What is a topological structure?

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10.02.2023

The goal is to understand what "topology" means. Everything new I say (if I do) will be joint work with Qi Zhu.

1 Topology is Cool!

The story of topological spaces goes back to ideas of Hausdorff in order to study "sets with a notion of closeness" and generalizes examples such as metric spaces from analysis, manifolds and schemes from geometry up to Polish spaces in logic.

By its very definition the notion of "topology" is tied to sets. In fact the definition proposed by Hausdorff (what we now call Hausdorff topological space) is introduced in the book "Principles of Set Theory". This naturally suggests the question how to define topologies in settings other than sets. Two key examples that are of particular relevance are:

1. Algebraic structures: We can wonder how to infuse topologies with groups, rings, modules, ... This way we get relevant examples such as group schemes or Lie groups or topological vector spaces.

2. Homotopy types: We can wonder how to add topologies to "homotopy types" meaning the ∞-category of spaces or similar ∞-categories. Let us see one example to understand its relevance. One classical result in topology is the Brouwer fixed point theorem, which states that every morphisms from the disc to itself has a fixed point. While there has been a general trend in homotopy theory to move from topological spaces to homotopy types, this example does not work as in the ∞-category of spaces as the disc is contractible and so the statement becomes trivial. So, in order to be able to state and prove a "Brouwer fixed point theorem" in a higher categorical setting, we need to "topologize homotopy types".

2 Naive Generalizations

Here is our first naive effort towards generalizing topologies. Let $\mathcal{C}$ be a category with a given functor $U : \mathcal{C} \to \mathcal{S}\mathrm{et}$, which we intuitively think of as the "underlying set". Define $\mathcal{T}\mathrm{op}(\mathcal{C})$ as the pullback of the diagram $\mathcal{C} \to \mathcal{S}\mathrm{et} \leftarrow \mathcal{T}\mathrm{op}$. Intuitively this is saying we have an object with a topology on the underlying set.

Let's check some examples:

- If $\mathcal{C} = \mathcal{S}\mathrm{et}$ and U the identity we get usual topologies. Good!

- If $\mathcal{C} = \mathcal{T}\mathrm{op}$, then we get something meaningless, a set with two topologies and morphisms continuous maps with respect to both topologies. X

- If $\mathcal{C} = \mathcal{A}\mathrm{b}$, then we get "topological abelian groups". This might appear reasonable, however, its a problematic category. For example, the category of topological abelian groups is not even an abelian category. Indeed, the identity map $\mathbb{R}_{disc} \to \mathbb{R}_{class}$ has no kernel or cokernel, but is not an isomorphism (the inverse is not continuous). We can see that the same applies to topological vector spaces.

- Let X be a set and $\mathcal{C} = \mathcal{S}\text{et}_{/X}$. Then $\mathcal{T}\text{op}(\mathcal{C})$ has objects non-continuous morphisms $T \to X$. We get similar results when looking at $\mathcal{S}\text{et}_{X/}$.

We have the same problem when mingling things with homotopy theory.

- Let $\mathcal{C} = \mathcal{S}$, the ∞-category of homotopy types. Then $\mathcal{T}\text{op}(\mathcal{S})$ is the ∞-category with objects in $\mathcal{S}$ with a topology on its path-components. In particular, if we take $\mathcal{S}^{cn}$ the full subcategory of connected objects, then $\mathcal{T}\text{op}(\mathcal{S}^{cn}) \simeq \mathcal{S}^{cn}$, meaning there exists no meaningful topology.

3 A first Generalization: Condensed stuff

As we just saw our first approach failed miserably, so let's try something else. Our approaches hinges on redefining what a topology is.

Here is the basic idea: Let $\mathcal{T}$ be a full subcategory of $\mathcal{T}\text{op}$. Then this induced a functor $\text{Hom}(\mathcal{T}, -) : \mathcal{T}\text{op} \to \text{Fun}(\mathcal{T}^{op}, \mathcal{S}\text{et})$, which takes a topological space X to the restricted presheaf $\text{Hom}(-, X) : \mathcal{T}^{op} \to \mathcal{S}\text{et}$.

1. If $\mathcal{T}$ is the full subcategory with one object being the one point space, then $\text{Hom}(\mathcal{T}, -)$ is just the forgetful functor to set.

2. If $\mathcal{T} = \mathcal{T}\text{op}$, then this is just the Yoneda embedding.

So, those are two extreme cases: one forgets everything about the topology, the other remembers everything about the topology (via the Yoneda lemma) and so what we want is to pick a good middle case that remembers some thing in a topologically effective manner.

