INTRODUCTION TO QUASICATEGORIES

CHARLES REZK

Contents

Date: November 29, 2022.

Note: this is a major revision of the previous version of these notes. I'll leave a copy of the previous version (last modified in June 2021) online for a while, but this version is much better.

Note: this is a draft, which can change daily, though it also sometimes goes months without any change. I am in the process of adding additional material on (co)Cartesian fibrations.

1. Introduction to ∞-categories

I'll give a brief discussion to motivate the notion of ∞-categories. **W 19 Jan**

1.1. **Groupoids.** Modern mathematics is based on sets. The most familiar way of constructing new sets is as sets solutions to equations. For instance, given a commutative ring R, we can consider the set $X(R)$ of tuples $(x, y, z) \in R^3$ which satisfy the equation $x^5 + y^5 = z^5$. We can express such sets as limits. For instance, $X(R)$ is the pullback of the diagram of sets

$$R \times R \xrightarrow{(x,y)\mapsto x^5+y^5} R \xleftarrow{z^5 \leftarrow\!\shortmid z} R.$$

Another way to construct new sets is by taking "quotients", e.g., as sets of equivalence classes of an equivalence relation. This is in some sense much more subtle than sets of solutions to equations: mathematicians did not routinely construct sets this way until they were comfortable with the set theoretic formalism introduced by the end of the 19th century.

Some sets of equivalence classes are nothing more than that. However, some have "higher" structure standing behind them, which is often encoded in the form of a *groupoid*[1]. Here are some examples.

- Given a topological space X, we can define an equivalence relation on the set of points, so $x \sim x'$ if and only if there is a continuous path connecting them. The set of equivalence classes is the set $\pi_0 X$ of path components. Standing behind this equivalence relation is the *fundamental groupoid* $\Pi_1 X$, whose objects are points of X, and whose morphisms are path-homotopy classes of paths between two points.
- Given any category C, there is an equivalence relation on the collection of objects, so that $X \sim Y$ if there exists an isomorphism between them. Equivalence classes are the isomorphism classes of objects. Standing behind this equivalence relation is the *core* of C (also called the *maximal subgroupoid*), which is a groupoid having the same objects as C, but having as morphisms only the isomorphisms in C.
- As a special case of the above, let $C = \mathrm{Vect}_F$ be the category of finite dimensional vector spaces and linear maps over some field F. Then isomorphism classes of objects correspond to non-negative integers, via the notion of dimension. The core $\mathrm{Vect}_F^{\mathrm{core}}$ is a groupoid whose objects are finite dimensional vectors spaces, and whose morphisms are *invertible* linear maps.

Note that many interesting problems are about describing isomorphism classes, e.g., classifying finite groups of a given order, or principal G-bundles on a space. In practice, one learns that when you try to classify some type of objects up to isomorphism, you will need to have a good handle on the isomorphisms between such objects, including the groups of automorphisms of such objects. So you will likely need to know about the groupoid, even if it is not the primary object of interest.

For instance, a problem such as: "classify principal G-bundles on a space M up to isomorphism" naturally leads you to consider the problem: "describe the groupoid $\mathrm{Bun}_G(M)$ of principal G-bundles on a space M". This kind of problem can be thought of as a more sophisticated analogue of one like: "find the set $X(R)$ of solutions to $x^5 + y^5 = z^5$ in the ring R". (In fact, the theory of "moduli stacks" exactly develops this analogy between the two problems.) To do this, you can imagine having a "groupoid-based mathematics", generalizing the usual set-based one. Here are some observations about this.

[1] I assume familiarity with basic categorical concepts, such as in Chapter 1 of [Rie16].

- We regard two sets as "essentially the same" if they are *isomorphic*, i.e., if there is a bijection $f\colon X \to X'$ between them. Any such bijection has a unique inverse bijection $f^{-1}\colon X' \to X$.

  On the other hand, we regard two categories as "essentially the same" if they are merely *equivalent*, i.e., if there is a functor $f\colon C \to C'$ which admits an inverse *up to natural isomorphism.* It is not the case that such an inverse up to natural isomorphism is itself unique. These same remarks apply in particular to equivalences of groupoids.

  Although any equivalence of categories admits some kind of inverse, the failure to be unique leads to complications. For example, one goal of every course in abstract linear algebra is to demonstrate and exploit an equivalence of categories

  $$f\colon \mathrm{Mat}_F \to \mathrm{Vect}_F.$$

  Here Vect_F is the category of finite dimensional vector spaces over F, while Mat_F is the *matrix* category, whose objects are non-negative integers, and whose morphisms $n \to m$ are $m \times n$-matrices with entries in F. The functor f is defined by an explicit construction, e.g., it sends the object n to the vector space F^n. However, there is no completely "natural" way to construct an inverse functor $f^{-1}\colon \mathrm{Vect}_F \to \mathrm{Mat}_F$: producing such an inverse functor requires making an arbitrary choice, for each abstract vector space V, of a basis for V.
- We can consider "solutions to equations" in groupoids (e.g., limits). However, the naive construction of limits of groupoids may not preserve equivalences of groupoids, thus, we need to consider "weak" or "homotopy" limits.

  For example, suppose M is a space which is a union of two open subsets U and V. The *homotopy pullback* of

  $$\mathrm{Bun}_G(U) \to \mathrm{Bun}_G(U \cap V) \leftarrow \mathrm{Bun}_G(V)$$

  is a groupoid, whose objects are triples (P, Q, α), where $P \to U$ and $Q \to V$ are G-bundles, and $\alpha\colon P|_{U\cap V} \xrightarrow{\sim} Q|_{U\cap V}$ is an isomorphism of G-bundles over $U \cap V$. The morphisms $(P, Q, \alpha) \to (P', Q', \alpha')$ are pairs $(f\colon P \to P', g\colon Q \to Q')$ are pairs of bundle maps which are compatible over $U \cap V$ with the isomorphisms α, α'. Compare this with the *strict pullback*, which consists of (P, Q) such that $P|_{U\cap V} = Q|_{U\cap V}$ as bundles. In particular, $P|_{U\cap V}$ and $Q|_{U\cap V}$ must be *identical sets.*

  A basic result about bundles is that $\mathrm{Bun}_G(M)$ is equivalent to this homotopy pullback. The strict limit may fail to be equivalent to this. Note that it is impossible to describe the strict pullback without knowing precisely what definition of G-bundle we are using: in this case we need to be able to say when two bundles are *equal*, rather than *isomorphic.* The homotopy pullback is however relatively insensitive to the precise definition of G-bundle. (The point being, there can exist many non-identical "precise definitions of G-bundle", because what we really care about in the end is understanding $\mathrm{Bun}_G(M)$ up to *equivalence*, rather than up to isomorphism.)

These kinds of issues persist when dealing with higher groupoids and categories.

1.2. **Higher groupoids.** There is a category Gpd of groupoids, whose objects are groupoids and whose morphisms are functors. However, there is even more structure here: there are *natural transformations* between functors $f, f'\colon G \to G'$ of groupoids. That is, $\mathrm{Fun}(G, G')$ forms not merely a set, but a category. We can consider the collection consisting of (0) groupoids, (1) *equivalences* between groupoids, and (2) natural isomorphisms between equivalences. This is an example of a *2-groupoid*[2]. There is no reason to stop at 2-groupoids: there are n-groupoids, the totality of which are an example of an $(n+1)$-groupoid. (In this hierarchy, 0-groupoids are sets, and 1-groupoids are groupoids.) We might as well take the limit, and consider ∞-groupoids.

[2]More precisely, a "quasistrict 2-groupoid".

It turns out to be difficult (though not impossible) to construct an "algebraic" definition of n-groupoid. The approach which in seems to work best in practice is to use homotopy theory. We start with the observation that every groupoid G has a *classifying space* BG. This is defined explicitly as a quotient space

$$G \quad \mapsto \quad BG := \left(\coprod_{x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n} \Delta^n_{\mathrm{top}} \right) / \sim,$$

where we glue in a topological n-simplex Δ^n_{top} for each n-fold sequence of composable arrows in G, modulo certain identifications. It turns out (i) the fundamental groupoid of BG is equivalent to G, and (ii) the higher homotopy groups π_k of BG are trivial, for $k \geq 2$. A space with property (ii) is said to be *1-truncated*. Furthermore, (iii) there is a bijection between *equivalence classes of groupoids up to equivalence* and *CW-complexes which are 1-types, up to homotopy equivalence*. (More is true, but I'll stop there for now.)

The conclusion is that groupoids and equivalences between them are modelled by 1-types and homotopy equivalences between them. This suggests that we should *define* n-groupoids as n-types (CW complexes with trivial homotopy groups in dimensions $> n$), with equivalences being homotopy equivalences. Removing the restriction on homotopy groups leads to modelling ∞-groupoids by CW-complexes up to homotopy equivalence.

There is a different approach, which we will follow. It uses the fact that the classiyfing space construction factors through a "combinatorial" construction, called the "nerve". That is, we have

$$(G \in \mathrm{Gpd}) \mapsto (NG \in s\mathrm{Set}) \mapsto (\|NG\| = BG \in \mathrm{Top}),$$

where NG is the *nerve* of the groupoid, and is an example of a *simplicial set*, while $\|X\|$ denotes the *geometric realization* of a *simplicial set* X. In fact, the nerve of a groupoid is a particular kind of simplicial set called a *Kan complex*. It is a classical fact of homotopy theory that Kan complexes model all homotopy types. Thus, we will choose our definitions so that ∞-groupoids are precisely the Kan complexes.

1.3. **∞-categories.** An ∞-category is a generalization of ∞-groupoid in which morphisms are no longer required to be invertible in any sense.

There are a number of approaches to defining ∞-categories. Here are two which build on top of the identification of ∞-groupoids with Kan complexes.

- A category C consists of a set $\mathrm{ob}\, C$ of objects, and for each pair of objects a set $\hom_C(x, y)$ of maps from x to y. If we replace the set $\hom_C(x, y)$ with a Kan complex (or more generally a simplicial set) $\mathrm{map}_C(x, y)$, we obtain a category *enriched* over Kan complexes (or simplicial sets). This leads to one model for ∞-categories: categories enriched over simplicial sets.
- The nerve construction makes sense for categories: given a category C, we have a simplicial set NC. In general, NC is not a Kan complex; however, it does land in a special class of simplicial sets, which are called quasicategories. This leads to another model for ∞-categories: *quasicategories*.

In this paper we focus on the second case: the quasicategory model for ∞-categories.

1.4. **Historical remarks.** Quasicategories were invented by Boardman and Vogt [BV73, §IV.2], under the name *restricted Kan complex*. They did not use them to develop a theory of ∞-categories. This development began with the work of Joyal, first published in [Joy02]. Much of the material in this course was developed first by Joyal, in published papers and unpublished manuscripts [Joy08a], [Joy08b], [JT08]. Lurie [Lur09] gives a thorough treatment of quasicategories (which he simply calls "∞-categories"), recasting and extending Joyal's work significantly. There is been much work since then which has refined our understanding even more.

There are significant differences between the ways that Joyal and Lurie develop the theory. In particular, they give different definitions of the notion of a "categorical equivalence" between simplicial sets, though they do in fact turn out to be equivalent [Lur09, §2.2.5]. The approach I follow here is essentially that of Joyal. It is also basically the same as the approach Lurie takes in his reworking of the foundations at kerodon.net.

I have tried to generally adopt the terminology and notation of [Lur09] in most places.

1.5. **Goal of this book.** The goal of this book is to give a reasonably approachable introduction to the subject of higher category theory. In particular, I am writing with the following ideas in mind.

- The prerequisites are merely some basic notions of category theory, as seen in a first year algebraic topology or algebraic geometry course. No advanced training in homotopy theory is assumed: in particular, no knowledge of simplicial sets or model categories is assumed. You will learn what you need to know about these by reading this book.
- The book is written in "lecture notes" style rather that "textbook" style. That is, I will try to avoid introducing a lot of theory in section 3 which is only to be used in section 42, even if that is the "natural" place for it. The goal is to introduce new ideas near where they are first used, so that motivations are clear.
- The structure of the exposition is organized around the following type of question: Here is a [definition we can make/theorem we can prove] for ordinary categories; how do we generalize it to quasicategories? In some cases the answer is easy. In others, it can require a significant detour.
- The exposition is largely from the bottom up, rather than from the top down. Thus, I attempt to give complete details about everything I prove, so that nothing is relegated to references. (The current document does not achieve this yet, but that is the plan; in some cases, such details will be put into appendices.)
- The idea is that, after you have read this book, you will be well-prepared to dip into the main references on quasicategories (e.g., Lurie's books) without too much difficulty. Note that this book is not meant to (and does not) supplant any such reference.

1.6. **Prerequisites.** I assume only familiarity with basic concepts of category theory, such as from [Lei14], or as discussed in the first few chapters of [Rie16]. Some categorical prerequisites: you should be at least aware of the following notions (or know where to turn to in order to learn them):

- categories, functors, and natural transformations;
- full subcategories;
- the Yoneda lemma;
- initial and terminal objects;
- limits and colimits;
- groupoids;
- products and coproducts;
- pushouts and pullbacks;
- adjoint functors.

It is also helpful, but not essential, to know a little algebraic topology (such as fundamental groups and groupoids, and the definition of singular homology, as described in Chs. 1–3 of Hatcher's textbook)

1.7. **References and other sources.** As noted, the material depends mainly on the work of Joyal and Lurie.

- Joyal's first paper [Joy02] on the subject explicitly introduces quasicategories as a model for ∞-categories. It is worth looking at.

- There are several versions of unpublished lecture notes by Joyal [Joy08a], [Joy08b], which develop the theory of quasicategories from scratch. Also note the paper by Joyal and Tierney [JT08], which gives a summary of some of this unpublished work.
- Lurie's "Higher topos theory" [Lur09] gives a complete development of ∞-categories, including many topics not even touched in this book. The main general material on ∞-categories is in Chapters 1–4, together with quite of bit of material from the appendices. It is also worth looking at Chapter 5, which develops the very important notions of accessible and presentable ∞-categories. The final two chapters apply these ideas to the theory of ∞-topoi.
- Lurie's "Higher algebra" [Lur12] treats a number of "advanced topics", including *stable ∞-categories* (the ∞-categorical foundations underlying derived categories in homological algebra and stable homotopy), various notions of monoidal structures on ∞-categories (via the theory of *∞-operads*), and other topics.
- After I came up with the first version of these notes, Cisinski published the book "Higher Categories and Homotopical Algebra". It covers much of the material in these notes (and much more), on roughly similar lines: in his book model categories play a more prominent role from the start than they do here.
- Bergner's "The homotopy theory of $(\infty, 1)$-categories" is a survey of various approaches to higher categories and their interrelationships.
- Groth's note "A short course on ∞-categories" provides a brief survey to some of the basic ideas about quasicategories and their applications. It is not a complete treatment, but it does get very quickly to some of the more advanced topics.
- Riehl and Verity . . .
- Kerodon . . .

1.8. **Things to add.** This is a place for me to remind myself of things I might add.

- A discussion of n-truncation and n-groupoids, including the equivalence of ordinary groupoids to 1-groupoids (so connecting with the introduction).
- Pointwise criterion for limits/colimits: Show that $S^{\triangleright} \to \mathrm{Fun}(D, C)$ is a colimit cone if each projection to $S^{\triangleright} \to \mathrm{Fun}(\{d\}, C) \approx C$ is one.

1.9. **Acknowledgements.** Thanks to all those who have submitted corrections and suggestions for improvements, including most notably: Lang (Robbie) Yin, Nima Rasekh, Zachary Halladay, Doron Grossman-Naples, Darij Grinberg, and Vigleik Angeltveit. I'd also like to thank the participants of courses I have given based on a version of these notes: (Math 595 at the University of Illinois in Fall 2016, again in Spring 2019, and yet again in Spring 2022).

Part 1. Simplicial sets and nerves of categories

2. Simplicial sets

In the subsequent sections, we will define quasicategories as a generalization of the notion of a category. To accomplish this, we will recharacterize categories as a particular kind of *simplicial set*. Relaxing this characterization will lead us to the definition of quasicategories. **F 21 Jan**

Simplicial sets were introduced as a combinatorial framework for the homotopy theory of spaces. There are a number of treatments of simplicial sets from this point of view. I recommend Greg Friedman's survey [Fri12] as a starting place for learning about this viewpoint. Here I'll focus on what we need in order to develop quasicategories.

2.1. **The simplicial operator category Δ.** We write Δ for the category whose

- **objects** are the finite and non-empty totally ordered sets $[n] := \{0 < 1 < \cdots < n\}$ for $n \geq 0$, and

- **morphisms** $\delta\colon [n] \to [m]$ are weakly monotone functions, i.e., such that $x \le y$ implies $\delta(x) \le \delta(y)$.

Note that we exclude the empty set from Δ. I'll refer to morphisms in Δ as **simplicial operators**. **simplicial operators**

Because $[n]$ is an ordered set, you can also think of it as a category: the objects are the elements of $[n]$, and there is a morphism (necessarily unique) $i \to j$ if and only if $i \le j$. Thus, morphisms in Δ are precisely functors between. We can, and will, also think of $[n]$ as the category "freely generated" by the picture

$$\boxed{0} \to \boxed{1} \to \cdots \to \boxed{n-1} \to \boxed{n}.$$

The point is that arbitrary non-identity morphisms $i \to j$ in $[n]$ can be expressed uniquely as iterated composites of the arrows $i \to i+1$ which are displayed in the picture.

I will often use the following notation for morphisms in Δ, which describes a simplicial operator by a list of its values:

$$\delta = \langle \delta_0 \cdots \delta_n \rangle \colon [n] \to [m] \quad \text{with } \delta_0 \le \cdots \le \delta_n \quad \text{represents the function} \quad k \mapsto \delta_k.$$

2.2. *Remark.* There are distinguished simplicial operators called **face** and **degeneracy** operators: **face operator** **degeneracy operator**

$$d^i := \langle 0, \dots, \widehat{i}, \dots, n \rangle \colon [n-1] \to [n], \quad 0 \le i \le n,$$

$$s^i := \langle 0, \dots i, i, \dots n \rangle \colon [n+1] \to [n], \quad 0 \le i \le n.$$

All maps in Δ can be obtained as a composition of face and degeneracy operators, and in fact Δ can be described as the category *generated* by the above symbols, subject to a set of *relations* called the "simplicial identities", which can be found in various places, e.g., [Fri12, Def. 3.2].

2.3. **Simplicial sets.** A **simplicial set** is a functor $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$, i.e., a contravariant functor (or "presheaf") from Δ to sets. It is typical to write X_n for $X([n])$, and call it the set of n-*simplices* in X. In these notes I'll call it the set of n**-dimensional cells** (or just n**-cells**) of X instead[3]. I will also sometimes speak of the set of all cells of X, i.e., of the disjoint union $\coprod_{n \ge 0} X_n$ of the sets X_n. **simplicial set** **n-dimensional cells** **n-cells**

The 0-dimensional cells of a simplicial set are also called **vertices**, while the 1-dimensional cells are also called **edges**. **vertices** **edges**

Given a cell $a \in X_n$ and a simplicial operator $\delta\colon [m] \to [n]$, I will write $a\delta \in X_m$ as shorthand for $X(\delta)(a)$. That is, I'll think of simplicial operators as acting on cells from the right; this is a convenient choice given that X is a *contravariant* functor. In this language, a simplicial set consists of

- a sequence of sets $X_0, X_1, X_2, \dots,$ and
- functions $a \mapsto af\colon X_n \to X_m$ for each simplicial operator $\delta\colon [m] \to [n]$, such that
- $a\,\mathrm{id} = a$, and $(a\delta)\gamma = a(\delta\gamma)$ for any cell a and simplicial operators δ and γ whenever this makes sense.

If I need a simplicial operator to act from the left, I'll write $\delta^*(a) := a\delta$. Thus, a simplicial operator $\delta\colon [m] \to [n]$ induces a function $\delta^*\colon X_n \to X_m$ for any simplicial set X.

Occasionally I'll use a subscript notation when speaking of the action of particular simplicial operators. So, given a simplicial operator of the form $\delta = \langle \delta_0 \cdots \delta_m \rangle \colon [m] \to [n]$, we can indicate the action of δ on cells using subscripts:

$$a_{\delta_0 \cdots \delta_m} := af = a\langle \delta_0 \dots \delta_m \rangle.$$

This is handy when m is small. In particular, applying simplicial operators of the form $\langle i \rangle \colon [0] \to [n]$ to an n-dimensional cell $a \in X_n$ gives vertices $a_0, \dots, a_n \in X_0$, which we call the "*vertices* of a", while applying simplicial operators of the form $\langle ij \rangle \colon [1] \to [n]$ for $0 \le i \le j \le n$ gives edges $a_{ij} \in X_1$, which we call the "*edges* of a".

[3]Because the word "simplices" also applies to the so called "standard n-simplices" defined below (3.1), and I would like to avoid confusion between them.

2.4. **The category of simplicial sets.** A simplicial set is a functor. So a **map of simplicial sets** (or **simplicial map**) is a natural transformation of functors. Explicitly, a map $\phi\colon X \to Y$ between simplicial sets is a collection of functions $\phi_n\colon X_n \to Y_n$, $n \geq 0$, which commute with simplicial operators: **map of simplicial sets** **simplicial map**

$$(\phi_n a)f = \phi_n(af) \qquad \text{for all simplicial operators } f \text{ and cells } a \text{ in } X\text{, when this makes sense.}$$

I'll write sSet for the category of simplicial sets and maps between them[4].

2.5. *Remark.* A simplicial set is not the same thing as an *abstract simplicial complex*, though there are some relationships between the two notions: see (6.19).

2.6. **Subcomplexes.** A **subcomplex** of a simplicial set X is a subset $Y \subseteq X$ of the set of cells which inherits the action by simplicial operators, and thus is a simplicial set in it's own right. If Y is a subcomplex of X, then the evident inclusion function $\phi\colon Y \to X$ is a map of simplicial sets. **subcomplex**

2.7. *Exercise.* Show that for any map $f\colon X \to Y$ of simplicial sets, the image $f(X) \subseteq Y$ of f is a subcomplex of Y.

2.8. **Discrete simplicial sets.** A simplicial set X is **discrete** if every simplicial operator f induces a bijection $f^*\colon X_n \to X_m$. **discrete simplicial set**

Every set S gives us a discrete simplicial set S^{disc}, defined so that $(S^{\text{disc}})_n = S$, and so that each simplicial operator acts according to the identity map of S. This construction defines a functor $\text{Set} \to s\text{Set}$ given on objects by $S \mapsto S^{\text{disc}}$.

2.9. *Exercise.* Show that for any set S and simplical set X there is a bijection $\text{Hom}_{s\text{Set}}(S^{\text{disc}}, X) \to \text{Hom}_{\text{Set}}(S, X_0)$.

2.10. *Exercise* (Discrete simplicial sets come from sets). Show that (i) every discrete simplicial set X is isomorphic to S^{disc} for the set $S = X_0$, and (ii) for every pair of sets S and T, the evident function $\text{Hom}_{\text{Set}}(S,T) \to \text{Hom}_{s\text{Set}}(S^{\text{disc}}, T^{\text{disc}})$ is a bijection.

Let $s\text{Set}^{\text{disc}}$ denote the full subcategory of sSet spanned by discrete simplicial sets. That is, objects of $s\text{Set}^{\text{disc}}$ are discrete simplicial sets, and morphisms of $s\text{Set}^{\text{disc}}$ are all simplicial maps between them. Then (2.10) means that that *the full subcategory of discrete simplicial sets is equivalent to the category of sets.* The equivalence is given by the functor $\text{Set} \to s\text{Set}$ defined on objects by $S \mapsto S^{\text{disc}}$, while the inverse equivalence is $s\text{Set} \to \text{Set}$ defined on objects by $X \mapsto X_0$.

For this reason, it is often convenient to (at least informally) "identify" sets with their corresponding discrete simplicial sets. Thus, given a set S we will abuse notation and also write S for the discrete simplicial set S^{disc} defined above.

2.11. *Exercise.* Show that for any simplicial set X, the discrete simplicial set $(X_0)^{\text{disc}}$ is isomorphic to a subcomplex of X.

3. Standard simplicies

Standard simplices are the basic building blocks of simplicial sets. They are exactly the "representable functors" on the simplicial indexing category Δ. A standard n-simplex may also be thought of as the "free simplicial set on a single n-cell".

3.1. **Standard n-simplex.** The **standard n-simplex** Δ^n is the simplicial set defined by **standard n-simplex**

$$\Delta^n := \text{Hom}_\Delta(-, [n]).$$

That is, the standard n-simplex is exactly the functor *represented* by the object $[n]$. Explicitly, this means that

$$(\Delta^n)_m = \text{Hom}_\Delta([m],[n]) = \{\text{simplicial operators } a\colon [m] \to [n]\},$$

[4]Lurie [Lur09] uses Set_Δ to denote the category of simplicial sets.

while the action of simplicial operators on cells of Δ^n is given by composition: $f\colon [m'] \to [m]$ sends $(a\colon [m] \to [n]) \in (\Delta^n)_m$ to $(af\colon [m'] \to [n]) \in (\Delta^n)_{m'}$.

The **generator** of Δ^n is the cell

generator of a cell

$$\iota_n := \langle 01 \dots n \rangle = \mathrm{id}_{[n]} \in (\Delta^n)_n$$

corresponding to the identity map of $[n]$.

The **Yoneda lemma** (applied to the category Δ) asserts that the function

Yoneda lemma

$$\mathrm{Hom}_{s\mathrm{Set}}(\Delta^n, X) \to X_n, \qquad g \mapsto g(\iota_n),$$

is a bijection for every simplicial set X. (If this fact is not familiar to you, prove it.)

3.2. *Exercise.* Prove the Yoneda lemma for Δ.

The Yoneda lemma can be stated this way: for each n-dimensional cell $a \in X_n$ there exists a *unique* map $f_a\colon \Delta^n \to X$ of simplicial sets which sends the generator to it, i.e., such that $f_a(\iota_n) = a$. We call the map f_a the **representing map** of the cell a.

representing map of a cell

I'll often use the bijection provided by the Yoneda lemma implicitly. In particular, instead of using notation such as f_a, I'll just write $a\colon \Delta^n \to X$ for the representing map of the cell $a \in X_n$, i.e., for the *unique* map of simplicial sets sending the generator ι_n of Δ^n to a. Thus with our notation we have $a = a(\iota_n)$, where the two appearances of "a" denote respectively the cell of X_n and the representing morphism $\Delta^n \to X$.

3.3. *Exercise.* Show that the representing map $f\colon \Delta^n \to X$ of $a \in X_n$ sends $\langle c_0 \dots c_k \rangle \in (\Delta^n)_k$ to $a\langle c_0 \dots c_k \rangle \in X_k$.

Note that if $X = \Delta^m$ is also a standard simplex, then the Yoneda lemma gives a bijection

$$\mathrm{Hom}_{s\mathrm{Set}}(\Delta^n, \Delta^m) \xrightarrow{\sim} (\Delta^m)_n = \mathrm{Hom}_\Delta([n],[m]).$$

The inverse of this bijection sends a simplicial operator $f\colon [n] \to [m]$ to the map $\Delta^f\colon \Delta^n \to \Delta^m$ of simplicial sets defined on cells $g \in (\Delta^n)_k = \mathrm{Hom}_\Delta([k],[n])$ by $g \mapsto fg$.

Here is another abuse of notation: I'll write $f\colon \Delta^n \to \Delta^m$ instead of Δ^f for the map induced by the simplicial operator f, as it is also the representing map of the corresponding n-dimensional cell $f \in (\Delta^m)_n$.

3.4. *Remark.* We can summarize the above remarks in the following way: the full subcategory of sSet spanned by the standard simplices is equivalent to the simplicial operator category Δ, via a functor which on objects sends $[n]$ to Δ^n.

3.5. **Teminal and initial simplicial sets.** The standard 0-simplex Δ^0 is the terminal object in sSet; i.e., for every simplicial set X there is a unique map $X \to \Delta^0$. Sometimes I'll write $*$ instead of Δ^0 for this object, and refer to it as "the point". Note that Δ^0 is a discrete simplicial set, corresponding to a singleton set.

The **empty simplicial set** $\varnothing$ is the functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$ sending each $[n]$ to the empty set. It is the initial object in sSet; i.e., for every simplicial set X there is a unique map $\varnothing \to X$.

empty simplicial set

3.6. *Exercise.* Show that a simplicial set X is isomorphic to the empty simplicial set if and only if X_0 is isomorphic to the empty set.

3.7. **Standard simplices on totally ordered sets.** The definition of the standard simplices Δ^n can be extended to simplicial sets "generated" by arbitrary totally ordered sets. Thus, from any totally ordered set S we get a simplicial set Δ^S with $(\Delta^S)_n = \{\text{order preserving } [n] \to S\}$.

Note that for any *non-empty and finite* totally ordered set $S = \{s_0 < s_1 < \dots < s_n\}$, there is a *unique* order preserving bijection $[n] \xrightarrow{\sim} S$ for a unique $n \geq 0$, so that there is a *unique* isomorphism

$\Delta^S \approx \Delta^n$ of simplicial sets. When $S = \varnothing$ we have $(\Delta^{\varnothing})_k = \varnothing$ for all k, so $\Delta^{\varnothing}$ is the empty simplicial set.

This notation is especially convenient for subsets $S \subseteq [n]$ with induced ordering, as the simplicial set Δ^S is in a natural way a subcomplex of Δ^n.

Furthermore, any simplicial operator $f\colon [m] \to [n]$ factors through its image $S = f([m]) \subseteq [n]$, giving a factorization

$$[m] \xrightarrow{f_{\text{surj}}} S \xrightarrow{f_{\text{inj}}} [n]$$

of maps between ordered sets, and thus a factorization $\Delta^m \xrightarrow{\Delta^{f_{\text{surj}}}} \Delta^S \xrightarrow{\Delta^{f_{\text{inj}}}} \Delta^n$ of the induced map Δ^f of simplicial sets, where Δ^S can be regarded as a subcomplex of Δ^n.

3.8. *Exercise.* Show that $\Delta^{f_{\text{inj}}}$ and $\Delta^{f_{\text{surj}}}$ respectively induce maps between simplicial sets which are (respectively) injective and surjective on sets of k-cells for all k. (The case of $\Delta^{f_{\text{inj}}}$ is formal, but the case of $\Delta^{f_{\text{surj}}}$ is not completely formal.)

3.9. **Pictures of standard simplices.** When we draw a "picture" of Δ^n, we draw a geometric n-simplex: the convex hull of $n+1$ points in general position, with vertices labelled by $0, \dots, n$. The faces of the geometric simplex correspond exactly to *injective* simplicial operators into $[n]$: these cells are called *non-degenerate*. For each non-degenerate cell f in Δ^n, there is an infinite collection of *degenerate* cells with the same "image" as f (when viewed as a simplicial operator with target $[n]$).

Here are some "pictures" of standard simplices, which show their non-degenerate cells. Note that we draw the 1-cells of Δ^n as arrows. This lets us easily see the total ordering on the vertices of Δ^n.

Δ^0: $\langle 0 \rangle$

Δ^1: $\langle 0 \rangle \longrightarrow \langle 1 \rangle$

Δ^2:
$$\begin{tikzcd} & \langle 1 \rangle \arrow[dd] \\ \langle 0 \rangle \arrow[ur] \arrow[dr] & \\ & \langle 2 \rangle \end{tikzcd}$$

Δ^3:
$$\begin{tikzcd} & \langle 1 \rangle \arrow[dd] \arrow[dr] & \\ \langle 0 \rangle \arrow[ur] \arrow[dr] \arrow[rr] & & \langle 3 \rangle \\ & \langle 2 \rangle \arrow[ur] & \end{tikzcd}$$

We'll extend the terminology of "degenerate" and "non-degenerate" cells to arbitrary simplicial sets in (19.5).

4. The nerve of a category

The nerve of a category is a simplicial set which retains all the information of the original category. In fact, the nerve construction provides a full embedding of Cat, the category of (small) categories, into sSet, which means that we are able to think of categories as being just a special kind of simplicial set.

4.1. **Construction of the nerve.** Given a category C, the **nerve** of C is the simplicial set NC **nerve**
defined so that

$$(NC)_n := \mathrm{Hom}_{\mathrm{Cat}}([n], C),$$

the set of functors from $[n]$ to C, and so that simplicial operators $f\colon [m] \to [n]$ act by precomposition: $a \mapsto af$ for an element $a\colon [n] \to C$ in $(NC)_n$.

4.2. *Example.* There is an evident isomorphism $N[n] \approx \Delta^n$, which is in fact the unique isomorphism between these two simplicial sets.

Given a functor $F\colon C \to D$ between categories, we obtain a map $NF\colon NC \to ND$ of simplicial sets, sending $(a\colon [n] \to C) \in (NC)_n$ to $(Fa\colon [n] \to D) \in (ND)_n$. Thus the nerve construction defines a functor $N\colon \mathrm{Cat} \to s\mathrm{Set}$.

4.3. **Structure of the nerve.** We observe the following, whose verification we leave to the reader.

- $(NC)_0$ is canonically identified with the set $\operatorname{ob} C$ of objects of C.
- $(NC)_1$ is canonically identified with the set $\operatorname{mor} C$ of morphisms of C.
- The operators $\langle 0\rangle^*, \langle 1\rangle^* \colon (NC)_1 \to (NC)_0$ assign to a morphism its source and target respectively.
- The operator $\langle 00\rangle^* \colon (NC)_0 \to (NC)_1$ assigns to an object its identity morphism.
- $(NC)_2$ is in bijective correspondence with the set of pairs (f, g) of composable morphisms, i.e., such that the target of f is the source of g. This bijection is given by sending $a \in (NC)_2$ to $(a_{01}, a_{12}) \in (NC)_1 \times (NC)_1$.
- The operator $\langle 02\rangle^* \colon (NC)_2 \to (NC)_1$ assigns, to a 2-cell corresponding to a pair (f, g) of composable morphisms, the composite morphism gf.

In particular, you can recover the category C from its nerve NC, up to isomorphism, since the nerve contains all information about objects, morphisms, identity morphisms, and composition of morphisms in C.

We have the following general description of n-dimensional cells in the nerve.

4.4. **Proposition.** *Let C be a category, with object set $\operatorname{ob} C$ and morphism set $\operatorname{mor} C$.*

(1) *There is a bijective correspondence*
$$(NC)_n \xrightarrow{\sim} \left\{ (g_1, \dots, g_n) \in (\operatorname{mor} C)^{\times n} \;\middle|\; \operatorname{target}(g_{i-1}) = \operatorname{source}(g_i) \right\},$$
which sends $(a\colon [n] \to C) \in (NC)_n$ to the sequence $(a\langle 0,1\rangle, \dots, a\langle n-1,n\rangle)$

(2) *With respect to the correspondence of (1), the map $f^* \colon (NC)_n \to (NC)_m$ induced by a simplicial operator $\delta\colon [m] \to [n]$ coincides with the function*
$$(g_1, \dots, g_n) \mapsto (h_1, \dots, h_m), \qquad h_k = \begin{cases} \mathrm{id} & \text{if } \delta(k-1) = \delta(k) \\ g_j g_{j-1} \cdots g_{i+1} & \text{if } \delta(k-1) = i < j = \delta(k). \end{cases}$$

Proof. For (1), one verifies that an inverse is given by the function which sends a sequence $(g_1, \dots, g_n)$ to $(a\colon [n] \to C) \in (NC)_n$ defined on objects by $a(k) = \operatorname{target}(g_{k-1}) = \operatorname{source}(g_k)$, and on morphisms by $a(\langle ij\rangle) = g_j g_{j-1} \cdots g_{i+1}$ for $i < j$. For (2), note that for $a \in (NC)_n$ corresponding to the tuple $(g_1, \dots, g_n)$ we can compute
$$(a\delta)\langle k-1, k\rangle = a\langle \delta(k-1), \delta(k)\rangle = \begin{cases} \mathrm{id} & \text{if } \delta(k-1) = \delta(k), \\ g_j g_{j-1} \cdots g_{i+1} & \text{if } \delta(k-1) = i < j = \delta(k). \end{cases}$$
□

4.5. *Remark.* You can probably see from the above remarks that most of the information in the nerve of C is redundant: we only needed $(NC)_k$ for $k = 0, 1, 2$ and certain simplicial operators between them to recover complete information about the category C.

4.6. *Exercise.* Show that for any discrete simplicial set X there exists a category C and an isomorphism $NC \approx X$.

4.7. **Characterization of nerves.** This leads to the question: given a simplicial set X, how can we detect that it is isomorphic to the nerve of some category?

4.8. **Proposition.** *A simplicial set X is isomorphic to the nerve of some category if and only if for all $n \geq 2$ the function*
$$\phi_n \colon X_n \to \left\{ (g_1, \dots, g_n) \in (X_1)^{\times n} \;\middle|\; g_{i-1}\langle 1\rangle = g_i\langle 0\rangle,\ 1 \leq i \leq n \right\}$$
which sends $a \in X_n$ to $(a_{0,1}, \dots, a_{n-1,n})$ is a bijection.

Proof. First, suppose $X = NC$ for some category C. Then the function ϕ_n is precisely the bijection of (4.4)(1). Thus, if X is isomorphic to the nerve of some category then its ϕ_n are bijections.

Now suppose X is a simplical set such that the ϕ_n are bijections. We *define* a category C, with

$$\operatorname{ob} C := X_0, \qquad \operatorname{mor} C := X_1,$$

following the discussion in (4.3). Thus, the source and target of $g \in X_1$ are g_0 and g_1 in X_0 respectively, the identity map of $x \in X_0$ is $x_{00} \in X_1$, while the composite of (g, h) such that $g_1 = h_0$ is a_{02}, where $a \in X_2$ is the *unique* 2-cell with $a_{01} = g$ and $a_{12} = h$. Note that such an a exists exactly because ϕ_2 is a bijection. We leave the remaining details (e.g., unit and associativity properties) to the reader, though we note that proving associativity requires consideration of cells in X_3. (Or look ahead to (8.10), where we carry out the argument explicitly in a slightly different context.)

Next, we claim that for $a \in X_n$, and for $0 \leq i \leq j \leq k \leq n$, we have that

$$a_{i,k} = a_{j,k} a_{i,j},$$

where $a_{i,k}, a_{i,j}, a_{j,k} \in X_1$ are images of a under face operators $[1] \to [n]$, and right-hand side represents composition of two morphisms in C. To see this, note first that for $b \in X_2$, we have $b_{0,2} = b_{1,2} b_{0,1}$ by construction of C. The general case follows from this by setting $b = a_{i,j,k} \in X_2$.

Now we can define maps $\psi_n \colon X_n \to (NC)_n$ by sending $a \in X_n$ to $\psi_n(a) \colon [n] \to C$ defined by $\psi_n(a)(i \to j) = a_{i,j}$, which is a functor $[n] \to C$ by the above remarks. These maps ψ_n are seen to be bijections using the bijections ϕ_n and (4.4), since $\psi_n(a)((i-1) \to i) = a_{i-1,i}$. If $\delta \colon [m] \to [n]$ is a simplicial operator, then we compute

$$\psi_m(a\delta)(i \to j) = (a\delta)_{i,j} = a_{\delta(i),\delta(j)} = (\psi_n(a))(\delta(i) \to \delta(j)) = (\psi_n(a)\delta)(i \to j),$$

whence ψ is a map of simplicial sets. We have thus constructed an isomorphism $\psi \colon X \to NC$ of simplicial sets, as desired. $\square$

4.9. **A characterization of maps between nerves.** Maps between nerves are the same thing as functors between categories.

4.10. **Proposition.** *The nerve functor* $N \colon \mathrm{Cat} \to s\mathrm{Set}$ *is fully faithful. That is, every simplicial set map* $g \colon NC \to ND$ *between nerves is of the form* $g = N(f)$ *for a unique functor* $f \colon C \to D$.

Proof. We need to show that $\mathrm{Hom}_{\mathrm{Cat}}(C, D) \to \mathrm{Hom}_{s\mathrm{Set}}(NC, ND)$ induced by the functor N is a bijection for all categories C and D. *Injectivity* is clear, as a functor f is determined by its effect on objects and morphisms, which is exactly the effect of $N(f)$ on 0- and 1-cells of the nerves.

For *surjectivity*, observe that for any map $g \colon NC \to ND$ of simplicial sets, we can define a candidate functor $f \colon C \to D$, defined on objects and morphisms by the action of g on 0-dimensional and 1-dimensional cells. That F has the correct action on identity maps follows from the fact that g commutes with the simplicial operator $\langle 00 \rangle \colon [1] \to [0]$. That f preserves composition uses (4.4) and the fact that g commutes with the simplicial operator $\langle 02 \rangle \colon [1] \to [2]$.

Note that given $g \colon NC \to ND$ and $f \colon C \to D$ as constructed above, the maps $g, N(f) \colon NC \to ND$ coincide on 0-dimensional and 1-dimensional cells by construction. It follows that $g = N(f)$ by using the exercise (4.11) below. Thus, we have shown that $N \colon \mathrm{Hom}_{\mathrm{Cat}}(C, D) \to \mathrm{Hom}_{s\mathrm{Set}}(NC, ND)$ is surjective as desired. $\square$

4.11. *Exercise* (Important: Maps to a nerve are determined by edges). Show that if D is a category and X is any simplicial set (not necessarily a nerve), then two maps $g, g' \colon X \to ND$ are equal if and only if $g_1 = g'_1 \colon X_1 \to (ND)_1$, i.e., g and g' are equal if and only if they coincide on edges. (Hint: use (4.4).)

5. Spines

In this section we will restate our characterization of simplicial sets which are isomorphic to nerves, in terms of a certain "extension" condition. To state this condition we need the notion of a "spine" of a standard n-simplex. **M 24 Jan**

5.1. **The spine of an n-simplex.** The **spine** of the n-simplex Δ^n is the simplicial set I^n defined by **spine**

$$(I^n)_k = \{\, \langle a_0 \cdots a_k\rangle \in (\Delta^n)_k \;\mid\; a_k \leq a_0 + 1 \,\}.$$

That is, a k-dimensional cell of I^n is a simplicial operator $a\colon [k] \to [n]$ whose image is of the form either $\{j\}$ or $\{j, j+1\}$. The action of simplicial operators on cells of I^n is induced by their action on Δ^n. (To see that this action is well defined, observe that for $a\colon [k] \to [n]$ in $(I^n)_k$ and $f\colon [p] \to [k]$, the image of the simplicial operator af is contained in the image of a.)

The spine I^n is by definition a subcomplex of Δ^n. Here is a picture of I^3 in Δ^3:

$$\begin{tikzcd} & \langle 1\rangle \arrow[dd] & \\ \langle 0\rangle \arrow[ur] & & \langle 3\rangle \\ & \langle 2\rangle \arrow[ur] & \end{tikzcd} \qquad \text{is the spine inside} \qquad \begin{tikzcd} & \langle 1\rangle \arrow[dd] \arrow[dr] & \\ \langle 0\rangle \arrow[ur] \arrow[dr] \arrow[rr] & & \langle 3\rangle \\ & \langle 2\rangle \arrow[ur] & \end{tikzcd}$$

Note that $I^0 = \Delta^0$ and $I^1 = \Delta^1$, but $I^n \neq \Delta^n$ for $n \geq 2$.

