Abstract

The concept of a topology is ubiquitous throughout mathematics. But even such a basic notion can come to its limits with various scenarios where the naive notion of a topology does causes several problems. In this talk we want to discuss certain ways of fixing these problems through condensed mathematics as well as cohesive/fractured structures on topoi. In particular, we will compare try to compare these notions which amounts to a fractured structure on condensed spaces.

Everything in this talk can and should probably be done in the ∞-world. However, just for the sake of this talk it makes no difference, so modulo the last section we will stick to a 1-categorical language for simplicity. The ∞-categorically minded person can however roughly just always replace **Set** by **An** and then basically everything said in this talk generalizes.

Anything new said in this talk (if anything) here is joint with Nima Rasekh.

Contents

1 Topology

We were all introduced to the notion of a topology early on - namely already in our first analysis lectures and for example used compactness to formulate the extreme value theorem from Bolzano and Weierstraß. Later we went on finally starting to study algebraic topology, e.g. through Stefan Schwede's lecture series, and learned about different invariants and constructions in topology. On the other hand, this notion of closeness also pertains most other parts of mathematics: be it algebraic geometry, number theory, functional analysis or probability theory. Nonetheless, even a notion as basic and ubiquitous as topology can have flaws.

1. Algebra: There are certain problems combining topology and algebra. For example, the category of topological abelian groups **TopAb** is not abelian since $\mathrm{id}_{\mathbb{R}} : \mathbb{R}_{\mathrm{disc}} \to \mathbb{R}_{\mathrm{eucl}}$ has trivial kernel and cokernel but is not an isomorphism. So it becomes difficult to perform homological category on this category. Similar examples would be Lie groups or group schemes.

2. Homotopy Theory: When working with homotopy types like the ∞-category of spaces/anima **An**, we can model spaces with actual spaces (i.e. CW complexes/Kan complexes). However, we forget much of its information and in particular the geometry/topology. For example, the topological spaces $\mathbb{R}$ and D^2 are quite different but as homotopy types they are both equivalent to a point $*$.

Moreso, a topology is by definition a collection of open sets on a set. In particular, we need an underlying set to talk about a topology. However, it sometimes seems helpful to ask for a topology on objects without a naturally underlying set.

Therefore, the quest for an axiomatization of certain properties of topology begins. We introduce two relevant notions: condensed mathematics and cohesive/fractured structures on a topos.

2 Some Topos Theory

The raison d'être of Grothendieck topoi is that they form the ideal universe in which one can do geometry. Indeed, a geometer wants to study geometric categories such as **Top**, **Mfld**, **Sch**, Var and so on but these categories are often ill-behaved. For example, **Mfld** and **Sch** do not admit all colimits. To remedy this problem the first naive attempt is to freely adjoin all colimits, i.e. to take the free cocompletion which amounts to taking the presheaf category:

$$\mathscr{C} \rightsquigarrow \mathbf{PSh}(\mathscr{C}).$$

However, these newly added colimits do not relate with the pre-existing colimits, so we need to demand additional conditions. Having done that, we end in a category of sheaves.

$$\mathscr{C} \rightsquigarrow \mathbf{PSh}(\mathscr{C}) \rightsquigarrow (a : \mathbf{PSh}(\mathscr{C}) \to \mathbf{Sh}(\mathscr{C}, J)).$$

This is the concept of a Grothendieck topos.

Definition 2.1. Let $\mathscr{C}$ be a category. Then, $\mathbf{PSh}(\mathscr{C}) = \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathbf{Set})$ is the **presheaf category** of $\mathscr{C}$.

Definition 2.2. A **Grothendieck topos** is a full subcategory $\mathscr{G} \hookrightarrow \mathbf{PSh}(\mathscr{C})$ for some category $\mathscr{C}$ which admits a left-exact[1] left adjoint $a : \mathbf{PSh}(\mathscr{C}) \to \mathscr{G}$.

Remark 2.3. Alternatively, a Grothendieck topos is a category which is equivalent to a sheaf category, i.e. a full subcategory of a presheaf category satisfying certain gluing conditions.

