

Using $A_i S = r_i S$



Using

$$A_i s = (r_i\cdot s)A_i + A_i\cdot s \qquad r_i = 1 - \hat{\alpha}_i A_i$$
$$r_i s = (r_i\cdot s)\, r_i \qquad r_i\cdot s = s - \alpha_i A_i\cdot s$$

we get

$$\begin{aligned} A_i\cdot(sm\otimes n - m\otimes sn) &= (r_i\cdot s) A_i\cdot m\otimes n + (A_i\cdot s)\, m\otimes n \\ &\quad + (r_i\cdot s)\, r_i\cdot m\otimes A_i\cdot n - A_i\cdot m\otimes sn \\ &\quad - r_i\cdot m\otimes (r_i\cdot s)A_i\cdot n - r_i\cdot m\otimes (A_i\cdot s)n \\ &= (r_i\cdot s)\, r_i\cdot m\otimes A_i\cdot n - r_i\cdot m\otimes (r_i\cdot s)A_i\cdot n \\ &\quad + (s - \hat{\alpha}_i A_i s) A_i\cdot m\otimes n - A_i\cdot m\otimes sn \\ &\quad + (A_i\cdot s)\, m\otimes n - (m - \hat{\alpha}_i A_i\cdot m)\otimes (A_i\cdot s)n \\ &= (r_i\cdot s)\, r_i\cdot m\otimes A_i\cdot n - r_i\cdot m\otimes (r_i\cdot s)A_i\cdot n \\ &\quad + s A_i\cdot m\otimes n - A_i m\otimes sn \\ &\quad - \left((A_i s)\,\hat{\alpha}_i A_i\cdot m\otimes n - \hat{\alpha}_i A_i\cdot m\otimes (A_i\cdot s)n\right) \\ &\quad + (A_i\cdot s)\, m\otimes n - m\otimes (A_i\cdot s)n \\ &\in \langle s' m\otimes n - m\otimes s' n \;:\; m,n\in M,N \rangle . \end{aligned}$$

Hence $A_i$ is well-defined.

xt, since $r_i = 1 - \hat{\alpha}_i A_i$, we have

$\cdot m \otimes n + r_i \cdot m \otimes A_i \cdot n$

$$= A_i \cdot m \otimes n + m \otimes A_i \cdot n - \hat{\alpha}_i A_i \cdot m \otimes A_i \cdot n$$

$$= A_i \cdot m \otimes n - A_i \cdot m \otimes \hat{\alpha}_i A_i \cdot n + m \otimes A_i \cdot n$$

$$= A_i \cdot m \otimes r_i \cdot n + m \otimes A_i \cdot n$$

$$= m \otimes A_i \cdot n + A_i \cdot m \otimes r_i \cdot n .$$

s gives the 2nd expression for $A_i \cdot (m \otimes n)$.

w for $s \in S$ and $i \in I$, we need to show

$$A_i \cdot (s \cdot (m \otimes n)) = (r_i \cdot s) \cdot (A_i \cdot (m \otimes n)) + (A_i \cdot s) \cdot (m \otimes n)$$

.h.s. $= (A_i s) \cdot m \otimes n + (r_i s) \cdot m \otimes A_i \cdot n$

h.s. $= (r_i \cdot s) A_i \cdot m \otimes n + (r_i s) r_i \cdot m \otimes A_i \cdot n + (A_i \cdot s) m \otimes n$

$$= (A_i s) \cdot m \otimes n + (r_i s) \cdot m \otimes A_i \cdot n$$

$$= \text{lhs}$$

m this, we see that $r_i = 1 - \hat{\alpha}_i A_i = A_i \hat{\alpha}_i - 1$ acts by

$$r_i \cdot (m \otimes n) = m \otimes n - \hat{\alpha}_i A_i \cdot m \otimes n - \hat{\alpha}_i r_i \cdot m \otimes A_i \cdot n$$

$$= r_i \cdot m \otimes n - r_i \cdot m \otimes \hat{\alpha}_i A_i \cdot n$$

$$= r_i \cdot m \otimes r_i \cdot n$$



This clearly induces an action of $W$ on $M \otimes_S N$. Thus we have proved that we indeed have an action of $\underline{A}$ on $M \otimes_S N$.

<u>On</u> <u>$\mathrm{Hom}_S(M,N)$</u>, define:

$$(s \cdot f)(m) = s f(m)$$

$$(A_i \cdot f)(m) = f(A_i \cdot m) + A_i \cdot f(r_i \cdot m)$$

$$= A_i \cdot f(m) - r_i \cdot f(A_i \cdot m)$$

Need to check that this is indeed an action. Clearly $s \cdot$ is o

First. since $r_i = 1 - \hat{\alpha}_i A_i$

and since $f$ is $S$-linear, we have

$$f(A_i \cdot m) + A_i \cdot f(r_i \cdot m) = f(A_i \cdot m) + A_i \cdot (f(m) - \hat{\alpha}_i f(A_i \cdot m))$$

$$= A_i \cdot f(m) + f(A_i \cdot m) - (A_i \hat{\alpha}_i) f(A_i \cdot m)$$

$$= A_i \cdot f(m) - r_i \cdot f(A_i \cdot m)$$

This shows that the two expressions for $A_i \cdot f$ are equal.

Now we show that

$f)(sm) = s(A_i f)(m)$.

$..s. = f((A_i s)\cdot m) + A_i \cdot f((\gamma_i s)\cdot m)$

$s. = s f(A_i\cdot m) + s A_i \cdot f(\gamma_i \cdot m)$

$= f(sA_i\cdot m) + (A_i\,\gamma_i(s) + A_i\cdot s) f(\gamma_i\cdot m)$

$= f(sA_i\cdot m) + A_i\cdot f(\gamma_i(s)\gamma_i\cdot m) + f((A_i\cdot s)\gamma_i\cdot m)$

: $\quad \gamma_i s = \gamma_i(s)\gamma_i$

$A_i s = sA_i + (A_i\cdot s)\gamma_i$

see that $\quad$ l.h.s. = r.h.s..

shows that $\quad A_i\cdot f \in Hom_S(M,N)$.

need to check

$A_i\cdot(s\cdot f) = (\gamma_i\cdot s)\cdot(A_i f) + (A_i\cdot s)\cdot f$

$m \mapsto s f(A_i\cdot m) + A_i s\cdot f(\gamma_i\cdot m)$

$m \longmapsto (\gamma_i\cdot s) f(A_i\cdot m) + (\gamma_i\cdot s) A_i\cdot f(\gamma_i\cdot m) + (A_i\cdot s) f(m)$

$= (\gamma_i\cdot s) f(A_i\cdot m) + A_i s\cdot f(\gamma_i\cdot m) - f((A_i\cdot s)\gamma_i\cdot m) + f((A_i\cdot s) m)$

$\quad A_i s = (\gamma_i\cdot s)A_i + A_i\cdot s \quad$ and $\quad sA_i = A_i s - (A_i\cdot s)\gamma_i$

see $\quad$ l.h.s. = r.h.s



Finally, for $\gamma_i = 1 - \hat{\alpha}_i A_i = A_i\hat{\alpha}_i - 1$, we see that

$$(\gamma_i\cdot f)(m) = f(m) - \hat{\alpha}_i f(A_i\cdot m) - \hat{\alpha}_i A_i\cdot f(\gamma_i\cdot m)$$
$$= f(\gamma_i\cdot m) - \hat{\alpha}_i A_i\cdot f(\gamma_i\cdot m)$$
$$= \gamma_i\cdot f(\gamma_i\cdot m)$$

Consequently,

$$(\omega\cdot f)(m) = \omega\cdot f(\omega^{-1}\cdot m)$$

This is certainly an action of $W$ on $Hom_S(M,N)$. Hence we have an action of $\underline{A}$ on $Hom_S(M,N)$.

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All these proofs seem to be longer than necessary.

But anyway, we have showned that

$$s\cdot(m\otimes n) = sm\otimes n$$
$$A_i\cdot(m\otimes n) = A_i\cdot m\otimes n + \gamma_i\cdot m\otimes A_i = m\otimes A_i\cdot n + A_i\cdot m\otimes \gamma$$
$$\omega\cdot(m\otimes n) = \omega\cdot m\otimes\omega\cdot n$$
$$(s\cdot f)(m) = sf(m)$$
$$(A_i\cdot f)(m) = f(A_i\cdot m) + A_i\cdot f(\gamma_i\cdot m) = A_i\cdot f(m) - \gamma_i\cdot f(A_i\cdot m$$
$$(\omega\cdot f)(m) = \omega\cdot f(\omega^{-1} m)$$

make $\quad M\otimes_S N \quad$ & $\quad Hom_S(M,N) \quad \underline{A}$ - modules again.

<u>ition</u>: Given $\underline{A}$ modules $M, N$ and $P$ (they are therefore also $S$-modules), the following canonical $S$-module maps are also $A$-module maps:

1. $Hom_S(S, M) \cong M$ , $S \otimes_S M = M = M \otimes_S S$
2. $M \otimes_S N = N \otimes_S M$
3. $M \otimes_S (N \otimes_S P) = (M \otimes_S N) \otimes_S P$
4. $Hom_S(M \otimes_S N, P) = Hom_S(M, Hom_S(N, P))$
5. $M \otimes_S Hom_S(N, P) \Rightarrow Hom_S(Hom_S(M, N), P)$
6. $Hom_S(M, N) \otimes_S P \Rightarrow Hom_S(M, N \otimes_S P)$

ition
<u>on</u>:

For an $\underline{A}$-module $P$, set

$$P^A = \{ p \in P : \ A_i \cdot p = 0 \quad \forall i \in I \}$$

osition
<u>tion</u> For $\underline{A}$-modules $M$ and $N$,

$$Hom_{\underline{A}}(M, N) = (Hom_S(M, N))^A .$$



<u>Example</u>: Regard $\underline{A}$ as an $\underline{A}$ module by left multiplications. Then our previous constructions define an $A$-module structure on $\underline{A} \otimes_S \underline{A}$. Set

Define: $\Delta : \underline{A} \longrightarrow \underline{A} \otimes_S \underline{A}$

by

$$\Delta a = a \cdot (1 \otimes 1)$$

Thus

$$\Delta w = w \otimes w$$

$$\Delta s = s \otimes 1 = 1 \otimes s$$

$$\Delta A_i = A_i \otimes 1 + \gamma_i \otimes A_i = 1 \otimes A_i + A_i \otimes \gamma_i$$

For any two $\underline{A}$ modules $M$ and $N$, since we have

$$a \cdot (m \otimes n) = a_{(1)} \cdot m \otimes a_{(2)} \cdot n$$

for $a \in S$ or $a = A_i$, $i \in I$, where $\Delta a = a_{(1)} \otimes a_{(2)}$,

we have

$$a \cdot (m \otimes n) = a_{(1)} \cdot m \otimes a_{(2)} \cdot n \qquad \forall a \in A$$

?

*ition*: In the finite case,

come back to this later

$$\Delta A_{w_0} = \sum_{w\in W} A_{w_0 w} \otimes w_0 A_w$$

$$= \sum_{w\in W} A_w \otimes w_0 A_{w_0 w}$$

: It is easy to show by induction on $\ell(w)$ that for any $w \in W$

$$\Delta A_w = A_w \otimes w + \sum_{v<w} A_v \otimes a_v$$

for some $a_v \in \underline{A}$. So

$$\Delta A_{w_0} = \sum_{w\in W} A_w \otimes a_w$$

with $a_{w_0} = w_0$. Now for any $i \in I$,

$$A_i A_{w_0} = 0$$

$$\Rightarrow \quad 0 = \Delta(A_i)\,\Delta(A_{w_0})$$

$$\Rightarrow \quad 0 = (A_i \otimes 1 + r_i \otimes A_i) \sum_{w\in W} A_w \otimes a_w$$

$$= \sum_{w\in W} \left(A_i A_w \otimes a_w + r_i A_w \otimes A_i a_w\right)$$

$$= \sum_{w\in W} \left(A_i A_w \otimes a_w + (1-\alpha_i A_i) A_w \otimes A_i a_w\right)$$

$$= \sum_{w\in W} A_i A_w \otimes r_i a_w - A_w \otimes A_i a_w$$



$$\Rightarrow \quad \sum_{w\in W} A_i A_w \otimes r_i a_w = \sum_{w\in W} A_w \otimes A_i a_w$$

Now $\quad$ lhs $= \sum_{r_i w > w} A_{r_i w} \otimes r_i a_w$

$$\Rightarrow \quad a_{r_i w} = -A_i a_w \quad \text{if } r_i w < w$$

$$\Rightarrow \quad a_{w_0} = w_0 A_{w_0 w}\,. \quad ?$$

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# Lecture 3 February 12, 1997

Recall that a Kac-Moody root datum consists of

- a generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ ;
- two finitely generated free $\mathbb{Z}$-modules $h_{\mathbb{Z}}^{\vee} = h_{\mathbb{Z}}^{*}$ and $h_{\mathbb{Z}}$ with a perfect pairing $\langle\ \rangle$ between them;
- two maps

$$I \longrightarrow h_{\mathbb{Z}}^{\vee} : \quad i \longmapsto \alpha_i$$

$$I \longrightarrow h_{\mathbb{Z}} : \quad i \longmapsto \alpha_i^{\vee}$$

such that

$$\langle \alpha_i, \alpha_j^{\vee} \rangle = a_{ji}$$

The weight lattice is $h_{\mathbb{Z}}^{\vee}$

the co-weight lattice is $h_{\mathbb{Z}}$

the root lattice is $Q \overset{\text{def}}{=} \bigoplus_{i\in I} \mathbb{Z}\alpha_i$

the co-root lattice is $Q^{\vee} \overset{\text{def}}{=} \bigoplus_{i\in I} \mathbb{Z}\alpha_i^{\vee}$

Say the datum is of the adjoint type if $Q \longrightarrow h_{\mathbb{Z}}^{\vee} : \alpha_i \longmapsto \alpha_i$ is an iso.

Say " " " " " symply connected type if $Q^{\vee} \longrightarrow h_{\mathbb{Z}} : \alpha_i^{\vee} \mapsto \alpha_i^{\vee}$ is an isom

For $sl(2,\mathbb{C})$, use $e, f, h$ for the standard generators s.t.

$$[h,e]=2e$$
$$[h,f]=-2f$$
$$[e,f]=h$$

Given a Kac-Moody root datum $(A, I, h_{\mathbb{Z}}^{\vee}, h_{\mathbb{Z}}, \langle\ \rangle)$, set

$$\underline{h} = \mathbb{C}\otimes_{\mathbb{Z}} h_{\mathbb{Z}}$$

and regard it as a commutative Lie algebra. Set $h_i = \alpha_i^{\vee}$

<u>Theorem</u> (see Kac?): For any Kac-Moody root datum, there exists a Lie algebra $\underline{g}$ over $\mathbb{C}$ (of Kac-Moody type) and Lie algebra homomorphisms

$$\phi: \underline{h} \longrightarrow \underline{g}$$
$$\phi_i: sl_2(\mathbb{C}) \longrightarrow \underline{g} \qquad \forall\ i\in I$$

such that

① $\phi_i(h) = \phi(h_i)$

$[\phi(h), \phi_i(e)] = \langle \alpha_i, h\rangle \phi_i(e) \qquad h\in \underline{h}$

$[\phi(h), \phi_i(f)] = -\langle \alpha_i, h\rangle \phi_i(f) \qquad i\in I$

$[\phi_i(e), \phi_j(f)] = 0 \qquad (i\neq j)$



② For each $i\in I$, $\underline{g}$ as an $sl_2(\mathbb{C})$ module via $\phi_i$ (using the adj. rep) is a direct sum of finite-dimensional $sl_2(\mathbb{C})$-module.

③ If $\underline{g}', \phi', \& \phi_i'$ are another such system, then $\exists$ a unique $\psi: \underline{g}\to\underline{g}'$ s.t. $\phi' = \psi\circ\phi$ and $\phi_i' = \psi\circ\phi_i$

Thus $(\underline{g}, \phi, \phi_i, i\in I)$ is unique.

<u>Definition</u>: ① An $sl_2(\mathbb{C})$-module $V$ over $\mathbb{C}$ is <u>integrable</u> if it is a direct sum of finite-dim. modules

② An $\underline{h}$-module $V$ over $\mathbb{C}$ is <u>integrable</u> if

$$V = \bigoplus_{\mu\in h_{\mathbb{Z}}^*} V_\mu$$

where

$$V_\mu = \{ v\in V:\ hv = \mu(v)v \quad \text{for all } h\in\underline{h} \}$$

③ A $\underline{g}$-module $V$ over $\mathbb{C}$ is <u>integrable</u> if it is $sl_2(\mathbb{C})$-integrable (via $\phi_i$, for each $i\in I$) and $\underline{h}$-integrable via $\phi$.

So the adjoint representation of $\underline{g}$ on $\underline{g}$ is integrable.

tion :

- $\phi : \underline{h} \to \underline{g}$ is injective, so call $\underline{h} \subset \underline{g}$ the Cartan subalgebra
- $\phi_i : sl_2(\mathbb{C}) \to \underline{g}$ is injective for each $i \in I$. Set

$$e_i = \phi_i(e)$$
$$f_i = \phi_i(f)$$
$$h_i = \phi_i(h) = \alpha_i^\vee$$

- $Z(\underline{g})$, the center of $\underline{g}$, is contained in $\underline{h}$.

- Set

$$n_+ = \langle e_i \rangle_{i\in I} = \text{Lie subalgebra generated by } \{e_i, i \in I\}$$
$$n_- = \langle f_i \rangle_{i \in I}$$

Then

$$\underline{g} = n_- + \underline{h} + n_+$$

— triangular decomposition

Every ideal of $\underline{g}$ contained totally in $n_-$ or $n_+$ is $0$.

$$b_+ \stackrel{\text{def}}{=} \underline{h} + n_+ = b \qquad (\text{Borel subalgebra})$$
$$b_- \stackrel{\text{def}}{=} \underline{h} + n_-$$



Fact: $\underline{g}$ is the Lie algebra over $\mathbb{C}$ with generators

$$h \in \underline{h} \qquad e_i, f_i \qquad i \in I$$

with relations

$$[h, h'] = 0$$
$$[h, e_i] = \langle \alpha_i, h \rangle e_i$$
$$[h, f_i] = -\langle \alpha_i, h \rangle f_i$$
$$[e_i, f_j] = \delta_{ij} h_i$$
$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0$$
$$(\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0 \qquad (i \neq j)$$

Warning $\quad \underline{h} \subsetneqq$ Centralizer of $\underline{h}$ in $\underline{g}$

The $Q$-grading of $\underline{g}$ :

For $\beta \in Q$, the root lattice, set

$$\underline{g}_\beta = \{ x \in \underline{g} : \quad [h, x] = \langle \beta, h \rangle x \quad \forall h \in \underline{h} \}$$

Then

$$\underline{g} = \bigoplus_{\beta \in Q} \underline{g}_\beta$$

and $[\underline{g}_{\beta_1}, \underline{g}_{\beta_2}] \subset \underline{g}_{\beta_1+\beta_2}$

ice that $\mathfrak{g}_0 = \mathfrak{h} \quad \mathfrak{g}_{\alpha_i} = \mathbb{C}e_i \quad \mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i \qquad i \in I.$

$$\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$$

$$Q_+ = \bigoplus_{i \in I} \mathbb{Z}_+ \alpha_i \subset Q \qquad \text{sub-semigroup}$$

$\beta, \nu \in Q$, say $\beta \geq \nu$ if $\beta - \nu \in Q_+$.

$$\mathfrak{n}_\pm = \bigoplus_{\beta \in Q_\pm} \mathfrak{g}_\beta$$

$$\Delta = \{\beta \in Q : \mathfrak{g}_\beta \neq 0 \quad \beta \neq 0\} \text{ — set of roots}$$

$$\Delta_+ = \Delta \cap Q_+ \; : \; \text{set of positive roots}$$

$$\Pi = \{\alpha_i : i \in I\} \quad \text{set of simple roots}$$

$$\Delta_- = -\Delta_+$$

$$\Delta_+ \cup \Delta_- = \Delta$$

$$\Delta_+ \cap \Delta_- = \emptyset$$

$$\mathfrak{n}_\pm = \bigoplus_{\beta \in \Delta_\pm} \mathfrak{g}_\beta$$



The principle $\mathbb{Z}$-grading of $\mathfrak{g}$ :

let $\rho^\vee \in Q^\vee$ be the [redacted] (?) element such that

$$\langle \alpha_i, \rho^\vee \rangle \equiv 1 \qquad \forall i \in I.$$

For $\beta \in Q$, the integer

$$ht(\beta) = \langle \beta, \rho^\vee \rangle$$

is called the height of $\beta$. For $n \in \mathbb{Z}$, set

$$\mathfrak{g}_n = \bigoplus_{\substack{\beta \in Q \\ ht(\beta) = n}} \mathfrak{g}_\beta$$

This is a $\mathbb{Z}$-grading for $\mathfrak{g}$.

The set of real roots

Need to define the Weyl group first. To define the Weyl gp, need to define the Kac-Moody group.

Compact involution of $\mathfrak{g}$ :

This is the conjugation-linear automorphism of $\mathfrak{g}$ s.t.

$$e_i \longleftrightarrow -f_i \qquad i \in I$$
$$h \longleftrightarrow -h \qquad h \in \mathfrak{h}_\mathbb{R} \overset{def}{=} \mathbb{R} \otimes_\mathbb{Z} \mathfrak{h}_\mathbb{Z} \subset \mathfrak{h}$$

## Kac-Moody group

(C): For $u \in \mathbb{C}$, set $t \in \mathbb{C}^\times$, set

$$x(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$
$$y(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \qquad \in SL_2(\mathbb{C}).$$
$$h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

A finite-dimensional representation of $SL_2(\mathbb{C})$ is said to be rational if its matrix entries are regular functions on $SL_2(\mathbb{C})$.

A representation of $SL_2(\mathbb{C})$ on a vector space $V$ over $\mathbb{C}$ is said to be differentiable if it is a direct sum of finitely many finite dimensional rational representations.

Fact: Integrable representations of $sl_2(\mathbb{C}) \longleftrightarrow$ differentiable rep. of $SL_2(\mathbb{C})$
(This is because $SL_2(\mathbb{C})$ is an algebraic group).



## The complex torus $H$

Define
$$H = Hom(h_{\mathbb{Z}}^{\vee}, \mathbb{C}^\times).$$

For $h \in h_{\mathbb{Z}}$ and $t \in \mathbb{C}^\times$, define $t^h \in H$ by
$$t^h(\lambda) = t^{\langle \lambda, h \rangle} \qquad \lambda \in h_{\mathbb{Z}}^{\vee}$$

Thus, for each such $h \in h_{\mathbb{Z}}$, the map
$$\mathbb{C}^\times \longrightarrow H: \qquad t \longmapsto t^h$$
is a homomorphisms. Moreover
$$t^{h+h'} = t^h \cdot t^{h'}.$$

A representation of $H$ on $V/\mathbb{C}$ is said to be differentiable if it is a direct sum of 1-dim rational representations of $H$.

Fact: Differentiable representations of $H \longleftrightarrow$ integrable representations of $\underline{h}$.

Next: The Kac-Moody group $G$ corresponding to the Kac-Moody root datum we started with at the ~~beg~~ beginning.

-Moody group $G$

iven the Kac-Moody root datum, there is a group $G$ with homomorphisms

$\phi : H \longrightarrow G$

$\phi_i : SL_2(\mathbb{C}) \to G \qquad i \in I$

$$\phi_i(h(t)) = \phi(t^{h_i})$$

$$\phi(t^h)\ \phi_i(x(u))\ \phi(t^{-h}) = \phi_i(x(t^{\langle \alpha_i, h\rangle} u))$$

$$\phi(t^h)\ \phi_i(y(u))\ \phi(t^{-h}) = \phi_i(y(t^{-\langle \alpha_i, h\rangle} u))$$

$$\phi_i(x(u))\ \phi_j(y(v)) = \phi_j(y(v))\ \phi_i(x(u)) \qquad i \neq j$$

$\exists$ representation $Ad$ of $G$ on $\mathfrak{g}$ such that under $\phi$ and $\phi_i$, $i \in I$, the corresponding representations of $H$ and $SL_2(\mathbb{C})$ on $\mathfrak{g}$ differentiate to the representions of $\mathfrak{h}$ and $Sl_2(\mathbb{C})$ on $\mathfrak{g}$ defined by $ad$.

if $(G', \phi'$ and $\phi_i')$ is another system with above properties, then $\exists$ a unique $\psi : G \to G'$ s.t.

$$\phi' = \psi \circ \phi \qquad \phi_i' = \psi \circ \phi_i$$



- $G$ is generated by the images of $\phi$ and $\phi_i$, $i \in I$
- A $G$-module is said to be differentiable if ~~or~~ it is differentiable as $H$ and $SL_2(\mathbb{C})$ modules under $\phi$ and each $\phi_i'$, $i \in I$. Thus,

  differentiable $G$-module $\leftrightarrow$ integrable $\mathfrak{g}$-module.

- $\exists$ faithful differential $G$-module (probably not $Ad$).
- $\phi : H \to G$ is injective. So we call $H \subset G$ the Cartan subgroup.

Have

$$Z(G) \overset{def}{=} \ker Ad \subset H$$

$$\ker \phi_i \subset Z(SL_2(\mathbb{C}))$$

## The Weyl group $W$

For each $i \in I$, set, $u \in \mathbb{C}$, set

$$x_i(u) = \phi_i(x(u)) = \phi_i\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in G$$

$$y_i(u) = \phi_i(y(u)) = \phi_i\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \in G$$

$$n_i = y_i(1)\, x_i(-1)\, y_i(1) = \phi_i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G$$

i $$\overbrace{n_i\, n_j\, n_i \cdots}^{m_{ij}} = \overbrace{n_j\, n_i\, n_j \cdots}^{m_{ij}}$$

s for $w = r_{i_1} \cdots r_{i_n}$ (red), let

$$n_w = n_{i_1} n_{i_2} \cdots n_{i_n}$$

n

$$n_w \cdot \mathfrak{g}_\beta = \mathfrak{g}_{w \cdot \beta}$$

t in general

$$n_w\, n_{w^{-1}} \neq id.$$

:

$$N = \langle n_i,\ H \rangle_{i \in I} \subset G$$

the subgroup of $G$ generated by $\{n_i,\ i \in I\}$ and $H$.

en

$$N/H \simeq W$$
$$n_i/H \longrightarrow r_i$$

larning: can have $H \subsetneq Z_G(H)$.



## The real roots

Note that
$$W \cdot \Delta = \Delta$$
so $W$ permutes the root system.

Set
$$\Delta^{re} = \bigcup_{i \in I} W \cdot \alpha_i$$
and call elements in $\Delta^{re}$ the real roots.

If $\beta = w \cdot \alpha_i \in \Delta^{re}$ for some $i \in I$, then
$$\mathfrak{g}_\beta = n_w \cdot \mathfrak{g}_{\alpha_i}$$
so
$$\dim_{\mathbb{C}} \mathfrak{g}_\beta = 1$$
and
$$\mathfrak{g}_{m\beta} = 0 \qquad \text{for } |m| > 1.$$

Also, define
$$r_\beta = w\, r_i\, w^{-1} \in W$$

Then ① $r_{w_1 \beta} = w_1\, r_\beta\, w_1^{-1}$ for any $w_1 \in W$

② $r_\beta \cdot \lambda = \lambda - \langle \lambda, \beta^\vee \rangle \beta \qquad \lambda \in h_{\mathbb{Z}}^*$

$r_\beta \cdot h = h - \langle \beta, h \rangle \beta^\vee \qquad h \in h_{\mathbb{Z}}.$

## Lecture 4 Feburary 19, 1997

I am moving the part on the Bruhat decomposition of $G/P$ to the end of Lecture 3. The main part of this lecture is on

**Equivariant Cohomology** (due to Borel)

- Let $L$ be a topological group. ($L = K$ or $T$ in our applications). A principal $L$-bundle is a topological space $E$ equipped with a continuous $\underset{\text{free}}{\wedge}$ right action of $L$

$$E \times L \longrightarrow L$$

and a projection $E \longrightarrow B$ s.t. locally $E = B' \times L$ where w/ $L$ acts on $B' \times L$ by $(b, g) \overset{g_1}{\longmapsto} (b, g g_1)$ ; $B' \underset{\text{open}}{\subseteq} B$.

Have $B = E/L$ with the quotient topology.

- The universal principal $L$-bundle $E_L$ is a principal $L$-bu[ndle] s.t. $E_L$ is contractible. Set $B_L = E_L/L$. It is called the Classifying space of $L$.

**Example:** $B_{S^1} = \mathbb{C}P^\infty$

$E_T = E_K$ (because $T \subset K$)

**Definition:** An $L$-space is a ~~g~~ topological space $X$ endowed with a continuous left $L$-action:

$$L \times X \to X: \quad (\ell, x) \mapsto \ell \cdot x = \ell x.$$

**Definition:** Given an $L$-space $X$, form the space

$$E_L \times^L X = (E_L \times X)/L$$

where $(e, x)\cdot \ell = (e\ell^{-1}, \ell x)$ is a free left $L$-action.

The $L$-equivariant cohomology of $X$ is by definition the singular homology of $E_L \times^L X$:

$$\underline{H^L(X)} = \underline{H^*(E_L \times^L X)}$$

**Structures on $H^L(X)$:**

1. It is a graded ring, where the grading is nothing but the grading on $H^*(E_L \times^L X)$. (And so is the ring structu

2. The fibration $E_L \times^L X \to E_L/L = B_L$ gives a ~~ring~~ grade ring homomorphism

$$H^*(B_L) \longrightarrow H^*(E_L \times^L X)$$

ie. $$H^L(pt) \longrightarrow H^L(X)$$

Thus $H^L(X)$ has a natural $H^L(pt)$-module structure

**Functoriality:** Given an $L$-map of $L$-spaces

$$f: X \to Y,$$

form the map

$$E_L \times^L X \longrightarrow E_L \times^L Y$$

$$[e, x] \longmapsto [e, f(x)]$$

~~It indu~~. Have commutative diagram

$$\begin{tikzcd} E_L \times^L X \arrow[r] \arrow[d, "\pi_x"'] & E_L \times^L Y \arrow[d, "\pi_Y"] & {[e, x]} \arrow[r, mapsto] \arrow[d, mapsto] & {[e, f(x)]} \arrow[d, mapsto] \\ B_L \arrow[r, "id"] & B_L & {[e]} \arrow[r, phantom, "="] & {[e]} \end{tikzcd}$$

have a graded ring homomorphism

$$f^* : \quad H^L(X) \longleftarrow H^L(Y)$$

ch is also a $H^*(B_L) = H^L(pt)$ - module map.

special case of $Y = pt$. with

$$f : \quad X \longrightarrow pt.$$

res

$$\pi_X : \quad E_L \times^L X \longrightarrow E_L \times^L pt = B_L,$$

$$f^* : \quad H^L(pt) \longrightarrow H^L(X)$$

just the one considered before.

## L-equivariant homology

This is the space

$$\mathrm{Hom}_{H^L(pt)}(H^L(X), H^L(pt)).$$

Than any L-space map

$$f : \quad X \to Y$$

induces

$$f_* : \quad \mathrm{Hom}_{H^L(pt)}(H^L(X), H^L(pt)) \to \mathrm{Hom}_{H^L(pt)}(H^L(Y), H^L(pt)).$$

**The restriction homomorphism** (or the evaluation at 0):

This is the map

$$v_0(X) : \quad H^L(X) \longrightarrow H^*(X)$$

induced by the map

$$E_L \times X \longrightarrow E_L \times^L X.$$

This is a ~~$\mathbb{Z}$-ring ho~~ graded $\mathbb{Z}$-ring homomorphism.

r any $L$-space map $f: X \to Y$, have commutative diagram

$$\begin{array}{ccc} H^L(X) & \xrightarrow{V_0(X)} & H^*(X) \\ f^* \uparrow & & \uparrow f^* \\ H^L(Y) & \xrightarrow{V_0(Y)} & H^*(Y) \end{array}$$

amples
iples :

1. $L$ acts freely on $X$. Then

$$H^L(X) = H^*(X/L)$$

Proof : Have the following fibre bundle with contractible fibre $E_L$.

$$\begin{array}{ccc} E_L \times^L X & \leftarrow & E_L \\ \downarrow & & \\ X/L & & \end{array}$$

Thus $H^L(X) = H^*(E_L \times^L X) = H^*(X/L)$. //

2. $L$ acts trivially on $X$. Then

$$H^L(X) = H^L(pt) \otimes H^*(X)$$

oof : Have $E_L \times^L X = B_L \times X$. //

Proposition : Have $H^*(B_T) = S(h_{\mathbb{Z}}^*)$.

Proof : For $\lambda \in h_{\mathbb{Z}}^*$, define

$$e^\lambda : T \longrightarrow \mathbb{C}^* ; \quad e^\lambda(e^h) = e^{\langle \lambda, h \rangle}, \quad h \in h_{\mathbb{Z}}.$$

If $E$ is a principal $T$-bundle, defi form the complex-line bundle $\mathcal{L}_\lambda = E \times_T \mathbb{C}$ by

$$[et, c] = [e, e^{-\lambda}(t) c] \qquad t \in T,\ e \in E,\ c \in \mathbb{C}$$

Then map

$$\lambda \longmapsto C_1(E \times_T \mathbb{C}) \quad , \text{ the first Chern class}$$

gives a homomorphism

$$S(h_{\mathbb{Z}}^*) \longrightarrow H^*(E/T).$$

In particular, take $E = E_T = E_K$. Then get

$$S(h_{\mathbb{Z}}^*) \longrightarrow H^*(E_T/T) = H^T(pt).$$

One can then show that this is an isomorphism of graded rings if $\lambda \in h_{\mathbb{Z}}^*$ is given deg $= 2$. //

**second $S$-module structure on $H^T(K/T)$**

Set $E_u = E_T = E_K$. The map

$$E_u \times^T (K/T) \longrightarrow E_u/T : \quad [e, kT] \longmapsto ekT$$

s another ring homomorphism, which we will denote by $\pi_R$ reasons that will be clear next time:

$$\pi_R : S \longrightarrow H^T(K/T).$$

-ks :

1. $\pi_R$, ~~will be~~ together with the map

$$\pi_L : S \longrightarrow H^T(K/T)$$

induced by $K/T \to pt$, will be the source and target maps for the Hopf algebroid structure on $H^T(K/T)$ that we will discuss next lecture.

~~Set $E_u^{(2)} = \{(e_1, e_2)$~~

2. Set

$$E_u^{(2)} = E_u \times_{E_u/K} E_u = \{(e_1, e_2) \in E \times E : e_1 K = e_2 K\}$$

$$\subset E_u \times E_u$$

It is a $(K \times K)$-inv. subset of $E_u \times E_u$. Since $K$ acts on $E_u$ freely, we have the identification

$$E_u^{(2)} \cong E_u \times K : \quad (e_1, e_2) \longmapsto (e_1, k) \quad \text{if } e_2 = e_1 k$$

Under this identification, the $T \times T$ action on $E_u^{(2)}$ becomes the action

$$(e, k) \xmapsto{(t_1, t_2)} (e t_1, t_1^{-1} k t_2)$$

of $T \times T$ on $E_u \times K$. (easy to check this:

$$(e, k) \longmapsto (e_1, e_1 k) \xmapsto{(t_1, t_2)} (e t_1, e_1 k t_2) \longmapsto (e t_1, e t_1 t_1^{-1} k t_2)$$
$$\longmapsto (e t_1, t_1^{-1} k t_2)$$

Thus we have

$$E_u^{(2)}/T \times T \cong E_u \times^T K/T$$

The map $E_u \times^T (K/T) \to E_u/T : [e, kT] \mapsto ekT$ now is just the projection from $E_u^{(2)}/T\times T$ to the 2nd factor $E_u$.

This will be used in the next lecture. //

osition.

<u>tion</u>. For any $T$-space $Y$, we have

$$H^T(K\times^T Y) = H^T(K/T)\otimes_S H^T(Y)$$

where the $S$-module structure on $H^T(K/T)$ is via the second ring homomorphism

$$\pi_K:\ S\longrightarrow H^T(K/T).$$

(The ~~#~~ $S$-module structure on $H^T(Y)$ is the usual one).

f:

Consider the following commutative square:

$$\begin{tikzcd}
(\times^T Y) \arrow[r, "p_1"] & E_u\times^K(K\times^T Y) \cong E_u\times^T Y \arrow[d, "g_1"] \\
K\times^T pt) \arrow[r, "g_2"] & E_u\times^K(K\times^T pt) \\
{}^T(K/T) & E_u/T \arrow[u, phantom, "\cong"]
\end{tikzcd}
\qquad : \qquad
\begin{tikzcd}
{[e,[k,y]]_T} \arrow[r, mapsto, "p_1"] \arrow[d, mapsto, "p_2"'] & {[e,[k,y]]_K} \arrow[r, mapsto, "\sim"] \arrow[d, mapsto, "g_1"] & {[ek,y]} \\
{[e,[k,pt]]_T} \arrow[r, mapsto, "g_2"] \arrow[d, mapsto, "\wr"] & {[e,[k,pt]]_K} \arrow[d, mapsto, "\wr"] & \\
{[e,k]} & {[ek]=[ek,pt]} &
\end{tikzcd}$$

notice that $g_1^*:\ S\longrightarrow H^T(Y)$ is the usual homo. (induced from $Y\to pt$).

t $\quad g_2^* = \pi_K:\ S\longrightarrow H^T(K/T)$ is the second homomorphism.

Now since the square is commutative, ie. $g_2\circ p_2 = g_1\circ p_1$, we get a ring homomorphism

$$H^T(K/T)\otimes_S H^T(Y) \longrightarrow H^T(K\times^T Y).$$

$$x\otimes y \longmapsto p_2^*(x)\ p_1^*(y)$$

assuming even col

To show that this is an isomorphism, we first notice that the fibration $p_1$ has fibre $K/T$ which is a CW-complex of only even dimensions. Thus ~~Lerray-Hirsch~~ Leray-Hirsch theorem tells us that $H^T(K\times^T Y)$ is a free module over $H^T(Y)$ with basis coming from $H^*(K/T)$. But the special case of $Y=pt$ says that $H^T(K/T)$ is a free $S=H^T(pt)$-module with basis coming from $H^*(K/T)$. Using a basis of $H^*(K/T)$, we see that the map

$$H^T(K/T)\otimes_S H^T(Y)\longrightarrow H^T(K\times^T Y)$$

is an isomorphism.

//

<u>on</u>: The map ~~[illegible]~~ morphism

$$\mathcal{E}: \quad H^T(K/T) \longrightarrow S$$

induced by $\quad T/T \hookrightarrow K/T$

is called the <u>co-unit</u> <u>map</u>

<u>n</u>: For any $K$-space $X$, the map

$$\Delta_X: \quad H^T(X) \longrightarrow H^T(K/T) \otimes_S H^T(X)$$

induced by the $T$-map

$$\mu_X: K \times^T X \longrightarrow X \quad : \quad [k, x] \mapsto kx ,$$

ie.

$$\Delta_X : H^T(X) \xrightarrow{\mu_X^*} H^T(K \times^T X) \cong H^T(K/T) \otimes_S H^T(X)$$

is called the <u>co-module</u> <u>map</u>

sition
<u>on</u>: For any $K$-space $X$, we have

$$(\mathcal{E} \otimes id) \circ \Delta_X = id|_{H^T(X)}$$

nd

$$(\Delta_{K/T} \otimes id) \circ \Delta_X = (id \otimes \Delta_X) \circ \Delta_X :$$

$$H^T(X) \longrightarrow H^T(K/T) \otimes_S H^T(K/T) \otimes_S H^T(X).$$

<u>Definition</u>: <u>A</u> <u>groupoid</u> <u>scheme</u> $(\mathcal{G}, \mathcal{S})$ consists of two schemes $\mathcal{G}$ and $\mathcal{S}$ and five morphisms:

$$p_L, p_R: \mathcal{G} \to \mathcal{S}$$
$$\ell: \quad \mathcal{S} \to \mathcal{G}$$
$$i: \quad \mathcal{G} \to \mathcal{G}$$
$$\mu: \quad \mathcal{G} \times_S \mathcal{G} \to \mathcal{G}$$

( ~~$\mathcal{G}$ is x right~~ fibre produ[ct]
$-\times_S$ refers to $p_L$, and
$\times_S -$ refers to $p_R$ )

They must satisfy:

$$p_L \circ \ell = id_{\mathcal{G}} = p_R \circ \ell$$
$$p_L \circ i = p_R \qquad p_R \circ i = p_L$$
$$p_L \circ \mu = p_L \circ p_1 \qquad p_R \circ \mu = R_R \circ p_1$$
$$\mu \circ (id_{\mathcal{G}}, \ell \circ p_R) = id_{\mathcal{G}}$$
$$\mu \circ (\ell \circ p_L, id_{\mathcal{G}}) = id_{\mathcal{G}}$$
$$\mu \circ (id_{\mathcal{G}}, i) = i \circ p_R \qquad \mu \circ (i, id_{\mathcal{G}}) = \ell \circ p_R$$
$$\mu \circ (id_{\mathcal{G}} \times \mu) = \mu \circ (\mu \times id_{\mathcal{G}})$$

- These imply $i \circ i = id_{\mathcal{G}}$.
- If $\mathcal{G} = \operatorname{spec} R$ and $\mathcal{S} = \operatorname{spec} S$, then
$$\mathcal{G} \times_S \mathcal{G} = \operatorname{spec}(R \times_S R).$$

end of lecture

D. Peterson — **Lecture 5** — Feb. 25, 1997 — Tuesday

Recall the concept of a groupoid:

A groupoid is a small category with every morphism invertible

Example: Let $G$ be a group acting on a space $X$. Then we can form a groupoid $(\mathcal{G}, \mathcal{S})$, where $\mathcal{S} = X$.

$$\mathcal{G} = \{ (x, g, y) : \quad x, y \in X, \quad x = g\,y \}$$

Multiplication is given by

$$(x, g, y)(x', g', y') = (x, gg', y') \qquad \text{if } y = x'$$

Source map:

$$\mathcal{G} \to S : \quad (x, g, y) \longmapsto y$$

Target map:

$$\mathcal{G} \to S : \quad (x, g, y) \longmapsto x$$

Inverse map:

$$\mathcal{G} \to \mathcal{G} : \quad (x, g, y) \longmapsto (y, g^{-1}, x)$$

Units:

$$S \longrightarrow \mathcal{G} : \quad x \longmapsto (x, e, x).$$

An action $\phi : \mathcal{G} \times_S X \to X$ of a groupoid scheme $(\mathcal{G}, S)$ on a scheme $X$ / $S$ with structure morphism $p_X : X \to S$ is one such that

① $\phi \circ (\mu \times id_X) = \phi \circ (id_{\mathcal{G}} \times \phi)$

② $p_X \circ \phi = p_2 \circ p_1 \qquad$ where $p_1 : \mathcal{G} \times X \to \mathcal{G}$. $(g, x) \mapsto$

③ $\phi \circ ((e \circ p_X) \times id_X) = id_X$.

e groupoid scheme $\mathcal{U} = \operatorname{Spec} H^T(K/T)$

Let $E_u$ be the principal $K$ (and thus also $T$)-bundle

For $n \geq 1$, let

$$E_u^n = E_u \times \cdots \times E_u \quad n \text{ times}$$

$$K^n = K \times \cdots \times K \quad n \text{ times}$$

$$T^n = T \times \cdots \times T \quad n \text{ times}$$

Set

$$E_u^{(n)} = \{(e_1, \cdots, e_n) \in E_u^n : e_1K = \cdots = e_nK\} \subset E_u^n.$$

As a subset of $E_u^n$, the set $E_u^{(n)}$ is invariant under the $K^n$-action, so $E_u^{(n)}$ is a principal $K^n$-bundle.