The key property of the spine is the following.

5.2. **Proposition.** *Given a simplicial set X, for every $n \geq 0$ there is a bijection*

$$\operatorname{Hom}(I^n, X) \xrightarrow{\sim} \left\{\, (a_1, \dots, a_n) \in (X_1)^{\times n} \;\middle|\; a_i\langle 1\rangle = a_{i+1}\langle 0\rangle \,\right\},$$

defined by sending $f\colon I^n \to X$ to $(f(\langle 01\rangle), f(\langle 12\rangle), \cdots, f(\langle n-1, n\rangle))$. (In the case $n = 0$, the target of the bijection is taken to be the set X_0 of vertices of X, and the bijection in this case sends $f \mapsto f\langle 0\rangle$.)

We will give the proof at the end of the next section (6.21), after we describe I^n as a colimit of a diagram of standard simplices. Speficically, I^n is obtained from a collection of 1-simplices by "gluing" them together at their ends.

5.3. **Nerves are characterized by unique spine extensions.** Given (5.2), we can now state our new characterization of nerves: they are simplicial sets such that every map $I^n \to X$ from a spine extends *uniquely* along $I^n \subseteq \Delta^n$ to a map from the standard n-simplex. That is, nerves are precisely the simplicial sets with "unique spine extensions".

5.4. **Proposition.** *A simplicial set X is isomorphic to the nerve of some category if and only if the restriction map* $\operatorname{Hom}(\Delta^n, X) \to \operatorname{Hom}(I^n, X)$ *along $I^n \subseteq \Delta^n$ is a bijection for all $n \geq 2$.*

Proof. Using the description of (5.2), we see that the restriction map $\operatorname{Hom}(\Delta^n, X) \to \operatorname{Hom}(I^n, X)$ is identical to the function given on n-cells of X by

$$(a \in X_n) \quad \mapsto \quad (a_{0,1}, \dots, a_{n-1,n}) \in (X_1)^{\times n}.$$

The conclusion is then immediate from our earlier characterization of nerves (4.8). $\square$

6. Colimits of simplicial sets and subcomplexes

Because we will work with simplicial sets so much, it is worthwhile to take some time to figure out how to describe colimits of functors to sSet. Because simplicial sets are built from sets, we start by recalling how to "compute" colimits of functors to Set.

6.1. **Colimits of sets.** Recall that any functor $F\colon C \to \mathrm{Set}$ from a small category to sets has a colimit. Here *small* means that the collections $\operatorname{ob} C$ and $\operatorname{mor} C$ of objects and morphisms of C are themselves sets[5]. Given any such functor there is a "simple formula" for its colimit. First consider the coproduct (i.e., disjoint union) $\coprod_{c\in\operatorname{ob} C} F(c)$ of the values of the functor. I'll write (c, x) for a typical element of this coproduct, with $c \in \operatorname{ob} C$ and $x \in F(c)$. Consider the relation $\sim$ on this defined by

$$(c, x) \sim (c', x') \quad \text{if } \exists\ \alpha\colon c \to c' \text{ in } C \text{ such that } F(\alpha)(x) = x'.$$

Define

$$X := \Big(\coprod_{c\in\operatorname{ob} C} F(c)\Big)/\approx,$$

the set obtained as the quotient by the equivalence relation "$\approx$" which is *generated* by the relation "$\sim$". For each object c of C we have a function $i_c\colon F(c) \to X$ defined by $i_c(x) := [c, x]$, sending x to the equivalence class of (c, x). Then the data $(X, \{i_c\})$ is a colimit of the functor F: i.e., for any set S and collection of functions

$$f_c\colon F(c) \to S \quad \text{for each } c \in \operatorname{ob} C, \quad \text{such that} \quad f_{c'} \circ F(\alpha) = f_c \quad \text{for all } \alpha\colon c \to c'$$

there exists a *unique* function $f\colon X \to S$ such that $f \circ i_c = f_c$.

6.2. *Exercise.* Verify that $(X, \{i_c\})$ is in fact a colimit of F.

We write $\operatorname{colim}_C F$ for the object X.

Note that if the relation "$\sim$" is not itself an equivalence relation, it can be difficult to figure out what "$\approx$" actually is: the simple formula may not be so simple in practice.

6.3. *Example.* A **pushout** is a colimit of a diagram whose shape is a "span": **pushout**

$$X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2.$$

Using the above recipe, we see that a pushout of sets is a quotient by an equivalence relation "$\approx$" of a set whose elements are pairs (k, a), where $k \in \{0, 1, 2\}$ and $a \in X_k$, where the $\approx$ is generated by the relation "$\sim$" with:

$$(k, x) \sim (k, x), \qquad (0, x) \sim (1, f_1(x)), \qquad (0, x) \sim (2, f_2(x)).$$

6.4. *Exercise.* Consider the following span in Set:

$$\mathbb{Z} \xleftarrow{\lfloor x\rfloor \leftarrow\!\shortmid x} \tfrac{1}{2}\mathbb{Z} \xrightarrow{x\mapsto\lceil x\rceil} \mathbb{Z},$$

where $\frac{1}{2}\mathbb{Z} := \{\, n/2 \in \mathbb{R} \mid n \in \mathbb{Z}\,\}$. Use the above recipe to compute its pushout.

6.5. *Exercise.* Consider the functor $F\colon \Delta \to \mathrm{Set}$ which sends $[n]$ to its underlying set $\{0, 1, \dots, n\}$, and has the obvious effect on functions. Compute $\operatorname{colim}_\Delta F$.

6.6. *Exercise.* If C is a groupoid (i.e., all morphisms of C are isomorphisms), then the relation $\sim$ is already an equivalence relation.

6.7. *Exercise.* Let S be a set with a relation $\sim$, and write $\approx$ for the relation generated by $\sim$. Show that for $s, s' \in S$, we have $s \approx s'$ if and only if there exists a sequence $s_0, \dots, s_n \in S$ with $n \geq 0$, such that $s_0 = s$, $s_n = s'$, and for each $0 \leq i < n$ we have either $s_i \sim s_{i+1}$ or $s_{i+1} \sim s_i$. (That is, elements are in the same equivalence class if and only if they are "connected" by a finite sequence of elements related by $\sim$.)

There are cases when things are more tractable.

[5]The point is that some categories are not small, such as Set itself, since there is no "set of all sets".

6.8. **Proposition.** *Let $\mathcal{A}$ be a collection of subsets of a set S. Regard $\mathcal{A}$ as a partially ordered set under "$\subseteq$", and hence as a category. Suppose $\mathcal{A}$ has the following property: for all $s \in S$, and $T, U \in \mathcal{A}$ such that $s \in T \cap U$, there exists $V \in \mathcal{A}$ such that $s \in V \subseteq T \cap U$. Then the tautological map*

$$\operatorname{colim}_{T \in \mathcal{A}} T \to \bigcup_{T \in \mathcal{A}} T$$

(sending $[(T,t)] \mapsto t$) is a bijection.

Proof. The tautological map is clearly surjective, so it remains only to show that $(T,t) \approx (T',t')$ if and only if $t = t'$. The remaining details are left for the reader, using (6.7). □

Note: an easy way to satisfy the hypothesis of (6.8) is to show that $\mathcal{A}$ is closed under pairwise intersection, i.e., that $T, U \in \mathcal{A}$ implies $T \cap U \in \mathcal{A}$.

6.9. **Colimits of simplicial sets.** Colimits of simplicial sets also exist, and are "computed pointwise" (or "dimensionwise").

6.10. **Proposition.** *Let $F \colon C \to s\mathrm{Set}$ be a functor from a small category to simplicial sets. This functor has a colimit $(X, \{i_c\})$, where X is a simplicial set and $i_c \colon F(c) \to X$ are simplicial maps, with the property that when we restrict to n-cells, the object $(X_n, \{i_{c,n} \colon F(c)_n \to X_n\})$ is a colimit of the functor $F_n \colon C \to \mathrm{Set}$ defined on objects by $F_n(c) := F(c)_n$.*

Proof. This is a standard exercise whose details we leave to the reader. □

We can summarize the main idea of the above claim by saying that if $X = \operatorname{colim}_{c \in C} F(c) \in s\mathrm{Set}$, then for each $n \geq 0$ we have a canonical bijection

$$X_n \xleftarrow{\sim} \operatorname{colim}_{c \in C} F(c)_n.$$

6.11. **Colimits of subcomplexes.** Recall (2.6) that a subcomplex of a simplicial set X is just a subfunctor, i.e., a collection of subsets $A_n \subseteq X_n$ which are closed under the action of simplicial operators, and thus form a simplicial set so that the inclusion $A \to X$ is a morphism of simplicial sets. We typically write $A \subseteq X$ when A is a subcomplex of X.

6.12. *Example.* Examples we have already seen include the spines $I^n \subseteq \Delta^n$ and the $\Delta^S \subseteq \Delta^n$ associated to subsets $S \subseteq [n]$.

For every set S of cells of a simplicial set, there is a smallest subcomplex which contains the set, namely the intersection of all subcomplexes containing S.

6.13. *Example.* For a vertex $x \in X_0$, we write $\{x\} \subseteq X$ for the smallest subcomplex which contains x. This subcomplex has exactly one n-dimensional cell for each $n \geq 0$, namely $x\langle 0 \cdots 0 \rangle$, and thus is isomorphic to Δ^0.

More generally, for a collection of vertices $a, b, c, \dots \in X_0$, we write $\{a, b, c, \dots\} \subseteq X$ for the smallest subcomplex which contains $a, b, c, \dots$. This subcomplex is a discrete simplicial set. This choice of notation is supported by our informal identification of discrete simplicial sets with sets (2.8).

The result (6.8) carries over to simplicial sets, where the role of subsets is replaced by subcomplexes.

6.14. **Proposition.** *Let $\mathcal{A}$ be a collection of subcomplexes of a simplicial set X. Regard $\mathcal{A}$ partially ordered set under "$\subseteq$", and hence as a category. Suppose $\mathcal{A}$ has the following property: for all $n \geq 0$, all $x \in X_n$, and all $K, L \in \mathcal{A}$ such that $x \in K_n \cap L_n$, there exists $M \in \mathcal{A}$ such that $x \in M_n$ and $M \subseteq K \cap L$. Then the tautological map*

$$\operatorname{colim}_{K \in \mathcal{A}} K \to \bigcup_{K \in \mathcal{A}} K$$

is a bijection.

Proof. We use the fact that colimits of simplicial sets are computed degreewise (6.10), together with the analogous statement we already proved about colimits of subsets (6.8). □

6.15. *Remark* (Pushouts of subcomplexes). A special case of (6.14) applied to simplicial sets which we will use often is the following. If K and L are subcomplexes of a simplicial set X, then so are both $K \cap L$ and $K \cup L$, and furthermore the evident commutative square

$$\begin{tikzcd} K \cap L \arrow[r, tail] \arrow[d, tail] & L \arrow[d, tail] \\ K \arrow[r, tail] & K \cup L \end{tikzcd}$$

is a pushout square in simplicial sets, i.e., $K \cup L \approx \operatorname{colim}[K \leftarrow K \cap L \rightarrow L]$. (Proof: (6.14) with $\mathcal{A} = \{K, L, K \cap L\}$.)

6.16. **Subcomplexes of standard simplices.** For each $S \subseteq [n]$ we have a subcomplex $\Delta^S \subseteq \Delta^n$, whose cells correspond simplicial operators to $[n]$ whose image is contained in S. The following says that every subcomplex of Δ^n is a union of such Δ^Ss.

6.17. **Lemma.** *Let $K \subseteq \Delta^n$ be a subcomplex. If $(f\colon [m] \to [n]) \in K_m$ with $f([m]) = S$, then $f \in (\Delta^S)_m$ and $\Delta^S \subseteq K$.*

This proof uses the following elementary fact.

6.18. **Lemma.** *Any order preserving surjection $f\colon S \to T$ between finite totally ordered sets admits an order preserving section, i.e., $s\colon T \to S$ such that $fs = \mathrm{id}_T$.*

Proof. Let $s(t) = \min\{\, s \in S \mid f(s) = t \,\}$. □

Proof of (6.17). Choose a section $t\colon S \to [m]$ of $f_{\text{surj}}\colon [m] \to S$ (6.18). Consider a cell $\overline{g} \in (\Delta^S)_k \subseteq (\Delta^n)_k$, represented by a map $\overline{g}\colon [k] \to [n]$ whose image is contained in S. We get a commutative diagram

$$\begin{tikzcd} & {[k]} \arrow[dl, "s:=tg"'] \arrow[d, "g"] \arrow[dr, "\overline{g}"] & \\ {[m]} \arrow[r, two heads, "f_{\text{surj}}"'] & S \arrow[r, tail, "f_{\text{inj}}"'] & {[n]} \end{tikzcd}$$

so $\overline{g} = fs$ and hence is a cell of the subcomplex K since f is. Thus $\Delta^S \subseteq K$, and it is immediate that $f \in (\Delta^S)_m$. □

6.19. *Remark.* Thus, a subcomplex $K \subseteq \Delta^n$ determines and is determined by a collection $\mathcal{K}$ of non-empty subsets of $[n]$ with the property that $T \subseteq S$ and $S \in \mathcal{K}$ implies $T \in \mathcal{K}$: namely,

$$\mathcal{K} = \left\{\, S \subseteq [n] \;\middle|\; \Delta^S \subseteq K \,\right\} \qquad \text{and} \qquad K = \bigcup_{S \in \mathcal{K}} \Delta^S.$$

In other words, a subcomplex of Δ^n is the "same thing" as an *abstract simplicial complex* whose vertex set is a subset of $[n]$.

We can sharpen (6.17): every subcomplex of Δ^n is a *colimit* of subcomplexes of the form Δ^S.

6.20. **Proposition.** *Let $K \subseteq \Delta^n$ be a subcomplex. Let $\mathcal{A}$ be the poset of all non-empty subsets $S \subseteq [n]$ such that $\Delta^S \subseteq K$. Then the tautological map*

$$\operatorname{colim}_{S \in \mathcal{A}} \Delta^S \to K$$

is an isomorphism.

Proof. We must show that for each $m \geq 0$, the map $\operatorname{colim}_{S\in\mathcal{A}}(\Delta^S)_m \to K_m$ is a bijection. Each $(\Delta^S)_m = \{\, f\colon [m] \to [n] \mid f([m]) \subseteq S \,\}$ is a distinct subset of $K_m \subseteq (\Delta^n)_m$; i.e., $S \neq S'$ implies $(\Delta^S)_m \neq (\Delta^{S'})_m$. In view of (6.14), it suffices to show that for each $f \in K_m$ there is a minimal S in $\mathcal{A}$ such that $f \in (\Delta^S)_m$. This is immediate from (6.17), which says that $f \in (\Delta^S)_m$ and $\Delta^S \subseteq K$ where $S = f([m])$, and it is obvious that this S is minimal with this property. □

6.21. **Proof of** (5.2)**.** Now we can prove our claim about maps out of a spine, using an explicit description of a spine as a colimit.

Proof of (5.2). Let $\mathcal{A}$ be the poset of all non-empty $S \subseteq [n]$ such that $\Delta^S \subseteq I^n$; i.e., subsets of $[n]$ of the form $\{j\}$ or $\{j, j+1\}$. Explicitly the poset $\mathcal{A}$ has the form

$$\{0\} \to \{0,1\} \leftarrow \{1\} \to \{1,2\} \leftarrow \{2\} \to \cdots \leftarrow \{n-1\} \to \{n-1,n\} \leftarrow \{n\}.$$

By (6.20), $\operatorname{colim}_{S\in\mathcal{A}} \Delta^S \to I^n$ is an isomorphism. Thus $\operatorname{Hom}(I^n, X) \approx \operatorname{Hom}(\operatorname{colim}_{S\in\mathcal{A}} \Delta^S, X) \approx \lim_{S\in\mathcal{A}} \operatorname{Hom}(\Delta^S, X)$, and an elementary argument gives the result. □

7. Limits of simplicial sets

The notion of colimit has a dual notion, namely that of limit.

7.1. **Product of simplicial sets.** The **product** of a collection $\{X_\alpha\}_{\alpha\in A}$ of simplicial sets is defined to be their product as functors. For instance, if X and Y are simplicial sets, the set of n-cells of the product $X \times Y$ is the set $X_n \times Y_n$ of pairs (a, b) of n-cells in X and Y, with simplicial operators f acting by $(a, b) \mapsto (af, bf)$. **product of simplicial sets**

Later on in these notes, we will need to identify a product $\Delta^m \times \Delta^n$ of standard simplices (or a subcomplex of this product) explicitly with a colimit of a diagram of standard simplices. The following exercise displays the simplest non-trivial example of this.

7.2. *Exercise* (Important: product of 1-simplices). Let $X = \Delta^1 \times \Delta^1$. Let $A, B \subseteq X$ be subcollections of cells in X defined so that

$$(f, g) \in \begin{cases} A & \text{if } f(i) \leq g(i) \text{ for all } i = 0, \dots, n, \\ B & \text{if } f(i) \geq g(i) \text{ for all } i = 0, \dots n. \end{cases}$$

where $f, g\colon [n] \to [1]$ are simplicial operators. Show that A and B are subcomplexes of X which are each isomorphic to Δ^2, and that $A \cap B$ is isomorphic to Δ^1. Use this to show that the following is a pushout square of simplicial sets.

$$\begin{tikzcd}
\Delta^1 \arrow[r, "\langle 02\rangle"] \arrow[d, "\langle 02\rangle"'] & \Delta^2 \arrow[d, "(\langle 011\rangle, \langle 001\rangle)"] \\
\Delta^2 \arrow[r, "(\langle 001\rangle, \langle 011\rangle)"'] & \Delta^1 \times \Delta^1
\end{tikzcd}$$

That is, in simplicial sets a "square" can be obtained by gluing two "triangles" along a common edge, as suggested by the following picture.

$$\begin{tikzcd}
(0,1) \arrow[r] & (1,1) \\
(0,0) \arrow[u] \arrow[ur] \arrow[r] & (1,0) \arrow[u]
\end{tikzcd}$$

8. Horns and inner horns

We now are going to give another (less obvious!) characterization of nerves, in terms of "extending inner horns", rather than "extending spines". It will be this characterization that we "weaken" to obtain the definition of a quasicategory.

8.1. **Definition of horns.** We define a collection of subcomplexes of the standard simplices, called "horns". For each $n \geq 1$, these are subcomplexes $\Lambda^n_j \subset \Delta^n$ for each $0 \leq j \leq n$. The **horn** Λ^n_j is the subcomplex of Δ^n defined by **horn**

$$(\Lambda^n_j)_k = \{\, f\colon [k] \to [n] \ \mid \ ([n] \smallsetminus \{j\}) \not\subseteq f([k]) \,\}.$$

Using the fact (6.20) that subcomplexes of Δ^n are always unions of Δ^Ss, we see that Λ^n_j is the union of "faces" $\Delta^{[n]\smallsetminus i}$ of Δ^n other than the jth face:

$$\Lambda^n_j = \bigcup_{i \neq j} \Delta^{[n]\smallsetminus i} \subset \Delta^n.$$

When $0 < j < n$ we say that $\Lambda^n_j \subset \Delta^n$ is an **inner horn**. We also say it is a **left horn** if $j < n$ and a **right horn** if $0 < j$. Sometimes I'll speak of an **outer horn**, meaning a horn Λ^n_j with $j \in \{0, n\}$, i.e., a non-inner horn. **inner horn** **left horn** **right horn** **outer horn**

8.2. *Example* (1-horns). The horns inside Δ^1 are just the vertices viewed as subcomplexes: $\Lambda^1_0 = \Delta^{\{0\}} = \{0\} \subset \Delta^1$ and $\Lambda^1_1 = \Delta^{\{1\}} = \{1\} \subset \Delta^1$. Neither is an inner horn, the first is a left horn, and the second is a right horn.

8.3. *Example* (2-horns). These are the three horns inside the 2-simplex.

$$\begin{tikzcd} & \langle 1\rangle & \\ \langle 0\rangle \arrow[ur, "\langle 01\rangle"] \arrow[rr, "\langle 02\rangle"'] & & \langle 2\rangle \end{tikzcd} \qquad \begin{tikzcd} & \langle 1\rangle \arrow[dr, "\langle 12\rangle"] & \\ \langle 0\rangle \arrow[ur, "\langle 01\rangle"] & & \langle 2\rangle \end{tikzcd} \qquad \begin{tikzcd} & \langle 1\rangle \arrow[dr, "\langle 12\rangle"] & \\ \langle 0\rangle \arrow[rr, "\langle 02\rangle"'] & & \langle 2\rangle \end{tikzcd}$$

$$\Lambda^2_0 \qquad\qquad \Lambda^2_1 \qquad\qquad \Lambda^2_2$$

Only Λ^2_1 is an inner horn, while Λ^2_0 and Λ^2_1 are left horns, and Λ^2_1 and Λ^2_2 are right horns. Note that Λ^2_1 is the same as the spine I^2.

8.4. *Exercise.* Visualize the four horns inside the 3-simplex. The simplicial set Λ^3_j actually kind of looks like a horn: you blow into the vertex $\langle j\rangle$, and sound comes out of the opposite missing face $\Delta^{[3]\smallsetminus j}$.

8.5. *Exercise.* Show that Λ^n_j is the largest subcomplex of Δ^n which does not contain the cell $d^j := \langle 0 \cdots \widehat{j} \cdots n\rangle \in (\Delta^n)_{n-1}$, the "face opposite the vertex j".

We note that inner horns always contain spines: $I^n \subseteq \Lambda^n_j$ if $0 < j < n$. This is also true for outer horns if $n \geq 3$, but not for outer horns with $n = 1$ or $n = 2$.

8.6. **The inner horn extension criterion for nerves.** We can now characterize nerves as those simplicial sets which admit "unique inner horn extensions". This is different than, but analogous to, the characterization in terms of unique spine extensions (5.4).

8.7. **Proposition.** *A simplicial set X is isomorphic to the nerve of a category, if and only if* $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_j, X)$ *is a bijection for all $n \geq 2$, $0 < j < n$.*

The proof will take up the rest of the section.

8.8. **Nerves have unique inner horn extensions.** First we show that nerves have unique inner horn extensions.

8.9. **Proposition.** *If C is a category, then for every* inner *horn $\Lambda^n_j \subset \Delta^n$ the evident restriction map*
$$\mathrm{Hom}(\Delta^n, NC) \to \mathrm{Hom}(\Lambda^n_j, NC)$$
is a bijection.

Proof. Since inner horns contain spines, we can consider restriction along $I^n \subseteq \Lambda^n_j \subseteq \Delta^n$. The composite
$$\mathrm{Hom}(\Delta^n, NC) \to \mathrm{Hom}(\Lambda^n_j, NC) \xrightarrow{r} \mathrm{Hom}(I^n, NC)$$
of restriction maps is a bijection (5.4), so r is a surjection. Thus, it suffices to show that r is injective.

Recall (4.11) that a map of simplicial sets $X \to NC$ to a nerve is uniquely determined by its values on edges. Thus we can reduce to the following: given a map $f\colon \Lambda^n_j \to NC$ from an inner horn to a nerve, the values f takes on arbitrary edges $\langle a,b\rangle \in (\Lambda^n_j)_1$ are uniquely determined by the values it takes on edges in the spine $(I^n)_1$, i.e., edges of the form $\langle a,a\rangle$ and $\langle a,a+1\rangle$. There are three cases.

$n=2$: The claim is immediate since $I^2 = \Lambda^2_1$.

$n=3$: Here is a picture of $I^3 \subset \Lambda^3_1 \subset \Delta^3$, showing all nondegenerate 0,1, and 2 cells.

$$\begin{tikzcd}
0 \arrow[dr] \arrow[drr, dotted, bend left] \arrow[dd, dotted, bend right] \arrow[d, phantom, "a"] & & \\
& 1 \arrow[dl] \arrow[r, dotted] \arrow[u, phantom, "c"] \arrow[d, phantom, "b"] & 3 \\
2 \arrow[urr, bend right] & &
\end{tikzcd}$$

Note that Λ^3_1 is the smallest subcomplex of Δ^3 containing the 2-cells $a = \langle 012\rangle, b = \langle 123\rangle, c = \langle 013\rangle$. Since the target of $f\colon \Lambda^3_1 \to NC$ is the nerve of a category, we can check by hand that the values of f on edges is determined by its value on I^3. In fact, only three edges of Λ^3_1 are not in its spine, and we have
$$f\langle 02\rangle = f\langle 12\rangle \circ f\langle 01\rangle, \quad f\langle 13\rangle = f\langle 23\rangle \circ f\langle 12\rangle, \quad f\langle 03\rangle = f\langle 13\rangle \circ f\langle 01\rangle,$$
using the presence of the 2-cells $\langle 012\rangle, \langle 123\rangle, \langle 013\rangle$ in Λ^3_1. A similar argument applies for $\Lambda^3_2 \subseteq \Delta^3$.

$n \geq 4$: In this case we have that the subcomplex $\Lambda^n_j \subset \Delta^n$ contains all 0, 1, and 2 dimensional cells of Δ^n. The argument that the value of a map $f\colon \Lambda^n_j \to NC$ on edges is determined by its value on the spine proceeds much as the case $n=3$: compute $f\langle x,y\rangle$ by induction on the value of $y-x \geq 1$.

□

8.10. **Nerves are characterized by unique inner horn extension.** Let X be an arbitrary simplicial set, and suppose it has unique inner horn extensions, i.e., each restriction map $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_j, X)$ is a bijection for all $0 < j < n$ with $n \geq 2$. **W 26 Jan**

Observe that unique extension along $\Lambda^2_1 \subset \Delta^2$, defines a "composition law" on the set X_1. That is, given $f, g \in X_1$ such that $f_1 = g_0$ in X_0,[6] there is a unique map
$$u\colon \Lambda^2_1 = \Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \xrightarrow{(f,g)} X \qquad \text{such that} \qquad \langle 01\rangle \mapsto f \in X_1,\ \langle 12\rangle \mapsto g \in X_1.$$

[6] Recall that $f_1 = f\langle 1\rangle$ and $g_0 = g\langle 0\rangle$, regarded as maps $\Delta^0 \to X$ and thus as elements of X_0, using the notation discussed in (2.3).

Let $\widetilde{u}\colon \Delta^2 \to X$ be the *unique* extension of u along $\Lambda^2_1 \subset \Delta^2$, and define the "composite"

$$g \circ f := \widetilde{u}_{02}.$$

Thus, the 2-cell $\widetilde{u}$ is uniquely characterized by: $\widetilde{u}_{01} = f$, $\widetilde{u}_{12} = g$, and $g \circ f$ is defined by: $\widetilde{u}_{02} = g \circ f$.

This composition law is automatically unital. Given $x \in X_0$, write $1_x := x\langle 00\rangle \in X_1$, so that $(1_x)_0 = x = (1_x)_1$. Then applying the composition law gives $1_x \circ f = f$ and $g \circ 1_x = g$. (Proof: consider the 2-cells $f\langle 011\rangle, g\langle 001\rangle \in X_2$, and use the fact that their representing maps $\Delta^2 \to X$ are the unique extensions of their restrictions to $\Lambda^2_1 \subset \Delta^2$.)

Now consider $\Lambda^3_1 \subset \Delta^3$. Recall (6.20) that Λ^3_1 is a union (and colimit) of $\Delta^S \subseteq \Delta^3$ such that $S \not\supseteq \{0,2,3\}$. A map $v\colon \Lambda^3_1 \to X$ can be pictured as

$$\begin{tikzcd}
0 \arrow[dr, "f"] \arrow[drr, bend left, "(h\circ g)\circ f"] \arrow[dd, bend right, "g\circ f"'] & & \\
& 1 \arrow[r, "h\circ g"] \arrow[dl, "g"] & 3 \\
2 \arrow[urr, bend right, "h"'] & &
\end{tikzcd}$$

so that the planar 2-cells in the picture correspond to non-degenerate 2-cells of Δ^3 which are contained in Λ^3_1, while the edges are labelled according to their images in X, using the composition law defined above. Let $\widetilde{v}\colon \Delta^3 \to X$ be an extension of v along $\Lambda^3_1 \subset \Delta^3$, and consider the restriction $w := \widetilde{v}\langle 023\rangle\colon \Delta^2 \to X$ to the face $\Delta^2 \approx \Delta^{\{0,2,3\}} \subset \Delta^3$. Then $w_{01} = g \circ f$, $w_{12} = h$, and $w_{02} = (h \circ g) \circ f$, and thus the existence of w demonstrates that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

In other words, the *existence* of extensions along $\Lambda^3_1 \subset \Delta^3$ implies that the composition law we defined above is associative. (We could carry out this argument using $\Lambda^3_2 \subset \Delta^3$ instead.)

Thus, given an X with unique inner horn extensions, we can construct a category C, so that objects of C are elements of X_0, morphisms of C are elements of X_1, and composition is given as above.

Next we construct a map $X \to NC$ of simplicial sets. There are obvious maps $\alpha_n\colon X_n \to (NC)_n$, corresponding to restriction along spines $I^n \subseteq \Delta^n$; i.e., $\alpha(x) = (x_{01}, \dots, x_{n-1,n})$. These maps are compatible with simplicial operators, so that they define a map $\alpha\colon X \to NC$ of simplicial sets. *Proof:* For any n-cell $x \in X_n$, *all* of its edges are determined by edges on its spine via the (associative) composition law: $x_{ij} = x_{j-1,j} \circ x_{j-2,j-1} \circ \cdots \circ x_{i,i+1}$, for all $0 \leq i \leq j \leq n$. Thus for $f\colon [m] \to [n]$ we have $\alpha(xf) = ((xf)_{01}, \dots, (xf)_{m-1,m}) = (x_{f_0 f_1}, \dots, x_{f_{m-1} f_m}) = (x_{01}, \dots, x_{n-1,n})_{f_0 \cdots f_m} = (\alpha x) f$.

Now we can prove that nerves are characterized by unique extension along inner horns.

Proof of (8.7). We have already shown (8.9) that nerves have unique extensions for inner horns. Consider a simplicial set X which has unique inner horn extension. By the discussion above, we obtain a category C and a map $\alpha\colon X \to NC$ of simplicial sets, which is clearly a bijection in degrees ≤ 1. We will show $\alpha_n\colon X_n \to (NC)_n$ is bijective by induction on n.

Fix $n \geq 2$, and consider the commutative square

$$\begin{tikzcd}
\mathrm{Hom}(\Delta^n, X) \arrow[r, "\sim"] \arrow[d, "\alpha_{\Delta^n}"'] & \mathrm{Hom}(\Lambda^n_1, X) \arrow[d, "\alpha_{\Lambda^n_1}"] \\
\mathrm{Hom}(\Delta^n, NC) \arrow[r, "\sim"'] & \mathrm{Hom}(\Lambda^n_1, NC)
\end{tikzcd}$$

The vertical maps are induced by post-composition with $\alpha\colon X \to NC$. The horizontal maps are induced by restriction along $\Lambda^n_1 \subset \Delta^n$, and are bijections (top by hypothesis, bottom by (8.9)).

Because Λ^n_1 is a colimit of standard simplices of dimension $< n$ (6.20), the map $\alpha_{\Lambda^n_1}$ is a bijection by the induction hypothesis. Therefore α_{Δ^n} is also a bijection. $\square$

Part 2. Quasicategories

9. Definition of ∞-category

We can now define the notion of a *quasicategory*, by removing the uniqueness part of the inner horn extension criterion for nerves. This notion first appeared in [BV73, §IV.2], where such a simplicial set was said to satisfy the "restricted Kan condition". The terminology "quasicategory" for such simplicial sets was introduced by André Joyal [Joy02]. These days they are usually referred to as ∞-*categories*, due to Jacob Lurie's use of this term in [Lur09]. Note that the term "∞-category" also gets used to refer to the general concept of which quasicategories are an examplar: i.e., there are models for ∞-categories other than quasicategories, such as Segal categories, complete Segal spaces, relative categories, and others. However, I will use the terms *quasicategory* and ∞-*category* interchangably in what follows.

9.1. **Identifying 1-categories with their nerves.** To distinguish the classical notion of category from ∞-categories, I will often refer to them as *1-categories*. Furthermore, from this point on, I will (at least informally) often not distinguish a 1-category C from its nerve. In particular, I may assert something like "let C be a simplicial set which is a 1-category", which should be read as "C is a simplicial set which is isomorphic to the nerve of some category". This should not lead to much confusion, due to the fact that the nerve functor is a fully faithful embedding of Cat into sSet (4.10). Recall that under this informal identification, the linearly ordered category $[n]$ is identified with the standard n-simplex Δ^n.

9.2. **Definition of quasicategory.** A **quasicategory** (i.e, an ∞**-category**) is a simplicial set C such that for every map $f\colon \Lambda^n_j \to C$ from an inner horn, there *exists* an extension of it to $g\colon \Delta^n \to C$. That is, C is a quasicategory if the function $\mathrm{Hom}(\Delta^n, C) \to \mathrm{Hom}(\Lambda^n_j, C)$ induced by restriction along $\Lambda^n_j \subset \Delta^n$ is *surjective* for all $0 < j < n$, $n \geq 2$, so there always exists a dotted arrow in any commutative diagram of the form **quasicategory** **∞-category**

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, tail] & C \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

Any 1-category (more precisely, the nerve of any 1-category) is an ∞-category. In fact, by what we have shown (8.7), a 1-category is precisely a quasicategory for which there exist *unique* extensions of inner horns.

Let C be a quasicategory. We refer to elements of C_0 (vertices) as the **objects** of C, and elements of C_1 (edges) as the **morphisms** of C. Every morphism $f \in C_1$ has a **source** and **target**, namely its vertices $f_0 = f\langle 0\rangle, f_1 = f\langle 1\rangle \in C_0$. For $f \in C_1$ we write $f\colon f_0 \to f_1$, just as we would for morphisms in a category. Likewise, for every object $x \in C_0$, there is a distinguished morphism $1_x\colon x \to x$, called the **identity morphism**, defined by $1_x = x_{00} = x\langle 00\rangle$. When C is (the nerve of) a 1-category, all the above notions coincide with the usual ones. Note, however, that we cannot generally define *composition* of morphisms in an ∞-category in the same way we do for a 1-category. **objects** **morphisms** **source** **target** **identity morphism**

We now describe some basic categorical notions which admit immediate generalizations to quasicategories. Many of these generalizations apply to arbitrary simplicial sets.

9.3. **Products of quasicategories.** Recall (7.1) that the product of simplicial sets X and Y is just the product of the functors. Thus, $(X \times Y)_n = X_n \times Y_n$, with the evident action of simplicial operators: $(x, y)f = (xf, yf)$.

9.4. **Proposition.** *The product of two quasicategories (as simplicial sets) is a quasicategory.*

Proof. Exercise, using the bijective correspondence between the sets of (i) maps $K \to X \times Y$ and (ii) pairs of maps $(K \to X, K \to Y)$. □

9.5. *Exercise.* If C and D are 1-categories, then $N(C \times D) \approx NC \times ND$. Thus, the notion of product of quasicategories generalizes that of categories.

9.6. **Coproducts of quasicategories.** The coproduct of simplicial sets X and Y is just the coproduct of functors, whence $(X \amalg Y)_n = X_n \amalg Y_n$, i.e., the set n-cells of the coproduct is the disjoint union of the sets of n-cells of X and Y. More generally, $\left(\coprod_s X_s\right)_n = \coprod_s (X_s)_n$ for an indexed collection $\{X_s\}$ of simplicial sets.

9.7. **Proposition.** *The coproduct of any indexed collection of quasicategories is a quasicategory.*

To prove this, we introduce the set of **connected components** of a simplicial set. Given a simplicial set X, define an equivalence relation $\approx$ on the set $\coprod_{n\geq 0} X_n$ of cells of X, generated by the relation **connected components**

$$a \sim a\delta \qquad \text{for all } n \geq 0,\ a \in X_n,\ \delta\colon [m] \to [n].$$

An equivalence class for $\approx$ is called a **connected component** of X, and we write $\pi_0 X$ for the set of connected components. This construction defines a functor $\pi_0 \colon s\mathrm{Set} \to \mathrm{Set}$. **connected component**

Note that the above recipe is exactly how we constructed the a colimit of a functor $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$. Thus, there is an identification

$$\mathrm{colim}_{\Delta^{\mathrm{op}}} X \approx \pi_0 X$$

between the set of connected components of X and the colimit of the functor $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$.

9.8. *Exercise* (Simplicial sets are coproducts of connected subcomplexes)*.* Show that each connected component C (i.e., equivalence class of cells) of a simplicial set X is a subcomplex of X, and that the evident map $\coprod_{C\in\pi_0 X} C \to X$ of simplicial sets induced by inclusions of connected components is an isomorphism.

9.9. *Exercise.* Show that any map $f\colon X \to Y$ of simplicial sets carries each connected component $C \subseteq X$ into some connected component of Y, and thus induces a map $\pi_0 X \to \pi_0 Y$.

9.10. *Exercise* (Connected components are path components)*.* Show that there is a canonical bijection

$$(X_0/\approx_1) \xrightarrow{\sim} \pi_0 X,$$

where the left-hand side denotes the set of equivalence classes in the vertex set X_0 with respect to the equivalence relation $\approx_1$ which is *generated* by the relation $\sim_1$ on X_0, defined by

$$a \sim_1 b \qquad \text{iff there exists } e \in X_1 \text{ such that } a = e_0, b = e_1.$$

We say that a simplicial set X is **connected** if $\pi_0 X$ is a singleton, i.e., if it consists of a single connected component. **connected**

9.11. *Exercise.* Show that every standard simplex Δ^n is connected, and that every horn Λ^n_j is connected.

Proof of (9.7)*.* Consider a disjoint union $X = \coprod_s X_s$ of quasicategories. Note that any connected component of X must be contained inside one of the X_ss, since for any cell of X_s, any simplicial operator applied to it must also be a cell of X_s. Therefore, since horns are connected (9.11), and maps preserve components (9.9), we see that the image of any map $f\colon \Lambda^n_j \to X$ must be contained in some connected component and therefore in some X_s. Thus if X_s is a quasicategory, the desired extension of f exists (if $0 < j < n$). □

9.12. *Exercise* (Important)*.* Show that the evident map $\pi_0(X \times Y) \to \pi_0 X \times \pi_0 Y$ induced by projections is a bijection.

9.13. *Exercise* (Colimits of functors to sets via π_0). Given a functor $F\colon C \to \mathrm{Set}$ from at ordinary category to sets, we can describe its colimit as the set of connected components of the nerve of its *point category*.

(1) Show that there is a 1-category C_F (the **point category** of F) so that: **point category**
- *objects* are pairs (c, x) where $c \in C_0$ and $x \in F(x)$, and
- *morphisms* $(c, x) \to (c', x')$ are maps $\alpha\colon c \to c' \in C_1$ such that $F(\alpha)(x) = x'$.

(2) Show that there is an isomorphism $\pi_0 N(C_F) \xrightarrow{\sim} \mathrm{colim}_C F$.

10. Subcategories

Recall that if C is a 1-category, then a *subcategory* C' of C consists of subsets $\mathrm{ob}\, C' \subseteq \mathrm{ob}\, C$ and $\mathrm{mor}\, C' \subseteq \mathrm{mor}\, C$ of objects and morphisms of C which is a 1-category in its own right. Furthermore, C' is a *full subcategory* if it contains all the morphisms in C which go between any two of its objects.

Both notions admit generalizations to ∞-categories.

10.1. **Subcategories of ∞-category.** A **subcategory** of an ∞-category C is a subcomplex $C' \subseteq C$ such that for all $n \geq 2$ and $0 < k < n$, every $f\colon \Delta^n \to C$ such that $f(\Lambda^n_k) \subseteq C'$ satisfies $f(\Delta^n) \subseteq C'$. That is, *all* inner horn extensions in C along "horns in C'" are themselves contained in C'. It is clear that a subcategory is in fact a quasicategory. **subcategory**

10.2. *Exercise* (Subcategories are determined by their morphisms). Let C be a quasicategory, and consider $S \subseteq C_1$ a collection of morphisms in C. Define $C'_n := \{\, a \in C_n \mid a_{ij} \in S \text{ for all } 0 \leq i \leq j \leq n \,\}$. Show that the C'_ns describe a subcomplex C' of C if and only if $f_{00}, f_{11} \in S$ for all $f \in S$. Show that furthermore C' is a subcategory if and only if, in addition, for all $u \in C_2$ we have that $u_{01}, u_{12} \in S$ implies $u_{02} \in S$.

10.3. *Remark.* In general, if $C' \subseteq C$ is a subcomplex and C and C' are quasicategories, it need not be the case that C' is a subcategory of C. See (12.5) below.

When C is an 1-category, a subcategory of C in the above sense is the same as a subcategory in the usual sense, which correspond exactly to subsets $S \subseteq C_1$ of morphisms for which (i) if $(x \xrightarrow{f} y) \in S$ then $\mathrm{id}_x, \mathrm{id}_y \in S$, and (ii) S is closed under composition, i.e., if $g \circ f$ is defined and $f, g \in S$, then $g \circ f \in S$.

10.4. **Full subcategories of an ∞-category.** We say that a subcomplex $C' \subseteq C$ of an ∞-category C is a **full subcategory** if for all n and all $a \in C_n$, we have that $a \in C'_n$ if and only if $a_i \in C'_0$ for all $i = 0, \dots, n$. **full subcategory**

10.5. *Exercise.* Show that a full subcategory $C' \subseteq C$ is in fact a subcategory as defined in (10.1), and thus in particular a full subcategory C' is itself a quasicategory.

Given an ∞-category C and a set $S \subseteq C_0$ of vertices, let

$$C'_n = \{\, a \in C_n \mid a_j \in S \text{ for all } j = 0, \dots, n \,\},$$

the set of n-dimensional cells whose vertices are in S. This is evidently a full subcategory of C, called the **full subcategory spanned by** S. **full subcategory spanned by** S

When C is a 1-category, a full subcategory of C in the above sense is the same as a full subcategory in the usual sense.

10.6. **Opposite of a quasicategory.** Given a 1-category C, the *opposite* category C^{op} has $\mathrm{ob}\, C^{\mathrm{op}} = \mathrm{ob}\, C$, and $\mathrm{Hom}_{C^{\mathrm{op}}}(x, y) = \mathrm{Hom}_C(y, x)$, and the sense of composition is reversed: $g \circ_{C^{\mathrm{op}}} f = f \circ_C g$.