The insight of Clausen and Scholze [Sch19] was to try the full subcategory of compact Hausdorff spaces $\mathcal{CH}\text{aus}$. Then we use the Yoneda embedding $\mathcal{CH}\text{aus} \to \text{Fun}(\mathcal{CH}\text{aus}^{op}, \mathcal{S}\text{et})$, but as usual we like to identity the representable presheaf of $U \cup V$ with the pushout $y_U \coprod_{y_{U \cap V}} y_V$. Hence, we focus on the subcategory of sheaves. That's what we call *condensed sets* denote $\mathcal{C}\text{ond}(\mathcal{S}\text{et})$.

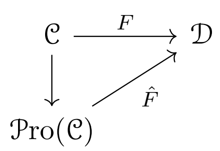
Remark 3.1. Depending on which set-theoretical assumptions one makes on the source and target this could also be known as *pyknotic sets* [BH19]. We will not go into this and just use the terminology condensed throughout.

Now, notice, the inclusion functor $\mathcal{T}\text{op} \to \mathcal{C}\text{ond}(\mathcal{S}\text{et})$ is not generally fully faithful, however, it is faithful and also full when restricted to compact Hausdorff spaces.

While very insightful from a conceptual perspective, we can use a more computational lens.

Definition 3.2. Let $\mathcal{C}$ be a category. A pro-object is a cofiltered diagram $F : I \to \mathcal{C}$, meaning I is cofiltered category. The collection of pro-objects in $\mathcal{C}$ assemble into a category, called the category of pro-objects $\mathcal{P}\text{ro}(\mathcal{C})$. It is formally defined as $\mathcal{I}\text{nd}(\mathcal{C}^{op})^{op}$, meaning the full subcategory of $\text{Fun}(\mathcal{C}, \mathcal{S}\text{et})^{op}$ consisting of objects given by filtered colimits of representables.

The definition already suggests a reasonable universal property. For every category $\mathcal{D}$ closed under cofiltered limits and functor $F : \mathcal{C} \to \mathcal{D}$ there exists a unique lift that preserves all cofiltered limits



The relevant example are pro-finite sets $\mathcal{P}\mathrm{ro}(\mathcal{F}\mathrm{in})$. Following the universal property described above, there exists a diagram

$$\begin{array}{ccc} \mathcal{F}\mathrm{in} & \xrightarrow{Disc} & \mathcal{T}\mathrm{op} \\ \downarrow & \nearrow_{D} & \\ \mathcal{P}\mathrm{ro}(\mathcal{F}\mathrm{in}) & & \end{array}$$

One of the key results [RZ10, Theorem 1.1.12] is that $D : \mathcal{P}\mathrm{ro}(\mathcal{F}\mathrm{in}) \to \mathcal{T}\mathrm{op}$ is fully faithful and the essential image is given by *Stone spaces* which can equivalently be described as totally disconnected compact Hausdorff spaces. In particular, the functor above factors as $\mathcal{P}\mathrm{ro}(\mathcal{F}\mathrm{in}) \to \mathcal{CH}\mathrm{aus}$, which induces a diagram

$$\mathrm{Fun}(\mathcal{CH}\mathrm{aus}^{op}, \mathcal{S}\mathrm{et}) \to \mathrm{Fun}(\mathcal{P}\mathrm{ro}(\mathcal{F}\mathrm{in})^{op}, \mathcal{S}\mathrm{et}).$$

This functor is evidently not an equivalence of any sort. However, if we restrict ourselves to sheaves for jointly surjective families of morphisms, then we in fact get an equivalence of categories [Yam22], hence, we can describe condensed sets also as "sheaves on profinite sets", which is very computationally feasible: A sheaf is a functor $F : \mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}^{op} \to \mathcal{S}\mathrm{et}$ such that

- $F(T_1 \coprod T_2) \to F(T_1) \times F(T_2)$ is a bijection for all profinite sets T_1, T_2, and
- for all parallel arrows of pro-finite sets $f, g : T_1 \to T_2$ with equalizer E, $T(E)$ is the equalizer of $T(f), T(g)$.

Hence, we think of *condensed sets* as our "new topological spaces". Given that how can we define topological objects?

Definition 3.3. Let $\mathcal{C}$ be a category. Let $\mathcal{C}\mathrm{ond}(\mathcal{C})$, the *category of condensed objects in* $\mathcal{C}$ to be the category of sheaves with value in $\mathcal{C}$.