Set

$$B^{(n)} = E_u^{(n)}/T^n$$

Then it is easy to check that $B^{(2)}$ is a groupoid over $B^{(1)} = E/T = B_T$ with the following structure maps:

(This is a subquotient of the coarse groupoid $E \times E$ over $E$).



- Sourse and target maps:

$$p_1 : B^{(2)} = E_u^{(2)}/T^2 \longrightarrow E/T : \quad [e_1, e_2] \mapsto [e_1]$$

$$p_2 : B^{(2)} = E_u^{(2)}/T^2 \longrightarrow E/T : \quad [e_1, e_2] \mapsto [e_2]$$

- identities:

$$d \,(= \text{diagonal}) : B^{(1)} = E/T \longrightarrow B^{(2)} : \quad [e] \mapsto [e, e]$$

- Inverse:

$$t \,(= \text{transposition}) : B^{(2)} \to B^{(2)} : \quad [e_1, e_2] \longmapsto [e_2, e_1]$$

- multiplication:

$$\mu : B^{(2)} \times_{B^{(1)}} B^{(2)} \cong B^{(3)} \longrightarrow B^{(2)} : \quad ([e_1, e_2], [e_2, e_3]) \mapsto [e_1,$$

We now pull back all the above structure maps on cohomology:

First notice that

$$E_u^{(2)} \cong E_u \times K$$

by

$$(e_1, e_2) \longmapsto (e_1, k) \quad \text{if } e_2 = e_1 k$$

Under this identification, the $T^2$ action on $E_u^{(2)}$ becomes

$$(e_1, k) \longmapsto (e_1, e_1 k) \longmapsto (e_1 t_1, e_2 k t_2)$$
$$\longmapsto (e_1 t_1, e_1 t_1 t_1^{-1} k t_2)$$
$$\longmapsto (e_1 t_1, t_1^{-1} k t_2)$$

Thus we get an induced ~~diff~~ identification

$$E_u^{(2)}/T^2 \cong E_u \times^T K/T$$
$$[e_1, e_2] \longmapsto [e_1, kT] \quad \text{if } e_2 = e_1 k$$

Similarly, we have

$$E_u^{(3)} \cong E_u \times K \times K \ : \ (e_1, e_1 k_1, e_1 k_1 k_2) \longmapsto (e_1, k_1, k_2)$$

and

$$(e_1 t_1, e_1 k_1 t_2, e_1 k_1 k_2 t_3) = (e_1 t_1, e_1 t_1 t_1^{-1} k_1 t_2, e_1 k_1 k_2\, t_2^{-1} k_2 t_3)$$
$$\longmapsto (e_1 t_1, t_1^{-1} k_1 t_2, t_2^{-1} k_2 t_3)$$

So

$$E_u^{(3)}/T^3 \cong \cancel{E_u \times^T K \times^T K/T}\, (E_u \times K \times K)/T^3$$

where the $T^3$ action on $E_u \times K \times K$ is

$$(e_1, k_1, k_2) \cdot (t_1, t_2, t_3) = (e_1 t_1, t_1^{-1} k_1 t_2, t_2^{-1} k_2 t_3)$$

But $\quad (E_u \times K \times K)/T^3 = E_u \times^T (K \times^T K/T)$

So we have the identifications

$$B^{(2)} = E_u \times^T K/T$$
$$B^{(3)} = E_u \times^T (K \times^T K/T).$$

Hence

$$H^*(B^{(2)}) = H^T(K/T)$$
$$H^*(B^{(3)}) = H^T(K \times^T K/T)$$
$$= H^T(K/T) \otimes_S H^T(K/T) \qquad \text{(from last time)}$$

where the last identification is due to the general fact we proved last time that for any $K$-space $Y$,

$$H^T(K \times^T Y) \cong H^T(K/T) \otimes_S H^T(Y).$$

We also have

$$H^*(B^{(1)}) = H^*(E/T) = S$$

Therefore, the pull-backs on cohomology of all the structure maps for the groupoid $B^{(2)}$ over $B^{(1)}$ give the groupoid structure on $\mathcal{U} = \operatorname{spec} H^T(K/T)$. We ummary

Set $R = H^T(K/T)$, $S = H^T(pt) = H(B_T) = H(B^{(1)})$

Then from:

$p_1 : B^{(2)} \to B^{(1)}$, $[e_1, e_2] \mapsto [e_1]$

$p_2 : B^{(2)} \to B^{(1)}$ ; $[e_1, e_2] \mapsto [e_2]$

$d : B^{(1)} \to B^{(2)}$ ; $[e] \mapsto [e, e]$

$t : B^{(2)} \to B^{(2)}$ : $[e_1, e_2] \mapsto [e_2, e_1]$

$\mu : B^{(3)} \to B^{(2)}$ : $[e_1, e_2, e_3] \mapsto [e_1, e_3]$

we get :

$\Pi_L = p_1^* : S \to R$

$\Pi_R = p_2^* : S \to R$

$\varepsilon = d^* : R \to S$

$C = t^* : R \to R$

$\Delta = \mu^* : R \to R \otimes_S R$

heorem : The above maps $\Pi_L, \Pi_R, \varepsilon, C$ and $\Delta$ make ($\mathcal{U} = \operatorname{Spec} R$, $\underline{h} = \operatorname{Spec} S$) into a groupoid scheme. Moreover, if $X$ is any $K$-space, the map

$$\Delta_X = \{ K \times^T X \to X : [k, x] \mapsto kx \}^* : H^T(X) \to H^T(K/T) \otimes H^T(X)$$

is the composition of an action of $(\mathcal{U}, \underline{h})$ on $\operatorname{Spec} H^T(X)$, (assuming that $H^*(X)$ is even)



Characteristic operators

Definition : A characteristic operator for $(K, T)$ is a rule that assigns to each $K$-space $X$ an $H^K(X)$-linear endomorphism $\phi_X : H^T(X) \to H^T(X)$ such that if $F : X \to Y$ is a $K$-map, then $F^* \circ \phi_Y = \phi_X \circ F^*$.

Remark : When $K = T$, any $H^T(X)$-linear endomorphism of $H^T(X)$ must be a multiplication operator by characters. This is why the name characteristic operators.

Fact : The set $\hat{A}$ of all characteristic operators is an $S$-algebra.

Definition We say that a characteristic operator is of compact support if there exists a compact subset $K_0 \subset K$ which is $T$-stable such that given any $K$-space $X$, a $T$-stable subset $X_0$ of $X$ and an element $z \in H^T(X)$ vanishing in $H^T(K_0 X_0)$, the element $\phi_X(z_0)$ must vanish in $H^T(X_0)$

Remark : In the finite case, can take $K_0 = K$ and every characteristic operator is compact.

**Definition – Notation:**

$\hat{A}_c$ = the $S$-subalgebra of $\hat{A}$ of all characteristic operators of compact support.

**Proposition:** For any characteristic operator $a$ and any $K$-space $X$, we have

$$\Delta_X \circ a = (a \otimes id) \circ \Delta_X : \ H^T(X) \longrightarrow H^T(K/T) \otimes_S H^T(X)$$

**Corollary 1** For a characteristic operator $a$, we have

$$a = 0 \iff a = 0 \text{ on } H^T(K/T)$$
$$\iff \varepsilon \circ a = 0 \in Hom_S(H^T(K/T), S)$$

**Proof:**

If $\varepsilon \circ a = 0 : H^T(K/T) \to S$, then for any $K$-space $X$,

$$a \text{ on } H^T(X) = (\varepsilon \otimes id) \circ \Delta_X \circ a \qquad \left(\text{because } (\varepsilon \otimes id) \circ \Delta_X = id_{H^T(X)}\right)$$
$$= (\varepsilon \otimes id) \cdot (a \otimes id) \circ \Delta_X \qquad (\text{by Proposition})$$
$$= (\varepsilon \circ a \otimes id) \circ \Delta_X$$
$$= 0 .$$

//



**Corollary 2** $\hat{A}$ has no $S$-torsion.

**Proof:** If $s \in S$ and $a \in \hat{A}$ are such that

$$sa = 0, \quad \text{and} \quad a \neq 0$$

then for any $z \in H^T(K/T)$

$$0 = (\varepsilon \circ sa)(z) = \varepsilon(s(a \cdot z))$$
$$= s\, \varepsilon(a \cdot z)$$

But since $a \neq 0$, we know by Corollary 1 that $\varepsilon \circ a \neq 0$ so $\exists\, z \neq 0$ s.t. $\varepsilon(a \cdot z) \neq 0 \in S$. Since $S$ is a polynomial algebra, it has no $S$-torsion. Thus $s = 0$.

This shows that $\hat{A}$ has no $S$-torsion

**Corollary 3** (added by me) (of Corollary 1).

The action of $a \in \hat{A}$ on $H^T(X)$ is expressed using

$$\Delta_X : H^T(X) \longrightarrow H^T(K/T) \otimes_S H^T(X)$$

and the map $\varepsilon \circ a : H^T(K/T) \longrightarrow S$ by

$$a \text{ on } H^T(X) = (\varepsilon \circ a \otimes id) \circ \Delta_X$$

**Remark:** Should think of $\hat{A}$ as the dual of $H^T(K/T)$ by $a \mapsto \varepsilon \circ a \in Hom(H^T(K/T), S)$.

## Integration over the fibre

Assume that $P: E \to B$ is a fibration over a pathwise connected base $B$ with $b_0 \in B$. Let $F = P^{-1}(b_0)$. Assume that this fibration is orientable. This means that the holonomy around $b_0$ acts trivially on $H^*(F)$. Since $B$ is pathwise connected, the weak homotopy type of $F$ is independent of the choice of $b_0$. Then we have ..., assuming $H^r(F) = 0$ for $r > n$.

$$\mathrm{Hom}_{\mathbb{Z}}(H^n(F), \mathbb{Z}) \longrightarrow \left(\mathrm{Hom}_{H^*(B)}(H^*(E), H^*B)) \text{ of degree } -n\right)$$

denoted by

$$\tau \longmapsto \int_\tau$$

obtained as follows, by using the Serre spectral sequence:

$$H^{m+n}(E) \longrightarrow E_\infty^{m,n} = E_2^{m,n} = H^m(B, H^n(F)) \xrightarrow{\tau} H^m(B, \mathbb{Z}).$$

Remark
1) This is just the identity map when $B = pt$.
2) It is functorial over pullbacks
3) It preserves certain Mayer-Vietoris sequences
4) Can do this for relative cohomology as well.



## The $A$-action on $H^*(E/T)$ for any principal $K$-bundle $E$

If $E$ is a principal $T$-bundle, then we have a ring homomorphism

$$ch: S \longrightarrow H^{even}(E/T), \quad \lambda \longmapsto C_1(\mathcal{L}_\lambda = E \times_T \mathbb{C}_{-\lambda}) \in H^2(E/T).$$

We call it the characteristic homomorphism. Using the characteristic homomorphism, we get an $S$-module structure on $H^*(E/T)$:

$$s \cdot z = ch(s)\, z$$

Now assume that $E$ is also a principal $K$-bundle, so thus also a $T$-bundle. Then we can use the $K$-action to define the following $W$-action on $H^*(E/T)$: for $w \in W$,

$$w \cdot z = w^* z$$

where $w: E/T \to E/T$: $w eT = e\omega T$. Because of the following basic property of the characteristic map,

$$w^* C_1(\mathcal{L}_\lambda) = C_1(\omega^* \mathcal{L}_\lambda) = C_1(\mathcal{L}_{\omega\lambda}) \Rightarrow$$

ie $\quad w^* ch(\lambda) = ch(\omega\lambda)$

we have, for any $\omega \in W$ and $s \in S$

$$\omega s = (\omega \cdot s)\, \omega$$

as operators on $H^*(E/T)$. Therefore we have an action of the smashed product algebra $\mathbb{C}W \ltimes S$ on $H^*(E/T)$.

Now for each $i \in I$, consider the fibre bundle

$$\begin{array}{c} E/T \\ \downarrow \pi_i \\ E/K_i \end{array}$$

which has fibre $K_i/T = P_i/B = \mathbb{C}P^1$ so it has a prefered orientation $\sigma_i \in \mathrm{Hom}_{\mathbb{Z}}(H^2(K_i/T), \mathbb{Z})$ namely the fundamental cycle.

Integration over the fibre gives

$$H^*(E/T) \longrightarrow H^{*-2}(E/K_i) \ ; \quad z \longmapsto \int_{\sigma_i} z$$

Now <u>define</u>

$$\underline{A_i} : \underline{H^*(E/T)} \longrightarrow \underline{H^{*-2}(E/T)} \ ; \quad A_i \cdot z = \pi_i^* \int_{\sigma_i} z .$$

<u>Proposition</u>: For any $z \in H^*(E/T)$,

$$\alpha_i \cdot (A_i \cdot z) = z - r_i \cdot z \qquad \circledast$$

<u>Proof</u>: We will check this over $\mathbb{Q}$ ( why? )

The fibration $\pi_i : E/T \to E/K_i$ gives a $H^*(E/K_i)$-module structure on $H^*(E/T)$. Since the fibre is $= \mathbb{C}P^1$, this is in



(over $\mathbb{Q}$)

fact a free $H^*(E/K_i)$-module, A basis of which is given by $1$ and $\frac{1}{2}ch(\alpha_i) \in H^2(E/T)$. For $z_0 \in H^*(E/K_i)$ we use the same letter to denote the pull back $\pi_i^* z_0 \in H^*(E/T)$. We will check $\circledast$ for $z = z_0$ and $z = \frac{1}{2}ch(\alpha_i)\, z_0$. Clearly $A_i \cdot z_0 = 0$ and $r_i \cdot z_0 = z_0$.

Thus $\circledast$ holds for $z = z_0$. Now for $z = \frac{1}{2}ch(\alpha_i)\, z_0$,

$$\alpha_i \cdot (A_i \cdot z) = \alpha_i \cdot \left( A_i \cdot \left(\frac{ch(\alpha_i)}{2}\right) z_0 \right)$$

<u>Lemma</u>. $A_i \cdot ch(\alpha_i) = 2$. ( a calculation over $\mathbb{C}P^1$ )

Assume lemma. Then

$$\alpha_i \cdot (A_i \cdot z) = \alpha_i \cdot z_0 = ch(\alpha_i)\, z_0$$

On the other hand,

$$\begin{aligned} z - r_i \cdot z_0 &= \tfrac{1}{2} ch(\alpha_i) z_0 - r_i \cdot \left(\tfrac{1}{2} ch(\alpha_i)\, z_0\right) \\ &= \tfrac{1}{2} ch(\alpha_i)\, z_0 - r_i \cdot \left(\tfrac{1}{2} ch(\alpha_i)\right) \ r_i \cdot z_0 \\ &= \tfrac{1}{2} ch(\alpha_i)\, z_0 + \tfrac{1}{2} ch(\alpha_i)\, z_0 \\ &= ch(\alpha_i)\, z_0 \end{aligned}$$

Hence $\circledast$ holds for $z = \frac{1}{2} ch(\alpha_i)\, z_0$ //

It is strange to carry the $\frac{1}{2}$ around. <u>Why necessary</u>?

//

Therefore we have

**Theorem:** For any principal $K$-bundle $E$, the following define an $\underline{A}$-action on $H^*(E/T)$:

$$s \cdot z = ch(s)\, z$$

$$\omega \cdot z = \omega^* z$$

$$A_i \cdot z = \pi_i^* \int_{\sigma_i} z$$

Moreover, the characteristic morphism

$$ch : S \longrightarrow H^{even}(E/T) : \quad \lambda \mapsto c_1(\mathcal{L}_\lambda)$$

is an $\underline{A}$-map, where $A_i$ acts on $s \in S$ by

$$A_i s = \frac{s - r_i s}{\alpha_i}$$

as before (see Lecture 2). //

**Example** $E = K$ with right action of $K$ by right multiplications. Then the $A_i$'s on $H^*(K/T)$ are the BGG-operators.

**Example:** If $E_1 \xrightarrow{f} E_2$ is a $K$-map, then $f^* : H^*(E_2/T) \to H^*(E_1/T)$ is clearly an $\underline{A}$-map.



**$\underline{A}$-action on $H^T(X)$ for $K$-space $X$:**

**Example:** Let $X$ be a $K$-space and let $E_u = E_K$ be the universal principal bundle of $K$. Let

$$E = E_u \times X$$

with the $K$-action given by

$$(e, x) \cdot k = (ek^{-1}, kx)$$

Then

$$E/T = E_u \times^T X$$

so get an action of $\underline{A}$ on $H^T(X)$.

If $f : X \to Y$ is a $K$-map, then

$$E_u \times X \longrightarrow E_u \times Y : \quad (e, x) \mapsto (e, f(x))$$

is a $K$-map, so

$$f^* : H^T(Y) \longrightarrow H^T(X)$$

is an $\underline{A}$-map. Finally, the $\underline{A}$-action on $H^T(X)$ is clearly $H^K(X)$-linear. Thus ~~we~~ we ~~have~~ can think of elements of $\underline{A}$ as characteristic operators.

**Property:** For any $K$-space $X$, the morphism

$$S \longrightarrow H^T(X) \qquad (= (X \to pt)^*)$$

is an $\underline{A}$-map.

**Proof:** This is the same as the characteristic morphism, via the [illegible] $S = H^T(pt) \to H^T(X)$

**Proposition**: For any $K$-space $X$, the multiplication map

$$H^T(X) \otimes_S H^T(X) \longrightarrow H^T(X)$$

is an $\underline{A}$-map.

## T-equivariant homology

• For a $T$-space $X$, the $T$-equivariant homology of $X$ is defined to be $Hom_S(H^T(X), S)$.

• Suppose that $X$ is a $K$-space. Then $H^T(X)$ is an $\underline{A}$-module. Since $S$ is also an $\underline{A}$-module, we know that $Hom_S(H^T(X), S)$ is then also an $\underline{A}$-module (see Lecture 2):

$$(s\cdot f)(z) = s f(z)$$

$$(A_i\cdot f)(z) = f(A_i\cdot z) + A_i\cdot f(r_i\cdot z) = A_i\cdot f(z) - r_i\cdot f(A_i\cdot z)$$

$$(\omega\cdot f)(z) = \omega\cdot f(\omega^{-1}\cdot z)$$

• If $F: X \to Y$ is a $K$-map, then we have shown that $F^*: H^T(Y) \to H^T(X)$ is an $\underline{A}$-map. Define

$$F_*: Hom_S(H^T(X), S) \longrightarrow Hom_S(H^T(Y), S)$$

by $(F_*f)(z_Y) = f(F^* z_Y)$. Then $F_*$ is an $\underline{A}$-map as well.



Let's check $F_*(A_i\cdot f) = A_i\cdot(F_*f)$. So let $z \in H^T(Y)$, need to show

$$(A_i\cdot f)(F^* z) = \cancel{F_*f}\big(A_i\cdot(F_*f)\big)(z)$$

Now

$$lhs = f(A_i\cdot F^* z) + A_i\cdot f(r_i\cdot F^* z)$$

$$rhs = (F_*f)(A_i\cdot z) + A_i\cdot F_*f(r_i\cdot z)$$

$$= f(F^*(A_i\cdot z)) + A_i\cdot f(F^*(r_i\cdot z))$$

Since $F^*$ is an $\underline{A}$-map, we indeed have $lhs = rhs$. //

Example: Suppose $Y$ is a $T$-space such that $H^r(Y) = 0$ for $r > n$. Then integration over the fibre for

$$\begin{array}{ccc} Y & \longrightarrow & E_u \times^T Y \\ & & \downarrow \\ & & E_u/T \end{array}$$

gives a map

$$\begin{array}{ccc} Hom_{\mathbb{Z}}(H^r(Y), \mathbb{Z}) & \longrightarrow & Hom_S(H^T(Y), S) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ c & \longmapsto & \int_c \end{array}$$

For each $i \in I$, we have a map

$$\mathrm{Hom}_{\mathbb{Z}}(H^n(Y), \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H^{n+2}(K_i \times^T Y), \mathbb{Z}) : \quad \tau \longmapsto \sigma_i * \tau$$

where $\sigma_i * \tau$ ~~the~~ $\in \mathrm{Hom}_{\mathbb{Z}}(H^{n+2}(K_i \times^T Y), \mathbb{Z})$ is the composition

$$H^{n+2}(K_i \times^T Y) \xrightarrow[\int_\tau]{} H^2(K_i/T) \xrightarrow[\sigma_i]{} \mathbb{Z}$$

using integration over the fibre first for the bundle

$$\begin{array}{ccc} Y & \longrightarrow & K_i \times^T Y \\ & & \downarrow \\ & & K_i/T . \end{array}$$

Consequently we have a map

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbb{Z}}(H^n(Y), \mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(H^{n+2}(K_i \times^T Y), \mathbb{Z}) & \longrightarrow & \mathrm{Hom}_S(H^T(K_i \times^T Y), S) \\ \tau & \longmapsto & \sigma_i * \tau & \longmapsto & \int_{\sigma_i * \tau} . \end{array}$$

Now suppose that $X$ is a $K$-space with $K$-action

$$\mu : K \times X \to X .$$

Assume that $F : Y \to X$ is a $T$-equivariant map.

Then for $\tau \in \mathrm{Hom}_{\mathbb{Z}}(H^n(Y), \mathbb{Z})$, we have $\int_\tau \in \mathrm{Hom}_S(H^T(Y), S)$, so

$$F_* \int_\tau \in \mathrm{Hom}_S(H^T(X), S)$$

and thus

$$A_i \cdot F_* \int_\tau \in \mathrm{Hom}_S(H^T(X), S)$$



On the other hand, we have

$$\begin{array}{ccccc} K_i \times^T Y & \xrightarrow{F_i} & K_i \times^T X & \xrightarrow{\mu} & X \\ [k_i, y] & \longmapsto & [k_i, f(y)] & \longmapsto & k_i \cdot f(y) \end{array}$$

and $\int_{\sigma_i * \tau} \in \mathrm{Hom}_S(H^T(K_i \times^T Y), S)$.

<u>Fact</u>

$$A_i \cdot F_* \int_\tau = \mu_* F_{i*} \int_{\sigma_i * \tau} \in \mathrm{Hom}_S(H^T(X), S).$$

<u>Proof</u>: ? . . . . . .

This fact will be used in the next lecture for $Y = X_w^0$, a Schubert variety in 6

End of Lecture 5

Next lecture: Schubert basis for $H^T(K/T)$.

# Lecture 6. Feburary 26, 1997

(The following is the beginning of lecture 4 given on Feb. 19)

## Schubert Cells in $G/P$

Recall that a closed subgroup $P$ of $G$ is called a standard parabolic subgroup if $P \supset B$.

Let $P \subset G$ be a standard parabolic subgroup. Then $\exists$ subset $J \subset I$ s.t.

$$P = B W_J B$$

where

$$W_J = \langle r_j \rangle_{j \in J}$$

is the subgroup of $W$ generated by $\{r_j, j \in J\}$.

Set

$$W_P = W_J$$

$$W^P = \{ u \in W : \; u < uv \; \text{ for all } v \in W_P, \; v \neq id \}$$

Thus $W^P$ is the set of minimum representatives of the coset space $W/W^P$. We have

$$G/P = \coprod_{w \in W^P} B w P$$

$$BwP = \mathbb{C}^{\ell(w)}$$

l is called the Schubert Cell corresponding to w.

Each $BwP$ is $T$-stable and

$$G/P = \coprod_{w \in W^P} BwP$$

kes $G/P$ into a CW-complex.

$$X_w^P = \text{closure of } BwP \text{ in } G/P$$

is a complex projective variety called the Schubert variety

e have

$$\underline{X_w^P} = \underline{\bigcup_{\substack{v \in W^P \\ v \leq w}} BvP}$$

r $w \in W^P$, let

$$i_w^P : X_w^P \hookrightarrow G/P$$

$[X_w^P] \in H_{2\ell(w)}(X_w^P, \mathbb{Z})$. Set

$$\underline{\sigma_w^P} = \underline{(i_w^P)_* [X_w^P]} \in H_{2\ell(w)}(G/P).$$



## Schubert Basis for $H_\cdot(G/P, \mathbb{Z})$ and $H^\cdot(G/P, \mathbb{Z})$

Fact: $\{\sigma_w^P : w \in W\}$ is a basis for ~~G/P~~ $H_\cdot(G/P, \mathbb{Z})$

Notation: The dual basis of $H^\cdot(G/P, \mathbb{Z})$ dual to $\{\sigma_w^P : w \in W\}$ is denoted by

$$\{\sigma_P^w : w \in W\}$$

Remark $H^{\text{odd}}_{\text{even}}(G/P) = 0$.

(Here starts lecture 6)

## Schubert Basis for $\mathrm{Hom}_S(H^T(G/P), S)$ and $H^T(G/P)$.

Definition: For $w \in W^P$, put

$$\sigma_{(w)}^P = (i_w^P)_* \int_{[X_w^P]} \in \mathrm{Hom}_S(H^T(G/P), S)$$

Then $\{\sigma_{(w)}^P : w \in W\}$ is a basis for $\mathrm{Hom}_S(H^T(G/P), S)$

There is then a unique basis

$$\{\sigma_P^{(w)} : w \in W\}$$

of $H^T(G/P)$ ~~st~~ (over $S$) s.t.

$$\langle \sigma_P^{(v)}, \sigma_{(w)}^P \rangle = \delta_{v,w}.$$

Both $\{\sigma_{(w)}^P\}$ & $\{\sigma_P^{(w)}\}$ are called Schubert basis

basis $\{\sigma_P^{(w)} : w \in W\}$ of $H^T(G/P)$ is characterized by

properties:

(1) $\deg(\sigma_P^{(w)}) = 2\ell(w)$

(2) Under evaluation at $0$:

$$\mathbb{Z} \otimes_S H^T(G/P) \longrightarrow H^*(G/P)$$

we have $\sigma_P^{(w)} \longmapsto \sigma_P^{w}$.

(3) $$\left[ i_w^P : X_w^P \to G/P \right]^* (\sigma_P^{(v)}) = 0 \quad \text{if } v \not\leq w.$$

, we look at

- The action of $\Delta$ on ~~the~~ $\mathrm{Hom}_S(H^T(G/P), S)$ in the basis $\{\sigma_{(w)}^P\}$
- The action of $\underline{A}$ on $H^T(G/P)$ in the basis $\{\sigma_P^{(w)}\}$
- The ring of cpt characteristic operators $\underline{\hat{A}_c}$ expressed in terms of the $\Delta$-action on $H^T(K/T) = H^T(G/B)$
- The Hopf algebroid structure on $H^T(K/T)$.



<u>Another set of elements $\{\psi_w^P : w \in W^P\}$ in $\mathrm{Hom}_S(H^T(G/P), S)$:</u>

For $w \in W$, consider the $T$-equivariant map

$$j_w^P : \quad pt \longrightarrow G/P : \quad pt \longmapsto wP$$

Set

$$\psi_w^P = (j_w^P)^* \in \mathrm{Hom}_S(H^T(G/P), S)$$

Of course $\psi_w^P = \psi_{w_1}^P$ if $w \in w_1 W_P$.

~~What~~ We think of $\psi_w^P$ as localizing at the $T$-fixed pt $wP$.

<u>Warning</u> $\{\psi_w^P : w \in W^P\}$ is NOT an $S$-basis for $\mathrm{Hom}_S(H^T(G/P)$

because $$\sigma_{(r_i)}^P = \frac{1}{\alpha_i}\psi_{id}^P - \frac{1}{\alpha_i}\psi_{r_i}^P.$$

<u>Remark</u> Expressing $\psi_w^P$ as a linear combination over $S'$ of the $\sigma_{(v)}^P$ we get the D-matrix in Kostant–Kumar. Will do this later.

<u>Properties</u>. Consider the $G$-equivariant map

$$\pi_P : G/B \longrightarrow G/P \; : \; gB \longmapsto gP.$$

Then

$$(\pi_P)_* \psi_w^B = \psi_w^P \qquad w \in W$$

$$(\pi_P)^* \psi_P^{(w)} = \psi_B^{(w)} \qquad w \in W^P.$$

ion of $\underline{A}$ on $\mathrm{Hom}_s(H^T(G/p), S)$ in the basis $\{\sigma^P_{(\omega)} : \omega \in W^P\}$.

ition
m 1: $A_i \cdot \sigma^P_{(\omega)} = \begin{cases} \sigma^P_{(r_i\omega)} & \text{if } \omega < r_i\omega \text{ and } r_i\omega \in W^P \\ 0 & \text{otherwise} \end{cases}$

f: let $i^P_\omega : X^P_\omega \hookrightarrow G/p$.

Recall that

$$\sigma^P_{(\omega)} = (i^P_\omega)_* \int_{[X^P_\omega]}$$

From the fact stated at the end of last lecture,

$$A_i \cdot \sigma^P_{(\omega)} = \mu_* \int_{\sigma_i * [X^P_\omega]} \in \mathrm{Hom}_S(G/p, S)$$

where

$$\mu : K_i \times^T X^P_\omega \longrightarrow G/p : \quad (k_i, x) \longmapsto k_i x$$

It follows that (?)

$$A_i \cdot \sigma^P_{(\omega)} = \begin{cases} \sigma^P_{(r_i\omega)} & \text{if } \omega < r_i\omega \text{ and } r_i\omega \in W^P \\ 0 & \text{otherwise} \end{cases}$$

//

ave: For $v, \omega \in W$,

$$v \cdot \psi^P_\omega = \psi^P_{v\omega}$$



Action of $\underline{A}$ on $H^T(G/p)$ in the basis $\{\sigma_p^{(\omega)} : \omega \in W^P\}$

Proposition 2: For $v \in W$, $\omega \in W^P$,

$$A_v \cdot \sigma_p^{(\omega)} = \begin{cases} \epsilon(v)\, \sigma_p^{(v\omega)} & \text{if } \ell(v^{-1}) + \ell(v\omega) = \ell(\omega) \\ & \iff v\omega \in \\ 0 & \text{otherwise} \end{cases}$$

Proof: let's first check that

$$A_i \cdot \sigma_p^{(\omega)} = \begin{cases} -\sigma_p^{(r_i\omega)} & \text{if } 1 + \ell(r_i\omega) = \ell(\omega) \quad (\text{i.e. } r_i \\ 0 & \text{otherwise} \end{cases}$$

From the previous Proposition 1, if $r_i\omega < \omega$ ($\Rightarrow$ $r_i(r_i\omega) > r$

$$A_i \cdot \sigma^P_{(r_i\omega)} = \sigma^P_{(\omega)}$$

But

$$(A_i f)(z) = A_i \cdot f(z) - r_i \cdot f(A_i \cdot z) \qquad z \in H^T(G/p)$$

by definition, so by letting $f = \sigma^P_{(r_i\omega)}$ and $z = \sigma_p^{(v)}$, we get

$$\delta_{\omega, v} = 0 - r_i \cdot \sigma^P_{(r_i\omega)}(A_i \cdot \sigma_p^{(v)})$$

or $\quad (A_i \cdot \sigma_p^{(v)}, \sigma^P_{(r_i\omega)}) = -\delta_{\omega, v}$

$$\Rightarrow \quad A_i \cdot \sigma_p^{(\omega)} = -\sigma_p^{(r_i\omega)}$$

otherwise follows.

//

rk: Recall that

$$\mathcal{E} = \psi^B_{id} = \sigma^B_{(id)} \in \mathrm{Hom}_S(H^T(G/B), S)$$

We can identify

$$\underline{A} = \mathrm{Hom}_S(H^T(G/B), S) \qquad (*)$$

by

$$a \longmapsto f_a : \quad f_a(z) = \mathcal{E}(a \cdot z)$$

Then this is an identification of $S$-modules, and for Proposition 2,

$$f_{A_w} = \epsilon(w)\, \sigma^B_{(w^{-1})}$$

ie.

$$A_w \longmapsto \epsilon(w)\, \sigma^B_{(w^{-1})}$$

us by Proposition 1, we see that under the identification $(*)$,

e (left) $\underline{A}$-action on $\mathrm{Hom}_S(H^T(G/B), S)$ becomes the (left) action

$\underline{A}$ on $\underline{A}$ by

$$a \cdot b = b(*a)$$

here, recall from lecture 2, that

$$*S = S$$
$$*w = w^{-1}$$
$$*A_w = \epsilon(w)\, A_{w^{-1}}$$

( The $*$ in Lecture 2 is defined to be $*S = S$, $*w = \epsilon(w) w^{-1}$, $*A_w = A_{w^{-1}}$. )



## The ring $\hat{\underline{A}}$ of characteristic operators again

Proposition Set

$$\mathcal{E} = \psi^B_{id} \in \mathrm{Hom}_S(H^T(G/B), S)$$
$$= \sigma_B^{(id)}.$$

So

$$\mathcal{E}(\sigma_B^{(w)}) = \delta_{w,id} \qquad w \in W.$$

Proposition:

(1) Every characteristic operator $a \in \hat{\underline{A}}$ can be uniquely written as

$$a = \sum_{w \in W} s_w A_w \qquad s_w \in S$$

In fact,

$$s_w = \mathcal{E}\left(a \cdot (\epsilon(w)\, \sigma_B^{(w^{-1})})\right)$$

( Recall $\epsilon(w) = (-1)^{\ell(w)}$ ).

(2) $a$ is compactly supported iff only finitely many $w$'s occur in the sum. (i.e. iff only finitely many $s_w$'s are

Proof (1). For any $a \in \hat{A}$, write

$$a' = a - \sum_{w \in W} \mathcal{E}\left(a \cdot (\epsilon(w)\sigma_B^{(w^{-1})})\right) A_w$$

Then $a' \in \hat{\underline{A}}$. Thus to show $a'=0$ it is enough to show that

$$\mathcal{E}(a'\cdot z)=0$$

for any $z \in H^T(G/B)$. (See Lecture 5). Since both $a'$ and $\mathcal{E}$ are $S$-linear, it is enough to show that

$$\mathcal{E}(a'\cdot \sigma_B^{(v)})=0$$

for all $v \in W$. Now

$$a'\cdot\sigma_B^{(v)} = a\cdot\sigma_B^{(v)} - \sum_{w\in W} \mathcal{E}\left(a\cdot \epsilon(w)\sigma_B^{(w^{-1})}\right) A_w\cdot\sigma_B^{(v)}$$

$$= a\cdot\sigma_B^{(v)} - \sum_{\substack{w\in W\\ \ell(w^{-1})+\ell(wv)=\ell(v)}} \mathcal{E}(a\cdot\epsilon(w)\sigma_B^{(w^{-1})})\, \epsilon(w)\, \sigma_B^{(wv)}$$

$$= a\cdot\sigma_B^{(v)} - \sum_{\substack{w\in W\\ \ell(w^{-1})+\ell(wv)=\ell(v)}} \mathcal{E}\left(a\cdot\sigma_B^{(w^{-1})}\right)\sigma_B^{(wv)}$$

*

$$\Delta\sigma_B^{(v)} = \sum_{\substack{u,w\in W\\ uw=v\\ \ell(v)=\ell(u)+\ell(w)}} \sigma_B^{(u)}\otimes\sigma_B^{(w)}$$

$$a = \cancel{\mathcal{E}\otimes id \otimes} \left((\mathcal{E}\circ a)\otimes id\right)\circ\Delta_*$$

(Corollary 3 in Lecture 5). $\Rightarrow a'\cdot\sigma_B^{(v)}=0$

$\Rightarrow a'=0$. //



Uniqueness is clear.

If $a$ has compact support, ~~we can then~~ since any compact subset of $K$ is contained in some $K_w$ where $K_w = K_{i_1}K_{i_2}\cdots K_{i_r}$ if $w = r_{i_1}r_{i_2}\cdots r_{i_r}$ [red], we see that there are only finitely many $w$'s involved in the expression

$$a = \sum_{w\in W} S_w A_w.$$

//

__Remark.__ We can think of $\underline{A}$ as $\mathrm{Hom}_S(H^T(K/T), S)$, or the $S$-dual of $H^T(K/T)$ via the pairing:

$$(a, z) \overset{\mathrm{def}}{=} \mathcal{E}(a\cdot z)$$

Let's check then that the $\underline{A}$ action on $\mathrm{Hom}_S(H^T(K/T), S)$ becomes the $\underline{A}$-action on $\underline{A}$ by left multiplications: For $a \in \underline{A}$, use $f_a \in \mathrm{Hom}_S(H^T(K/T), S)$ to denote the element given by

$$f_a(z) = (a, z) = \mathcal{E}(a\cdot z).$$

For $i \in I$, we have, by definition want to check

$$A_i\cdot f_a = f_{A_i a}$$

The Hopf Algebroid Structure on $H^T(K/T)$

Overview: Recall that from Lecture 5 that $H^T(K/T)$ is a Hopf algebroid over $S$. We now express the structure maps for this Hopf algebroid in the basis $\{\sigma_B^{(w)}: w \in W\}$.

First, recall that we have ring homomorphisms

$$\pi_L: \quad S \longrightarrow H^T(K/T)$$

$$\pi_R: \quad S \longrightarrow H^T(K/T).$$

This gives two $S$-module structures on $H^T(K/T)$. The map $\pi_L$ is nothing but the characteristic homomorphism $ch$ in Lecture 5.

The map $\pi_R$ is a little more mysterious. It gives the 2nd $S$-mod. str on $H^T(K/T)$ in Lec. 4.

Proposition: The elements $\{\sigma_B^{(w)}: w \in W\}$ is also a basis for the second $S$-module on $H^T(K/T)$ defined by $\pi_R$.

Remark: I (Lu) suspect that $\pi_R$ has a lot to do with the Bruhat-Poisson structure on $K/T$.



The next theorem expresses the structure maps for the Hopf algebroid structure on $H^T(K/T)$ in the basis $\{\sigma_B^{(w)}: w \in W\}$.

Theorem: (Recall notation from Lecture 5):

1) For $\lambda \in h_{\mathbb{Z}}^*$,

$$\pi_R(\lambda) = \pi_L(\lambda) + \sum_{i \in I} \langle \lambda, \check{\alpha}_i \rangle \sigma_B^{(r_i)}$$

2) $\varepsilon(\sigma_B^{(w)}) = \delta_{w,id}$

3) $\Delta \sigma_B^{(w)} = \sum_{\substack{u,v \in W \\ w = uv \,(red)}} \sigma_B^{(u)} \otimes \sigma_B^{(v)}$ ( $w = uv\,(red)$ means $w = uv$, $\ell(w) = \ell(u) + \ell(v)$ )

4) $c(\sigma_B^{(w)}) = \epsilon(w)\, \sigma_B^{(w^{-1})}$

5) For any $K$-space $X$ and $\sigma \in H^T(X)$

$$\Delta_X(\sigma) = \sum_{w \in W} \epsilon(w) \sigma_B^{(w^{-1})} \otimes (A_w \cdot \sigma) \in H^T(K/T) \otimes_S H^T(X)$$

Proof. Next page

first prove 5). 5) is due to the general fact if algebra $A$ acts on a space $M$, then using a basis $\cdots a_i, \cdots$ of $A$ and the dual basis $z_1, \cdots z_i, \cdots$ of $A^*$, co-module map is nothing but

$$\Delta_M: \quad M \longrightarrow A^* \otimes M:$$

$$\Delta_M(m) = \sum_i z_i \otimes a_i \cdot m$$

) in our example, we are identifying $H^T(K/T)$ w/ $\underline{A}^*$ the pairing

$$(a, z) = \varepsilon(a \cdot z) \qquad a \in \underline{A}, \quad z \in H^T(K/T)$$

or this pairing, we have $\{A_\omega : \omega \in W\}$ as a basis for $\underline{A}$ dual basis in $H^T(K/T)$ is $\{\epsilon(\omega)\sigma_B^{(\omega^{-1})} : \omega \in W\}$ (see e 6-8).. Thus for any $\sigma \in H^T(X)$

$$\Delta_X(\sigma) = \sum_{\omega \in W} \epsilon(\omega)\sigma_B^{(\omega^{-1})} \otimes (A_\omega \cdot \sigma)$$

erson gave the following proof in class:



Since $\{\epsilon(\omega)\sigma_B^{(\omega^{-1})} : \omega \in W\}$ is a basis for $H^T(K/T)$, we know

$$\Delta_X(\sigma) = \sum_{\omega \in W} \epsilon(\omega)\sigma_B^{(\omega^{-1})} \otimes \phi_\omega$$

for some $\phi_\omega \in H^T(X)$ for each $\omega \in W$. Need to show $\phi_\omega = A_\omega \cdot \sigma$

To do this, let $v \in W$, and calculate $A_v \cdot \sigma$. We have

$$A_v \cdot \sigma = (\varepsilon \otimes id)\, \Delta_X\, (A_v \cdot \sigma)$$

$$= (\varepsilon\, A_v \otimes id)\, \Delta_X(\sigma) \qquad \text{(see Lecture 5. Cor 1)}$$

$$= \sum_{\omega \in W} \varepsilon\left(A_v \cdot \epsilon(\omega)\sigma_B^{(\omega^{-1})}\right) \otimes \phi_\omega$$

$$= \varepsilon\left(A_v \cdot \epsilon(v)\sigma_B^{(v^{-1})}\right) \phi_v$$

$$= \phi_v.$$

This finishes the proof of 5).