This concept also admits a generalization to quasicategories, which we define using a non-trivial involution $\mathrm{op}\colon \Delta \to \Delta$ of the category Δ. This is the functor which on objects sends $[n] \mapsto [n]$, and on morphisms sends $\langle f_0, \dots, f_n \rangle\colon [n] \to [m]$ to $\langle m - f_n, \dots, m - f_0 \rangle$, i.e., $\mathrm{op}(f)(x) = m - f(n - x)$.

10.7. *Remark.* You can visualize this involution as the functor which "reverses the ordering" of the totally-ordered sets $[n]$. Note that the totally ordered set "$[n]$ with the order of its elements reversed" isn't actually an object of Δ, but rather is uniquely isomorphic to $[n]$, via the function $x \mapsto n - x$.

The **opposite** of a simplicial set $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ is the composite functor $X^{\mathrm{op}} := X \circ \mathrm{op}$. We have a unique isomorphism $(\Delta^n)^{\mathrm{op}} \approx \Delta^n$, and this isomorphism restricts to an isomorphism $(\Lambda^n_j)^{\mathrm{op}} \approx \Lambda^n_{n-j}$ of subcomplexes, so that the opposite of an inner horn is another inner horn. As a consequence, the opposite of a quasicategory is a quasicategory. It is straightforward to verify that $(NC)^{\mathrm{op}} = N(C^{\mathrm{op}})$, so the notion of opposite quasicategory generalizes the notion of opposite category. The functor $\mathrm{op}\colon \Delta \to \Delta$ satisfies $\mathrm{op} \circ \mathrm{op} = \mathrm{id}_\Delta$, so $(X^{\mathrm{op}})^{\mathrm{op}} = X$. **opposite**

11. Functors and natural transformations

11.1. **Functors.** A **functor** between ∞-category is merely a map $f\colon C \to D$ between the simplicial sets. **functor**

We write qCat for the 1-category of ∞-categories (=quasicategories) and functors between them. Clearly $\mathrm{qCat} \subset s\mathrm{Set}$ is a full subcategory. Because the nerve functor is a full embedding of Cat into qCat, the notions of functor for categories and quasicategories coincide when both are viewed as kinds of simplicial sets.

11.2. *Exercise* (Mapping property of a full subcategory). Let C be a quasicategory, and $C' \subseteq C$ the full subcategory spanned by some subset $S \subseteq C_0$. Show that a functor $f\colon D \to C$ factors through a functor $f'\colon D \to C' \subseteq C$ if and only if $f(D_0) \subseteq S$.

11.3. **Natural transformations.** Given functors $F, G\colon C \to D$ between categories, a *natural transformation* $\phi\colon F \Rightarrow G$ is a choice, for each object c of C, of a map $\phi(c)\colon F(c) \to G(c)$ in D, such that $g(\alpha) \circ \phi(c) = \phi(c') \circ f(\alpha)$ for every morphism $\alpha\colon c \to c'$ in C, i.e., such that the square

$$\begin{tikzcd} F(c) \arrow[r, "\phi(c)"] \arrow[d, "f(\alpha)"'] & G(c) \arrow[d, "g(\alpha)"] \\ F(c') \arrow[r, "\phi(c')"'] & G(c') \end{tikzcd}$$

commutes in D.

There is a standard convenient reformulation of this: a natural transformation $\phi\colon F \Rightarrow G$ is the same thing as a functor

$$\Phi\colon C \times [1] \to D,$$

from the product of C with $[1]$, so that $\Phi|C \times \{0\} = F$, $\Phi|C \times \{1\} = G$, and $\Phi|\{c\} \times [1] = \alpha(c)$ for each $c \in \operatorname{ob} C$.

11.4. *Remark.* In the above I have made implicit use of the evident isomorphisms $C \times \{0\} \approx C \approx C \times \{1\}$, and of $\{c\} \times [1] \approx [1]$. I will do this frequently in what follows, usually without comment, and implicitly identify $X \times \Delta^0$ or $\Delta^0 \times X$ with X.

This reformulation admits a straightforward generalization to quasicategories, using that $N[1] = \Delta^1$. A **natural transformation** $\phi\colon f_0 \Rightarrow f_1$ of functors $f_0, f_1\colon C \to D$ between quasicategories is *defined* to be a map **natural transformation**

$$\phi\colon C \times \Delta^1 \to D$$

of simplicial sets such that $\phi|C \times \{i\} = f_i$ for $i = 0, 1$. For ordinary categories this coincides with the classical notion, since $N(C) \times \Delta^1 \approx N(C) \times N([1]) \approx N(C \times [1])$ (9.5).

12. Examples of ∞-categories

There are many ways to produce ∞-categories, as we will see. Unfortunately, "hands-on" constructions of quasicategories which are not 1-categories are relatively rare. Here I give a few reasonably explicit examples to play with. **F 28 Jan**

12.1. **The ∞-category of 1-categories.** This is an example of a quasicategory in which *objects* are (small) categories, *morphisms* are functors between categories, and *2-dimensional cells* are certain kinds of natural isomorphisms of functors.

Define a simplicial set Cat_1 so that $(\mathrm{Cat}_1)_n$ has elements described by data $x := (C_i, F_{ij}, \zeta_{ijk})$ where

(0) for each $i \in [n]$, C_i is a (small) category,
(1) for each $i \leq j$ in $[n]$, $F_{ij} \colon C_i \to C_j$ is a functor, and
(2) for each $i \leq j \leq k$ in $[n]$, $\zeta_{ijk} \colon F_{ik} \Rightarrow F_{jk}F_{ij}$ is a natural *isomorphism* of functors $C_i \to C_k$,

such that

(a) for each i in $[n]$, $F_{ii} \colon C_i \to C_i$ is the identity functor Id_{C_i} of C_i,
(b) for each $i \leq j$ in $[n]$, $\zeta_{iij} \colon F_{ij} \Rightarrow F_{ij}\,\mathrm{Id}_{C_i}$ and $\zeta_{ijj} \colon F_{ij} \Rightarrow \mathrm{Id}_{C_j}\,F_{ij}$ are the identity natural isomorphism of F_{ij}, and
(c) for each $i \leq j \leq k \leq \ell$, the diagram

$$\begin{tikzcd} F_{i\ell} \arrow[r, Rightarrow, "\zeta_{ij\ell}"] \arrow[d, Rightarrow, "\zeta_{ik\ell}"'] & F_{j\ell}F_{ij} \arrow[d, Rightarrow, "\zeta_{jk\ell}F_{ij}"] \\ F_{k\ell}F_{ik} \arrow[r, Rightarrow, "F_{k\ell}\zeta_{ijk}"'] & F_{k\ell}F_{jk}F_{ij} \end{tikzcd} \tag{12.2}$$

of natural isomorphisms of functors commutes, i.e., we have an identity $(\zeta_{jk\ell}F_{ij})\zeta_{ij\ell} = (F_{k\ell}\zeta_{ijk})\zeta_{ik\ell}$ of natural transformations $F_{i\ell} \to F_{k\ell}F_{jk}F_{ij}$ of functors $C_i \to C_\ell$.

For a simplicial operator $\delta \colon [m] \to [n]$ define

$$(C_i, F_{ij}, \zeta_{ijk})\delta = (C_{\delta(i)}, F_{\delta(i)\delta(j)}, \zeta_{\delta(i)\delta(j)\delta(k)}).$$

12.3. *Remark.* Note that condition (c) is actually implied by condition (b) whenever $i = j$, or $j = k$, or $k = \ell$.

12.4. *Example.* An 0-cell in Cat_1 is a category. A 1-cell in Cat_1 is a functor. A 2-cell in Cat_1 is a natural isomorphism $F_{02} \Rightarrow F_{12}F_{01}$ from a functor to a composite of two functors.

I claim that Cat_1 is a quasicategory. Fillers for $\Lambda^2_1 \subset \Delta^2$ always exist: a map $\Lambda^2_1 \to \mathrm{Cat}_1$ amounts to a choice of functors $(C_0 \xrightarrow{F_{01}} C_1 \xrightarrow{F_{12}} C_2)$, and an extension to Δ^2 can be given by setting $F_{02} = F_{12}F_{01}$ and $\zeta_{012} = \mathrm{id}_{F_{02}}$. Note that this is not the only possibly extension: we can take F_{02} to be any functor which is naturally isomorphic to $F_{12}F_{01}$, and ζ_{012} can be any such natural isomorphism.

Fillers for $\Lambda^3_1 \subset \Delta^3$ and $\Lambda^3_2 \subset \Delta^3$ always exist, and are unique: finding a filler amounts to choosing isomorphisms $\zeta_{023} = \zeta_{ik\ell}$ (for Λ^3_1) or $\zeta_{013} = \zeta_{ij\ell}$ (for Λ^3_2) making (12.2) commute, which can be done uniquely because in either case the other three isomorphisms are given. All fillers for all horns $\Lambda^n_j \subset \Delta^n$ in higher dimensions $n \geq 4$ exist and are unique: there is no additional data to supply in these cases, and all properies of the data are automatically satisfied.

12.5. *Exercise.* Note that the (nerve of) the 1-category Cat of small categories and functors between them is isomorphic to the subcomplex of Cat_1 whose cells are $(C_i, F_{ij}, \zeta_{ijk})$ such that $F_{ik} = F_{jk}F_{ij}$ and $\zeta_{ijk} = \mathrm{id}_{F_{ik}}$. Show that this subcomplex is a quasicategory which not a subcategory in the sense of (10.1). ("The 1-category of 1-categories is not a subcategory of the ∞-category of 1-categories.")

12.6. *Remark.* Suppose D is a category. Then a functor $D \to \mathrm{Cat}_1$ is basically the same thing as what is called a *pseudofunctor* $D \to \mathrm{Cat}$.

12.7. **Automorphism ∞-groupoid of a group.** Here is a "smaller" variant of the above example. Given a group G, let $(M_G)_n$ be the set whose elements are data $x := (\phi_{ij}, u_{ijk})$ where

(1) for each $i \leq j$ in $[n]$, $\phi_{ij} \colon G \to G$ is a group automorphism, and
(2) for each $i \leq j \leq k$ in $[n]$, $u_{ijk} \in G$ such that $\phi_{jk}\phi_{ij}(g) = u_{ijk}\phi_{ik}(g)^{-1}u_{ijk}^{-1}$ for all $g \in G$,

such that

(a) for each i in $[n]$, $\phi_{ii} \colon G \to G$ is the identity map id_G,
(b) for each $i \leq j$, u_{iij} and u_{ijj} are the identity element of G, and
(c) for each $i \leq j \leq k \leq \ell$, we have the identity

$$\phi_{k\ell}(u_{ijk})u_{ik\ell} = u_{jkl}u_{ij\ell}.$$

For a simplicial operator $\delta \colon [m] \to [n]$ define

$$(\phi_{ij}, u_{ijk})\delta = (\phi_{\delta(i)\delta(j)}, u_{\phi(i)\phi(j)\phi(k)}).$$

Thus M_G has exactly one 0-cell, while a 1-cell is an automorphism of G, and a 2-cell is a collection of three automorphisms together with an element $u = u_{012} \in G$ so that $\phi_{12}\phi_{02}(g) = u\phi_{02}(g)u^{-1}$ for all $g \in G$.

12.8. *Exercise.* Show that the above defines a simplicial set which is a quasicategory (in fact, a Kan complex as defined below (12.12), and therefore an ∞-groupoid as we will see later).

12.9. **Singular complex of a space.** Given a topological space, we can extract from it an ∞-category (in fact, an ∞-groupoid) sometimes called its *fundamental ∞-groupoid*, or more conventionally its *singular complex*.

The **topological n-simplex** is **topological n-simplex**

$$\Delta^n_{\mathrm{top}} := \Big\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\Big|\; \sum x_i = 1,\ x_i \geq 0 \Big\},$$

the convex hull of the standard basis vectors $e_0, \dots, e_n$ in $(n+1)$-dimensional Euclidean space, viewed as a subspace. These fit together as the values of a functor $\Delta_{\mathrm{top}} \colon \Delta \to \mathrm{Top}$ to the category of topological spaces and continuous maps, with $\Delta_{\mathrm{top}}([n]) = \Delta^n_{\mathrm{top}}$. A simplicial operator $\delta \colon [m] \to [n]$ sends $(x_0, \dots, x_m) \in \Delta^m_{\mathrm{top}}$ to $(y_0, \dots, y_n) \in \Delta^n_{\mathrm{top}}$ with $y_j = \sum_{f(i)=j} x_i$. That is, it sends a standard basis vector e_i to $e_{\delta(i)}$, and is extended to the simplex by linearity.

For a topological space T, we define its **singular complex** $\mathrm{Sing}\, T$ to be the simplicial set with cells $[n] \mapsto \mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{top}}, T)$, the set of continuous maps from the topological n-simplex, with the action of simplicial operators induced by restriction. **singular complex**

Define **topological horns** **topological horns**

$$(\Lambda^n_j)_{\mathrm{top}} := \big\{ x \in \Delta^n_{\mathrm{top}} \;\big|\; \exists\, i \in [n] \smallsetminus \{j\} \text{ such that } x_i = 0 \big\} \subset \Delta^n_{\mathrm{top}},$$

and observe that continuous maps $(\Lambda^n_j)_{\mathrm{top}} \to T$ correspond in a natural way with maps $\Lambda^n_j \to \mathrm{Sing}\, T$.

12.10. *Exercise.* Prove the previous statement, by showing that $(\Lambda^n_j)_{\mathrm{top}}$ is a colimit in topological spaces of a functor $\Delta^S \mapsto \Delta^S_{\mathrm{top}}$, on the poset of subcomplexes $\Delta^S \subseteq \Lambda^n_j$, and use (6.20).

There exists a continuous retraction $r_n \colon \Delta^n_{\mathrm{top}} \to (\Lambda^n_j)_{\mathrm{top}}$, and thus we see that

$$\mathrm{Hom}(\Delta^n, \mathrm{Sing}\, T) \to \mathrm{Hom}(\Lambda^n_j, \mathrm{Sing}\, T)$$

is surjective for every horn (not just inner ones): any continous map $f \colon (\Lambda^n_j)_{\mathrm{top}} \to T$ can be extended to a map $f' \colon \Delta^n_{\mathrm{top}} \to T$ by setting $f' := fr_n$.

12.11. *Exercise.* Describe a continuous retraction $r \colon \Delta^n_{\mathrm{top}} \to (\Lambda^n_j)_{\mathrm{top}}$.

12.12. *Remark* (Kan complexes). A simplicial set X which has extensions for all horns is called a **Kan complex**. Thus, $\operatorname{Sing} T$ is a Kan complex, and so in particular is a quasicategory (and as we will see below, a "quasigroupoid" (14.14)). **Kan complex**

12.13. **Eilenberg-MacLane object.** In algebraic topology, an *Eilenberg-MacLane space* is a space (called $K(A,d)$) with only one non-trivial homotopy group A in dimension d. One of their key features is that they serve as representing objects for cohomology: homotopy classes of maps $X \to K(A,d)$ correspond to elements of the cohomology group $H^d(X;A)$. They have an explicit construction in terms of simplicial homotopy theory, which provide us with explicit constructions of quasicategories.

Fix an abelian group A and an integer $d \geq 0$. We define a simplicial set $K = K(A,d)$, so that K_n is a set whose elements are data $a = (a_\delta)$ consisting of

(a) for each simplicial operator $\delta\colon [d] \to [n]$, an element $a_\delta \in A$, such that
(b) $a_\delta = 0$ if δ is *not* an injective function $[d] \to [n]$, and such that
(c) for each $\gamma\colon [d+1] \to [n]$ we have $\sum_{j=0}^{d+1} (-1)^j a_{\gamma d^j} = 0$, where $d^j := \langle 0, \dots, \widehat{j}, \dots, d+1\rangle \colon [d] \to [d+1]$, the injective simplical operator which omits j from its image.

Note that the condition in (c) is automatically satisfied when γ is not injective, given (b): if $\gamma(v) = \gamma(v+1)$, then γd^j is non-injective when $j \neq v, v+1$, and $\gamma d^v = \gamma d^{v+1}$.

For a simplicial operator $\epsilon\colon [m] \to [n]$ and $a \in K(A,d)_n$ we define

$$(a\epsilon)_\delta := a_{\epsilon\delta} \qquad \text{for } \delta\colon [d] \to [m].$$

12.14. *Exercise.* Verify the above formulas define a simplicial set.

When $d = 0$, the object $K(A,0)$ is seen to be a discrete simplicial set, equal to A in each dimension.

12.15. *Exercise.* Show that $K(A,1)$ is isomorphic to the nerve of a groupoid, namely the nerve of the group A regarded as a category with one object.

In general, the object $K(A,d)$ is a Kan complex, and hence a quasicategory (and in fact a quasigroupoid as we will see). This is demonstrated in the following exercises.

12.16. *Exercise.* Given a simplicial set X, a **normalized d-cocycle** with values in an abelian group A is a function $f\colon X_d \to A$ such that **normalized d-cocycle**

(1) $f(xs^i) = 0$ for all $x \in X_{d-1}$ and $0 \leq i \leq d-1$, where $s^i = \langle 0, \dots, i, i, \dots, d-1\rangle \colon [d] \to [d-1]$, and
(2) $\sum (-1)^i f(xd^i) = 0$ for all $x \in X_{d+1}$ and $0 \leq i \leq d+1$, where $d^i = \langle 0, \dots, \widehat{i}, \dots, d\rangle \colon [d-1] \to [d]$.

Show that there are inverse bijections

$$\phi\colon \operatorname{Hom}_{s\mathrm{Set}}(X, K(A,d)) \longleftrightarrow Z^d_{\mathrm{norm}}(X;A) : \psi$$

from the set of simplicial maps to the set of normalized d-cocycles, where ϕ sends $g\colon X \to K(A,d)$ to its restriction $f := g_d\colon X_d \to K(A,d)_d = A$ on d-cells, and ψ sends $f \in Z^d_{\mathrm{norm}}(X;d)$ to the simplicial map g given $g_n(x)_\delta := f(x\delta)$ for $x \in X_n$ and $\delta \in \operatorname{Hom}_\Delta([d],[n])$.

12.17. *Exercise.* Show that $K(A,d)$ is a Kan complex, i.e., that $\operatorname{Hom}(\Delta^n, K(A,d)) \to \operatorname{Hom}(\Lambda^n_j, K(A,d))$ is surjective for all horns $\Lambda^n_j \subset \Delta^n$. In fact, this map is *bijective* unless $n = d$. (Hint: use the previous exercise to transform the question into one about extending normalized cocycles, and note that there are four distinct cases to check, namely $n < d$, $n = d$, $n = d+1$, and $n > d+1$.)

12.18. *Remark.* Eilenberg-MacLane objects are an example of a *simplicial abelian group*: the map $+\colon K \times K \to K$ defined in each dimension by $(a+b)_\delta = a_\delta + b_\delta$ is a map of simplicial sets which satisfies the axioms of an abelian group, reflecting the fact that $Z^d_{\mathrm{norm}}(X;A)$ is an abelian group.

As the terminology of "normalized cocycle" suggests, these objects are closely related to cohomology of topological spaces. In particular, if T is a topological space, and $X = \operatorname{Sing} T$ is its singular complex, then its singular cohomology group $H^d(T; A)$ is a quotient of the group $\operatorname{Hom}(X, K(A, d)) \approx Z^d_{\text{norm}}(X; A)$ of normalized cohains (by a subgroup of "normalized boundaries").

13. Homotopy category of an ∞-category

Our next goal is to define the notion of an *isomorphism* in an ∞-category. This notion behaves much like that of *homotopy equivalence* in topology. We will define isomorphism by means of the *homotopy category* of an ∞-category. If we think of an ∞-category as "an ordinary category with higher structure", then its homotopy category is the ordinary category obtained by "flatting out the higher stucture". **M 31 Jan**

13.1. **The fundamental category of a simplicial set.** The homotopy category of a quasicategory is itself a special case of the notion of the *fundamental category* of a simplicial set, which we turn to first.

A **fundamental category** for a simplicial set X consists of (i) a category hX, and (ii) a map $\alpha\colon X \to N(hX)$ of simplicial sets, such that for every category C, the map **fundamental category**

$$\alpha^*\colon \operatorname{Hom}(N(hX), NC) \to \operatorname{Hom}(X, NC)$$

induced by restriction along α is a bijection. This is a universal property which characterizes the fundamental category up to unique isomorphism, if it exists.

13.2. **Proposition.** *Every simplicial set has a fundamental category.*

Proof sketch. Given X, we construct hX by generators and relations. First, consider the **free category** F, whose objects are the set X_0, and whose morphisms are finite "composable" sequences $[a_n, \dots, a_1]$ of edges of X_1. Thus, morphisms in F are "words", whose "letters" are edges a_i with $(a_{i+1})_0 = (a_i)_1$, and composition is concatenation of words; the element $[a_n, \dots, a_1]$ is then a morphism $(a_1)_0 \to (a_n)_1$. We must also suppose that there is an empty sequence $[]_x$ in F for each vertex $x \in X_0$; these correspond to identity maps in F. **free category**

Then hX is defined to be the minimal quotient category of F subject to the following relations on the set of morphisms:

- $[a] \sim []_x$ for each $x \in X_0$ where $a = x_{00} \in X_1$, and
- $[g, f] \sim [h]$ whenever there exists $a \in X_2$ such that $a_{01} = f$, $a_{12} = g$, and $a_{02} = h$.

The map $\alpha\colon X \to N(hX)$ sends $x \in X_n$ to the equivalence class of $[x_{n-1,n}, \dots, x_{0,1}]$. Given this, verifying the desired universal property of α is formal.

(We will give another construction of the fundamental category in (17.28).) □

13.3. *Exercise.* Complete the proof of (13.2) by showing that $\alpha^*\colon \operatorname{Hom}(N(hX), NC) \to \operatorname{Hom}(X, NC)$ is a bijection for any category C.

As a consquence, the fundamental category construction describes a functor $h\colon s\mathrm{Set} \to \mathrm{Cat}$, which is left adjoint to the nerve functor $N\colon \mathrm{Cat} \to s\mathrm{Set}$, so that there is an isomorphism

$$\operatorname{Hom}_{\mathrm{Cat}}(hX, C) \xrightarrow{\sim} \operatorname{Hom}_{s\mathrm{Set}}(X, NC)$$

natural in the simplicial set X and category C.

In general, the fundamental category of a simplicial set is not an easy thing to get a hold of explicitly, because it is difficult to give an explicit description of a "quotient category" induced by a relation on its morphisms. We will not be making much use of it. When C is a quasicategory, there is a more concrete construction of hC, which in this context is called the *homotopy category* of C. *Warning:* Sometimes people will not distinguish "fundamental category" from "homotopy category" as I have here, and just call either the homotopy category. I will use the notation "hC" for either.

13.4. **The homotopy relation on morphisms.** Fix a quasicategory C. For $x, y \in C_0$, let $\hom_C(x, y) := \{\, f \in C_1 \mid f_0 = x,\ f_1 = y \,\}$ denote the set of "morphisms" in C from x to y. We write 1_x for the element $x_{00} \in \hom_C(x, x)$.

Define relations $\sim_\ell$, $\sim_r$ on $\hom_C(x, y)$ (called **left homotopy** and **right homotopy**) by **left homotopy** **right homotopy**

- $f \sim_\ell g$ iff there exists $a \in C_2$ with $a_{01} = 1_x$, $a_{02} = f$, $a_{12} = g$,
- $f \sim_r g$ iff there exists $b \in C_2$ with $b_{12} = 1_y$, $b_{01} = f$, $b_{02} = g$.

Pictorally:

$$f \sim_\ell g\colon \quad \begin{tikzcd} x \arrow[dr, "f"] \arrow[dd, bend right, "1_x"'] \arrow[dd, phantom, "a" near start, shift left=4ex] & \\ & y \\ x \arrow[ur, "g"'] & \end{tikzcd} \qquad\qquad f \sim_r g\colon \quad \begin{tikzcd} & y \arrow[dd, bend left, "1_y"] \\ x \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[r, phantom, "b"] & {} \\ & y \end{tikzcd}$$

Note that $f \sim_\ell g$ in $\hom_C(x, y)$ if and only if $g \sim_r f$ in $\hom_{C^{\mathrm{op}}}(y, x)$.

13.5. *Remark.* If C is an ordinary category, then the left homotopy and right homotopy relations reduce to the equality relation on the set $\mathrm{Hom}_C(x, y)$.

13.6. **Proposition.** *The relations $\sim_\ell$ and $\sim_r$ are equal to each other, and are an equivalence relation on* $\hom_C(x, y)$.

Proof. Given $f, g, h\colon x \to y$ in a quasicategory C, we will prove

(1) $f \sim_\ell f$,
(2) $f \sim_\ell g$ and $g \sim_\ell h$ imply $f \sim_\ell h$,
(3) $f \sim_\ell g$ implies $f \sim_r g$,
(4) $f \sim_r g$ implies $g \sim_\ell f$.

Statements (3) and (4) combine to show that $\sim_\ell$ is symmetric, and thus with (1) and (2) that $\sim_\ell$ is an equivalence relation. Statements (3) and (4) and symmetry imply that $\sim_r$ and $\sim_\ell$ coincide. The idea is to use the inner-horn extension condition for C to produce the appropriate relations.

Staetement (1) is exhibited by $f_{001} \in C_2$.

$$\begin{tikzcd} x \arrow[dr, "f"] \arrow[dd, bend right, "1_x"'] \arrow[dd, phantom, "f_{001}", shift left=4ex] & \\ & y \\ x \arrow[ur, "f"'] & \end{tikzcd}$$

Statements (2), (3), and (4) are demonstrated by the following diagrams, which present a map from an inner horn of Δ^3 (respectively Λ^3_1, Λ^3_1, and Λ^3_2) to C constructed from the given data. The restriction of any extension to Δ^3 along the remaining face (respectively $\Delta^{\{023\}}$, $\Delta^{\{023\}}$, and $\Delta^{\{013\}}$) gives the conclusion.

$$\begin{tikzcd} x \arrow[dr, "1_x"] \arrow[drr, bend left, "f"] \arrow[dd, bend right, "1_x"'] \arrow[dd, phantom, "x_{000}", shift left=4ex] & & \\ & x \arrow[r, "g"] \arrow[dl, "1_x"] \arrow[ur, phantom, "a"] \arrow[dr, phantom, "b"] & y \\ x \arrow[urr, bend right, "h"'] & & \end{tikzcd} \quad \begin{tikzcd} x \arrow[dr, "1_x"] \arrow[drr, bend left, "g"] \arrow[dd, bend right, "f"'] \arrow[dd, phantom, "a", shift left=4ex] & & \\ & x \arrow[r, "g"] \arrow[dl, "g"] \arrow[ur, phantom, "g_{001}"] \arrow[dr, phantom, "g_{011}"] & y \\ y \arrow[urr, bend right, "1_y"'] & & \end{tikzcd} \quad \begin{tikzcd} x \arrow[dr, "f"] \arrow[drr, bend left, "g"] \arrow[dd, bend right, "1_x"'] \arrow[dd, phantom, "f_{001}", shift left=4ex] & & \\ & y \arrow[r, "1_y"] \arrow[ur, phantom, "b"] \arrow[dr, phantom, "f_{011}"] & y \\ x \arrow[ur, "f"] \arrow[urr, bend right, "f"'] & & \end{tikzcd}$$

□

We now define $f \approx g$ to mean $f \sim_\ell g$ (equivalently $f \sim_r g$). We speak of **homotopy classes** $[f]$ of morphisms $f \in \hom_C(x, y)$, meaning equivalence classes under $\approx$. **homotopy classes**

13.7. **Composition of homotopy classes of morphisms.** Next we observe that we can compose homotopy classes. Given $f \in \hom_C(x, y)$, $g \in \hom_C(y, z)$, $h \in \hom_C(x, z)$, we say that h is a **composite** of (g, f) if there exists a 2-cell $a \in C_2$ with $a\langle 01\rangle = f$, $a\langle 12\rangle = g$, $a\langle 02\rangle = h$; thus composition is a three-fold relation on $\hom(x, y) \times \hom(y, z) \times \hom(x, z)$. The composition relation is compatible with the homotopy relation in the following sense. **composite**

13.8. **Lemma.** *If $f \approx f'$, $g \approx g'$, h a composite of (g, f), and h' a composite of (g', f'), then $h \approx h'$.*

Proof. Since $\approx$ is an equivalence relation, it suffices prove the special cases (a) $f = f'$, and (b) $g = g'$. We prove case (b), as case (a) is analogous.

Let $a \in C_2$ exhibit $f \sim_\ell f'$, and let $b, b' \in C_2$ exhibit h as a composite of (g, f) and h' as a composite of (g, f') respectively. The inner horn $\Lambda^3_2 \to C$ defined by

$$\begin{tikzcd}
0 \arrow[dr, "f"] \arrow[drr, bend left, "h"] \arrow[dd, bend right, "1_x"'] \arrow[drr, phantom, "b"] \arrow[dd, phantom, shift left=6, "a"] & & \\
 & 2 \arrow[r, "g"] & 3 \\
1 \arrow[ur, "f'"'] \arrow[urr, bend right, "h'"'] \arrow[urr, phantom, "b'"] & &
\end{tikzcd}$$

extends to $u \colon \Delta^3 \to C$, and $u|\Delta^{\{0,1,3\}}$ exhibits $h \sim_\ell h'$. □

Thus, composites of (g, f) live in a unique homotopy class of morphisms in C, which only depends on the homotopy classes of g and f. I will write $[g] \circ [f]$ for the homotopy class containing composites of (g, f).

I'll leave the following as exercises; the proofs are much like what we have already seen.

13.9. **Lemma.** *Given $f \colon x \to y$, we have $[f] \circ [1_x] = [f] = [1_y] \circ [f]$.*

13.10. **Lemma.** *If $[g] \circ [f] = [u]$, $[h] \circ [g] = [v]$, then $[h] \circ [u] = [v] \circ [f]$.*

13.11. *Exercise.* Prove (13.9) and (13.10).

13.12. **The homotopy category of a quasicategory.** For any quasicategory, we define its **homotopy category** hC, with object set $\mathrm{ob}(hC) := C_0$, and with morphism sets $\hom_{hC}(x, y) := \hom_C(x, y)/\approx$, with composition defined by $[g] \circ [f]$. The above lemmas (13.9) and (13.10) exactly imply that hC is a category. **homotopy category**

We define a map $\pi \colon C \to N(hC)$ of simplicial sets as follows. On vertices, π is the identity map $C_0 = N(hC)_0 = \mathrm{ob}\, hC$. On edges, the map is defined by the tautological quotient maps $\hom_C(x, y) \to \hom_C(x, y)/\approx$ sending $f \mapsto [f]$. The map π sends an n-cell $a \in C_n$ to the unique $\pi(a) \in N(hC)_n$ such that $\pi(a)_{i-1,i} = \pi(a_{i-1,i})$. These functions are seen to be compatible with simplicial operators using the following exercise.

13.13. *Exercise.* Let C be a quasicategory and $a \in C_n$ an n-cell, and define $f_i := a_{i-1,i} \in C_1$ for $i = 1, \dots, n$ and $g := a_{0,n} \in C_1$. Show that $[f_n] \circ \cdots \circ [f_1] = [g]$ in the homotopy category hC.

Note that if C is a 1- category, then $f \approx g$ if and only if $f = g$. Thus, $\pi \colon C \to N(hC)$ is an isomorphism of simplicial sets if and only if the quasicategory C is isomorphic to the nerve of a 1-category, which must be its homotopy category.

The following says that the homotopy category of a quasicategory is its fundamental category, justifying the notation "hC".

13.14. **Proposition.** *Let C be a quasicategory and D a small category, and let $\phi \colon C \to N(D)$ be a map of simplicial sets. Then there exists a unique map $\psi \colon N(hC) \to N(D)$ such that $\psi\pi = \phi$.*

Proof. We first show existence, by constructing a suitable map ψ, which being a map between nerves can be described as a functor $hC \to D$. On objects, let ψ send $x \in \mathrm{ob}(hC) = C_0$ to $\phi(x) \in \mathrm{ob}(D) = (ND)_0$. On morphisms, let ψ send $[f] \in \hom_{hC}(x, y)$ to $\phi(f) \in \hom_D(\phi(x), \phi(y)) \subseteq (ND)_1$. Observe that the function on morphisms is well-defined since if $f \sim_\ell f'$, exhibited by some $a \in C_2$, then $\phi(a) \in (ND)_2$ exhibits the identity $\phi(f) = \phi(f')\phi(1_x) = \phi(f')$ in D. It is straightforward to show that ψ so defined is actually a functor, and that $\psi\pi = \phi$ as maps $C \to N(D)$.

The functor ψ defined above is the unique solution: the value of ψ on objects and morphisms is uniquely determined, and $\pi\colon C_k \to (hC)_k$ is bijective for $k = 0$ and surjective for $k = 1$. $\square$

In particular, the homotopy category construction provides a left adjoint to the nerve functor, so we have a pair of adjoint functors

$$h\colon \mathrm{qCat} \rightleftarrows \mathrm{Cat} : N.$$

13.15. *Exercise.* Understand the homotopy categories of the various examples of quasicategories described in (12). In particuar, in each case describe the sets of morphisms in the homotopy category between any two objects.

13.16. *Exercise* (Easy but important: homotopy category of a product). Show that formation of the homotopy category commutes with arbitrary products: for any collection $\{C_i\}$ of quasicategories, there is an evident isomorphism

$$h \prod_i C_i \approx \prod_i hC_i.$$

In particular, we have $h(C \times D) \approx hC \times hD$ for any quasicategories C and D.

13.17. **A criterion for composition.** We have observed that for morphisms $f\colon x \to y$ and $g\colon y \to z$ in a quasicategory that we can define a composite "$g \circ f$" using extension along $\Lambda^2_1 \subset \Delta^2$, and that though such compositions are not unique, they are unique up to homotopy, so we get a well-defined homotopy class $[g] \circ [f]$. The following proposition says that every element in this homotopy class is obtained from this construction.

13.18. **Proposition.** *If $f\colon x \to y$, $g\colon y \to z$, and $h\colon x \to z$ are morphisms in a quasicategory C, then $[h] = [g] \circ [f]$ if and only if there exists $u\colon \Delta^2 \to C$ such that*

$$u|\Delta^{\{0,1\}} = f, \qquad u|\Delta^{\{1,2\}} = g, \qquad u|\Delta^{\{0,2\}} = h.$$

Thus, every *morphism in the homotopy class of h can be interpreted as a composite of g with f.*

Proof. Clearly if u exists then $[h] = [g] \circ [f]$. Conversely, suppose given f, g, h with $h \in [g] \circ [f]$, and choose some $a\colon \Delta^2 \to C$ with $a_{01} = f$ and $a_{12} = g$, whence $[g] \circ [f] = [h']$ for $h' = a_{02}$. Since $h \in [h']$ there is a $b \in C_2$ witnessing the relation $h' \sim_r h$, and using this we can construct a map $\Lambda^3_2 \to C$ according to the diagram

$$\begin{tikzcd}
0 \arrow[dr, "h'"] \arrow[drr, bend left, "h"] \arrow[dd, bend right, "f"'] & & \\
\arrow[r, phantom, "a"] & 2 \arrow[r, "1_z"] \arrow[u, phantom, "b"] \arrow[d, phantom, "g_{011}"] & 3 \\
1 \arrow[ur, "g"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

Extend to a map $v\colon \Delta^3 \to C$; then $u = v|\Delta^{\{0,1,3\}}$ exhibits h as a composite of (g, f) as desired. $\square$

13.19. *Exercise* (Subcategories of a quasicategory vs of its homotopy category). Let $C' \subseteq C$ be a subcategory (10.1) of a quasicategory C. Show that if $f, g\colon x \to y$ are morphisms of C which are homotopic in C, then $f \in C_1'$ if and only if $g \in C_1'$. Use this to show that there is a bijective correspondence

$$(\text{subcategories of } C) \leftrightarrow (\text{subcategories of } hC),$$

and also a bijective correspondence

$$(\text{full subcategories of } C) \leftrightarrow (\text{full subcategories of } hC).$$

14. Isomorphisms in a quasicategory

Let C be a quasicategory. We say that an edge $f \in C_1$ is an **isomorphism** if its image in the homotopy category hC is an isomorphism in the usual sense of category theory. Another term you will see for such edges is **equivalence**, but I prefer to use the term isomorphism here, since this is the direct generalization of the notion of isomorphisms in a 1-category.

isomorphism
equivalence

Explicitly, a morphism $f\colon x \to y$ in a quasicategory is an isomorphism if and only if there exists an edge $g\colon y \to x$ such that $[g] \circ [f] = [1_x]$ and $[f] \circ [g] = [1_y]$, where equality is in the homotopy category hC.

14.1. *Example.* Consider $f \in C_1$. If we can produce $g \in C_1$ and $a, b \in C_2$ such that

$$a_{01} = f = b_{12}, \qquad a_{12} = g = b_{01}, \qquad a_{02} = x_{00}, \qquad b_{02} = y_{00}:$$

$$\begin{tikzcd}
& x \arrow[rr, "1_x"] \arrow[dr, "f"] \arrow[d, phantom, "b"] & \arrow[d, phantom, "a"] & x \\
y \arrow[ur, "g"] \arrow[rr, "1_y"'] & & y \arrow[ur, "g"'] &
\end{tikzcd}$$

then $[g] \circ [f] = [1_x]$ and $[f] \circ [g] = [1_y]$, so f isomorphism. The converse also holds: if f is an isomorphism, then there exist $g \in C_1$ and $a, b \in C_2$ as above, which can be proved using (13.18).

14.2. *Example* (Identity maps are isomorphims). For every $x \in C_0$ the identity map $1_x\colon x \to x$ is an isomorphism: for instance, use $a = b = x_{000}$ in the above diagram.

14.3. *Exercise.* Show that any functor $f\colon C \to D$ between quasicategories sends isomorphisms to isomorphisms.

14.4. **Preinverses and postinverses.** Let C be a quasicategory. Given $f\colon x \to y \in C_1$, a **postinverse**[7] of f is a $g\colon y \to x \in C_1$ such that $[g] \circ [f] = [1_x]$, and a **preinverse**[8] of f is an $e\colon y \to x \in C_1$ such that $[f] \circ [e] = [1_y]$. An **inverse** is an $f' \in C_1$ which is both a postinverse and a preinverse. The following is trivial, but very handy.

postinverse
preinverse
inverse

14.5. **Proposition.** *In a quasicategory C consider $f \in C_1$. The following are equivalent.*

- *f is an isomorphism.*
- *f admits an inverse f'.*
- *f admits a postinverse g and a preinverse e.*
- *f admits a postinverse g and g admits a postinverse h.*
- *f admits a preinverse e and e admits a preinverse d.*

If these equivalent conditions apply, then $f \approx d \approx h$ and $f' \approx e \approx g$, and all of them are isomorphisms.

Proof. All of these are equivalent to the corresponding statements about morphisms in the homotopy category hC, where they are seen to be equivalent to each other by elementary arguments. □

Note that inverses to a morphism in a quasicategory are generally *not unique*, though necessarily they are *unique up to homotopy.*

[7] or *left inverse*, or *retraction*,
[8] or *right inverse*, or *section*,

14.6. *Exercise.* Consider the quasicategory of categories Cat_1 as in (12.1). Show that a functor $F\colon C \to C'$ corresponding to a morphism of Cat_1 is an isomorphism of Cat_1 (in the above sense) if and only if F is an equivalence of categories.

14.7. **Quasigroupoids.** A **quasigroupoid**[9] is a quasicategory C such that hC is a groupoid, i.e., a quasicategory in which every morphism is an isomorphism. **quasigroupoid**

14.8. *Exercise.* If every morphism in a quasicategory admits a preinverse, then it is a quasigroupoid. Likewise if every morphism admits a postinverse.

14.9. **The core of a quasicategory.** For an ordinary category A, the **core** (or **interior**, or **maximal subgroupoid**) of A is the subcategory $A^{\mathrm{core}} \subseteq A$ consisting of all the objects, and all the *isomorphisms* between the objects. **core** **interior** **maximal subgroupoid**

For a quasicategory C, we define the core[10] $C^{\mathrm{core}} \subseteq C$ to be the subcomplex consisting of cells all of whose edges are all isomorphisms. That is, C^{core} is defined so that the diagram

$$\begin{tikzcd} C^{\mathrm{core}} \arrow[r, tail] \arrow[d] & C \arrow[d, "\pi"] \\ (hC)^{\mathrm{core}} \arrow[r, tail] & hC \end{tikzcd}$$

is a pullback of simplicial sets. Observe that $N(A^{\mathrm{core}}) = (NA)^{\mathrm{core}}$ for a category A.

14.10. **Proposition.** *Given a quasicategory C, its core C^{core} is a subcategory and a quasigroupoid, and every subcomplex of C which is a quasigroupoid is contained in C^{core}.*

Proof. First we show that C^{core} is a subcategory (10.1). Suppose $f\colon \Delta^n \to C$ such that $f(\Lambda^n_k) \subseteq C^{\mathrm{core}}$ for $n \geq 2$ and $0 < k < n$. When $n \geq 3$ then $(\Lambda^n_k)_1 = (\Delta^n)_1$, so clearly $f(\Delta^n) \subseteq C^{\mathrm{core}}$. When $n = 2$ we have that $f(\Delta^2) \subseteq C^{\mathrm{core}}$ because the composite of two isomorphisms is an isomorphism.

It follows that C^{core} is a quasicategory. Since the inverse of an isomorphism is an isomorphism, it is straightforward to see that C^{core} is a quasigroupoid. The final statement is clear: if $G \subseteq C$ is a subcomplex which is a quasigroupoid, then every edge in G has in inverse in G, and hence an inverse in C. □

14.11. *Exercise.* Show that if C is a quasicategory, there is an isomorphism $(hC)^{\mathrm{core}} \approx h(C^{\mathrm{core}})$.

14.12. **Kan complexes.** Recall that a Kan complex (12.12) is a simplicial set which has the extension property with respect to all horns, not just inner horns. That is, K is a Kan complex iff

$$\mathrm{Hom}(\Delta^n, K) \to \mathrm{Hom}(\Lambda^n_j, K)$$

is surjective for all $0 \leq j \leq n$, $n \geq 1$.

14.13. *Exercise.* Show that every simplicial set X has extensions for 1-dimensional horns, so that for any simplicial set X, every $\Lambda^1_j \to X$ extends over $\Lambda^1_j \subset \Delta^1$, where $j \in \{0,1\}$. Thus, X is a Kan complex if and only if it has extensions just for the horns inside simplices of dimension ≥ 2.

14.14. **Proposition.** *Every Kan complex is a quasigroupoid.*

Proof. It is immediate that a Kan complex K is a quasicategory. To show K is a quasigroupoid, note that the extension condition for $\Lambda^2_0 \subset \Delta^2$ implies that every morphism in hK admits a postinverse. Explicitly, if $f\colon x \to y$ is an edge in K, let $u\colon \Lambda^2_0 \to K$ with $u_{01} = f$ and $u_{02} = f_{00} = 1_x$, so there is an extension $v\colon \Delta^2 \to K$ and $g := v_{12}$ satisfies $gf \approx 1_x$. Use (14.8). □

[9] This is the same as what Lurie [Lur09] and many others call an ∞-*groupoid*.