Let us see very concretely how this can help us solve a problem we had before.

Example 3.4. As we saw before the map $\mathbb{R}_{disc} \to \mathbb{R}_{class}$ has a trivial cokernel, meaning it is surjective. However, what happens when consider them as condensed abelian groups? By construction, the condensed abelian groups are given by $S \mapsto \mathrm{Hom}(S, \mathbb{R}_{disc})$ and $S \mapsto \mathrm{Hom}(S, \mathbb{R}_{class})$. When $S = *$ they precisely recover the original topological spaces with the identity and so $Q(*)$ is trivial again, however, for a general S, $\mathrm{Hom}(S, \mathbb{R}_{disc})$ is given by locally constant morphisms, whereas $\mathrm{Hom}(S, \mathbb{R}_{class})$ is given by the continuous ones, so we have

$$Q(S) = \{f : S \to \mathbb{R} : f \text{ continuous}\}/\{f : S \to \mathbb{R} : f \text{ locally constant}\}$$

This result is not a coincidence and in fact we have the following.

Theorem 3.5 ([Sch19])**.** *The category of condensed abelian groups is an abelian category.*

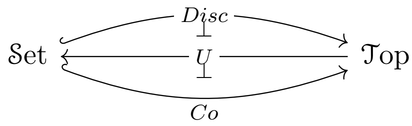
Similarly, we can now define a good notion of topologized homotopy types.

Definition 3.6. A condensed homotopy type/space is a condensed object in the ∞-category of spaces (this is what Scholze now calls "condensed anima").

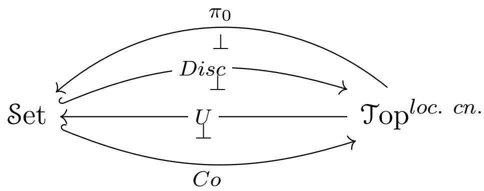
Notice from the definition it follows immediately that $\mathcal{C}\mathrm{ond}(\mathcal{S})$ is an ∞-topos, with all its bells and whistles, in particular it is locally Cartesian closed, it satisfies descent and has nice "univalent universes".

4 Theoretical Framework: Local & Cohesive Topos

This last example brings us to a second way of imposing some topological structure into our setting: locality and cohesion. Given a set how can we topologize it? There are two standard ways: the discrete topology and the codiscrete topology, both of which are fully faithful. This assembles into the following very nice diagram



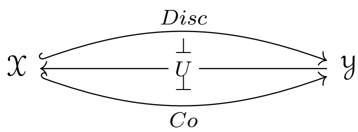
Finally, with some minor conditions on $\mathcal{T}\mathrm{op}$ we can further compute path-components of a topological spaces getting the following diagram.



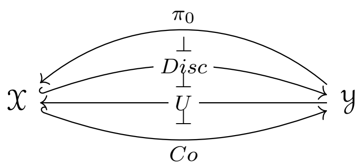
Notice here, $Disc, Co$ are fully faithful and π_0(when it exists) commutes with products. What we would like is a formalization of this data. This goes back to ideas of Lawvere who was working on topos theory [Law07]. The category of topological spaces are messed up and so what we would want to do is to replace Top with a better category which fits into this diagram.

Remark 4.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be two topoi. Recall that $\mathcal{Y}$ is over $\mathcal{X}$, or $\mathcal{X}$ is a base topos, if there exists a left adjoint $f^* : \mathcal{X} \to \mathcal{Y}$ commuting with finite limits.

Definition 4.2. A topos $\mathcal{Y}$ over $\mathcal{X}$ via U is *local*



with Co, $Disc$ fully faithful, and moreover is *cohesive* if there exists a further left adjoint

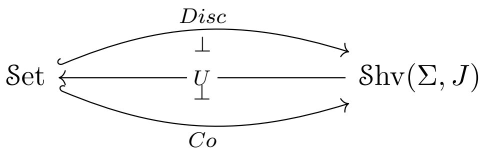


with π_0 product preserving.

Remark 4.3. If $\mathcal{X} = \mathcal{S}\mathrm{et}$ and $U : \mathrm{Hom}_{\mathcal{Y}}(1, -)$ then we just say $\mathcal{Y}$ is local or cohesive.