<u>Remark</u>: What is quoted as Cor 1 in Lecture 5 is the fact that the action of $\underline{A}$ on $H^T(X)$ is obtained by the comodule map

$$\Delta_X: \; H^T(X) \longrightarrow H^T(K/T) \otimes_S H^T(X)$$

by ~~$\Delta_X(\sigma) =$~~ $a \cdot \sigma = (a, \sigma^{(1)})\, \sigma^{(2)}$ if $\Delta\sigma = \sigma^{(1)} \otimes \sigma^{(2)}$ and $(a, z) = \varepsilon(a \cdot z)$ is the pairing between $H^T(K/T)$ & $\underline{A}$. This is just like in the Hopf algebra case.

e now prove 3). This is just a special case of 4) for $X = K/T$. Indeed,

f 4), we get

$$\Delta \sigma_B^{(\omega)} = \sum_{u_1 \in W} \epsilon(u_1) \sigma_B^{(u_1^{-1})} \otimes A_{u_1} \cdot \sigma_B^{(\omega)}$$

$$A_{u_1} \cdot \sigma_B^{(\omega)} = \begin{cases} \epsilon(u_1)\, \sigma_B^{(u_1 \omega)} & \text{if } \ell(u_1^{-1}) + \ell(u_1\omega) = \ell(\omega) \\ 0 & \text{otherwise} \end{cases}$$

$$\Delta \sigma_B^{(\omega)} = \sum_{\substack{u_1 \in W \\ \omega = u_1^{-1} \cdot (u_1\omega) \text{ (red)}}} \epsilon(u_1) \sigma_B^{(u_1^{-1})} \otimes \epsilon(u_1) \sigma_B^{(u_1 \omega)}$$

$$= \sum_{\substack{u = u_1^{-1} \in W \\ v = u_1 \omega \in W \\ \omega = uv \text{ (red)}}} \epsilon(\cancel{u}) \sigma_B^{(u)} \otimes \sigma_B^{(v)}$$

s finishes the proof of 3).

) is clear from definition since $\mathcal{E} = \sigma_{(id)}^{B}$.

t remains to prove 1) and 4).

To prove 1), we need the following Lemma.

**Lemma**: For any $\sigma \in H^T(K/T)$,

$$\sigma = \sum_{\omega \in W} \pi_R\big(\mathcal{E}(A_\omega \cdot \sigma)\big)\, \epsilon(\omega)\, \sigma_B^{(\omega^{-1})}$$

**Proof** Write

$$\sigma = \sum_{\omega \in W} \pi_R(S_\omega)\ \epsilon(\omega)\, \sigma_B^{(\omega^{-1})}$$

for some $S_\omega \in S$ for each $\omega \in W$. ~~Us~~

Using $\quad \mathcal{E} \circ \pi_R = id_S$

and $\quad (A_v,\ \epsilon(\omega)\, \sigma_B^{(\omega^{-1})}) \quad \big(= \mathcal{E}(A_v \cdot \epsilon(\omega) \sigma_B^{(\omega^{-1})})\big) = \delta_{v,\omega}$

we get

$$\underset{\substack{\| \\ (A_v,\, \sigma)}}{\mathcal{E}(A_v \cdot \sigma)} = \sum_{\omega \in W} \mathcal{E}\pi_R(S_\omega)\ \big(A_v,\ \epsilon(\omega) \sigma_B^{(\omega^{-1})}\big) = \mathcal{E}\pi_R(S_v)$$
$$= S_v$$

$$\Rightarrow \quad \sigma = \sum_{\omega \in W} \pi_R\big(\mathcal{E}(A_v \cdot \sigma)\big)\ \epsilon(\omega)\, \sigma_B^{(\omega^{-1})}.$$

This proves the Lemma.

Rk: In proving the Lemma, we used the fact that $S$-valued the pairing $(\ \ )$ between $\underline{A}$ and $H^T(K/T)$ defined by

$$(a, \sigma) = \mathcal{E}(a\cdot\sigma)$$

satisfies
$$(\pi_R(s)\, a, \sigma) = \mathcal{E}(\pi_R(s))\,(a,\sigma) = s\,(a,\sigma)$$

and
$$\mathcal{E}(\pi_R(s)) = s \qquad \forall\, s \in S.$$

It says that $\mathcal{E}: H^T(K/T) \to S$ is not only an $S$-map for the first $S$-module structure on $H^T(K/T)$, (defined by $\pi_L$) but also for the 2nd $S$-module structure ~~de~~ on $H^T(K/T)$ defined ~~by H~~ ~~for at~~ by $\pi_R$.

Is this really true? Recall that $\pi_R: S \to H^T(K/T)$ is the pullback of the map

$$(E_u \times K)/(T\times T) \longrightarrow E/T$$
$$[e,k] \longmapsto ek.$$

It is not clear why $\mathcal{E}: H^T(K/T) \to S$ is a $\pi_R(s)$-linear. //

Now we prove 1): By Lemma

$$\pi_L(\lambda) = \sum_{w\in W} \pi_R\left(\mathcal{E}(A_w\cdot \pi_L(\lambda))\right)\, \epsilon(w)\, \sigma_B^{(w^{-1})}$$

But
$$A_w\cdot \pi_L(\lambda) = \pi_L(A_w\cdot\lambda) = \begin{cases} \pi_L(\lambda) & w = id \\ \langle \lambda, \check{\alpha}_i\rangle & w = r_i \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \pi_L(\lambda) = \pi_R\left(\mathcal{E}(\pi_L(\lambda))\right) + \sum_{i\in I} \pi_R\left(\mathcal{E}(\langle\lambda,\check{\alpha}_i\rangle)\right)(-1)\,\sigma_B^{(r_i)}$$
$$= \pi_R(\lambda) - \sum_{i\in I} \langle \lambda, \check{\alpha}_i\rangle\, \sigma_B^{(r_i)}$$

$$\Rightarrow \quad \pi_R(\lambda) = \pi_L(\lambda) + \sum_{i\in I} \langle\lambda,\check{\alpha}_i\rangle\, \sigma_B^{(r_i)}$$

<u>Remark</u>: This is an interesting formula. Understand what this says for Kostant's Harmonic form ~~of~~ $s^{r_i}$ later.

It remains to prove 4), i.e.

$$c(\sigma_B^{(w)}) = \epsilon(w)\, \sigma_B^{(w^{-1})}$$

The following is the proof given by Peterson. It is kind of stra

: first prove that

$$c(\sigma_A^{(\omega)}) = \pm \sigma_B^{(\omega^{-1})}$$

& determine the sign later. For $\omega \in W$, let

$$E_\omega^{(2)} = \{ (e, ek) : e \in E_\omega,\ k \in K_\omega \}$$

en

$$H^*(E_\omega^{(2)}/T \times T) \simeq H^T(X_\omega^B)$$

Why? This is saying that we do not distinguish $X_\omega^B$ & its Bott-Samelson resolution ?)

Recall that $t : E^{(2)} \to E^{(2)} : (e_1, e_2) \mapsto (e_2, e_1)$

So $t(E_\omega^{(2)}) = E^{(2)}_{\omega^{-1}}$

& $R_\omega = \{ \sigma \in H^T(E^{(2)}/T \times T) : \deg \sigma = 2\ell(\omega)$ and

$\sigma|_{E_v^{(2)}/T\times T} = 0$ for $v \in W$ s.t. $v \neq \omega \}$

e know $R_\omega = \mathbb{Z}\, \sigma_B^{(\omega)}$

$C(R_\omega) = R_{\omega^{-1}}$

$\Rightarrow \quad c(\sigma_B^{(\omega)}) = \pm \sigma_B^{(\omega^{-1})}$.



Now show that $c(\sigma_B^{(\omega)}) = \epsilon(\omega)\, \sigma_B^{(\omega^{-1})}$.

$\omega = id$ OK.

$\omega = r_i$ OK.

For $\ell(\omega) \geq 2$, assume sign $= \epsilon(v)$ for $\ell(v) < \ell(\omega)$.

Since $(C \otimes C) \cdot T \circ \Delta = \Delta \circ C$

where $T(\sigma \otimes \sigma') = \sigma' \otimes \sigma$

we get, from

$$\Delta(\sigma_B^{(\omega)}) = \sum_{\omega = uv,\ l(\omega)} \sigma_B^{(u)} \otimes \sigma_B^{(v)}$$

that

$$\Delta(c\sigma_B^{(\omega)}) = \sum_{\omega = uv,\ l(\omega)} c(\sigma_B^{(v)}) \otimes c(\sigma_B^{(u)})$$

$$= c(\sigma_B^{(\omega)}) \otimes 1 + 1 \otimes c(\sigma_B^{(\omega)})$$

$$+ \sum_{\substack{\omega = uv,\ l(\omega) \\ u \neq 1 \\ v \neq 1}} \epsilon(u)\epsilon(v)\, \sigma_B^{(v^{-1})} \otimes \sigma_B^{(u^{-1})}$$

But $\Delta(\epsilon(\omega)\sigma_B^{(\omega^{-1})}) = \epsilon(\omega)\sigma_B^{(\omega^{-1})} \otimes 1 + 1 \otimes \epsilon(\omega)\sigma_B^{(\omega^{-1})}$ + same sum

$\Rightarrow$ must have $\Delta(\sigma_B^{(\omega)}) = \epsilon(\omega)\, \sigma_B^{(\omega^{-1})}$.

This proves 4).

This completes the proof of the theorem //

rable $\underline{A}$-modules ($\Leftrightarrow$ actions of $\mathcal{U} = \mathrm{Spec}\, H^T(K/T)$)

tion: Let $X$ be an affine scheme over $\underline{h} = \mathrm{Spec}\, S$ with structure homomorphism ~~$\mathcal{O}(x) \Rightarrow$~~ $\pi_X : S \to \mathcal{O}(X)$.
An $\underline{A}$-module structure on $\mathcal{O}(X)$ is said to be **integrable** if for all $s \in S$ and $p \in \mathcal{O}(X)$.

i) $s \cdot p = \pi_X(s)\, p$

ii) $\pi_X : S \to \mathcal{O}(X)$ and $m : \mathcal{O}(X) \otimes_S \mathcal{O}(X) \to \mathcal{O}(X)$

are both $\underline{A}$-module maps

iii) For each $p \in \mathcal{O}(X)$, $A_w \cdot p = 0$ for all but finitely many $w \in W$.

ple
$\mathcal{U}$ as a scheme over $\underline{h} = \mathrm{Spec}\, S$ with structure homomorphism $\pi_L$ (?) Is this an example?
Maybe not, because in $H^T(K/T) \otimes_S H^T(K/T)$ we use $\pi_R$ to defined the the $S$-mod. str. on the first copy of $H^T(K/T)$)

OK. Because in the multiplication map $H^T(K/T) \otimes_S H^T(K/T)$, even the $S$-str. on the first copy is defined by $\pi_L$.

Integrable $\underline{A}$-module str on $\mathcal{O}(X)$

$\Updownarrow$

action $\phi : \mathcal{U} \times_{\underline{h}} X \to X$.

One way:
If $\phi : \mathcal{U} \times_{\underline{h}} X \to X$ is an action, have

$$\phi^* : \mathcal{O}(X) \longrightarrow H^T(K/T) \otimes \mathcal{O}(X)$$

Then for $a \in \underline{A}$, define $p \in P$

$$a \cdot p = m \cdot (\pi_X(\epsilon(a \cdot p_{(1)})) \otimes \cancel{id) \phi^*(p)}.\, p_{(2)}) \quad \text{if } \phi^* p = p_{(1)} \otimes p_{(2)}$$

The other way, given $\underline{A}$-action on $\mathcal{O}(X)$, define

$$\phi^*(p) = \sum_{w \in W} c(\sigma^{(w)}_{C/B}) \otimes (A_w \cdot p)$$

This is the dual map given the action

$$\phi : \mathcal{U} \times_{\underline{h}} X \longrightarrow X.$$

Next, we look at the 2nd action of $\underline{A}$ on $H^T(K/T)$.

<u>Notation</u>: The action of $\underline{A}$ on $H^T(K/T)$ that we have been talking about all way along will from now on be denoted by $a_L$. The 2nd action that we will introduce now will be denoted by $a_R$.

## ...e second action of $\underline{A}$ on $H^T(K/T)$

...ine a second action of $\underline{A}$ on $H^T(K/T)$ by

$$a_R \cdot \; = \; C \circ (a_L \cdot) \circ C$$

...perties

1) $a_L \circ b_R = b_R \circ a_L \qquad \forall\, a, b \in \underline{A}$

2) $\Delta \circ a_L = (a_L \otimes id) \circ \Delta$

$\Delta \circ b_R = (id \otimes b_R) \circ \Delta$

3) for $s \in S$, $a \in \underline{A}$ and $z \in H^T(K/T)$

$S_L \cdot z = \pi_L(s) z$

$S_R \cdot z = \pi_R(s) z$

- $a_L \cdot \pi_L(s) = \pi_L(a \cdot s)$

$a_R \cdot \pi_R(s) = \pi_R(a \cdot s)$

$a \cdot \mathcal{E}(z) = \sum \mathcal{E}(a_{(1)L}\, a_{(2)R} \cdot z) \qquad$ if $\Delta a = a_{(1)} \otimes a_{(2)}$

$(\Rightarrow) \quad \omega \cdot \mathcal{E}(z) = \mathcal{E}(\omega_L\, \omega_R \cdot z)$



Thus, in the basis $\{\sigma_B^{(\omega)} : \omega \in W\}$,

- $$A_{vR} \cdot \sigma_B^{(\omega)} = \begin{cases} \sigma_B^{(\omega v^{-1})} & \text{if } \ell(\omega v^{-1}) + \ell(v) = \ell(\omega) \\ 0 & \text{otherwise} \end{cases}$$

- Any $z \in H^T(K/T)$ can be written as

$$z = \sum_{\omega \in W} \Big( \pi_L\big(\mathcal{E}(A_{\omega R} \cdot z)\big) \Big)\, \sigma_B^{(\omega)}$$

- Any $a \in \widehat{A}$ can be written as

$$a = \sum_{\omega \in W} \mathcal{E}\big(a_R \cdot \sigma_{G/B}^{(\omega)}\big)\, A_\omega$$

- $\forall\, \omega \in W$,

$$\mathcal{E} \circ A_{\omega R} = \sigma_{(\omega)}^{B}$$

$$\mathcal{E} \circ \omega_R = \psi_\omega^{B}$$

(Recall: $\mathcal{E} \circ A_{\omega L} = \epsilon(\omega)\sigma_{(\omega^{-1})}^{B}$ see Page 6-8 ).

End of Lecture 6

lecture 7 March 4, 1997

formulas from last time:

$$A_{vR}\cdot\sigma_B^{(\omega)} = \begin{cases}\sigma^{(\omega v^{-1})} & \text{if } \ell(\omega v^{-1})+\ell(v)=\ell(\omega)\\ 0 & \text{otherwise}\end{cases}$$

for any $a\in\underline{\hat{A}}$

$$a = \sum_{\omega\in W}\varepsilon(a_R\cdot\sigma_B^{(\omega)})\,A_\omega$$

$z\in H^T(K/T)$

$$z = \sum_{u\in W}\pi\big(\varepsilon(A_{uR}\cdot z)\big)\,\sigma_B^{(u)}$$

$$\varepsilon\circ A_{\omega R} = \sigma^B_{(\omega)}$$

$$\varepsilon\circ \omega_R = \psi^B_\omega$$

Given $\omega\in W$, $\exists\ d_{u,\omega}\in S\theta$ of degree $\ell(\omega)$ for each $u\le\omega$ s.t.

$$\omega = \sum_{u\le\omega} d_{u,\omega}A_u$$

Moreover $$d_{\omega\omega} = \prod_{\substack{\alpha\in\Delta_+^{red}\\ \omega^{-1}\alpha<0}}(-\alpha) = \epsilon(\omega)\prod_{\substack{\alpha\in\Delta_+^{red}\\ \omega^{-1}\alpha<0}}\alpha$$



Proof: Induction on $\ell(\omega)$:

$\ell(\omega)=0 \quad \omega=id. \quad id = id.$

$\ell(\omega)=1 \quad \omega=r_i \quad r_i = 1-\alpha_i A_i \quad$ OK.

Assume $\omega = r_i\omega_1 > \omega_1$, Assume

$$\omega_1 = \sum_{u\le\omega_1} d_{u,\omega_1}A_u \qquad d_{u,\omega_1}\in S^{\ell(\omega)}(\mathfrak{h}^*)$$

Then

$$\omega = r_i\omega_1 = (1-\alpha_i A_i)\sum_{u\le\omega_1} d_{u,\omega_1}A_u$$

$$= \sum_{u\le\omega_1} d_{u,\omega_1}A_u - \sum_{u\le\omega_1}\alpha_i\,(A_i\, d_{u,\omega_1})\,A_u$$

Since $$A_i\, d_{u,\omega_1} = (r_i\cdot d_{u,\omega_1})\,A_i + A_i\cdot d_{u,\omega_1}$$

$$\Rightarrow\quad \omega = \sum_{u\le\omega_1} d_{u,\omega_1}A_u - \sum_{u\le\omega_1}\alpha_i\,(r_i\cdot d_{u,\omega_1})\,A_iA_u + \alpha_i\,(A_i\cdot d_{u,\omega_1}$$

$$= \sum_{u\le\omega_1}(d_{u,\omega_1} - \alpha_i A_i\cdot d_{u,\omega_1})A_u - \sum_{u\le\omega_1}\alpha_i(r_i\cdot d_{u,\omega_1})A_iA_u$$

$$= \sum_{u\le\omega_1}(r_i\cdot d_{u,\omega_1})\,A_u - \sum_{\substack{u\le\omega_1\\ r_iu>u}}\alpha_i\,(r_i\cdot d_{u,\omega_1})\,A_{r_iu}$$

$d_{u,w} = r_i \cdot d_{u,w_1}$ if $u \le w_1$

$d_{r_iu,w} = -\alpha_i (r_i \cdot d_{u,w_1})$ if $u \le w_1$, $r_i u > u$

shows that $d_{u,w} \in S^{\ell(w)}(\mathfrak{h}_{\mathbb{R}}^*)$ for any $u \le w$.

eover,

$$d_{r_iw_1,w} = -\alpha_i (r_i \cdot d_{w_1,w_1})$$

ume $\quad d_{w_1,w_1} = \epsilon(w_1) \prod_{\substack{\alpha \in \Delta_+^{re} \\ w_1^{-1}\alpha < 0}} \alpha$

n

$$d_{w,w} = -\alpha_i (r_i \cdot d_{w_1,w_1}) = \epsilon(w) \prod_{\substack{\alpha \in \Delta_+^{re} \\ w^{-1}\alpha < 0}} \alpha$$

//

k : Sara Billey's formula gives an express for each $d_{u,w}$. Will come back to this later.

//



Corollary

1) $\psi_w^B = \sum_{u \le w} d_{u,w}\, \sigma_{(w)}^B$

2) $\bigcap_{w \in W} \ker \psi_w^B = 0$.

3) $H^T(K/T)$ is reduced, i.e. the only nilpotent element

4) $H^T(G/P) \leftarrow (H^T(K/T))^{(W_P)_R}$ is also reduced.

Proof 1) follows from

$$\mathcal{E} \circ A_{wR} = \sigma_{(w)}^B$$

$$\mathcal{E} \circ W_R = \psi_{(w)}^B$$

2) If $z \in \bigcap_{w \in W} \ker \psi_w^B$, then $\psi_w^B(z) = 0 \quad \forall w$.

Since the matrix $D = (d_{u,w})$ is b upper-triangular

it is invertible $\Rightarrow \sigma_{(w)}^B(z) = 0$

But $\{\sigma_{(w)}^B\}$ is a basis for $\mathrm{Hom}_S(H^T(K/T), S)$

$\Rightarrow z = 0$.

If $z \in H^T(K/T)$ is s.t. $z^m = 0$ for some $m \geq 1$.

then for each $w \in W$

$$\mathcal{E}(W_R \cdot z^m) = 0$$

But

$$W_R \cdot z^m = (W_R \cdot z)^m$$

$$\Rightarrow \quad \mathcal{E}((W_R \cdot z)^m) = 0$$

$$(\mathcal{E}(W_R \cdot z))^m = 0$$

$$\Rightarrow \quad \mathcal{E}(W_R \cdot z) = 0$$

ie. $\quad z \in \ker \psi^0_w \quad \forall w$

$$\Rightarrow \quad z = 0.$$

Clear.

//



**Proposition** The action $a_R \cdot$ of $\underline{A}$ on $H^T(K/T)$ descends to an action on $H^\cdot(K/T)$ via the map

$$\mathbb{Z} \otimes_S H^T(K/T) \longrightarrow H^\cdot(K/T).$$

where the $S$-module structure on $H^T(K/T)$ is defined by $\pi_L$

**Proof:** This is because the $S$ action defined by $\pi_L$ commute with $a_R \cdot$ for any $a \in \underline{A}$.

//

**Remark.** The induced action of $\overset{A \cup G}{\underline{A}}$ on $H^\cdot(K/T)$ is by the BGG-operators.

ue Constants" for the multiplication on $H^T(K/T)$

r $u, v, \omega \in W$, define $a_\omega^{u,v} \in S$ by

$$\Delta A_\omega = \sum_{u,v\in W} a_\omega^{u,v} A_u \otimes A_v$$

($\Delta$ cocommutative $\Rightarrow a_\omega^{uv} = a_\omega^{vu}$)

$\underline{\text{tion}}$:

$$\sigma_B^{(u)} \sigma_B^{(v)} = \sum_{\omega\in W} \pi_L(a_\omega^{u,v}) \sigma_B^{(\omega)}$$

: We know that

$$\sigma_B^{(u)}\sigma_B^{(v)} = \sum_{\omega\in W} \pi_L(\varepsilon(A_{\omega R} \cdot \sigma_B^{(u)}\sigma_B^{(v)}))\ \sigma_B^{(\omega)}$$

$$A_{\omega R} \cdot (\sigma_B^{(u)}\sigma_B^{(v)}) = \sum_{u',v'\in W} \pi_R(a_\omega^{u',v'}) (A_{u'R} \cdot \sigma_B^{(u)}) (A_{v'R} \cdot \sigma_B^{(v)})$$

$$\begin{aligned}\varepsilon(A_{\omega R}\cdot(\sigma_B^{(u)}\sigma_B^{(v)})) &= \sum_{u',v'\in W} a_\omega^{u'v'} \varepsilon(A_{u'R}\cdot\sigma_B^{(u)})\, \varepsilon(A_{v'R}\cdot\sigma_B^{(v)}) \\ &= \sum_{u',v'\in W} a_\omega^{u'v'} (\sigma_{(u')}^B, \sigma_B^{(u)})(\sigma_{(v')}^B, \sigma_B^{(v)}) \\ &= \sum_{u',v'\in W} a_\omega^{u',v'} \delta_{u',u}\, \delta_{v',v} \\ &= a_\omega^{u,v}\end{aligned}$$



$$\Rightarrow \quad \sigma_B^{(u)}\sigma_B^{(v)} = \sum_{\omega\in W} \pi_L(a_\omega^{u,v})\, \sigma_B^{(\omega)}$$

//

Special properties of the $a_\omega^{u,v}$'s:

① $a_\omega^{u,v} = 0$ unless $u \le \omega$, $v \le \omega$

Proof: This is seen from the definition:

$$\begin{aligned}\Delta A_i &= \cancel{1\otimes A_i}\ A_i \otimes 1 + r_i \otimes A_i \\ &= A_i \otimes 1 + (1-\alpha_i A_i)\otimes A_i \\ &= 1\otimes A_i + A_i\otimes 1 - A_i \otimes \alpha_i A_i \\ &= 1\otimes A_i + A_i \otimes 1 - A_i \otimes \alpha_i A_i\end{aligned}$$

$$\begin{aligned}\Delta A_i A_j &= (1\otimes A_i + A_i\otimes 1 - A_i\otimes\alpha_i A_i)(1\otimes A_j + A_j\otimes 1 - A_j\otimes\alpha_j A_j) \\ &= 1\otimes A_iA_j + A_j\otimes A_i + A_i\otimes A_j + A_iA_j\otimes 1 \\ &\quad - A_i\otimes\alpha_i A_i A_j - A_iA_j\otimes\alpha_i A_i - A_iA_j\otimes\alpha_i A_i \alpha_j A_j \\ &\quad - A_j\otimes A_i\alpha_j A_j - A_iA_j\otimes\alpha_j A_j + A_iA_j \otimes \alpha_i A_i \alpha_j A_j\end{aligned}$$

so clear from induction on $\ell(\omega)$.

//

$a_w^{u,v}$ is a homogeneous polynomial of degree $\ell(u)+\ell(v)-\ell(w)$ in $S$

$$\deg \sigma_B^{(u)}\sigma_B^{(v)} = \deg(\pi a_w^{u,v}\sigma_B^{(w)})$$

$$2\ell(u)+2\ell(v) = 2(\deg a_w^{u,v} \text{ in } S) + 2\ell(w)$$

$$\Rightarrow \deg(a_w^{u,v} \text{ in } S) = \ell(u)+\ell(v)-\ell(w)$$

//

ition
For $w, v \in W$ $\quad v \le w$

$$d_{v,w} = a_w^{v,w}$$

where, recall $d_{v,w} \in S$ are defined by

$$w = \sum_{v\le w} d_{v,w} A_v$$

e

$$w = \sum_{v \le w} a_w^{v,w} A_v .$$



Proof: Write

$$w \otimes w = \sum_{u_1,u_2 \le w} S_w^{u_1,u_2} A_{u_1}\otimes A_{u_2}$$

$$\Rightarrow w\, \varepsilon(w_R \cdot \sigma_B^{(w)}) = \sum_{u_1,u_2 \le w} S_w^{u_1,u_2} A_{u_1}\, \varepsilon(A_{u_2 R}\cdot \sigma_B^{(w)})$$

$$= \sum_{u_1,u_2\le w} S_w^{u_1,u_2} A_{u_1}\, \delta_{u_2,w}$$

$$= \sum_{u_1 \le w} S_w^{u_1,w} A_{u_1}$$

But $\quad \varepsilon(w_R\cdot\sigma_B^{(w)}) = d_{w,w}$

$$\Rightarrow \quad d_{ww}\, w = \sum_{u_1\le w} S_w^{u_1,w} A_{u_1}$$

$$\Rightarrow \quad S_w^{u_1,w} = d_{w,w}\, d_{u_1,w}$$

On the other hand,

$$w = \sum d_{u,w} A_u$$

$$\Rightarrow \quad w\otimes w = \sum_{u_1,u_2}\left(\sum_v d_{v,w}\, a_v^{u_1,u_2}\right) A_{u_1}\otimes A_{u_2}$$

$$\Rightarrow S_w^{u_1,u_2} = \sum_v d_{v,w}\, a_v^{u_1,u_2}$$

$$\Rightarrow S_w^{u_1,w} = \sum_v d_{v,w}\, a_v^{u_1,w} = d_{w,w}\, a_w^{u_1,w}$$

By $S^{u_1,w} = d_{w,w}\, d_{u_1,w}$ & $d_{w,w} \neq 0$, get

$$d_{u_1,w} = a_w^{u_1,w}$$

//

Very stange proof).

osition
ion: For $w \in W$,

$$\sum_{w \in uv[\mathrm{id}]} \epsilon(u)\sigma_B^{(u^{-1})}\sigma_B^{(v)} = \delta_{w,\mathrm{id}} \qquad ①$$

$$\sum_{w \in uv[\mathrm{id}]} \sigma_B^{(u)}\, \epsilon(v)\sigma_B^{(v)} = \delta_{w,\mathrm{id}} \qquad ②$$

$$① \Leftrightarrow m \circ (c \otimes \mathrm{id}) \circ \Delta = \varepsilon$$

$$② \Leftrightarrow m \circ (\mathrm{id} \otimes c) \circ \Delta = \varepsilon$$

rk
This will also be true for quantum cohomology! //



Remark: Def Fix $e_0 \in E_u$. Define

$$i: K/T \longrightarrow E_u/T: \qquad kT \longmapsto e_0 kT$$

Then $\quad i \times i: \quad K/T \times K/T \longrightarrow E_u^{(2)}/T \times T$

Consequenty.

$$(i \times i)^*: \; H^T(K/T) \longrightarrow H^*(K/T) \otimes_{\mathbb{Z}} H^*(K/T)$$

We have

$$(i \times i)^* \sigma_B^{(w)} = \sum_{w = u v[\mathrm{id}]} \epsilon(u)\sigma_B^{(u^{-1})} \otimes \sigma_B^{(v)}$$

$$= \sum_{w \in uv[\mathrm{id}]} \epsilon(u)\, \sigma_B^{u^{-1}} \otimes \sigma_B^{v}.$$

## The Finite Case

Proposition In the finite case, we have

$$A_L = \mathrm{End}_{A_R}(H^T(K/T))$$

$$A_R = \mathrm{End}_{A_L}(H^T(K/T))$$

$H^T(K/T)$ is a free $A_L$ (as well as $A_R$) module with one generator $\sigma_B^{(\omega_0)}$, where $\omega_0$ is the longest element in $W$. If $\phi \in End_{A_L}(H^T(K/T))$ then $\exists$ $a \in A$ s.t.

$$\phi(\sigma_B^{(\omega_0)}) = a_R \cdot \sigma_B^{(\omega_0)}$$

Claim: $\forall z \in H^T(K/T)$,

$$\phi(z) = a_R \cdot z.$$

Proof: For any $z \in H^T(K/T)$, $\exists$ $b \in A$ st.

$$z = b_L \cdot \sigma_B^{(\omega_0)}$$

$$\begin{aligned} \Rightarrow \phi(z) &= \phi(b_L \cdot \sigma_B^{(\omega_0)}) \\ &= b_L \cdot \phi(\sigma_B^{(\omega_0)}) \qquad (\phi \in End_{A_L}) \\ &= b_L \cdot a_R \cdot \sigma_B^{(\omega_0)} \\ &= a_R \cdot b_L \cdot \sigma_B^{(\omega_0)} \\ &= a_R \cdot z. \end{aligned}$$

//.



## The space $H^T(K)$ with $K$ acting on $K$ by conjugations

Consider now $K$ as a $K$-space by conjugations. The map

$$p: K \to K/T$$

is $T$-equivariant (but not $K$-equivariant). Thus

$$p^*: H^T(K/T) \longrightarrow H^T(K)$$

is an $S$-module map:

$$p^*(\pi_L(s)\, z) = \pi(s)\, p^*(z)$$

where

$$\pi = [K \to pt]^*: S \longrightarrow H^T(K).$$

Now $A$ acts on both $H^T(K)$ & $H^T(K/T)$ by characteristic operat. But since $p$ is not a $K$-map, $p^*$ does not intertwine the $A$-actions on $H^T(K)$ & on $H^T(K/T)$. We have, nevertheless, the following.

ition. For $a \in \underline{A}$ with $\Delta a = a_{(1)} \otimes a_{(2)}$, and for $\forall\, z \in H^T(K/T)$

$$a \cdot p^*(z) = p^*(a_{(1)L}\, a_{(2)R} \cdot z)$$

n particular, for $s \in S$ and $w \in W$

$$\pi(s)\, p^*(z) = p^*(\pi_L(s) z) = p^*(\pi_R(s) z)$$

$$w \cdot p^*(z) = p^*(w_L\, w_R \cdot z)$$

$$A_w \cdot p^*(z) = p^*\Big(\sum_{\substack{u \in W \\ v \in W}} \pi_L(a_w^{uv})\, A_{uL}\, A_{vR} \cdot z\Big)$$



Proposition: For any $K$-space $X$ with action map

$$\mu_X : \quad K \times X \longrightarrow X$$

the pull back

$$\mu_X^{\#} : \quad H^T(X) \longrightarrow H^T(K \times X)$$

is the composition

$$H^T(X) \xrightarrow{\Delta_X} H^T(K/T) \otimes_S H^T(X) \xrightarrow{p^* \otimes id} H^T(K) \otimes_S H^T(X)$$

$$= H^T(K \times X).$$

//

The Pontryagin action of the ring $H_*(K)$:

$\mu_K : K \times K \longrightarrow K : \quad (k_1, k_2) \longmapsto k_1 k_2$

gives a map

$$\mu_{K*} : \quad H_*(K) \otimes H_*(K) \longrightarrow H_*(K).$$

This defines a ring structure on $H_*(K)$. Now for any $K$-sp

$X$ w/ $\quad \mu_X : \quad K \times X \longrightarrow X$

get

$$\mu_{X*} : \quad H_*(K) \otimes H_*(X) \longrightarrow H_*(X)$$

defines an action of $H_*(K)$ on $H_*(X)$.

at the special case $X = K/T$ w/

$$\mu_X = \mu_{K/T} : \quad K \times K/T \longrightarrow K/T$$

$\underline{A}_R$ acts on $H_*(K/T)$, and this action commutes with the Pontryagin action of $H_*(K)$ on $H_*(K/T)$

Define a ring structure on $H_*(K/T)$ by

$$\sigma_v \sigma_w = \begin{cases} \sigma_{vw} & \text{if } \ell(v)+\ell(w)=\ell(vw) \\ 0 & \end{cases}$$

then

$$\mu_{K/T\,*} (\underset{H_*(K)}{\sigma} \times \underset{H_*(K/T)}{\sigma'}) = p_*(\sigma)\,\sigma'$$

- Consequently,

$$p_* : H_*(K) \longrightarrow H_*(K/T)$$

a ring homomorphism.



Theorem (Peterson–Kac) Over any field $\mathbb{F}$.

1) $p^*(H^*(K/T), \mathbb{F})$ is a Hopf subalgebra of $H^*(K, \mathbb{F})$.

2) $p_*(H_*(K/T), \mathbb{F}) = H_*(K/T, \mathbb{F})^S$

$$= \{\sigma : \lambda \cap \sigma = 0 \quad \forall \lambda \in \hat{h}_\mathbb{Z}\}$$

3) If $m_{ij} = \infty$ for all $i \neq j$, then

$p^*(H^*(K/T), \mathbb{Q})$ = the dual of a tensor algebra as a Hopf algebra.

icare Duality in the finite case

efine $\Delta$-module homomorphism

$$PD: \quad H^T(G/P) \longrightarrow \mathrm{Hom}_S(H^T(G/P), S).$$

$$PD(z)(y) = \int_{[G/P]} yz \in S$$

he case $P = B$:

$$\int_{[G/B]} = \varepsilon \circ A_{\omega_0 R}$$

eneral,

$$\int_{[G/P]} \sigma_P^{(\omega)} = \delta_{\omega, \omega_0\omega_P}$$

e $\omega_P$ is the longest element in $W_P$, so $\omega_0\omega_P$ is the longest
nent in $W^P$.

all that (from Lecture 2)

$$\Delta A_{\omega_0} = \sum_{\omega \in W} A_\omega \otimes \omega_0 A_{\omega_0 \omega}$$



$$PD(\sigma_P^{(\omega)}) = \omega_0 \cdot \sigma^P_{(\omega_0\omega\omega_P)}$$

Also

$$\omega_{0L}\, \omega_{0R} \cdot \sigma_B^{(\omega)} = \epsilon(\omega)\, \sigma_B^{(\omega_0\omega\omega_0)}$$

It follows that $PD$ is an $S$-module isomorphism.

The Euler Class:

For $z \in H^T(G/P)$, ~~defn~~ consider the operator $M_z$ on $H^T(G/P)$ by $y \longmapsto zy$. The Euler Class $\chi_{G/P} \in H^T(G/P)$ is defined by the property:

$$\mathrm{trace}\, M_z = \int_{[G/P]} \chi_{G/P}\, z$$

Proposition:

$$\chi_{G/P} = \sum_{\omega \in W^P} \sigma_P^{(\omega)} \left( \omega_0 \cdot \sigma_P^{(\omega_0\omega\omega_P)} \right)$$

Proof: By the definition of trace and using the dual basis $\{\sigma^P_{(\omega)}\}$ of $\{\sigma_P^{(\omega)}\}$, we have

$$M_z = \sum_{w \in W^P} \left( \sigma^P_{(w)},\ z\,\sigma^{(w)}_P \right)$$

$$\sigma^P_{(w)} = PD\left( w_0 \cdot \sigma_P^{(w_0 w w_P)} \right)$$

$$\therefore M_z = \sum_{w \in W^P} \left( PD\left(w_0 \cdot \sigma_P^{(w_0 w w_P)}\right),\ z\,\sigma_P^{(w)} \right)$$

$$= \sum_{w \in W^P} \int_{[G/P]} z\,\sigma_P^{(w)} \left( w_0 \cdot \sigma_P^{(w_0 w w_P)} \right)$$

$$\chi_{G/P} = \sum_{w \in W^P} \sigma_P^{(w)} \left( w_0 \cdot \sigma_P^{(w_0 w w_P)} \right).$$

//

ill use $PD$ to denote its inverse as well.

$$\chi_{G/P} = \sum_{w \in W^P} \sigma_P^{(w)}\, PD(\sigma^P_{(w)})$$



<u>Lemma</u> For $v, w \in W^P$

$$\sigma_P^{(v)}\, PD(\sigma^P_{(w)}) = 0$$

unless $v \leq w$.

<u>Proof</u>

So $\chi_{G/P}$ is the trace of a rank 1 upper triangular mat

Also $\sigma_P^{(w)}\, PD(\sigma^P_{(w)}) = w \cdot PD(\sigma^P_{(1d)})$

<u>Facts</u> 1) $\chi_{G/P}$ has image $\prod_{\alpha > 0,\ w_0 w_P \alpha > 0} \alpha_R \cdot 1$ in $H^T(K/T)$

2) $\chi_{G/P}$ is $W$-invariant under the left action

3) Image of $\chi_{G/P}$ in $H^*(G/P)$ is $|W^P|\ \sigma_P^{w_0 w_P}$

: facts on the classifying spaces

$$\begin{tikzcd}
H^*(B_T) \arrow[r, hook, "\pi_n"] & H^T(G/B) & \\
H^*(B_T)^{W_P} \arrow[u, hook] \arrow[r, hook] & H^T(G/B)^{(W_P)_R} \cong & H^T(G/P) \arrow[ul]
\end{tikzcd}$$

-$\mathbb{Q}$, we have

$$H^*(B_T)^{W_P} \cong H^*(B_{K\cap P}) \cong H^T(G/P).$$

Fact

1) $H^*(B_{K\cap P}, \mathbb{Q}) = \left(\mathbb{Q}\otimes_{\mathbb{Z}} H^T(G/P)\right)^W$ (?)

2) $S \otimes_{H^*(B_K)} H^*(B_{K\cap P}) \cong H^T(G/P)$

$$\mathbb{Z} \otimes_S H^T(G/P) \cong H^*(G/P)$$



Open Problems

① In what sense does the diagonal map

$$K \to K\times K : \quad k \mapsto (k,k)$$

correspond to the co-product

$$\Delta : \underline{A} \to \underline{A} \otimes_S \underline{A}$$

( Given homomorphism $K_1 \to K_2$ with $T_1 \to T_2$, $N_1 \to N_2$.
can easily calculate

$$H^{T_2}(K_2/T_2) \to H^{T_1}(K_1/T_1) )$$

② Conjecture : For each $u, v, w \in W$, the $\epsilon(uvw)\, a_w^{u,v}$ is a polynomial in the $\alpha_i$'s $i \in I$ with $\mathbb{Z}_+$-coefficien

True for: ① $\ell(u)+\ell(v)=\ell(w)$ — (Kumar)

② $v=w$ or $u=w$ — Sara Billey

③ Similar models for K-theory (done?). Coborclism.

$H^T(G/P) \longrightarrow K^T(G/P)$.

BGG-operators $\longrightarrow$ Demazure operators

Find combinatorial intepretation of the coefficients of
$\epsilon(uvw)\, a_w^{u,v}$

Find combinatorial interpretation of the structure constants
of $H^{S^1}(Grass(k,n))$ with $S^1$ acting by $\exp(t\check{\rho})$.

Prove Little-Richardson Rule for $\sigma$ where $\sigma$ is a
diagram automorphism of f. dim. $G$ and $\sigma$ is
admissible, i.e. $\langle \alpha_{\sigma^k(i)}, \check{\alpha}_i \rangle \neq 0 \Rightarrow \sigma^k(i) = 1$.

( In this case $G^\sigma$ has the structure of a Kac-Moody gp.
$\lambda \in h_{\mathbb{Z}}^*$ $\sigma(\lambda) = \lambda$ $\lambda$ minuscule, $\alpha \in \Delta$,
$\Rightarrow 0 \le \langle \lambda, \check{\alpha} \rangle \le 1$
$\Rightarrow H^*(G/P_\lambda) \longrightarrow H^*(G^\sigma/(G^\sigma \cap P))$ ? )

tudy more of the Bruhat Graph

$(G/B)^T \longleftrightarrow W$

ertices : $W$
dges $w \to w r_\alpha$ $\alpha > 0$
$\sim$ $T$-stable curves ($\cong \mathbb{P}^1$) in $G/P$

Full subgraphs corresp.
to $X_w^P$: w/
vertices $v \le w$
$v \to v r_\alpha$ iff $v, v r_\alpha \le w$



Theorem ( Carrell- Peterson )
The Kazdan-Lusztig Polynomial $P_{vw} = 1$
$\Leftrightarrow$ for the graph, the have the same #
of edges emanate from each point.