[10] Lurie (along with many others) uses the notation $C^{\simeq}$ for what we are calling C^{core}.

This proposition has a converse: *quasigroupoids are precisely the Kan complexes.* This is a very important technical result, and it is not trivial; it is the main result of [Joy02]. We will prove this later as (35.2).

Recall (12.9) that we observed that the singular complex $\operatorname{Sing} T$ of a topological space is a Kan complex, and therefore a quasigroupoid. It is reasonable to think of $\operatorname{Sing} T$ as the **fundamental quasigroupoid** of the space T. **fundamental quasigroupoid**

14.15. *Exercise* (for topologists). Show that if T is a topological space, then $h \operatorname{Sing} T$, the homotopy category of the singular complex of T, is precisely the usual fundamental groupoid of T.

14.16. **Quasigroupoids, components, and isomorphism classes.** We say that two objects in a quasicategory are **isomorphic** if there exists an isomorphism between them. This is an equivalence relation on C_0, and thus we speak of **isomorphism classes** of objects. **isomorphic** **isomorphism classes**

Recall (9.10) that the set of connected components of a simplicial set is given by

$$\pi_0 X \approx \Big((\coprod_{n\geq 0} X_n)/\sim \Big) \approx (X_0/\sim_1),$$

the equivalence classes of cells of X under the equivalence relation generated by "related by a simplicial operator", or equivalently the equivalence classes of vertices of X under the equivalence relation generated by "connected by an edge" . Note that if T is a topological space, then elements of $\pi_0 \operatorname{Sing} T$ correspond exactly to path components of T.

For *quasigroupoids*, π_0 recovers the set of isomorphism classes of objects.

14.17. **Proposition.** *If C is a quasicategory, then*

$$\pi_0\big(C^{\text{core}}\big) \approx \{\textit{isomorphism classes of objects of } C\}.$$

Proof. Straightforward: edges in C^{core} are precisely the isomorphisms in C. □

14.18. *Exercise.* Show that for a quasicategory C, $\pi_0(C^{\text{core}}) \approx \pi_0(h(C^{\text{core}})) \approx \pi_0((hC)^{\text{core}})$.

15. Function complexes and the functor quasicategory

Given ordinary categories C and D, the *functor category* $\operatorname{Fun}(C, D)$ has **W 2 Feb**

- as objects, the functors $C \to D$, and
- as morphisms $f \to f'$, natural transformations of functors.

Furthermore, for any category A there is a bijective correspondence between sets of functors

$$\big\{A \times C \to D\big\} \iff \big\{A \to \operatorname{Fun}(C, D)\big\}.$$

Explicitly, a functor $\phi\colon A \to \operatorname{Fun}(C, D)$ corresponds to $\widetilde{\phi}\colon A \times C \to D$, given on objects by $\widetilde{\phi}(a, c) = \phi(a)(c)$ for $a \in \operatorname{ob} A$ and $c \in \operatorname{ob} C$, and on morphisms by $\widetilde{\phi}(\alpha, \gamma) = \phi(a')(\gamma) \circ \phi(\alpha)(c) = \phi(\alpha)(c') \circ \phi(a)(\gamma)\colon \phi(a)(c) \to \phi(a')(c')$ for $\alpha\colon a \to a' \in \operatorname{mor} A$ and $\gamma\colon c \to c' \in \operatorname{mor} C$.

The generalization of the functor category to quasicategories admits a similar adjunction, and in fact can be defined for arbitrary simplicial sets.

15.1. **Function complexes.** Given simplicial sets X and Y, we may form the **function complex** $\operatorname{Fun}(X, Y)$. This is a simplicial set with **function complex**

$$\operatorname{Fun}(X, Y)_n = \operatorname{Hom}(\Delta^n \times X, Y),$$

so that the action of a simplicial operator $\delta\colon [m] \to [n]$ on $\operatorname{Fun}(X, Y)$ is induced by

$$\operatorname{Hom}(\delta \times \operatorname{id}_X, Y)\colon \operatorname{Hom}(\Delta^n \times X, Y) \to \operatorname{Hom}(\Delta^m \times X, Y).$$

In particular, the set $\operatorname{Fun}(X, Y)_0$ of vertices of the function complex is precisely the set of maps $X \to Y$ of simplicial sets.

15.2. *Remark.* There are many alternate notations for the function complex. A common one is $\mathrm{Map}(X,Y)$, because it is the "object of maps" from X to Y, and is sometimes called the *mapping space.*

15.3. **Proposition.** *The function complex construction defines a functor*

$$\mathrm{Fun}\colon s\mathrm{Set}^{\mathrm{op}} \times s\mathrm{Set} \to s\mathrm{Set}.$$

Proof. Left as an exercise. □

By construction, for each n, there is a bijective correspondence

$$\{\Delta^n \times X \to Y\} \longleftrightarrow \{\Delta^n \to \mathrm{Fun}(X,Y)\}.$$

In fact, we can replace Δ^n with an arbitrary simplicial set.

15.4. **Proposition.** *For simplicial sets X, Y, Z, there is a bijection*

$$\mathrm{Hom}(X \times Y, Z) \xrightarrow{\sim} \mathrm{Hom}(X, \mathrm{Fun}(Y,Z))$$

natural in all three variables.

Proof. The bijection sends $f\colon X \times Y \to Z$ to $\widetilde{f}\colon X \to \mathrm{Fun}(Y,Z)$ defined so that for $x \in X_n$, the element $\widetilde{f}(x) \in \mathrm{Fun}(Y,Z)_n$ is represented by the composite

$$\Delta^n \times Y \xrightarrow{x\times \mathrm{id}} X \times Y \xrightarrow{f} Z.$$

The inverse of this bijection sends $g\colon X \to \mathrm{Fun}(Y,Z)$ to $\widetilde{g}\colon X \times Y \to Z$, defined so that for $(x,y) \in X_n \times Y_n$, the element $\widetilde{g}(x,y) \in Z_n$ is represented by

$$\Delta^n \xrightarrow{(\mathrm{id},y)} \Delta^n \times Y \xrightarrow{g(x)} Z.$$

The proof amounts to showing that both $\widetilde{f}$ and $\widetilde{g}$ are in fact maps of simplicial sets, and that the above constructions are in fact inverse to each other. This is left as an exercise, as is the proof of naturality. □

15.5. *Exercise* (Important). Show, using the previous proposition, that there are natural isomorphisms

$$\mathrm{Fun}(X \times Y, Z) \approx \mathrm{Fun}(X, \mathrm{Fun}(Y,Z)).$$

of simplicial sets. (Hint: show that both objects represent *isomorphic* functors $s\mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$, and apply the Yoneda lemma.)

15.6. *Remark.* The construction of the function complex is not special to simplicial sets. The construction of $\mathrm{Fun}(X,Y)$ (and its properties as described above) works the same way in any category of functors $C^{\mathrm{op}} \to \mathrm{Set}$, where C is a small category (e.g., $C = \Delta$). In this general setting, the role of the standard n-simplices is played by the representable functors $\mathrm{Hom}_C(-,c)\colon C^{\mathrm{op}} \to \mathrm{Set}$.

15.7. **Functor quasicategories.** When C and D are quasicategories, then the vertices of the function complex $\mathrm{Fun}(C,D)$ are precisely the functors $C \to D$, and the edges of $\mathrm{Fun}(C,D)$ are precisely the natural transformations. It is thus reasonable to hope that the generalization of functor category of functor category to quasicategories is precisely the function complex. In fact, this is the case for the functor category between ordinary categories.

15.8. *Exercise.* Show that for ordinary categories C and D that $N\,\mathrm{Fun}(C,D) \approx \mathrm{Fun}(NC,ND)$. (Hint: use that $N([n]) = \Delta^n$, and the fact that the nerve preserves finite products (9.5).)

15.9. *Exercise.* Give an example of quasicategories C and D such that categories $h\,\mathrm{Fun}(C,D)$ and $\mathrm{Fun}(hC,hD)$ are not isomorphic (i.e., the homotopy category of the functor quasicategory need not be the same as the functor category of the homotopy categories). (Hint: you can choose C to be an ordinary category, but not D.)

In remains to show that *a function complex between quasicategories is again a quasicategory.* To prove this, we will need a to take a detour to develop some technology about "weakly saturated" classes of maps and "lifting properties". After this, we will complete the proof as (22.4).

Part 3. Lifting properties

16. Weakly saturated classes and inner-anodyne maps

Quasicategories are defined by an "extension property": they are the simplicial sets C such that any map $K \to C$ extends over L, whenever $K \subset L$ is an inner horn inclusion $\Lambda^n_j \subset \Delta^n$. The set of inner horns "generates" a larger class of maps (which will be called the class of *inner anodyne* maps), which "automatically" shares the extension property of the inner horns. This class of inner anodyne maps is called the *weak saturation* of the set of inner horns.

For instance, we will observe (16.14) that the spine inclusions $I^n \subset \Delta^n$ are inner anodyne, so that quasicategories admit "spine extensions", i.e., any $I^n \to C$ extends over $I^n \subset \Delta^n$ to a map $\Delta^n \to C$.

These kinds of extension properties are going to play a major role in what follows, and now is a good time to develop some theory to handle them.

16.1. **Weakly saturated classes.** Consider a category (such as sSet) which has all small colimits. A **weakly saturated class** is a class $\mathcal{A}$ of morphisms in the category, which **weakly saturated class**

(1) contains all isomorphisms,
(2) is closed under cobase change,
(3) is closed under composition,
(4) is closed under transfinite composition,
(5) is closed under coproducts, and
(6) is closed under retracts.

Given a class of maps S, its **weak saturation** $\overline{S}$ is the smallest weakly saturated class containing S. **weak saturation**

I need to explain some of the elements of this definition.

- **Closed under cobase change** is also called **closed under pushout**: it means that if f' is the pushout of $f\colon X \to Y$ along some map $g\colon X \to Z$, then $f \in \mathcal{A}$ implies $f' \in \mathcal{A}$: **Closed under cobase change** **closed under pushout**

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

- **Closed under composition** means that if $g, f \in \mathcal{A}$ and gf is defined, then $gf \in \mathcal{A}$. **Closed under composition**
- We say that $\mathcal{A}$ is **closed under countable composition** if given a countable sequence of composable morphisms, i.e., maps **closed under countable composition**

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots$$

such that each $f_k \in \mathcal{A}$ for all $k \in \mathbb{Z}_{>0}$, the induced map $X_0 \to \operatorname{colim}_k X_k$ to the colimit is in $\mathcal{A}$.

The notion **closed under transfinite composition** is a generalization of this, in which $\mathbb{N}$ is replaced by an arbitrary ordinal λ (i.e., a well-ordered set). This means that for any ordinal λ and any functor $X\colon \lambda \to s\mathrm{Set}$, if for every $i \in \lambda$ with $i \neq 0$ the evident map **closed under transfinite composition**

$$(\operatorname{colim}_{j<i} X(j)) \to X(i)$$

is in $\mathcal{A}$, then the induced map $X(0) \to \operatorname{colim}_{j\in\lambda} X(j)$ is in $\mathcal{A}$.

- **Closed under coproducts** means that if $\{f_i\colon X_i \to Y_i\}$ is a set of maps in $\mathcal{A}$, then $\coprod_i f_i\colon \coprod_i X_i \to \coprod_i Y_i$ is in $\mathcal{A}$.

Closed under coproducts

- We say that f is a **retract** of g if there exists a commutative diagram in C of the form

retract

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}"] & X' \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"'] \\ Y \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & Y' \arrow[r] & Y \end{tikzcd}$$

This is really a special case of the notion of a retract of an object in the functor category $\mathrm{Fun}([1], s\mathrm{Set})$. We say that $\mathcal{A}$ is **closed under retracts** if for every diagram as above, $g \in \mathcal{A}$ implies $f \in \mathcal{A}$.

closed under retracts

16.2. *Remark.* This list of properties is not minimal: (3) is the special case of (4) when $\lambda = [2]$, and (5) can be deduced from (2) and (4). *Exercise:* Show this.

16.3. *Example.* Consider the category of sets. The class of all surjective maps is weakly saturated, and in fact is the weak saturation of $\{\, \{0,1\} \to \{1\} \,\}$. Likewise, the class of injective maps is weakly saturated, and in fact is the weak saturation of $\{\, \varnothing \to \{1\} \,\}$.

16.4. *Example.* The classes of injections and surjections of simplicial sets are weakly saturated classes. Later we will identify the class of monorphisms of simplicial sets as the weak saturation of the set of "cell inclusions" (20.5).

16.5. **Proposition.** *Fix a collection $\mathcal{C}$ a simplicial sets (e.g., the class of quasicategories). Let $\mathcal{A}$ be the class of maps of simplicial sets $i\colon A \to B$ such that every map $f\colon A \to C$ to an element $C \in \mathcal{C}$ admits an extension to $g\colon B \to C$ such that $gi = f$. Then $\mathcal{A}$ is a weakly saturated class.*

16.6. *Exercise.* Give a proof of (16.5). It is highly recommended that you work through the argument if you haven't seen it before.

16.7. *Remark.* There is a dual notion of a **weakly cosaturated class**: a weakly cosaturated class is the same thing as a weakly saturated class in the opposite category, and is characterized by being closed under properties formally dual to (1)–(6).

weakly cosaturated class

16.8. **Classes of "anodyne" morphisms.** We use the following notation for sets of types of horns:

$$\begin{aligned} \mathrm{InnHorn} &:= \{\, \Lambda^n_k \subset \Delta^n \mid 0 < k < n,\ n \geq 2 \,\}, && \text{(inner horns)}, \\ \mathrm{LHorn} &:= \{\, \Lambda^n_k \subset \Delta^n \mid 0 \leq k < n,\ n \geq 1 \,\}, && \text{(left horns)}, \\ \mathrm{RHorn} &:= \{\, \Lambda^n_k \subset \Delta^n \mid 0 < k \leq n,\ n \geq 1 \,\}, && \text{(right horns)}, \\ \mathrm{Horn} &:= \{\, \Lambda^n_k \subset \Delta^n \mid 0 \leq k \leq n,\ n \geq 1 \,\}, && \text{(horns)}. \end{aligned}$$

The weak saturation of each of these sets will play an important role in what follows. Right now, we focus on the smallest of these classes, namely the weak saturation $\overline{\mathrm{InnHorn}}$ of the set of inner horns, which is called the class of **inner anodyne**[11] morphisms. The weak saturations of the other sets are the classes of "left anodyne" and "right anodyne" maps (30.1), and plain old "anodyne" morphisms (**??**), about which we have more to say later. Note that anodyne morphisms are always monomorphisms, since monomorphisms of simplicial sets themselves form a weakly saturated class.

inner anodyne

[11]The "anodyne" terminology for the weak saturation of a set of horns was introduced by Gabriel and Zisman [GZ67]. "Anodyne" derives from ancient Greek, meaning "without pain". We leave it to the reader to decide whether this choice of terminology is appropiate.

16.9. **Proposition.** *If C is a quasicategory and $A \subseteq B$ is an inner anodyne inclusion, then any $f\colon A \to C$ admits an extension to $g\colon B \to C$ so that $g|A = f$.*

Proof. We know that the class $\mathcal{A}$ of maps $i\colon A \to B$ such that every map from A to a quasicategory extends along i is weakly saturated (16.5). The claim follows immediately since InnHorn $\subseteq \mathcal{A}$ by definition. □

16.10. *Exercise* (Easy but important). Show that every inner anodyne map induces a bijection on vertices. (Hint: show that the class of maps of simplicial sets which are a bijection on vertices is weakly saturated.)

16.11. **Examples of inner anodyne morphisms.** It is crucial to be able to prove that certain explicit maps are inner anodyne.

Let $S \subseteq [n]$. The associated **generalized horn** is the subcomplex $\Lambda^n_S \subset \Delta^n$ defined by **generalized horn**

$$\Lambda^n_S := \bigcup_{i \in S} \Delta^{[n] \smallsetminus i},$$

i.e., the union of codimension one faces of the n-simplex indexed by elements of S. In particular, $\Lambda^n_{[n]\smallsetminus\{j\}}$ is the usual horn Λ^n_j. I'll generalize this notation to arbitary totally ordered sets, so $\Lambda^T_S = \bigcup_{i \in S} \Delta^{T \smallsetminus i}$ when $S \subseteq T$.

We call $\Lambda^n_S \subset \Delta^n$ a **generalized inner horn** if S is not an "interval" in $[n]$, i.e., if there exist $s < t < s'$ with $s, s' \in S$ and $t \notin S$. **generalized inner horn**

16.12. **Lemma.** *All generalized inner horn inclusions $\Lambda^n_S \subset \Delta^n$ are inner anodyne.*

There is a slick proof of this given by Joyal [Joy08a, Prop. 2.12], which we present in the appendix (77.1).

16.13. *Example.* Consider $\Lambda^3_{\{0,3\}}$, which can be pictured as the solid diagram in

$$\begin{tikzcd}
0 \arrow[dr] \arrow[dd, bend right] \arrow[drr, dotted, bend left] & & \\
& 2 \arrow[r] & 3 \\
1 \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

We can get from this to Δ^3 in two steps:

$$\begin{tikzcd}
\Lambda^{\{0,2,3\}}_{\{0,3\}} \arrow[r, tail] \arrow[d, tail] & \Delta^{\{0,2,3\}} \arrow[d, tail] & \\
\Lambda^3_{\{0,3\}} \arrow[r, tail] & \Lambda^3_2 \arrow[r, tail] & \Delta^3
\end{tikzcd}$$

The square is a pushout of subcomplexes since $\Lambda^3_{\{0,3\}} \cap \Delta^{\{0,2,3\}} = \Lambda^{\{0,2,3\}}_{\{0,3\}}$, and the map along the top is isomorphic to $\Lambda^2_1 \subset \Delta^2$, an inner horn inclusion. This proves that $\Lambda^3_{\{0,3\}} \subset \Delta^3$ is inner anodyne.

Recall that every standard n-simplex contains a spine $I^n \subseteq \Delta^n$.

16.14. **Lemma.** *The spine inclusions $I^n \subset \Delta^n$ are inner anodyne for all n. Thus, for a quasicategory C, any $I^n \to C$ extends to $\Delta^n \to C$.*

Proof. This is proved in [Joy08a, Prop. 2.13]. We give the proof in the appendix (77.2). □

16.15. *Example.* To show that $I^3 \subset \Delta^3$ is inner anodyne, observe that we can get from I^3 to a generalized inner horn in two steps by attaching 2-cells along inner horn inclusions:

$$\begin{tikzcd}
\Lambda^{\{0,1,2\}}_{\{0,2\}} \arrow[r, tail] \arrow[d, tail] & \Delta^{\{0,1,2\}} \arrow[d, tail] & \Lambda^{\{1,2,3\}}_{\{1,3\}} \arrow[r, tail] \arrow[d, tail] & \Delta^{\{1,2,3\}} \arrow[d, tail] \\
I^3 \arrow[r, tail] & I^3 \cup \Delta^{\{0,1,2\}} \arrow[r, equal] & I^3 \cup \Delta^{\{0,1,2\}} \arrow[r, tail] & \Lambda^3_{\{0,3\}}
\end{tikzcd}$$

since $I^3 \cap \Delta^{\{0,1,2\}} = \Lambda^{\{0,1,2\}}_{\{0,2\}}$ and $(I^3 \cup \Delta^{\{0,1,2\}}) \cap \Delta^{\{1,2,3\}} = \Lambda^{\{1,2,3\}}_{\{1,3\}}$.

16.16. *Exercise.* Use (16.14) to show that the tautological map $\pi\colon C \to N(hC)$ from a quasicategory to (the nerve of) its homotopy category is surjective in every dimension.

17. Lifting calculus and inner fibrations

We have defined quasicategories by an "extension property": in general, we say that X satisfies the extension property for $f\colon A \to B$ if for any diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & X \\
B \arrow[ur, dotted, "s"'] &
\end{tikzcd}$$

there exists a morphism s making the diagram commute. In this section, we discuss a "relative" version of this, called a "lifting property".

17.1. **The lifting relation.** Given morphisms $f\colon A \to B$ and $g\colon X \to Y$ in a category, a **lifting problem** for (f, g) is a pair of morphisms (u, v) such that $vf = gu$. That is, a lifting problem is any commutative square of solid arrows of the form **lifting problem**

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r, "v"'] \arrow[ur, dotted, "s"] & Y
\end{tikzcd}$$

A **lift** for the lifting problem is a morphism s such that $sf = u$ and $gs = v$, i.e., a dotted arrow making the diagram commute. **lift**

We may thus define the **lifting relation** on morphisms in our category: we write "$f \boxslash g$" if *every* lifting problem for (f, g) admits a lift[12]. Equivalently, $f \boxslash g$ exactly if **lifting relation**

$$\mathrm{Hom}(B, X) \xrightarrow{s \mapsto (sf, gs)} \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B, Y)$$

is a surjection, where the target is the set of pairs $(u\colon A \to X, v\colon B \to Y)$ such that $gu = vf$ (i.e., the target is exactly the set of lifting problems for (f, g)).

When $f \boxslash g$ holds, one sometimes says f has the **left lifting property** relative to g, or that g has the **right lifting property** relative to f. Or we just say that f **lifts against** g. **left lifting property** **right lifting property** **f lifts against g.**

We extend the notation to classes of maps, so "$\mathcal{A} \boxslash \mathcal{B}$" means: $a \boxslash b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Note: I will also sometimes speak of a **lifting problem of type** $f \boxslash g$, by which I mean a pair (u, v) which is a lifting problem for (f, g). **lifting problem of type $f \boxslash g$**

17.2. *Exercise.* Show that $f \boxslash f$ if and only if f is an isomorphism.

Given a class of morphisms $\mathcal{A}$, define the **right complement** $\mathcal{A}^{\boxslash}$ and **left complement** ${}^{\boxslash}\mathcal{A}$ by **right complement** **left complement**

$$\mathcal{A}^{\boxslash} = \{\, g \mid a \boxslash g \text{ for all } a \in \mathcal{A} \,\}, \qquad {}^{\boxslash}\mathcal{A} = \{\, f \mid f \boxslash a \text{ for all } a \in \mathcal{A} \,\}.$$

[12]Sometimes one sees the notation "$f \perp g$" or "$f \pitchfork g$" used instead. Our notation is taken from [Rie14, §11].

17.3. **Proposition.** *For any class $\mathcal{A}$, the left complement ${}^{\boxslash}\mathcal{A}$ is a weakly saturated class, and the right complement $\mathcal{A}^{\boxslash}$ is a weakly cosaturated class.*

17.4. *Exercise* (Important). Prove (17.3). (This is a "relative" version of the proof of (16.5).)

17.5. *Exercise* (Easy). Prove that if $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A}^{\boxslash} \supseteq \mathcal{B}^{\boxslash}$ and ${}^{\boxslash}\mathcal{A} \supseteq {}^{\boxslash}\mathcal{B}$. Use this to show $\mathcal{A}^{\boxslash} = ({}^{\boxslash}(\mathcal{A}^{\boxslash}))^{\boxslash}$ and ${}^{\boxslash}\mathcal{A} = {}^{\boxslash}(({}^{\boxslash}\mathcal{A})^{\boxslash})$.

17.6. *Exercise* (for those who know a little homological algebra). Fix an abelian category $\mathcal{C}$ (e.g., the category of modules over some ring R). Let $\mathcal{P}$ be the class of morphisms in $\mathcal{C}$ of the form $0 \to P$ where P is projective, and let $\mathcal{B}$ be the class of epimorphisms in $\mathcal{C}$. Show that $\mathcal{P} \boxslash \mathcal{B}$; also, show that $\mathcal{B} = \mathcal{P}^{\boxslash}$ if $\mathcal{C}$ has enough projectives.

17.7. *Exercise.* In the setting of the previous exercise, identify the class ${}^{\boxslash}\mathcal{B}$.

17.8. **Inner fibrations.** A map p of simplicial sets is an **inner fibration** if $\mathrm{InnHorn} \boxslash p$. The class of inner fibrations $\mathrm{InnFib} = \mathrm{InnHorn}^{\boxslash}$ is thus the right complement of the set of inner horns. Note that C is a quasicategory if and only if $C \to *$ is an inner fibration. **inner fibration**

Because InnFib is a right complement, it is weakly cosaturated (16.7). In particular, it is closed under composition. This implies that if $p\colon C \to D$ is an inner fibration *and* D is a quasicategory, then C is also a quasicategory. Also note that since the left complement of InnFib is weakly saturated (17.3), we have $\overline{\mathrm{InnHorn}} \boxslash \mathrm{InnFib}$.

17.9. *Exercise.* Show that if $f\colon C \to D$ is any functor from a quasicategory C to a *category* D, then f is an inner fibration. In particular, all functors between categories are automatically inner fibrations. (Hint: use the fact that all inner horns mapping to a category have *unique* extensions to simplices.)

17.10. *Exercise.* Show that any inclusion $C' \subseteq C$ of a subcomplex of a quasicategory is an inner fibration if and only if C' is a subcategory (10.1) of C.

17.11. *Exercise.* Let $p\colon C \to D$ be a functor between quasicategories, and let $p^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ be the restriction of p to cores (14.9). Show that if p is an inner fibration then p^{core} is also an inner fibration. (Hint. There are two distinct cases of lifting problems $(\Lambda^n_k \subset \Delta^n) \boxslash p^{\mathrm{core}}$, namely $n = 2$ and $n \geq 3$.)

17.12. *Exercise.* Consider a pullback square of simplicial sets

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' \arrow[r, "\pi"'] & Y \end{tikzcd}$$

such that π is a *surjective* map. Show that if p' is an inner fibration then so is p.

17.13. *Example* (Campbell's example). Here is an example of an inner fibration whose target is not a quasicategory. Let $H^\ell := \Delta^2/\Delta^{\{0,1\}}$, i.e., the pushout of the diagram $\Delta^2 \hookleftarrow \Delta^{\{0,1\}} \to \Delta^0$ of simplicial sets, and let $f\colon \Delta^1 \to H^\ell$ be the composite $\Delta^1 \xrightarrow{\langle 02\rangle} \Delta^2 \xrightarrow{\pi} H^\ell$, where π is the evident projection. Then f is an inner fibration. To see this, note that the base-change of f along the projection map π is the inclusion $\Lambda^2_0 \subseteq \Delta^2$, which is an inner fibration since both source and target are categories (17.10). Thus f is an inner fibration by (17.12).

However, H^ℓ is not a quasicategory: there is a map $\Delta^2_1 \to H^\ell$ which does not extend over $\Lambda^2_1 \subset \Delta^2$ (*Exercise:* find it).

The map f has been observed by Alexander Campbell [Cam19] to be a counterexample to a number of plausible-sounding statements, some of which we will discuss later (40.18).

17.14. *Exercise.* Show that the map $f\colon \Delta^1 \to H^\ell$ of (17.13) is not inner anodyne. (Hint: (17.2).)

17.15. **Factorizations.** It turns out that we can always factor any map of simplicial sets into an inner anodyne map followed by an inner fibration. This is a consequence of the following general observation. **F 4 Feb**

17.16. **Proposition** ("Small object argument")**.** *Let S be a* set *of morphisms in* sSet*. Then every map f between simplicial sets admits a factorization $f = pj$ with $j \in \overline{S}$ and $p \in S^{\boxslash}$.*

The proof of this proposition is by means of what is known as the "small object argument". I'll give the proof in the next section. For now we record a consequence.

17.17. **Corollary.** *For any set S of morphisms in* sSet*, we have that $\overline{S} = {}^{\boxslash}(S^{\boxslash})$.*

Proof. That $\overline{S} \subseteq {}^{\boxslash}(S^{\boxslash})$ is immediate from (17.3). Given f such that $f \boxslash S^{\boxslash}$, use the small object argument (17.16) to choose $f = pj$ with $j \in \overline{S}$ and $p \in S^{\boxslash}$. We have a commutative diagram of solid arrows

$$\begin{tikzcd} \bullet \arrow[r, "\mathrm{id}"'] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}"] & \bullet \arrow[r, "\mathrm{id}"'] \arrow[d, "j"'] & \bullet \arrow[d, "f"'] \\ \bullet \arrow[r, dotted, "s"] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r, "p"] & \bullet \end{tikzcd}$$

A map s exists making the diagram commute, because $f \boxslash p$, so there is a lift in

$$\begin{tikzcd} \bullet \arrow[r, "j"] \arrow[d, "f"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "\mathrm{id}"'] \arrow[ur, dotted, "s"] & \bullet \end{tikzcd}$$

The diagram exhibits f as a retract of j, whence $f \in \overline{S}$ since weak saturations are closed under retracts. $\square$

17.18. *Remark* (Retract trick)*.* The proof of the corollary is called the "retract trick": given $f = pj$, $f \boxslash p$ implies that f is a retract of j, while $j \boxslash f$ implies that f is a retract of p.

In the case we are currently interested in, we have that $\overline{\mathrm{InnHorn}} = {}^{\boxslash}\mathrm{InnFib}$ and $\overline{\mathrm{InnHorn}}^{\boxslash} = \mathrm{InnFib}$, so we get the following.

17.19. **Proposition.** *An map f of simplicial sets can be factored $f = pj$, where j is inner anodyne and p is an inner fibration.*

17.20. **Weak factorization systems.** A **weak factorization system** in a category is a pair $(\mathcal{L}, \mathcal{R})$ of classes of maps such that **weak factorization system**

- every map f admits a factorization $f = r\ell$ with $r \in \mathcal{R}$ and $\ell \in \mathcal{L}$, and
- $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

Thus, in any weak factorization the "left" class $\mathcal{L}$ is weakly saturated and the "right" class $\mathcal{R}$ is weakly cosaturated. The small object argument implies that $(\overline{S}, S^{\boxslash})$ is a weak factorization in sSet for every set of maps S. In particular, $(\overline{\mathrm{InnHorn}}, \mathrm{InnFib})$ is a weak factorization system.

17.21. *Exercise* (for those who know some homological algebra)*.* In an abelian category, let $\mathcal{A}$ be the class of monomorphisms with projective cokernel, and let $\mathcal{B}$ be the class of epimorphisms. Show that the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorization system if and only if the category has enough projectives. (This exercise is related to (17.6).)

17.22. *Exercise* (due to Goodwillie)*.* Classify all weak factorization systems in the category of sets. (There are exactly six.)

17.23. **Uniqueness of liftings.** The relation $f \boxslash g$ says that lifting problems admit solutions, but not that the solutions are unique. However, we can incorporate uniqueness into the lifting calculus if our category has pushouts.

Given a map $f\colon A \to B$, let $f^\vee := (\mathrm{id}_B, \mathrm{id}_B)\colon B \amalg_A B \to B$ be the "fold" map, i.e., the unique map such that the composition with either of the canonical maps $B \to B \amalg_A B$ is f. It is straightforward to show that for a map $g\colon X \to Y$ we have that $\{f, f^\vee\} \boxslash g$ if and only if in every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dotted, "s"] & Y \end{tikzcd}$$

there exists a *unique* lift s.

17.24. *Example.* Consider the category of topological spaces. Let $\mathcal{A}$ be the class of morphisms of the form $A \times \{0\} \to A \times [0,1]$, where A is an arbitrary space. Then $(\mathcal{A} \cup \mathcal{A}^\vee)^\boxslash$ contains all covering maps (by the "Covering Homotopy Theorem").

A weak factorization system $(\mathcal{L}, \mathcal{R})$ in which liftings are always unique is called an **orthogonal factorization system**. **orthogonal factorization system**

17.25. *Exercise.* Show that in an orthogonal factorization system, the factorizations $f = r\ell$ with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$ are unique up to *unique isomorphism*.

17.26. *Exercise.* Show that $(\{\text{surjections}\}, \{\text{injections}\})$ is an orthogonal factorization system for Set.

17.27. *Exercise.* Let S be a set of maps of simplicial sets, and consider the weak factorization system $(\overline{S \cup S^\vee}, (S \cup S^\vee)^\boxslash)$. Show that this is in fact an orthogonal factorization system. (Hint: show that that for any map g, the class of maps f such that any the lifting problem of type $f \boxslash g$ has a *unique* solution is a weakly saturated class.)

17.28. *Example* (The fundamental category via an orthogonal factorization system). In simplicial sets, the projection map $C \to *$ is in the right complement to $S := \mathrm{InnHorn} \cup \mathrm{InnHorn}^\vee$ if and only if C is *isomorphic* to a nerve of a category (8.7). The small object argument using S, applied to a projection $X \to *$, thus produces a morphism $\pi\colon X \to Y$ in $\overline{S}$ with Y the nerve of a category.

Uniqueness of liftings in this case implies that $\pi\colon X \to Y$ has precisely the universal property of the fundamental category of X defined in (13.1): given $f\colon X \to C$ with C a category, a unique extension of f over $X \to Y$ exists. Thus, the small object argument applied to S gives another construction of the fundamental category (13.1) of an arbitrary simplicial set S.

17.29. *Exercise.* Prove that if $f\colon X \to Y$ is any inner anodyne map, then the induced functor $h(f)\colon hX \to hY$ between fundamental categories is an isomorphism. (Hint: use the universal property of fundamental categories to construct an inverse to $h(f)$.)

18. The small object argument

In this section we give the proof of (17.16), i.e., that given a fixed set $S = \{s_i\colon A_i \to B_i\}$ of maps of simplical sets, we can factor any map $f\colon X \to Y$ as $f = pj$ with $j \in \overline{S}$ and $p \in S^\boxslash$. For the reader: it may be helpful to first work through the special case where $Y = \Delta^0$ (the terminal object in simplicial sets). Also, it is worth noting that we will only rarely need to know any details of this proof in the subsequent text.

18.1. **A factorization construction.** Given any map $f\colon X \to Y$, we first produce a factorization

$$X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y, \qquad (Rf)(Lf) = f$$

as follows. Consider the set

$$[S,f] := \{\, (s_i,u,v) \;\mid\; s_i \in S,\ fu = vs_i \,\} = \left\{ \begin{tikzcd} A_i \arrow[r, "u"] \arrow[d, "s_i"'] & X \arrow[d, "f"] \\ B_i \arrow[r, "v"'] & Y \end{tikzcd} \right\}$$

of all commutative squares which have an arrow from S on the left-hand side, and f on the right-hand side, i.e., the set of all lifting problems of type $s_i \boxslash f$ for some $s_i \in S$. We define Ef, Lf, and Rf using the diagram

$$\begin{tikzcd} \coprod\limits_{(s_i,u,v)} A_i \arrow[rr, "(u)"] \arrow[d, "\coprod s_i"'] & & X \arrow[d, "Lf"'] \arrow[drr, "f"] & & \\ \coprod\limits_{(s_i,u,v)} B_i \arrow[rr] \arrow[rrrr, bend right, "(v)"'] & & Ef \arrow[rr, "Rf"] & & Y \end{tikzcd}$$

where the the coproducts are indexed by the set $[S,f]$, and the square is a pushout. Note that $Lf \in \overline{S}$ by construction. However, we do not expect that Rf in $S^{\boxslash}$.

We can iterate the construction:

$$\begin{tikzcd} X \arrow[d, equal] \arrow[dr, "Lf"] \arrow[drr, "L^2 f"] \arrow[drrr, "L^3 f"] \arrow[drrrrr, bend left, "L^\omega f"] & & & & & \\ X \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[r, "LRf"] \arrow[dl, "Rf"] & E^2 f \arrow[r, "LR^2 f"] \arrow[dll, "R^2 f"] & E^3 f \arrow[r] \arrow[dlll, "R^3 f"] & \cdots \arrow[r] & E^\omega f \arrow[dlllll, bend left, "R^\omega f"] \\ Y & & & & & \end{tikzcd}$$

Here each triple $(E^\alpha f, L^\alpha f, R^\alpha f)$ is obtained by factoring the "R" map of the previous one, so that

$$E^{\alpha+1} f := E(R^\alpha f), \quad L^{\alpha+1} f := L(R^\alpha f) \circ (L^\alpha f), \quad R^{\alpha+1} f := R(R^\alpha f). \tag{18.2}$$

Taking direct limits gives a factorization $X \xrightarrow{L^\omega f} E^\omega f \xrightarrow{R^\omega f} Y$ of f, with $E^\omega f = \operatorname{colim}_{n\to\infty} E^n f$.

We can go even further, using the magic of transfinite induction, and define compatible factorizations $(E^\lambda f, L^\lambda f, R^\lambda f)$ for each ordinal[13] λ. For successor ordinals $\alpha+1$ use the prescription of (18.2), while for limit ordinals β take a direct limit $E^\beta f := \operatorname{colim}_{\alpha<\beta} E^\alpha f$ as in the construction of $E^\omega f$ above.

It is immediate that every $L^\alpha f \in \overline{S}$, because weak saturations are closed under transfinite composition. The maps $R^\alpha f$ are not generally contained in $S^{\boxslash}$, though they do satisfy a "partial lifting property": whenever $\alpha < \beta$ there exists *by construction* a dotted arrow making

$$\begin{tikzcd} A_i \arrow[r, "u'"] \arrow[d, "s_i"'] & E^\alpha f \arrow[r] & E^{\alpha+1} f \arrow[r] & E^\beta f \arrow[d, "R^\beta f"] \\ B_i \arrow[rrr, "v"'] \arrow[urr, dotted] & & & Y \end{tikzcd}$$

commute, for any u' and v making the square commute. This is so exactly because $E^{\alpha+1} f$ was obtained from $E^\alpha f$ by "formally adjoining" a solution to *every* such lifting problem. Thus, we get

[13] For a treatment of ordinals, see for instance the chapter on sets in [TS14].

a solution to a lifting problem (u, v) of s_i against $R^\beta f$ whenever the map $u\colon A_i \to E^\beta f$ can be factored through one of the maps $E^\alpha f \to E^\beta f$ with $\alpha < \beta$.

The "small object argument" amounts to the following.

Claim. There exists an ordinal κ such that for every domain A_i of a map in S, every map $A_i \to E^\kappa f$ factors through some $E^\alpha f \to E^\kappa f$ with $\alpha < \kappa$.

Given this, it follows from the "partial lifting property" that $S \boxslash R^\kappa f$, and so we obtain the desired factorization: $f = (R^\kappa f) \circ (L^\kappa f)$ with $L^\kappa f \in \overline{S}$ and $R^\kappa f \in S^{\boxslash}$.

It remains to prove the claim, which we will do by choosing κ to be a *regular cardinal* which is "bigger" than all the simplicial sets A_i.

18.3. **Regular cardinals.** The **cardinality** of a set X is the smallest ordinal λ such that there exists a bijection between X and λ; we write $|X|$ for this. Ordinals which can appear this way are called **cardinals**. For instance, the first infinite ordinal ω is the countable cardinal. **cardinality** **cardinals**

Note: the class of infinite cardinals is an unbounded subclass of the ordinals, so is well-ordered and can be put into bijective correspondence with ordinals. The symbol $\aleph_\alpha$ denotes the αth infinite cardinal, e.g., $\aleph_0 = \omega$.

Say that λ is a **regular cardinal**[14] if it is an infinite cardinal, and if for every set A of ordinals such that (i) $\alpha < \lambda$ for all $\alpha \in A$, and (ii) $|A| < \lambda$, we have that $\sup A < \lambda$. For instance, ω is a regular cardinal, since any finite collection of finite ordinals has a finite upper bound. Not every infinite cardinal is regular[15]; however, there exist arbitrarily large regular cardinals[16]. **regular cardinal**

Every ordinal α defines a category, which is the poset of ordinals strictly less than α. Colimits of functors $Y\colon \kappa \to \mathrm{Set}$ with κ a regular cardinal have the following property: the map

$$\operatorname{colim}_{\alpha<\kappa} \operatorname{Hom}(X, Y_\alpha) \to \operatorname{Hom}(X, \operatorname{colim}_{\alpha<\kappa} Y_\alpha) \tag{18.4}$$

is a bijection whenever $|X| < \kappa$. This generalizes the familiar case of $\kappa = \omega$: any map of a finite set into the colimit of a countable sequence factors through a finite stage.

18.5. *Exercise.* Prove that (18.4) is a bijection when $|X| < \kappa$.

18.6. **κ-small simplicial sets.** Given a regular cardinal κ, we say that a simplicial set is **κ-small** if it is isomorphic to the colimit of some functor $F\colon C \to s\mathrm{Set}$, such that (i) $|\mathrm{ob}\, C|, |\mathrm{mor}\, C| < \kappa$, and (ii) each $F(c)$ is isomorphic to a standard simplex Δ^n. Morally, we are saying that a simplicial set is κ-small if it can be "presented" with fewer than κ generators and fewer than κ relations. **κ-small**

Given a functor $Y\colon \kappa \to s\mathrm{Set}$ and a κ-small simplicial set X, we have a bijection as in (18.4). (This is sometimes phrased as: *κ-small simplicial sets are κ-compact.*) Thus, to prove the claim about the small object argument, we simply choose a regular cardinal κ greater than $\sup\{|(A_i)_n|\}$, where $(A_i)_n$ is the set of n-cells of the simplicial set A_i, which ranges over the set of domains of morphisms in S.

18.7. *Example.* The standard simplices Δ^n, as well as any subcomplex such as the horns Λ^n_j, are ω-small: this is a consequence of (6.20). Thus, when we carry out the small object argument for $S = \mathrm{InnHorn}$, we can take $(E^\omega f, L^\omega f, R^\omega f)$ to be the desired factorization.

18.8. *Remark.* The small object argument can be carried out in a very large class of categories, including the *locally presentable categories.* This class includes familiar algebraic examples, such as categories of groups, rings, lie algebras, modules, etc., along with many others. With a little more care the argument can sometimes be carried out even more generally (possibly under additional hypotheses on S), for instance in the category of topological spaces

[14]In the terminology of [TS14, §3.7], a regular cardinal is one which is equal to its own cofinality.

[15]For instance, $\aleph_\omega = \sup\{\, \aleph_k \mid k < \omega \,\}$ is not regular.

[16]For instance, every successor cardinal $\aleph_{\alpha+1}$ is regular.

18.9. **Functoriality.** The construction $f \mapsto (X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y)$ is a functor $\mathrm{Fun}([1], s\mathrm{Set}) \to \mathrm{Fun}([2], s\mathrm{Set})$, and it follows that so is $f \mapsto (X \xrightarrow{L^\alpha f} E^\alpha f \xrightarrow{R^\alpha f} Y)$ for any α. Because the choice of regular cardinal κ depends only on S, not on the map f, we see that the small object argument actually produces a *functorial factorization* of a map into a composite of an element of $\overline{S}$ with an element $S^{\boxslash}$. We will have use of this later in (80).