[1] I.e. finitelimit preserving.

Example 2.4. The prime example of a Grothendieck topos is the category of sheaves $\mathbf{Sh}(X)$ on a space X.

All of this generalizes to the world of ∞-categories roughly by replacing the 1-category **Set** by the ∞-category **An**. Topoi have amazing properties![2]

3 Condensed Mathematics

We motivate condensed mathematics via the Yoneda formalism. Let $\mathscr{T}$ be a full subcategory of **Top**. Consider the restricted Yoneda embedding:

$$\mathbf{Top} \to \mathrm{Fun}(\mathscr{T}^{\mathrm{op}}, \mathbf{Set}),\ (X \mapsto \mathrm{Hom}_{\mathbf{Top}}(-, X) : \mathscr{T}^{\mathrm{op}} \to \mathbf{Set}).$$

There are the two following extreme cases:

1. If $\mathscr{T} = *$, then this is the forgetful functor $\mathbf{Top} \to \mathbf{Set}$.
2. If $\mathscr{T} = \mathbf{Top}$, then this is the Yoneda embedding よ $: \mathbf{Top} \to \mathbf{PSh}(\mathbf{Top})$.

It's an insights from Clausen-Scholze and Barwick-Haine that $\mathscr{T} = \mathbf{CHaus}$ works really well and it's an explicit computation that the restricted Yoneda embedding is faithful in this case (even after passing to sheaves).

Definition 3.1 (Clausen-Scholze 2019)**.** A **condensed set** is a sheaf of sets on **CHaus** with coverings the jointly surjective families of maps.

Unravelled, a condensed set is a functor $X : \mathbf{CHaus}^{\mathrm{op}} \to \mathbf{Set}$ satisfying the following properties:

(i) $X(\emptyset) = *$,

(ii) For $S, T \in \mathbf{CHaus}$ the natural map

$$X(S \amalg T) \to X(S) \times X(T)$$

is bijective.

(iii) For any surjection $S' \to S$ in **CHaus** with projections $p_1, p_2 : S' \times_S S' \to S'$ the map

$$X(S) \to \{x \in X(S') : p_1^*(x) = p_2^*(x) \quad \text{in } X(S' \times_S S')\}$$

is bijective.

The same definition with sheaves of abelian groups defines **condensed abelian groups**.

Theorem 3.2. The category of condensed abelian groups is a nice abelian category.

Proof. See [Sch19, Theorem 1.10]. □

Definition 3.3. A **condensed anima** is a (hypercomplete) sheaf of anima on **CHaus** with coverings the jointly surjective families of maps. We denote the ∞-category of condensed anima by **Cond(An)**.

Remark 3.4. It's probably not the best terminology to call this a *condensed space* since condensed points to a topological aspect and so does a space. But really we want to use the condensed part to model the topology while we are really interested in the homotopy type for the latter being the soul of the model, i.e. the anima.

[2] E.g. they admit (sub-)object classifiers and have (weak) descent.

4 Cohesion

We will begin with the more classical notion of a cohesive topos first developed by William Lawvere [Law07]. Urs Schreiber generalized it to the ∞-world in his *Differential cohomology in a cohesive infinity-topos* [Sch13]. The word cohesion is inspired from chemistry which describes how molecules stick together. In that sense mathematical cohesion is supposed to describe how points 'cohere' or 'stick together'.

Let us start with a set, then there are two universal ways of endowing a topology on this set. Either we take the finest topology or we take the coarsest topology on this set. In jargon, we take the discrete resp. the codiscrete topology. These yield two fully faithful functors which result in a triple of adjoint functors

$$\mathbf{Top} \;\begin{array}{c} \xleftarrow{\;\mathrm{Disc}\;} \\ \xrightarrow{\;U\;} \\ \xleftarrow[\;\mathrm{CoDisc}\;]{} \end{array}\; \mathbf{Set}.$$

If we choose **Top** nice enough, e.g. only taking locally path-connected spaces, then there is a further left adjoint $\pi_0 \dashv \mathrm{Disc}$ which yields the adjunction quadruple

$$\mathbf{Top} \;\begin{array}{c} \xrightarrow{\;\pi_0\;} \\ \xleftarrow{\;\mathrm{Disc}\;} \\ \xrightarrow{\;U\;} \\ \xleftarrow[\;\mathrm{CoDisc}\;]{} \end{array}\; \mathbf{Set}.$$

This is the prime example of cohesion.