Study directed Bruhat graphs:
$w \xrightarrow{\check{\alpha}} w r_\alpha$ if $w < w r_\alpha$.

End of Lecture 7

# Lecture 8. March 11, 1997 Tuesday

all picture for the next two lectures

:t $K$: compact simple Lie group

$\Omega K$: base preserving algebraic loops in $K$

en roughly $T \subset K$ acts on $\Omega K$ by conjugation:

$$(t \cdot k)(z) = t\, k(z)\, t^{-1}$$

ughly, the diagonal embedding

$$\Omega K \longrightarrow \Omega K \times \Omega K$$

ves a co-product

$$H_T(\Omega K) \longrightarrow H_T(\Omega K) \otimes_S H_T(\Omega K)$$

d the multiplication map for the group structure on $\Omega K$:

$$\Omega K \times \Omega K \longrightarrow \Omega K$$

ves a product

$$H_T(\Omega K) \otimes_S H_T(\Omega K) \longrightarrow H_T(\Omega K)$$

fact, $H_T(\Omega K)$ is a commutative & cocommutative Hopf lgebra over $S$. We will identify this Hopf algebra structure using $A_{af}$. In fact, we have a map

$$\Omega K \rightarrow G_{af}/B_{af}$$

which gives

$$H_T(\Omega K) \longrightarrow H_T(G_{af}/B_{af}) = A_{af}$$

Under this, we will identify

$$H_T(\Omega K) = Z_{af}(S) \qquad (\text{centralizer of } S \text{ in } A_{af})$$

and describe $Z_{af}(S)$ using the affine Weyl group $W_{af}$.

---

<u>otation</u>: For a variety $X$ over $\mathbb{C}$, use

$$\hat{X} = \mathrm{Mor}(\mathbb{C}^\times, X)$$

et $G$ be a finite-dimensional connected simple algebraic group over $\mathbb{C}$. We then have the finite root datum

$$I,\ \alpha_i,\ \alpha_i \in h_{\mathbb{Z}}^\vee \quad \alpha_i^\vee \in h_{\mathbb{Z}}.\ \Delta_+,\ \Pi,\ W,\ \underline{g},\ \underline{h},\ \underline{b} \ \cdots$$

et $\theta$ be the highest root. From these we form the following Kac-Moody root datum:

- $(h_{\mathbb{Z}})_{af} = h_{\mathbb{Z}} \qquad (h_{\mathbb{Z}}^\vee)_{af}^\vee = h_{\mathbb{Z}}^\vee$
- $I_{af} = I \cup \{0\}$
- $Q_{af} = \bigoplus_{i \in I_{af}} \mathbb{Z}\alpha_i = \mathbb{Z}\alpha_0 + Q = \mathbb{Z}\delta + Q \qquad \delta = \alpha_0 + \theta$

$$Q_{af} \longrightarrow (h_{\mathbb{Z}})_{af}: \quad \begin{array}{ll} \alpha_0 \longmapsto -\theta & \\ \alpha_i \longmapsto \alpha_i & i \neq 0 \quad i \in I \end{array}$$

$$\Pi_{af} = \Pi \cup \{-\theta\}$$

$$\Pi_{af}^\vee = \Pi^\vee \cup \{-\theta^\vee\}$$

$$\underset{\hat{Q}_{aff}}{\delta} = \alpha_0 + \theta \longmapsto 0 \quad \in (h_{\mathbb{Z}})_{af}^\vee = h_{\mathbb{Z}}^\vee$$

$$\text{ie. } \langle \delta, h \rangle = 0 \quad \forall\, h \in h_{\mathbb{Z}}.$$

Corresponding to this root datum, we have the following Kac-Moody group lie algebra $\underline{g}_{af}$:

$$\underline{g}_{af} = \underline{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] = \hat{\underline{g}}$$

$$e_i = e_i \otimes 1$$

$$f_i = f_i \otimes 1$$

$$e_0 = e_{-\theta} \otimes t$$

$$f_0 = e_\theta \otimes t^{-1} \qquad \Rightarrow [e_0, f_0] = [e_{-\theta} \otimes 1, e_\theta \otimes 1] = \cdots$$

<u>Roots</u> are in $Q_{af}$. They are all those in $Q_{af}$ of the form

$$\alpha + n\delta \qquad n \in \mathbb{Z},\ \alpha \in \Delta,\ \text{or } \alpha = 0$$

<u>The</u> <u>root</u> <u>spaces</u> are

$$(\underline{g}_{af})_{\alpha + n\delta} = \begin{cases} \underline{g}_\alpha \otimes t^n & \text{if } \alpha \in \Delta,\ n \in \mathbb{Z} \\ \underline{h} \otimes t^n & \alpha = 0 \quad n \in \mathbb{Z} \end{cases}$$

so $\quad \Delta^{re} = \{\alpha + n\delta : \alpha \in \Delta,\ n \in \mathbb{Z}\}$

and all $n\delta$'s $n \in \mathbb{Z}$, are "imaginary roots". They have multiplicity $= \dim_{\mathbb{C}} \underline{h}$.

he positive roots are

$$(\Delta_{af})_+ = \{\alpha + n\delta : n > 0 \text{ or } n \geq 0 \ \alpha \in \Delta_+\}$$

$$(\Delta_{af})_+^{re} = \{\alpha + n\delta : n \geq 0 \ \alpha \in \Delta_+\}$$

<u>affine Weyl group $W_{af}$</u>:

By definition,

$$W_{af} = W \ltimes \Gamma$$

the semi-direct product, where $\Gamma = Q^\vee$ with

$$Q^\vee \to \Gamma : h \longmapsto t_h.$$

$$w t_h w^{-1} = t_{w \cdot h}$$

$$t_h t_{h'} = t_{h+h'}$$

e reason why this is the same as the group generated by the reflections $r_0, r_i, i \in I$ is because

$$t_{\theta^\vee} = r_0 r_\theta$$

r $w \in W$,

$$w \cdot (\alpha + n\delta) = w \cdot \alpha + n\delta \qquad (\Rightarrow w\delta = \delta)$$

$$t_h \cdot (\alpha + n\delta) = \alpha + n\delta - \langle \alpha, h \rangle \delta$$

$$(\text{so} \quad t_h \cdot \alpha = \alpha - \langle \alpha, h \rangle \delta, \quad t_h \cdot (n\delta) = n\delta - \langle \alpha, h \rangle \delta.)$$

The Kac-Moody group:

$$G_{af} = \widehat{G} = \mathrm{Mor}(\mathbb{C}^\times, G) \qquad (\text{Laurent series in } t)$$

Set

$$P_0 = \mathrm{Mor}(\mathbb{C}, G) \qquad (\text{power series in } t)$$

$$B_{af} = \{ g \in \mathrm{Mor}(\mathbb{C}, G) : g(0) \in B \} \subset P_0$$

$$U_{af}^+ = \{ g \in \mathrm{Mor}(\mathbb{C}, G) : g(0) \in U^+ \}$$

$$K_{af} = \{ g \in G_{af} : g(S^1) \subset K \}$$

$$\Omega K = \{ k \in K_{af} : k(1) = id \}$$

$$T_{af} = T$$

$$G = \text{const. loops} \subset G_{af}$$

$K_{af}$ acts on $\Omega K$ by

$$k \cdot k' = k k' k(1)^{-1}$$

Then

$$i_\Omega : \Omega K \longrightarrow G_{af}/P_0 \qquad k \longmapsto k.* \qquad * = P_0$$

is a $K_{af}$-equivariant map. This map is also a home

because $\quad G_{af} = K_{af} B_{af} \qquad K_{af} \cap B_{af} = T$

$$= (\Omega K) K B_{af} = (\Omega K) P_0$$

compact involution on $G_{af}$ :

$$(\omega_{K_{af}})(g)(t) = \omega_K(g(\bar{t}^{-1})) \qquad g \in Mor(\mathbb{C}^\times, G) = G_{af}$$

re $\omega_K : G \to G$ is the compact involution on $G$ corresponding
K.

normalizer $N_{af}$ of $H_{af} = H$ in $G_{af}$ is

$$N_{af} = \widehat{N} = Mor(\mathbb{C}^\times, N)$$

re, recall, $N$ is the normalizer of $H$ in $G$, so also have

$$N_{af} = \text{semi-direct product of } N \text{ and } \Omega T$$

$$\Omega T = \{ g \in \Omega K : g(S^1) \subset T \}$$

$g \in \Omega T$ must be a homomorphism from $S^1$ to $T$.

s $\quad \Omega T \cong \Gamma = Q^\vee$

ere $\quad Q^\vee \xrightarrow{\sim} \Omega T : \quad h \longmapsto \hat{h} : \quad \hat{h}(z) = z^h \quad z \in \mathbb{C}^\times$

s way we have an also see

$$W \ltimes \Gamma \xrightarrow{\sim} W_{af} : (w, t_h) \mapsto (w \cdot \hat{h}^{-1} H) \in N_{af}/H.$$



The nil-Hecke rings $\underline{A}$ and $\underline{A}_{af}$.

Since $(\mathfrak{h}_\mathbb{R})_{af} = \mathfrak{h}_\mathbb{R}$, we have $S_{af} = S$.

let $\underline{A}$ be the nil-Hecke ring defined by $W$

let $\underline{A}_{af}$ " " " " " " " $W_{af}$

Then we have the embedding

$$\underline{A} \hookrightarrow \underline{A}_{af} : \quad \begin{array}{ll} s \longmapsto s & \\ A_i \longmapsto A_i & i \in I \ (i \neq 0) \end{array}$$

Recall that if $\beta = w\alpha_i \in \Delta^{re}$ with $i \in I$, then we define

$$A_{\beta^\vee} = w^* A_{\alpha_i} w^{-1} = w A_i w^{-1}$$

$$r_\beta = 1 - \beta A_{\beta^\vee}$$

(It is not obvious how to write $A_{\beta^\vee}$ in terms of the $A_i$'s.)

Define a ring homomorphism

$$ev : \underline{A}_{af} \to \underline{A} : \quad \begin{array}{l} ev|_S = id \\ ev(A_{\beta^\vee}) = A_{\bar{\beta}^\vee} \\ ev(wt_h) = w \end{array} \quad \text{where if } \beta = \alpha + n\delta,\ \bar{\beta} = \alpha$$

This is well-defined.

The embedding $\underline{A} \hookrightarrow \underline{A}_{af}$ is a section of $ev$.

Now identify

$$\Omega K \xrightarrow[i_\Omega]{\sim} G_{af}/P_0$$

we see that $\Omega K$ is a Kac-Moody $G/P$, so we have all we discussed before, namely:

- set $W_{af}^- = W_{af}^{P_0}$
- For each $x \in W_{af}^-$, have Schubert variety $X_x^\Omega$ and inclusion

$$i_x^\Omega : \quad X_x^\Omega \longrightarrow \Omega K$$

so have Schubert basis

$$\sigma_x^\Omega \in H_{2\ell(x)}(\Omega K)$$

$$\sigma_\Omega^x \in H^{2\ell(x)}(\Omega K)$$

$$\sigma_{(x)}^\Omega \in \mathrm{Hom}_S(H^T(\Omega K), S)$$

$$\sigma_\Omega^{(x)} \in H^T(\Omega K)$$

- Also for $x \in W_{af}$, have $\psi_x^\Omega \in \mathrm{Hom}_S(H^T(\Omega K), S)$. (It possible that $\psi_x^\Omega = \psi_y^\Omega$ for $x \neq y$)

- Have $\underline{A}_{af}$-module structures on $H^T(\Omega K)$ and $\mathrm{Hom}_S(H^T(\Omega K), S)$.

In the Schubert basis

$$A_x \cdot \sigma_{(y)}^\Omega = \begin{cases} \sigma_{(xy)}^\Omega & \text{if } xy \in W_{af}^- \quad \ell(xy) = \ell(x) + \ell(y) \\ 0 & \text{otherwise} \end{cases}$$

Define.

$$H_T(\Omega K) = \mathrm{Hom}_S(H^T(\Omega K), S) = S\text{-span of } \{\sigma_{(x)}^\Omega, \ x \in W_{af}^-\}$$
$$\subset \mathrm{Hom}_S(H^T(\Omega K), S)$$

In our special case at hand, not only do we have $G_{af}/B_{af} \to G_{af}/P_0$ but also: $\Omega K \hookrightarrow G_{af}/B_{af}$. Thus have

$$H_T(\Omega K) \longrightarrow \underline{A}_{af}.$$

Next time, write the images of $\sigma_{(x)}^\Omega$, for $x \in W_{af}^-$, in $\underline{A}_{af}$ under the above embedding and identify $H_T(\Omega K)$ as a subalgebra of $\underline{A}_{af}$.

End of Lecture 8

On $W_{af}^-$, next page.

## About $W_{af}^-$ and $W_{af}/W$

ll that $W_{af}^- = W_{af}^{P_0}$ is the set of minimal representatives the coset space $W_{af}/W_{P_0} = W_{af}/W$. $(W_{P_0} = W)$.

$$x \in W_{af}^- \iff x < xr_i \quad \forall\, i \in I \ (i \neq 0)$$

$$\iff x \cdot \alpha_i > 0 \quad \forall\, i \in I.$$

ite $x = \omega t_{-h}$. Then

$$x \cdot \alpha_i = \omega \cdot t_{-h} \cdot \alpha_i$$

$$= \omega \cdot (\alpha_i + \langle h, \alpha_i \rangle \delta)$$

$$= \omega \alpha_i + \langle h, \alpha_i \rangle \delta$$

$x \in W_{af}^- \iff \omega\alpha_i + \langle h, \alpha_i\rangle \delta > 0 \quad \forall\, i \in I$

$\iff \langle h, \alpha_i \rangle \geq 0$ and when $\langle h, \alpha_i \rangle = 0$ must have $\omega \alpha_i > 0$

$\iff$ $h$ is dominant and when $\langle h, \alpha_i \rangle = 0$ must have $\omega < \omega r_i$



Now for $h$ dominant, set

$$W_h = \text{the subgroup of } W \text{ generated by } \langle r_i : \langle h, \alpha_i \rangle = 0 \rangle$$

$$= \{ \omega \in W : \omega h = h \}$$

Set $P_h = B W_h B \supset B$ parabolic.

Then $W_h = W_{P_h}$. ~~Clearly~~, ~~As before~~, Let $W^h = W^{P_h}$ be the set of minimal representatives of the coset space $W/W_h$, ie.

$$\omega \in W^h \iff \omega < \omega r_i \quad \forall\, r_i \in W_h$$

so $\omega \in W^h \iff$ For each $i$ with $\langle h, \alpha_i \rangle = 0$ have $\omega < \omega r_i$.

Thus we have proved

$\underline{\underline{W_{af}^-}} = \{ \omega t_{-h} :$ $h$ dominant (ie. $\langle h, \alpha_i \rangle \geq 0$ $\forall$ i) and $\omega \in W^h \}$

$= \{ \omega t_{-h} :$ $h$ dominant and if $\langle h, \alpha_i \rangle = 0$ for $i \in I$ must have $\omega \alpha_i > 0 \}$

he map
$$W_{af} \longrightarrow W_{af}/W$$
$$wt_{-h} \longmapsto wt_{-h}/W$$
s of course a bijection.

Now another model for $W_{af}/W$ is $\Gamma = Q^\vee$:
$$\Gamma \xrightarrow{\sim} W_{af}/W$$
$$t_h \longmapsto t_h/W$$
n other words, for each coset $W_{af}/W$ has a unique
ranslation element $t_{-h}$ in it, namely
$$wt_{-h}/W = wt_{-h}w^{-1}/W = t_{-w\cdot h}/W$$

us:

① each coset $W_{af}/W$ has a unique minimal representive.

② each coset $W_{af}/W$ has a unique translation element

is a representative.

③ let $x \in W_{af}^-$. Then $x$ is the minimal representative
or the coset $xW$. We know that $x$ must be of the form
$=wt_{-h}$ where $h$ is dominant & $w \in W^h$. The translation
element in this coset is $t_{-w\cdot h}$, so $wt_{-h} \le t_{-w\cdot h}$



④ When $h$ is dominant and regular, we have
$$wt_{-h} \in W_{af}^-$$
for all $w \in W$. So for different $w_1, w_2 \in W$, the two ~~tet~~ elements $w_1t_{-h}$ & $w_2t_{-h}$ lie in two different coset in $W_{af}/W$.

⑤ A special case is when
$$x \in W_{af}^- \cap \Gamma$$
This is the case iff the minimal representative for $xW$, namely $x$ itself, coincides with the translational element representative of $xW$. Write ~~$x = wt$~~
$x = wt_{-h}$ where $h$ is dominant & $w \in W^h$.

Then $\quad x = t_{-w\cdot h} \iff wt_{-h} = t_{-w\cdot h} \iff w = 1$

So
$$W_{af}^- \cap \Gamma = \{ t_{-h} : h \text{ is dominant} \}$$

let's now calculate the length $\ell(t_{-h})$ when $h$ is dominant.

Recall that $\alpha+n\delta>0 \iff$ either $n>0$ or $n=0$ $\alpha>0$.

Now we need to see wh for $\alpha+n\delta>0$, when ~~can~~ do we have

$$t_{-h}\cdot(\alpha+n\delta)<0$$

Now $t_{-h}\cdot(\alpha+n\delta) = \alpha + (n+\langle h,\alpha\rangle)\delta$

( $n>0$ $\alpha<0$, then $t_{-h}\cdot(\alpha+n\delta)<0$ for $n=0,1,\cdots -\langle h,\alpha\rangle-1$

( $n>0$ $\alpha=0$ $t_{-h}\cdot(\alpha+n\delta)=n\delta>0$

$n>0$ $\alpha>0$ $t_{-h}\cdot(\alpha+n\delta)>0$

$n=0$ $\alpha>0$ $t_{-h}\cdot(\alpha+n\delta)>0$

the only case when $\alpha+n\delta>0$ and $t_{-h}\cdot(\alpha+n\delta)<0$

when $\alpha=-\beta<0$ (so $\beta>0$)

$$n=0,1,\cdots\langle h,\beta\rangle-1$$

number of such element is $\sum_{\beta>0}\langle h,\beta\rangle = \langle h,2\rho\rangle$

ence

$$\underline{\ell(t_{-h})} = \underline{\langle h,2\rho\rangle} = \underline{\sum_{\beta>0}\langle h,\beta\rangle}$$

for h dominant



Let's notice that

the sum of all $\{\alpha+n\delta>0 : t_{-h}(\alpha+n\delta)<0\}$

$$= \sum_{\beta>0}\left(-\beta - \beta+\delta + (-\beta+2\delta) + \cdots + (-\beta+(\langle h,\beta\rangle-1)\delta)\right)$$

$$= \sum_{\beta>0}\left(-\langle h,\beta\rangle\beta + \tfrac{1}{2}\langle h,\beta\rangle(\langle h,\beta\rangle-1)\,\delta\right)$$

⑦ For any ~~xxx~~ $x=\omega t_{-h}\in W_{af}^{-}$, $t = t_{-h_1}\in\Gamma^{-}=W_{af}^{-}\cap\Gamma$ we have

$$xt = \omega t_{-(h+h_1)}\in W_{af}^{-} \quad \text{and}$$

$$\ell(xt) = \ell(x)+\ell(t)$$

?

⑧ · Can prove that, for $x=\omega t_{-h}\in W_{af}^{-}$,

for $\alpha+n\delta>0$ be st. $x\cdot(\alpha+n\delta) = \omega\alpha+(n+\langle h,\alpha\rangle)\delta<0$

$\iff$ either $\alpha<0$ $\omega\alpha>0$ and $n=1,2,\cdots -\langle\alpha,h\rangle-1$

or $\alpha<0$ $\omega\alpha<0$ and $n=1,2,\cdots,-\langle\alpha,h\rangle$

~~Thus~~ In otherwords

$$\{\alpha+n\delta>0 : \omega t_{-h}\cdot(\alpha+n\delta)<0\} = \{-\beta+n\delta : \beta>0\ \omega\beta<0\ n=1,\cdots\langle\beta,h\rangle-1\}$$
$$\cup\{-\beta+n\delta : \beta>0\ \omega\beta>0\ n=1,2,\cdots\langle\beta,h\rangle\}$$

Consequently,

$$\ell(\omega t_{-h}) = \langle 2\rho,h\rangle - \ell(\omega)$$

## Lecture 9 March 12. 1997 Wednesday

Recall the $\mathcal{A}_{af}$ - action on $\mathrm{Hom}_S(H^T(\Omega K), S)$:

$$A_x \cdot \sigma^{\Omega}_{(y)} = \begin{cases} \sigma^{\Omega}_{(xy)} & \text{if } xy \in W^-_{af} \quad \ell(x)+\ell(y)=\ell(xy) \\ 0 & \text{otherwise} \end{cases}$$

$$w \cdot \psi_t = \psi_{wt} \qquad t, t' \in \Gamma \quad w \in W$$
$$t' \cdot \psi_t = \psi_{t't}$$

Define

$$H_T(\Omega K) = \sum_{x \in W_{af}} S\, \sigma^{\Omega}_{(x)}$$

as the S-span $\mathcal{A}_{af}$ -submodule of $\mathrm{Hom}_S(H^T(\Omega K), S)$ spanned over $S$ by $\{\sigma^{\Omega}_{(x)} : x \in W^-_{af}\}$. For $x \in W^-_{af}$, set

$$F_x = \sum_{\substack{y \in W^-_{af} \\ y \le x}} S\, \sigma^{\Omega}_{(y)}$$

Then

$$i^{\Omega}_x : X^{\Omega}_x \to \Omega K$$

gives

$$\mathrm{Hom}_S(H^T(X^{\Omega}_x), S) \xrightarrow{\sim} F_x$$
$$\mathrm{Hom}_S(F_x, S) \xrightarrow{\sim} H^T(X^{\Omega}_x).$$

## re on $F_x$

① $\{1 \otimes \psi_t : t \in \Gamma, \ t \leq x w_0\}$ is a free $S$-basis for $\mathrm{Frac}(S) \otimes_S F_x$

where $\mathrm{Frac}(S)$ = the fractional field of $S$

$x \in \Gamma_- \Leftrightarrow$ the minimal rep. of $xW$ coincides with ~~its~~ translational representative of $xW$.

② Set

$$\Gamma_- = \Gamma \cap W_{af}^- = \{ t_{-h} : h \in \underline{h}_{\mathbb{Z}} \text{ dominant} \}$$

see end of lecture 8 on $W_{af}^- \rightleftarrows W_{af}/W \simeq \Gamma$.

Then:

- $X_t^\Omega$ is $K$-stable, so $F_t$ is an $\underline{A}$-submodule of $H_T(\Omega K)$
- $\sigma_{(t)}^\Omega \in [H_T(\Omega K)]^{\underline{A}}$ i.e. $\sigma_{(t)}^\Omega$ is $\underline{A}$-invariant

**Proof** To show that $X_t^\Omega$ is $K$-stable, it is enough to show

~~$P_0 B$~~ $P_0 t P_0 \subset X_t^\Omega \Leftrightarrow t^{-1} B t \in P_0$

But for any $\alpha \in \Delta_+$

$$t_{-h} \cdot \alpha = \alpha + \langle h, \alpha \rangle \delta \in \Delta(P_0 / b_{af}) \quad (\text{i.e. a root for } P_0)$$

>nignment $\Rightarrow t^{-1} B t \in P_0 \Rightarrow X_t^\Omega$ is $K$-stable $\Rightarrow F_t$ is $\underline{A}$-submod. of $H_T(\Omega K)$

fa

Next, need to show that $\forall i \in I$, $A_i \cdot \sigma_{(t)}^\Omega = 0$

on's

But $A_i \cdot \sigma_{(t)}^\Omega = 0$ unless $r_i t \in W_{af}^-$ w. $\ell(r_i t) = \ell(t)+1$. So just need to show that

~~$r_i t \notin W_{af}^-$ for any $i \in I$~~ This is not possible. Suppose $r_i t \in W_{af}^-$ for some $i$.

Then $r_i$ must satisfy " $\langle h, \alpha_j \rangle = 0$ for some $j \in I \Rightarrow r_i \alpha_j > 0$ "

Since $r_i \alpha_i < 0$, must have ~~$\langle h, \alpha \rangle$~~ $\langle h, \alpha_i \rangle > 0$.

If $\ell(r_i t) = \ell(t)+1$, then $t < r_i t$ or $t^{-1} < t^{-1} r_i \Rightarrow t^{-1} \alpha_i > 0$

But $t^{-1} \cdot \alpha_i = t_h \cdot \alpha_i = \alpha_i - \langle h, \alpha_i \rangle \delta$. Since $\langle h, \alpha_i \rangle > 0 \Rightarrow t_h \cdot \alpha_i < 0$

Contradiction. Hence $A_i \cdot \sigma_{(t)}^\Omega = 0 \ \forall i \in I$ //

## Hopf algebra structure on $H_T(\Omega K)$

**Proposition**: $H_T(\Omega K)$ is a Hopf algebra over $S$, commutative and cocommutative.

**Proof** (outline) and structure maps.

- The $T$-equivariant multiplication map

$$m : \Omega K \times \Omega K \longrightarrow \Omega K$$

induces the ~~co~~ product map:

$$\mu : H_T(\Omega K) \otimes H_T(\Omega K) \longrightarrow H_T(\Omega K)$$

Since

$$m(X_x^\Omega \times X_t^\Omega) \subset X_{xt}^\Omega$$

we actually have

$$\mu : F_x \otimes F_t \longrightarrow F_{xt}$$

- The diagonal imbedding

$$\Omega K \longrightarrow \Omega K \times \Omega K$$

induces the co-product:

$$\Delta : H_T(\Omega K) \longrightarrow H_T(\Omega K) \otimes H_T(\Omega K)$$

clearly

$$\Delta F_x \subset F_x \otimes F_x$$

- co-commutativity is clear. As for commutativity of $\mu$, one can give a couple of reasons. One reason is that over $Frac(S)$, $F_X$ has basis $\{1\otimes\psi_t : t\in\Gamma \quad t\leq xw_0\}$ and $\psi_t\psi_{t'} = \psi_{t'}\psi_t = \psi_{t't}$.

Another reason is because $\Omega K$ is a double loop space so its (at least ordinary) homology is commutative.

- <u>unit</u>: $\psi_{id}$
- <u>antipode</u>: $c(F_t) = F_{\omega(t)}$ where $\omega$ is the diagram

automorphism defined by

$$\omega\cdot\alpha_i = -\alpha_{\omega(i)} \quad \text{as } i\neq 0 \qquad \omega(0)=0$$

$$(\Rightarrow \quad \omega(w) = w_0 w w_0 \quad \text{for } w\in W \quad \text{and} \quad \omega(t_h) = t_{-w_0 h} \;)$$

In terms of the $\psi_t$'s, the Hopf algebra structure is easier to express:

$$\varepsilon(\psi_t) = 1$$
$$c(\psi_t) = \psi_{t^{-1}}$$
$$\Delta\psi_t = \psi_t\otimes\psi_t$$
$$\psi_t\psi_{t'} = \psi_{tt'}$$
$$\psi_{id} = 1$$

In the following, we describe a model for $H_T(\Omega K)$.

<u>The map</u>

$$\underline{j}: \quad \underline{H_T(\Omega K)} \longrightarrow \underline{A_{af}}$$

First, we have the general fact that if $X$ is a $T$-space and

$$\phi: \quad \Omega K\times X \longrightarrow X$$

is a $T$-equivariant map (with $T$ acting on $\Omega K$ by conjugation and on $\Omega K\times X$ by the diagonal action), then each $\sigma\in H_T(\Omega K)\subset \mathrm{Hom}_S(H^T(\Omega K), S)$ defines the following composition map

$$H^T(X) \xrightarrow{\phi^*} H^T(\Omega K)\otimes_S H^T(X) \xrightarrow{c(\sigma)\otimes id} S\otimes_S H^T(X) = H^T(X).$$

If $\phi$ defines an action of $\Omega K$ on $X$, then these composition maps define an $H_T(\Omega K)$-module structures on $H^T(X)$.

Now assume that $X$ is a $K_{af}$-space. By restriction to $T$ and $\Omega K$, it is both a $T$-space and an $\Omega K$-space and the action map

$$\phi: \quad \Omega K\times X \to X$$

is $T$-equivariant. Thus each $\sigma\in H_T(\Omega K)$ ~~becomes~~ defines an operator on $H^T(X)$. This is functorial in $X$, so we get a characteristic operator. In other words, we have a map

$$j: \quad H_T(\Omega K) \longrightarrow \hat{A}_{af}.$$

A calculation shows that $j(\psi_e) = t$. Thus $j(\sigma)$ is compactly supported;(?) so ie $j(\sigma) \in \underline{A}_{af}$. It is obvious that $j$ is a ring homomorphism. Since $H_T(\Omega K)$ is commutative and since $j$ is an $S$-map,(?) we have

$$j(H_T(\Omega K)) \subset Z_{\underline{A}_{af}}(S), \text{ centralizer of } S \text{ in } \underline{A}_{af}$$

$(t \in W_{af} \subset \underline{A}_{af}$ commute w/ $S)$

Set

$$\underline{\underline{A}_\Omega} = \underline{Z_{\underline{A}_{af}}(S)}$$

It is a commutative $S$-algebra. Thus we have an $S$-algebra homomorphism

$$j : H_T(\Omega K) \longrightarrow \underline{A}_\Omega = Z_{\underline{A}_{af}}(S)$$

Will show that it is in fact an isomorphism.

<u>Connection between $j : H_T(\Omega K) \to \underline{A}_{af}$ and $j_\Omega : \Omega K \to G_{af}/B_{af} : k \mapsto kB_{af}$.</u>

Have commutative diagram

$$\begin{tikzcd} H_T(\Omega K) \arrow[r, "j"] \arrow[d, "\wr"] & \underline{A}_{af} \arrow[d] \\ \mathrm{Hom}_S(H^T(\Omega K), S) \arrow[r, "(j_\Omega)_*"'] & \mathrm{Hom}_S(H^T(G_{af}/B_{af}), S) \end{tikzcd} \qquad \begin{array}{c} a \\ \downarrow \\ \varepsilon \circ a_R = \varepsilon \circ c(a)_L \end{array}$$

Before we find $j(\sigma^n_{\ldots})$, we collect some facts about the action of $H_T(\Omega K)$ on $H^T(X)$ for a $K_{af}$-space $X$.

~~<u>Lemma 1.</u> For any $K_{af}$-space $X$, the action of $\underline{A}_{af}$ on $H^T(X)$ factors through $\underline{A}$ via the map~~ ( Is this right ? )

~~$$ev : \underline{A}_{af} \longrightarrow \underline{A}$$~~

~~where, recall,~~

~~$$ev|_S = id$$
$$ev|_{A_{\tilde{r}}} = A_{r^\vee}$$
$$ev|_{wt_\lambda} = w$$~~

//

<u>Lemma 2</u> For $\sigma \in H_T(\Omega K)$,

$$(id \otimes ev)\Delta \circ j(\sigma) = j(\sigma) \otimes 1 \qquad ?$$

<u>Proof.</u> This is roughly due to the fact that

$$\Omega K \hookrightarrow K_{af} : k \mapsto (k, 1).$$

//

Now for any $\underline{A}_{af}$-module $M$ and $\underline{A}$-module $N$, set $M *_S N = M \otimes_S ev^* N$, as an $\underline{A}_{af}$-module. Then by Lemma 2,

$$j(\sigma)\cdot(m\otimes n) = j(\sigma)\cdot m \otimes n$$

Apply this to the action map

$$F : H_T(\Omega K) \otimes_S H^T(X) \longrightarrow H^T(X)$$

<u>Proposition</u>. The above action map is an $\Delta_{af}$-module map

<u>Proof</u>: For $\sigma \in H_T(\Omega K)$ and $z \in H^T(X)$, we know by ~~the above~~ ~~discussion that~~, ~~for $\omega \in WX$~~

$$F(\sigma\otimes z) = j(\sigma)\cdot z \qquad ?$$

~~so for $\omega$~~ so for $\omega \in W$

$$\omega\cdot F(\sigma\otimes z) = \omega\cdot j(\sigma)\cdot z$$

In particular

$$\omega\cdot F(\psi_t\otimes z) = \omega\cdot t\cdot z = \omega t \omega^{-1}\cdot\omega\cdot z = (\omega\cdot t)\cdot(\omega\cdot z)$$

On the other hand //

$$F(\omega\cdot(\psi_t\otimes z)) = F(\omega\cdot\psi_t\otimes\omega\cdot z) = F(\omega\cdot(\psi_t\otimes z))$$

Also

$$t'\cdot F(\psi_t\otimes z) = t't\cdot z = F(\psi_{t't}\otimes z) = F(t'\cdot(\psi_t\otimes z))$$

//



<u>Proposition</u>: The multiplication map

$$H_T(\Omega K)\otimes_S H_T(\Omega K) \longrightarrow H_T(\Omega K)$$

is an $\Delta_{af}$-map

<u>Proof</u>: This is because

$$\sigma\sigma' = j(\sigma)\cdot\sigma'.$$

//

More generally, for any $\Delta_{af}$-module $M$, the map

$$\phi : H_T(\Omega K)\otimes_S M \longrightarrow M$$
$$\sigma\otimes m \longmapsto j(\sigma)\cdot m$$

is always an $\Delta_{af}$-module map.

//

$$H_T(\Omega K) = \Delta_{af}\cdot 1$$

$$H_T(\Omega K) \longrightarrow \mathrm{Hom}_S(ev^*M, M) \qquad ?$$

What is this ?

( Pages 9-7 & 9-8 need to be rewritten, reorganized.?)

now look at $j(\sigma^{\Lambda}_{(x)})$.

Introduce the ideal $I \subset \Delta_{af}$ : (left ideal)

$$I = \sum_{\substack{w \in W \\ w \neq id}} \Delta_{af} A_w$$

: is the ideal of annihilators of $1 \in H_T(\Omega K)$ for the action $\Delta_{af}$ on $H_T(\Omega K)$.

osition. For $x \in W_{af}^-$

$$j(\sigma^{\Lambda}_{(x)}) = A_x \mod I$$

f

$$j(\sigma^{\Lambda}_{(x)}) \cdot 1 = \sigma^{\Lambda}_{(x)} 1 = \sigma^{\Lambda}_{(x)} = A_x \cdot \sigma^{\Lambda}_{(id)} = A_x \cdot 1$$

$$\Rightarrow \quad j(\sigma^{\Lambda}_{(x)}) - A_x \in I$$

//

1: $A_{xw_0} = j(\sigma^{\Lambda}_{(x)}) A_{w_0}$ where $w_0$ = longest in $W$

of: $j(\sigma^{\Lambda}_{(x)}) A_{w_0} = (A_x + a) A_{w_0} = A_x A_{w_0} = A_{xw_0}$ $(a \in I)$ //

( $\ell(x) + \ell(w_0) = \ell(xw_0)$ holds for all $x \in W_{af}^-$ )

2: For any $x \in W_{af}^-$, $t \in \Gamma$.

$$\sigma^{\Lambda}_{(x)} \sigma^{\Lambda}_{(t)} = \sigma^{\Lambda}_{(xt)}$$

$$F_x F_t = F_{xt}$$

This is due to the following general fact.
For any parabolic $P$,
$\forall x \in W^P$ $y \in W_P$
$\ell(xy) = \ell(x) + \ell(y)$

Proof. Since $\sigma^{\Lambda}_{(t)} \in [H_T(\Omega K)]^{\Delta}$, have

$$\begin{aligned}
\sigma^{\Lambda}_{(x)} \sigma^{\Lambda}_{(t)} &= j(\sigma^{\Lambda}_{(x)}) \cdot \sigma^{\Lambda}_{(t)} \\
&= (A_x + a) \cdot \sigma^{\Lambda}_{(t)} && a \in I \\
&= A_x \cdot \sigma^{\Lambda}_{(t)} && (a \cdot \sigma^{\Lambda}_{(t)} = 0) \\
&= \sigma^{\Lambda}_{(xt)}.
\end{aligned}$$

( We are saying $\ell(x) + \ell(t) = \ell(xt)$ automatically ? )

//

**Proposition**: $H_T(\Omega K) \otimes_S \underline{A} \longrightarrow \Delta_{af}$, $\sigma \otimes a \longmapsto j(\sigma) a$

is an $\Delta_{af}$-module isomorphism, where $\Delta_{af}$ acts on $\underline{A}$ via $ev: \Delta_{af} \longrightarrow \underline{A}$

Proof :

or: $j: H_T(\Omega K) \xrightarrow{\sim} \underline{A}_\Omega$ is an isomorphism. //

thus we have a direct sum decomposition

$$\underline{A}_{af} = \underline{A}_\Omega + I$$

as an $\underline{A}_\Omega$ - module.

<u>Structures</u> <u>on</u> <u>$\underline{A}_\Omega$</u>

· First, by identifying

$$\underline{A}_\Omega \cong \underline{A}_{af}/I$$

we get an <u>$\underline{A}_{af}$ - module</u> structure on $\underline{A}_\Omega$, i.e., for $a \in \underline{A}_{af}$ and $a' \in \underline{A}_\Omega$. $a \cdot a' \in \underline{A}_\Omega$ is the unique element of $\underline{A}_\Omega$ st

$$a \cdot a' - aa' \in I$$

· By definition, $Z(\underline{A}_{af}) \subset \underline{A}_\Omega = Z_{\underline{A}_{af}}(S)$, and the action of $\underline{A}_{af}$ on $\underline{A}_\Omega$ is $Z(\underline{A}_{af})$ - linear

· For each $x \in W_{af}^-$. $j(\sigma_{(x)}^{\Omega})$ is the unique element in $\underline{A}_\Omega$ such that

$$j(\sigma_{(x)}^{\Omega}) \in A_x + I.$$

In other words, $j(\sigma_{(x)}^{\Omega}) = A_x \cdot 1$ for the action of $\underline{A}_{af}$ on $\underline{A}_\Omega$.

· We can calculate the action of $\underline{A}_{af}$ on $\underline{A}_\Omega$ as follows

<u>Proposition</u>: For $s \in S$, $a \in \underline{A}_\Omega$, $w \in W$, $t \in \Gamma$ and $\beta \in \Delta^{re}$

$$s \cdot a = sa = as$$

$$wt \cdot a = wtaw^{-1}$$

$$A_{\beta^\vee} \cdot a = A_{\beta^\vee} a - r_\beta \, a \, A_{\bar\beta^\vee} \qquad (\beta = \alpha + n\delta, \ \bar\beta = \alpha)$$

$$= A_{\beta^\vee} a \, r_{\bar\beta} + a A_{\bar\beta^\vee}$$

$$( \ A_{\alpha_0^\vee} \cdot \bar{a}_0^\vee = -\check{\theta} \ )$$

The proof of this proposition is not trivial. Need calculat

Introduce <u>Hopf</u> <u>algebra</u> (over $S$) <u>structure</u> <u>on</u> <u>$\underline{A}_\Omega$</u>:

$$\pi(s) = s$$
$$\varepsilon(t) = 1$$
$$c(t) = t^{-1}$$
$$\Delta(t) = t \otimes t$$

<u>Theorem</u>: The map $j: H_T(\Omega K) \longrightarrow \underline{A}_\Omega$ is an isomorphism of both $\underline{A}_{af}$-modules and Hopf algebra

End of Lect

Lecture 10 March 19, 1997 Wed

**$\Omega$ - integrable $\underline{A}_{af}$ - modules**

We first recall the definition of integrable $\underline{A}$ - modules where $\underline{A}$ is $\underline{A}_{af}$ or $\underline{A}_{finite}$, that was given at the end of lecture 6:

An integrable $\underline{A}$ - module is an $\underline{A}$ - module structure on $\mathcal{O}(X)$, where $X$ is an affine scheme over $\underline{h} = \operatorname{spec} S$ with structure homomorphism $\pi_X : S \longrightarrow \mathcal{O}(X)$ such that

(1) $s \cdot p = \pi_X(s)\, p \qquad \forall s \in S \quad p \in \mathcal{O}(X)$

(2) $\pi_X : S \longrightarrow \mathcal{O}(X)$ is an $\underline{A}$ - module map

(3) $m : \mathcal{O}(X) \otimes_S \mathcal{O}(X) \longrightarrow \mathcal{O}(X)$ is an $\underline{A}$ - module m

(4) For each $p \in \mathcal{O}(X)$, $A_w \cdot p = 0$ for all but finitely many $w \in W$.

Now back to our notation where $\underline{A}$ denotes the A nil - Hecke ring for the finite Weyl group $W$. Then condi (4) is not needed.

<u>Definition</u>: An $\Omega$-integrable $\underline{A}_{cf}$-module is by definition an an affine scheme $X$ over $\underline{h} = \operatorname{spec} S$, with structure homomorphism $\pi_X : S \to \mathcal{O}(X)$, and an $\underline{A}_{cf}$-module structure on $\mathcal{O}(X)$ such that

(1) $X$ is an integrable $\underline{A}$-module by restricting the action of $\underline{A}_{cf}$ to $\underline{A}$;

(2) $m : \mathcal{O}(X) *_S \mathcal{O}(X) \longrightarrow \mathcal{O}(X)$ is an $\underline{A}_{cf}$-map.

(part of the requirement for ① is in ② as well)

Question: Is (2) weaker than asking $m : \mathcal{O}(X) \otimes_S \mathcal{O}(X) \to \mathcal{O}(X)$ being an $\underline{A}_{cf}$-map? This seems to be just a different requirement. So the notion of $\Omega$-integrable $\underline{A}_{cf}$-module seems different from that of an integrable $\underline{A}_{cf}$-module.