19. Degenerate and non-degenerate cells

We have noted that monomorphisms of simplicial sets form a weakly saturated class. Here we identify an important set of maps called Cell, whose weak saturation is precisely the class of monomorphisms. We do so by getting a very explicit handle on monomorphisms of simplicial sets. This will involve the notion of *degenerate* and *non-degenerate* cells of a simplicial set.

19.1. **Boundary of a standard simplex.** For each $n \geq 0$, define

$$\partial\Delta^n := \bigcup_{k \in [n]} \Delta^{[n] \setminus \{k\}} \subset \Delta^n,$$

the union of all codimension-one faces of the n-simplex. This means that

$$(\partial\Delta^n)_k = \{\, f\colon [k] \to [n] \;\mid\; f([k]) \neq [n] \,\},$$

so k-cells of $\partial\Delta^n$ correspond exactly to the *non-surjective* simplicial operators to $[n]$. We call $\partial\Delta^n$ the **boundary** of Δ^n. Note that $\partial\Delta^0 = \varnothing$ and $\partial\Delta^1 = \Delta^{\{0\}} \amalg \Delta^{\{1\}} \approx \Delta^0 \amalg \Delta^0$. **boundary**

19.2. *Exercise.* Show that $\partial\Delta^n$ is the largest subcomplex of Δ^n which does not contain the "generator" $\iota_n = \langle 0 \dots n \rangle \in (\Delta^n)_n$. In other words, $\partial\Delta^n$ is the maximal *proper* subcomplex of Δ^n.

19.3. *Exercise.* Show that if C is a category, then the evident maps $\mathrm{Hom}(\Delta^n, C) \to \mathrm{Hom}(\partial\Delta^n, C)$ defined by restriction are isomorphisms when $n \geq 3$, but not necessarily when $n \leq 2$.

19.4. **Trivial fibrations and monomorphisms.** Let Cell be the set consisting of the inclusions $\partial\Delta^n \subset \Delta^n$ for $n \geq 0$. The resulting right complement is $\mathrm{TrivFib} := \mathrm{Cell}^{\boxslash}$, the class of **trivial fibrations** (also sometimes called **acyclic fibrations**). By the small object argument (17.16), we obtain a weak factorization system $(\overline{\mathrm{Cell}}, \mathrm{TrivFib})$. In particular, we can always factor any map as $f = pj$, where $j \in \overline{\mathrm{Cell}}$ and p is a trivial fibration. **trivial fibrations** **acyclic fibrations**

Since the elements of Cell are monomorphisms, and the class of all monomorphisms is weakly saturated, we see that all elements of $\overline{\mathrm{Cell}}$ are monomorphisms. We are going to prove the converse, i.e., we will show that $\overline{\mathrm{Cell}}$ is precisely equal to the class of monomorphisms.

19.5. **Degenerate and non-degenerate cells.** I'll write $\Delta^{\mathrm{surj}}, \Delta^{\mathrm{inj}} \subset \Delta$ for the subcategories of the category Δ of simplicial operators, consisting of all the objects and the *surjective* and *injective* order-preserving maps respectively. Recall (3.7) that every simplicial operator factors *uniquely* as $f = f^{\mathrm{inj}} f^{\mathrm{surj}}$, a surjection followed by an injection.

A cell $a \in X_n$ is said to be **degenerate** if there exists a *non-injective* simplicial operator $f \in \Delta$ and a cell b in X such that $a = bf$. In view of the factorization $f = f^{\mathrm{inj}} f^{\mathrm{surj}}$, we see that a is degenerate if and only if there exists a *non-identity surjective* simplicial operator $f \in \Delta^{\mathrm{surj}}$ and a cell b in X such that $a = bf$. **degenerate**

Likewise, a cell $a \in X_n$ is said to be **non-degenerate** if it is not degenerate, i.e., if whenever $a = bf$ we must have $f \in \Delta^{\mathrm{inj}}$. Equivalently, a is non-degenerate if whenever $a = bf$ with $f \in \Delta^{\mathrm{surj}}$ we must have $f = \mathrm{id}$. **non-degenerate**

We write $X_n = X_n^{\mathrm{deg}} \amalg X_n^{\mathrm{nd}}$ for the decomposition of X_n into complementary subsets of degenerate and non-degenerate cells. Note that if $f\colon A \to X$ is a map of simplicial sets, then $f(A_n^{\mathrm{deg}}) \subseteq X_n^{\mathrm{deg}}$, while $f^{-1}(X_n^{\mathrm{nd}}) \subseteq A_n^{\mathrm{nd}}$. Also note that neither X_n^{deg} nor X_n^{nd} assemble to give a subcomplex of X (unless X is empty).

The property of being degenerate or non-degenerate is preserved in subcomplexes.

19.6. **Proposition.** *If X is a simplicial set and $A \subseteq X$ is a subcomplex, then $A_n^{\mathrm{nd}} = X_n^{\mathrm{nd}} \cap A_n$ and $A_n^{\mathrm{deg}} = X_n^{\mathrm{deg}} \cap A_n$.*

Proof. The first statement is a consequence of the second, since subsets of degenerate and non-degenerate cells are complementary. It is clear that $A_n^{\mathrm{deg}} \subseteq X_n^{\mathrm{deg}} \cap A_n$. Conversely, suppose $a \in X_n^{\mathrm{deg}} \cap A_n$, so $a \in A_n$ and $a = bg$ for some non-identity $g\colon [n] \to [k] \in \Delta^{\mathrm{surj}}$ and $b \in X_k$. Any surjection in Δ has a section (6.18), so there exists $s\colon [k] \to [n]$ such that $gs = 1_{[k]}$. Then $b = bgs = as \in A_k$, whence $a \in A_n^{\mathrm{deg}}$ as desired. □

19.7. *Exercise* (easy). For any simplicial set X, we have $X_0^{\mathrm{deg}} = \varnothing$ and $X_0^{\mathrm{nd}} = X_0$, while X_1^{deg} is the image of $\langle 00\rangle^*\colon X_0 \to X_1$ (which is an injective function) and X_1^{nd} is its complement.

19.8. *Example.* Here are all cells in the standard 2-simplex up to dimension 3, with the non-degenerate ones indicated by a box.

$(\Delta^2)_0$	$(\Delta^2)_1$	$(\Delta^2)_2$	$(\Delta^2)_3$
$\boxed{\langle 0\rangle}$	$\langle 00\rangle$	$\langle 000\rangle$	$\langle 0000\rangle$
$\boxed{\langle 1\rangle}$	$\langle 11\rangle$	$\langle 111\rangle$	$\langle 1111\rangle$
$\boxed{\langle 2\rangle}$	$\langle 22\rangle$	$\langle 222\rangle$	$\langle 2222\rangle$
	$\boxed{\langle 01\rangle}$	$\langle 001\rangle$ $\langle 011\rangle$	$\langle 0001\rangle$ $\langle 0011\rangle$ $\langle 0111\rangle$
	$\boxed{\langle 02\rangle}$	$\langle 002\rangle$ $\langle 022\rangle$	$\langle 0002\rangle$ $\langle 0022\rangle$ $\langle 0222\rangle$
	$\boxed{\langle 12\rangle}$	$\langle 112\rangle$ $\langle 122\rangle$	$\langle 1112\rangle$ $\langle 1122\rangle$ $\langle 1222\rangle$
		$\boxed{\langle 012\rangle}$	$\langle 0012\rangle$ $\langle 0112\rangle$ $\langle 0122\rangle$

19.9. *Exercise.* Describe the degenerate and non-degenerate cells of all the standard n-simplices Δ^n.

19.10. *Exercise.* For every $n \geq 0$, let $\Delta^n/\partial\Delta^n$ be the pushout of the diagram $\Delta^n \leftarrowtail \partial\Delta^n \to \Delta^0$, where $\partial\Delta^n \rightarrowtail \Delta^n$ is the usual inclusion and $\partial\Delta^n \to \Delta^0$ is the unique map to the terminal object. Describe all degenerate and non-degenerate cells of $\Delta^n/\partial\Delta^n$.

19.11. *Exercise.* Show that if C is an ordinary category, then a cell $a \in N(C)_k$ of the nerve with $k > 0$ is non-degenerate if and only if it is represented by a composable sequence of *non-identity* maps $c_0 \to \cdots \to c_k$ in the category C.

19.12. *Exercise.* Let X be a simplicial set. Show that

$$X_n^{\mathrm{deg}} = \{\, af \;\mid\; a \in X_k,\ f\colon [n] \to [k],\ k < n \,\}.$$

19.13. **Simplicial sets are canonically free with respect to surjective operators.** The key observation is that degenerate cells in a simplicial set are precisely determined by knowledge of the non-degenerate cells.

19.14. **Proposition** (Eilenberg-Zilber lemma)**.** *Let a be a cell of X. Then there exists a* unique *pair (b, σ) consisting of a* non-degenerate *cell b and a map σ in Δ^{surj} such that $a = b\sigma$.*

Proof. [GZ67, §II.3]. First note that for degenerate a such a pair (b, σ) exists by definition, while for nondegenerate a we can take the pair (a, id).

Given $\sigma\colon [n] \to [m]$, let $\Gamma(\sigma) = \left\{\, \delta\colon [m] \to [n] \;\middle|\; \sigma\delta = \mathrm{id}_{[m]} \,\right\}$ denote the set of sections of σ. The sets $\Gamma(\sigma)$ is non-empty when $\sigma \in \Delta^{\mathrm{surj}}$ (6.18). We note the following elementary observation, whose proof is left for the reader:

$$\text{If } \sigma, \sigma' \in \Delta^{\mathrm{surj}} \text{ are such that } \Gamma(\sigma) = \Gamma(\sigma'), \text{ then } \sigma = \sigma'.$$

Let $a \in X_n$ be such that $a = b_i\sigma_i$ for $b_i \in X^{\mathrm{nd}}_{m_i}$, $\sigma_i \in \Delta^{\mathrm{surj}}([n],[m_i])$, for $i = 1, 2$. We want to show that $m_1 = m_2$, $b_1 = b_2$, and $\sigma_1 = \sigma_2$.

Pick any $\delta_1 \in \Gamma(\sigma_1)$ and $\delta_2 \in \Gamma(\sigma_2)$. Then we have

$$b_1 = b_1\sigma_1\delta_1 = a\delta_1 = b_2\sigma_2\delta_1, \qquad b_2 = b_2\sigma_2\delta_2 = a\delta_2 = b_1\sigma_1\delta_2,$$

so b_1 and b_2 are related by the simplicial operators $\sigma_2\delta_1$ and $\sigma_1\delta_2$. Since b_1 and b_2 are non-degenerate, both $\sigma_2\delta_1\colon [m_1] \to [m_2]$ and $\sigma_1\delta_2\colon [m_2] \to [m_1]$ must be injective. This implies $m_1 = m_2$, and since the only order-preserving injective map $[m] \to [m]$ is the identity map, we must have $\sigma_2\delta_1 = \mathrm{id} = \sigma_1\delta_2$, from which it follows that $b_1 = b_2$. This also shows that $\delta_1 \in \Gamma(\sigma_2)$ and $\delta_2 \in \Gamma(\sigma_1)$. Since δ_1 and δ_2 were arbitrarily chosen sections, we have shown $\Gamma(\sigma_1) = \Gamma(\sigma_2)$, and therefore $\sigma_1 = \sigma_2$. □

We can reinterpret the Eilenberg-Zilber lemma as follows.

19.15. **Corollary.** *For any simplicial set X, the evident maps*

$$\coprod_{j\geq 0} X^{\mathrm{nd}}_j \times \mathrm{Hom}_{\Delta^{\mathrm{surj}}}([n],[j]) \to X_n$$

defined by $(j, x, \sigma) \mapsto x\sigma$ are bijections. Furthermore, these bijections are natural with respect to surjective simplicial operators $[n'] \to [n]$, and are also natural with respect to monomorphisms $X \to X'$ of simplicial sets.

Proof. The bijection is a restatement of (19.14). For the second statement, note that if $\tau\colon [n'] \to [n]$ is a surjective simplicial operator, then $(k, x, \sigma\tau) \mapsto (x\sigma)\tau$. The third statement is straightforward. □

Thus the restricted functor $X|(\Delta^{\mathrm{surj}})^{\mathrm{op}}\colon (\Delta^{\mathrm{surj}})^{\mathrm{op}} \to \mathrm{Set}$ is *canonically isomorphic* to a coproduct of representable functors $\mathrm{Hom}_{\Delta^{\mathrm{surj}}}(-,[k])$ indexed by the nondegenerate cells of X. Or more simply: simplicial sets are *canonically free* with respect to surjective simplicial operators.

19.16. *Exercise.* Let X be a simplicial set such that each X_k is a finite set for all $0 \leq k \leq n$. Give a formula for the size of $|X_n|$ as a function of the $\left|X^{\mathrm{nd}}_k\right|$ with $0 \leq k \leq n$.

The following exercises show that the subcomplexes of a simplicial set X can be completely characterized by the sets of non-degenerate cells of X that they contain, generalizing what we showed about subcomplexes of standard simplices (6.19).

19.17. *Exercise.* Let $X^{\mathrm{nd}} = \coprod_{n\geq 0} X^{\mathrm{nd}}_n$ be the set of non-degenerate cells of X. For $x, y \in X^{\mathrm{nd}}$ write $y \leq x$ if there exists $f \in \Delta$ such that $y = xf$. Show that "$\leq$" is a partial order on the set X^{nd}, which we can call the *face relation*.

19.18. *Exercise.* Show that if $xf = yg$ for some $x, y \in X^{\mathrm{nd}}$, $f \in \Delta$ and $g \in \Delta^{\mathrm{surj}}$, then $y \leq x$.

19.19. *Exercise.* Let $S \subseteq X^{\mathrm{nd}}$ be a subset of non-degenerate cells which is downward-closed under the face relation, i.e., $y \leq x$ and $x \in S$ implies $y \in S$. Show that there exists a unique subcomplex $A \subseteq X$ such that $A^{\mathrm{nd}} = S$. (Hint: the cells of A are of the form xg where $x \in S$ and $g \in \Delta$.)

19.20. *Remark.* A simplicial set can be recovered up to isomorphism if you only know (i) its sets of non-degenerate cells, and (ii) the faces of the non-degenerate cells. From this you can reconstruct the the degenerate cells using (19.15). Simplicial operators on degenerate cells are computed using the fact that any simplicial operator factors into a surjection followed by an injection.

Warning. The faces of a non-degenerate cell can be degenerate; this happens for instance for $\Delta^n/\partial\Delta^n$ in (19.10) when $n \geq 2$. If X is such that all faces of non-degenerate cells are also non-degenerate, then we get a functor $X^{\mathrm{nd}}\colon (\Delta^{\mathrm{inj}})^{\mathrm{op}} \to \mathrm{Set}$, and the full simplicial set X can be recovered from X^{nd}. For instance, this is so for the standard simplices Δ^n, as well as any subcomplexes of such. Functors $(\Delta^{\mathrm{inj}})^{\mathrm{op}} \to \mathrm{Set}$ are the combinatorial data behind the notion of a Δ-*complex*, as seen in Hatcher's textbook on algebraic topology [Hat02, Ch. 2.1].

The following exercises give a different point of view of this principle.

19.21. *Exercise.* Fix an object $[n]$ in Δ, and consider the category $\Delta^{\text{surj}}_{[n]/}$, which has

- **objects** the surjective morphisms $\sigma\colon [n] \to [k]$ in Δ, and
- **morphisms** commutative triangles in Δ of the form

$$\begin{tikzcd} & {[k]} \arrow[d, "\tau"] \\ {[n]} \arrow[ur, "\sigma"] \arrow[r, "\sigma'"'] & {[k']} \end{tikzcd}$$

Show that the category $\Delta^{\text{surj}}_{[n]/}$ is *isomorphic* to the poset $\mathcal{P}(\underline{n})$ of subsets of the set $\underline{n} = \{1, \dots, n\}$. In particular, $\Delta^{\text{surj}}_{[n]/}$ is a lattice (i.e., has finite products and coproducts, called *meets* and *joins* in this context).

19.22. *Exercise.* Let X be a simplicial set. Given $n \geq 0$ and $\sigma\colon [n] \to [k]$ in Δ^{surj}, let $X_n^\sigma := \sigma^*(X_k)$, the image of the operator σ^* in X_n. Show that $X_n^{\sigma\vee\sigma'} = X_n^\sigma \cap X_n^{\sigma'}$, where $\sigma \vee \sigma'$ is join in the lattice $\Delta^{\text{surj}}_{[n]/}$. Conclude that for each $x \in X_n$ there exists a maximal σ such that $x \in X_n^\sigma$.

20. The skeletal filtration

20.1. **Skeleta.** Given a simplicial set X, the k**-skeleton** $\operatorname{Sk}_k X \subseteq X$ is the subcomplex with n-cells **M 7 Feb** **k-skeleton**

$$(\operatorname{Sk}_k X)_n = \bigcup_{0\leq j\leq k} \{\, yf \mid y \in X_j,\ f\colon [n] \to [j] \in \Delta \,\}.$$

It is immediate that this defines a subcomplex of X, which is in fact the smallest subcomplex containing all cells of dimensions $\leq k$. Note that $\operatorname{Sk}_{k-1} X \subseteq \operatorname{Sk}_k X$ and $X = \bigcup_k \operatorname{Sk}_k X$, and that a map $X \to Y$ of simplicial sets restricts to a map $\operatorname{Sk}_k X \to \operatorname{Sk}_k Y$. The skeleta constructions define functors $\operatorname{Sk}_k\colon s\mathrm{Set} \to s\mathrm{Set}$.

In view of (19.15), we see that

$$(\operatorname{Sk}_k X)_n \approx \coprod_{0\leq j\leq k} X_j^{\text{nd}} \times \operatorname{Hom}_{\Delta^{\text{surj}}}([n],[j]).$$

The complement of the set of cells of $\operatorname{Sk}_{k-1} X$ in $\operatorname{Sk}_k X$ consists precisely of the *nondegenerate* k-cells of X together with their associated degenerate cells (in dimensions $> k$).

20.2. *Example.* The $(n-1)$-skeleton of the stardand n-simplex is precisely what we have called its boundary: $\operatorname{Sk}_{n-1}\Delta^n = \partial\Delta^n$. The only cells of Δ^n not contained in its boundary are the generator $\iota = \langle 0 \dots n\rangle \in (\Delta_n)_n$ together with the degenate cells associated to it.

20.3. **Proposition.** *The evident square*

$$\begin{tikzcd} \coprod_{a\in X_k^{\text{nd}}} \partial\Delta^k \arrow[r] \arrow[d, tail] & \operatorname{Sk}_{k-1} X \arrow[d, tail] \\ \coprod_{a\in X_k^{\text{nd}}} \Delta^n \arrow[r] & \operatorname{Sk}_k X \end{tikzcd}$$

is a pushout of simplicial sets. More generally, for any subcomplex $A \subseteq X$, the evident square

$$\begin{tikzcd} \coprod_{a\in X_k^{\text{nd}}\smallsetminus A_k^{\text{nd}}} \partial\Delta^k \arrow[r] \arrow[d, tail] & A \cup \operatorname{Sk}_{k-1} X \arrow[d, tail] \\ \coprod_{a\in X_k^{\text{nd}}\smallsetminus A_k^{\text{nd}}} \Delta^k \arrow[r] & A \cup \operatorname{Sk}_k X \end{tikzcd}$$

is a pushout.

Proof. In each of the above squares, the complements of the vertical inclusions coincide precisely. In particular, the complement of the inclusion $(A \cup \mathrm{Sk}_{k-1} X)_n \subseteq (A \cup \mathrm{Sk}_k X)_n$ is in bijective correspondence with $(X_k^{\mathrm{nd}} \smallsetminus A_k^{\mathrm{nd}}) \times \mathrm{Hom}_{\Delta^{\mathrm{surj}}}([n],[k])$, and thus the square is a pushout (20.4). $\square$

In the proof, we used the following fact which generalizes (6.15), which is worth recording.

20.4. **Lemma.** *If*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "i"'] & X \arrow[d, tail, "j"] \\ Y' \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback of simplicial sets such that (i) j is a monomorphism, and (ii) f induces in each degree n a bijection $Y'_n \smallsetminus i(X'_n) \xrightarrow{\sim} Y_n \smallsetminus j(X_n)$, then the square is a pushout square.

Proof. Verify the analogous statement for a pullback square of sets. $\square$

20.5. **Corollary.** $\overline{\mathrm{Cell}}$ *is precisely the class of monomorphisms.*

Proof. We know all elements of $\overline{\mathrm{Cell}}$ are monomorphisms. Any monomorphism is isomorphic to an inclusion $A \subseteq X$ of a subcomplex, so we only need show that such inclusions are contained in $\overline{\mathrm{Cell}}$. Since $X \approx \mathrm{colim}_k\, A \cup \mathrm{Sk}_k X$, (20.3) exhibits the inclusion as a countable composite of pushouts along coproducts of elements of Cell. $\square$

20.6. **Geometric realization.** Recall the singular complex functor Sing: Top $\to$ sSet (12.9). This functor has a left adjoint $\|-\| : s\mathrm{Set} \to \mathrm{Top}$, called **geometric realization**, constructed explicitly by **geometric realization**

(20.7)
$$\|X\| := \mathrm{Cok}\left[\coprod_{f\colon [m]\to[n]} X_n \times \Delta^m_{\mathrm{top}} \rightrightarrows \coprod_{[p]} X_p \times \Delta^p_{\mathrm{top}} \right];$$

that is, take a collection of topological simplices indexed by cells of X, and make identifactions according to the simplicial operators in X. (Here the symbol "Cok" represeents taking a "coequalizer", i.e., the colimit of a diagram of shape $\bullet \rightrightarrows \bullet$.)

20.8. *Exercise.* Describe the two unlabelled maps in (20.7). Then show that $\|-\|$ is in fact left adjoint to Sing.

Because geometric realization is a left adjoint, it commutes with colimits. It is straightforward to check that $\|\Delta^n\| \approx \Delta^n_{\mathrm{top}}$, and that $\|\partial\Delta^n\| \approx \partial\Delta^n_{\mathrm{top}}$, where the latter is the subspace

$$\partial\Delta^n_{\mathrm{top}} := \left\{ (x_0,\dots,x_n) \in \Delta^n_{\mathrm{top}} \;\middle|\; \exists k \text{ such that } x_k = 0 \right\} \subset \Delta^n_{\mathrm{top}}.$$

Applying this to the skeletal filtration, we discover that there are pushouts

$$\begin{tikzcd} \coprod_{a\in X_k^{\mathrm{nd}}} \partial\Delta^k_{\mathrm{top}} \arrow[r] \arrow[d, tail] & \|\mathrm{Sk}_{k-1} X\| \arrow[d, tail] \\ \coprod_{a\in X_k^{\mathrm{nd}}} \Delta^k_{\mathrm{top}} \arrow[r] & \|\mathrm{Sk}_k X\| \end{tikzcd}$$

of spaces, and that $\|X\| = \bigcup \|\mathrm{Sk}_k X\|$ with the direct limit topology. Thus, $\|X\|$ is presented to us as a *CW-complex*, whose cells are in an evident bijective correspondence with the set of non-degenerate cells of X.

21. Pushout-product and pullback-hom

We are going to prove several "enriched" versions of lifting properties associated to inner anodyne maps and inner fibrations. As a consequence we'll be able to prove (22.4) that function complexes of quasicategories are themselves quasicategories, and thus completing our detour.

21.1. **Definition of pushout-product and pullback-hom.** Given maps $f\colon A \to B$, $g\colon K \to L$ and $h\colon X \to Y$ of simplicial sets, we define new maps $f\Box g$ and $g^{\Box h}$ called the **pushout-product**[17] and the **pullback-hom**[18]. The pushout-product $f\Box g\colon (A \times L) \amalg_{A\times K} (B \times K) \to B \times L$ is the unique map fitting in the commutative diagram **pushout-product** **pullback-hom**

$$\begin{tikzcd} A \times K \arrow[r, "\mathrm{id}\times g"] \arrow[d, "f\times\mathrm{id}"'] & A \times L \arrow[d] \arrow[ddr, bend left, "f\times\mathrm{id}"] & \\ B \times K \arrow[r] \arrow[drr, bend right, "\mathrm{id}\times g"'] & (A \times L) \amalg_{A\times K} (B \times K) \arrow[dr, "f\Box g"] & \\ & & B \times L \end{tikzcd}$$

in which the square is a pushout. The pullback-hom $h^{\Box g}\colon \mathrm{Fun}(L,X) \to \mathrm{Fun}(K,X) \times_{\mathrm{Fun}(K,Y)} \mathrm{Fun}(L,Y)$ is the unique map fitting in the commutative diagram

$$\begin{tikzcd} \mathrm{Fun}(L,X) \arrow[dr, "h^{\Box g}"] \arrow[drr, bend left, "\mathrm{Fun}(g,X)"] \arrow[ddr, bend right, "\mathrm{Fun}(L,h)"'] & & \\ & \mathrm{Fun}(K,X) \times_{\mathrm{Fun}(K,Y)} \mathrm{Fun}(L,Y) \arrow[r] \arrow[d] & \mathrm{Fun}(K,X) \arrow[d, "\mathrm{Fun}(\mathrm{id},h)"] \\ & \mathrm{Fun}(L,Y) \arrow[r, "\mathrm{Fun}(g,\mathrm{id})"'] & \mathrm{Fun}(K,Y) \end{tikzcd}$$

in which the square is a pullback.

21.2. *Remark.* Usually we form the pushout-product $f\Box g$ when f and g are monomorphisms of simplicial sets, in which case $f\Box g$ is also a monomorphism. In this case, the cells $(b,\ell) \in B \times L$ which are *not* in the image of $f\Box g$ are exactly those such that $b \in B \smallsetminus f(A)$ *and* $\ell \in L \smallsetminus g(K)$.

21.3. *Remark* (Important!). On vertices, the pullback-hom $h^{\Box g}$ is just the "usual" map $\mathrm{Hom}(L,X) \to \mathrm{Hom}(K,X) \times_{\mathrm{Hom}(K,Y)} \mathrm{Hom}(L,Y)$ sending $s \mapsto (sg, hs)$. Thus, $h^{\Box g}$ is surjective on vertices if and only if $g \boxslash h$.

We think of the pullback-hom as encoding an "enriched" version of the lifting problem for (g,h). Thus, the target of $h^{\Box g}$ is an object which "parameterizes familes" of lifting problems of type $g \boxslash h$, while the source of $h^{\Box g}$ "parameterizes families" of such lifting problems which are equipped with a choice of lift.

21.4. *Remark.* The pushout-product construction is symmetric: $f\Box g$ is isomorphic to $g\Box f$ in the arrow category $\mathrm{Fun}([1], s\mathrm{Set})$. Ultimately, this is because product is symmetric. The pullback-hom construction however is not symmetric.

[17] This is sometimes called the *box-product*. Some also call it the *Leibniz-product*, as its form is that of the Leibniz rule for boundary of a product space: $\partial(X \times Y) = (\partial X \times Y) \cup_{\partial X\times\partial Y} (X \times \partial Y)$ (which is itself reminiscent of the original Leibniz rule $D(fg) = (Df)g + f(Dg)$ of calculus).

[18] Sometimes called the *box-power* or *pullback-power*. A common alternate notation is $g \pitchfork h$. This may also be called the *Leibniz-hom*, though I don't know what rule of calculus it is related to.

The product/function complex adjunction gives rise to the following relationship between lifting problems, which we may refer to as **adjunction of lifting problems**. **adjunction of lifting problems**

21.5. **Proposition.** *We have that* $(f\Box g) \boxslash h$ *if and only if* $f \boxslash (h^{\Box g})$.

Proof. Compare the two lifting problems using the product/map adjunction.

$$\begin{tikzcd} (A\times L)\amalg_{A\times K}(B\times K) \arrow[r, "{(u,v)}"] \arrow[d, "f\Box g"'] & X \arrow[d, "h"] \\ B\times L \arrow[r, "w"'] \arrow[ur, dotted, "s"] & Y \end{tikzcd} \iff \begin{tikzcd} A \arrow[r, "\widetilde{u}"] \arrow[d, "f"'] & \mathrm{Fun}(L,X) \arrow[d, "h^{\Box g}"] \\ B \arrow[r, "{(\widetilde{v},\widetilde{w})}"'] \arrow[ur, dotted, "\widetilde{s}"] & \mathrm{Fun}(K,X)\times_{\mathrm{Fun}(K,Y)}\mathrm{Fun}(L,Y) \end{tikzcd}$$

On the left-hand side are maps

$$u\colon A\times L\to X,\qquad v\colon B\times K\to X,\qquad w\colon B\times L\to Y,\qquad s\colon B\times L\to X,$$

while on the right-hand side are maps

$$\widetilde{u}\colon A\to \mathrm{Fun}(L,X),\qquad \widetilde{v}\colon B\to\mathrm{Fun}(K,X),\qquad \widetilde{w}\colon B\to\mathrm{Fun}(L,Y),\qquad \widetilde{s}\colon B\to\mathrm{Fun}(L,X).$$

The data of (u,v,w) giving a commutative square as on the left corresponds bijectively to data $(\widetilde{u},\widetilde{v},\widetilde{w})$ giving a commutative square as on the right. Similarly, lifts s correspond bijectively to lifts $\widetilde{s}$. $\Box$

It is important to note the special cases where one or more of $A=\varnothing$, $K=\varnothing$, or $Y=*$ hold. For instance, if $K=\varnothing$ and $Y=*$, the proposition implies

$$(A\times L\xrightarrow{f\times L} B\times L)\boxslash(X\to *)\quad\text{iff}\quad (A\xrightarrow{f}B)\boxslash(\mathrm{Fun}(L,X)\to *).$$

This is the kind of case we are interested in for proving that $\mathrm{Fun}(K,C)$ is a quasicategory whenever C is. The more general statement of the proposition is a kind of "relative" version of the thing we want; it is especially handy for carrying out inductive arguments.

21.6. *Exercise* (if you like monoidal categories). Let $\mathcal{C}:=\mathrm{Fun}([1],s\mathrm{Set})$, the "arrow category" of simplicial sets. Show that $\Box\colon\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ defines a symmetric monoidal structure on $\mathcal{C}$, with unit object $(\varnothing\subset\Delta^0)$. Furthermore, show that this is a *closed* symmetric monoidal structure, with $-\Box g$ left adjoint to $(-)^{\Box g}\colon\mathcal{C}\to\mathcal{C}$.

21.7. **Inner anodyne maps and pushout-products.** The key fact we want to prove is the following.

21.8. **Proposition.** *We have that* $\overline{\mathrm{InnHorn}}\Box\overline{\mathrm{Cell}}\subseteq\overline{\mathrm{InnHorn}}$, *i.e., that* $i\Box j$ *is inner anodyne whenever* i *is inner anodyne and* j *is a monomorphism.*

To set up the proof we need the following.

21.9. **Proposition.** *For any sets of maps* S *and* T, *we have* $\overline{S}\Box\overline{T}\subseteq\overline{S\Box T}$.

Proof. Let $\mathcal{F}=(S\Box T)^{\boxslash}$. From the small object argument we have that $\overline{S\Box T}={}^{\boxslash}\mathcal{F}$ (17.17), so we will show $(\overline{S}\Box\overline{T})\boxslash\mathcal{F}$. First we show that $(\overline{S}\Box T)\boxslash\mathcal{F}$. Consider

$$\begin{aligned}\mathcal{A}&:=\{\,a\mid (a\Box T)\boxslash\mathcal{F}\,\}\\ &\approx\{\,a\mid a\boxslash(\mathcal{F}^{\Box T})\,\}\end{aligned}$$

by correspondence between lifting problems for pushout-products and pullback-homs (21.5). Thus $\mathcal{A}$ is a left complement, and so is weakly saturated. Since $S\subseteq\mathcal{A}$ then $\overline{S}\subseteq\mathcal{A}$, i.e., $(\overline{S}\Box T)\boxslash\mathcal{F}$. The same idea applied to

$$\mathcal{B}:=\{\,b\mid(\overline{S}\Box b)\boxslash\mathcal{F}\,\}\approx\left\{\,b\;\middle|\;b\boxslash(\mathcal{F}^{\Box\overline{S}})\,\right\},$$

gives $\overline{T}\subseteq\mathcal{B}$, whence $(\overline{S}\Box\overline{T})\boxslash\mathcal{F}$. $\Box$

21.10. **Lemma.** *We have* $\mathrm{InnHorn}\square\mathrm{Cell} \subseteq \overline{\mathrm{InnHorn}}$.

Proof. This is a calculation, given in [Joy08a, App. H], and presented in the appendix (78.3). $\square$

Proof of (21.8). We have that

$$\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}\square\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}.$$

The first inclusion is (21.9), while the second is an immediate conseqence of $\mathrm{InnHorn}\square\mathrm{Cell} \subseteq \overline{\mathrm{InnHorn}}$ (21.10). $\square$

21.11. *Example.* Let's carry out a proof of (21.10) explicitly in one case, by showing that $(\Lambda^2_1 \subset \Delta^2)\square(\partial\Delta^1 \subset \Delta^1)$ is inner anodyne. This map is the inclusion

$$(\Lambda^2_1 \times \Delta^1) \cup_{\Lambda^2_1\times\partial\Delta^1} (\Delta^2 \times \partial\Delta^1) \subset \Delta^2 \times \Delta^1,$$

whose target is a "prism", and whose source is a "trough". To show this is in $\overline{\mathrm{InnHorn}}$, we'll give an explicit procedure for constructing the prism from the trough by succesively attaching simplices along inner horns.

Note that $\Delta^2 \times \Delta^1 = N([2] \times [1])$, so we are working inside the nerve of a poset, whose elements (objects) are "ij" with $i \in \{0, 1, 2\}$ and $j \in \{0, 1\}$. Here is a picture of the trough, showing all the non-degenerate cells as the planar 2-cells of the graph.

$$\begin{tikzcd}
01 \arrow[r] \arrow[rr, bend left] & 11 \arrow[r] & 21 \\
00 \arrow[u] \arrow[ur] \arrow[r] \arrow[rr, bend right] & 10 \arrow[u] \arrow[ur] \arrow[r] & 20 \arrow[u]
\end{tikzcd}$$

The complement of this in the prism consists of three non-degenerate 3-cells, five non-degenerate 2-cells (two of which form the "lid" of the trough, while the other three are in the interior of the prism), and one non-degenerate edge cell (separating the two 2-cells which form the lid).

The following chart lists all non-degenerate cells in the complement of the trough, along with their codimension one faces (in order). The "$\surd$" marks cells which are contained in the trough.

$\langle 00,21\rangle$	$\langle 00,20,21\rangle$	$\langle 00,01,21\rangle$	$\langle 00,10,21\rangle$	$\langle 00,11,21\rangle$	$\langle 00,10,20,21\rangle$	$\langle 00,10,11,21\rangle$	$\langle 00,01,11,21\rangle$
$\surd\langle 21\rangle$	$\surd\langle 20,21\rangle$	$\surd\langle 01,21\rangle$	$\surd\langle 10,21\rangle$	$\surd\langle 11,21\rangle$	$\surd\langle 10,20,21\rangle$	$\surd\langle 10,11,21\rangle$	$\surd\langle 01,11,21\rangle$
$\surd\langle 10\rangle$	$\langle 00,21\rangle$	$\langle 00,21\rangle$	$\langle 00,21\rangle$	$\langle 00,21\rangle$	$\langle 00,20,21\rangle$	$\langle 00,11,21\rangle$	$\langle 00,11,21\rangle$
	$\surd\langle 00,20\rangle$	$\surd\langle 00,01\rangle$	$\surd\langle 00,10\rangle$	$\surd\langle 00,11\rangle$	$\langle 00,10,21\rangle$	$\langle 00,10,21\rangle$	$\langle 00,01,21\rangle$
					$\surd\langle 00,10,20\rangle$	$\surd\langle 00,10,11\rangle$	$\surd\langle 00,01,11\rangle$

Note that the cells $\langle 00, 21\rangle$, $\langle 00, 10, 21\rangle$, and $\langle 00, 11, 21\rangle$ of the complement appear multiple times as faces. We can attach simplices to the domain in the following order:

$$①\langle 00, 10, 21\rangle, \quad ②\langle 00, 10, 20, 21\rangle, \quad ③\langle 00, 10, 11, 21\rangle, \quad ④\langle 00, 01, 11, 21\rangle.$$

In each case, the intersection of the simplex with (domain+previously attached simplices) is an inner horn. This directly exhibits $(\Lambda^2_1 \subset \Delta^2)\square(\partial\Delta^1 \subset \Delta^1)$ as an inner anodyne map.

22. Function complexes of quasicategories are quasicategories

22.1. **Enriched lifting properties.** We record the immediate consequences of $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (21.8).

22.2. **Proposition.**

(1) *If* $i\colon A \to B$ *is inner anodyne and* $j\colon K \to L$ *a monomorphism, then*

$$i\square j\colon (A \times L) \cup_{A\times K} (B \times K) \to B \times L$$

is inner anodyne.

(2) *If $j\colon K \to L$ is a monomorphism and $p\colon X \to Y$ is an inner fibration, then*
$$p^{\square j}\colon \operatorname{Fun}(L,X) \to \operatorname{Fun}(K,X) \times_{\operatorname{Fun}(K,Y)} \operatorname{Fun}(L,Y)$$
is an inner fibration.

(3) *If $i\colon A \to B$ is inner anodyne and $p\colon X \to Y$ is an inner fibration, then*
$$p^{\square i}\colon \operatorname{Fun}(B,X) \to \operatorname{Fun}(A,X) \times_{\operatorname{Fun}(A,Y)} \operatorname{Fun}(B,Y)$$
is a trivial fibration.

These can be summarized as
$$\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}, \qquad \mathrm{InnFib}^{\square\overline{\mathrm{Cell}}} \subseteq \mathrm{InnFib}, \qquad \mathrm{InnFib}^{\square\overline{\mathrm{InnHorn}}} \subseteq \mathrm{TrivFib}.$$
Statement (1) is just restating (21.8). The other two statements follow from (1) using the adjunction of lifting problems for pushout-products and pullback-homs (21.5), together with the facts that $\mathrm{InnFib} = \mathrm{InnHorn}^{\boxslash}$ and $\mathrm{TrivFib} = \mathrm{Cell}^{\boxslash}$. For instance, (2) follows from the observation that $i \boxslash p^{\square j}$ iff $(i\square j) \boxslash p$, and that $i \in \overline{\mathrm{InnHorn}}$ and $j \in \overline{\mathrm{Cell}}$ imply $i\square j \in \overline{\mathrm{InnHorn}}$. Likewise (3) follows a similar argument using that $j \boxslash p^{\square i}$ iff $(i\square j) \boxslash p$.

We are going to use these consequences all the time. To announce that I am using any of these, I will simply assert "$\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$" without other explanation; sometimes, to indicate an application of statements (2) and (3), I will call it "enriched lifting". The following gives the most general statement, of which (21.8) amounts to the special case of $S = U = \mathrm{InnHorn}$ and $T = \mathrm{Cell}$.

22.3. **Proposition.** *Let S, T, and U be sets of morphisms in $s\mathrm{Set}$. Write $\overline{S}$, $\overline{T}$, and $\overline{U}$ for the weak saturations of these sets, and let $S\mathrm{Fib} := S^{\boxslash}$, $T\mathrm{Fib} := T^{\boxslash}$, and $U\mathrm{Fib} := U^{\boxslash}$ denote the respective right complements. If $S\square T \subseteq \overline{U}$, then*
$$\overline{S}\square\overline{T} \subseteq \overline{U}, \qquad U\mathrm{Fib}^{\square\overline{T}} \subseteq S\mathrm{Fib}, \qquad U\mathrm{Fib}^{\square\overline{S}} \subseteq T\mathrm{Fib}.$$
Proof. Exercise using (21.5). □

There are many useful special cases of (22.2), obtained by taking the domain of a monomorphism to be empty, or the target of an inner fibration to be terminal.

- If $i\colon A \to B$ is inner anodyne, so is $i \times \mathrm{id}_L\colon A \times L \to B \times L$.
- If $p\colon X \to Y$ is an inner fibration, then so is $\operatorname{Fun}(L,p)\colon \operatorname{Fun}(L,X) \to \operatorname{Fun}(L,Y)$.
- If $j\colon K \to L$ is a monomorphism and C a quasicategory, then $\operatorname{Fun}(j,C)\colon \operatorname{Fun}(L,C) \to \operatorname{Fun}(K,C)$ is an inner fibration.
- If $i\colon A \to B$ is inner anodyne and C a quasicategory, then $\operatorname{Fun}(i,C)\colon \operatorname{Fun}(B,C) \to \operatorname{Fun}(A,C)$ is a trivial fibration.

In particular, we can now prove that function complexes between quasicategories are quasicategories.

22.4. **Theorem.** *For C a quasicategory and L a simplicial set,* $\operatorname{Fun}(L,C)$ *is a quasicategory.*

Proof. Immediate from the above remarks, but let's spell this out with a little detail. Because $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$, we have (21.8) that
$$(\Lambda^n_j \subset \Delta^n)\square(\varnothing \subseteq K) = (\Lambda^n_j \times K \to \Delta^n \times K)$$
is inner anodyne for any K and $0 < j < n$. Thus, for any diagram
$$\begin{tikzcd} \Lambda^n_j \times K \arrow[r] \arrow[d] & C \\ \Delta^n \times K \arrow[ur, dotted] & \end{tikzcd}$$
with C a quasicategory, a dotted arrow exists. By adjunction, this is the same as saying we can extend $\Lambda^n_j \to \operatorname{Fun}(K,C)$ along $\Lambda^n_j \subset \Delta^n$. That is, we have proved that $\operatorname{Fun}(K,C)$ is a quasicategory. □

There are more examples of this type of enriched lifting property, including the following which will be important for us.

22.5. **Proposition.** *We have that*

$$\overline{\mathrm{Cell}}\Box\varnothing\mathrm{Cell} \subseteq \overline{\mathrm{Cell}}, \qquad \textit{and} \qquad \mathrm{TrivFib}^{\Box\overline{\mathrm{Cell}}} \subseteq \mathrm{TrivFib}.$$

22.6. *Exercise.* Prove (22.5). (Hint: (20.5).)

22.7. *Exercise.* Show that every trivial fibration fibration admits a section.

22.8. *Remark.* Several weakly saturated classes $\overline{S}$ that appear these notes have the property that $S\Box\mathrm{Cell} \subseteq \overline{S}$, and thus analogues of the above remarks will hold for such classes. In addition to the inner anodyne maps $\overline{\mathrm{InnHorn}}$ (21.8) and monomorphisms $\overline{\mathrm{Cell}}$ (22.5), these will include anodyne maps $\overline{\mathrm{Horn}}$ (**??**), left anodyne and right anodyne maps $\overline{\mathrm{LHorn}}$ and $\overline{\mathrm{RHorn}}$ (63.2), as well as the class $\mathrm{CatEq} \cap \overline{\mathrm{Cell}}$ (41.3) of monomorphisms which are categorical equivalences (which will be defined in (24.5)).