Definition 4.1. Let $\mathscr{X}$ be a topos[3] over a topos $\mathscr{Y}$ via the map $f_* : \mathscr{X} \to \mathscr{Y}$.

(i) If f_* admits fully faithful adjoints

$$\mathscr{X} \;\begin{array}{c} \xleftarrow{\;f^*\;} \\ \xrightarrow{\;f_*\;} \\ \xleftarrow[\;f^!\;]{} \end{array}\; \mathscr{Y},$$

then $\mathscr{X}$ is called **local** over $\mathscr{Y}$.

(ii) It $\mathscr{X}$ is local over $\mathscr{Y}$ and f^* admits a further left adjoint $f_!$ which is product-preserving, then $\mathscr{X}$ is called **cohesive** over $\mathscr{Y}$.

$$\mathscr{X} \;\begin{array}{c} \xrightarrow{\;f_!\;} \\ \xleftarrow{\;f^*\;} \\ \xrightarrow{\;f_*\;} \\ \xleftarrow[\;f^!\;]{} \end{array}\; \mathscr{Y}$$

So if $\mathscr{X}$ is cohesive over $\mathscr{Y}$, then $\mathscr{Y}$ embeds in two ways into $\mathscr{X}$, namely via f^* and $f^!$.

Definition 4.2. Let $\mathscr{X}$ be an topos and let $*_{\mathscr{X}}$ be a terminal object in $\mathscr{X}$. Then, the **global sections functor** is given by

$$\Gamma = \mathrm{Hom}_{\mathscr{X}}(*_{\mathscr{X}}, -) : \mathscr{X} \to \mathbf{Set},\; \mathcal{F} \mapsto \mathrm{Hom}_{\mathscr{X}}(*_{\mathscr{X}}, \mathcal{F}).$$

[3]Everything can be done a bit more generally than over topoi, so over motivating example is still valid.

Since Γ is limit-preserving, it admits a left-adjoint Disc : $\mathbf{Set} \to \mathscr{X}$ by the Adjoint Functor Theorem. In fact, it can be described explicitly as $\mathrm{Disc} = *_{\mathscr{X}} \otimes - : \mathbf{Set} \to \mathscr{X}$.

Definition 4.3. Let $\mathscr{X}$ be a topos.

(i) If $\Gamma : \mathscr{X} \to \mathbf{Set}$ is local, then $\mathscr{X}$ is a **local topos**.

(ii) If $\Gamma : \mathscr{X} \to \mathbf{Set}$ is cohesive, then $\mathscr{X}$ is a **cohesive topos**.

Remark 4.4. If $\mathscr{X}$ is a cohesive topos, then we will employ the following notation:

$$\mathscr{X} \;\; \begin{array}{c} \xrightarrow{\Pi} \\ \xleftarrow{\text{Disc}} \\ \xrightarrow{\Gamma} \\ \xleftarrow[\text{CoDisc}]{} \end{array} \;\; \mathbf{Set}$$

The notation follows our geometric intuition:

- The functor Π is like the set of connected components of a space.
- The functor Disc is like the discrete 'topology/cohesion'.
- The functor Γ is like the global sections functor.
- The functor CoDisc is like the codiscrete 'topology/cohesion'.

Compare this with the motivating example!

Example 4.5. The homotopy theorist's favourite cohesion might be

$$\mathbf{sSet} \;\; \begin{array}{c} \xrightarrow{\pi_0} \\ \xleftarrow{\text{const}} \\ \xrightarrow{(-)_0} \\ \xleftarrow[E]{} \end{array} \;\; \mathbf{Set}.$$

where E denotes the bar construction.