Set $\quad \mathfrak{a} = \operatorname{Spec} H_T(\Omega K)$. Then $\mathfrak{a}$ is an integrable $\underline{A}$-module. ~~By~~ We know from Lecture 9 (page 9-9) that $m : H_T(\Omega K) *_S H_T(\Omega K) \to H_T(\Omega K)$ is an $\underline{A}_{cf}$-module map, so $\mathfrak{a}$ is an $\Omega$-integrable $\underline{A}_{cf}$-module.

<u>Proposition</u>: An $\Omega$-integrable $\underline{A}_{cf}$-module structure on $\mathcal{O}(X)$ is equivalent to

(1) an integrable $\underline{A}$-module structure $\mathcal{O}(X)$; and

(2) an $\underline{A}$-module map $g : H_T(\Omega K) \to \mathcal{O}(X)$.

More explicitly, given an $\underline{A}_{cf}$-module str. on $\mathcal{O}(X)$, by restriction to $\underline{A}$ we get an integrable $\underline{A}$-module str. on $\mathcal{O}(X)$, and the map

$$g : H_T(\Omega K) \longrightarrow \mathcal{O}(X) : \qquad g(\sigma) = j(\sigma)\cdot 1$$

Conversely, given (1) and (2), the $\underline{A}_{cf}$-module str. on $\mathcal{O}(X)$ is defined by

$$\big(j(\sigma)\, a\big)\cdot p = g(\sigma)\,(a\cdot p)$$

<u>Proof</u>: Assume that the $\underline{A}_{cf}$-module str. on $\mathcal{O}(X)$ is given. We need to show that the map $g$ is an $\underline{A}$-map, i.e., for $a \in \underline{A}$ and $\sigma \in H_T(\Omega K)$, need to show

$$g(a\cdot\sigma) = a\cdot g(\sigma)$$

Now
$$g(a\cdot\sigma) = j(a\cdot\sigma)\cdot 1$$
$$a\cdot g(\sigma) = a\cdot j(\sigma)\cdot 1 = (a\, j(\sigma))\cdot 1$$

Thus we need to show

$$( j(a\cdot\sigma) - a\,j(\sigma) )\cdot 1 = 0 \quad \in \mathcal{O}(X)$$

But we know that the action of $\Delta$ on $H_T(\Omega K)$ is characterized by the fact that

$$j(a\cdot\sigma) - a\,j(\sigma) \in I = \sum_{\substack{w\in W \\ w\neq id}} \Delta_{cf} A_w$$

Since for any $i \in I$,

$$A_i \cdot 1 = A_i \cdot \pi_x(1) = \pi_x(A_i \cdot 1) = 0 \quad \in \mathcal{O}(X)$$

we see that $b \cdot 1 = 0$ for any $b \in I$. Thus

$$( j(a\cdot\sigma) - a\,j(\sigma) )\cdot 1 = 0$$

or $j : H_T(\Omega K) \to \mathcal{O}(X)$ is an $\Delta$-map.

Conversely, assume that we are given an integrable $\Delta$-module structure on $\mathcal{O}(X)$ and an $\Delta$-map $j : H_T(\Omega K) \to \mathcal{O}(X)$. Define for $\sigma \in H_T(\Omega K)$ and $a \in \Delta$, $p \in \mathcal{O}(X)$

$$(j(\sigma)\,a)\cdot p = j(\sigma)(a\cdot p)$$

Need to show that this gives an $\Omega$-integrable $\Delta_{cf}$-mod. str. on $\mathcal{O}(X)$.

First need to show that this is indeed an action of $\Delta_{cf}$.

This must follow from the fact that

$$H_T(\Omega K) \otimes_S \Delta \longrightarrow \Delta_{cf}, \quad \sigma \otimes a \longmapsto j(\sigma)a$$

is an $\Delta_{cf}$-module map. (?) In order to show

$$m : \mathcal{O}(X) \otimes_S \mathcal{O}(X) \longrightarrow \mathcal{O}(X)$$

is an $\Delta_{cf}$-module map. only need to show

$$m\,(j(\sigma)\cdot(p_1 \otimes p_2)) = j(\sigma)\cdot(p_1 p_2)$$

But $\quad j(\sigma)\,(p_1 p_2) = j(\sigma)\, p_1 p_2$

and (Remark after Lemma 2 on Page 9-7)

$$m(\,j(\sigma)\cdot(p_1 \otimes p_2)) = m(\,j(\sigma)\cdot p_1 \otimes p_2) = m(j(\sigma)p_1 \otimes p_2)$$
$$= j(\sigma)\,p_1 p_2$$

So

$$m(\,j(\sigma)\cdot(p_1 \otimes p_2)) = j(\sigma)\cdot(p_1 p_2).$$

//

Need to fill in the proof of why $(j(\sigma)c)\cdot p \overset{\text{def}}{=} j(\sigma)(c\cdot p)$ define an $\Delta_{cf}$-action.

In more geometrical terms, let

$$\mathcal{U} = \operatorname{spec} H^T(K/T)$$

We said in Lecture 6 that an integrable $A$-module should be thought of as an action $\phi: \mathcal{U} \times_{\mathfrak{h}} X \to X$. In this language, an $\Omega$-integrable $A_{af}$-module str. on $\mathcal{O}(X) \iff$ pairs $(\phi, f)$ where $\phi$ is an action of $\mathcal{U}$ on $X$ and $f: X \to \mathfrak{a}$ is a $\mathcal{U}$-equivariant map.

<u>The polynomials $j_x^y$,</u> $x \in W_{af}^-$ $y \in W_{af}$

For $x \in W_{af}^-$, introduce $j_x^y \in S$, $y \in W_{af}$, by

$$\underline{j(\sigma_{(x)}^{\Omega})} = \underline{\sum_{y \in W_{af}} j_x^y A_y}$$

In terms of the map

$$j_\Omega : \Omega K \longrightarrow G_{af}/B_{af}$$

we have

$$\underline{j_\Omega^* \sigma_{G_{af}/B_{af}}^{(y)}} = \underline{\sum_{x \in W_{af}^-} j_x^y \sigma_\Omega^{(x)}}$$

<u>Immediate properties of the polynomial $j_x^y$'s:</u> $x \in W_{af}^-$, $y$

<u>Property 1</u>: $\deg j_x^y = 2(\ell(y) - \ell(x))$

Recall that for $x \in W_{af}^-$, then ~~represen. element~~ distinguished elem. the coset $x W_0$ ...

This is because

$$\deg(\sigma_{(x)}^{\Omega}) = -2\ell(x)$$
$$\deg A_y = -2\ell(y)$$

<u>Property 2</u>: $j_x^y = \delta_{xy}$ if $y \in W_{af}^-$

<u>Property 3</u>: $j_x^y = 0$ unless $y \le t \le x w_0$ for some $t \in \Gamma$

$$\Big( \Rightarrow \deg j_x^y = 2(\ell(y) - \ell(x)) \le 2(\ell(x w_0) - \ell(x))$$
$$= 2(\ell(x) + \ell(w_0) - \ell(x)) = 2\ell(w_0) \Big)$$

<u>Proof</u>: Since

$$j_\Omega(X_x^\Omega) \subset \pi_{p_0}^{-1}(X_x^{G_{af}/B_{af}}) = X_{x w_0}^{G_{af}/B_{af}}$$

and since $j_{\Omega *}(\psi_t) = \psi_t$ by definition, we have $j_\Omega^*(z) = 0$ in $H^T(X_x^\Omega)$ if $\psi_t(z) = 0$ for all $t \in \Gamma$ w/ ...

Property 3 now follows from this //

<u>osition</u>: For $x, z \in W_{af}^-$

$$\sigma_{(x)}^{\Omega}\,\sigma_{(z)}^{\Omega} = \sum_{\substack{y \in W_{af} \\ yz \in W_{af}^- \\ \ell(y)+\ell(z) = \ell(yz)}} j_x^y\, \sigma_{(yz)}^{\Omega}$$

<u>of</u>:

$$\sigma_{(x)}^{\Omega}\,\sigma_{(z)}^{\Omega} = j(\sigma_{(x)}^{\Omega}) \cdot \sigma_{(z)}^{\Omega}$$

$$= \sum_{y \in W_{af}} j_x^y\, A_y \cdot \sigma_{(z)}^{\Omega}$$

$$= \sum_{\substack{y \in W_{af} \\ yz \in W_{af}^- \\ \ell(y)+\ell(z) = \ell(yz)}} j_x^y\, \sigma_{(yz)}^{\sigma}$$

//

<u>ecture</u>: The $j_x^{y}$'s are polynomials in the $\alpha_i$'s with coefficients in $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$

<u>ark 1</u>. Can show $j_x^y \in \mathbb{Z}_+ \oplus$ when $\ell(y) = \ell(x)$ by making connection with quantum cohomology: these are the Gromov-Witten invariants.



<u>Remark 2</u>: We proved last time that $\forall\, x \in W_{af}^-$ and $t \in \Gamma^- = W_{af}^- \cap \Gamma$,

$$\sigma_{(x)}^{\Omega}\,\sigma_{(t)}^{\Omega} = \sigma_{(xt)}^{\Omega}.$$

On the other hand, since $H_T(\Omega K)$ is commutative, we hav

$$\sigma_{(x)}^{\Omega}\,\sigma_{(t)}^{\Omega} = \sigma_{(t)}^{\Omega}\,\sigma_{(x)}^{\Omega} = j(\sigma_{(t)}^{\Omega}) \cdot \sigma_{(x)}^{\Omega}$$

It follows that, for $h$ dominant

$$j(\sigma_{(t_{-\lambda})}^{\Omega}) = \sum_{w \in W} A_{t_{-w \cdot h}}$$

Since $\sigma_{(t_{-\lambda})}^{\Omega}$ is $\underline{A}$-invariant, we know that $j(\sigma_{(t_{-\lambda})}^{\Omega})$ is in the center of $A_{af}$.

## <u>An integral formula</u>

Define

$$ev_1 : K_{af}/T \longrightarrow K/T$$
$$ev_1(kT) = k(1)T$$

__roposition__ For $x, y \in W_{af}^{-}$ and $w \in W$.

$$\dot{j}_x^{y\omega(w^{-1})} = \langle \sigma^{(yw_0)}_{G_{af}/B_{af}} \; ev_1^*(w_{0L} \cdot \sigma^{(w)}_{G/B}), \; \sigma^{G_{af}/B_{af}}_{(xw_0)} \rangle$$

$$= \int_{[\Sigma_\Omega^{yw_0} \cap \bar{\Sigma}^\Omega_{xw_0}]} ev_1^*(w_{0L} \cdot \sigma^{(w)}_{G/B})$$

here $\Sigma_\Omega^{yw_0} = \overline{B_{af}^{-} \, yw_0 \cdot B_{af}}$ $\quad \bar{\Sigma}^\Omega_{xw_0} = \overline{B_{af} \, xw_0 \cdot B_{af}}$

nd $\omega(w^{-1}) = w_0 w w_0^{-1}$ is the diagram automorphism.

__marks__ 1. $w_{0L} \cdot \sigma^{(w)}_{G/B}$ restricts to $\sigma^{w}_{G/B}$ under the restriction map

$H^T(K/T) \longrightarrow H(K/T)$

2. This formula for $\ell(w)=1$ will be used later to show that

$H_*(\Omega K) \simeq \mathcal{g} H^*(G/B)$

__oof__ The proof, given on the next page, uses various formulas we have proved so far.



Proof:

$$\langle \sigma^{(yw_0)}_{G_{af}/B_{af}} \, ev_1^*(w_{0L} \cdot \sigma^{(w)}_{G/B}), \; \sigma^{G_{af}/B_{af}}_{(xw_0)} \rangle$$

$$= \varepsilon\left( (A_{xw_0})_R \cdot \left(\sigma^{(yw_0)}_{G_{af}/B_{af}} \, ev_1^*(w_{0L} \cdot \sigma^{(w)}_{G/B})\right)\right)$$

(definition of $\langle \;\; \rangle$)

$$= \varepsilon\left( \dot{j}(\sigma^\Omega_{(x)})_R \cdot A_{w_0 R} \cdot \left(\sigma^{(yw_0)}_{G_{af}/B_{af}} \, ev_1^*(w_{0L} \cdot \sigma^{(w)}_{G/B})\right)\right)$$

($A_{xw_0} = \dot{j}(\sigma^\Omega_{(x)}) A_{w_0}$ from Lectu

$$= \varepsilon\left( \dot{j}(\sigma^\Omega_{(x)})_R \cdot \left( \sum_{v \in W} ((A_{w_0 v})_R \cdot \sigma^{(yw_0)}_{G_{af}/B_{af}}\right) ((w_0 A_v)_R \cdot ev_1^* \ldots$$

($\Delta A_{w_0} = \sum_{v \in W} A_{w_0 v} \otimes w_0 A_v$ from Le

$$= \varepsilon\Big(\dot{j}(\sigma^\Omega_{(x)})_R \cdot \Big( \sum_{\substack{v \in W \\ \ell(yw_0 v^{-1} w_0) + \ell(w_0 v) = \ell(yw_0)}} \sigma^{(y w_0 v^{-1} w_0)}_{G_{af}/B_{af}} \; ev_1^*(w_{0R} A_{vR} w_{0L} \cdot \sigma^{(w)}_{G/B})\Big)\Big)$$

$\ell(w_0 v^{-1} w_0) + \ell(w_0 v) = \ell(w_0)$ automatically satisfied. (Formula for $A_{w_0 v}$ from Lecture 6 & beginning of Lecture 7), $ev_1^*$ comm

$$= \varepsilon\Big( \dot{j}(\sigma^\Omega_{(x)})_R \cdot \sum_{\substack{v \in W \\ \ell(wv^{-1}) + \ell(v) = \ell(w)}} \sigma^{(y\,\omega(v)^{-1})}_{G_{af}/B_{af}} \; ev_1^*(w_{0R} w_{0L} \cdot \sigma^{(wv^{-1})}_{G/B})\Big)$$

$$= \varepsilon\Big( \sum_{\substack{v\in W \\ \ell(wv^{-1})+\ell(v)=\ell(w)}} \big(\dot{\jmath}(\sigma^{\Omega}_{(x)})_R \cdot \sigma^{(y\,\omega(v)^{-1})}_{G_{af}/B_{af}}\big)\; ev_1^*(\omega_{0R}\omega_{0L}\cdot \sigma^{(wv^{-1})}_{G/B}) \Big)$$

( $(\mathrm{id}\otimes ev)\Delta\, \dot{\jmath}(\sigma) = \dot{\jmath}(\sigma)\otimes 1$ on page 1-7 of Lecture 9 )

$$= \varepsilon\Big( \sum_{\substack{v\in W \\ \ell(wv^{-1})+\ell(v)=\ell(w)}} \varepsilon\big(\dot{\jmath}(\sigma^{\Omega}_{(x)})_R \cdot \sigma^{(y\,\omega(v)^{-1})}_{G_{af}/B_{af}}\big)\; \varepsilon(ev_1^*(\omega_{0R}\omega_{0L}\cdot \sigma^{(wv^{-1})}_{G/B})) \Big)$$

$\varepsilon$ is a homom.

$$= \sum_{\substack{v\in W \\ \ell(wv^{-1})+\ell(v)=\ell(w)}} \varepsilon\big(\dot{\jmath}(\sigma^{\Omega}_{(x)})_R \cdot \sigma^{(y\,\omega(v)^{-1})}_{G_{af}/B_{af}}\big)\; \delta_{v,w}$$

$\varepsilon(ev_1^*(\omega_{0R}\omega_{0L}\cdot \sigma^{(wv^{-1})}_{G/B})) = \varepsilon(\sigma^{(wv^{-1})}_{G/B}) = \delta_{v,w}$

why?

$$= \varepsilon\big(\dot{\jmath}(\sigma^{\Omega}_{(x)})_R \cdot \sigma^{(y\,\omega(w)^{-1})}_{G_{af}/B_{af}}\big)$$

$$= \langle \dot{\jmath}(\sigma^{\Omega}_{(x)}),\; \sigma^{(y\,\omega(w)^{-1})}_{G_{af}/B_{af}} \rangle$$

$$= \dot{\jmath}_x^{\,y\,\omega(w)^{-1}}$$

?P

The fact that this is then equal to the integral is almost by definition of the ~~bas~~ Schubert basis & of the pairing $\langle\ \rangle$. //

**Remark** $X^{\Omega}_x$ is rational & irreducible (?).

**The basis $\{\sigma_{[x]} : x\in W^-_{af}\}$ for $H_T(\Omega K)$**

For $x\in W^-_{af}$, set

$$\sigma_{[x]} = \epsilon(x)\; c(\sigma^{\Omega}_{(x)}) \in H_T(\Omega K)$$

This is an $S$-basis for $H_T(\Omega K)$.

The automorphism $\nu$ of $A_{af}$ is used to obtain properties of this basis:

$$\nu|_{\Delta} = \mathrm{id}|_{\Delta} \qquad \nu|_{\Lambda_n} = c$$

Can check that

$$\nu(a) = (-1)^{\frac{1}{2}\deg a}\, w_0\,\omega(a)\,w_0 \qquad a\in W_{af}$$

where, recall, $\omega(w) = w_0 w w_0$ $\quad \omega(t_h) \stackrel{?}{=} t_{\omega(h)} =$

Also have

$$\nu(a)\cdot c(\sigma) = c(a\cdot\sigma)$$

**Fact 1:** $\forall\ x \in W_{af}^-$

$$\sigma_{[x]} = w_0 \cdot \sigma^{\Omega}_{(\omega(x))}$$

Proof:

$$\sigma_{[x]} = \epsilon(x)\ c(\sigma^{\Omega}_{(x)})$$
$$= \epsilon(x)\ c(A_x \cdot 1)$$
$$= \epsilon(x)\ \nu(A_x) \cdot 1$$
$$= \epsilon(x)\ (-1)^{\ell(x)}\ w_0\ \omega(A_x)\ w_0 \cdot 1$$
$$= \epsilon(x)\ (-1)^{\ell(x)}\ w_0\ A_{\omega(x)} \cdot 1$$
$$= w_0 \cdot (\sigma^{\Omega}_{(\omega(x))})$$

//

**Fact 2** For $x \in W_{af}$, $y \in W_{af}^-$

$$\nu(A_x) \cdot \sigma_{[y]} = \begin{cases} \epsilon(x)\ \sigma_{[xy]} & \text{if } xy \in W_{af}^-\ \ \ell(x)+\ell(y)=\ell(xy) \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Follows from $\sigma_{[x]} = \epsilon(x)\ \nu(A_x) \cdot 1$

//

**Fact 3** For $t \in \Gamma^-$, $x, z \in W_{af}^-$

$$\sigma_{[t]} = \sigma^{\Omega}_{(\omega(t))}$$



**Fact 4** For $x, z \in W_{af}^-$

$$\sigma_{[x]}\ \sigma_{[z]} = \sum_{\substack{y \in W_{af} \\ yz \in W_{af}^- \\ \ell(y)+\ell(z)=\ell(yz)}} \epsilon(xy)\ j_x^y\ \sigma_{[yz]}$$

## Ideals in $H_T(\Omega K)$ and $A_{af}$

**Proposition** If $M$ is an $A_{af}$-submodule of $H_T(\Omega K)$, then

1) $M$ is an ideal of $H_T(\Omega K)$ which is stab[le] under $\underline{A}$;

2) $j(M)\,\underline{A} = \underline{A}\,j(M)$ is a 2-sided ideal [of] $A_{af}$

**Proof** Assume that $M$ is an $A_{af}$-submodule of $H_T(\Omega K)$. Then it is automatically $\underline{A}$-stable. If $\sigma \in H_T(\Omega K)$ an[d] $m \in M$, we have

$$\sigma m = j(\sigma) \cdot m$$

Since $M$ is $A_{af}$-stable $\Rightarrow j(\sigma) \cdot m \in M \Rightarrow \sigma m \in M$

Hence $M \subset H_T(\Omega K)$ is an ideal. Now for $i \in I$ and $m \in M$,

$$A_i\, j(m) = j(m)\, A_i + j(A_i \cdot m)\, r_i$$

$$\Rightarrow \quad \underline{A}\, j(M) \subset j(M)\, \underline{A}.$$

Also have

$$j(m)\, A_i = A_i\, j(m) - r_i\, j'(A_i \cdot m)$$

$$\Rightarrow \quad j'(M)\, \underline{A} \subset \underline{A}\, j'(M)$$

$$\Rightarrow \quad j'(M)\, \underline{A} = \underline{A}\, j'(M)$$

Thus $j(M)$ is stable under both left and right multiplications by elements in both $j(H_T(\Omega K))$ and $\underline{A}$. Hence $j(M)$ is a 2-sided ideal of $\underline{A}_{af}$

//

<u>xamples of ideals of $H_T(\Omega K)$</u>:

For $\beta \in \Delta_+^{re}$, let

$$K(\beta) = \sum_{\substack{x \in W_{af}^- \\ x \cdot \beta < 0}} S\, \sigma_{[x]}$$

Since $l(zx) = l(z) + l(x)$ and $x \cdot \beta < 0 \Rightarrow (zx) \cdot \beta < 0$, the formular for $\nu\cancel{(A_z)\,\sigma_{[y]}}$ in Fact 2 on Page 10-14 implies that $K(\beta)$ is an $A_{af}$-stable submodule of $H_T(\Omega K)$. Hence it is an $\underline{A}$-stable ideal of $H_T(\Omega K)$.

The sum of these things will be the kernel of the map from $H_T(\Omega K)$ to $QH(G/B)$.

<u>Future Lectures</u>:

- Compare $H_*(\Omega K)$ and $QH^*(G/B)$
- Compare moduli spaces and intersection of Schubert varieties;

  the stable Bruhat order
- Compare $QH^*(G/B)$ and $QH^*(G/P)$
- Compare: $\sigma_{G/B}^{r_i} *$ in $QH^*(G/B)$

  $\sigma_{[r_i t_{-h}]} *$ in $H_T(\Omega K)$

  $\sigma_{G/B}^{[r_i]} \cdot$ in $H^T(G/B)$.

End of Lecture

Lecture 11 March 26, 1997 Wed

Today we study curves $\mathbb{P}^1 \to G/P$

Fact Since $G/P$ is projective and thus proper, we have

$$\mathrm{Mor}(\mathbb{P}^1, G/P) = \mathrm{Mor}(\mathbb{P}^1 \setminus \{\text{a finite set}\}, G/P)$$

In particular

$$\mathrm{Mor}(\mathbb{P}^1, G/P) = \mathrm{Mor}(\mathbb{C}^\times, G/P) \qquad (\mathbb{C}^\times = \mathbb{P}^1 \setminus \{0, \infty\})$$

Lemma Let $G'$ be a linear algebraic group. Then every principal $G'$-bundle over $\mathbb{A}^1 = \mathbb{C}$ is trivial, so it admits a section.

Proof W.L.O.G., assume that $G'$ is connected.

Let $\begin{array}{c} G' \to E \\ \downarrow \\ \mathbb{A}^1 \end{array}$ be a principal $G'$-bundle.

Let $B' \subset G'$ be a Borel subgroup of $G$. Then have

bundle $\begin{array}{c} E/B' \\ \downarrow \\ \mathbb{A}^1 \end{array}$ with fibre $G/B'$ which always admits

a rational section. Since $G/B'$ is proper, we actually have a morphism $s: \mathbb{A}^1 \to E/B'$. Now form the principal



$B'$-bundle $\begin{array}{c} E_{new} = \mathbb{A}^1 \times_{E/B'} E \\ \downarrow \\ \mathbb{A}^1 = \mathbb{A}^1 \times_{E/B'} E/B' \end{array}$ It remains to show that $E_{new}$ has a section.

Consider the normal series of $B'$:

$$B' = B_K \supset B_{K-1} \supset \cdots \supset B_0 = 0 \qquad \dim B_r = r$$

$B_r/B_{r-1}$ is abelian so is either $G_a = (\mathbb{C}, \text{additive})$ or $G_m = (\mathbb{C}^\times, \text{multiplicative})$

Case 1 — $G_m$. Since the only line bundle over $\mathbb{A}^1$ is the trivial one, the associated line bundle over $\mathbb{A}^1$ is trivial

Case 2 — $G_a$. Since $\mathbb{A}^1$ is affine, $H^1(\mathbb{A}^1, \mathbb{C}_{\mathbb{A}^1}) = 0$ which is the obstruction for a $G_a$-bundle to be trivial.
$(H^1(\mathbb{A}^1, G_a) = H^1(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) = 0)$.

//

Recall notation: for a variety $X$ over $\mathbb{C}$,

$$\tilde{X} = \mathrm{Mor}(\mathbb{C}^\times, X)$$

<u>Theorem 1</u> The map

$$\tilde{\pi}_P : \check{G} \longrightarrow \widetilde{G/P} = \mathrm{Mor}(\mathbb{P}^1, G/P)$$

$$g(t) \longmapsto g(t)P$$

is surjective.

<u>Proof</u> Given $\phi \in \mathrm{Mor}(\mathbb{A}^1, G/P) = \mathrm{Mor}(\mathbb{P}^1, G/P)$, form the principal $P$-bundle over $\mathbb{A}^1$:

$$E = \{ (t, g) \in \mathbb{A}^1 \times G : \phi(t) = gP \}$$

with $P$ acting on the copy of $G$ from the right by right multiplications. By Lemma, $E$ admits a section, i.e.

$\exists\ s : \mathbb{A}^1 \to E : \quad s(t) = (t, g(t)) \in E$

Thus $g(t) \in \mathrm{Mor}(\mathbb{A}^1, G) \subset \check{G}$ is a lift of $\phi$. Similarly, can show that can also lift $\phi$ to some $g'(t) \in \mathrm{Mor}(\mathbb{P}^1 \setminus \{0\}, G)$.

//

Next, we study the degrees of the curve $\tilde{\pi}_P(g) \in \mathrm{Mor}(\mathbb{P}^1, G/P)$ for $g \in \check{G} = \mathrm{Mor}(\mathbb{C}^\times, G)$.



<u>Recall notation</u>

① For a variety $X$ over $\mathbb{C}$, have

$$\check{X} = \mathrm{Mor}(\mathbb{C}^\times, X)$$

and

$$(\check{X})_0 = \{ \phi \in \check{X} : \phi|_{S^1} \text{ is trivial in } \pi_1(X) \}$$

For example, for $\mathfrak{sl}(2,\mathbb{C})$,

$$\check{B} = \left\{ \begin{pmatrix} a(t) & b(t) \\ 0 & d(t) \end{pmatrix} : \begin{array}{l} a, b, d \in \mathbb{C}[t, t^{-1}] \\ ad = 1 \end{array} \right\}$$

Now $a, d \in \mathbb{C}[t, t^{-1}]$ (Laurent polynomials) and $ad = 1 \Rightarrow \quad a = \lambda t^k \quad d = \frac{1}{\lambda} t^{-k}$

But must have $k = 0$ in order for $g(t) = \begin{pmatrix} a(t) & b(t) \\ 0 & d(t) \end{pmatrix} \in (\check{B})_0$.

Thus

$$(\check{B})_0 = \left\{ \begin{pmatrix} \lambda & b(t) \\ 0 & \frac{1}{\lambda} \end{pmatrix} : \lambda \in \mathbb{C}^\times,\ b \in \mathbb{C}[t, t^{-1}] \right\}$$

This is true in general:

$$(\check{B})_0 = H \ltimes \check{U}_+$$

<u>Remark</u>: Compare with $B_{af} = \left\{ \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} : \begin{array}{l} a, b, c, d \in \mathbb{C}[t] \\ ad - bc = 1 \\ c(0) = 0 \end{array} \right\}$

Very different from $\cancel{U_+}$ $(\check{B})_0$.

② $\pi_P : \quad Q^\vee \longrightarrow H_2(G/P)$

$$\pi_P(\alpha_i^\vee) = \begin{cases} \sigma_{r_i}^{G/P} & \text{if } r_i \notin W_P \\ 0 & \text{if } r_i \in W_P \end{cases}$$

heorem 2

(A). Let $w_1, w_2 \in W$ and $h_1, h_2 \in Q^\vee$.

If $g \in B_{af}^- w_1 t_{h_1} (\tilde{B})_0 \cap B_{af} w_2 t_{h_2} (\tilde{B})_0$,

then $\phi := \tilde{\pi}_B(g) \in Mor(\mathbb{P}^1, G/B)$

satisfies

$$\phi_*[\mathbb{P}^1] = \pi_B(h_2 - h_1)$$
$$\phi(\infty) \in B_- w_1 \cdot B$$
$$\phi(0) \in B w_2 \cdot B$$

(B) We have two disjoint unions.

$$\tilde{G} = \coprod_{x \in W_{af}} B_{af}^- x (\tilde{B})_0 = \coprod_{y \in W_{af}} B_{af}\, y\, (\tilde{B})_0 .$$

Here, recall

$$B_{af} = \{ g \in Mor(\mathbb{P}^1 \backslash \{\infty\}, G) : g(0) \in B \}$$
$$B_{af}^- = \{ g \in Mor(\mathbb{P}^1 \backslash \{0\}, G) : g(\infty) \in B^- \}$$

Proof of (A) Since $g \in B_{af}^- w_1 t_{h_1} (\tilde{B})_0$, we can write

$$g(t) = b_-(t)\, n_1\, t^{-h_1}\, a_1\, u_1(t) \qquad t \in \mathbb{C}^\times$$

where $b_-(t) \in B_{af}^-$, $u_1(t) \in \tilde{U}_+$, $a_1 \in H$ and $n_1$ is a representative of $w_1$ in $G$. Then by definition

$$\phi(t) = g(t) \cdot B = b_-(t)\, w_1 \cdot B \qquad t \in \mathbb{C}^\times$$

Since $b_- \in Mor(\mathbb{P}^1 \backslash \{0\}, G)$ and $b_-(\infty) \in B_-$, we have

$$\phi(\infty) \in B_- w_1 \cdot B$$

Similarly,

$$\phi(0) \in B\, w_2 \cdot B .$$

It remains to calculate $\phi_*[\mathbb{P}^1] \in H_2(G/B)$. We do this by calculating

$$\langle \phi_*[\mathbb{P}^1], \lambda \rangle$$

for every dominant integral $\lambda \in \underline{h}^*$ considered as element in $H^2(G/B)$. So let $\lambda$ be a such and le $V(\lambda)$ be the irreducible highest weight module of $G$ with highest weight $\lambda$ and highest weight vector $V_\lambda^+ \in V(\lambda)$. Then we have the morphism

$$J : G/B \longrightarrow \mathbb{P}(V(\lambda)), \quad g\cdot B \longmapsto \mathbb{C}\, g\cdot V_\lambda^+$$

and $\lambda \in H^2(G/B)$ is the pullback by $J$ of the standard generator of $H^2(\mathbb{P}(V(\lambda)))$. Thus

$$\langle \phi_*[\mathbb{P}^1], \lambda \rangle = \text{the degree of } J\circ\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}(V(\lambda)).$$

Using
$$g(t) = b_0(t)\, n_1\, t^{-h_1}\, a_1\, u_1(t) \qquad t\in\mathbb{C}^\times$$
we have
$$g(t)\cdot V_\lambda^+ = t^{-\langle\lambda, h_1\rangle}\, a_1^\lambda\, b_0(t)\, n_1\cdot V_\lambda^+ \qquad t\in\mathbb{C}^\times$$
so in any chosen homogeneous coordinates. we can write

$$(J\circ\phi)(t) = [\, V_0(t), V_1(t), \cdots, V_k(t)\,]$$

where each $V_j(t) \in \mathbb{C}[t, t^{-1}]$ and has degree at most $-\langle\lambda, h_1\rangle$ and the degree $-\langle\lambda, h_1\rangle$ occurs. Similarly, using the fact that

$$g\in B_{af}^+\, w_2\, t_{h_2}(\bar{B})_0$$
we see that the minimal degree of the $V_j(t)$'s is $-\langle\lambda\, h_2\rangle$.

Thus

$$\text{degree of } J\circ\phi = \text{max. deg} - \text{min. deg}$$
$$= \langle\lambda, h_2 - h_1\rangle .$$

Hence
$$\langle \phi_*[\mathbb{P}^1], \lambda\rangle = \langle \lambda, h_2 - h_1\rangle$$
$\Rightarrow$
$$\phi_*[\mathbb{P}^1] = h_2 - h_1 .$$

This finishes the proof of (A).

<u>Proof of (B)</u> First assume we have the unions, ie.

$$\tilde{G} = \coprod_{x\in W_{af}} B_{af}^-\, x\, (\bar{B})_0 = \coprod_{y\in W_{af}} B_{af}\, y\, (\bar{B})_0 \qquad (*)$$

We prove the disjointness. So assume

$$g\in \left(B_{af}^-\, x_1\, (\bar{B})_0\right) \cap \left(B_{af}^-\, x_1'\, (\bar{B})_0\right)$$

Then also
$$g\in B_{af}\, y\, (\bar{B})_0$$
for some $y$. Write

$$x_1 = w_1 t_{h_1}, \quad x_1' = w_1' t_{h_1'} \quad y = w_2 t_{h_2}$$

Then by (A), the curve $\pi_0(g) = \varphi$ satisfies
$$\phi_*[\mathbb{P}^1] = h_2 - h_1 = h_2' - h_1'$$
$\Rightarrow$ $h_1 = h_1'$. Also $\phi(\infty) \in B_- w_1\cdot B \cap B_- w_1'\cdot B$

$\Rightarrow$ $w_1 = w_1'$ Hence $x_1 = x_1'$. This shows the first union in $(*)$ is disjoint. Similarly is the 2nd.

Now we need to show

$$\tilde{G} \subset \coprod_{y \in W_{af}} B_{af}\, y\, (\tilde{B})_0$$

Since $\{U_{\pm\alpha_i},\ i \in I_{af}\}$ generate $\tilde{G}$, it suffices to show that $\coprod_{y\in W_{af}} B_{af}\, y\, (\tilde{B})_0$ is stable under the left multiplication by $U_{\pm\alpha_i}$ $\forall\, i \in I_{af}$. Clearly ok for $U_{\alpha_i} \subset B_{af}$. Only need to show

$$\left(U_{-\alpha_i} \setminus \{id\}\right) \coprod_{y \in W_{af}} B_{af}\, y\, (\tilde{B})_0 \subset \coprod_{y\in W_{af}} B_{af}\, y\, (\tilde{B})_0 .$$

Now we know:

$$U_{-\alpha_i} \setminus \{id\} \subset B_{af}\, r_i\, U_{\alpha_i}$$

<u>Case 1</u>: $\overline{y^{-1}\cdot\alpha_i} > 0$. $\Rightarrow$ $U_{y^{-1}\cdot\alpha_i} \subset (\tilde{B})_0$

$\Rightarrow$ $U_{\alpha_i}\, y\, (\tilde{B})_0 \subset y\, (\tilde{B})_0$

$\Rightarrow$ $(U_{-\alpha_i} \setminus \{id\})\, B_{af}\, y\, (\tilde{B})_0 \subset B_{af}\, y\, (\tilde{B})_0$ ok

Case 2. $\overline{y^{-1}\cdot\alpha_i} < 0$ $\Rightarrow$ $U_{\alpha_i} \setminus \{id\} \subset U_{-\alpha_i}\, r_i H\, U_{-\alpha_i}$

$$\Rightarrow B_{af}\, r_i\, U_{\alpha_i}^{\setminus\{id\}}\, y\, (\tilde{B})_0 \subset B_{af}\, r_i\, U_{-\alpha_i}\, (r_i H)\, U_{-\alpha_i}\, y\, (\tilde{B})_0$$

$$\subset B_{af}\, r_i\, (r_i H)\, y\, (\tilde{B})_0$$

$$= B_{af}\, y\, (\tilde{B})_0$$

//

End of proof of Thm 2.

<u>Definition</u>: For $w_1, w_2 \in W^P$, $z \in H_2(G/P)$, set

$M^{w_1,w_2}_{G/P,z}$ = the variety of all $\phi \in Mor(\mathbb{P}^1, G/P)$ s.t.

$$\phi_*[\mathbb{P}^1] = z$$
$$\phi(\infty) \in B \cdot w_1 \cdot P$$
$$\phi(0) \in B\, w_2 \cdot P$$

It is a smooth irreducible variety of dimension =

Insert 11-10.5 (i)(ii)(iii) attached to the back

$$\dim M^{w_1,w_2}_{G/P,z} = \ell(w_2) - \ell(w_1) + \langle z, c_1(TG/P) \rangle$$

<u>Connection of $M^{w_1,w_2}_{G/P,z}$ to Schubert cells in $G_{af}/B_{af}$</u>:

Introduce

$$W^{\pm}_{af} = \{ x \in W_{af} :\ \beta \in \Delta^{re}_{+},\ x\cdot\beta < 0 \Rightarrow \pm\bar{\beta} > 0$$

so $W^{-}_{af}$ is as before the minimal coset representatives of $W_{af}/W$. It is easy to see that

$$W^{-}_{af}\, w_0 \subset W^{+}_{af}$$

where $w_0 \in W$ is the longest element of $W$.

So $\exists\ h \in \Gamma$ s.t. $b_0 \in (\tilde B)_0$

$$b^-(t)\, x_1(t) = b'^-(t)\, x_1(t)\, t^h\, b_0(t)$$

or $\quad b^- x_1 \in b'^- x_1\, t_h\, (\tilde B)_0$

or $\quad b^- x_1 \in B_{af}^- x_1 (\tilde B)_0 \cap B_{af}^- x_1 t_h (\tilde B)_0$

By the disjointness of the union

$$\tilde G = \bigsqcup_{x \in W_{af}} B_{af}^- x\, (\tilde B)_0$$

must have $t_h = id$ or $b(t) \in (\tilde B)_0$. Hence $g_2 \cdot (\tilde B)_0 = g' \cdot (\tilde B)_0$.

This shows that $\tilde\pi_B$ is injective. (Is this argument rigorous enough?)

Now suppose $\phi \in M^{w_1, w_2}_{G/B,\, \pi_B(h_2 - h_1)}$. Let $g' \in \tilde G$ be any element such that $\tilde\pi_B(g') = \phi$. Then by Theorem (B), there must exist $x_1' = w_1 t_{h_1'}$ and $x_2' = w_2 t_{h_2'} \in W_{af}$ s.t.

$$g' \in \cancel{B_{af}^- x_1' (\tilde B)_0 \cap B_{af} x_2' \cdot (\tilde B)_0}$$

$$g' \in B_{af}^- x_1' (\tilde B)_0 \cap B_{af} x_2' (\tilde B)_0$$

let $\quad g = g' t_{h - h_1'}$

Then $\pi_B(g) = \pi_B(g') = \phi$ but now $g' \in B_{af}^- x_1 (\tilde B)_0 \cap B_{af} x_2' t_{h-h_1'} (\tilde B)_0$



But since

$$\phi_*[\mathbb{P}^1] = \pi_B(h_2 - h_1)$$

we must have $\quad x_2' t_{h-h_1'} = x_2$.

Hence $\quad g' \in B_{af}^- x_1 (\tilde B)_0 \cap B_{af} x_2 (\tilde B)_0 \quad$ or

$$g' \cdot (\tilde B)_0 \in \left(B_{af}^- x_1 \cdot (\tilde B)_0\right) \cap B_{af} x_2 \cdot (\tilde B)_0 .$$

This shows that $\pi_B$ is onto. Hence $\pi_B$ is bijective.

//

<u>Remark</u> Note that in the definition of $M^{w_1, w_2}_{G/P, \tau}$, we consider a reparametrization of a curve $\phi$ or a shift of $\phi$ by an element in $\tilde H$ as a new curve.

ct: For $x = w t_h \in W_{af}^{\pm}$, have

move in lec 12

$$\ell(x) = \pm \ell_s(x)$$

where $\ell_s(x)$, the stable length of $x$, is defined to be

$$\ell_s(w t_h) = \ell(w) + \langle 2\rho, h \rangle .$$

eorem 3 Let $x_1 = w_1 t_{h_1}$, $x_2 = w_2 t_{h_2}$ be in $W_{af}^{+}$. Then we

ve in have an natural inverse isomorphism between smooth

e 12 varieties:

$$B_{af}^{-} x_1 \cdot B_{af} \cap B_{af} x_2 \cdot B_{af} \underset{\pi_+}{\overset{\pi_-}{\rightleftarrows}} M^{w_1, w_2}_{G/B,\, \pi_B(h_2 - h_1)}$$

given by

$$\pi_-(g \cdot B_{af}) = \widehat{\pi}_B(g) \qquad \text{if } g \in B_{af}^{-} x_1$$

$$\pi_+(\widehat{\pi}_B(g)) = g \cdot B_{af} \qquad \text{if } g \in B_{af} x_2 .$$

mark: Note that the intersection $B_{af}^{-} x_1 \cdot B_{af} \cap B_{af} x_2 \cdot B_{af}$ is smooth and has dimension =

$$\begin{aligned} \ell(x_2) - \ell(x_1) &= \ell(w_2) + \langle 2\rho, h_2 \rangle - \ell(w_1) - \langle 2\rho, h_1 \rangle \\ &= \ell(w_2) - \ell(w_1) + \langle 2\rho, h_2 - h_1 \rangle \\ &= \dim M^{w_1, w_2}_{G/B,\, \pi_B(h_2 - h_1)} \end{aligned}$$

End of Lecture 11



Thus we have defined a map, for any $x_1 = w_1 t_{h_1}$, $x_2 = w_2 t_{h_2} \in W$

$$\widehat{\pi}_B : B_{af}^{-} x_1 (\widehat{B})_0 \cap B_{af} x_2 (\widehat{B})_0 \longrightarrow M^{w_1, w_2}_{G/B,\, \pi_B(h_2 - h_1)}$$

Since $\widehat{\pi}_B(g(\widehat{B})_0) = \widehat{\pi}_B(g)$, we get a well-defined map, still denoted by $\widehat{\pi}_B$:

$$\widehat{\pi}_B : B_{af}^{-} x_1 \cdot (\widehat{B})_0 \cap B_{af} x_2 \cdot (\widehat{B})_0 \longrightarrow M^{w_1, w_2}_{G/B,\, \pi_B(h_2 - h_1)}$$

Proposition: The map

$$\widehat{\pi}_B : B_{af}^{-} x_1 \cdot (\widehat{B})_0 \cap B_{af} x_2 \cdot (\widehat{B})_0 \longrightarrow M^{w_1, w_2}_{G/B,\, \pi_B(h_2 - h_1)}$$

is bijective

Proof: We can in fact prove that

$$\widehat{\pi}_B \big|_{B_{af}^{-} x_1 \cdot (\widehat{B})_0} : B_{af}^{-} x_1 \cdot (\widehat{B})_0 \longrightarrow M^{w_1, w_2}_{G/B,\, \pi_B(h_2 - h_1)}$$

is injective. Indeed, if $g = b^- x_1$ and $g' = b^{-\prime} x_1$, where $b^-, b^{-\prime} \in B_{af}^{-}$ are such that $\widehat{\pi}_B(g(\widehat{B})_0) = \widehat{\pi}_B(g' \cdot (\widehat{B})_0)$, ie. $\widehat{\pi}_B(g) = \widehat{\pi}(g')$, then $b^-(t) x_1(t) \cdot B = b^{-\prime}(t) x_1(t) \cdot B$.
Here $x_1(t)$ is a representative of $x_1$. Hence $\exists\, b(t) \in \widehat{B}$ s.t. $b^-(t) x_1(t) = b^{-\prime}(t) x_1(t) b(t)$. But

$$\widehat{B} = \widehat{H} \ltimes \widehat{U}_+ = \Gamma \times H \ltimes \widehat{U}_+ = \Gamma \times (\widehat{B})_0 .$$

# Lecture 12, April 8, 1997 Tuesday

Recall last lecture ...