22.9. **Composition functors.** We can use the above theory to construct "composition functors". If C is an ordinary category, the operation of composing a sequence of n maps can be upgraded to a functor

$$\mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1], C) \to \mathrm{Fun}([1], C)$$

which on *objects* describes composition of a sequence of maps. The source of this functor is the evident inverse limit in Cat of

$$\mathrm{Fun}([1], C) \xrightarrow{\langle 1\rangle^*} \mathrm{Fun}([0], C) \xleftarrow{\langle 0\rangle^*} \mathrm{Fun}([1], C),$$

which is isomorphic to $\mathrm{Fun}(I^2, C)$.

We can generalize this to quasicategories, with the proviso that the composition functor we produce is not uniquely determined. We use the following observation: any trivial fibration admits a section (22.7).

Let C be a quasicategory. Then map $r\colon \mathrm{Fun}(\Delta^2, C) \to \mathrm{Fun}(I^2, C)$ induced by restriction along $I^2 \subseteq \Delta^2$ is a trivial fibration by (22.2), since $I^2 \subset \Delta^2$ is an inner-horn inclusion. Therefore r admits a section s, so we get a diagram

$$\begin{tikzcd} \mathrm{Fun}(I^2, C) \arrow[r, bend left, "s"] & \mathrm{Fun}(\Delta^2, C) \arrow[l, "r"] \arrow[r, "r'"] & \mathrm{Fun}(\Delta^{\{0,2\}}, C) \end{tikzcd}$$

where r' is restriction along $\Delta^{\{0,2\}} \subset \Delta^2$. The composite $r's$ can be thought of as a kind of "composition" functor. It is not unique, since s isn't, but we'll see (25.13) that this is ok: all functors constructed this way are "naturally isomorphic" to each other.

The same argument gives rise to a (non-unique) "n-fold composition functor"

$$\mathrm{Fun}([1], C) \times_C \cdots \times_C \mathrm{Fun}([1], C) \to \mathrm{Fun}([1], C),$$

whose source is isomorphic to $\mathrm{Fun}(I^n, C)$, using that spine inclusions are inner anodyne (16.14).

22.10. **A useful variant.** The proof of (21.8) actually proves something a little stronger.

22.11. **Proposition** ([Joy08a, §2.3.1], [Lur09, §2.3.2])**.** *We have that* $\overline{\{\Lambda^2_1 \subset \Delta^2\}\Box\mathrm{Cell}} = \overline{\mathrm{InnHorn}}$.

Proof. We give a proof in the appendix (78.4). □

A consequence of this is another characterization of quasicategories.

22.12. **Corollary.** *A simplicial set C is a quasicategory if and only if $f\colon \mathrm{Fun}(\Delta^2, C) \to \mathrm{Fun}(\Lambda^2_1, C)$ is a trivial fibration.*

Proof. First notice that $(\partial\Delta^k \subset \Delta^k) \boxslash f$ for all $k \geq 0$ iff $(\partial\Delta^k \subset \Delta^k)\square(\Lambda^2_1 \subset \Delta^2) \boxslash (C \to *)$ for all $k \geq 0$, since $f = (C \to *)^{\square\{\Lambda^2_1 \subset \Delta^2\}}$. Therefore $f \in \mathrm{TrivFib} = \mathrm{Cell}^{\boxslash}$ if and only if $(C \to *) \in (\mathrm{Cell}\square\{\Lambda^2_1 \subset \Delta^2\})^{\boxslash}$. The conclusion immediately follows using (22.11). □

Part 4. Categorical equivalence of quasicategories

The notion of equivalence of categories is one the most important of the basic concepts in the theory of categories. When two categories are equivalent, we can for many purposes regard them as being "essentially the same", even when they are *not isomorphic*. Here we will set up the analgous notion of equivalence for quasicategories.

23. Natural isomorphisms

23.1. **Natural isomorphisms of functors.** Let C and D be quasicategories. Recall that a natural transformation between functors $f_0, f_1 \colon C \to D$ is defined to be a morphism $\alpha\colon f_0 \to f_1$ in the functor quasicategory $\mathrm{Fun}(C, D)$, or equivalently a map $\widetilde{\alpha}\colon C \times \Delta^1 \to D$ such that $\widetilde{\alpha}|C \times \{i\} = f_i$, $i = 0, 1$. **W 9 Feb**

Say that $\alpha\colon f_0 \to f_1$ is a **natural isomorphism** if a is an isomorphism in the functor quasicategory $\mathrm{Fun}(C, D)$. Thus, α is a natural isomorphism iff there exists a natural transformation $\beta\colon f_1 \to f_0$ such that $\beta\alpha \approx 1_{f_0}$ and $\alpha\beta \approx 1_{f_1}$, where "$\approx$" is homotopy between morphisms in the quasicategory $\mathrm{Fun}(C, D)$. **natural isomorphism**

This notion of natural isomorphism corresponds with the usual one for ordinary categories, since in that case homotopy of morphisms is the same as equality of morphisms.

Observe that "there exists a natural isomorphism $f_0 \to f_1$" is an equivalence relation on the set of all functors $C \to D$, as this relation precisely coincides with "there exists an isomorphism $f_0 \to f_1$" in the category $h\,\mathrm{Fun}(C, D)$. We then say that f_0 and f_1 are **naturally isomorphic** fuctors. **naturally isomorphic**

Furthermore, the "naturally isomorphic" relation is compatible with composition: if f, f' are naturally isomorphic and g, g' are naturally isomorphic, then so are gf and $g'f'$. You can read this off from the fact the operation of composition of functors extends to a functor $\mathrm{Fun}(D, E) \times \mathrm{Fun}(C, D) \to \mathrm{Fun}(C, E)$ between quasicategories, and so induces a functor

$$h\,\mathrm{Fun}(D, E) \times h\,\mathrm{Fun}(C, D) \approx h\big(\mathrm{Fun}(D, E) \times \mathrm{Fun}(C, D)\big) \to h\,\mathrm{Fun}(C, E).$$

(This uses (13.16) to identify the homotopy category of the product with the product of homotopy categories.)

23.2. **Pointwise criterion for natural isomorphisms.** Recall that if C and D are ordinary categories, a natural transformation $\alpha\colon f_0 \to f_1$ between functors $f_0, f_1\colon C \to D$ is a natural isomorphism iff and only if α is a "pointwise isomorphism" (or "objectwise isomorphism"); i.e., if for each object c of C the evident map $\alpha(c)\colon f_0(c) \to f_1(c)$ is an isomorphism in D. That natural isomorphisms are "pointwise isomorphisms" is immediate. The opposite implication follows from the fact that a natural transformation between functors of ordinary values can be completely recovered from its "values on objects". Thus, given $\alpha\colon f_0 \to f_1$ such that each $\alpha(c)\colon f_0(c) \to f_1(c)$ is an isomorphism, we may explicitly construct an inverse transformation $\beta\colon f_1 \to f_0$ by setting $\beta(c) := \alpha(c)^{-1}\colon f_1(c) \to f_0(c)$. Note that this β is in fact the *unique* inverse to α (since inverses to morphisms are unique when they exist).

One of these directions is straightforward for quasicategories.

23.3. **Proposition.** *Let C and D be quasicategories. If $\alpha\colon C \times \Delta^1 \to D$ is a natural isomorphism between functors $f_0, f_1\colon C \to D$, then for each object c of C the induced map $\alpha(c)\colon f_0(c) \to f_1(c)$ is an isomorphism in D.*

Proof. The map $\mathrm{Fun}(C, D) \to \mathrm{Fun}(\{c\}, D) = D$ induced by restriction along $\{c\} \subseteq C$ is a functor between quasicategories, so it takes isomorphisms to isomorphisms (14.3). It sends α to $\alpha(c)$. □

The converse to this proposition is also true: *A natural transformation $\alpha\colon C \times \Delta^1 \to D$ of functors between quasicategories is a natural isomorphism if and only if each of the maps $\alpha(c)$ are isomorphisms in D.* Unfortunately, this is much more subtle to prove, as it requires using the existence of inverses to the $\alpha(c)$s to produce an inverse to α, which though it exists is not at all unique. We will prove this converse later as (37.2).

23.4. *Remark.* An immediate consequence of the pointwise criterion is that if D is a *quasigroupoid*, then so is $\mathrm{Fun}(C, D)$.

23.5. *Remark.* The pointwise criterion can be reformulated in terms of homotopy categories. The homotopy category construction takes quasicategories to categories, and takes functors to functors. Furthermore, given a natural transformation $\alpha\colon f_0 \to f_1$ of functors $f_0, f_1\colon C \to D$ between quasicategories (i.e., a functor $\alpha\colon C \times \Delta^1 \to D$ such that $(\alpha|C \times \{j\}) = f_j$), we obtain an induced transformation $h\alpha\colon hf_0 \to hf_1$ of functors $hf_0, hf_1\colon hC \to hD$ between their homotopy categories (so that the value of $h\alpha$ at an object $c \in \mathrm{ob}\, hC = C_0$ is the homotopy class of the edge $\alpha(\{c\} \times \Delta^1) \subseteq D$). Then the pointwise criterion asserts that α is a natural isomorphism of functors between quasicategories if and only if $h\alpha$ is a natural isomorphism of functors between ordinary categories.

24. Categorical equivalence

We are now in position to define the correct generalization of the notion of "equivalence" of categories. This will be called *categorical equivalence* of quasicategories, and will be a direct generalization of the classical notion.

Given this, we use it to define a notion of categorical equivalence which applies to arbitrary maps of simplicial sets which may not be quasicategories. Finally, we will show that the two definitions agree for maps between quasicategories.

24.1. **Categorical equivalences between quasicategories.** A **categorical inverse** to a functor $f\colon C \to D$ between quasicategories is a functor $g\colon D \to C$ such that gf is naturally isomorphic to 1_C and fg is naturally isomorphic to 1_D. We provisionally say that a functor f between quasicategories is a **categorical equivalence** if it admits a categorical inverse.

categorical inverse

categorical equivalence

24.2. *Remark.* Categorical equivalence between quasicategories is a kind of "homotopy equivalence", where homotopies are natural isomorphisms between functors.

If C and D are nerves of ordinary categories, then natural isomorphisms between functors in our sense are precisely natural isomorphisms between functors in the classical sense, so that categorical equivalence between nerves of categories coincides precisely with the usual notion of equivalence of categories.

If quasicategories are equivalent, then their homotopy categories are equivalent.

24.3. **Proposition.** *If $f\colon C \to D$ is a categorical equivalence between quasicategories, then $h(f)\colon hC \to hD$ is an equivalence of categories.*

Proof. Immediate, given that natural isomorphisms $f \Rightarrow g\colon C \to D$ induce natural isomorphisms $h(f) \Rightarrow h(g)\colon hC \to hD$. □

Note: the converse is not at all true. For instance, there are many examples of quasicategories which are not equivalent to Δ^0, but whose homotopy categories are: e.g., $\mathrm{Sing}\, T$ for any non-contractible simply connected space T (12.9), or $K(A, d)$ for any non-trivial abelian group A and $d \geq 2$ (12.13).

24.4. *Exercise* (Categorical inverses are unique up to natural isomorphism). Let $f\colon C \to D$ be a functor between quasicategories, and suppose $g, g'\colon D \to C$ are both categorical inverses to f. Show that g and g' are naturally isomorphic.

24.5. **General categorical equivalence.** We can extend the notion of categorical equivalence to maps between arbitrary simplicial sets. Say that a map $f\colon X \to Y$ between arbitrary simplicial sets is a **categorical equivalence** if for every quasicategory C, the induced functor $\mathrm{Fun}(f, C)\colon \mathrm{Fun}(Y, C) \to \mathrm{Fun}(X, C)$ of quasicategories admits a categorical inverse. **categorical equivalence**

We claim that on functors between quasicategories this general definition of categorical equivalence coincides with the provisional notion described earlier.

24.6. **Lemma.** *For a functor $f\colon C \to D$ between quasicategories, the two notions of categorical equivalence described above coincide. That is, the following are equivalent:*

(1) *f admits a categorical inverse.*
(2) *For every quasicategory E, the functor $\mathrm{Fun}(f, E)\colon \mathrm{Fun}(D, E) \to \mathrm{Fun}(C, E)$ admits a categorical inverse.*

To prove this, we will need the following observation. The construction $X \mapsto \mathrm{Fun}(X, E)$ is a functor $s\mathrm{Set}^{\mathrm{op}} \to s\mathrm{Set}$, and so in particular induces a natural map

$$\gamma_0\colon \mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathrm{Fun}(Y, E), \mathrm{Fun}(X, E))$$

of sets, which sends $f\colon X \to Y$ to $\mathrm{Fun}(f, E)\colon \mathrm{Fun}(Y, E) \to \mathrm{Fun}(X, E)$. The observation we need is that this construction admits an "enrichment", to a map

$$\gamma\colon \mathrm{Fun}(X, Y) \to \mathrm{Fun}(\mathrm{Fun}(Y, E), \mathrm{Fun}(X, E)),$$

which coincides with γ_0 on vertices. The map γ in question is adjoint to the "composition" map $\mathrm{Fun}(X, Y) \times \mathrm{Fun}(Y, E) \to \mathrm{Fun}(X, E)$. (*Exercise:* Describe explicitly what γ does to n-dimensional cells.) We say that the functor $\mathrm{Fun}(-, E)$ is an *enriched* functor, as it gives not merely a map between hom-sets (i.e., acts on vertices in function complexes), but a map between function complexes.

Proof. (1) $\Longrightarrow$ (2). When C, D, and E are quasicategories so are the function complexes between them (22.4). In this case, the above map γ takes functors $C \to D$ to functors $\mathrm{Fun}(D, E) \to \mathrm{Fun}(C, E)$ between quasicategories, natural transformations of such functors to natural transformations, and natural isomorphisms of such functors to natural isomorphisms. Using this observation, it is straightforward to show that a categorical inverse $g\colon D \to C$ to $f\colon C \to D$ gives rise to a categorical inverse $\mathrm{Fun}(g, E)\colon \mathrm{Fun}(C, E) \to \mathrm{Fun}(D, E)$ to the induced functor $\mathrm{Fun}(f, E)\colon \mathrm{Fun}(D, E) \to \mathrm{Fun}(C, E)$.

(2) $\Longrightarrow$ (1). Conversely, suppose $f\colon C \to D$ is a categorical equivalence in the general sense, so that $f^* = \mathrm{Fun}(f, E)$ admits a categorical inverse for every quasicategory E, which implies that each functor

$$h(f^*)\colon h\,\mathrm{Fun}(D, E) \to h\,\mathrm{Fun}(C, E)$$

is an equivalence of ordinary categories (24.3). In particular, it follows that f^* induces a bijection of sets

$$\widehat{f^*}\colon \pi_0(\mathrm{Fun}(D, E)^{\mathrm{core}}) \xrightarrow{\sim} \pi_0(\mathrm{Fun}(C, E)^{\mathrm{core}}).$$

Recall that $\pi_0(\mathrm{Fun}(D, E)^{\mathrm{core}}) \approx \pi_0((h\,\mathrm{Fun}(D, E))^{\mathrm{core}})$ is precisely the set of *natural isomorphism classes* of functors $D \to E$.

Taking $E = C$, surjectivity of $\widehat{f^*}$ implies that there is a functor $g \in \mathrm{Fun}(D, C)_0$ together with a natural isomorphism $gf \to \mathrm{id}_C$ in $\mathrm{Fun}(C, C)_1$. Taking $E = D$, we note that since

$$f^*(\mathrm{id}_D) = \mathrm{id}_D f = f\,\mathrm{id}_C \approx fgf = f^*(fg),$$

injectivity of $\widehat{f^*}$ implies that $\mathrm{id}_D \approx fg$, i.e., there exists a natural isomorphism $\mathrm{id}_D \to fg$ in $\mathrm{Fun}(D, D)_1$. Thus, we have shown that g is a categorical inverse of f, as desired. $\square$

24.7. *Remark.* The definition of categorical equivalence we are using here is *very* different to the definition adopted by Lurie in [Lur09, §2.2.5]. It is also slightly different from the definition of "weak categorical equivalence" used by Joyal [Joy08a, 1.20]. Lurie adopts a definition closely related to Joyal's in `https://kerodon.net` [Lur21]. As we will show soon (27.13), weak categorical equivalence and the definition used in kerodon are equivalent to our definition of categorical equivalence. The discussion around [Lur09, 2.2.5.8] show's that Joyal's definitions is equivalent to the one used in [Lur09], and so they are both equivalent to the one we have used.

25. Trivial fibrations and inner anodyne maps

Inner anodyne maps and trivial fibrations are particular kinds of categorical equivalences.

25.1. **Trivial fibrations to the terminal simplicial set.** Recall that a trivial fibration $p\colon X \to Y$ of simplicial sets is a map such that $(\partial\Delta^k \subset \Delta^k) \boxslash p$ for all $k \geq 0$. That is, $\mathrm{TrivFib} = \mathrm{Cell}^{\boxslash}$, so p is a trivial fibration if and only if $\mathrm{Cell} \boxslash p$.

25.2. *Exercise.* Consider an indexed collection of trivial fibrations $p_i\colon X_i \to Y_i$. Show that $p := \coprod p_i\colon \coprod X_i \to \coprod Y_i$ is a trivial fibration. (Hint: see proof of (9.7).)

25.3. **Proposition.** *Let X be a simplicial set and $p\colon X \to *$ be a trivial fibration whose target is the terminal simplicial set. Then*

1. *X is a Kan complex (and thus a quasigroupoid),*
2. *for any simplicial set K, $p'\colon \mathrm{Fun}(K, X) \to *$ is a trivial fibration, and*
3. *p is a categorical equivalence.*

Proof. Since $\mathrm{Horn} \subset \overline{\mathrm{Cell}}$, it is immediate that X is a Kan complex, proving (1).

Statement (2) is an immediate consequence of enriched lifting (22.3) applied to $\overline{\mathrm{Cell}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$ (22.5), as this implies that for $i\colon \varnothing \to K$, the pullback-hom map

$$p^{\square i} = \mathrm{Fun}(K, X) \to \mathrm{Fun}(\varnothing, X) \times_{\mathrm{Fun}(\varnothing,*)} \mathrm{Fun}(K, *) \approx *$$

is a trivial fibration.

Next note that X only has one isomorphism class of objects. To show this, note that since X is a quasigroupoid, it suffices to produce for any pair of objects $a, b \in X_0$ a morphism $a \to b$. This amounts to producing a lift in

$$\begin{tikzcd} \partial\Delta^1 = \{0,1\} \arrow[r, "{(a,b)}"] \arrow[d, tail] & X \arrow[d, "p"] \\ \Delta^1 \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

which exists because $(\partial\Delta^1 \subset \Delta^1) \in \mathrm{Cell}$.

To prove (3), first note that X is non-empty, since $\mathrm{Hom}(\Delta^0, X) \to \mathrm{Hom}(\partial\Delta^0, X) = *$ is surjective. Choose any $s \in \mathrm{Hom}(\Delta^0, X)$, so $ps = \mathrm{id}_{\Delta^0}$. The composite $sp\colon X \to X$ is an object of $\mathrm{Fun}(X, X)$, and we want to show it is naturally isomorphic to id_X. But by (2), $\mathrm{Fun}(X, X) \to *$ is a trivial fibration, so all objects of $\mathrm{Fun}(X, X)$ are isomorphic. □

We will prove a partial converse to this later (40.11): *a quasicategory C is categorically equivalent to $*$ if and only if $C \to *$ is a trivial fibration.*

25.4. **Preisomorphisms.** We need a way to produce categorical equivalences between simplicial sets which are not necessarily quasicategories. **F 11 Feb**

Let X be a simplicial set. Say that an edge $a \in X_1$ is a **preisomorphism** if it projects to an isomorphism under $\alpha\colon X \to hX$, the tautological map to the (nerve of the) fundamental category (13.1). If X is actually a quasicategory, the preisomorphisms are just the isomorphisms (since in that case the fundamental category is the same as the homotopy category). Note that degenerate edges are always preisomorphisms, since they go to identity maps in the fundamental category. **preisomorphism**

25.5. **Proposition.** *An edge $a \in X_1$ is a preisomorphism if and only if for every map $g\colon X \to C$ to a quasicategory C, the image $g(a)$ is an isomorphism in C.*

Proof. Isomorphisms in C are exactly the edges which are sent to isomorphisms under $\gamma\colon C \to hC$. Given this the proof is straightforward, using the fact that the formation of fundamental categories is functorial, and that hX is itself a category and hence a quasicategory. □

As a consequence, any map $X \to Y$ of simplicial sets takes preisomorphisms to preisomorphisms. In particular, any map *from* a quasicategory takes isomorphisms to preisomorphisms. We will use this observation below.

25.6. *Example.* Consider the simplicial set Z described by the picture

$$\begin{tikzcd} y \arrow[r, "y_{00}"] \arrow[d, "g"'] \arrow[dr, phantom, "b" near start] & y \arrow[d, "h"] \\ x \arrow[r, "x_{00}"'] \arrow[ur, "f"] \arrow[ur, phantom, "a" near end, shift right=2ex] & x \end{tikzcd}$$

which shows all its non-degenerate cells: $x, y \in Z_0$, $f, g, h \in Z_1$, $a, b \in Z_2$. It is not a quasicategory. However, a map $\phi\colon Z \to C$ to a quasicategory corresponds exactly a choice of:

- objects $\phi(x), \phi(y) \in C_0$,
- morphisms $\phi(f)\colon \phi(x) \to \phi(y)$ and $\phi(g), \phi(h)\colon \phi(y) \to \phi(x)$ in C_1, and
- 2-cells $\phi(a), \phi(b) \in C_2$ exhibiting $f \sim_\ell h$ and $g \sim_\ell f$ respectively.

In particular, $\phi(g)$ is a preinverse of $\phi(f)$ and $\phi(h)$ is a postinverse of $\phi(f)$, so all of these edges are isomorphisms in C. Therefore all edges of Z are preisomorphisms.

25.7. *Example.* Here is a variant of the previous example. Consider the simplicial set Z' described by the picture

$$\begin{tikzcd} y \arrow[r, "y_{00}"] \arrow[d, "g"'] \arrow[dr, phantom, "b" near start] & y \arrow[d, "g"] \\ x \arrow[r, "x_{00}"'] \arrow[ur, "f"] \arrow[ur, phantom, "a" near end, shift right=2ex] & x \end{tikzcd}$$

with six non-degenerate cells: $x, y \in Z'_0$, $f, g \in Z'_1$, $a, b \in Z'_2$. Again, Z' is not a quasicategory.

A map $\phi\colon Z' \to C$ to a quasicategory corresponds exactly to a choice of:

- objects $\phi(x), \phi(y) \in C_0$,
- morphisms $\phi(f)\colon \phi(x) \to \phi(y)$ and $\phi(g)\colon \phi(y) \to \phi(x)$ in C_1, and
- 2-cells $\phi(a), \phi(b) \in C_2$ exhibiting $f \sim_\ell g$ and $g \sim_\ell f$ respectively.

Thus as in the previous example, every edge of Z' is a preisomorphism.

25.8. *Exercise.* Let Z and Z' be as in (25.6) and (25.7). Consider the maps $i\colon \Delta^1 \to Z$ and $i'\colon \Delta^1 \to Z'$ which in either case represent the edge labelled f. Show that an edge in a quasicategory C is an isomorphism if and only if its representing map $\Delta^1 \to C$ extends along i, and if and only if its representing map extends along i'.

Say that vertices in a simplicial set X are **preisomorphic** if they can be connected by a chain of preisomorphisms (which can point in either direction). Clearly, any map $g\colon X \to C$ to a quasicategory takes preisomorphic vertices of X to isomorphic objects of C. **preisomorphic**

We can apply this to function complexes. If two maps $f_0, f_1\colon X \to Y$ are preisomorphic (viewed as vertices in $\mathrm{Fun}(X, Y)$), then for any quasicategory C, the induced functors $\mathrm{Fun}(f_0, C), \mathrm{Fun}(f_1, C)\colon \mathrm{Fun}(Y, C) \to \mathrm{Fun}(X, C)$ are naturally isomorphic. To see this, consider

$$\Delta^1 \xrightarrow{a} \mathrm{Fun}(X, Y) \xrightarrow{b} \mathrm{Fun}(\mathrm{Fun}(Y, C), \mathrm{Fun}(X, C))$$

where b is adjoint to the composition map $\mathrm{Fun}(Y, C) \times \mathrm{Fun}(X, Y) \to \mathrm{Fun}(X, C)$. If a represents a preisomorphism $f_0 \to f_1$ in $\mathrm{Fun}(X, Y)$, then ba represents an isomorphism $\mathrm{Fun}(f_0, C) \to \mathrm{Fun}(f_1, C)$, since the target of b is a quasicategory. As a consequence we get the following.

25.9. **Lemma.** *If $f: X \to Y$ and $g: Y \to X$ are maps of simplicial sets such that gf is preisomorphic to id_X in $\mathrm{Fun}(X,X)$ and fg is preisomorphic to id_Y in $\mathrm{Fun}(Y,Y)$, then f and g are categorical equivalences.*

It is important to note that this is a sufficient condition for a map to be a categorical equivalence, but not a necessary one: there are many categorical equivalences of simplicial sets to which the lemma cannot be applied (see (26.3) below).

25.10. **Trivial fibrations are always categorical equivalences.**

25.11. **Proposition.** *Every trivial fibration between simplicial sets is a categorical equivalence.*

Here is some notation. Given maps $f: A \to Y$ and $g: B \to Y$, we write $\mathrm{Fun}_{/Y}(f,g)$ or $\mathrm{Fun}_{/Y}(A,B)$ for the simplicial set defined by the pullback square

$$\begin{tikzcd} \mathrm{Fun}_{/Y}(A,B) \arrow[r] \arrow[d] & \mathrm{Fun}(A,B) \arrow[d, "g_*=\mathrm{Fun}(A,g)"] \\ \{f\} \arrow[r] & \mathrm{Fun}(A,Y) \end{tikzcd}$$

Note that vertices of $\mathrm{Fun}_{/Y}(A,B)$ correspond exactly to "sections of g over f", i.e., to $s: A \to B$ such that $gs = f$. You can think of $\mathrm{Fun}_{/Y}(A,B)$ as a simplicial set which "parameterizes" sections of g over f. I'll call this the **relative function complex over** Y. **relative function complex over Y**

25.12. *Exercise.* Show that n-dimensional cells of $\mathrm{Fun}_{/Y}(A,B)$ correspond to maps $a: \Delta^n \times A \to B$ such that $ga = \pi(\mathrm{id} \times f)$, where $\pi: \Delta^n \times Y \to Y$ is the projection.

Proof of (25.11). Fix a trivial fibration $p: X \to S$. We regard both X and S as objects *over* S, via p and id_S, and consider various relative function complexes over S.

Note that since p is a trivial fibration, so are $\mathrm{Fun}(X,p) = p^{\Box(\varnothing\subset X)}$ and $\mathrm{Fun}(S,p) = p^{\Box(\varnothing\subset X)}$ by enriched lifting $\overline{\mathrm{Cell}}\Box\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$. The maps

$$\mathrm{Fun}_{/S}(S,X) \to \mathrm{Fun}_{/S}(S,S) = * \qquad \text{and} \qquad \mathrm{Fun}_{/S}(X,X) \to \mathrm{Fun}_{/S}(X,S) = *$$

are (by construction) base changes of $\mathrm{Fun}(S,p)$ and $\mathrm{Fun}(X,p)$ respectively, and so are also trivial fibrations since TrivFib is closed under base change. It follows from (25.3) that both $\mathrm{Fun}_{/S}(S,X)$ and $\mathrm{Fun}_{/S}(X,X)$ are quasigroupoids which are categorically equivalent to the terminal object (and so are non-empty and such that all objects are isomorphic). Note that these are isomorphic to subcomplexes of simplicial sets $\mathrm{Fun}(S,X)$ and $\mathrm{Fun}(X,X)$ respectively, which however need not be quasicategories. However all edges of $\mathrm{Fun}_{/S}(S,X)$ and $\mathrm{Fun}_{/S}(X,X)$ are necessarily preisomorphisms in $\mathrm{Fun}(S,X)$ and $\mathrm{Fun}(X,X)$.

Since $\mathrm{Fun}_{/S}(S,X) \to *$ is a trivial fibration we can pick a vertex s of $\mathrm{Fun}_{/S}(S,X)$, and this s can be regarded as a map $s: S \to X$ such that $ps = \mathrm{id}_S$. Pick any isomorphism $a: \mathrm{id}_X \to sp$ in $\mathrm{Fun}_{/S}(X,X)$, which is hence a preisomorphism in $\mathrm{Fun}(X,X)$. Thus, we have exhibited maps p and s whose composites are preisomorphic to identity functors, and therefore they are categorical equivalences by (25.9). $\Box$

25.13. *Remark* ("Uniqueness" of sections of trivial fibrations). Suppose that $p: C \to D$ is a trivial fibration between quasicategories. As we have noted, the relative function complex $\mathrm{Fun}_{/D}(D,C)$ "parameterizes sections of p". Since this is a quasigroupoid equivalent to the terminal quasicategory (25.11), not only is p a categorical equivalence, but also

- p admits a section, which is a categorical inverse to p, and
- any two sections of p are naturally isomorphic.

We will often make use of this observation.

25.14. *Exercise.* Let $p\colon C \to D$ be a functor between categories. Show that p is a trivial fibration if and only if (i) it is surjective on objects, and (ii) $\hom_C(x,y) \to \hom_D(px,py)$ is a bijection for all objects $x, y \in C_0$.

25.15. *Exercise.* Let $p\colon C \to D$ be a trivial fibration between categories. Show that $S := \mathrm{Fun}_{/D}(D,C)$ is an category which is equivalent to the terminal category, and that the set of objects of S is in bijective correspondence wtih the set of sections of the map $p\colon C_0 \to D_0$.

25.16. **Inner anodyne maps are always categorical equivalences.**

25.17. **Proposition.** *Every inner anodyne map between simplicial sets is a categorical equivalence.*

Proof. Let $j\colon X \to Y$ be a map in $\overline{\mathrm{InnHorn}}$, and let C be any quasicategory. The induced map $\mathrm{Fun}(j,C)\colon \mathrm{Fun}(Y,C) \to \mathrm{Fun}(X,C)$ is a trivial fibration by enriched lifting and $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (22.2), and therefore is a categorical equivalence. □

25.18. **Every simplicial set is categorically equivalent to a quasicategory.**

25.19. **Proposition.** *Fix a simplicial set X.*

1. *There exists a quasicategory C and an inner anodyne map $f\colon X \to C$, which is therefore a categorical equivalence.*
2. *For any two $f_i\colon X \to C_i$ as in (1), there exists a categorical equivalence $g\colon C_1 \to C_2$ such that $gf_1 = f_2$.*
3. *Any two categorical equvialences $g_1, g_2\colon C_1 \to C_2$ such that $g_if_1 = f_2$ are naturally isomorphic.*

To prove this we introduce more notation. Given maps $f\colon X \to A$ and $g\colon X \to B$, we write $\mathrm{Fun}_{X/}(f,g)$ or $\mathrm{Fun}_{X/}(A,B)$ for the simplicial set defined by the pullback square

$$\begin{tikzcd}
\mathrm{Fun}_{X/}(A,B) \arrow[r] \arrow[d] & \mathrm{Fun}(A,B) \arrow[d, "f^*=\mathrm{Fun}(f,B)"] \\
\{g\} \arrow[r] & \mathrm{Fun}(X,B)
\end{tikzcd}$$

This is the **relative function complex under** X. **relative function complex under X**

25.20. *Exercise.* Show that n-cells of $\mathrm{Fun}_{X/}(A,B)$ correspond to maps $a\colon \Delta^n \times A \to B$ such that $a(\mathrm{id}\times f) = g\pi$, where $\pi\colon \Delta^n \times X \to X$ is the projection

Proof of (25.19). (1) By the small object argument (17.16), we can factor $X \to *$ into $X \xrightarrow{j} C \xrightarrow{p} *$ where $j \in \overline{\mathrm{InnHorn}}$ and $p \in \mathrm{InnFib}$. The inner anodyne map j is the desired categorical equivalence to a quasicategory.

(2) For $i, j \in \{1,2\}$, we have a restriction map $f_{i,j}^*\colon \mathrm{Fun}(C_i,C_j) \to \mathrm{Fun}(X,C_j)$, which is necessarily a trivial fibration by enriched lifting since $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$. Therefore the maps $\mathrm{Fun}_{X/}(C_i,C_j) \to *$ (obtained by base-change from the $f_{i,j}^*$) are all trivial fibrations, i.e., each $\mathrm{Fun}_{X/}(C_i,C_j)$ is a quasigroupoid with only one isomorphism class of objects (25.3). As in the proof of (25.11) we construct $g\colon C_1 \to C_2$ and $g'\colon C_2 \to C_1$ which are categorically inverse to each other; details are left to the reader.

(3) The maps g_1, g_2 correspond to vertices in $\mathrm{Fun}_{X/}(C_1,C_2)$, which as we have observed is a quasigroupoid with only one isomorphism class of objects. □

Thus, we can always "replace" a simplicial set X by a categorically equivalent quasicategory C. Although such C is not unique, it is unique up to categorical equivalence.

You can think of such a replacement $X \to C$ of X as a quasicategory "freely generated" by the simplicial set X, an idea which is validated by the fact that $\mathrm{Fun}(j,D)\colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(X,D)$ is a trivial fibration for every quasicategory D, and so is in particular a categorical equivalence. In the next section I will look at some specific instances of this idea.

26. Some examples of categorical equivalences

26.1. **Free monoid on one generator.** Let F denote the free monoid on one generator g. This is a category with one object x, and morphism set $\{ g^n \mid n \geq 0 \}$.

Associated to the generator g is a map

$$\gamma\colon S^1 := \Delta^1/\partial\Delta^1 \to N(F)$$

sending the image of the generator $\iota \in (\Delta^1)_1$ in S^1 to g. (We use "L/K" as a shorthand for "$L \amalg_K *$" whenever $K \subseteq L$. The object S^1 is called the "simplicial circle", which has exactly two nondegenerate cells, one in dimension 0 and one in dimension 1.)

It is not hard to see that F is "freely generated" as a category by S^1, in the sense that $h(S^1) = F$ (the fundamental category of S^1 is F). It turns out that $N(F)$ is actually "freely generated as a *quasicategory*" by S^1.

26.2. **Proposition.** *The map $\gamma\colon S^1 \to N(F)$ is a categorical equivalence, and in fact is inner anodyne.*

Proof. This is an explicit calculation. Note that a general cell in $N(F)_d$ corresponds to a sequence $(g^{m_1}, \dots, g^{m_d})$ of elements of the monoid F, where $m_1, \dots, m_d \geq 0$. Let $a_k \in N(F)_k$ denote the k-cell corresponding to the sequence $(g, g, \dots, g)$, and let $Y_k \subseteq N(F)$ denote the subcomplex which is the image of the representing map $a_k\colon \Delta^k \to N(F)$. For $f\colon [d] \to [k]$ we compute that $a_k f = (g^{m_1}, \dots, g^{m_d})$ where $m_i = f(i) - f(i-1)$, so that

$$(Y_k)_d = \{ a_k f \mid f\colon [d] \to [k] \} = \{ (g^{m_1}, \dots, g^{m_d}) \mid m_1 + \dots + m_d \leq k \},$$

Clearly $Y_{k-1} \subseteq Y_k$ for all k and $N(F) = \bigcup_{k\geq 1} Y_k$, with $Y_1 \approx S^1$ and $Y_2 \approx Y_1 \cup_{\Lambda^2_1} \Delta^2$. Furthermore we have the following:

- A simplicial operator $f\colon [d] \to [k]$ (i.e., element of $(\Delta^k)_d$) is such that $a_k f$ is in the subcomplex Y_{k-1} of Y_k if and only if $f(d) - f(0) < k$, if and only if either $f(d) < k$ or $f(0) > 0$, i.e., if and only if f is in the subcomplex $\Lambda^k_{\{0,k\}} = \Delta^{\{0,\dots,k-1\}} \cup \Delta^{\{1,\dots,k\}}$ of Δ^k.
- Every cell y of Y_k not in Y_{k-1} is the image under a_k of a *unique* cell in Δ^k. That is, for $f\colon [d] \to [k]$, we have $m_1 + \dots + m_d = f(d) - f(0)$, which is equal to k if and only if $f(0) = 0$ and $f(d) = k$, and if this is the case then $f(i) = m_1 + \dots + m_i$, so f is uniquely determined by $a_k f = (g^{m_1}, \dots, g^{m_d})$.

In other words, the square

$$\begin{tikzcd}
\Lambda^k_{\{0,k\}} \arrow[r] \arrow[d, tail] & Y_{k-1} \arrow[d, tail] \\
\Delta^k \arrow[r, "a_k"'] & Y_k
\end{tikzcd}$$

is a pullback, and a_k induces in each dimension d a bijection $(\Delta^k)_d \smallsetminus (\Lambda^k_{\{0,k\}})_d \xrightarrow{\sim} (Y_k)_d \smallsetminus (Y_{k-1})_d$. It follows (20.4) that the square is a pushout.

The inclusion $\Lambda^k_{\{0,k\}} \subset \Delta^k$ is a *generalized inner horn*, and we have noted this is inner anodyne when $k \geq 2$ (16.12). It follows that each $Y_{k-1} \to Y_k$ is inner anodyne for $k \geq 2$, whence $S^1 \to N(F)$ is inner anodyne, since it is a transfinite composition of the $Y_{k-1} \to Y_k$. $\square$

26.3. *Remark.* This gives an explicit example of a categorical equivalence to which (25.9) does not apply: γ does not admit an "inverse up to preisomorphims". There is only one map $\delta\colon N(F) \to S^1$, namely the composite $N(F) \to * \to S^1$, and it is clear that neither $\gamma\delta\colon N(F) \to N(F)$ nor $\delta\gamma\colon S^1 \to S^1$ are preisomorphic to identity functors.

26.4. **Free categories.** We can generalize the above to free monoids with arbitrary sets of generators, and in fact to free categories. Let S be a **1-dimensional** simplicial set, i.e., one such that $S = \mathrm{Sk}_1 S$. These are effectively the same thing as *quivers* (i.e., directed graphs which are allowed to have parallel edges and loops): S_0 corresponds to the set of vertices of the quiver, and S_1^{nd} corresponds to the set of edges of the quiver. **1-dimensional**

Let $F := hS$. We call F the **free category** on the 1-dimensional simplicial set S. In this case, the morphisms of the fundamental category are precisely the words in the edges S_1^{nd} of the quiver (including empty words for each vertex, corresponding to identity maps). That is, it is precisely the free category described in the proof of (13.2). **free category**

26.5. **Proposition.** *The evident map $\gamma\colon S \to N(F)$ is a categorical equivalence, and in fact is inner anodyne.*

Proof. This is virtually the same as the proof of (26.2). In this case, $Y_k \subseteq N(F)$ is the subcomplex generated by all $a\colon \Delta^k \to N(F)$ such that each spine-edge $a_{i-1,i}$ is in S_1^{nd}, and Y_k is obtained by attaching a generalized horn of type $\Lambda^k_{\{0,k\}} \subset \Delta^k$ to Y_{k-1} for each such a. □

As a consequence, it is "easy" to construct functors $F \to C$ from a free category to a quasicategory: start with a map $S \to C$, which amounts to specifying vertices and edges in C corresponding to elements S_0 and S_1^{nd}, and extend over $S \subseteq F$. The evident restriction map $\mathrm{Fun}(F, C) \to \mathrm{Fun}(S, C)$ is a categorical equivalence, and in fact a trivial fibration. In other words, free categories are also "free quasicategories".

26.6. *Exercise.* Describe the ordinary category $A := h\Lambda^3_0$ "freely generated" by Λ^3_0. Show that the tautological map $\Lambda^3_0 \to N(A)$ is inner anodyne.

26.7. **Free commutative monoids.** Let F be the free monoid on one generator again, with generator corresponding to simplicial circle $S^1 = \Delta^1/\partial\Delta^1 \subset N(F)$. Thus $F^{\times n}$ is the free commutative monoid on n generators. Recall that the nerve functor preserves products, so $N(F^{\times n}) \approx N(F)^{\times n}$. We obtain a map **M 14 Feb**

$$\delta = \gamma^{\times n}\colon (S^1)^{\times n} \to N(F^{\times n})$$

from the "simplicial n-torus".

26.8. **Proposition.** *The map $\delta\colon (S^1)^{\times n} \to N(F^{\times n})$ is a categorical equivalence, and in fact is inner anodyne.*

Proof. This is a consequence of the fact that if $j\colon A \to B$ is inner anodyne and K an arbitrary simplicial set, then $j \times K \times\colon A \times K \to B \times K$ is inner anodyne (because $\overline{\mathrm{InnHorn}}\Box\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$). It follows that $A^{\times n} \to B^{\times n}$ is a composite of inner anodyne maps, and so is inner anodyne and thus a categorical equivalence (25.17). Also use the fact that the nerve construction preserves products (9.5), so $N(F^{\times n}) = N(F)^{\times n}$. □

26.9. *Exercise.* Let $S^1 \vee S^1 \subset (S^1)^{\times 2}$ be the subcomplex obtained as the evident "one-point union" of the two "coordinate circles"; i.e., $S^1 \vee S^1 = (S^1 \times \{*\}) \cup (\{*\} \times S^1)$. Suppose given a map $\phi\colon S^1 \vee S^1 \to C$ to a quasicategory C, corresponding to a choice of object $x \in C_0$ together with two morphisms $f, g\colon x \to x$ in C_1. Show that there exists an extension of ϕ along $S^1 \vee S^1 \subset N(F^{\times 2})$ if and only if $[f][g] = [g][f]$ in hC.

26.10. *Remark.* The analogue of the above exercise for $n = 3$ isn't true. That is, consider the subcomplex $S^1 \vee S^1 \vee S^1 \subset (S^1)^{\times 3}$ which is a one-point union of three circles, suppose we have $S^1 \vee S^1 \vee S^1 \to C$ corrposding to three morphisms $f, g, h\colon x \to x$ in C, and suppose we also know that $[f][g] = [g][f]$, $[g][h] = [h][g]$, and $[f][h] = [h][f]$ in hC. Then you can show that there exists an extension to a map $K \to C$ as in (26.9), where $K \subseteq (S^1)^{\times 3}$ is the subcomplex $(S^1 \times S^1 \times \{*\}) \cup (S^1 \times \{*\} \times S^1) \cup (\{*\} \times S^1 \times S^1)$. However, there need not exist an extension to a map $(S^1)^{\times 3} \to C$, and thus there may not exist an extension to a map $N(F^{\times 3}) \to C$. (For an explicit example where this fails, take $C = \operatorname{Sing} T$, where $T \subseteq (S^1_{\text{top}})^{\times 3}$ is the subspace of the topological 3-torus consisting of tuples (x_1, x_2, x_3) such that at least one x_i is the basepoint of S^1_{top}.)