Example 4.6. There is cohesion in global homotopy theory: Let G be a compact Lie group, then there is cohesion

$$(\mathbf{Top_{Glo}})_{/\mathbb{B}G} \;\; \begin{array}{c} \xrightarrow{\Pi_G} \\ \xleftarrow{\Delta_G} \\ \xrightarrow{\Gamma_G} \\ \xleftarrow[\nabla_G]{} \end{array} \;\; G\text{-}\mathbf{Top}.$$

which we will not further elaborate on. See [Rez14, Chapter 5].

5 Fractured Structure

Cohesion is nice but sometimes the topos in question is simply not cohesive. This happens in the condensed setting! So instead one may try to weaken our notion and try to obtain a slightly more general concept that contains more objects but still keeps many of the nice properties of cohesion. This will be the notion of a fractured structure developed by Lurie [Lur18] and Carchedi [Car20].

Definition 5.1. Let $\mathscr{X}$ be a topos. A subcategory $j_! : \mathscr{X}^{\text{corp}} \hookrightarrow \mathscr{X}$ is a **fracture subcategory** if it satisfies the following conditions:

(i) If $X \in \mathscr{X}^{\mathrm{corp}}$ and $f : X \to Y$ in $\mathscr{X}$ is an isomorphism, then f belongs to $\mathscr{X}^{\mathrm{corp}}$.

(ii) The category $\mathscr{X}^{\mathrm{corp}}$ admits pullbacks and these are preserved by $j_!$.

(iii) The inclusion functor $j_! : \mathscr{X}^{\mathrm{corp}} \hookrightarrow \mathscr{X}$ admits a right adjoint $j^* : \mathscr{X} \to \mathscr{X}^{\mathrm{corp}}$ which is conservative and preserves small colimits.

$$\mathscr{X}^{\mathrm{corp}} \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathscr{X}$$

(iv) For every map $U \to V$ in $\mathscr{X}^{\mathrm{corp}}$ the diagram

$$\begin{array}{ccc} j^*U & \longrightarrow & j^*V \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

given by the counit $j_! j^* \Rightarrow \mathrm{id}_{\mathscr{X}}$ is a pullback in $\mathscr{X}$.

A **fractured topos** is a pair $\mathscr{X}^{\mathrm{corp}} \hookrightarrow \mathscr{X}$ where $\mathscr{X}$ is a topos and $\mathscr{X}^{\mathrm{corp}}$ is a fracture subcategory of $\mathscr{X}$.

It will turn out that $\mathscr{X}^{\mathrm{corp}}$ is a topos, so then the condition that j^* preserves small colimits is equivalent to it admitting a right adjoint. So we then obtain a triple adjunction

$$\mathscr{X}^{\mathrm{corp}} \quad \overset{j_!}{\longrightarrow},\ \overset{j^*}{\longleftarrow},\ \underset{j_*}{\longrightarrow} \quad \mathscr{X}$$

Remark 5.2. Intuitively, a fractured structure is locally a cohesive structure. This is not completely true but at least almost, namely in the following sense.

Let $j_! : \mathscr{X}^{\mathrm{corp}} \hookrightarrow \mathscr{X}$ be a fractured topos and $X \in \mathscr{X}^{\mathrm{corp}}$. Since $j^* : \mathscr{X} \to \mathscr{X}^{\mathrm{corp}}$ preserves small colimits, it admits a right adjoint $j_* : \mathscr{X}^{\mathrm{corp}} \to \mathscr{X}$ by the Adjoint Functor Theorem. In particular, this yields a triple adjunction for slice categories[4]

$$\mathscr{X}^{\mathrm{corp}}_{/X} \quad \overset{(j_!)_{/X}}{\longrightarrow},\ \overset{(j^*)_{/X}}{\longleftarrow},\ \underset{(j_*)_{/X}}{\longrightarrow} \quad \mathscr{X}_{/X}$$

Here, $(j_!)_{/X}$ is fully faithful and preserves fiber products by definition. It preserves the terminal object $\mathrm{id}_X \in \mathscr{X}_{/X}$ and so it preserves finite limits. So the requirements for the Adjoint Functor Theorem are almost fulfilled to yield another left adjoint of $(j_!)_{/X}$. It is furthermore preserves products, then this would result in a quadruple adjunction realizing a cohesive structure.