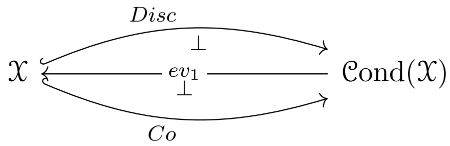
As we just noted topological spaces don't actually fit into this diagram as it is not a topos, but it's not too hard to find nice examples. Let's start with an old example similar to condensed stuff. in [Joh79] of a topos very close to topological spaces, that is in fact local.

Example 4.4. Let Σ be the full subcategory of $\mathcal{T}\mathrm{op}$ with two objects 1 and the 1-point compactification of $\mathbb{N}$. Then the category of sheaves $\mathcal{S}\mathrm{hv}(\Sigma, J)$, where J has the canonical topology permits a faithful functor from $\mathcal{T}\mathrm{op}$, which is fully faithful when restricting to sequential spaces, meaning spaces which have the universal property that continuous maps out of them are determined by preserving convergent sequences [Joh79, Lemma 2.1]. Moreover, based on [Joh79] we can show $\mathcal{S}\mathrm{hv}(\Sigma, J)$ is local, meaning there is a diagram



The historical example suggests also the example relevant to us.

Example 4.5 ([BH19]). Notice, $\mathcal{C}\mathrm{ond}(\mathcal{X})$ is a topos over $\mathcal{X}$ via composition $\mathcal{X} \to \mathrm{Fun}(\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}^{op}, \mathcal{X}) \to \mathcal{C}\mathrm{ond}(\mathcal{X})$. Now, $\mathcal{C}\mathrm{ond}(\mathcal{X})$ is local over $\mathcal{X}$ as we have the diagram of adjunctions



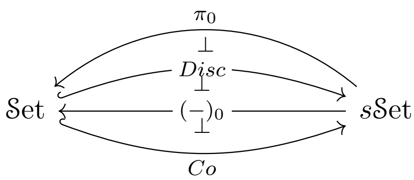
where the functors $Co, Disc$ are in fact fully faithful. In fact, we can more explicitly describe $Co(X)(K) = \prod_{|K|} X$. This fact resembles the example we saw above.

That's a good first step. Can we advance this to a cohesive structure? No!

Example 4.6. ([BH19, Example 2.2.14]) Take the functor $Disc : \mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in} \to \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{et})$. It is given as $Disc(S) = \coprod_S 1$, where 1 is the terminal condensed set. On the other hand, take a cofiltered diagram of finite sets, then the limit in $\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{et})$ will be the corresponding Stone space given via the embedding $\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in} \to \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{et})$, meaning it is not of the form $\coprod_S 1$.

Here is the homotopy theorists favorite example.

Example 4.7. $s\mathcal{S}\mathrm{et}$ is cohesive. Explicitly, this means we have the diagram

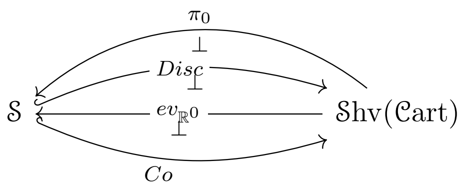


Here $Disc(S)$ is given by $Disc(S)_n = S$ (the discrete Kan complex with vertex set S) and $Co(S)$ is given by $Co(S)_n = S^{n+1}$ (the contractible Kan complex with vertex set S).

Indirectly, homotopy theorists use all the time this cohesive structure on $s\mathcal{S}\mathrm{et}$ when doing homotopy theory. Notice of course, here there is nothing special about $\mathcal{S}\mathrm{et}$ and one could use any other category (or ∞-category) to get a similar diagram of adjunctions. Another elegant example cohesion that I will not discuss in further detail, but is worth mentioning, is global equivariant homotopy theory, whose cohesion was established by Rezk [Rez14].

Finally, let us come back to one of the original objections. In [Shu18] Shulman defines "real-cohesion", which is a specific class of cohesive ∞-topoi.

Example 4.8. Let $\mathcal{C}\mathrm{art}$ be the 1-category with objects $\mathbb{R}^n$ and morphisms continuous maps. Take the evident Grothendieck topology of finitely jointly surjective morphisms in $\mathbb{R}^n$ and use that to define the ∞-category of sheaves $\mathcal{S}\mathrm{hv}(\mathcal{C}\mathrm{art})$. Then $\mathcal{S}\mathrm{hv}(\mathcal{C}\mathrm{art})$ is cohesive [Sch13, Proposition 4.3.2], meaning we have the diagram



Notice, in particular in $\mathcal{S}\mathrm{hv}(\mathcal{C}\mathrm{art})$ the object $\mathbb{R}^1$ is contractible and we can use that to abstractly prove Brouwers fixed point theorem, as we now have a disc $\mathbb{R}^2$ and a "topological circle" as the coequalizer of $\mathrm{id}, +1 : \mathbb{R} \to \mathbb{R}$. Indeed it holds in every cohesive ∞-topos which has this property, meaning it is real-cohesive.