The fact

$$\tilde{G} = \coprod_{\substack{x \in W_{af} \\ \text{disjoint}}} B_{af}^- \, x(\tilde{B})_0 = \coprod_{\substack{y \in W_{af} \\ \text{disjoint}}} B_{af} \, y(\tilde{B})_0$$

is a special case of the following general fact:

**Fact:** If $V$ is a subgroup of $\tilde{G}$ such that for each $\alpha \in \Delta_+^{re}$, either $U_\alpha \subset V$ or $U_{-\alpha} \subset V$, then we have two disjoint unions:

$$G_{af} = \tilde{G} = \coprod_{x \in W_{af}} B_{af}^- \, x \, V = \coprod_{y \in W_{af}} B_{af} \, y \, V$$

Two decompositions for any Kac-Moody group:

$\forall \, x \in W$ (of the K-M group in question) $\ni y$

0) $U_- = \left(U_- \cap (x^{-1} B_+ x)\right)\left(U_- \cap (x^{-1} B_- x)\right)$

1) $(U_+ \cap x B_- x^{-1}) \times (B_- x \cdot B) \xrightarrow{\sim} x B_- \cdot B$

2) $(U_+ \cap x B_- x^{-1}) \times \left((B_- x \cdot B) \cap B y \cdot B\right) \xrightarrow{\sim} x B_- \cdot B \cap B y \cdot B$

$\Rightarrow B_- x \cdot B \cap B y \cdot B \neq \phi \iff x \leq y$,

and in this case, $B_- x \cdot B \cap B y \cdot B$ is a non-singular irreduci[ble] affine variety of dimension $= \ell(y) - \ell(x)$.

~~Recall~~ In order to prove Theorem 3 stated at the end of last lecture, we need the following facts. Recall that

$$W_{af}^+ = \{ x \in W_{af} : \ \beta \in \Delta_+^{re}, \ x \cdot \beta < 0 \ \Rightarrow \bar{\beta} > 0 \}$$

**Proposition 1:** The following are equivalent:

(0) $x \in W_{af}^+$

(1) $B_{af} \cap x^{-1} B_{af}^- x \subset (\tilde{B})_0$

(2) $x B_{af} x^{-1} \cap B_{af}^- \subset x(\tilde{B})_0 x^{-1}$

(3) $B_{af} \, x \, B_{af} \subset B_{af} \, x \, (\tilde{B})_0$

(1') $(\tilde{B})_0 \cap x^{-1} B_{af} x \subset B_{af}$

(2') $x(\tilde{B})_0 x^{-1} \cap B_{af} \subset x \, B_{af} \, x^{-1}$

(3') $B_{af}^- \, x \, (\tilde{B})_0 \subset B_{af}^- \, x \, B_{af}$

**Proof** The equivalence between (0) + (1) is clear because (2) says that $\forall$ if $\beta \in \Delta_+^{re}$ and $x\beta < 0$ then $\bar{\beta} > 0$.

It is also clear that (1) is equivalent to (2) because $x$ 1) $x^{-1}$ = 2)

Now assume (1). We want to prove (3). It is enough to show that

$$x B_{af} \subset B_{af}\, x\, (\bar{B})_0 .$$

Let $x b \in x B_{af}$. Write $b = b_1 b_2$ where

$$b_1 \in B_{af} \cap x^{-1} B_{af}\, x \quad , \quad b_2 \in B_{af} \cap x^{-1} B_{af}^- \,x$$

Then

$$xb = x b_1 b_2 = (x b_1 x^{-1})\; x\; b_2$$

Now $x b_1 x^{-1} \in B_{af}$ and $b_2 \in (\bar{B})_0$ by 1). Hence $x b \in B_{af}\, x (\bar{B})_0$.

This shows that $\big((0) \Leftrightarrow\big)(1)\big(\Leftrightarrow (2)\big) \Rightarrow (3)$. Now assume 3). We want to prove (1).



<u>Proposition 2</u>: For $x_1, x_2 \in W_{af}^{+}$, the two maps

$$\phi_1 : B_{af}^- x_1 \cdot B_{af} \longrightarrow B_{af}^- x_1 \cdot (\bar{B})_0 : \; b^- x_1 \cdot B_{af} \longmapsto b^- x_1 \cdot (\bar{B})_0$$

$$\phi_2 : B_{af}\, x_2 \cdot (\bar{B})_0 \longrightarrow B_{af}\, x_2 \cdot B_{af} : \; b^+ x_2 \cdot (\bar{B})_0 \longmapsto b^+ x_2 \cdot B_{af}$$

are both well-defined. Moreover, their restrictions to the following intersections give isomorphisms that are mutually inverses of each other

$$B_{af}^- x_1 \cdot B_{af} \cap B_{af}\, x_2 \cdot B_{af} \underset{\phi_2}{\overset{\phi_1}{\rightleftarrows}} B_{af}^- x_1 \cdot (\bar{B})_0 \cap B_{af}\, x_2 \cdot (\bar{B})_0$$

<u>Proof</u>: $\phi_1$ is well-defined because $B_{af}^- \cap x_1 B_{af}\, x_1^{-1} \subset x_1 (\bar{B})_0\, x_1^{-1}$ ((2) in P

$\phi_2$ is well-defined because $B_{af} \cap x_2 (\bar{B})_0\, x_2^{-1} \subset x_2 B_{af}\, x_2^{-1}$ ((2') in 1

Since $B_{af}\, x_2\, B_{af} \subset B_{af}\, x_2 (\bar{B})_0\, x_2^{-1}$ ((3) in Prop.1), we have

$$\phi_1 \big( B_{af}^- x_1 \cdot B_{af} \cap B_{af}\, x_2 \cdot B_{af} \big) \subset B_{af}^- x_1 \cdot (\bar{B})_0 \cap B_{af}\, x_2 \cdot (\bar{B})_0$$

In more details, suppose that

$$m_1 = b^- x_1 \cdot B_{af} = b^+ x_2 \cdot B_{af} \in B_{af}^- x_1 \cdot B_{af} \cap B_{af}\, x_2 \cdot B_{af}$$

where $b^- \in B_{af}^-$, $b^+ \in B_{af}$. Then $\exists\, b \in B_{af}$ s.t.

$$b^- x_1 = b^+ x_2\, b .$$

Write $b = b_1 b_2$ where $b_1 \in B_{af} \cap x_2^{-1} B_{af}\, x_2$ , $b_2 \in B_{af} \cap x_2^{-1} B_{af}^-\, x$

Then $b^- x_1 = b^+ (x_2 b_1 x_2^{-1})\; x_2\, b_2$. We know that $x_2 b_1 x_2^{-1} \in B$ by definition of $b_1$ and that $b_2 \in (\bar{B})_0$ by (1) of Prop.1. Thu

$\phi(m_1) = b^- x_1 \cdot (\bar{B})_o = (b^+ x_2 b_1 x_1^{-1})\, x_2 \cdot (\bar{B})_o \in B_{af}\, x_1 \cdot (\bar{B})_o$ .

Moreover, by the definition of $\phi_2$, we have

$$\begin{aligned}\phi_2(\phi_1(m_{1})) &= (b^+ x_2 b_1 x_2^{-1})\, x_2 \cdot B_{af} \\ &= b^+ x_2 b_1 \cdot B_{af} \\ &= b^+ x_2 \cdot B_{af} \qquad (\text{since } b_1 \in B_{af}) \\ &= m_1\end{aligned}$$

This shows that $\phi_1$ is injective and $\phi_2$ is onto (when restricted to the intersections). Similarly we can show that

$$\phi_2(B_{af}^- x_1 \cdot (\bar{B})_o \cap B_{af} x_2 \cdot (\bar{B})_o) = B_{af}^- x_1 \cdot B_{af} \cap B_{af} x_2 \cdot B_{af}$$

and $\phi_1(\phi_2(m_2)) = m_2$ for $m_2 \in B_{af}^- x_1 \cdot (\bar{B})_o \cap B_{af} x_2 \cdot (\bar{B})_o$. let's write out the details again: suppose that

$$m_2 = b^- x_1 \cdot (\bar{B})_o = b^+ x_2 \cdot (\bar{B})_o \in B_{af}^- x_1 \cdot (\bar{B})_o \cap B_{af} x_2 \cdot (\bar{B})_o$$

where $b^- \in B_{af}^-$ and $b^+ \in B_{af}$. Then $\exists\ b_0 \in (\bar{B})_o$ s.t.

$$b^+ x_2 = b^- x_1 b_0 .$$

Write $b_0 = b_1 b_2$ where $b_1 \in (\bar{B})_o \cap x_1^{-1} B_{af}^- x_1$ and $b_2 \in (\bar{B})_o \cap x_1^{-1} B_{af} x_1$.
Then $b^+ x_2 = b^- (x_1 b_1 x_1^{-1})\, x_1 b_2$. Now $x_1 b_1 x_1^{-1} \in B_{af}^-$ by definition and $b_2 \in B_{af}$ by (1') of Prop. 1. Hence $b^+ x_2 \in B_{af}^- x_1 B_{af}$, or $\phi_2(m_2) = b^+ x_2 \cdot B_{af} \in B_{af}^- x_1 \cdot B_{af}$. In other words,

$$\phi_2(B_{af}^- x_1 \cdot B_{af} \cap B_{af} x_2 \cdot B_{af}) \subset B_{af}^- x_1 \cdot (\bar{B})_o \cap B_{af} x_2 \cdot (\bar{B})_o .$$



Moreover, by the definition of $\phi_1$, we have

$$\begin{aligned}\phi_1(\phi_2(m_2)) &= b^- (x_1 b_1 x_1^{-1})\, x_1 \cdot (\bar{B})_o \\ &= b^- x_1 b_1 \cdot (\bar{B})_o \\ &= b^- x_1 \cdot (\bar{B})_o \qquad (\because\ b_1 \in (\bar{B})_o) \\ &= m_2 .\end{aligned}$$

This shows that when restricted to the intersections, both $\phi_1$ & $\phi_2$ are isomorphisms and that they are the inverses of each other.

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We can now prove Theorem 3 stated in Lecture 11. We restate

**Theorem 3**: Let $x_1 = w_1 t_{h_1}$ and $x_2 \in W_2 t_{h_2}$ be in $W_{af}^+$. Then we have mutually inverse isomorphisms

$$B_{af}^- x_1 \cdot B_{af} \cap B_{af} x_2 \cdot B_{af} \underset{\pi_+}{\overset{\pi_-}{\rightleftarrows}} M^{w_1, w_2}_{G/B\ \pi_B(h_2 - h_1)}$$

defined by

$$\pi_-(g \cdot B_{af}) = \bar{\pi}_B(g) \quad \text{if } g \cdot B_{af} \in B_{af}^- x_1 \text{ s.t. } g \cdot B_{af} \in L.$$

$$\pi_+(\bar{\pi}_B(g)) = g \cdot B_{af} \quad \text{if } g \in B_{af} x_2 \text{ s.t. } \bar{\pi}_B(g) \in RHS.$$

**Proof.** This is just Proposition 2 and the Prop. on 11-10's (1) combined, ie.

$$B_{af}^- x_1 \cdot B_{af} \cap B_{af} x_2 \cdot B_{af} \cong B_{af}^- x_1 \cdot (\bar{B})_o \cap B_{af} x_2 \cdot (\bar{B})_o \cong M^{w_1, w_2}_{G/B,\ \pi_B}$$

/

## The Stable Bruhat order $\overset{s}{\leq}$ and the stable length $\ell_s$

Say $h \in Q^\vee$ is sufficiently dominant if $\langle \rho_i, h\rangle \gg 0$ for each $i \in I$

**Definition** .

1) For $x, y \in W_{af}$, write "$x \overset{s}{\leq} y$" and say "$x$ is $\leq y$ under the stable Bruhat order" if $x t_h \leq y t_h$ for sufficiently dominant $h$.

2) For $x = w t_h \in W_{af}$, define the stable length of $x$ to be
$$\ell_s(x) = \ell(w) + \langle 2\rho, h\rangle$$

**Facts** 1) For any $w \in W$ and $h$ dominant, have $\overset{= w t_h}{x} \in W_{af}^{+}$

2) For any given $x \in W_{af}$, have $x t_h \in W_{af}^{+}$ for sufficiently dominant $h$.

**Proof** Clearly 2) follows from 1). We only prove 1). If $\alpha < 0$ is a root for the finite $\mathfrak{g}$, then for any $n > 0$.
$$x\cdot(\alpha + n\delta) = w\alpha + (n - \langle h, \alpha\rangle)\delta$$
Since $\langle h, \alpha\rangle \leq 0$, have $n - \langle h, \alpha\rangle \geq n > 0$. Thus always has $x\cdot(\alpha + n\delta) > 0$. This shows that if $\beta = \alpha + n\delta > 0$ is such that $x\cdot\beta < 0$ must have $\alpha > 0$. Thus $x \in W_{af}^{+}$

//



**Proposition** $\underset{x_1}{\underset{\shortparallel}{w_1 t_{h_1}}} \overset{s}{\leq} \underset{x_2}{\underset{\shortparallel}{w_2 t_{h_2}}} \iff M^{w_1, w_2}_{G/B,\, \pi_B(h_2 - h_1)} \neq \phi$

$\iff B_{af}^{-} x_1 \cdot (\overline{B})_0 \cap B_{af} x_2 \cdot (\overline{B})_0 \neq$

**Proof** : We have proved in Lecture 11 that
$$M^{w_1, w_2}_{G/B,\, \pi_B(h_2 - h_1)} \doteq B_{af}^{-} x_1 \cdot (\overline{B})_0 \cap B_{af} x_2 \cdot (\overline{B})_0$$
Now suppose $x_1 \overset{s}{\leq} x_2$. Then $\exists$ sufficiently dominant $h$ s.t. $x_1 t_h$, $x_2 t_h \in W_{af}^{+}$ and $x_1 t_h \leq x_2 t_h$

This implies
$$B_{af}^{-} x_1 t_h \cdot B_{af} \cap B_{af} x_2 t_h \cdot B_{af} \neq \phi$$
But by Theorem 3, since $x_1 t_h$, $x_2 t_h \in W_{af}^{+}$, we have
$$M^{w_1, w_2}_{G/B,\, \pi_B(h_2 - h_1)} \cong B_{af}^{-} x_1 t_h \cdot B_{af} \cap B_{af} x_2 t_h \cdot B_{af} \neq \phi$$
Conversely, if $M^{w_1, w_2}_{G/B,\, \pi_B(h_2 - h_1)} \neq \phi$, then for $h$ sufficiently dominant so that $x_1 t_h$, $x_2 t_h \in W_{af}^{+}$, we have
$$B_{af}^{-} x_1 t_h \cdot B_{af} \cap B_{af} x_2 t_h \cdot B_{af} = M^{w_1, w_2}_{G/B,\, \pi_B(h_2 - h_1)} \neq \phi$$
Thus $x_1 t_h \leq x_2 t_h$. Hence $w_1 t_{h_1} \overset{s}{\leq} w_2 t_{h_2}$

//

<u>Proposition</u>: For $x_1, x_2 \in W_{af}^+$, have $x_1 \overset{s}{\leq} x_2 \iff x_1 \leq x_2$

<u>Proof</u>: Suppose $x_1, x_2 \in W_{af}^+$. Then

$$x_1 \overset{s}{\leq} x_2 \iff M^{w_1,w_2}_{G/B,\, \pi_a(h_2 - h_1)} = B_{af}^- x_1 \cdot B_{af} \cap B_{af} x_2 \cdot B_{af} \neq \phi$$
$$\iff x_1 \leq x_2$$

//

<u>Proposition</u>: For $w_1, w_2 \in W$, have $w_1 \overset{s}{\leq} w_2 \iff w_1 \leq w_2$

<u>Proof 1</u>: Since $M^{w_1,w_2}_{G/B,\, \pi_a(0)} = B^- w_1 \cdot B \cap B w_2 \cdot B$

we have

$$w_1 \overset{s}{\leq} w_2 \iff B^- w_1 \cdot B \cap B w_2 \cdot B \neq \phi$$
$$\iff w_1 \leq w_2$$

//

<u>Proof 2</u>: We first prove that $\underline{W \subset W_{af}^+}$.

Suppose $\beta = \alpha + n\delta > 0$ is s.t. $w \cdot \beta < 0$. Then we must have $\alpha > 0$: for if $\alpha < 0$, then $n > 0$, and thus

$$w \cdot \beta = w\alpha + n\delta > 0$$

Contradition. Hence $\alpha > 0$. Hence $W \subset W_{af}^+$

//



<u>Question</u>: Given $w \in W$, for which $x \in W_{af}$ do we have $w \overset{s}{<} x$ and $\ell_s(x) = \ell_s(w) + 1 = \ell(w) + 1$ ?

<u>Answer</u>: Iff $x$ is one of the following two forms:

either $x = w r_\alpha$ where $\alpha \in \bar{\Delta}_+$ (positive roots for the finite $\mathfrak{g}$) and $\ell(x) = \ell(w) + 1$

or $x = w r_\alpha t_{\alpha^\vee} = w r_{\alpha+\delta}$ where $\alpha \in \bar{\Delta}_+$ and $\ell(w r_\alpha) = \ell(w) - \langle 2\rho, \alpha^\vee \rangle + 1$.

<u>Proof</u>: Later.

Next time: $M_{G/P,\, \tau}$. $(W_P)_{af}$ $(W^P)_{af}$

End of Lecture 12

<u>Fact</u>: If $w t_h \in W_{af}^+ \Rightarrow h \in Q^\vee$ is dominant.

<u>Proof</u>: Proof by contradiction: Suppose $h$ is not dominant. Then $\exists\ i$ s.t. $\langle \alpha_i, h \rangle < 0$. Must have $\langle \alpha_i, h \rangle \leq -2$.
Let $\beta = -\alpha_i + \delta \in \Delta_+^{re}$. Then $x \cdot \beta = -w\alpha_i + (1 + \langle \alpha_i, h \rangle)\delta$
But $\bar{\beta} = -\alpha_i < 0$. Contradictory to $x \in W_{af}^+ \Rightarrow h$ domin

# Lecture 13. April 9. 1997 Wednesday

We first collect some facts about $\ell_s$ and $\overset{s}{\leq}$. Then talk ab
$(W_p)_{af}$ and $(W^p)_{af}$.

**Proposition $\ell_s$:** The following are true about $\ell_s$:

(1) $\ell_s(w) = w \qquad \forall w \in W$

(2) $\ell_s(x t_h) = \ell_s(x) + \langle 2\rho, h \rangle \qquad \forall x \in W_{af},\ h \in Q^\vee$

(3) $\ell_s(x w_0) = \ell(w_0) - \ell_s(x) \qquad \forall x \in W_{af},\ w_0 =$ longest in W

(4) $-\ell(x) \leq \ell_s(x) \leq \ell(x) \qquad \forall x \in W_{af}$

$\ell_s(x) = \ell(x) \iff x \in W_{af}^+$

$\ell_s(x) = -\ell(x) \iff x \in W_{af}^-$

(5) For any $x, y \in W_{af}$

$$\ell_s(xy) = \ell_s(y) + \sum_{\substack{\beta \in \Delta_+^{re} \\ x \cdot \beta < 0}} \operatorname{sign}(\overline{y^{-1} \cdot \beta})$$

where
$$\operatorname{sign} \alpha = \begin{cases} 1 & \text{if } \alpha \in \bar{\Delta}_+ \\ -1 & \text{if } \alpha \in -\bar{\Delta}_+ \end{cases}$$

Recall

$\bar{\Delta}_+ =$ the set of roots of the finite $\mathfrak{g}$.

<u>Proof</u>: (1) and (2) are clear from the definition.

(3): Write $x = w t_h$. Then

$$\ell_s(x w_0) = \ell(w t_h w_0) = \ell(w w_0 t_{w_0 h})$$

$$= \ell(w w_0) + \langle 2\rho, w_0 h\rangle$$

$$= \ell(w_0) - \ell(w) - \langle 2\rho, h\rangle$$

$$= \ell(w_0) - \ell_s(x)$$

(4) We break the proof of (4) into a few parts: We first prove that $\ell_s(x) = \ell(x)$ for $x \in W_{af}^+$: Assume $x = w t_h \in W_{af}^+$. Then by the definition of $W_{af}^+$, if $\alpha + n\delta > 0$ is s.t.

$$x\cdot(\alpha + n\delta) = w\alpha + (n - \langle\alpha, h\rangle)\delta < 0$$

we must have $\alpha > 0$. Thus ( ~~[illegible]~~ )

if $w\alpha < 0$, then $n$ can only take values $0, 1, \cdots, \langle\alpha, h\rangle$

and if $w\alpha > 0$, then $n$ can only take values $0, 1, \cdots \langle\alpha, h\rangle - 1$.

do we need to know that the number $\langle\alpha, h\rangle \geq 0$ ? Easy to see that if $x = w t_h \in W_{af}^+$, we must have $\langle h, \alpha_i\rangle \geq -1$ for each $i \in I$

Thus the set

$$A = \{\alpha + n\delta > 0 : \ x\cdot(\alpha + n\delta) < 0\}$$

is contained in the set

$$B = \{\alpha + n\delta : \ \alpha > 0 \ \ w\alpha < 0 \ \ n = 0, 1, \cdots \langle\alpha, h\rangle\}$$
$$\cup \{\alpha + n\delta : \ \alpha > 0 \ \ w\alpha > 0 \ \ n = 0, 1, \cdots \langle\alpha, h\rangle - 1\}$$

Clearly $B \subset A$. Thus $A = B$. Hence

$$\ell(x) = \# B = \sum_{\substack{\alpha > 0 \\ w\alpha < 0}} (\langle\alpha, h\rangle + 1) + \sum_{\substack{\alpha > 0 \\ w\alpha > 0}} \langle\alpha, h\rangle$$

$$= \sum_{\alpha > 0} \langle\alpha, h\rangle + \sum_{\substack{\alpha > 0 \\ w\alpha < 0}} 1$$

$$= \langle 2\rho, h\rangle + \ell(w)$$

$$= \ell_s(x).$$

This shows

$$\ell_s(x) = \ell(x) \qquad \text{for } x \in W_{af}^+.$$

We have proved (Lecture 8) that

$$-\ell_s(x) = \ell(x) \qquad \text{for } x \in W_{af}^-.$$

To prove that

$$\ell_s(x) \leq \ell(x) \qquad \text{for all } x \in W_{af}$$

we need the following Lemma:

<u>Lemma</u>: Suppose that $h_1 \in Q^\vee$ is dominant and regular. Then for all $x \in W_{af}$, we have

$$\ell(x t_{h_1}) \leq \ell(x) + \langle 2\rho, h_1\rangle$$

We will prove the Lemma later. Let's assume the Len

for now. Let $x \in W_{af}$ be arbitrary. Let $h_1$ be suffici dominant so that $x t_{h_1} \in W_{af}^+$. Then, we have

$$\begin{aligned} \ell_s(x) &= \ell_s(x t_{h_1}) - \langle 2\rho, h_1 \rangle \\ &= \ell(x t_{h_1}) - \langle 2\rho, h_1 \rangle \\ &\leq \ell(x) + \langle 2\rho, h_1 \rangle - \langle 2\rho, h_1 \rangle \qquad (\text{Lemma}) \\ &= \ell(x). \end{aligned}$$

This shows that $\ell_s(x) \leq \ell(x)$ for all $x \in W_{af}$.

Now if $\ell(x) = \ell_s(x) = \ell(\omega) + \langle 2\rho, h \rangle$ for $x = \omega t_h \in W_{af}$, then since the set

$$B = \{\alpha + n\delta : \ \alpha > 0, \ \omega\alpha < 0, \ n = 0, 1, \cdots \langle \alpha, h \rangle\} \ \cup$$
$$\{\alpha + n\delta : \ \alpha > 0 \ \ \omega\alpha < 0 \ \ n = 0, 1, \cdots \langle \alpha, h \rangle - 1\}$$

? ( if $\langle \alpha, h \rangle < 0$, then the first set in the union is taken to be $\phi$. Similarly for the 2nd set ) is obviously contained in the set

$$A = \{ \alpha + n\delta > 0 : \quad x \cdot (\alpha + n\delta) < 0 \}$$

But $\# B = \ell(\omega) + \langle 2\rho, h \rangle \implies B = A$. So for every $\beta \in \Delta_+^{re} \in A$ have $\bar{\beta} > 0$. This shows that $x \in W_{af}^+$.

Similarly we can show $\ell_s(x) \geq -\ell(x)$ $\forall x \in W_{af}$ and $\ell_s(x) = -\ell(x) \iff x \in W_{af}^-$. The finishes the proof of (4) (except for the lemma). (Something is not right here)



We now prove (5): $\forall x, y \in W_{af}$ — Do not trust the proof!

$$\ell_s(xy) = \ell_s(y) + \sum_{\substack{\beta \in \Delta_+^{re} \\ x \cdot \beta < 0}} \operatorname{sign}(\overline{y^{-1} \cdot \beta})$$

Write $x = \omega_1 t_{h_1}$, $y = \omega_2 t_{h_2}$. Then

$$\begin{aligned} \ell_s(xy) &= \ell_s(\omega_1 \omega_2 t_{\omega_2^{-1} h_1 + h_2}) \\ &= \ell(\omega_1 \omega_2) + \langle 2\rho, \omega_2^{-1} h_1 + h_2 \rangle \end{aligned}$$

So

$$\ell_s(xy) - \ell_s(y) = \ell(\omega_1 \omega_2) - \ell(\omega_2) + \langle 2\omega_2 \rho, h_1 \rangle$$

so need to show

$$\ell(\omega_1 \omega_2) - \ell(\omega_2) + \langle 2\omega_2 \rho, h_1 \rangle = \sum_{\substack{\beta \in \Delta_+^{re} \\ x \cdot \beta < 0}} \operatorname{sign}(\overline{y^{-1} \cdot \beta})$$

Notice the special case: $x = \omega_1$, $y = \omega_2$. we are saying

$$\ell(\omega_1 \omega_2) - \ell(\omega_2) = \sum_{\substack{\bar{\Delta}_1 \ni \beta > 0 \\ \omega_1 \cdot \beta < 0}} \operatorname{sign}(\omega_2^{-1} \cdot \beta)$$

This is a statement about the finite Weyl group and can be proved by induction on $\ell(\omega_2)$, for example. We assume this. Thus need to show

$$\langle 2\omega_2 \rho, h_1 \rangle = \sum_{\substack{\beta \in \Delta_+^{re} \\ x \cdot \beta < 0}} \operatorname{sign}(\overline{y^{-1} \cdot \beta}) - \sum_{\substack{\bar{\Delta}_1 \ni \alpha > 0 \\ \omega_1 \cdot \alpha < 0}} \operatorname{sign}(\omega_2^{-1} \cdot \alpha)$$

let

$$A = \{\beta = \alpha + n\delta > 0 : \quad x\cdot\beta < 0\}$$

$$= \{\beta = \alpha + n\delta > 0 : \quad \omega_1\alpha + (n - \langle\alpha, h_1\rangle)\delta < 0\}$$

For $\beta = \alpha + n\delta \in A$, have

$$y^{-1}.\beta = \omega_1^{-1}\alpha + (n + \langle\omega_1^{-1}\alpha, h_1\rangle)\delta$$

So

$$\overline{y^{-1}.\beta} = \omega_1^{-1}\alpha$$

Break $A$ as a disjoint union

$$A = A_1 \cup A_2 \cup A_3 \cup A_4$$

where

$$A_1 = \{\beta = \alpha + n\delta > 0 : \quad \alpha > 0 \quad \omega_1\alpha > 0, \ \omega_1\alpha + (n - \langle\alpha, h_1\rangle)\delta < 0\}$$

$$A_2 = \{ \cdots \quad \cdots : \quad \alpha > 0 \quad \omega_1\alpha < 0 \quad \cdots \}$$

$$A_3 = \{ \cdots \quad \cdots : \quad \alpha < 0 \quad \omega_1\alpha > 0 \quad \cdots \}$$

$$A_4 = \{ \cdots \quad \cdots : \quad \alpha < 0 \quad \omega_1\alpha < 0 \quad \cdots \}$$

So

$$A_1 = \{\beta = \alpha + n\delta > 0 : \ \alpha > 0, \ \omega_1\alpha > 0, \ n = 0, 1, \cdots \langle\alpha, h_1\rangle - 1\}$$

$$A_2 = \{\beta = \alpha + n\delta > 0 : \ \alpha > 0, \ \omega_1\alpha < 0, \ n = 0, 1, \cdots \langle\alpha, h_1\rangle\}$$

$$A_3 = \{\beta = \alpha + n\delta > 0 : \ \alpha < 0, \ \omega_1\alpha > 0, \ n = 1, \cdots, \langle\alpha, h_1\rangle - 1\}$$

$$A_4 = \{\beta = \alpha + n\delta > 0 : \ \alpha < 0 \ \ \omega_1\alpha < 0 \ \ n = 1, \cdots, \langle\alpha, h_1\rangle\}$$



Note that

$$\sum_{\alpha \in \overline{\Delta}_+} \mathrm{sign}(\omega_1^{-1}\alpha) = \sum_{\substack{\alpha > 0 \\ \omega_1^{-1}\alpha > 0}} 1 + \sum_{\substack{\alpha > 0 \\ \omega_1^{-1}\alpha < 0}} (-1) = 2\rho - 2(\rho - \omega_1\rho) = 2\omega_1\rho$$

Similarly,

$$\sum_{\substack{\beta \in \Delta_+^{re} \\ x\cdot\beta < 0}} \mathrm{sign}(\overline{y^{-1}.\beta}) = \sum_{\beta \in A_1 \cup A_2 \cup \cancel{A_3} \cup A_4} \mathrm{sign}(\overline{y^{-1}\beta})$$

So

$$\sum_{\substack{\beta \in \Delta_+^{re} \\ x\cdot\beta < 0}} \mathrm{sign}(\overline{y^{-1}\beta}) = \sum_{\substack{\alpha \in \overline{\Delta}_+ \\ \omega_1\alpha < 0}} \mathrm{sign}(\omega_1^{-1}\alpha) = \sum_{\beta \in A_1 \cancel{\cup A_3}} \mathrm{sign}(\overline{y^{-1}\beta})$$

$$= \langle 2\omega_1\rho, h_1 \rangle$$

This shows (5). (This is not a good proof. Need to come back may not even be correct

This proves the Proposition except for the Lemma.

<u>Lemma</u> Suppose that $h_1 \in Q^\vee$ is dominant and regular. Then for all $x \in W_{af}$, we have

$$\ell(x t_{h_1}) \leq \ell(x) + \langle 2\rho, h_1 \rangle$$

shorter proof: $\ell(x t_{h_1}) = \ell(x) + \cdots$

$= \ell(x) + \cdots$

<u>Proof</u> Set

$$A_* = \{\beta \in \Delta_+^{re}, \ \cancel{\beta = \alpha + n\delta > 0} : \quad x\cdot\beta < 0\}$$

(same ... domain ...

$$A_1 = \{\nu \in \Delta_+^{re} : \quad x t_{h_1} \nu < 0\}$$

$$= \{\nu = \alpha + n\delta > 0 : \quad x \cdot t_{h_1} \cdot (\alpha + n\delta) < 0\}$$

$$A_1=\{\alpha+n\delta>0 : \quad x t_{h_1}\cdot(\alpha+n\delta)<0\}$$
$$=\{\alpha+n\delta>0 : \quad x\cdot(\alpha+(n-\langle h_1,\alpha\rangle)\delta)<0\}$$

Write $A_1$ as

$$A_1 = B_1 \cup B_2$$

where:

$$B_1= A_1 \cap \{\alpha+n\delta : \quad \alpha+(n-\langle\alpha,h_1\rangle)\delta>0\}$$
$$B_2= A_1 \cap \{\alpha+n\delta : \quad \alpha+(n-\langle\alpha,h_1\rangle)\delta<0\}$$

The map

$$B_1 \longrightarrow A : \quad \alpha+n\delta \longmapsto \alpha+(n-\langle\alpha,h_1\rangle)\delta$$ Not necessary!

is injective : indeed, if

$$\alpha+(n-\langle\alpha,h_1\rangle)\delta = \alpha'+(n'-\langle\alpha',h_1\rangle)\delta$$
$$\Rightarrow \quad \alpha=\alpha' \quad \text{and} \quad n-\langle\alpha,h_1\rangle = n'-\langle\alpha',h_1\rangle$$
$$\Rightarrow \quad \alpha=\alpha', \quad n=n'. \qquad \text{Hence } \# B_1 \leq \# A = \ell(x)$$

Define ~~a map~~ the inclusion map

$$B_2 \longrightarrow C = \{\alpha+n\delta>0 : \quad \alpha+n\delta-\langle\alpha,h_1\rangle\delta<0\}$$
$$=\{\alpha+n\delta>0 : \quad \alpha>0 \ , n=0,1,\cdots,\langle\alpha,h_1\rangle-1\}$$

It is clearly that $\quad \# C = \sum_\alpha \langle\alpha,h_1\rangle = \langle 2\rho,h_1\rangle$

$$\Rightarrow \quad \# B_2 \leq \# C = \langle 2\rho, h_1\rangle$$



Hence

$$\ell(x t_{h_1}) = \# A = \# B_1 + \# B_2 \leq \# A + \# C = \ell(x) + \langle 2\rho, h_1\rangle$$

In the next proposition, we collect some facts about $\overset{s}{\leq}$ :

<u>Proposition $\overset{s}{\leq}$</u>

(1) For $x,y\in W_{af}$, we have

$$x \overset{s}{\leq} y \iff x t_h \leq y t_h \quad \text{for sufficiently dominant } h$$
$$\iff y t_{-h} \leq x t_{-h} \quad \text{" " "}$$
$$\iff x t_h \overset{s}{\leq} y t_h \quad \text{for all } h$$
$$\iff y w_0 \overset{s}{\leq} x w_0 \quad \text{where } w_0 = \text{the longest in}$$

(2) For $z\in W_{af}^+$, we have

(2a) $\quad x\leq z \Rightarrow x \overset{s}{\leq} z$

(2b) $\quad z\overset{s}{\leq} y \Rightarrow z \leq y$

(3) For $z\in W_{af}^-$, we have

(3a) $\quad x\leq z \Rightarrow z \overset{s}{\leq} x$

(3b) $\quad y\overset{s}{\leq} z \Rightarrow z \leq y$

(4) For $x,y\in W_{af}^+$, $\quad x\overset{s}{\leq} y \iff x\leq y$

For $x,y\in W_{af}^-$, $\quad x\overset{s}{\leq} y \iff y\leq x$

∎

Proof: (1) Only need to prove that

$$x \overset{s}{\le} y \iff y t_{-h} \le x t_{-h} \quad \text{for sufficiently dominant } h$$
$$\iff y w_0 \overset{s}{\le} x w_0$$

lemma 1: If $x, y \in W_{af}$, then $x \le y \iff x w_0 \le y w_0$

lemma 2: $x \overset{s}{\le} y \iff w_0 y \le w_0 x$

Proof of lemma 2: If $\phi \in M^{w_1, w_2}_{G/B,\ \pi_0(h_1 - h_2)}$, than $\phi_1$ defined by

$$\phi_1(t) := \phi(\tfrac{1}{t}) w_0 \cdot B$$

is in $M^{w_0 w_2,\ w_0 w_1}_{G/B,\ \pi_0(h_1 - h_2)}$. This shows $x \overset{s}{\le} y \iff w_0 y \le w_0 x$.

I can not prove (1).



Proposition 1: Suppose that $h \in Q^\vee$. Then

$$h \in Q^\vee_+ := \sum_{i \in I} \mathbb{Z}_+ \alpha_i^\vee \iff id \overset{s}{\le} w t_h \quad \forall w \in W$$
$$\iff w \overset{s}{\le} w_0 t_h \quad \forall w \in W$$
$$\iff x \overset{s}{\le} x t_h \quad \forall x \in W_{af}$$

Proposition 2: For $\beta \in \Delta^{re}_+$ and $x \in W_{af}$

$$r_\beta x \overset{s}{<} x \iff \ell_s(r_\beta x) < \ell_s(x)$$
$$\iff \overline{x^{-1}\beta} < 0$$
$$x \overset{s}{<} r_\beta x \iff \ell_s(x) < \ell_s(r_\beta x)$$
$$\iff \overline{x^{-1} \cdot \beta} > 0$$

Proposition 3: For $\beta \in \Delta^{re}_+$ and $\overline{\beta} > 0$, and $x \in W_{af}$

$$x r_\beta \overset{s}{<} x \iff \ell_s(x r_\beta) < \ell_s(x)$$
$$\iff x \cdot \beta < 0$$
$$x \overset{s}{<} x r_\beta \iff \ell_s(x) < \ell_s(x r_\beta)$$
$$\iff x \cdot \beta > 0$$

<u>Proposition 4</u>: $x \overset{s}{<} y \Rightarrow \ell_s(x) < \ell_s(y)$

<u>Proposition 5</u>: If $x \overset{s}{<} y$, then there exists a sequence of the form

$$x = x_0 \overset{s}{<} x_1 \overset{s}{<} x_2 \overset{s}{<} \cdots \overset{s}{<} x_n = y$$

with $n \geq 0$ and $\ell(x_k) = \ell_s(x) + k$ for $0 \leq k \leq n$.

<u>Proposition 6</u>: The following are equivalent: For $w \in W$ and $x \in W_{af}$.

(a) $w \overset{s}{<} x$ and $\ell_s(x) = \ell(w) + 1$

(b) $x$ is one of the following 2 cases:

① $x = w r_\alpha$, $\alpha \in \bar{\Delta}_+$ and $\ell(x) = \ell(w) + 1$

② $x = w r_\alpha t_{\alpha^\vee} = w r_{\alpha+\delta}$, where $\alpha \in \bar{\Delta}_+$ and

$$\ell(x) = \ell(w) - \langle 2\rho, \alpha^\vee \rangle + 1$$

This is related to multiplication by $H^2$. in the quantum cohomology.



We now turn to $\underline{(W_P)_{af}}$ and $\underline{(W^P)_{af}}$:

Fix a standard parabolic subgroup $P$ of $G$. Let

$$\Delta_+(P) = \{ \alpha \in \bar{\Delta}_+ : \mathfrak{g}_{-\alpha} \in P \}$$

$$Q_P^\vee = \sum_{\alpha \in \Delta_+(P)} \mathbb{Z}\alpha^\vee$$

Set

$$(W_P)_{af} = \{ w t_h : w \in W_P, \ h \in Q_P^\vee \}$$

This is the Weyl group of ~~Levi-factor~~ $\tilde{L}_P$, ~~of~~ where $L_P$ is the Levi-factor of $P$.

<u>Examples</u>: 1) $P = B$. $W_P = id$ $(W_P)_{af} = id$

2) $P = G$ $W_P = W$ $(W_P)_{af} = W_{af}$

3) $P = P_i$. $W_P = \langle 1, r_i \rangle$,

$$(W_P)_{af} = \langle r_{\alpha_i}, r_{\delta - \alpha_i} \rangle$$

In general, $(W_P)_{af}$ is a Coxeter group; It is a subgroup of $W_{af}$, but not a Coxeter subgroup, as seen in the example of $P = P_i$.