Thus, this is a situation where the "higher structure" of a quasicategory plays a role. When C is an ordinary category, it is easy to show that the desired extension does always exist. However, for a general quasicategory C, three pairwise-commuting endomorphisms of an object do not generally give rise to a functor $N(F^{\times 3}) \to C$ from the free commutative monoid on 3 generators.

26.11. **Finite groups are not finitely generated.** If A is any ordinary category, then $\operatorname{Sk}_2 N(A)$ "freely generates $N(A)$ as a category", in the sense that $h(\operatorname{Sk}_2 N(A)) \approx A$, or equivalently that $\operatorname{Fun}(N(A), N(B)) \to \operatorname{Fun}(\operatorname{Sk}_2 N(A), N(B))$ is an isomorphism for any category B. However, it is often the case that *no finite dimensional simplicial set* "freely generates $N(A)$ as a quasicategory". In fact, this is the case for every non-trivial finite group.

26.12. *Example.* Let G be the finite group of order 2, with generator g, and consider $C = NG$, the nerve of G viewed as a category with one object. Then C has exactly one non-degenerate cell in each dimension:

$$C_k^{\text{nd}} = \{a_k := (g, \dots, g)\}.$$

Thus, to give a map $\phi\colon C \to D$ to some other quasicategory requires describing the infinite list of data $\phi(a_k) \in D_k$, $k \geq 0$, which must necessarily satisfy a number of compatibilities in order that ϕ. It turns out that there is no way to reduce this to a finite list of data and conditions, in the following sense: there is *no finite dimensional simplicial set* which is categorically equivalent to C.

To prove this, relies on some topology, together with a fact to be proved later[19]: if $f\colon X \to Y$ is any categorical equivalence of simplicial sets, then the induced map $\|f\|: \|X\| \to \|Y\|$ of geometric realizations must be a homotopy equivalence of spaces. The realization $\|NG\|$ is the *classifying space* of G. The topology we need is that the cohomology of this space is non-zero in arbitrarily large dimensions: $H^{2k}(\|NG\|, \mathbb{Z}) \approx \mathbb{Z}/2 \not\approx 0$ for all $k > 0$. On the other hand, if X is a simplicial set with $X = \operatorname{Sk}_d X$, then $H^k(\|X\|, \mathbb{Z}) = 0$ for all $k > d$.

A similar observation applies for any non-trivial *finite* group G. Thus, non-trivial finite groups are never "freely generated as a quasicategory" by finite dimensional complexes. In the world of ∞-categories, finite groups are fundamentally infinite dimensional objects.

27. The homotopy category of quasicategories

27.1. **The homotopy category of** qCat**.** The homotopy category $h\text{qCat}$ of quasicategories is defined as follows. The objects of $h\text{qCat}$ are the quasicategories. Morphisms $C \to D$ in $h\text{qCat}$ are natural isomorphism classes of functors. That is,

$$\operatorname{Hom}_{h\text{qCat}}(C, D) := \text{isomorphism classes of objects in } h\operatorname{Fun}(C, D) = \pi_0\big(\operatorname{Fun}(C, D)^{\text{core}}\big).$$

That this defines a category results from the fact that composition of functors passes to a functor $h\operatorname{Fun}(D, E) \times h\operatorname{Fun}(C, D) \to h\operatorname{Fun}(C, E)$, and thus is compatible with natural isomorphism.

It comes with an obvious functor $\text{qCat} \to h\text{qCat}$. Note that a map $f\colon C \to D$ of quasicategories is a categorical equivalence if and only if its image in $h\text{qCat}$ is an isomorphism.

27.2. *Remark.* We can similarly define a category $h\text{Cat}$, whose objects are ordinary categories and whose morphisms are *isomorphism classes* of functors. The nerve functor evidently induces a full embedding $h\text{Cat} \to h\text{qCat}$.

27.3. *Warning.* Although we use the phrase "homotopy category", the definition of $h\text{qCat}$ given above is not an example of the notion of the homotopy category of a quasicategory defined in (13): qCat is a (large) ordinary category, so is isomorphic to its own homotopy category in that

[19]I don't know if this will actually get proved later. It is proved in [GJ09].

sense. Here we are using the equivalence relation on morphisms(=functors) defined by natural isomorphism.

We define $h\mathrm{Kan} \subset h\mathrm{qCat}$ to be the full subcategory of the homotopy category spanned by quasicategories which are Kan complexes.

For future reference, we note that $h\mathrm{qCat}$ and $h\mathrm{Kan}$ have finite products, which just amount to the usual products of simplicial sets.

27.4. **Proposition.** *The terminal simplicial set Δ^0 is a terminal object in $h\mathrm{qCat}$. If C_1, C_2 are quasicategories, then the projection maps exhibit $C_1 \times C_2$ as a product in $h\mathrm{qCat}$.*

Proof. This is straightforward. The key observation for the second statement is the fact that isomorphism classes of objects in a product of quasicategories correspond to pairs of isomorphism classes in each (9.12), and the fact that $\mathrm{Fun}(X, C_1 \times C_2) \xrightarrow{\sim} \mathrm{Fun}(X, C_1) \times \mathrm{Fun}(X, C_2)$. □

27.5. *Exercise* (Products of categorical equivalenes). Let $f\colon X \to Y$ and $f'\colon X' \to Y'$ be categorical equivalences of simplicial sets. Show that $f \times f'\colon X \times X' \to Y \times Y'$ is a categorical equivalence. (Hint: reduce to the case where one of the maps is identity.)

27.6. **The 2-out-of-6 and 2-out-of-3 properties.** A class of morphisms $\mathcal{W}$ in a category is said to satisfy the **2-out-of-6 property** if (i) $\mathcal{W}$ contains all identity maps, and (ii) given sequence (h, g, f) of maps such that the composites gf and hg are defined, if $gf, hg \in \mathcal{W}$ then also $f, g, h, hgf \in W$. **2-out-of-6 property**

A class of morphisms $\mathcal{W}$ in a category is said to satisfy the **2-out-of-3 property** if (i) $\mathcal{W}$ contains all identity maps, and (ii) given a sequence (g, f) of maps such that the composite gf is defined, if any two of (f, g, gf) are in $\mathcal{W}$, so is the third. **2-out-of-3 property**

27.7. *Example.* In any category, the class of isomorphisms satisfies 2-out-of-6 property and the 2-out-of-3 property. The class of identity maps satisfies 2-out-of-3, but does not generally satisfy 2-out-of-6. In fact, the class of isomorphisms is the *smallest* class which satisfies 2-out-of-6.

27.8. **Proposition.** *If $\mathcal{W}$ satisfies 2-out-of-6, then it satisfies 2-out-of-3.*

Proof. Given f, g such that gf is defined, apply 2-out-of-6 to the composable sequences (id, g, f), (g, id, f), (g, f, id). □

27.9. *Exercise.* Given a functor $f\colon C \to D$ between categories, let $\mathcal{W}$ be the class of maps in C that f takes to isomorphisms in D. Show that $\mathcal{W}$ satisfies 2-out-of-6, and thus 2-out-of-3.

27.10. *Example* (2-out-of-6 for equivalences of categories). In Cat, the category of small categories and functors, the class of equivalences satisfies 2-out-of-6, and thus 2-out-of-3.

To see this, first suppose (h, g, f) is a triple of functors such that there are natural isomorphisms $gf \approx \mathrm{id}$ and $hg \approx \mathrm{id}$. Then, since (i) natural isomorphism is an equivalence relation on functors and (ii) is compatible with composition, we see that

$$h = h\,\mathrm{id} \approx h(gf) = (hg)f \approx \mathrm{id}\, f = f,$$

and thus that g is an equivalence since $hg \approx \mathrm{id}$ and $gh \approx gf \approx \mathrm{id}$.

Next, note that composites of equivalences are equivalences, by a straightforward argument: if g and f are equivalences and composable, and g' and f' are categorical inverses to them, then $f'g'$ is easily seen to be a categorical inverese to gf.

Now suppose that (h, g, f) are such that gf and hg are categorical equivalences. Choose categorical inverses u and v for these, so that

$$gfu \approx \mathrm{id}, \quad ugf \approx \mathrm{id}, \quad hgv \approx \mathrm{id}, \quad vhg \approx \mathrm{id}\,.$$

Apply the above remarks to the triples (ug, f, ug), (vh, g, fu), (gv, h, gv), and (ugv, hgf, vgu) to show that f, g, h are equivalences, where we use that

$$fug \approx (vhg)fug = vh(gfu)g \approx vhg \approx \mathrm{id}, \quad gvh \approx gvh(gfu) = g(vhg)fu \approx gfu \approx \mathrm{id}\,.$$

It follows that the composite hgf is also an equivalence.

Alternately, we can apply (27.9) to the tautological functor $\mathrm{Cat} \to h\mathrm{Cat}$, which sends a functor to an isomorphism in $h\mathrm{Cat}$ if and only if it is an equivalence.

27.11. **Proposition.** *The class* CatEq *of categorical equivalences in* sSet *satisfies 2-out-of-6, and thus 2-out-of-3.*

Proof. It is immediate that the identity map of a simplicial set is a categorical equivalence.

Next consider functors f, g, h between *quasicategories* such that gf and hg are are defined and are categorical equivalences. Then f, g, h and hgf are categorical equivalences by an argument which is word-for-word the same as in (27.10).

For the general case, we reduce to the quasicategory case by applying $\mathrm{Fun}(-, C)$, where C is an arbitrary quasicategory. □

27.12. **Other characterizations of categorical equivalence.** It turns out that we can replace the condition in the definition of categorical equivalence with some seemingly weaker conditions.

27.13. **Proposition.** *Let $f: X \to Y$ be a map of simplicial sets. The following are equivalent.*

(1) *The map f is a categorical equivalence: i.e., for every quasicategory C, the functor $\mathrm{Fun}(Y, C) \to \mathrm{Fun}(X, C)$ induced by restriction along f admits a categorical inverse.*
(2) *For every quasicategory C, the map $h\,\mathrm{Fun}(Y, C) \to h\,\mathrm{Fun}(X, C)$ induced by restriction along f is an equivalence of ordinary categories.*
(3) *For every quasicategory C, the map $\pi_0(\mathrm{Fun}(Y, C)^{\mathrm{core}}) \to \pi_0(\mathrm{Fun}(X, C)^{\mathrm{core}})$ induced by restriction along f is a bijection of sets.*

Proof. (1) ⇒ (2) is immediate from (24.3), while (2) ⇒ (3) is immediate, since an equivalence of ordinary categories induces a bijection on isomorphism classes of objects. We prove that (3) implies (1).

In the case that f is a map between *quasicategories*, this is really what the second half of the proof of (24.6) actually shows. That is, we let C be either X or Y, and use the bijections

$$\widehat{f}^*: \pi_0(\mathrm{Fun}(Y, X)^{\mathrm{core}}) \xrightarrow{\sim} \pi_0(\mathrm{Fun}(X, X)^{\mathrm{core}}), \qquad \widehat{f}^*: \pi_0(\mathrm{Fun}(Y, Y)^{\mathrm{core}}) \xrightarrow{\sim} \pi_0(\mathrm{Fun}(X, Y)^{\mathrm{core}}),$$

to (a) produce a $g: Y \to X$ such that $gf \approx \mathrm{id}_X$, and (b) show that $fgf \approx f\,\mathrm{id}_X = \mathrm{id}_Y f$ implies $fg \approx \mathrm{id}_Y$.

We reduce the case of a general map f to that of a map f' between quasicategories as follows. Use factorization to construct a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

so that u and v are inner anodyne (and so categorical equivalences), and X' and Y' are quasicategories. If we apply $\mathrm{Fun}(-, C)$ to the square with C a quasicategory, the vertical maps become trivial fibrations, and hence induce bijections on isomorphism classes of objects. Therefore $\mathrm{Fun}(f, C)$ induces a bijection on isomorphism classes of objects if and only if $\mathrm{Fun}(f', C)$ does. □

Joyal [Joy08a, 1.20] singles out statement (2) of (27.13) as his basic notion of equivalence, which he calls **weak categorical equivalence**[20]. In kerodon.net Lurie singles out statement (3) as the basic notion of equivalence. We see that either of these are equivalent to the definition of categorical equivalence we are using. **weak categorical equivalence**

[20]This is not to be confused with "weak homotopy equivalence", which we will talk about later (52).

Finally, we note one more criterion for categorical equivalence between quasicategories, which is in some sense "dual" to the one given above.

27.14. **Proposition.** *Let $f: C \to D$ be a functor between quasicategories. The following are equivalent*

(1) *The map f is a categorical equivalence.*
(2) *For every simplicial set X, the map $f_*: \mathrm{Fun}(X, C) \to \mathrm{Fun}(Y, D)$ induced by composition with f is a categorical equivalence.*
(3) *For every simplicial set X, the map $(hf)_*: h\,\mathrm{Fun}(X, C) \to h\,\mathrm{Fun}(X, D)$ induced by composition with f is an equivalence of categories.*
(4) *For every simplicial set X, the map $\pi_0(\mathrm{Fun}(X, C)^{\mathrm{core}}) \to \pi_0(\mathrm{Fun}(X, D)^{\mathrm{core}})$ induced by composition with f is a bijection of sets.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are straightforward. So we need to prove (4) implies (1). The proof is analogous to the argument of the proof of (27.13): let X be either C or D, and use the bijections

$$\widehat{f}_*: \pi_0(\mathrm{Fun}(D, C)^{\mathrm{core}}) \xrightarrow{\sim} \pi_0(\mathrm{Fun}(D, D)^{\mathrm{core}}), \qquad \widehat{f}_*: \pi_0(\mathrm{Fun}(C, C)^{\mathrm{core}}) \xrightarrow{\sim} \pi_0(\mathrm{Fun}(C, D)^{\mathrm{core}})$$

to (a) produce a $g: D \to C$ such that $fg \approx \mathrm{id}_D$, and (b) show that $fgf \approx \mathrm{id}_D f = f \,\mathrm{id}_C$ implies $gf \approx \mathrm{id}_C$. □

27.15. **The homotopy 2-category of** qCat**.** A **2-category** E is a category which is itself "enriched" over Cat. That is, **2-category**

- for each pair of objects $x, y \in \mathrm{ob}\, E$, there is a category $\underline{\mathrm{Hom}}_E(x, y)$, so that
- the objects of $\underline{\mathrm{Hom}}_E(x, y)$ are precisely the set $\mathrm{Hom}_E(x, y)$ of morphisms of E, and
- there are "composition functors" $\underline{\mathrm{Hom}}_E(y, z) \times \underline{\mathrm{Hom}}_E(x, y) \to \underline{\mathrm{Hom}}_E(x, z)$ for all $x, y, z \in \mathrm{ob}\, E$ which on objects is just ordinary composition of morphisms in E, which
- is unital and associative in the evident sense.

One refers to the objects of $\underline{\mathrm{Hom}}_E(x, y)$ as **1-morphisms** $f: x \to y$ of E, and the morphisms of $\underline{\mathrm{Hom}}_E(x, y)$ as **2-morphisms** $\alpha: f \Rightarrow g$ of E. The **underlying category** of E consists of the objects and 1-morphisms only. **1-morphisms** **2-morphisms** **underlying category**

The standard example of a 2-category is Cat, the category of categories, with objects=categories, 1-morphisms=functors, 2-morphisms=natural transformations.

We can enlarge the category qCat of quasicategories to a **homotopy 2-category** $h_2\mathrm{qCat}$, so that **homotopy 2-category**

$$\mathrm{Hom}_{h_2\mathrm{qCat}}(C, D) := h\,\mathrm{Fun}(C, D).$$

That is,

- **objects** of $h_2\mathrm{qCat}$ are quasicategories,
- **1-morphisms** of $h_2\mathrm{qCat}$ are functors between quasicategories,
- **2-morphisms** of $h_2\mathrm{qCat}$ are *isomorphism classes* of natural transformations of functors.

Note that qCat sits inside $h_2\mathrm{qCat}$ as its underlying category. Thus, $h_2\mathrm{qCat}$ contains all the information of qCat. On the other hand $h\mathrm{qCat}$ is obtained from $h_2\mathrm{qCat}$ by first identifying 1-morphisms (functors) which are 2-isomorphic (i.e., naturally isomorphic), and then throwing away the 2-morphisms. Thus, $h_2\mathrm{qCat}$ contains all the information of $h\mathrm{qCat}$.

27.16. *Exercise* (for topologists). Define a *homotopy 2-category* $h_2\mathrm{Top}$ of spaces, so that objects are topological spaces, 1-morphisms are continuous maps, and 2-morphisms are "homotopy classes of homotopies" between maps.

Part 5. Joins, slices, and Joyal's extension and lifting theorems

In this part we describe and apply two methods to construct new quasicategories from old, called "joins" and "slices". They are both generalizations of constructions which can be carried out on categories: the most familiar of these classical constructions is *slice category* $C_{/x}$ associated to an object x of a category C, in which objects of the slice $C_{/x}$ are morphisms $c \to x$ in C, and morphisms of $C_{/x}$ are commutative triangles in C.

With these constructions in hand, we will be able to define notions of *limit* and *colimit* of a functor to a quasicategory. We will also be able to prove some of the results we have deferred up until now, including the equivalence of quasigroupoids and Kan complexes (35.2) and the pointwise criterion for natural isomorphisms (37.2). Much of the material in this part comes from Joyal's seminal paper [Joy02].

28. Joins

28.1. **Join of categories.** If A and B are ordinary categories, we can define a category $A \star B$ called the join. This has **W 16 Feb**

$$\mathrm{ob}(A \star B) = \mathrm{ob}\, A \amalg \mathrm{ob}\, B, \qquad \mathrm{mor}(A \star B) = \mathrm{mor}\, A \amalg (\mathrm{ob}\, A \times \mathrm{ob}\, B) \amalg \mathrm{mor}\, B,$$

so that we put in a *unique* map from each object of A to each object of B. Explicitly,

$$\mathrm{Hom}_{A\star B}(x,y) := \begin{cases} \mathrm{Hom}_A(x,y) & \text{if } x,y \in \mathrm{ob}\, A, \\ \mathrm{Hom}_B(x,y) & \text{if } x,y \in \mathrm{ob}\, B, \\ \{*\} & \text{if } x \in \mathrm{ob}\, A,\ y \in \mathrm{ob}\, B, \\ \varnothing & \text{if } x \in \mathrm{ob}\, B,\ y \in \mathrm{ob}\, A, \end{cases}$$

with composition defined so that the evident inclusions $A \to A \star B \leftarrow B$ are functors, and in fact are isomorphisms to full subcategories of $A \star B$. (Check that this really defines a category, and that A and B are identified with full subcategories of $A \star B$.)

28.2. *Example.* We have that $[p] \star [q] \approx [p+1+q]$.

28.3. *Exercise* (Functors from a join of categories). Show that functors $f\colon A \star B \to C$ are in bijective correspondence with triples $(f_A\colon A \to C,\ f_B\colon B \to C,\ \gamma\colon f_A \circ \pi_A \Rightarrow f_B \circ \pi_B)$, where f_A and f_B are functors, and γ is a natural transformation of functors $A \times B \to C$, where $\pi_A\colon A \times B \to A$ and $\pi_B\colon A \times B \to B$ denote the evident projection functors.

28.4. *Exercise* (Functors to a join of categories). Show that functors $f\colon C \to A \star B$ are in bijective correspondence with triples of functors $(\pi\colon C \to [1],\ f_{\{0\}}\colon C^{\{0\}} \to A,\ f_{\{1\}}\colon C^{\{1\}} \to B)$, where $C^{\{j\}} := \pi^{-1}(\{j\}) \subseteq C$ is the fiber of π over $j \in \mathrm{ob}[1]$, i.e., the subcategory of C consisting of objects which π sends to j and morphisms which π sends to id_j.

28.5. *Exercise.* Describe an isomorphism $(A \star B)^{\mathrm{op}} \approx B^{\mathrm{op}} \star A^{\mathrm{op}}$.

28.6. **Cones on categories.** An important special case are the *left cone* and *right cone* of a category, defined by $A^{\triangleleft} := [0] \star A$ and $A^{\triangleright} := A \star [0]$. For instance, the right cone $A^{\triangleright}$ is the category obtained by adjoining one additional object v to A, as well as a unique map $x \to v$ for each object x of $A^{\triangleright}$. In this case, v becomes a terminal object for $A^{\triangleright}$, and we can say that $A \mapsto A^{\triangleright}$ *freely adjoins a terminal object* to A. (Note that a terminal object of A will not be terminal in $A^{\triangleright}$ anymore.) Likewise, $A \mapsto A^{\triangleleft}$ *freely adjoins an initial object* to A.

Limits and colimits of functors can be characterized using cones: if $f\colon A \to C$ is a functor, a *colimit* of f is a functor $\widehat{f}\colon A^{\triangleright} \to C$ which is initial among functors which extend f, and likewise, a *limit* of f is a functor $\widehat{f'}\colon A^{\triangleleft} \to C$ which is terminal among functors which extend f.

28.7. *Remark.* It is worthwhile to spell this out in detail. Given a functor $f\colon A \to C$, to describe a functor $g\colon A^{\triangleright} \to C$ which *extends* f, it suffices to give

(1) an object $g(v)$ in C,
(2) for each object $a \in \operatorname{ob} A$ a morphism $g(a \to v)\colon f(a) = g(a) \to g(v)$ in C, such that
(3) for each morphism $\alpha\colon a \to a'$ in A we have an equality $g(a' \to v) \circ f(\alpha) = g(a \to v)$ of morphisms $f(a) \to g(v)$ in C.

$$\begin{tikzcd} a \arrow[dd, "\alpha"'] & & f(a) \arrow[dr, "g(a\to v)"] \arrow[dd, "f(\alpha)"'] & \\ & \Longrightarrow & & g(v) \\ a' & & f(a') \arrow[ur, "g(a'\to v)"'] & \end{tikzcd}$$

Given extensions $g, g'\colon A^{\triangleright} \to C$ of f, we may consider natural transformations $\phi\colon g \to g'$ which *extend the identity transformation* of f. Explicitly, such a transformation ϕ is exactly determined by

(1) a morphism $\phi(v)\colon g(v) \to g'(v)$ in C such that
(2) for each object $a \in \operatorname{ob} A$ we have an equality $g'(a \to v) = \phi(v) \circ g(a \to v)$ of morphisms $f(a) \to g'(v)$ in C.

$$\begin{tikzcd} & & & g(v) \arrow[dd, "\phi(v)"] \\ a & \Longrightarrow & f(a) \arrow[ur, "g(a\to v)"] \arrow[dr, "g'(a\to v)"'] & \\ & & & g'(v) \end{tikzcd}$$

An extension $\widehat{f}\colon A^{\triangleright} \to C$ of f is a *colimit* of f if for every g extending f there exists a unique map $\phi(v)\colon \widehat{f}(v) \to g(v)$ in C such that $g(a \to v) = \phi(v) \circ \widehat{f}(a \to v)$ for all $a \in \operatorname{ob} A$. The object $\widehat{f}(v)$ is what is colloquially known as "the colimit of f", although the full data of a colimit of f is actually the functor $\widehat{f}$. We will call the functor $\widehat{f}$ a *colimit cone* in what follows.

28.8. **Ordered disjoint union.** As noted above (28.2), the join operation on categories effectively descends to Δ. We will call this the **ordered disjoint union**. It is a functor $\sqcup\colon \Delta \times \Delta \to \Delta$, **ordered disjoint union** defined so that $[p] \sqcup [q] := [p + 1 + q]$, to be thought of as the disjoint union of underlying sets, ordered so that the subsets $[p]$ and $[q]$ retain their ordering, and elements of $[p]$ come before elements of $[q]$.

It is handy to extend this to the category Δ_+, the full subcategory of ordered sets obtained by adding the empty set $[-1] := \varnothing$ to Δ. The functor $\sqcup$ extends in an evident way to $\sqcup\colon \Delta_+ \times \Delta_+ \to \Delta_+$.

This extended functor makes Δ_+ into a (strict, but nonsymmetric) monoidal category, with unit object $[-1]$.

Note that for any map $f\colon [p] \to [q_1] \sqcup [q_2]$ in Δ_+, there is a *unique* decomposition $[p] = [p_1] \sqcup [p_2]$ such that $f = f_1 \sqcup f_2$ for some (necessarily unique) $f_i\colon [p_i] \to [q_i]$ in Δ_+. (We need an object $[-1]$ to be able to say this, even if $p, q_1, q_2 \geq 0$; if $f([p]) \subseteq [q_1]$ then $p_2 = -1$.)

28.9. **Join of simplicial sets.** Let X and Y be simplicial sets. The **join** of X and Y is a simplicial **join** set $X \star Y$ defined as follows. It has n-dimensional cells

$$(X \star Y)_n := \coprod_{[n]=[n_1]\sqcup[n_2]} X_{n_1} \times Y_{n_2},$$

where $[n_1], [n_2] \in \operatorname{ob} \Delta_+$, and we declare $X_{-1} = * = Y_{-1}$ to be a one-point set. The action of simplicial operators is defined in the evident way, using the observation of the previous paragraph: for $(x, y) \in X_{n_1} \times Y_{n_2} \subseteq (X \star Y)_n$ and $f\colon [m] \to [n]$, we have $(x, y)f = (xf_1, yf_2) \in X_{m_1} \times Y_{m_2} \subseteq (X \star Y)_m$, where $f = f_1 \sqcup f_2$, $f_j\colon [m_j] \to [n_j]$ is the unique decomposition of f over $[n] = [n_1] \sqcup [n_2]$.

28.10. *Exercise.* Check that the above defines a simplicial set.

In particular,

$$\begin{aligned}(X \star Y)_0 &= X_0 \amalg Y_0, \\ (X \star Y)_1 &= X_1 \amalg X_0 \times Y_0 \amalg Y_1, \\ (X \star Y)_2 &= X_2 \amalg X_1 \times Y_0 \amalg X_0 \times Y_1 \amalg Y_2,\end{aligned}$$

and so on.

Note that there are evident maps $X \to X \star Y \leftarrow Y$, which give isomorphisms from X and Y to subcomplexes of $X \star Y$, and these subcomplexes are disjoint from each other, so that we have a monomorphism $X \amalg Y \rightarrowtail X \star Y$.

There are isomorphisms

$$(X \star Y) \star Z \xrightarrow{\sim} X \star (Y \star Z),$$

natural in X, Y, Z: on either side, the set of n-cells can described as $\coprod_{[n]=[n_1]\sqcup[n_2]\sqcup[n_3]} X_{n_1} \times Y_{n_2} \times Z_{n_3}$. Together with the evident isomorphisms $\varnothing \star X \approx X \approx X \star \varnothing$, the join gives a monoidal structure on sSet with unit object $\Delta^{-1} := \varnothing$. Note that $\star$ is *not* symmetric monoidal, though it is true that $(Y \star X)^{\mathrm{op}} \approx X^{\mathrm{op}} \star Y^{\mathrm{op}}$. (*Exercise:* verify this.)

28.11. **Joins of simplices.** We have the (unique) isomorphism

$$\Delta^p \star \Delta^q \approx \Delta^{p+1+q}.$$

Furthermore, if $f\colon [p'] \to [p]$ and $g\colon [q'] \to [q]$ are simplicial operators, then the induced map $f \star g\colon \Delta^{p'} \star \Delta^{q'} \to \Delta^p \star \Delta^q$ between joins of simplices is uniquely isomorphic to $(f \sqcup g)\colon \Delta^{p'+1+q'} \to \Delta^{p+1+q}$.

In particular, if $S \subseteq [p]$ and $T \subseteq [q]$ are subsets, giving rise to subcomplexes $\Delta^S \subseteq \Delta^p$ and $\Delta^T \subseteq \Delta^q$, then the evident map $\Delta^S \star \Delta^T \to \Delta^p \star \Delta^q \approx \Delta^{p+1+q}$ realizes the inclusion of the subcomplex $\Delta^{S\sqcup T} \subseteq \Delta^{p+1+q}$ associated to the subset $S \sqcup T \subseteq [p] \sqcup [q] = [p+1+q]$. This makes it relatively straightforward to describe the join of subcomplexes of standard simplices.

28.12. **Left and right cones of simplicial sets.** An important example of joins of simplicial sets are the cones. The **left cone** and **right cone** of a simplicial set X are **left cone** **right cone**

$$X^{\lhd} := \Delta^0 \star X, \qquad X^{\rhd} := X \star \Delta^0.$$

Note that outer horns are examples of cones:

$$(\partial\Delta^n)^{\lhd} = \Delta^0 \star \partial\Delta^n \approx \Lambda^{n+1}_0, \qquad (\partial\Delta^n)^{\rhd} = \partial\Delta^n \star \Delta^0 \approx \Lambda^{n+1}_{n+1}.$$

I will often write v for the **cone point**, i.e., the vertex of $\Delta^0 \star X$ or $X \star \Delta^0$ which corresponds to the unique vertex of the Δ^0-factor. **cone point**

It is straightforward to show that the nerve takes joins of categories to joins of simplicial sets: $N(A \star B) \approx N(A) \star N(B)$, and thus $N(A^{\lhd}) \approx (NA)^{\lhd}$ and $N(A^{\rhd}) \approx (NA)^{\rhd}$.

28.13. *Exercise.* Show that the nerve of a join of categories is isomorphic to the join of their nerves.

28.14. *Exercise.* The outer horns Λ^2_2 and Λ^2_0 are actually nerves of categories, sometimes called the *walking span* and *walking cospan*. Show that there are isomorphisms

$$(\partial\Delta^1)^{\lhd} \approx \Lambda^2_0, \qquad (\partial\Delta^1)^{\rhd} \approx \Lambda^2_0$$

and

$$(\Lambda^2_2)^{\lhd} \approx \Delta^1 \times \Delta^1 \approx (\Lambda^2_0)^{\rhd}.$$

28.15. **The join of quasicategories is a quasicategory.** Here is a handy rule for constructing maps into a join (compare (28.4)). Note that every join admits a canonical map $\pi\colon X \star Y \to \Delta^0 \star \Delta^0 \approx \Delta^1$, namely the join applied to the projections $X \to \Delta^0$ and $Y \to \Delta^0$.

28.16. **Lemma** ([Joy08a, Prop. 3.5], compare (28.4))**.** *Maps $f\colon K \to X \star Y$ are in bijective correspondence with the set of triples*

$$\big(\pi\colon K \to \Delta^1, \quad f_{\{0\}}\colon K^{\{0\}} \to X, \quad f_{\{1\}}\colon K^{\{1\}} \to Y\big),$$

where $K^{\{j\}} := \pi^{-1}(\{j\}) \subseteq K$, the pullback of $\{j\} \to \Delta^1$ along π.

Proof. This is a straightforward exercise. In one direction, the correspondence sends f to $(\overline{\pi}f, f|K^{\{0\}}, f|K^{\{1\}})$, where $\overline{\pi}\colon X \star Y \to \Delta^0 \star \Delta^0 = \Delta^1$. □

28.17. **Proposition.** *If C and D are quasicategories, so is $C \star D$.*

Proof. Use the previous lemma (28.16), together with the observations (which we leave as an exercise) that for any map $\pi\colon \Lambda^n_j \to \Delta^1$ from an *inner* horn, the preimages $\pi^{-1}(\{0\})$ and $\pi^{-1}(\{1\})$ are either inner horns, standard simplices, or are empty, and for any map $\pi\colon \Delta^n \to \Delta^1$ from a standard simplex, the preimages are either a standard simplex or empty. □

28.18. *Exercise.* Let $f\colon [m] \to [n]$ be any simplicial operator. Show that the induced map $f\colon \Delta^m \to \Delta^n$ on standard simplices is uniquely isomorphic to a join of maps $f_0 \star f_1 \star \cdots \star f_n$, with $f_j\colon \Delta^{m_j} \to \Delta^0$, where each $m_j \geq -1$.

28.19. *Exercise.* Show that (28.16) implies the following: there is an adjoint pair of functors

$$i^*\colon s\mathrm{Set}_{/\Delta^1} \rightleftarrows s\mathrm{Set}_{/\partial\Delta^1} : i_*$$

where the left adjoint i^* is the functor defined by pullback along the inclusion $\partial\Delta^1 \rightarrowtail \Delta^1$, and the right adjoint i_* sends $p\colon K \to \partial\Delta^1$ to $(K^{\{0\}} \star K^{\{1\}} \to \Delta^1)$, where $K^{\{j\}} := p^{-1}(j) \subseteq K$. This gives another characterization of join, as "direct image along i".

29. Slices

29.1. **Slices of categories.** Given an ordinary category C, and an object $x \in \operatorname{ob} C$, we may form the **slice categories** $C_{x/}$ and $C_{/x}$, (also called *undercategory* and *overcategory*, or *slice-over category* and *slice-under category*). **slice categories**

For instance, the slice-over category $C_{/x}$ is the category whose *objects* are maps $\alpha\colon c \to x$ with target x, and whose *morphisms* $(\alpha\colon c \to x) \to (\alpha'\colon c' \to x)$ are maps $\beta\colon c \to c'$ such that $\alpha'\beta = \alpha$.

This can be reformulated in terms of joins. Let "T" denote the terminal category (isomorphic to $[0]$). Note that $\operatorname{ob} C_{/x}$ corresponds to the set of functors $\alpha\colon [0] \star T \to C$ such that $\alpha|T = x$, and $\operatorname{mor} C_{/x}$ corresponds to the set of functors $\beta\colon [1] \star T \to C$ such that $\beta|T = x$.

More generally, given a functor $f\colon A \to C$ of categories, we obtain slice categories $C_{f/}$ and $C_{/f}$ defined as follows. The category $C_{/f}$ has

- **objects:** functors $\alpha\colon [0] \star A \to C$ such that $\alpha|A = f$,
- **morphisms** $f \to f'$**:** functors $\beta\colon [1] \star A \to C$ such that $\beta|A = f$.

Likewise, the category $C_{f/}$ has

- **objects:** functors $\alpha\colon A \star [0] \to C$ such that $\alpha|A = f$,
- **morphisms** $f \to f'$**:** functors $\beta\colon A \star [1] \to C$ such that $\beta|A = f$.

29.2. *Exercise.* Describe composition of morphisms in $C_{/f}$ and $C_{f/}$.

29.3. *Exercise.* Show that $(C_{f/})^{\mathrm{op}} \approx (C^{\mathrm{op}})_{/f^{\mathrm{op}}}$ (isomorphism of categories).

29.4. *Exercise.* Fix a functor $f\colon A \to C$, and let B be a category. Describe bijections

$$\{\text{functors } \alpha\colon B \to C_{/f}\} \leftrightarrow \{\text{functors } \beta\colon B \star A \to C \text{ s.t. } \beta|A = f\}$$

and

$$\{\text{functors } \alpha\colon B \to C_{f/}\} \leftrightarrow \{\text{functors } \beta\colon A \star B \to C \text{ s.t. } \beta|A = f\}.$$

29.5. *Remark.* The notions of limits and colimits can be formulated very compactly in terms of the general notion of slices, as shown in (29.6) below. We will directly generalize this formulation to define limits and colimits for quasicategories. Compare (28.7).

29.6. *Exercise.* Prove for a functor $f\colon A \to C$ between categories, that a colimit of f amounts to the same thing as an initial object of $C_{f/}$, and that a limit of f amounts to the same thing as a terminal object of $C_{/f}$.

29.7. **Joins and colimits of simplicial sets.** The join functor $\star\colon s\mathrm{Set} \times s\mathrm{Set} \to s\mathrm{Set}$ is in some ways analogous to the product functor $\times$, e.g., it is a monoidal functor. **F 18 Feb**

The product operation $(-) \times (-)$ on simplicial sets commutes with colimits in each input, and the functors $X \times -$ and $- \times X$ admit right adjoints (in both cases, the right adjoint is $\mathrm{Fun}(X, -)$). The join functor does not commute with colimits in each variable, but *almost* does so, as the only obstruction is the value on the initial object

More precisely, the functors $X \star -$ and $- \star X\colon s\mathrm{Set} \to s\mathrm{Set}$ do not preserve the initial object, since $X \star \varnothing \approx X \approx \varnothing \star X$. However, (the identity map of) X is tautologically the initial object of $s\mathrm{Set}_{X/}$, the slice category of simplicial sets under X.

29.8. **Proposition.** *For every simplicial set X, the induced functors*

$$X \star -, \quad - \star X\colon s\mathrm{Set} \to s\mathrm{Set}_{X/}$$

preserve colimits.

Proof. This follows from the degreewise formula for the join, which has the form:

$$(X \star Y)_n = X_n \amalg (X_{n-1} \times Y_0) \amalg \cdots \amalg (X_0 \times Y_{n-1}) \amalg Y_n = X_n \amalg (\text{terms which are “linear” in } Y).$$

That is, for each $n \geq 0$ the functor $Y \mapsto (X \star Y)_n\colon s\mathrm{Set} \to \mathrm{Set}_{X_n/}$ is seen to be colimit preserving, since each functor $X_k \times (-)\colon \mathrm{Set} \to \mathrm{Set}$ is colimit preserving. □

29.9. *Exercise* (Trivial, but important). Show that the functors $X \star -$ and $- \star X\colon s\mathrm{Set} \to s\mathrm{Set}$ preserve *pushouts*.

29.10. **Slices of simplicial sets.** We have seen that the functors

$$S \star -\colon s\mathrm{Set} \to s\mathrm{Set}_{S/} \qquad \text{and} \qquad - \star T\colon s\mathrm{Set} \to s\mathrm{Set}_{T/}$$

preserve colimits, and therefore we predict that they admit right adjoints. These exist, and are called **slice** functors, denoted **slice**

$$(f\colon S \to X) \mapsto X_{f/}\colon s\mathrm{Set}_{S/} \to s\mathrm{Set}$$

and

$$(g\colon T \to X) \mapsto X_{/g}\colon s\mathrm{Set}_{T/} \to s\mathrm{Set}.$$

I will sometimes distinguish these as **slice-under** and **slice-over**, respectively[21]. Explicitly, there are are bijective correspondences **slice-under** **slice-over**

$$(29.11) \qquad \left\{ \begin{tikzcd} S \arrow[r, "f"] \arrow[d] & X \\ S \star K \arrow[ur, dotted] & \end{tikzcd} \right\} \iff \{K \dashrightarrow X_{f/}\}, \qquad \left\{ \begin{tikzcd} T \arrow[r, "g"] \arrow[d] & X \\ K \star T \arrow[ur, dotted] & \end{tikzcd} \right\} \iff \{K \dashrightarrow X_{/g}\}.$$

[21] In kerodon [Lur21, 4.3.5], Lurie refers to these as *coslice* and *slice* respectively.

Here we write "$S \to S \star K$" and "$T \to K \star T$" for the inclusions $S \star \varnothing \subseteq S \star K$ and $\varnothing \star T \subseteq K \star T$, using the canonical isomorphisms $S \star \varnothing = S$ and $\varnothing \star T = T$. I will refer to the above correspondence as the **join/slice adjunction**. **join/slice adjunction**

Taking $K = \Delta^n$ we obtain the formulas

$$(X_{f/})_n = \mathrm{Hom}_{s\mathrm{Set}_{S/}}(S \star \Delta^n, X), \qquad (X_{/g})_n = \mathrm{Hom}_{s\mathrm{Set}_{T/}}(\Delta^n \star T, X),$$

which we regard as the definition of slices. (I.e., these formulas specify the n-cells of the slices, and naturality in "Δ^n" specifies the action of simplicial operators.)

29.12. *Exercise.* Given this explicit definition of slices in terms of their cells and the action of simplicial operators, verify the bijective corrrespondences (29.11).

In particular, we note the special cases associated to $x\colon \Delta^0 \to X$:

$$\mathrm{Hom}_{s\mathrm{Set}}(K, X_{x/}) = \mathrm{Hom}_{s\mathrm{Set}_{\Delta^0/}}(\Delta^0 \star K, X) \approx \mathrm{Hom}_{s\mathrm{Set}_*}((K^{\triangleleft}, v), (X, x)),$$

$$\mathrm{Hom}_{s\mathrm{Set}}(K, X_{/x}) = \mathrm{Hom}_{s\mathrm{Set}_{\Delta^0/}}(K \star \Delta^0, X) \approx \mathrm{Hom}_{s\mathrm{Set}_*}((K^{\triangleright}, v), (X, x)).$$

The notation (X, x) with $x \in X_0$ represents a **pointed simplical set**, the category of which is $s\mathrm{Set}_* := s\mathrm{Set}_{\Delta^0/}$. The cones $K^{\triangleleft}$ and $K^{\triangleright}$ are pointed by their cone point v. **pointed simplical set**

The slice construction for simplicial sets agrees with that for categories, so we won't need to distinguish them.

29.13. **Proposition.** *The nerve preserves slices, i.e., if $f\colon A \to C$ is a functor between 1-categories, then $N(C_{f/}) \approx (NC)_{Nf/}$ and $N(C_{/f}) \approx (NC)_{/Nf}$.*

Proof. Left as an exercise. □

29.14. **Slice as a functor.** The function complex construction $\mathrm{Fun}(-,-)$ is a functor in two variables, contravariant in the first and covariant in the second. The slice constructions also behave something like a functor of two variables, though it is a little more complicated, because the slice constructions also depend on a map between the two objects. A precise statement is that every diagram on the left gives rise to commutative diagrams as on the right.

$$\begin{tikzcd} S \arrow[r, "f"] & X \arrow[d, "p"] \\ T \arrow[u, "j"] \arrow[r, "pfj"'] & Y \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} X_{/f} \arrow[r, "p_*"] \arrow[d, "j^*"'] & Y_{/pf} \arrow[d, "j^*"] \\ X_{/fj} \arrow[r, "p_*"'] & Y_{/pfj} \end{tikzcd} \qquad \begin{tikzcd} X_{/} \arrow[r, "p_*"] \arrow[d, "j^*"'] & Y_{pf/} \arrow[d, "j^*"] \\ X_{fj/} \arrow[r, "p_*"'] & Y_{pfj/} \end{tikzcd}$$

The notation here is not great, and the whole business of joins and slices can get pretty confusing because of this.

29.15. *Remark.* A very precise formulation is that each kind of slice defines a functor $\mathrm{Tw}(s\mathrm{Set}) \to s\mathrm{Set}$ from the **twisted arrow category** of simplicial sets, whose objects are maps f of simplicial sets, and whose morphisms are pairs $(j, p)\colon f \to pfj$, where j and p are themselves maps of simplicial sets. **twisted arrow category**

Let's spell this out in terms of the correspondence between "maps into slices" and "maps from joins". Given $T \xrightarrow{j} S \xrightarrow{f} X \xrightarrow{p} Y$, consider "restriction map" $X_{f/} \to Y_{pfj/}$. The composite of a map $u\colon K \to X_{f/}$ with this restriction map is described in terms of the bijection of (29.11) as follows. The map u corresponds to a dotted arrow in

$$\begin{tikzcd} T \arrow[r, "j"] \arrow[d, tail] & S \arrow[r, "f"] \arrow[d, tail] & X \arrow[r, "p"] & Y \\ T \star K \arrow[r, "j \star K"'] & S \star K \arrow[ur, dotted, "\widetilde{u}"'] & & \end{tikzcd}$$

The composite $K \xrightarrow{u} X_{f/} \to Y_{pfj/}$ corresponds to $p\widetilde{u}(j \star K)$.