5 Better Theoretical framework: Fractured Topos

In that last section we saw there is a limit on how well we can understand $\mathcal{C}\mathrm{ond}(\mathcal{X})$ based on $\mathcal{X}$. Concretely, as we will see, we would hope to use such categorical frameworks to study cohomologies. So how about other frameworks and sub-categories?

There are further framework proposed by Lurie [Lur18] based on ideas of Carchedi [Car20] and very much motivated by certain ∞-topoi arising in geometry. This requires us to move away from base topoi towards more general adjunctions.

Definition 5.1. A triple of adjunctions of ∞-topoi

$$\mathcal{X} \; \overset{f_!}{\longleftarrow} \; \xrightarrow{\;f^*\;} \; \underset{f_*}{\longleftarrow} \; \mathcal{Y}$$

is a fracture structure if for all objects X in $\mathcal{X}$, the induced functor $f^* : \mathcal{X}_{/X} \to \mathcal{Y}_{/X}$ is local, which concretely means $f_!, f_*$ are fully faithful and $f_!$ preserves finite limits.

Relevant examples are not as easy to see, but can be found in [Clo21].

Example 5.2. ([Clo21, Example 4.1.8]) For A any commutative ring, Aff_A^{fp} denote the category of finitely presented affine schemes. Consider the triple $(Aff_A^{fp}, (Aff_A^{fp})^{Zar,emb}, \tau_{Zar})$ consists of the Zariski open embeddings, and τ_{Zar} is the Zariski topology. Then for any given ∞-topos $\mathcal{X}$, the induced adjunction

$$\mathcal{S}\mathrm{hv}_{\mathcal{X}}((Aff_A^{fp})^{Zar,emb}) \; \overset{Disc}{\hookrightarrow} \; \overset{\perp}{\longleftarrow} \; \underset{Co}{\hookrightarrow} \; \mathcal{S}\mathrm{hv}_X(Aff_A^{fp})$$

gives us a fractured structure.

Notice in this example we are trying to understand Zariski sheaves by focusing on diagrams based on embeddings and this is a guiding principle for us. This example motivates us to pursue the following direction when studying condensed objects.

Definition 5.3. Let $\mathcal{CH}\mathrm{aus}^{inj}$ be the (wide but not full) subcategory of $\mathcal{CH}\mathrm{aus}$ with the same objects but with morphisms injections. Notice, $\mathcal{CH}\mathrm{aus}^{inj}$ comes with an evident Grothendieck topology given by finitely jointly surjective collection of morphisms. We denote by $\mathcal{C}\mathrm{ond}^{inj}(\mathcal{C})$, the category of $\mathcal{C}$-valued sheaves based on this topology.

The evident inclusion functor $\mathcal{I}\mathrm{nc} : \mathcal{CH}\mathrm{aus}^{inj} \to \mathcal{CH}\mathrm{aus}$ induces a diagram of adjunctions

$$\mathrm{Fun}((\mathcal{CH}\mathrm{aus})^{op}, \mathcal{X}) \; \overset{\mathcal{I}\mathrm{nc}_!}{\longleftarrow} \; \xrightarrow{\;\mathcal{I}\mathrm{nc}^*\;} \; \underset{\mathcal{I}\mathrm{nc}_*}{\longleftarrow} \; \mathrm{Fun}((\mathcal{CH}\mathrm{aus}^{inj})^{op}, \mathcal{X})$$

Now, it is a direct computation that $\mathcal{I}\mathrm{nc}^*$ and $\mathcal{I}\mathrm{nc}_*$ preserves sheaves. This means the triple adjunctions restrict to a triple adjunction of categories of sheaves

$$\mathcal{C}\mathrm{ond}(\mathcal{X}) \; \overset{\mathcal{I}\mathrm{nc}_!}{\longleftarrow} \; \xrightarrow{\;\mathcal{I}\mathrm{nc}^*\;} \; \underset{\mathcal{I}\mathrm{nc}_*}{\longleftarrow} \; \mathcal{C}\mathrm{ond}^{inj}(\mathcal{X})$$

What can we say about this?