[4]The passage to slice categories is what we mean by the word 'locally'.

6 Comparison of Fractured Structures with Condensed Mathematics

We have presented two ways of doing topology and now we want to compare these.

Lemma 6.1. The ∞-topos $\mathbf{Cond}(\mathbf{An})$ is not a cohesive ∞-topos.

Let $\mathbf{CHaus}^{\mathrm{inj}}$ denote the wide subcategory of compact Hausdorff spaces with injections as maps and finitely jointly surjective morphisms as covers. Then, the inclusion $i : \mathbf{CHaus}^{\mathrm{inj}} \to \mathbf{CHaus}$ induces via Kan extension a triple adjunction

$$\begin{tikzcd} \mathbf{PSh}(\mathbf{CHaus}) \arrow[r, "i^*"] & \mathbf{PSh}(\mathbf{CHaus}^{\mathrm{inj}}). \arrow[l, bend right, "i_!"'] \arrow[l, bend left, "i_*"] \end{tikzcd}$$

We denote by $\mathbf{Cond}^{\mathrm{inj}}(\mathbf{An})$ the category of sheaves on $\mathbf{CHaus}^{\mathrm{inj}}$, then one can show that the diagram extends to a triple adjunction of sheaf categories

$$\begin{tikzcd} \mathbf{Cond}(\mathbf{An}) \arrow[r, "i^*"] & \mathbf{Cond}^{\mathrm{inj}}(\mathbf{An}). \arrow[l, bend right, "i_!"'] \arrow[l, bend left, "i_*"] \end{tikzcd}$$

Lemma 6.2. The functor i^* is not a geometric morphism.

So it doesn't even make sense to ask whether $\mathbf{Cond}(\mathbf{An})$ is cohesive over $\mathbf{Cond}^{\mathrm{inj}}(\mathbf{An})$ via i^*.

Theorem 6.3. There is a triple adjunction

$$\begin{tikzcd} \mathbf{Cond}(\mathbf{An}) \arrow[r, "i^*"] & \mathbf{Cond}^{\mathrm{inj}}(\mathbf{An}) \arrow[l, bend right, "i_!"'] \arrow[l, bend left, "i_*"] \end{tikzcd}$$

yielding a fractured structure on the ∞-topos of condensed anima.

Proof Sketch. Use/Study the machinery of admissibility structures by Lurie [Lur18, Chapter 20.2, 20.3, 20.6], define a suitable admissibility structure on $\mathbf{CHaus}$ and check compatibilities with everything floating around. □

7 Outlook

We have the following remaining goals for the project:

- Can we generalize $\mathbf{An}$ to an arbitrary ∞-topos $\mathscr{X}$?
- Can we use this fractured structure to obtain certain results in the computation of cohomology groups? The hope is to get a formal argument for rather ε-heavy proofs by Clausen-Scholze.

And as always in mathematics, this is only a fractured part of what can be possible...

References

[Car20] David Joseph Carchedi. Higher orbifolds and Deligne-Mumford stacks as structured infinity-topoi. *Mem. Amer. Math. Soc.*, 264(1282):v+120, 2020.

[Law07] F. William Lawvere. Axiomatic cohesion. *Theory Appl. Categ.*, 19:No. 3, 41–49, 2007.

[Lur18] Jacob Lurie. Spectral algebraic geometry. Unpublished notes, February 2018.

[Rez14] Charles Rezk. Global homotopy theory and cohesion. 2014. Unpublished notes.

[Sch13] Urs Schreiber. Differential cohomology in a cohesive infinity-topos. *arXiv preprint*, 2013. arXiv:1310.7930.

[Sch19] Peter Scholze. Lectures on condensed mathematics. 2019. Unpublished notes.