**The length function $\ell_P(y)$:**

As a Coxeter group, $(W_P)_{af}$ has a length function

$$\ell_P(y) = \#\{\beta > 0 : \bar{\beta}^\vee \in Q_P^\vee, \ y\cdot\beta < 0\}$$

**Define** $(W^P)_{af}$ :

$$(W^P)_{af} = \{x \in W_{af} : \beta > 0 \ \ \bar{\beta}^\vee \in Q_P^\vee \ \Rightarrow \ x\cdot\beta > 0\}$$

**Proposition** :

$$W_{af} = (W^P)_{af} \ (W_P)_{af}$$

ie. each $z \in W_{af}$ can be uniquely written as a product

$$z = x\, y$$

where

$$x \in (W^P)_{af}, \quad y \in (W_P)_{af}$$

**Define** $\hat{\pi}_P : W_{af} \longrightarrow (W^P)_{af}, \ z \longmapsto x$

The next proposition gives various properties of $\hat{\pi}_P$ :

(Note: Our $\hat{\pi}_P$ is what Peterson calls $\pi_P$ in class)



**Proposition $\hat{\pi}_P$**

1) $\hat{\pi}_P(W) = W^P \subset (W^P)_{af} \subset (W_{af})^P$

where $(W_{af})^P$ is the set of minimal representatives for $W_{af}/W_P$

2) $\hat{\pi}_P(W_{af}^-) \subset W_{af}^-$

3) $\hat{\pi}_P(z) \leq z$ for all $z \in W_{af}$

4) For any $z, z' \in W_{af}$, $h \in Q^\vee$, have

- $\hat{\pi}_P(z t_h) = \hat{\pi}_P(z)\, \hat{\pi}_P(t_h)$
- $\ell_s(\hat{\pi}_P(z t_h)) = \ell_s(\hat{\pi}_P(z)) + \langle C_P, h \rangle$

where

$$C_P = \rho + w_P\rho = \sum_{\substack{\alpha \in \bar{\Delta}_+ \\ w_0 w_P \cdot \alpha < 0}} \alpha \qquad (w_P = \text{longest in } W$$

- $z \overset{s}{\leq} z' \ \Rightarrow \ \hat{\pi}_P(z) \overset{s}{\leq} \hat{\pi}_P(z')$
- $\hat{\pi}_P(r_\beta z) < \hat{\pi}_P(z) \iff \overline{z^{-1}\cdot\beta} \in \Delta(\mathfrak{g}/\mathfrak{p}) \ \ (\subset -\bar{\Delta}_+)$
- $\hat{\pi}_P(r_\beta z) = \hat{\pi}_P(z) \iff \overline{z^{-1}\cdot\beta} \in Q_P^+$
- $\hat{\pi}_P(r_\beta z) > \hat{\pi}_P(z) \iff \overline{z^{-1}\cdot\beta} \in -\Delta(\mathfrak{g}/\mathfrak{p}) \ \ (\subset \bar{\Delta}_+$

**Proposition:** For $y \in (W_P)_{af}$.

$$\ell_{s,P}(y) = \ell_s(y)$$

where $\ell_{s,P}$ is the stable length function for $(W_P)_{af}$

**Proposition:** For $x \in (W^P)_{af}$, $y \in (W_P)_{af}$.

$$\ell_s(xy) = \ell_s(x) + \ell_s(y)$$

$$\ell(x) + \ell_s(y) \leq \ell(xy) \qquad ?$$

**Proposition:** Any given $x \in (W^P)_{af}$, can put

$$x t_h \in W_{af}^+ , \quad x t_{-h} \in W_{af}^-$$

for sufficiently dominant $h \in (Q^\vee)^{W_P}$, ie.

$\langle \rho_i, h \rangle \gg 0$ for all $i \in I$ such that $r_i \notin W_P$.



**Notation:**

$(\tilde{P})_0$ = the identity component of $\tilde{P}$

$$\mathcal{M}_P = \tilde{G}/(\tilde{P})_0$$

$$*_P = (\tilde{P})_0 \in \mathcal{M}_P$$

$$\mathring{\pi}_P : \mathcal{M}_B \longrightarrow \mathcal{M}_P : g *_B \mapsto g \cdot *_P$$

Have action of $\Gamma$ on $\mathcal{M}_P$:

$$\mathcal{M}_P \times \Gamma \to \mathcal{M}_P : \quad (g \cdot *_P) \cdot t = g t \cdot *_P$$

This action is trivial if $t \in \{t_h : h \in Q_P^\vee\}$.

Set, for $z \in W_{af}$,

$$\mathcal{M}_{P,z}^{\pm} = B_{af}^{\pm} z \cdot *_P$$

**Proposition:** For $z \in W_{af}$ and $t \in \Gamma$

$$\mathcal{M}_{P,z}^{\pm} = \mathcal{M}_{P,\hat{\pi}_P(z)}^{\pm}$$

$$(\mathcal{M}_{P,z}^{\pm}) \cdot t = \mathcal{M}_{P,zt}^{\pm} .$$

and for $x_1, x_2 \in (W^P)_{af}$,

$$\mathcal{M}_{P,x_1}^{-} \cap \mathcal{M}_{P,x_2}^{+} \neq \phi \iff x_1 \overset{s}{\leq} x_2$$

## The moduli space $\mathcal{M}_\tau = \mathcal{M}_{\tau,P}$ :

**Definition**: Given a scheme $V/\mathbb{C}$ and a morphism

$$f: V \times_{\mathbb{C}} \mathbb{P}^1 \longrightarrow G/P.$$

we say that $f$ is <u>of type $\tau$</u>, for $\tau \in H_2(G/P)$, if for any $\mathbb{C}$-valued point $v$ of $V$, the map $f_v : \mathbb{P}^1 \to G/P$ defined by

$$\mathbb{P}^1 = \mathbb{C} \times_{\mathbb{C}} \mathbb{P}^1 \xrightarrow{v \times id} V \times_{\mathbb{C}} \mathbb{P}^1 \xrightarrow{f} G/P$$

satisfies

$$(f_v)_* [\mathbb{P}^1] = \tau.$$

The universal property of $(\mathcal{M}_\tau, ev)$ :

**Proposition**: Fix $\tau \in H_2(G/P)$. There exists a pair $(\mathcal{M}_\tau, ev)$ where $\mathcal{M}_\tau$ is a reduced scheme of finite type over $\mathbb{C}$ and $ev: \mathcal{M}_\tau \times_{\mathbb{C}} \mathbb{P}^1 \to G/P$ is a morphism over $\mathbb{C}$ s.t.

1) $ev$ is of type $\tau$ ;

2) if $V$ is any reduced scheme of finite type over $\mathbb{C}$ and $f: V \times_{\mathbb{C}} \mathbb{P}^1 \to G/P$ is a morphism over $\mathbb{C}$, then $\exists !$ morphism $\hat{f} : V \to \mathcal{M}_\tau$ over $\mathbb{C}$ s.t.

$$f = ev \circ (\hat{f} \times id) .$$



Thus $(\mathcal{M}_\tau, ev)$ is unique up to a unique isomorphism. Moreover, $\mathcal{M}_\tau$ is a quasi-projective, and it is either empty or else smooth and of dim

$$\dim \mathcal{M}_\tau = \dim G/P + \langle c_1 T_{G/P}, \tau \rangle$$

Here we outline a proof of the fact that the Zariski tangent space to $\mathcal{M}_\tau$ at $\phi \in \mathcal{M}_\tau$ always has the above dimension: Suppose

$$\phi : \mathbb{P}^1 \to G/P$$

is s.t. $\phi_* [\mathbb{P}^1] = \tau$. Then

$$T_\phi \mathcal{M}_\tau = \Gamma(\mathbb{P}^1, \phi^* T_{G/P})$$

Now as sheaves over $G/B$, we have

$$0 \to \mathfrak{a} \longrightarrow \mathfrak{b} \longrightarrow T_{G/P} \longrightarrow 0$$

where $\mathfrak{b}$ can be taken as the trivial sheaf of sections of the trivial vector bundle defined by $\mathfrak{g}$, and $\mathfrak{a}$ is the kernel sheaf. Pulling back to $\mathbb{P}^1$ by $\phi$, we have

$$0 \to \phi^*\mathfrak{a} \to \phi^*\mathfrak{b} \to \phi^* T_{G/P} \to 0$$

Thus we have the long exact sequence

$$0 \to H^0(\mathbb{P}^1, \phi^*\mathfrak{a}) \to H^0(\mathbb{P}^1, \phi^*\mathfrak{b}) \to H^0(\mathbb{P}^1, \phi^* T_{G/P}) \to$$

$$\to H^1(\mathbb{P}^1, \phi^*\mathfrak{a}) \to H^1(\mathbb{P}^1, \phi^*\mathfrak{b}) \to H^1(\mathbb{P}^1, \phi^* T_{G/P}) \to 0$$

Since $\mathfrak{b}$ is trivial as a vector bundle, have

$$H^1(\mathbb{P}^1, \phi^*\mathfrak{b}) = 0$$

$$\Rightarrow \quad H^1(\mathbb{P}^1, \phi^* T_{G/P}) = 0$$

$$\Rightarrow \quad \dim \Gamma(\mathbb{P}^1, \phi^* T_{G/P}) = \dim H^0(\mathbb{P}^1, \phi^* T_{G/P})$$

$$= \chi(\phi^* T_{G/P})$$

Using the general fact that for any vector bundle $E$ over $\mathbb{P}^1$,

$$\chi(E) = \dim E + \langle c_1(E), [\mathbb{P}^1] \rangle$$

We get

$$\dim \Gamma(\mathbb{P}^1, \phi^* T_{G/P}) = \dim G/P + \langle c_1(\phi^* T_{G/P}), [\mathbb{P}^1] \rangle$$

$$= \dim G/P + \langle \phi^* c_1(T_{G/P}), [\mathbb{P}^1] \rangle$$

$$= \dim G/P + \langle c_1(T_{G/P}), \phi_*[\mathbb{P}^1] \rangle$$

$$= \dim G/P + \langle c_1(T_{G/P}), \tau \rangle$$

//



Now for $\tau \in H_2(G/P)$, $v, w \in W^P$, set

$$\mathcal{M}_\tau^{v,w} = B_- v \cdot P \times_{G/P} \mathcal{M}_\tau \times_{G/P} B w \cdot P$$

By a theorem of Kleiman, we have

<u>Proposition</u>:

(1) $\mathcal{M}_\tau^{id, w_0 w_P}$ is open and dense in $\mathcal{M}_\tau$

(2) $\mathcal{M}_\tau^{v,w}$ is quasi-projective, and

$$\dim \mathcal{M}_\tau^{v,w} = \langle c_1(T_{G/P}), \tau \rangle - \ell(v) + \ell(w).$$

<u>Kleiman's Theorem</u>: Suppose $X$ is a homogeneous $G$-space and

$$\sigma_Y : Y \to X$$
$$\sigma_Z : Z \to X$$

are smooth maps. Then for generic $g_1, g_2 \in G$, the set

$$g_1 \cdot Y \times_X g_2 \cdot Z = \{ (y, z) : g_1 \cdot \sigma_Y(y) = g_2 \, \sigma_Z(z) \}$$

is a regular reduced variety of $\dim = \dim Y + \dim Z - \dim X$

End of Lecture 13

## Lecture 14, Tuesday, April 15, 1997

Today we introduce two rings for each parabolic $P$:

1. $R_P' = \mathfrak{g}H^T(G/P)_{(\mathfrak{g})}$ : $T$-equivariant quantum cohomology of $G/P$ with the quantum parameter $\mathfrak{g}$ inverted

2. $R_P = \mathfrak{g}H^T(G/P)$ : $T$-equivariant quantum cohomology of $G/P$.

Definition: $R_P'$ is a free $S$-module on symbols $\sigma_P^{(x)}$, $x \in (W^P)_{af}$ with $\mathbb{Z}$-grading

$$\deg(s\,\sigma_P^{(x)}) = \deg s + 2\,\ell_s(x)$$

<u>The $\Delta_{af}$ module structure on $R_P'$</u>

The $S$-module structure on $R_P'$ extends to an $\Delta_{af}$-module structure on $R_P'$ by

$$\nu(A_i)\cdot\sigma_P^{(x)} = \begin{cases} -\sigma_P^{(r_i x)} & \text{if } \overline{x^{-1}\cdot\alpha_i} \in \Delta(\mathfrak{g}/\mathfrak{p}) \\ 0 & \text{otherwise} \end{cases}$$

where $\nu$ is the automorphism of $\Delta_{af}$ defined at the end of lecture 10 (page 10-13).



<u>The map $\Psi_P : H_T(\Omega K) \longrightarrow R_P'$</u>:

It is the $S$-module map defined by

$$\Psi_P(\sigma^{\Omega}_{[x]}) = \begin{cases} \sigma_P^{(x)} & \text{if } x \in (W^P)_{af} \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in W_{af}^-$

It should be easy to check that

1) $\Psi_P(\sigma) = j(\sigma)\cdot\sigma_P^{(id)} \in R_P' \qquad \forall\, \sigma \in H_T(\Omega K)$

2) $\Psi_P$ is an $\Delta_{af}$ - map.

Theorem: There exists a unique commutative $S$-algebra structure on $R_P'$ such that

1) $\sigma_P^{(id)} = 1$

2) $R_P'$ is an $\Omega$-integrable $\Delta_{af}$-module with the structure homomorphism $S \longrightarrow R_P'$ : $s \longmapsto s\,\sigma_P^{(id)}$ and the $\Delta_{af}$-module structure defined above.

The definition of an $\Omega$-integrable $\mathcal{A}_{\mathcal{f}}$-module is given in Lecture 10. Recall that a proposition in Lecture 10 says that an $\Omega$-integrable $\mathcal{A}_{\mathcal{f}}$-module is equivalent to an affine scheme $X$ over $\underline{h} = \operatorname{spec} S$ with a structure morphism $\pi_x : S \to \mathcal{O}(x)$ and

1) an $\mathcal{A}$-module structure on $\mathcal{O}(x)$

2) an $S$-map $f: H_T(\Omega K) \to \mathcal{O}(x)$

such that

1) $s \cdot p = \pi_x(s) p \qquad \forall\ s \in S \quad p \in \mathcal{O}(x)$

2) $\pi_x$ is an $\mathcal{A}$-module map

4) $m: \mathcal{O}(x) \otimes_S \mathcal{O}(x) \to \mathcal{O}(x)$ is an $\mathcal{A}$-module map

5) $f: H_T(\Omega K) \to \mathcal{O}(x)$ is an $\mathcal{A}$-module map

Recall that we have used the notation

$$\mathcal{U} = \operatorname{spec} H^T(K/T)$$

$$\mathcal{Q} = \operatorname{spec} H_T(\Omega K).$$

Conditions 1)-4) say that $X = \operatorname{spec} \mathcal{O}(x)$ is a $\mathcal{U}$-space, where $\mathcal{U}$ is a groupoid, and condition 5) says that $\operatorname{spec} f : X \to \mathcal{Q}$ is a $\mathcal{U}$-space morphism.



## The geometrical models

The following is from Dale's lecture at Kac's seminar on April 18, 1997.

- $\mathcal{U} = \operatorname{spec} H^T(K/T)$:

$$G^\vee,\ \mathfrak{g}^\vee,\ \underline{h} = (\underline{h}^\vee)^* \subset (\mathfrak{g}^\vee)^* \qquad \text{etc.}$$

For each $i \in I$, let $f_i^{\vee *} \in (\mathfrak{g}^\vee)^*$ be such that $\langle f_i^{\vee *}, f_j^\vee \rangle = \delta_{ij}$ and with weight $\alpha_i^\vee$. Let

$$E = \sum_{i \in I} f_i^{\vee *} \in (\mathfrak{g}^\vee)^*$$

Set

$$\mathcal{U} = \{ (E+h, u) \in (E+\underline{h}) \times U_-^\vee : \ u^{-1} \cdot (E+h) \in (\underline{n}_+^\vee)^\perp$$

Notice that $\mathcal{U}$ can be identified with the following subset of $(E+\underline{h}) \times U_- \times (E+\underline{h})$:

$$\mathcal{U} = \{ (E+h_1,\ u,\ E+h_2) : \quad u^{-1} \cdot (E+h_1) = E + h_2 \}$$

It thus has a groupoid structure as a subgroupoid of the direct product groupoid $(E+\underline{h}) \times U_- \times (E+\underline{h})$:

- $\mathcal{A} = \operatorname{Spec} H_T(\Omega K) = \mathcal{B}^{\vee\, E+\mathfrak{h}}$

$$= \{ (E+h, b) \in (E+\mathfrak{h}) \times \mathcal{B}^\vee : \ b\cdot(E+h) = E+h \}$$

- Action of $\mathcal{U}$ on $\mathcal{B}^{\vee\, E+\mathfrak{h}}$:

$$\mathcal{U} \times_{\mathfrak{h}} \mathcal{B}^{\vee E+\mathfrak{h}} \longrightarrow \mathcal{B}^{\vee E+\mathfrak{h}}:$$

$$(E+h,\ u,\ E+h') \cdot (E+h',\ b) = (E+h,\ ubu^{-1})$$

- The variety $Y^{E+\mathfrak{h}}$:

$$Y^{E+\mathfrak{h}} = \{ (E+h,\ g\cdot B^\vee) \in (E+\mathfrak{h}) \times G^\vee/B^\vee : \ g^{-1}\cdot(E+h) \perp [\mathfrak{n}_+^\vee, \mathfrak{n}_+^\vee] \}$$

Have

$$\mathcal{B}^{\vee E+\mathfrak{h}} \longrightarrow Y^{E+\mathfrak{h}}:$$

$$(E+h, b) \longmapsto (E+h,\ b w_0 \cdot B^\vee)$$

$\mathcal{U}$ acts on $Y^{E+\mathfrak{h}}$:

$$\mathcal{U} \times_{\mathfrak{h}} Y^{E+\mathfrak{h}} \longrightarrow Y^{E+\mathfrak{h}}: \quad ($$

$$(u+h,\ u,\ E+h') \cdot (E+h',\ gB^\vee) = (E+h,\ ug\cdot B^\vee)$$

The inclusion $\quad \mathcal{B}^{\vee E+\mathfrak{h}} \longrightarrow Y^{E+\mathfrak{h}}$

is a $\mathcal{U}$-equivariant.

$$\cancel{Y_P^{E+\mathfrak{h}}} \cong \operatorname{Spec} R_P' = Y_P^{+E+\mathfrak{h}} \cap Y_G^{-E+\mathfrak{h}} \subset Y^{E+h} \quad (\mathcal{U}\text{-subset})$$



<u>The subring $\Lambda_P' \subset R_P'$</u>:

For $h \in Q^\vee$, so $\pi_P(h) \in H_2(G/P)$, set

$$q_{\pi_P(h)} = \sigma_P^{(\hat\pi_P(t_h))} \in R_P',$$

and

$$\Lambda_P' = \mathbb{Z}\{ q_{\pi_P(h)} : h \in Q^\vee \} = \mathbb{Z}[H_2(G/P)]$$

<u>Fact</u>:

- $\Lambda_P' \subset R_P'$ is a subring with

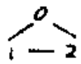
$$q_z\, q_{z'} = q_{z+z'}$$

$$\deg q_z = 2 \langle c_1 T_{G/P},\ z \rangle$$

- $q_{\pi_P(h)} \cdot \sigma_P^{(x)} = \sigma_P^{(x\, \hat\pi_P(t_h))} = \sigma_P^{(\hat\pi_P(x t_h))}$

- The $\mathbb{A}_{af}$-module structure on $R_P'$ is $\Lambda_P'$-linear.

<u>Example</u> ($\hat\pi_P(t_h)$ is not necessarily translational):

$sl_3$ with extended Dynkin diagram . Let $W_P = \langle r_1 \rangle$

and $\quad t = t_\theta = r_0 r_\theta = \underbrace{r_0 r_2 r_1}_{(\hat W^P)_{af}} \underbrace{r_2}_{\in (W_P)_{af}}$

$\Rightarrow \hat\pi_P(t) = r_0 r_2 r_1.$ //

ct: $\{\sigma_p^{(w)} : w \in W^P\}$ is a basis of $R_p'$ over $S \times \Lambda_p'$

oposition: Formulas for multiplications* in $R_p'$, and $A_{af}$ on $R_p'$:

$$A_i \cdot (s\sigma) = (A_i \cdot s)\sigma + (r_i \cdot s)(A_i \cdot \sigma)$$

$$A_i \cdot \sigma_p^{(w)} = \begin{cases} -\sigma_p^{(r_i w)} & \text{if } w^{-1}\alpha_i \in \Delta(\mathfrak{g}/\mathfrak{p}) \\ 0 & \text{otherwise} \end{cases}$$

$$(A_i \cdot \sigma) * \sigma' = A_i \cdot [\sigma * (r_i \cdot \sigma')] + \sigma * (A_i \cdot \sigma')$$

: operator $A_0'$

Assume that $G$ is simple. $\alpha_0 = \delta - \theta$ $\quad \Pi_{af} = \Pi \cup \{0\}$

$$A_0' = \nu(A_0) = -w_0 A_0 w_0$$

where $w_0 \in W$ is the longest element.

osition

$$A_0' \cdot (s\sigma) = -(A_{\theta^\vee} \cdot s)\sigma + (r_\theta \cdot s)(A_0' \cdot \sigma)$$

$$A_0' \cdot \sigma_p^{(w)} = \begin{cases} -\delta_{\pi_p(w^{-1}\theta^\vee)}^{-1} \, \sigma_p^{(\hat{\pi}_p(r_\theta w))} & \text{if } w^{-1}\theta \in \Delta(\mathfrak{g}/\mathfrak{p}) \\ 0 & \text{otherwise} \end{cases}$$

$$(A_0' \cdot \sigma) * \sigma' = A_0' \cdot (\sigma * (r_\theta \cdot \sigma')) - \sigma * (A_{\theta^\vee} \cdot \sigma')$$



**Theorem**: $\psi_p : H_*(\Omega K) \to R_p'$ is a homomorphism of $S$-algebras.

and $\quad \psi_p(\sigma) * \sigma' = j(\sigma) \cdot \sigma'$

for $\sigma \in H_*(\Omega K)$ and $\sigma' \in R_p'$

**Proof**: This is a direct consequence of $R_p'$ being $\Omega$-integrable. //

**The structure constants $J_{p,z}^{x,y}$, $x, y, z \in (W^P)_{af}$**

For $x, y, z \in (W^P)_{af}$, define structure constants $J_{p,z}^{x,y} \in S$ by

$$\sigma_p^{(x)} * \sigma_p^{(y)} = \sum_{z \in (W^P)_{af}} J_{p,z}^{x,y} \sigma_p^{(z)}$$

**Facts.**

(1) $\deg J_{p,z}^{x,y} = 2(\ell_S(x) + \ell_S(y) - \ell_S(z))$

(2) $J_{p,z}^{x,y} = J_{B,z}^{x,y}$

(3) $J_{p,\, z\pi_p(tt')}^{x\hat{\pi}_p(t),\, y\hat{\pi}_p(t')} = J_{p,z}^{x,y}$

(4) For $x, y, z \in (W^P)_{af}$ with $x \in W_{af}^-$.

$$J_{p,z}^{x,y} = \begin{cases} \epsilon(xyz)\, j_x^{zy^{-1}} & \text{if } \ell(yz^{-1}) + \ell_S(z) = \ell_S(y) \\ 0 & \text{otherwise} \end{cases}$$

**Multiplication by $H^2$ in $R_B'$:**

**Theorem:** For $i \in I$ and $w \in W$

$$\sigma_B^{(r_i)} * \sigma_B^{(w)} = \sum_{\substack{\alpha \in \bar{\Delta}_+ \\ \ell(w r_\alpha) = \ell(w)+1}} \langle \rho_i, \alpha^\vee \rangle \sigma_B^{(w r_\alpha)}$$

$$+ \sum_{\substack{\alpha \in \bar{\Delta}_+ \\ \ell(w r_\alpha) = \ell(w)+1-\langle 2\rho, \alpha^\vee \rangle}} \langle \rho_i, \alpha^\vee \rangle \, q_{\pi_B(\alpha^\vee)} \sigma_B^{(w r_\alpha)}$$

$$- (\rho_i - w \cdot \rho_i)\, \sigma_B^{(w)}$$

emark: One way of looking at the above formula is

$$\sigma_B^{(r_i)} + \rho_i = (\rho_i)_R + \sum_{\substack{\alpha \in \bar{\Delta}_+ \\ \ell(r_\alpha) = \langle 2\rho, \alpha^\vee \rangle - 1}} \langle \rho_i, \alpha^\vee \rangle \, q_{\pi_B(\alpha^\vee)} A_{r_\alpha},$$

where the left hand side is a multiplication operator on $R_B'$ and the right hand side is an element in $\underline{A}_{af}$ considered as an operator on $R_B'$. The right hand side is a commuting family of elements in $\underline{A}_{af}$.



**A fact with no classical analog:**

$$\underline{A}_{af} \otimes_{\Lambda_B^-} \Lambda_B' \cong \operatorname{End}_{[R_B']^W} R_B' \qquad \text{where } \Lambda_B^- = \sum_{\substack{h \in Q^\vee \\ h \text{ dominant}}} \mathbb{Z}\, q_h$$

$$[R_B']^W \cong \operatorname{End}_{\underline{A}_{af} \otimes_{\Lambda_B^-} \Lambda_B'} R_B'$$

$$\operatorname{End}_{\underline{A} \otimes \Lambda_B'} R_B' \cong \underline{A}_R \otimes \Lambda_B'$$

**The ring $R_P$**

Define
$$R_P = \sum_{\substack{x \in (W^P)_{af} \\ x \overset{s}{\geq} id}} S\, \sigma_P^{(x)}$$

(Recall that $x = w t_h \overset{s}{\geq} id \iff h \in Q_+^\vee$). It is clear from the way $\underline{A}$ acts that $R_P$ is an $\underline{A}$-stable submodule of $R_P'$

**Fact:** For $z \in W_{af}$,

$$\sum_{\substack{x \in (W^P)_{af} \\ x \overset{s}{\geq} z}} S\, \sigma_P^{(x)}$$

is an $R_P$-submodule of $R_P'$

let $\Lambda_P = \Lambda_P' \cap R_P = \sum_{\tau \in \pi_P(Q_+^\vee)} \mathbb{Z}\, q_\tau$

Then

$$R_P \otimes_{\Lambda_P} \Lambda_P' \cong R_P'$$

and $\{\sigma_P^{(w)} : w \in W^P\}$ is an $S \otimes \Lambda_P$-basis of $R_P$.

The augumentation homomorphism is defined to be

$$\varepsilon : \Lambda_P \to \mathbb{Z} : \quad \varepsilon(q_\tau) = \delta_{\tau,0}$$

act: The map

$$R_P \otimes_{\Lambda_P} \mathbb{Z} \xrightarrow{\sim} H^T(G/P)$$

$$\sigma_P^{(w)} \otimes 1 \longmapsto \sigma_P^{(w)}$$

is an isomorphism as $\underline{A}$-modules and $S$-algebras

Thus it is reasonable to call $R_P$ the $T$-equivariant quantum cohomology of $G/P$. It specializes to the $T$-equivariant cohomology of $G/P$ when the quantum parameters $\Lambda_P$ go to $0$.



Poincare Duality (compare with the non-quantum case treated in Lecture 7)

Define the $S \otimes \Lambda_P$-linear map

$$\int : R_P \to S \otimes \Lambda_P$$

by

$$\int \sigma_P^{(w)} = \delta_{w,\, w_0 w_P} \qquad \text{for } w \in W^P.$$

Theorem: $\int \sigma_P^{(v)} * \left(w_0 \cdot \sigma_P^{(w_0 w w_P)}\right) = \delta_{v,w}$

Corollary. Have an isomorphism

$$PD : R_P \cong \mathrm{Hom}_{S \otimes \Lambda_P}(R_P, S \otimes \Lambda_P)$$

defined by

$$PD(\varphi)(\varphi') = \int \varphi * \varphi'.$$

or concretely

$$PD\left(\sigma_P^{(w)}\right) = w_0 \cdot \sigma_P^{(w_0 w w_P)}$$

## The Euler Class $\chi_{G/P}$

$$\chi_{G/P} \stackrel{def}{=} PD^{-1}\left(tr_{R_P/S\otimes\Lambda_P}\right)$$

where

$$tr_{R_P/S\otimes\Lambda_P} \in Hom_{S\otimes\Lambda_P}(R_P, S\otimes\Lambda_P)$$

is defined by

$$tr_{R_P/S\otimes\Lambda_P}(\phi) = \text{trace over } S\otimes\Lambda_P \text{ of } \left(\ell_\phi : \phi' \mapsto \phi * \phi'\right)$$

In other words,

$$tr_{R_P/S\otimes\Lambda_P}(\phi) = \int \phi * \chi_{G/P}$$

Write

$$\sigma_P^{(v)} * \sigma_P^{(w)} = \sum_{u\in W^P} b_u^{v,w}\, \sigma_P^{(u)}$$

Then

$$tr_{R_P/S\otimes\Lambda_P}(\sigma_P^{(v)}) = \sum_{w\in W^P} b_w^{v,w}$$

$$= \sum_{w\in W^P} \int \sigma_P^{(v)} * \sigma_P^{(w)} * \left(\omega_0 \cdot \sigma_P^{(\omega_0 w \omega_P)}\right)$$

$$\Rightarrow \quad \chi_{G/P} = \sum_{w\in W^P} \sigma_P^{(w)} * \left(\omega_0 \cdot \sigma_P^{(\omega_0 w \omega_P)}\right)$$



## Facts

1) $\phi * \chi_{G/P} = 0 \iff \phi$ is nilpotent

2) $\chi_{G/P}$ annihilates $\Omega_{R_P/S\otimes\Lambda_P}$ ← who is this ?

Example: For $sl(3)$ and $P = B$. $\chi_{G/B}$ is invertible $\iff$ $\zeta_1\zeta_2(\zeta_1+\zeta_2)$ is invertible.

End of Lecture 14

# Lecture 15 Wed. April 16, 1997

ore facts on $R_B$:

ct 1: For $w \in W$,

$$\sum_{\substack{u,v \in W \\ uv = w \text{ (red)}}} \epsilon(u)\, \sigma_B^{(u^{-1})} * \sigma_B^{(v)} = \delta_{w,1}$$

$$\sum_{\substack{u,v \in W \\ uv = w \text{ (red)}}} \sigma_B^{(u)} * \epsilon(v)\, \sigma_B^{(v^{-1})} = \delta_{w,1}$$

Remark: Recall from Lecture 7 that similar identities hold for $H^T(K/T)$. They can now be considered as a corollary of this fact here about $\mathcal{G}H^T(K/T)$. Does this follow from any Hopf algebroid structure on $\mathcal{G}H^T(K/T)$?

act 2: For $\sigma \in R_B$,

$$\sigma = \sum_{w \in W} \left[ A_{w_0} \cdot \left( \sigma * \left( w_0 \cdot \sigma_B^{(w_0 w)} \right) \right) \right] * \epsilon(w)\, \sigma_B^{(w)}$$

hat does this mean? This is not expressing $\sigma$ in the basis $\{\epsilon(w)\, \sigma_B^{(w)} : w \in W\}$ of $R_B$ as an $S \otimes \Lambda_B$-module.

Fact 3: $R_B$ is a free $(R_B)^\Delta$-module with basis $\{\sigma_B^{(w)} : w \in W$

Fact 4: $(R_B)^\Delta$ is a polynomial ring on the $\mathcal{G}_{\pi_B(\alpha_i^\vee)}$'s and the $\sigma_B^{(r_i)} + \rho_i$ for $i \in I$

Fact 5: $(R_B)^\Delta \longrightarrow \mathbb{Z} \otimes_S R_B$ is onto over $\mathbb{Q}$ (?)

The $S$-subalgebra $R_P^-$ of $R_P'$:

Define
$$R_P^- = \operatorname{Im} \psi_P = \sum_{x \in (W^P)_{af} \cap W_{af}^-} S\, \sigma_P^{(x)}$$

Then
$$R_B^- \simeq H_T(\Omega K)$$

but in general
$$H_T(\Omega K) \twoheadrightarrow R_P^-$$

We have:

- $R_P^- = \underline{A}_{af} \cdot \sigma_P^{(id)}$
- Every $A_{af}$-submodule of $R_P'$ is an $R_P^-$-submodule of

- $R_P^- \otimes_{\Lambda_P^-} \Lambda_P' \equiv R_P'$

where $\Lambda_P^- = \Lambda_P' \cap R_P^- = \sum_{\substack{h \in Q^\vee \\ h \text{ is dominant}}} \mathbb{Z}\, \delta_{\pi_P(h)}$

emark : Working with the case when $G$ is simple, connected but not necessarily simply connected so $\Omega K$ is no longer connected. we get the following fact: Assume that $a_i = 1$ for all $i \in I$ in $\theta = \sum_{i \in I} a_i \alpha_i$. Let $P = P_{\rho_i}$ so $W_P = \langle r_j \rangle_{j \in I,\, j \neq i}$. Let $w = w_0 w_P$. Let $Q$ be a standard parabolic. Then

$$\sigma_Q^{\hat{\pi}_Q(w)} * (w \cdot \sigma_Q^{(v)}) = \delta_{\pi_Q(\rho_i^\vee - v^{-1}\rho_i^\vee)}\, \sigma_Q^{(\hat{\pi}_Q(wv))}$$

for all $v \in W^Q$. Consequently $\sigma^{(\hat{\pi}_Q(w))}$ is invertible in $R_Q'$ ( no clue ! ~~wh~~ How is $Q$ related to $P = P_{\rho_i}$ ! )

Example: $G = SL_3$ (?) $W_P = r_1$ $G/P = \mathbb{P}^2$. $\sigma^{-21} * \sigma^{31} * \sigma^{-21} = \delta^2$ (?).



## A Filtration :

For $h \in Q^\vee$, define an $\underline{A}$-submodule $F_{P,\bar{h}}$ of $R_P^-$ (depen[ding] only on $h$ mod $Q_P^\vee$ ) by

$$F_{P,\bar{h}} = R_P^- \cap \delta_{-\pi_P(h)} R_P = \sum_{\substack{x \in (W^P)_{af} \cap W_{af}^- \\ x \overset{\leq}{\geq} \hat{\pi}_P(t_{-h})}} S\, \sigma_P^{(x)}$$

( a finite sum ) . Then

$$F_{P,\bar{h}} * F_{P,\bar{h}'} \subset F_{P,\overline{h+h'}}$$

Remark: In the geometric models to be given later, elemen[ts] of $F_{P,\bar{h}}$ correspond to trivializing certain line bund[les] on the (Peterson) variety $Y$ ( a $\mathcal{Y}$ ).

Fact : When $h \in Q^\vee$ is dominant.

$$F_{P,\bar{h}} = \psi_P ( F_{t_{-\omega(h)}} )$$

where $F_{t_{-\omega(h)}}$ is the Bruhat-Filtration in $H_T(\Omega K)$ in Lecture 9 and $\omega(h) = -w_0 \cdot h$ is the diagram automorph[ism]. Have

- $F_{P,\bar{h}} * F_{P,\bar{h}'} = F_{P,\overline{h+h'}}$ .
- $\sigma * F_{P,\bar{h}} \subset F_{P,\bar{h}'} \iff \sigma \in F_{P,\overline{h'-h}}$ .

**More on $G/B$ and $G/P$:**

Fix parabolic $P$ and $Q$ s.t. $G \supset P \supset Q$. Recall a (classical) fact on $H^*(G/Q)$: the fibration

$$\begin{array}{ccc} P/Q & \longrightarrow & G/Q \\ & & \downarrow \\ & & G/P \end{array}$$

gives rise to a filtration on $H^*(G/Q)$ such that

$$Gr\, H^*(G/Q) \cong H^*(P/Q) \otimes H^*(G/P)$$

An analogous statement is true for quantum cohomology.

Consider the $S$-algebra

$$R^{P,Q} = \sum_{\substack{x \in (W^P)_{af} \\ y \in (W_Q)_{af} \\ x \succeq id}} S\, \sigma_Q^{(xy)}$$

Let

$$R^{P,Q}_{\leq n} = \sum_{\substack{x \in (W^P)_{af} \\ y \in (W_Q)_{af} \\ x \succeq id \\ \ell_S(y) \leq n}} S\, \sigma_Q^{(xy)}$$



**Fact:** $R^{P,Q}_{\leq m}\, R^{P,Q}_{\leq n} \subset R^{P,Q}_{\leq m+n}$

Define

$$\bar{R}^{P,Q} = gr\, R^{P,Q} = \sum_{n \in \mathbb{Z}} \bar{R}^{P,Q}_n$$

where

$$\bar{R}^{P,Q}_n = R^{P,Q}_{\leq n} \Big/ R^{P,Q}_{\leq (n-1)}$$

**Fact**

$$\overset{\curvearrowright R_P}{R_{G/P}} \otimes_{\mathbb{Z}} (\mathbb{Z} \otimes_S R'_{P/Q}) \cong \bar{R}^{P,Q}$$

Define

$$R_-^{P,Q} = \sum_{\substack{x \in (W^P)_{af} \\ x \succeq id \\ y \in (W_Q)_{af} \cap (W_P)^-_{af}}} S\, \sigma_Q^{(xy)} \qquad R^{P,Q}_{-,n} = \sum_{\substack{\text{same} \\ \ell_S(y) \leq n}} S\sigma$$

**Fact**

$$gr\, R_-^{P,Q} \cong \overset{\shortparallel\, R_P}{R_{G/P}} \otimes \left[ Im\left( H_* \Omega_0(K \cap P) \to \mathbb{Z} \otimes_S R'_Q \right) \right]$$

**Corollary** If $\mathbb{Z} \otimes_S R_{P/Q}$ and $\mathbb{Z} \otimes_S R_{G/Q}$ are reduced, then $\mathbb{Z} \otimes R_{G/Q}$ is reduced

**Fact**

- For $G = SL(n, \mathbb{C})$ every $R_{G/P} = R_P$ is reduced
- Other cases where every $R_P$ is reduced are: $G_2, B_2,$

<u>Remark</u> (from informal lecture in the common room after the lecture).

Look at the case $G \supset P \supset B$. The fact

$$gr\, R_-^{P,B} \simeq R_P \otimes (\mathrm{Im}(H_*\Omega_0(K\cap P) \to \mathbb{Z} \otimes_S R_B')) \quad (*)$$

has the following meaning in terms of the geometric models: Recall the (Peterson) variety $Y \subset G^\vee/B^\vee$. It contains $2^\ell$ - $T$-fixed points $\{w_P : P \text{ parabolic}\}$. Label them by $y_P$. Set

$$Y_P^+ = Y \cap B_-^\vee w_P \cdot B^\vee$$

$$Y_P^- = Y \cap B_+^\vee w_P \cdot B^\vee \qquad (B_+^\vee = B^\vee)$$

Then
$$R_P = \mathcal{O}(Y_P^+)$$

$$H_*(\Omega_0(K\cap P)) \simeq \mathcal{O}(Y_P^-)$$

Can think of $gr\, R_-^{P,B}$ as the subring of $\mathcal{O}(Y_G^- \cap Y_B^+)$ that are regular at $y_P$ (not quite sure this is true) so $(*)$ says that near $y_P$, the variety $Y$ looks like $Y_P^+ \times Y_P^-$ //



## The quantum cohomology $QH^*(G/P)$

What we present here is adequate for $G/P$ but is not the most general case.

For $n \geq 3$, consider the open subscheme $V_n(\mathbb{C})$ of $(\mathbb{P}_{\mathbb{C}}^1)^n$.

$$V_n(\mathbb{C}) = \{(z_1, \cdots, z_n) \in (\mathbb{P}_{\mathbb{C}}^1)^n : z_i \neq z_j \quad i \neq j$$
$$z_1 = \infty \quad z_2 = 0 \quad z_3 = 1\}$$

For $\tau \in H_2(G/P)$, let

$$\mathcal{M}_\tau = \{\phi : \mathbb{P}^1 \to G/P : \phi_*[\mathbb{P}^1] = \tau\}$$

$$\mathcal{M}_{n,\tau} = \mathcal{M}_\tau \times V_n(\mathbb{C})$$

So
$$\dim \mathcal{M}_{n,\tau} = \langle c_1(T_{G/P}), \tau \rangle + \dim G/P + n - 3$$

Set
$$ev : \mathcal{M}_{n,\tau} \to (G/P)^n.$$

$$ev(\phi, z_1, \cdots, z_n) = (\phi(z_1), \phi(z_2), \cdots, \phi(z_n))$$

Roughly speaking, $\mathcal{M}_{n,\tau}$ admits a compactification $\overline{\mathcal{M}_{n,\tau}}$ which admits a fundamental class $[\overline{\mathcal{M}}_{n,\tau}]$. (Manin-Kontsevich, Now for $\phi_1 \otimes \cdots \otimes \phi_n \in H^*((G/p)^n) = H(G/p)^{\otimes n}$,

have
$$\int_{\overline{\mathcal{M}}_{n,\tau}} ev^*(\phi_1 \otimes \cdots \otimes \phi_n) \in \mathbb{Z} \quad (\text{or } \mathbb{C}\,?)$$

Using Poincare duality, can regard above as giving a $\mathbb{Z}$-linear map

$$J_{n,\tau}: \quad \otimes^{n-1} H^*(G/p) \longrightarrow H^*(G/p)$$

of degree $= -2\,[\,\langle c_1(TG/p), \tau\rangle + (n-3)\,]$. In other words, for any $n$ subvariety $X_1, \cdots, X_n$ of $G/p$ with

$$\sum_{i=1}^{n} \operatorname{codim} X_i = 2 \dim_{\mathbb{C}} \overline{\mathcal{M}}_{n,\tau}$$

we have

$$\langle J_{n,\tau}(PD^{-1}[X_1] \otimes \cdots \otimes PD^{-1}[X_{n-1}]), [X_n] \rangle$$

$$= \#\,(\,\overline{\mathcal{M}}_{n,\tau} \times_{(G/p)^n} (g_1 X_1 \times \cdots \times g_n X_n))(\mathbb{C})$$

for all $(g_1, \cdots, g_n)$ in a dense open subset of $(G(\mathbb{C}))^n$.

These numbers are the Gromov-Witten invariants.



<u>Fact</u>: For $\phi \in H^2(G/p)$ and $n \geq 4$

$$J_{n,\tau}(\phi_1 \otimes \cdots \otimes \phi_{n-2} \otimes \phi) = \langle \phi, \tau\rangle J_{n-1,\tau}(\phi_1 \otimes \cdots \otimes \phi_{n-2}).$$

Now let $\mathcal{D} = \mathbb{Q}[[\varepsilon]]$ with indeterminant $\varepsilon$. Given $\nu \in \varepsilon(H^*(G/p) \otimes_{\mathbb{Z}} \mathcal{D})$, can make $H^*(G/p) \otimes_{\mathbb{Z}} \mathcal{D}$ into a commutative associative $\mathcal{D}$-algebra with unit $\sigma_P^{id}$ with quantum product $*_\nu$ by

$$\sigma *_\nu \sigma' = \sum_{n,\tau} J_{n,\tau}(\sigma \otimes \sigma' \otimes \frac{\nu^{n-3}}{(n-3)!})$$

where $\nu^{n-3} = \nu \otimes \cdots \otimes \nu$ ($(n-3)$-times).