Lemma 5.4. $\mathcal{C}\mathrm{ond}(\mathcal{X})$ *is not cohesive over* $\mathcal{C}\mathrm{ond}^{inj}(\mathcal{X})$. *It is not even local. It is not even over.*

Proof. The functor $\mathcal{I}\mathrm{nc}_!$ does not preserve the terminal object. Indeed, the terminal object in $\mathcal{C}\mathrm{ond}^{inj}(\mathcal{X})$ is not representable. □

Given that all other approaches failed here we are left with checking the fractured structure.

Theorem 5.5 (Hopefully, R.–Zhu)**.** *Let* $\mathcal{X}$ *be an* ∞*-topos. Then*

$$\mathcal{C}\mathrm{ond}(\mathcal{X}) \underset{\mathcal{I}\mathrm{nc}_*}{\overset{\mathcal{I}\mathrm{nc}_!}{\leftleftarrows}} \xrightarrow{\ \mathcal{I}\mathrm{nc}^*\ } \mathcal{C}\mathrm{ond}^{inj}(\mathcal{X})$$

gives us the structure of a fractured ∞*-topos.*

Sketch of Proof. Here is an idea of a proof. We can generate certain (many) fractured structures by choosing an appropriate collection of morphisms in our chosen ∞-topos. This idea has been formalized by Lurie via *admissibility structures*.

Hence the steps of the proof are given by:

1. Defining and studying admissibility structures [Lur18, Definition 20.2.1.1]. This is a class of maps containing equivalences, closed under pullbacks, and satisfying "backhand 2-out-of-3".

2. Showing that admissibility structures on a small ∞-category corresponds to a fractured structure on its category of presheaves [Lur18, Theorem 20.2.4.1].

3. Define *local admissibility structures* as a mild variation [Lur18, Definition 20.3.2.1].

4. Show that local admissibility structures compatible with a certain Grothendieck topology (known as Geometric sites [Lur18, Definition 20.3.4.1]) correspond to a fractured structure on the corresponding ∞-category of sheaves [Lur18, Theorem 20.3.4.4].

□

6 Why do we care? Cohomologies?

There is of course a certain conceptual satisfaction to having such nice comparison, but are there any concrete benefits? So, here is a (currently aspirational) benefit: cohomology!

Let C be a compact topological space. There are several ways to define "cohomology".

1. Singular Cohomology: Classical algebraic topology via cochains.

2. Cech Cohomology: Taking Cech covers, then nerves and then using that compute cohomology.

3. Sheaf Cohomology: Taking the category of sheaves in abelian groups on X and then computing sheaf cohomology, by deriving the global sections functor $\Gamma : \mathcal{S}\mathrm{hv}(X) \to \mathcal{A}\mathrm{b}$ with respect to an injective resolution.

Now, finally, given that we have a functor $\mathcal{T}\mathrm{op} \to \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{et})$, we can take the image of every compact Hausdorff space S in $\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{et})$. Using the fact that $\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{et})$ is a topos, we can define a fourth cohomology theory.

4. Condensed Cohomology: Taking the cohomology of the global section in the topos $\mathcal{C}\text{ond}(\mathcal{S}\text{et})$ at the object S. Concretely, taking a hyper-cover of representables $S_\bullet \to S$ and then evaluating the cohomology of the resulting complex

$$0 \to \Gamma(S_0, \mathbb{Z}) \to \Gamma(S_1, \mathbb{Z}) \to ...$$

In [Sch19, Theorem 3.2] Scholze proves that sheaf cohomology of a compact Hausdorff space coincides with the condensed cohomology of its associated condensed set. This has also been observed by Haine, who in fact generalized it to locally compact spaces [Hai22]. In particular the proof by Scholze is very hardcore direct computation, involving surprising amount of ϵ...

The current idea is to exploit the fractured structure to gain a more conceptual understanding regarding the condensed structure via the following series of conjectures.

1. Showing that condensed cohomology coincides with computing cohomology in the ∞-topos given by injections.

2. Establish a similar result for sheaf cohomology.

3. Proving that sheaf cohomology and condensed cohomology coincide in $\mathcal{C}\text{ond}^{inj}(\mathcal{X})$, benefiting from the fact that all morphisms in $\mathcal{C}\text{ond}^{inj}$ are injections, and so in particular $\mathcal{CH}\text{aus}^{inj}_{/X}$ is a poset.

This would allow us give a conceptual understanding of this result, but also generalize it to other coefficients, as it shifts the focus of the proof away from the value to the diagram.

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