In particular, for $\phi \in H^2(G/p)$, define

$$\sigma *_{\varepsilon\phi} \sigma' = \sum_{\tau} J_{3,\tau}(\sigma \otimes \sigma') \exp \varepsilon\langle \phi, \tau\rangle$$

The "potential function for $J_{3,\tau}$" satisfy WDVV-equation.

<u>The small quantum cohomology</u>:

Make $H^*(G/p) \otimes_{\mathbb{Z}} \Lambda_p$ into a $\Lambda_p$-algebra $QH^*(G/p)$ by

$$\sigma * \sigma' = \sum_{\tau \in \pi_P(Q_+^\vee)} q_\tau \, J_{3,\tau}(\sigma \otimes \sigma')$$

<u>Theorem</u>

(1) $*$ is associative

(2) $qH^*(G/P)$ is $\mathbb{Z}$-graded.

3) For $i \in I$, $w \in W$

$$\sigma_B^{r_i} * \sigma_B^{w} = \sum_{\substack{\alpha \in \tilde{\Delta}_+ \\ \ell(w r_\alpha) = \ell(w)+1}} \langle \rho_i, \alpha^\vee \rangle \sigma_B^{w r_\alpha}$$

$$+ \sum_{\substack{\alpha \in \tilde{\Delta}_+ \\ \ell(w r_\alpha) = \ell(w)+1-\langle 2\rho, \check{\alpha} \rangle}} \langle \rho_i, \alpha^\vee \rangle \, q_{\pi_B(\alpha^\vee)} \, \sigma_B^{w r_\alpha}$$

The proof of (1) is due to various people.

The proof of (3) is a not too hard geometric argument like the one given by Dale in Vogan's seminar.



<u>Relation between $qH^*(G/P)$ & $qH^*(G/B)$:</u>

Let $\tau \in H_2(G/P)$. Then there exists a unique $h \in Q^\vee$ s.t.

$$\pi_P(h) = \tau$$

and $-1 \leq \langle \alpha, h \rangle \leq 0$ for all $\alpha \in -\Delta(\mathfrak{p}/\mathfrak{b})$.

Define a standard parabolic $P_1 \subset P$ by

$$\Delta(\mathfrak{p}_1/\mathfrak{b}) = \{ \alpha \in \Delta(\mathfrak{p}/\mathfrak{b}) : \langle \alpha, h \rangle = 0 \}$$

There have birational morphisms

$$\mathcal{M}_{\pi_B(h), G/B} \longrightarrow \mathcal{M}_{\pi_{P_1}(h), G/P_1} \times_{G/P_1} G/B$$

$$\mathcal{M}_{\pi_{P_1}(h), G/P_1} \longrightarrow \mathcal{M}_{\tau, G/P}$$

This gives a commutative diagram:

$$\begin{tikzcd}
\otimes^{n+1} H^*(G/P) \arrow[r, "\text{can.}"] \arrow[d, "J_{n,\tau}"] & \otimes^{n+1} H^*(G/P_1) \arrow[r, "\text{can}"] \arrow[d, "J_{n,\pi_{P_1}(h)}"] & \otimes^{n+1} H^*(G/B) \arrow[d, "J_{n,\pi_B(h)}"] \\
H^*(G/P) & H^*(G/P_1) \arrow[l, "\text{integration}"', "\text{over fibre } P/P_1"] \arrow[r, hook, "\text{can}"] & H^*(G/B)
\end{tikzcd}$$

This will be used in Lecture 16 to prove $\mathbb{Z}\mathcal{W}_s R_P \cong qH^*(G/P)$.

End of Lecture 15

## Lecture 16 April 22, 1997

Recall last time:

- Defined $qH^*(G/P)$ from
$$J_{n,z}: \otimes^{n+1} H^*(G/P) \longrightarrow H^*(G/P)$$
- $\sigma * \sigma'$ from $J_{2,z}$'s
- Gave formula for $\mathbb{D}^{r_i} *$ in $qH^2(G/B)$
- Comparison of $qH^*(G/B)$ and $qH^*(G/P)$

Today: Compare $R_P$ and $qH^*(G/P)$.

**Theorem 1** We have an isomorphism
$$\mathbb{Z} \otimes_S R_P \cong qH^*(G/P)$$
$$1 \otimes q_t \sigma_P^{(\omega)} \longmapsto q_t \sigma_P^{\omega}$$

**Proof** First the case of $G/B$: The map $1 \otimes q_t \sigma_B^{(\omega)} \to q_t \sigma_B^{\omega}$ is bijective. Since both sides are generated by $H^2$ and there is no torsion remains to check formulas for multiplications by $H^2$ on each side. We wrote these formulas down in Lectures 14 & 15.
For any $P$, use the commutative diagram given at the end of lecture 15. //

<u>em 2</u> We have an isomorphism

$$R_p \otimes_{\Lambda_p} \mathbb{Z} \xrightarrow{\sim} H^T(G/p)$$

$$s\sigma_p^{(w)} \otimes 1 \longmapsto s\,\sigma_p^{(w)}$$

<u>f</u> For $G/B$, directly from the multiplication formular by $H^2$.
For $G/p$, take $z=0$ and $h=0$ in the commutative diagram at the end of lecture 15.

//

ve have the commutative diagram

$$\begin{tikzcd} & R_p \arrow[dl] \arrow[dr] & \\ \mathfrak{z}H^*(G/p) \arrow[dr, "\mathfrak{z}=0"'] & & H^T(G/p) \arrow[dl, "s\to"] \\ & H^*(G/p) & \end{tikzcd}$$

now on, will denote

$$R_p = \mathfrak{z}H^T(G/p)$$

$$R_p' = \mathfrak{z}H^T(G/p)_{(\mathfrak{z})}$$

(quantum cohomology with the quantum parameter inverted)



<u>The homomorphism $\Psi_p$:</u>

$$\begin{tikzcd} H_T(\Omega K) \arrow[r, "\Psi_p"] \arrow[d] & \mathfrak{z}H^T(G/p)_{(\mathfrak{z})} \arrow[d] \\ H_*(\Omega K) \arrow[r, "\overline{\Psi}_p"] & \mathfrak{z}H(G/p)_{(\mathfrak{z})} \end{tikzcd}$$

When $P=B$, $\overline{\Psi}_B$ is an isomorphism if we also invert the translational element in $H_*(\Omega K)$. Using $\overline{\Psi}_B$, we get structure constants for $H_*(\Omega K)$ as Gromov-Witten invariants, which, since they are numbers of certain curves, are non-negative integers.

<u>Results of Bott:</u>

For certain $h \in \mathfrak{t}^*$, construct $K/T \xrightarrow{\phi_h} \Omega K$ by

$$\phi_h(kT)^{(t)} = t^h k t^{-h} k^{-1}$$

- $\exists\, h$ s.t. $\mathrm{Im}\, \phi_{h*}(H_\cdot(K/T))$ generates $H_*(\Omega K)$
- Can find $\mathrm{Im}\,[\mathrm{Prim}\, H^*(\Omega K)]$ in $H^*(K/T)$.
- related to $H^T(K/T) \otimes_S H_T(\Omega K) \longrightarrow H_T(\Omega K)$

or $\quad \mathrm{spec}\, H^T(K/T) \times_{\mathfrak{h}} \mathrm{Spec}\, H_T(\Omega K) \longleftarrow \mathrm{spec}\, H_T(\Omega K)$.

## ometrical Models

Will construct geometrical models for

- the groupoid scheme $\mathcal{U} = \operatorname{spec} H^T(G/B)$ (finite $G$)
- scheme $\mathcal{U}_{G/P} = \operatorname{spec} H^T(G/P)$ with a groupoid $\mathcal{U}$-action;
- group schemes: $\hat{\mathcal{A}} = \operatorname{spec} H_T(\Omega K)$ (do not assume $G$ is simply connected)

  $\mathcal{A} = \operatorname{spec} H_T(\Omega_0 K)$

- $\mathfrak{z}\mathcal{U} = \operatorname{spec} \mathfrak{z}H^T(G/B)$

  $\mathfrak{z}\mathcal{U}_{G/P} = \operatorname{spec} \mathfrak{z}H^T(G/P)$ — All equipped with groupoid $\mathcal{U}$-actions

  $(\mathfrak{z}\mathcal{U}_{G/P})_{\mathfrak{z}} = \operatorname{spec} \mathfrak{z}H^T(G/P)_{(\mathfrak{z})}$

- The variety $\mathcal{Y}$ (used to be denoted by $Y$)
  - It is a projective scheme over $\underline{h}$ "with pieces $\mathfrak{z}\mathcal{U}_{G/P}$".
  - It has an "open piece" $\mathcal{Y}_{(\mathfrak{z})}$ where the $\ell$ distinguished line bundles ~~over~~ ~~are~~ have nonvanishing sections (:) (will explain later)
  - Has $\mathbb{Z}$-points $y_P \in \mathcal{Y}(\mathbb{Z})$ for each parabolic $P$
- — $\mathbb{G}_m = \operatorname{spec} \mathbb{Z}[t, t^{-1}]$ acts on all and gives gradings
- Have homomorphism $\hat{\mathcal{A}} \to \mathcal{A}$ as group schemes



- $\mathcal{A}$, as a groupoid scheme, acts on $\mathcal{Y}$, and can identify ~~&~~ the $\mathcal{A}$-orbit through $y_G$ with $\mathcal{Y}_{(\mathfrak{z})}$.
- Have natural morphisms

  $\mathcal{U}_{G/P} \longrightarrow \mathfrak{z}\mathcal{U}_{G/P}$ (corresponding to $\mathfrak{z}H^T(G/P) \xrightarrow{q=0} H^T(G/P)$)

  $\mathfrak{z}\mathcal{U}_{G/P} \longrightarrow \mathcal{Y}$ — imbeddings

  $(\mathfrak{z}\mathcal{U}_{G/P})_{(\mathfrak{z})} \longrightarrow \mathcal{Y}_{(\mathfrak{z})}$

- $\mathfrak{z}\mathcal{U}_{G/P} \cap \mathfrak{z}\mathcal{U}_{G/P'} = \phi$ if $P \neq P'$

  but $\mathfrak{z}\mathcal{U}_{G/P} \cap \mathcal{A} = (\mathfrak{z}\mathcal{U}_{G/P})_{(\mathfrak{z})}$

(In the Peterson lingo, "the quantum cohomologies ~~for~~ do not see each other, but they all see the homology of $\Omega K$").

Now we turn to the first model for $\mathcal{U}$:

e first model $\mathcal{U}$ of Spec $H^T(G/B)$:

let $e = \sum_{i \in I} e_{\alpha_i}$

mma: For any $h \in \underline{h}$, the fixed points of the vector field $V_{e+h}$ on $G/B$ defined by $e+h$ all lie in the ~~big~~ cell $\cdot B$

eorem

$$\mathcal{U} = \operatorname{spec} H^T(G/B) = (G/B)^{e+\underline{h}}$$

$$\doteq \{(e+h, x) \in (e+\underline{h}) \times G/B : V_{e+h}(x) = 0\}$$

$$= \{(e+h, u_- \cdot B) : \underbrace{u_-^{-1} \cdot (e+h)}_{\uparrow \text{Adjoint action}} \in \underline{b}\}$$

ut since $U_-$ stabalizes $e + \underline{b}_-$, when $u_-^{-1} \cdot (e+h) \in \underline{b}$, we have

$$u_-^{-1} \cdot (e+h) \in \underline{b} \cap (e + \underline{b}_-) = e + \underline{h}$$

$$\mathcal{U} = \{(e+h, u_- \cdot B) : u_-^{-1} \cdot (e+h) \in e + \underline{h}\}$$

$$= \{(e+h, u_-, e+h') : u_-^{-1} \cdot (e+h) = e+h'\}.$$



The groupoid structure on $\mathcal{U}$:

- $\mathcal{U} \underset{t}{\overset{s}{\rightrightarrows}} \underline{h}$: $(e+h, u_-, e+h') \overset{s}{\longmapsto} e+h, \quad \overset{t}{\longmapsto} e+h'$
- $\mathcal{U} \times_h \mathcal{U} \to \mathcal{U}$: $(e+h, u_-, e+h') \cdot (e+h', u', e+h'') = (e+h, u_- u', \ldots$
- $\underline{h} \longrightarrow \mathcal{U}$: $h \longmapsto (e+h, 1, e+h)$ (identities)
- inverse: $\mathcal{U} \to \mathcal{U}$: $(e+h, u_-, e+h') \mapsto (e+h', u_-^{-1}, e+h$

As a model for Spec $H^T(G/B)$, we must have two $W$-actions on $\mathcal{U}$ which gave $\omega_L$ & $\omega_R$ on $H^T(G/B)$. We now identify these two actions, in the next lecture.

End of Lecture 16

## Lecture 17. April 23, 1997

The following works for the general Kac-Moody case:

Set
$$e = \sum_{i \in I} e_i \in \underline{n}_+$$
$$f_i^{(n)} = \frac{f_i^n}{n!} \in U(\underline{n}_-)$$

Then
$$U(\underline{n}_-)_{\mathbb{Z}} = \langle f_i^{(n)} \rangle_{i \in I,\, n \geq 0}$$

and using the action of $2\rho^\vee$ we can give $U(\underline{n}_-)_{\mathbb{Z}}$ a $\mathbb{Z}$-gradi[ng]
with $\deg f_i = -2$.

Define
$$\mathcal{O}(U_-) = \mathrm{Hom}_{\mathbb{Z}}\left(U(\underline{n}_-)_{\mathbb{Z}}, \mathbb{Z}\right) \quad \text{(graded dual)}$$

and we use $U_-$ to denote the groupscheme defined by $\mathcal{O}(U_-)$

<u>Lemma</u>: For any $w \in W$, there exists a scheme morphism
$$U_w : \underline{h} \longrightarrow U_-$$

s.t.
$$u_w(h) \cdot (e + h) = e + w \cdot h$$

We have
$$u_{vw}(h) = u_v(w \cdot h)\, u_w(h)$$

and

$$U_{r_i}(h) = \exp\left(\langle \alpha_i, h\rangle f_i\right) = y_i\left(\langle \alpha_i, h\rangle\right)$$

where, recall, $\phi_i : SL(2,\mathbb{C}) \to G$ and for $u \in \mathbb{C}$ ($\mathbb{Z}$ ?)

$$x_i(u) = \phi_i\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

$$y_i(u) = \phi_i\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$$

<u>The finite case</u>

In this case, a theorem of Kostant says that the element $U_w(h) \in U_-$ is unique for any given $w \in W$ and $h \in \underline{h}$.

Example: For $\mathfrak{g} = \mathfrak{sl}(3)$, $h = \operatorname{diag}(x_1, x_2, x_3)$, have

$$U_{r_1}(h) = \begin{pmatrix} 1 & 0 & 0 \\ x_1 - x_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad U_{r_2}(h) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x_2 - x_3 & 1 \end{pmatrix}$$

$$U_{r_1 r_2}(h) = \begin{pmatrix} 1 & 0 & 0 \\ x_1 - x_3 & 1 & 0 \\ 0 & x_2 - x_3 & 1 \end{pmatrix} \qquad U_{r_2 r_1}(h) = \begin{pmatrix} 1 & 0 & 0 \\ x_1 - x_2 & 1 & 0 \\ (x_1 - x_2)(x_1 - x_3), & x_1 - x_3 & 1 \end{pmatrix}$$

$$U_{r_1 r_2 r_1}(h) = \begin{pmatrix} 1 & 0 & 0 \\ x_1 - x_3 & 1 & 0 \\ (x_1 - x_2)(x_1 - x_3), & x_2 - x_3 & 1 \end{pmatrix}$$

<u>Fact</u>: $U_{w_0}(t\rho^\vee) = \exp(tf)$, $t \in \mathbb{C}$, where $\{e, f, 2\rho^\vee\}$ is a TDS.

//

<u>The affine case</u>:

In this case, define $U_{r_i}(h) = y_i(\langle \alpha_i, h\rangle)$ for $i \in I_{af}$ and use

$$U_{vw}(h) = U_v(w \cdot h)\, U_w(h)$$

to extend to any $w$. This is well-defined because of the braid relations: assume that $2 < m_{ij} < \infty$ & $\langle \alpha_j, \alpha_i^\vee \rangle = -1$ for $i \neq j$. Then

$m_{ij} = 3$: $\quad$ set $a = \langle \alpha_i, h\rangle$, $b = \langle \alpha_j, h \rangle$

$$U_{iji} = U_{r_i}(r_j r_i h)\, U_{r_j}(r_i h)\, U_{r_i}(h) = y_i(b)\, y_j(a+b)\, y_j(a)$$

$$U_{jij} = y_j(a)\, y_i(a+b)\, y_j(b)$$

$m_{ij} = 4$:

$$U_{ijij} = y_i(a)\, y_j(a+b)\, y_i(a+2b)\, y_j(b)$$

$$U_{jiji} = y_j(b)\, y_i(a+2b)\, y_j(a+b)\, y_i(a)$$

$m_{ij} = 6$:

$$U_{ijijij} = y_i(a)\, y_j(3a+b)\, y_i(2a+b)\, y_j(3a+2b)\, y_i(a+b)\, y_j(b)$$

$$U_{jijiji} = y_j(b)\, y_i(a+b)\, y_j(3a+2b)\, y_i(2a+b)\, y_j(3a+b)\, y_i(a)$$

The fact that they are equal is due to Kostant's theorem ($U_w(h)$ is unique). These are called Universal exponential solutions to the Yang-Baxter Equations by Fomin & Kirillov in their paper in Lett. Math. Phys. (1996) 273-284.

What about $m_{ij} = \infty$ ? This is what is needed in the affine case!

Remark: In the affine case, the element $U_\omega(h) \in U_-$ is not necessarily unique for a given $(\omega, h)$. For example, when $t \cdot h = h$, have $U_t(h) \in Z(e+h) \cap U_-$.

The action of $W$ on $(e+\underline{h}) \times Z$, and on $S \otimes \mathcal{O}(Z)$

Suppose that $U_-$ acts on a scheme $Z$. Then $W$ acts on $(e+\underline{h}) \times Z$ by

$$\omega \cdot ((e+h, z)) = (e+\omega h, U_\omega(h) \cdot z)$$

Assume that $Z$ is affine. Then $W$ acts on $\mathcal{O}((e+\underline{h}) \times Z) = S \otimes \mathcal{O}(Z)$:

$$(\omega \cdot p_1)(e+h, z) = p_1(\omega^{-1} \cdot (e+h, z)) \qquad p_1 = s \otimes p \in S \otimes \mathcal{O}(Z).$$

Lemma: For $s \otimes p \in S \otimes \mathcal{O}(Z)$,

$$r_i \cdot (s \otimes p) = \sum_{n \geq 0} \left(r_i \cdot (\alpha_i^n s)\right) \otimes f_i^{(n)} \cdot p$$

Proof: By definition,

$$r_i \cdot (s \otimes p)(e+h, z) = (s \otimes p)(r_i \cdot (e+h, z))$$
$$= (s \otimes p)(e + r_i \cdot h, U_{r_i}(h) \cdot z)$$
$$= s(r_i \cdot h)\, p(U_{r_i}(h) \cdot z)$$
$$= (r_i \cdot s)(h)\, (U_{r_i}(h)^{-1} \cdot p)(z)$$
$$= (r_i \cdot s)(h)\, (\exp(-\langle \alpha_i, h\rangle f_i) \cdot p)(z)$$
$$= (r_i \cdot s)(h) \left(\sum_{n \geq 0} \frac{-\langle \alpha_i, h\rangle^n}{n!} f_i^n \cdot p\right)(z)$$
$$= \sum_{n \geq 0} \left((-\alpha_i)^n r_i \cdot s\right)(h)\, (f_i^{(n)} \cdot p)(z)$$
$$= \sum_{n \geq 0} r_i \cdot (\alpha_i^n s)(h)\, (f_i^{(n)} \cdot p)(z)$$

$$\Rightarrow \quad r_i \cdot (s \otimes p) = \sum_{n \geq 0} \left(r_i \cdot (\alpha_i^n s)\right) \otimes f_i^{(n)} \cdot p$$

Consequently, we get an integrable $\underline{A}$-module structure on $S \otimes \mathcal{O}(Z)$ by

$$A_i \cdot (s \otimes p) = \frac{1}{\alpha_i}(1 - r_i) \cdot (s \otimes p)$$
$$= (A_i \cdot s) \otimes p + \sum_{n \geq 1} r_i \cdot (\alpha_i^{n-1} s) \otimes f_i^{(n)} \cdot p$$

For each $p \in \mathcal{O}(Z)$, this is a finite sum.

## The groupoid scheme $\mathcal{U}' = (e+\underline{h}) \times U_-$:

Define

$$p_L = p_1: \quad \mathcal{U}' \to e+\underline{h}: \quad (e+h, u) \mapsto e+h$$

and

$$p_R: \quad \mathcal{U}' \to e+\underline{h}: \quad (e+h, u) \mapsto \text{proj. of } u^{-1}\cdot(e+h) \text{ to } e+\underline{h} \text{ in } e+\underline{b}_- = e+\underline{h} + \underline{n}_-.$$

These are the source and target maps for the groupoid structure on $\mathcal{U}'$. Other structure maps:

identities $i'$: $e+\underline{h} \hookrightarrow \mathcal{U}'$: $\quad e+h \mapsto (e+h, 1)$

multiplication: $\mu'$: $\mathcal{U}' \times_{e+\underline{h}} \mathcal{U}' \to \mathcal{U}'$:

$$(e+h, u)\cdot(e+h', u') = (e+h, uu')$$

$$\text{if} \quad p_r(e+h, u) = e+h' = p_L(e+h', u').$$

inverse: $\quad \iota: \mathcal{U}' \to \mathcal{U}'$: $\quad (e+h, u) \mapsto (p_R(e+h, u), u^{-1})$.

The idea now is to embed $\mathcal{U}$ as a subgroupoid scheme of $\mathcal{U}'$. Here the groupoid scheme str. on $\mathcal{U}$ is the one defined in lecture 5. To this end, we ~~look~~ use the integrable $\underline{A}$-module st. on $\mathcal{U}'$.

## The groupoid morphism $\mathcal{U} \to \mathcal{U}'$:

Consider the $W_L$ action on $(e+\underline{h})\times U_-$:

$$w_L\cdot(e+h, u) = (e+wh, u_w(h)u)$$

It satisfies

$$p_R \cdot w_L = p_R$$

by the definition of $u_w$. By the discussion on Page 17-4, we have an integrable $\underline{A}_L$-module structure on $\mathcal{O}(\mathcal{U}')$. In other word we have a groupoid action

$$\phi: \begin{tikzcd} \mathcal{U}\times_{\underline{h}}\mathcal{U}' \arrow[r] \arrow[d, "p_R\cdot p_1"'] & \mathcal{U}' \arrow[d, "p_R"] \\ e+\underline{h} \arrow[r, "\simeq"] & e+\underline{h} \end{tikzcd}$$

Also have

$$\begin{tikzcd} \mathcal{U}\times_{\underline{h}}\mathcal{U}'\times_{\underline{h}}\mathcal{U}' \arrow[r, "\mathrm{id}\times\mu'"] \arrow[d, "\phi\times\mathrm{id}"'] & \mathcal{U}\times_{\underline{h}}\mathcal{U}' \arrow[d, "\phi"] \\ \mathcal{U}'\times_{\underline{h}}\mathcal{U}' \arrow[r, "\mu'"] & \mathcal{U}' \end{tikzcd}$$

where $\mu'$: $\mathcal{U}'\times_{\underline{h}}\mathcal{U}' \to \mathcal{U}'$ is the multiplication morphism for $\mathcal{U}$

These imply that the following composition is a morphism of groupoid schemes over $\underline{h}$,

$$\mathcal{U} = \mathcal{U} \times_{\underline{h}} \underline{h} \xrightarrow{id \times i'} \mathcal{U} \times_{\underline{h}} \mathcal{U}' \xrightarrow{\phi} \mathcal{U}'$$

where $i' : \underline{h} \hookrightarrow \mathcal{U}'$ is the identity morphism for $\mathcal{U}'$.

<u>The groupoid isomorphism $\mathcal{U} \cong \mathcal{U}' \times_{e+\underline{b}_-} (e+\underline{h})$</u>

Define $\quad p_R' : \mathcal{U}' \longrightarrow e + \underline{b}_- ; \quad (e+h, u) \mapsto u^{-1}\cdot(e+h) \in e+\underline{b}_-$.

Form

$$\mathcal{U}' \times_{e+\underline{b}_-} (e+\underline{h}) := \mathcal{U}''$$

using $p_R'$ and $e + \underline{h} \hookrightarrow e + \underline{b}_-$ (the inclusion). We think of $\mathcal{U}' \times_{e+\underline{b}_-} (e+\underline{h}) = \mathcal{U}''$ as the subset of $\mathcal{U}'$:

$$\{ (e+h, u) : \quad u^{-1}\cdot(e+h) \in e+h \}$$

We claim that the morphism $\mathcal{U} \to \mathcal{U}'$ factors through $\mathcal{U}''$.

To prove this, we look at

$$\mathcal{O}(\mathcal{U}') \longrightarrow \mathcal{O}(\mathcal{U}) = H^T(G/B).$$

For each $w \in W$, recall that we have $\psi_w : H^T(G/B) \longrightarrow S$.

The map

$$\mathcal{O}(\mathcal{U}') \longrightarrow \mathcal{O}(\mathcal{U}) = H^T(G/B) \xrightarrow{\psi_w} S$$

corresponds to the scheme morphism

$$\underline{h} \longrightarrow \mathcal{U}' : \quad h \mapsto (e+h, \ u_{w^{-1}}(h)^{-1})$$

Since

$$(u_{w^{-1}}(h)^{-1})^{-1}\cdot(e+h) = u_{w^{-1}}(h)\cdot(e+h) = e + w^{-1}\cdot h$$

we see that

$$(e+h, \ u_{w^{-1}}(h)^{-1}) \in \mathcal{U}''.$$

Since $\{\psi_w : w \in W\}$ is a basis for $\mathrm{HFms}(H^T(K/T), S)$, we conclude that the morphism $\mathcal{U} \to \mathcal{U}'$ factors through $\mathcal{U}''$ to give $\quad \mathcal{U} \longrightarrow \mathcal{U}''$.

<u>Theorem</u> $\quad \mathcal{U} \xrightarrow{\sim} \mathcal{U}''$

as groupoid schemes over $\underline{h}$.

End of Lecture

LECTURE 18, April 29, 1997

Last time we had morphisms of groupoid schemes over $\underline{h}$

$$\begin{tikzcd}
\operatorname{Spec} H^T(K/T) = \mathcal{U} \arrow[rr] \arrow[dr, dashed] & & (e+\underline{h}) \times \mathcal{U}_- =: \mathcal{U}' \\
& {[(e+\underline{h}) \times \mathcal{U}_-] \underset{e+\underline{b}_-}{\times} (e+\underline{h}) = \mathcal{U}''} \arrow[ur] &
\end{tikzcd}$$

Consider the corresponding ring homomorphism

$$(*) \quad \mathcal{O}(\mathcal{U}') = S \otimes \mathcal{O}(\mathcal{U}_-) \longrightarrow \mathcal{O}(\mathcal{U}) = H^T(K/T).$$

<u>Definition</u> $w \in W$ is called <u>$G^\vee$-abelian</u> if the following equivalent conditions hold.

(1) $r_i r_j r_i$, where $a_{ij} = -1$, does not occur as a consecutive subexpression for any reduced expression of $w$.

(2) $\mathcal{U}_-^\vee \cap w \mathcal{B}^\vee w^{-1}$ is commutative.

<u>Lifting of $\sigma^{(w)}_{G/B}$ for $G^\vee$-abelian $w$ to $\mathcal{O}(\mathcal{U}_-)$</u>

Consider the quotient of $\mathcal{U}(\underline{n}_-)_{\mathbb{Z}}$ by the 2-sided ideal generated by $\{f_i^{(n)} \mid i \in I,\ n \geq 2\}$.

The resulting ring $\underbrace{\mathcal{U}(\underline{n}_-)_{\mathbb{Z}}}^{\text{with identity}} / \langle f_i^{(2)};\ i \in I \rangle$ is given by generators $\{f_i \mid i \in I\}$ and relations:

$$f_i f_i = 0, \quad f_i f_j f_i = 0 \text{ if } a_{ij} = -1,$$

$$\text{and } f_i f_j = f_j f_i \quad \text{if } a_{ij} = 0.$$

For $G^\vee$-abelian $w$ with reduced expression $r_{i_1} \cdots r_{i_N}$, put

$$f_w = f_{i_1} \cdots f_{i_N}.$$

These $f_w$ define a basis of $\mathcal{U}(\underline{n}_-)_{\mathbb{Z}} / \langle f_i^{(2)};\ i \in I \rangle$.

The dual basis gives us elements in $\mathcal{O}(\mathcal{U}_-)$

$$f_w^* \in \operatorname{Hom}(\mathcal{U}(\underline{n}_-)_{\mathbb{Z}} / \langle f_i^{(2)};\ i \in I \rangle, \mathbb{Z}) \subset \operatorname{Hom}(\mathcal{U}(\underline{n}_-)_{\mathbb{Z}}, \mathbb{Z}) = \mathcal{O}(\mathcal{U}_-)$$

<u>Claim</u>: Under the homomorphism $(*)$

$$S \otimes \mathcal{O}(\mathcal{U}_-) \longrightarrow H^T(G/B),$$
$$1 \otimes f_w^* \longmapsto \sigma^{(w)}_{G/B}.$$

<u>Proof</u>: Write $f_w^*$ for $1 \otimes f_w^*$. The statement is clear for the identity elements: $f_1^* \longmapsto \sigma^{(1)}_{G/B}$.

Suppose $r_i w \le w$. Then $r_i w$ is again $G^\vee$-abelian, and we have

$$(r_i \cdot f_w^*)(h, u) = f_w^*(u_{r_i}(h)u) = \alpha_i(h) f_{r_i w}^*(u) + f_w^*(u)$$

Therefore $\quad x_i f_w^* = \alpha_i f_{r_i w}^* + f_w^*$

$$A_i \cdot f_w^* = -f_{r_i w}^*.$$

Similarly, $A_j \cdot f_w^* = 0$ if $w \le r_j w$.

Define $x \in H^T(G/B)$ by

$$f_w^* \longmapsto \sigma_{G/B}^{(w)} + x.$$

Then $\quad A_i \cdot f_w^* \longmapsto A_i \sigma_{G/B}^{(w)} + A_i x.$

We can assume by induction that

$f_{r_i w} \longmapsto \sigma_{G/B}^{(r_i w)}$ whenever $r_i w \le w$.

Therefore $A_i \cdot x = 0$ in this case.

Also $A_j \cdot x = 0$ for $r_j w \ge w$, by the above.

So $x = 0$. $\square$

## Miniscule representations

*Definition.* A representation is miniscule if the following equivalent conditions hold.

(1) all weights lie in the same $W$-orbit

(2) the representation has highest weight $\lambda$ such that $0 \le \langle \lambda, \alpha^\vee \rangle \le 1$ for all $\alpha \in \phi^+$

Let $V = V(\lambda)$ be a miniscule representation of $G$ with highest weight $v^+ \in V(\lambda)$. The stabilizer of the $\lambda$ weight space is the parabolic subgroup $P = P_\lambda = B W_\lambda B$ (where $W_\lambda$ is the stabilizer of $\lambda$ in $W$).

The weights of $V(\lambda)$ are precisely $\{w.\lambda \mid w \in W^P\}$.

*Lemma* All $w \in W^P$, for $P$ as above, are $G^\vee$-abelian, and $\{v_w = f_w \cdot v^+ \mid w \in W^P\}$ gives a basis of $V(\lambda)$.

Proof: $W^P$ is characterised as

$$W^P = \{ w \in W \mid \alpha \in \phi^+, w.\alpha^\vee < 0 \Rightarrow \langle \lambda, \alpha^\vee \rangle = 1 \}$$

Therefore $\mathcal{U}^\vee_- \cap w \mathcal{B}^\vee w^{-1}$ (for $w \in W^P$) is generated by 1-parameter subgroups $\mathcal{U}^\vee_{-\alpha^\vee} = \exp \mathfrak{g}^\vee_{-\alpha^\vee}$ for which $\langle \lambda, \alpha^\vee \rangle = 1$. Any two such subgroups $\mathcal{U}^\vee_{-\alpha}, \mathcal{U}^\vee_{-\beta}$ commute, since $\langle \lambda, \alpha^\vee + \beta^\vee \rangle = 2$ and thus $\alpha^\vee + \beta^\vee$ is not a root of $\mathfrak{g}^\vee$ (by condition (2) for miniscule $\lambda$). So $w$ is $G^\vee$-abelian.

That $f_w \cdot v^+ \in V_{w.\lambda}$ is proved inductively. Let $w = r_i w'$ with $l(w) = l(w') + 1$. Then $w' \in W^P$ and $f_w \cdot v^+ = f_i f_{w'} \cdot v^+ \in V_{w'.\lambda - \alpha_i}$.
On the other hand $r_i w'.\lambda = w'.\lambda - \langle \alpha_i^\vee, w'.\lambda \rangle \alpha_i$.
We have $\langle \alpha_i^\vee, w'.\lambda \rangle = \langle (w')^{-1} \alpha_i^\vee, \lambda \rangle = 1$, since $r_i w' = w$ lies in $W^P$ and takes the positive weight $(w')^{-1}\alpha_i^\vee$ to $-\alpha_i^\vee$. Thus $w.\lambda = w'.\lambda - \alpha_i$ and $f_w \cdot v^+ \in V_{w.\lambda}$. (and $f_w \cdot v^+$ is nonzero). □

Corollary: All matrix coefficients in $\mathcal{O}(\mathcal{U}_-)$ of the miniscule representation $V(\lambda)$ go to Schubert basis elements in $H^T(G/\mathcal{B})$ under the homomorphism (*) (matrix coefficients with respect to $\{v_w\}$, that is)

Proof This follows since $f_i^{(2)}$ acts on $V(\lambda)$ by 0. □

Example Consider the standard representation $V(\varrho_1)$ of $SL_3$. It is clearly miniscule. The homomorphism $\mathcal{O}(\mathcal{U}_-) \to H^T(G/\mathcal{B})$ gives rise to the 'tautological' element

$$u = \begin{pmatrix} 1 & & \\ \sigma^{(r_1)}_{G/\mathcal{B}} & 1 & \\ \sigma^{(r_2 r_1)}_{G/\mathcal{B}} & \sigma^{(r_2)}_{G/\mathcal{B}} & 1 \end{pmatrix} \in \mathcal{U}_-\left(H^T(G/\mathcal{B})\right)$$

Similarly the structure maps $\pi_L$ and $\pi_R : \mathcal{O}(\underline{h}) \to H^T(G/\mathcal{B})$ correspond to

$$h_L = \begin{pmatrix} \pi_L(\varrho_1) & & \\ & \pi_L(\varrho_2 - \varrho_1) & \\ & & \pi_L(-\varrho_2) \end{pmatrix}, \quad h_R = \begin{pmatrix} \pi_R(\varrho_1) & & \\ & \pi_R(\varrho_2 - \varrho_1) & \\ & & \pi_R(-\varrho_2) \end{pmatrix}$$

in $\underline{h}\left(H^T(G/\mathcal{B})\right)$.

Then the following relation holds.

$$\begin{pmatrix} 1 & & \\ \sigma_{G/B}^{(\pi_1)} & 1 & \\ \sigma_{G/B}^{(\pi_2\pi_1)} & \sigma_{G/B}^{(\pi_2)} & 1 \end{pmatrix} \cdot (e+h_L) = e+h_R$$

This implies the factorization

$$\begin{tikzcd} \mathcal{O}(\underline{k}+\underline{h}) \times \mathcal{U}_-) \arrow[rr] \arrow[dr] & & H^T(K/T) \\ & \mathcal{O}(\mathcal{U}'') \arrow[ur, dashed] & \end{tikzcd}$$

from before explicitly.

<u>Remarks</u> The map $S \otimes \mathcal{O}(\mathcal{U}_-) \to H^T(G/B)$ gives rise to (after applying $\otimes \mathbb{Z}$ and dualizing) a map $H_*(G/B) \longrightarrow \mathcal{U}(\underline{n}_-)$. So to any representation $V$ with highest weight $v^+$ one can define a subspace of $V$ by applying the image of $H_*(G/B)$ in $\mathcal{U}(\underline{n}_-)$ to $v^+$. If $v^+$ is of weight $\lambda$ then the map $H_*(G/B) \xrightarrow{\cdot v^+} V$ factors through $H_*(G/B) \to H_*(G/P_\lambda)$. It seems natural to ask whether the resulting map $H_*(G/P_\lambda) \longrightarrow V$ is injective. If $\lambda$ is miniscule then this map is in fact bijective.

There is also a similar construction for $H^*(\Omega K)$. It will be shown later that $H^*(\Omega K) \cong \mathcal{U}(\underline{n}_+^e)$. Therefore one can apply it to the lowest weight vector $v^-$ of a representation $V$ to obtain a subspace of that representation. If $V$ is miniscule we again recover all of $V$. (in types ADE). This is seen as follows.

Let $\underline{s}_+$ be the centralizer in $\underline{n}_+^{af}$ of $e+e_0 = e+te_{-\theta}$. Any representation of $G_{af}$ with miniscule highest weight $\varrho_i$ is isomorphic to $V(\varrho_0^{af})$, the representation with highest weight $\varrho_0$ (since there is an admissible graph automorphism of the extended Dynkin diagram taking the vertex $i$ to the 0 vertex). We have the following commutative diagram

$$\begin{tikzcd} V^*(\varrho_i) & V^*(\varrho_i^{af}) \arrow[l, twoheadrightarrow] \arrow[r, phantom, "\cong"] & V^*(\varrho_0^{af}) \\ \mathcal{U}(\underline{n}_+^e) \arrow[u] & & \mathcal{U}(\underline{s}_+) \arrow[ll, "ev_0"'] \arrow[u, "\cdot v^-"'] \end{tikzcd}$$

By a theorem in the Kac-Moody case, the map $\mathcal{U}(\underline{s}_+) \longrightarrow V^*(\rho_0^{af})$ on the right hand side is bijective. Hence the composition is surjective and so is $\mathcal{U}(\underline{n}_+^e) \longrightarrow V^*(\rho_i)$.

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From now on let us assume that $G$ is finite-dimensional and $\mathbb{F}$ a field.

__Lemma__: We have the following inclusion of $\mathbb{F}$-valued points (not schematically)

$$Z_G(e) \subseteq B.$$

__Proof__: Suppose $g \in Z_G(e)$. Then, by the Bruhat decomposition, $g = b_1 n b_2$ for $b_1, b_2 \in B$ and $n \in N_G(T)$. We have $b_1 n b_2 . e = e$, hence

$$n b_2 . e = b_1^{-1} . e .$$

Let $w \in W$ be the Weyl group element represented by $n$. Then the left hand side of the above equation lies in the sum of weight spaces $\bigoplus_{\alpha \in w.\Delta_+} \mathfrak{g}_\alpha$, while the right hand side has nonzero components in all the $\mathfrak{g}_{\alpha_i}$, for $\alpha_i \in \Pi$. Thus $\Pi \subseteq w.\Delta_+$, which implies that $w = \mathrm{id}$.

$\square$

Consider the morphism

$$\phi : (e + \underline{h}) \times \mathcal{U}_- \longrightarrow e + \underline{b}_-$$
$$(e+h, u) \longmapsto u^{-1}.(e+h).$$

Let $X := (e+\underline{h}) \times \mathcal{U}_-$ and $Y := e + \underline{b}_-$. Then $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are graded polynomial rings over $\mathbb{Z}$ in $N = \#\Delta_+$ generators, where the grading is given as follows. For $\mathcal{O}(X) = S \otimes \mathcal{O}(\mathcal{U}_-)$ let $S$ be graded as usual by $\deg \underline{h}^* = 2$, and $\mathcal{O}(\mathcal{U}_-)$ by $\deg \mathfrak{g}_{-\alpha}^* = 2ht(\alpha)$. The grading on $\mathcal{O}(Y) = \mathcal{O}(\underline{b}_-)$ is given by $\deg \mathfrak{g}_{-\alpha}^* = 2(ht(\alpha)+1)$. Then we get that

$$\phi^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$$

is a homomorphism of graded polynomial rings.

Choose homogeneous generators of $\mathcal{O}(Y)$ and $\mathcal{O}(X)$. So $\mathcal{O}(Y) = \mathbb{Z}[y_1, \dots, y_N]$ and $\mathcal{O}(X) = \mathbb{Z}[x_1, \dots, x_N]$.

<u>Lemma</u>: $\phi^*(y_1), \dots, \phi^*(y_N)$ form a regular sequence in $\mathcal{O}(X) \otimes \mathbb{F}$.

<u>Proof</u> Let $J = \langle \phi^*(y_1), \dots, \phi^*(y_N) \rangle$.
Since the $\phi^*(y_i)$ are homogeneous elements in a graded ring it suffices to show that the depth of $J$ (or equivalently $\sqrt{J}$) equals $N$. The following claim will imply that $\sqrt{J} = \langle x_1, \dots, x_N \rangle$ and hence this lemma.

<u>Claim</u>: Let $h \in \underline{h}(\mathbb{F})$ and $u \in \mathcal{U}_-(\mathbb{F})$, then

$$u^{-1}.(e+h) = e \quad \Rightarrow \quad u = 1$$

<u>Proof</u> Consider the semisimple part of $u.e = e+h$. Since the semisimple part of $e$ is zero it must be zero. On the other hand it must be conjugate to $h$. Hence $h = 0$, and $u.e = e$. So $u \in Z_G(e)(\mathbb{F})$ which by a previous lemma is contained in $\mathbb{B}(\mathbb{F})$. Therefore $u = 1$. $\square$

We aim to prove the following

<u>Theorem</u> The map

$$\mathcal{O}((e+\underline{h}) \times \mathcal{U}_-) \underset{e+\underline{b}}{\times} (e+\underline{h})) \longrightarrow H^T(G/\mathbb{B})$$

is an isomorphism.