A particular special case which we will see a lot of are the "restriction" or "forgetful" maps

$$X_{/f} \to X \qquad \text{and} \qquad X_{f/} \to X$$

induced by sequence $\varnothing \to S \xrightarrow{f} X \to \Delta^0$, using that $X_{/\varnothing} = X = X_{\varnothing/}$. For instance, $X_{/f} \to X$ sends an n-cell $x \in (X_{/f})_n$ corresponding to $\widetilde{x}\colon \Delta^n \star S \to X$ extending f to the n-cell of X represented by the map $\widetilde{x}|(\Delta^n \star \varnothing)$ defined as the composite

$$\Delta^n = \Delta^n \star \varnothing \to \Delta^n \star S \xrightarrow{x} X.$$

Another special case of interest are the "projection" functors

$$X_{/f} \to Y_{/pf} \qquad \text{and} \qquad X_{f/} \to Y_{pf/}$$

induced by the sequence $\varnothing \to S \xrightarrow{f} X \xrightarrow{p} Y$. For instance, $X_{/f} \to Y_{/fp}$ sends an n-cell $x \in (X_{/f})_n$ corresponding to $\widetilde{x}\colon \Delta^n \star S \to X$ extending f to the n-cell of $Y_{pf/}$ represented by $p\widetilde{x}\colon \Delta^n \star S \to Y$.

29.16. *Exercise.* Let $f\colon S \to X$ and $g\colon T \to X$ be maps of simplicial sets. Describe and prove bijections between the following sets of solutions to lifting problems:

$$\left\{\begin{tikzcd} & X_{f/} \arrow[d] \\ T \arrow[r, "g"'] \arrow[ur, dotted] & X \end{tikzcd}\right\} \iff \left\{\begin{tikzcd} S \amalg T \arrow[r, "{(f,g)}"] \arrow[d, tail] & X \\ S \star T \arrow[ur, dotted] & \end{tikzcd}\right\} \iff \left\{\begin{tikzcd} & X_{/g} \arrow[d] \\ S \arrow[r, "f"'] \arrow[ur, dotted] & X \end{tikzcd}\right\}$$

Here $X_{f/} \to X$ and $X_{/g} \to X$ are the evident restriction maps, and $S \amalg T \to S \star T$ is the tautological inclusion.

30. Slices of quasicategories

In this section we show that, given a quasicategory C and an object $x \in C_0$, both $C_{/x}$ and $C_{x/}$ are also quasicategories.

30.1. **Left and right anodyne maps.** We recall the sets of *left horns*

$$\text{LHorn} := \{\, \Lambda^n_k \subset \Delta^n \mid 0 \leq k < n,\ n \geq 1 \,\} = \text{InnHorn} \cup \{\, \Lambda^n_0 \subset \Delta^n \mid n \geq 1 \,\}$$

and *right horns*

$$\text{RHorn} := \{\, \Lambda^n_k \subset \Delta^n \mid 0 < k \leq n,\ n \geq 1 \,\} = \text{InnHorn} \cup \{\, \Lambda^n_n \subset \Delta^n \mid n \geq 1 \,\}.$$

The associated weak saturations $\overline{\text{LHorn}}$ and $\overline{\text{RHorn}}$ are the **left anodyne** and **right anodyne** maps. The associated right complements **left anodyne** **right anodyne**

$$\text{LFib} := \text{LHorn}^{\boxslash}, \qquad \text{RFib} := \text{RHorn}^{\boxslash}$$

are the **left fibrations** and **right fibrations**. Note that **left fibrations** **right fibrations**

$$\overline{\text{InnHorn}} \subseteq \overline{\text{LHorn}} \cap \overline{\text{RHorn}} \qquad \text{and} \qquad \text{LFib} \cup \text{RFib} \subseteq \text{InnFib}.$$

These classes correspond to each other under the opposite involution $(-)^{\text{op}}\colon s\text{Set} \to s\text{Set}$; i.e., $\text{LHorn}^{\text{op}} = \text{RHorn}$, $\text{LFib}^{\text{op}} = \text{RFib}$.

30.2. **Proposition.** *Let C be a quasicategory and $x \in C_0$. The evident maps $C_{x/} \to C$ and $C_{/x} \to C$ which "forget x" (i.e., induced by the sequence $\varnothing \to \{x\} \to C$) are left fibration and right fibration respectively. In particular, they are inner fibrations, and so $C_{x/}$ and $C_{/x}$ are also quasicategories.*

Proof. I claim that $\pi\colon C_{/x} \to C$ is a right fibration. Explicitly, this map sends the n-cell $a\colon \Delta^n \to C_{/x}$, which corresponds to $\widetilde{a}\colon \Delta^n \star \Delta^0 \to C$ such that $\widetilde{a}|(\varnothing \star \Delta^0) = x$, to the n-cell represented by $\widetilde{a}|(\Delta^n \star \varnothing) \to C$. Using the join/slice adjunction, there is a bijective correspondence between lifting problems

$$\begin{tikzcd} \Lambda^n_j \arrow[r, "f"] \arrow[d, tail] & C_{/x} \arrow[d, "\pi"] \\ \Delta^n \arrow[r, "g"'] \arrow[ur, dotted] & C \end{tikzcd} \qquad \iff \qquad \begin{tikzcd} \varnothing \star \Delta^0 \arrow[r, tail] \arrow[rr, bend left, "x"] & (\Lambda^n_j \star \Delta^0) \cup_{\Lambda^n_j \star \varnothing} (\Delta^n \star \varnothing) \arrow[r, "(\widetilde{f},g)"] \arrow[d, tail] & C \\ & \Delta^n \star \Delta^0 \arrow[ur, dotted] & \end{tikzcd}$$

Note that there is a unique isomorphism $\Delta^n \star \Delta^0 \approx \Delta^{n+1}$. For any subset $S \subset [n]$, this isomorphism identifies the subcomplex $\Delta^S \star \Delta^0 \subset \Delta^n \star \Delta^0$ with $\Delta^{S\cup\{n+1\}} \subset \Delta^{n+1}$, while $\Delta^S \star \varnothing \subset \Delta^n \star \Delta^0$ is identified with $\Delta^S \subseteq \Delta^{n+1}$. Since $\Lambda^n_j = \bigcup_{k\in[n]\smallsetminus j} \Delta^{[n]\smallsetminus k}$, we see that

(1) the subcomplex $(\Lambda^n_j \star \Delta^0) \cup_{\Lambda^n_j \star \varnothing} (\Delta^n \star \varnothing)$ of $\Delta^n \star \Delta^0$ is the horn $\Lambda^{n+1}_j \subset \Delta^{n+1}$, and
(2) the subcomplex $\varnothing \star \Delta^0$ of $\Delta^n \star \Delta^0$ is the vertex $\{n+1\} \subseteq \Delta^{n+1}$.

Thus, the right hand diagram above is isomorphic to

$$\begin{tikzcd} \{n+1\} \arrow[r, tail] \arrow[rr, bend left, "x"] & \Lambda^{n+1}_j \arrow[r] \arrow[d, tail] & C \\ & \Delta^{n+1} \arrow[ur, dotted] & \end{tikzcd}$$

If C is a quasicategory, then an extension exists in this for $0 < j \leq n$, and thus a lift exists in the original lifting problem, whence $\pi\colon C_{/x} \to C$ is a right fibration.

Since right fibrations are inner fibrations, the composite $C_{/x} \to C \to *$ is an inner fibration, and thus $C_{/x}$ is a quasicategory.

The case of $C_{x/} \to C$ is similar, using the correspondence

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, tail] & C_{x/} \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & C \end{tikzcd} \qquad \iff \qquad \begin{tikzcd} \{0\} \arrow[r, tail] \arrow[rr, bend left, "x"] & \Lambda^{n+1}_{j+1} \arrow[r] \arrow[d, tail] & C \\ & \Delta^{n+1} \arrow[ur, dotted] & \end{tikzcd}$$

$\square$

30.3. *Exercise.* Let $p\colon C \to D$ be an inner fibration between simplicial sets. Show that p is a left fibration if and only if the induced maps $C_{x/} \to D_{px/}$ on slices are trivial fibrations for all $x \in C_0$, and that p is a right fibration if and only if the induced maps $C_{/x} \to D_{/px}$ on slices are trivial fibrations for all $x \in C_0$.

31. Initial and terminal objects

We can now give definitions of initial and terminal objects in a quasicategory, and to prove a few of their properties. For instance, we will prove that initial and terminal objects, if they exist, are unique up to isomorphism (31.7), and that a functor taking values in initial or terminal objects is initial or terminal in its functor category (31.9).

31.1. **Initial and terminal objects.** An **initial object**[22] of a quasicategory C is an $x \in C_0$ such that every $f\colon \partial\Delta^n \to C$ (for all $n \geq 1$) such that $f|\{0\} = x$, there exists an extension $f'\colon \Delta^n \to C$. A **terminal object** of C is an initial object of C^{op}. That is, a $y \in C_0$ such that every $f\colon \partial\Delta^n \to C$ with $f|\{n\} = y$ extends to Δ^n. Thus, initial and terminal objects are ones such that every extension problem of the the following types admits a solution. **initial object** **terminal object**

$$\begin{tikzcd} \{0\} \arrow[r, tail] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r] \arrow[d, tail] & C \\ & \Delta^n \arrow[ur, dotted, "\exists"'] & \end{tikzcd} \qquad \begin{tikzcd} \{n\} \arrow[r, tail] \arrow[rr, bend left, "y"] & \partial\Delta^n \arrow[r] \arrow[d, tail] & C \\ & \Delta^n \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

Let's spell out the first parts of the definition of initial object applied to $x \in C_0$:

- The condition for $n = 1$ says that for every object c in C there exists $f\colon x \to c$,
- The condition for $n = 2$ says that for every triple of maps $f\colon x \to c$, $g\colon c \to c'$, and $h\colon x \to c'$, we must have $[h] = [g][f]$. In particular (taking $f = 1_x$), we see there is at most one homotopy class of maps from x to any object.

If C is the nerve of an ordinary category, then $\mathrm{Hom}(\Delta^n, C) \xrightarrow{\sim} \mathrm{Hom}(\partial\Delta^n, C)$ for all $n \geq 3$ (19.3). Thus, for ordinary categories, the above definition coincides with the usual notion of initial object.

For general quasicategories, we see that an initial object $x \in C_0$ necessarily satisfies $\mathrm{Hom}_{hC}(x, y) \approx *$ for all $y \in C_0$, so that x represents an initial object in the homotopy category hC, but this is not sufficient to be initial in C: there are also an infinite sequence of "higher" conditions that an initial object of a quasicategory must satisfy.

We will now reformulate these notions using slice categories.

31.2. **Reformulation of initial/terminal via slices.** We can restate the definition of initial/terminal object using the "forgetful" functor of the relevant slice. **M 21 Feb**

31.3. **Proposition.** *If C is a quasicategory, then $x \in C_0$ is initial if and only if $C_{x/} \to C$ is a trivial fibration, and terminal if and only if $C_{/x} \to C$ is a trivial fibration.*

Proof. This is an application of the join/slice adjunction. Applied to $\partial\Delta^n \subset \Delta^n$ with $n \geq 0$ and $C_{x/} \to C$, this has the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "f"] \arrow[d, tail] & C_{x/} \arrow[d, "\pi"] \\ \Delta^n \arrow[r, "g"'] \arrow[ur, dotted] & C \end{tikzcd} \iff \begin{tikzcd} \Delta^0 \star \varnothing \arrow[r, tail] \arrow[rr, bend left, "x"] & (\Delta^0 \star \partial\Delta^n) \cup_{\varnothing\star\partial\Delta^n} (\varnothing \star \Delta^n) \arrow[r, "(\tilde{f}, g)"] \arrow[d, tail] & C \\ & \Delta^0 \star \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

Note that under the unique isomorphism $\Delta^0 \star \Delta^n \approx \Delta^{n+1}$, the subcomplex $\Delta^0 \star \partial\Delta^n$ corresponds to Λ^n_0. Thus the right-hand diagram is isomorphic to

$$\begin{tikzcd} \{0\} \arrow[r, tail] \arrow[rr, bend left, "x"] & \partial\Delta^{n+1} \arrow[r] \arrow[d, tail] & C \\ & \Delta^{n+1} \arrow[ur, dotted] & \end{tikzcd}$$

So $C_{x/} \to C$ is in $\mathrm{TrivFib} = \mathrm{Cell}^{\boxslash}$ if and only if x is an initial object of C, as desired. $\square$

[22] We use Joyal's definition of initial and terminal object [Joy02, §4] here. Lurie's definition [Lur09, 1.2.12.1] is different, but is equivalent to what we use, by [Lur09, 1.2.12.5] and (31.3).

31.4. *Remark.* This implies that if x is initial, then $C_{x/} \to C$ is a categorical equivalence. Later (40.8) we'll be able to show the converse, so that x is initial *if and only if* $C_{x/} \to C$ is a categorical equivalence.

31.5. *Exercise.* Given an initial object x of C, construct a functor $f\colon C^{\triangleleft} \to C$ such that $f|_C = \mathrm{id}_C$ and $f(v) = x$, where $v \in (C^{\triangleleft})_0$ is the cone point.

31.6. **Uniqueness of initial and terminal objects.** A crucial fact about initial and terminal objects in an ordinary category is that they are *unique up to unique isomorphism.* One way to formulate this is as follows: given a category C, let $C^{\mathrm{init}} \subseteq C$ be the full subcategory spanned by the initial objects. Then one of two cases applies: either there are no initial objects, so C^{init} is empty, or there is at least one initial object, and C^{init} is equivalent to the terminal category $[0]$.

This leads to an analogous formulation for quasicategories.

31.7. **Proposition.** *Let C be a quasicategory. Let C^{init} and C^{term} denote respectively the full subcategories spanned by initial objects and terminal objects. Then (i) either C^{init} is empty or is categorically equivalent to the terminal quasicategory Δ^0, and (ii) either C^{term} is empty or is categorically equivalent to the terminal quasicategory Δ^0.*

Proof. Since $C^{\mathrm{term}} = ((C^{\mathrm{op}})^{\mathrm{init}})^{\mathrm{op}}$, we just need to consider the case of initial objects. By definition of initial object, any $f\colon \partial\Delta^n \to C^{\mathrm{init}}$ with $n \geq 1$ can be extended to $g\colon \Delta^n \to C$, and the image of g must lie in the full subcategory C^{init} since all of its vertices do. If $C^{\mathrm{init}} \neq \varnothing$, then this extension condition also holds for $n = 0$, whence $C^{\mathrm{init}} \to \Delta^0$ is a trivial fibration, and thus C^{init} is categorically equivalent to Δ^0 by (25.1). □

There are some seemingly obvious facts about initial and terminal objects that we can't prove just yet. For instance:

- Given a quasicategory C with object x, an object $\widetilde{f} \in (C_{x/})_0$ of the slice under x is initial if and only if the corresponding morphism $f\colon x \to c$ in C is an isomorphism. Likewise, an object $\widetilde{g} \in (C_{/x})_0$ of the slice over x is terminal if and only if the corresponding morphism $g\colon c \to x$ in C is an isomorphism.
- In a quasicategory, every object which is isomorphic to an initial object is initial, and any object isomorphic to a terminal object is terminal.

Proofs will be given as (35.8) and (35.12).

31.8. **Initial and terminal objects in functor categories.** Here is a sample of a property of initial/terminal objects that we can now prove. A functor between ordinary categories whose values are all initial (or terminal) objects is itself initial (or terminal) as an object of the functor category. The same holds with categories replaced by quasicategories.

31.9. **Proposition.** *Consider a map $f\colon X \to C$ from a simplicial set to a quasicategory, and suppose $f(X) \subseteq C^{\mathrm{init}}$ (resp. $f(X) \subseteq C^{\mathrm{term}}$); i.e., for all $x \in X_0$ the object $f(x) \in C_0$ is initial (resp. terminal) in C. Then the functor f is initial (resp. terminal) viewed as an object of* $\mathrm{Fun}(X, C)$.

In particular, if C has an initial (or terminal) object c_0, then the "constant" map (defined as the composite $X \to \{c_0\} \to C$) is an initial (or terminal) object of $\mathrm{Fun}(X, C)$. Thus, if C has an initial (or terminal) object, then so does $\mathrm{Fun}(X, C)$.

31.10. *Remark.* In other words, there is an inclusion $\mathrm{Fun}(X, C^{\mathrm{init}}) \subseteq \mathrm{Fun}(X, C)^{\mathrm{init}}$ of the full subcategories of "objectwise initial functors" and "initial functors" in $\mathrm{Fun}(X, C)$. Once we know that any object isomorphic to an initial object is initial (35.12), it will follow that when C^{init} is non-empty then $\mathrm{Fun}(X, C^{\mathrm{init}}) = \mathrm{Fun}(X, C)^{\mathrm{init}}$. To see this, pick an initial object $c_0 \in C^{\mathrm{init}}$ and let $f_0\colon X \to C$ be the constant map with image $\{c_0\} \subseteq C$. Since any two initial objects are isomorphic, every $f \in \mathrm{Fun}(X, C)_0^{\mathrm{init}}$ is naturally isomorphic to f_0, and therefore $f(x)$ is isomorphic to $f_0(x) = c_0$ for every $x \in X_0$. By (35.12), $f(x)$ must be initial in C, so $f \in \mathrm{Fun}(X, C^{\mathrm{init}})_0$.

On the other hand, it is possible for $\mathrm{Fun}(X, C)^{\mathrm{init}}$ to be non-empty when C^{init} is empty, even when X and C are ordinary categories.

31.11. *Exercise.* Give an example of ordinary categories X and C such that C^{init} is empty but $\mathrm{Fun}(X, C)^{\mathrm{init}}$ is non-empty. (*Hint:* think small.)

Proof. (31.9) Assume $f(x) \in C_0$ is initial in C for all $x \in X_0$. Suppose given $g\colon \partial\Delta^n \to \mathrm{Fun}(X, C)$ with $n \geq 1$ and $g|\{0\} = f$. We want to show that there exists an extension $g'\colon \Delta^n \to \mathrm{Fun}(X, C)$ of g along $\partial\Delta^n \subset \Delta^n$. We convert this to the adjoint lifting problem:

$$\begin{tikzcd}
\{0\} \times X \arrow[r] \arrow[rr, bend left, "f"] & \partial\Delta^n \times X \arrow[r, "\widetilde{g}"] \arrow[d, tail] & C \\
& \Delta^n \times X \arrow[ur, dotted, "\widetilde{g}'"'] &
\end{tikzcd}$$

The strategy is to construct the extension by inductively constructing extensions $\widetilde{g}_k\colon F_k \to C$ where $F_k = (\partial\Delta^n \times X) \cup \mathrm{Sk}_k(\Delta^n \times X)$, $k \geq 0$ is the skeletal filtration (20.3) of the inclusion $\partial\Delta^n \times X \to \Delta^n \times X$. That is, we need to inductively construct lifts $\widetilde{g}_k$ in

$$\begin{tikzcd}
\coprod_{h \in F_k^{\mathrm{nd}} \smallsetminus F_{k-1}^{\mathrm{nd}}} \partial\Delta^k \arrow[r, "(h|\partial\Delta^k)"] \arrow[d, tail] & F_{k-1} \arrow[r, "\widetilde{g}_{k-1}"] \arrow[d, tail] & C \\
\coprod_{h \in F_k^{\mathrm{nd}} \smallsetminus F_{k-1}^{\mathrm{nd}}} \Delta^k \arrow[r, "(h)"'] & F_k \arrow[ur, dotted, "\widetilde{g}_k"'] &
\end{tikzcd}$$

for all $k \geq 0$. For $k = 0$ we have $F_0 = F_{-1} = \partial\Delta^n \times X$, since $n \geq 1$ so $(\partial\Delta^n \times X)_0 = (\Delta^n \times X)_0$.

For $k \geq 1$, note that a k-dimensional cell $h = (a, b)\colon \Delta^k \to \Delta^n \times X$ is not contained in in the subcomplex $\partial\Delta^n \times X$ if and only if $a \in (\Delta^n)_k \smallsetminus (\partial\Delta^n)_k$, i.e., if the corresponding simplicial operator $a\colon [k] \to [n]$ is surjective. Therefore such $a\colon \Delta^k \to \Delta^n$ sends the vertex $0 \in (\Delta^k)_0$ to $0 \in (\Delta^n)_0$. Therefore, each composite

$$\partial\Delta^k \xrightarrow{h|\partial\Delta^k} F_{k-1} = (\partial\Delta^n \times X) \cup \mathrm{Sk}_{k-1}(\Delta^n \times X) \xrightarrow{\widetilde{g}_{k-1}} C$$

sends the vertex 0 to $\widetilde{g}_{k-1}(0, b(0)) = \widetilde{g}(0, b(0)) = f(0, b(0))$, which by hypothesis is an inital object of C. Therefore an extension of $(\widetilde{g}_{k-1}h)|\partial\Delta^k$ along $\partial\Delta^k \subset \Delta^k$ exists as desired. □

32. Joins and slices in lifting problems

Recall that for an object x in a quasicategory C, the slice objects $C_{x/}$ and $C_{/x}$ are also quasicategories. It turns out that the conclusion remains true for more general kinds of slices of quasicategories.

32.1. **Proposition.** *Let $f\colon S \to C$ be a map of simplicial sets, and suppose C is a quasicategory. Then both $C_{f/}$ and $C_{/f}$ are quasicategories.*

The proof is just like that of (30.2): we will show below (32.15) that $C_{f/} \to C$ is a left fibration and $C_{/f} \to C$ is a right fibration.

To set this up, we need a little technology about how joins interact with lifting problems.

32.2. **Pushout-joins.** We define an analogue of the pushout-product for the the join. Given maps $i\colon A \to B$ and $j\colon K \to L$ of simplicial sets, the **pushout-join** (or **box-join**) $i \circledast j$ is the map **pushout-join** **box-join**

$$i \circledast j\colon (A \star L) \amalg_{A\star K} (B \star K) \xrightarrow{(i\star L, B\star j)} B \star L.$$

32.3. *Warning.* Unlike the pushout-product, the pushout-join is not symmetric, since the join is not symmetric: $i \circledast j \not\approx j \circledast i$.

32.4. *Example* (Pushout-joins of horns and cells). We have already observed examples of pushout-joins in the proof of (30.2), namely

$$(\Lambda^n_j \subset \Delta^n) \circledast (\varnothing \subset \Delta^0) \approx (\Lambda^{n+1}_j \subset \Delta^{n+1}), \quad (\varnothing \subset \Delta^0) \circledast (\Lambda^n_j \subset \Delta^n) \approx (\Lambda^{1+n}_{1+j} \subset \Delta^{1+n}),$$

and also

$$(\varnothing \subset \Delta^0) \circledast (\partial\Delta^n \subset \Delta^n) \approx (\partial\Delta^{1+n} \subset \Delta^{1+n}), \quad (\partial\Delta^n \subset \Delta^n) \circledast (\varnothing \subset \Delta^0) \approx (\partial\Delta^{n+1} \subset \Delta^{n+1})$$

in the proof of (31.3). These generalize to arbitrary horns and cells. The pushout-join of a horn with a cell is always a horn:

$$(\Lambda^n_j \subset \Delta^n) \circledast (\partial\Delta^k \subset \Delta^k) \approx (\Lambda^{n+1+k}_j \subset \Delta^{n+1+k}),$$
$$(\partial\Delta^k \subset \Delta^k) \circledast (\Lambda^n_j \subset \Delta^n) \approx (\Lambda^{k+1+n}_{k+1+j} \subset \Delta^{k+1+n}).$$

Also, the pushout-join of a cell with a cell is always a cell:

$$(\partial\Delta^n \subset \Delta^n) \circledast (\partial\Delta^k \subset \Delta^k) \approx (\partial\Delta^{n+1+k} \subset \Delta^{n+1+k}).$$

We leave proofs as an exercise for the reader.

32.5. *Exercise.* Prove the isomorphisms asserted in (32.4). (*Hint:* use (28.11).)

32.6. *Remark.* Both pushout-product and pushout-join are special cases of a general construction: given any functor $F\colon s\mathrm{Set} \times s\mathrm{Set} \to s\mathrm{Set}$ of two variables, you get a corresponding "pushout-F" functor: $F_\square\colon \mathrm{Fun}([1], s\mathrm{Set}) \times \mathrm{Fun}([1], s\mathrm{Set}) \to \mathrm{Fun}([1], s\mathrm{Set})$.

32.7. **Pullback-slices.** Just as the pushout-product is associated to the pullback-hom, so the pushout-join is associated to two kinds of **pullback-slices**. Given a sequence of maps $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$, we define the map **pullback-slices**

$$f^{\circledast_p j}\colon X_{/p} \to X_{/pj} \times_{Y_{/fpj}} Y_{/fp},$$

where the maps defining the pullback and the components of $f^{\circledast_p j}$ are the evident maps induced from the sequence, as described in (29.14). In a similar way, we define the map

$$f^{j \circledast_p}\colon X_{p/} \to X_{pj/} \times_{Y_{fpj/}} Y_{fp/}.$$

32.8. *Remark.* When $Y = *$, these pullback-slice maps are just the restriction maps $X_{/p} \to X_{/pj}$ and $X_{p/} \to X_{pj/}$. When $T = \varnothing$, these pullback-slice maps have the form $X_{/p} \to X \times_Y Y_{/fp}$ and $X_{p/} \to X \times_Y Y_{fp/}$. When both $Y = *$ and $T = \varnothing$, we get $X_{/p} \to X$ and $X_{p/} \to X$.

32.9. *Remark.* Both pullback-hom and pullback-slices are special cases of a general construction: given any functor $F\colon \mathrm{Tw}(s\mathrm{Set}) \to s\mathrm{Set}$ from the twisted arrow category (29.15), you get a corresponding "pullback-F" functor $F^\square\colon \mathrm{Tw}(s\mathrm{Set}) \to s\mathrm{Set}$. In the case of pullback-hom, the F in question is a composite functor $\mathrm{Tw}(s\mathrm{Set}) \to s\mathrm{Set}^{\mathrm{op}} \times s\mathrm{Set} \xrightarrow{\mathrm{Fun}} s\mathrm{Set}$.

32.10. **Joins, slices, and lifting problems.** The pushout-join and pullback-slice interact with lifting problems in much the same way that pushout-product and pullback-hom do.

32.11. **Proposition.** *Given $i\colon A \to B$, $j\colon K \to L$, and $h\colon X \to Y$, the following are equivalent.*

(1) $(i \boxast j) \boxslash h$.
(2) $i \boxslash (h^{\boxast_q j})$ *for all* $q\colon L \to X$.
(3) $j \boxslash (h^{i \boxast_p})$ *for all* $p\colon B \to X$.

Proof. A straightforward exercise. The equivalence of (1) and (2) looks like:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X_{/q} \arrow[d, "h^{\boxast_q j}"] \\ B \arrow[r, "{(v,w)}"'] \arrow[ur, dotted, "s"] & X_{/qj} \times_{Y_{/hqj}} Y_{/hq} \end{tikzcd} \iff \begin{tikzcd} \varnothing \star L \arrow[r] \arrow[rr, bend left, "q"] & (A \star L) \cup_{A \star K} (B \star K) \arrow[r, "{(\widetilde{u},\widetilde{v})}"] \arrow[d, "i \boxast j"'] & X \arrow[d, "h"] \\ & B \star L \arrow[r, "\widetilde{w}"'] \arrow[ur, dotted, "\widetilde{s}"] & Y \end{tikzcd}$$

□

Now we can set up "join/slice analogues" of the "enriched lifting theory" we have seen for products and function complexes.

32.12. **Proposition.** *Let S and T be sets of maps in sSet. Then $\overline{S} \boxast \overline{T} \subseteq \overline{S \boxast T}$.*

Proof. This is formal and nearly identical to the proof of the weak saturation result for box-products (21.9). □

32.13. **Proposition.** *We have*

$$\overline{\mathrm{Cell}} \boxast \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}, \qquad \overline{\mathrm{RHorn}} \boxast \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}, \qquad \textit{and} \qquad \overline{\mathrm{Cell}} \boxast \overline{\mathrm{LHorn}} \subseteq \overline{\mathrm{InnHorn}}.$$

Proof. Immediate from (32.4) and (32.12). □

32.14. **Proposition.** *Given $T \xrightarrow{i} S \xrightarrow{f} X \xrightarrow{p} Y$, consider the pullback-slice maps*

$$\ell\colon X_{f/} \to X_{fi/} \times_{Y_{pfi/}} Y_{pf/}, \qquad r\colon X_{/f} \to X_{/fi} \times_{Y_{/pfi}} Y_{/pf}.$$

We have the following.

(1) $i \in \overline{\mathrm{Cell}}$, $p \in \mathrm{TrivFib}$ *implies* $\ell, r \in \mathrm{TrivFib}$.
(2) $i \in \overline{\mathrm{Cell}}$, $p \in \mathrm{InnFib}$ *implies* $\ell \in \mathrm{LFib}$, $r \in \mathrm{RFib}$.
(3) $i \in \overline{\mathrm{RHorn}}$, $p \in \mathrm{InnFib}$ *implies* $\ell \in \mathrm{TrivFib}$.
(4) $i \in \overline{\mathrm{LHorn}}$, $p \in \mathrm{InnFib}$ *implies* $r \in \mathrm{TrivFib}$.

Proof. Exercise, using (32.13). □

We are especially interested in the special case when $X = C$ is a quasicategory and $Y = *$. **W 23 Feb**

32.15. **Corollary.** *Given $T \xrightarrow{j} S \xrightarrow{f} C$ with C a quasicategory, consider the pullback-slice maps*

$$\ell\colon C_{f/} \to C_{fj/}, \qquad r\colon C_{/f} \to C_{/fj}.$$

We have the following.

(1) $j \in \overline{\mathrm{Cell}}$ *implies* $\ell \in \mathrm{LFib}$, $r \in \mathrm{RFib}$.
(2) $j \in \overline{\mathrm{RHorn}}$ *implies* $\ell \in \mathrm{TrivFib}$.
(3) $j \in \overline{\mathrm{LHorn}}$ *implies* $r \in \mathrm{TrivFib}$.

In particular, (1) when $T = \varnothing$ gives

(1') $\ell\colon C_{p/} \to C$ *is a left fibration and* $r\colon C_{/p} \to C$ *is a right fibration.*

As a consequence we have proved (32.1).

Here is another useful special case when $T = \varnothing$: slices preserve trivial fibrations.

32.16. **Corollary.** *Given $S \xrightarrow{f} X \xrightarrow{p} Y$ where p is a trivial fibration, all of the maps in*

$$X_{f/} \to X \times_Y Y_{pf/} \to Y_{pf/} \qquad \text{and} \qquad X_{/f} \to X \times_Y Y_{/pf} \to Y_{/pf}$$

are trivial fibrations.

Proof. The two pullback-slice maps are trivial fibrations by (32.14). The projections are each base changes of the trivial fibration f, and so are trivial fibrations. ☐

We'll also meet the following consequence now and again: joins preserve monomorphisms.

32.17. **Proposition.** *If $i\colon A \to B$ is a monomorphism of simplical sets, then so are $S\star i\colon S\star A \to S\star B$ and $i \star S\colon A \star S \to B \star S$ for any S.*

Proof. The map $S \star i$ is the composite

$$S \star A \to (S \star A) \cup_{\varnothing\star A} (\varnothing \star B) \xrightarrow{(\varnothing\subseteq S)\circledast i} S \star B.$$

the second map is a monomorphism by $\overline{\text{Cell}} \circledast \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$ (32.13), while the first map is a cobase change of the monomorphism i.

Note: that join preserves monomorphisms can also be proved directly from the definition of join (28.9), or the identification of join as a right adjoint (28.19). ☐

32.18. **Composition functors for slices.** Here is a nice consequence of the above. Let C be a quasicategory and $f\colon x \to y$ a morphism in it; we represent f by a map $\Delta^1 \to C$ of simplicial sets, which we also call f. We obtain two restriction functors

$$C_{/x} \xleftarrow{r_0} C_{/f} \xrightarrow{r_1} C_{/y}$$

associated to the inclusions $\{0\} \subset \Delta^1 \supset \{1\}$. The first inclusion $\{0\} \subset \Delta^1$ is a left-horn inclusion, and thus by (32.15) the restriction map r_0 is a trivial fibration (and thus a categorical equivalence), and hence we can choose a section $s\colon C_{/x} \to C_{/f}$ of r_0, which is in fact a categorical inverse to r_0.

The resulting composite $r_1 s\colon C_{/x} \to C_{/y}$ can be thought of as a functor realizing the operation which sends an object $(c \xrightarrow{g} x)$ of $C_{/x}$ to an object "$(c \xrightarrow{f\circ g} y)$" of $C_{/y}$ defined by "composing f and g" (but remember that such composition is not uniquely defined in a quasicategory C; the choice of section s gives a collection of such choices for all g.)

32.19. *Exercise.* Show that if C is a category, then r_0 is an isomorphism, and that $r_1 s$ is precisely the functor $C_{/x} \to C_{/y}$ described above.

33. Limits and colimits in quasicategories

33.1. **Definition of limits and colimits.** Now we can define the notion of a limit and colimit of a functor between quasicategories (and in fact of a map from a simplicial set to a quasicategory). Given a map $f\colon S \to C$ where C is a quasicategory, a **colimit** of f is defined to be an initial object of the slice quasicategory $C_{f/}$. Explicitly, a colimit of $f\colon S \to C$ is a map $\widehat{f}\colon S \star \Delta^0 = S^{\triangleright} \to C$ extending f, such that for $n \geq 1$ a lift exists in every diagram of the form **colimit**

$$\begin{tikzcd} S \star \{0\} \arrow[r, tail] \arrow[rr, bend left, "\widehat{f}"] & S \star \partial\Delta^n \arrow[r] \arrow[d, tail] & C \\ & S \star \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

Sometimes it is better to call $\widehat{f}$ a **colimit cone** for f, in which case the restriction $\widehat{f}|\varnothing \star \Delta^0$ to the cone point is an object in C which can be called a "colimit of f". **colimit cone**

Similarly, a **limit** of f is a terminal object of $C_{/f}$. Explicitly, this is a map $\widehat{f}\colon \Delta^0 \star S = S^{\triangleleft} \to C$ extending p such that for $n \geq 1$ a lift exists in every diagram of the form **limit**

$$\begin{tikzcd} \{n\} \star S \arrow[r, tail] \arrow[rr, bend left, "\widehat{f}"] & \partial\Delta^n \star S \arrow[r] \arrow[d, tail] & C \\ & \Delta^n \star S \arrow[ur, dotted] & \end{tikzcd}$$

Again, we will also sometimes refer to $\widehat{f}\colon \Delta^0 \star S = S^{\triangleleft} \to C$ as a **limit cone** for f, while the restriction $\widehat{f}|\Delta^0 \star \varnothing$ can be called the "limit of f". **limit cone**

33.2. *Example.* Consider the empty simplicial set $S = \varnothing$ and the unique map $f\colon \varnothing \to C$. Then $C_{f/} = C$, so a colimit of f is precisely the same as an initial object of C. Likewise, a limit of f is precisely the same as a terminal object of C.

33.3. *Example.* Consider $S = \Lambda^2_0$, which is the nerve of a category which we can draw as the picture $\big(1 \leftarrow 0 \rightarrow 2\big)$. Then $(\Lambda^2_0)^{\triangleright} \approx \Delta^1 \times \Delta^1$ (28.14) is also an ordinary category; explicitly it has the form of a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr] & 1 \arrow[d] \\ 2 \arrow[r] & v \end{tikzcd}$$

where v is the "cone vertex". A colimit cone $(\Lambda^2_0)^{\triangleright} \to C$ is called a **pushout diagram** in C. **pushout diagram**

Similar considerations give $(\Lambda^2_2)^{\triangleleft} \approx \Delta^1 \times \Delta^1$. A limit cone $(\Lambda^2_2)^{\triangleleft} \to C$ is called a **pullback diagram** in C. **pullback diagram**

33.4. *Exercise* (Colimits in full subcategories). Let $C' \subseteq C$ be an inclusion of a full subcategory. Show that if $f\colon S \to C$ has a colimit $\widehat{f}$ in C, and if the image of $\widehat{f}$ is contained in C', then the restricted functor $\widehat{f}\colon S^{\triangleright} \to C'$ is a colimit of the restricted functor $f\colon S \to C'$.

33.5. **Uniqueness of limits and colimits.** Limits and colimits are unique if they exist.

33.6. **Proposition.** *Let $f\colon K \to C$ be a map to a quasicategory, and let $(C_{f/})^{\mathrm{colim}} \subseteq C_{f/}$ and $(C_{/f})^{\mathrm{lim}} \subseteq C_{/f}$ denote the full subcategories spanned by colimit cones and limit cones respectively. Then (i) either $(C_{f/})^{\mathrm{colim}}$ is empty or is categorically equivalent to Δ^0, and (ii) either $(C_{/f})^{\mathrm{lim}}$ is empty or is categorically equivalent to Δ^0.*

Proof. This is just the uniqueness of initial and terminal objects (31.7), since $(C_{f/})^{\mathrm{colim}} = (C_{f/})^{\mathrm{init}}$ and $(\mathbb{C}_{/f})^{\mathrm{lim}} = (C_{/f})^{\mathrm{term}}$. □

We have noted above (31.3) that an object x in a quasicategory C is initial iff $C_{x/} \to C$ is a trivial fibration, and terminal iff $C_{/x} \to C$ is a trivial fibration. There is a similar characterization of limit and colimit cones.

33.7. **Proposition.** *Let C be a quasicategory. Let $\widetilde{f}\colon K^{\triangleright} \to C$ be a map, and write $f := \widetilde{f}|K$. Then $\widetilde{f}$ is a colimit diagram if and only if $C_{\widetilde{f}/} \to C_{f/}$ is a trivial fibration.*

Likewise, let $\widetilde{g}\colon K^{\triangleleft} \to C$ be a map, and write $g := \widetilde{g}|K$. Then $\widetilde{g}$ is a limit diagram if and only if $C_{/\widetilde{g}} \to C_{/g}$ is a trivial fibration.

Proof. I'll just do the case of colimits.

We make an elementary observation about iterated slices (see (33.8) below). There is an isomorphism $(C_{f/})_{\widetilde{f}/} \approx C_{\widetilde{f}/}$, where the symbol "$\widetilde{f}$" refers to both a morphism $\widetilde{f}\colon K^{\triangleright} \to C$ (on the right-hand side of the isomorphism) and the corresponding object $\widetilde{f} \in (C_{f/})_0$ (on the left-hand side of the isomorphism). The point is that in either simplical set, a k-dimensional cell corresponds to a map $K \star \Delta^0 \star \Delta^k \to C$ which restricts to $\widetilde{f}$ on $K \star \Delta^0 \star \varnothing$.

Using this, the statement amounts to the special case for initial and terminal objects (31.3), applied to the restriction functor $(C_{f/})_{\widetilde{f}/} \to C_{f/}$. □

33.8. *Exercise* (Iterated slices). Let $f\colon A \star B \to C$ be a map of simplicial sets. Describe isomorphisms

$$C_{f/} \approx (C_{f_A/})_{\widetilde{f_B}/}, \qquad C_{/f} \approx (C_{/f_B})_{/\widetilde{f_A}},$$

where $f_A\colon A \to C$ and $f_B\colon B \to C$ are the evident restrictions of f to subcomplexes, and $\widetilde{f_A}\colon A \to C_{/f_B}$ and $\widetilde{f_B}\colon B \to C_{f_A/}$ are the adjoints to f.

33.9. **Limits and colimits in slices.** Given a map $f\colon S \to C$ to a quasicategory, we have "forgetful functors" $\pi\colon C_{/f} \to C$ and $\pi\colon C_{f/} \to C$ from the slices to C.

The following proposition says that an initial object of C implies a compatible initial object of $C_{/f}$, and a terminal object of C implies a compatible terminal object of $C_{f/}$. Note that when C is an ordinary category this is entirely straightforward: e.g., given an initial object c_0 of C, there is a a *unique* cone $\widetilde{f}\colon S^{\triangleleft} \to C$ extending a given $f\colon S \to C$ which sends the cone vertex to c_0, and it's an easy exercise to show that $\widetilde{f}$ represents an initial object of the slice $C_{/f}$.

33.10. **Proposition.** *Let $f\colon S \to C$ be a map from a simplicial set to a quasicategory.*

(1a) *If $x \in (C_{/f})_0$ is an object such that $\pi(x) \in C_0$ is initial in C, then x is initial in $C_{/f}$.*
(1b) *If C has an initial object then so does $C_{/f}$.*
(2a) *If $x \in (C_{f/})_0$ is an object such that $\pi(x) \in C_0$ is terminal in C, then x is terminal in $C_{f/}$.*
(2b) *If C has a terminal object then so does $C_{f/}$.*

Proof. (See [Lur09, 1.2.13.8].) I'll only prove (1a) and (1b), as the other parts are analogous.

To prove (1a), let $x \in (C_{/f})_0$ and $y = \pi(x) \in C_0$; we need to show that if y is initial then so is x. To show that x is initial we must produce a lift in any diagram of the form

$$\begin{tikzcd} \Delta^0 \star \varnothing \arrow[r, tail] \arrow[rr, bend left, "x"] & (\Delta^0 \star \partial\Delta^n) \cup_{\varnothing \star \partial\Delta^n} (\varnothing \star \Delta^n) \arrow[r] \arrow[d, tail] & C_{/f} \\ & \Delta^0 \star \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

for $n \geq 0$, using the identification $(\varnothing \subset \Delta^0) \circledast (\partial\Delta^n \subset \Delta^n) \approx (\partial\Delta^{n+1} \subset \Delta^{n+1})$. This lifting problem is equivalent to one of the form

$$\begin{tikzcd} \Delta^0 \star \varnothing \star S \arrow[r, tail] \arrow[rr, bend left, "x'"] & (\Delta^0 \star \partial\Delta^n \star S) \cup_{\varnothing \star \partial\Delta^n \star S} (\varnothing \star \Delta^n \star S) \arrow[r] \arrow[d, tail] & C \\ & \Delta^0 \star \Delta^n \star S \arrow[ur, dotted] & \end{tikzcd}$$

(because $(-) \star S$ preserves *pushouts* (29.9)), which in turn is equivalent to one of the form

$$\begin{tikzcd}
S \arrow[r, tail] \arrow[rr, bend left, "x''"] & \partial\Delta^n \star S \arrow[r] \arrow[d, tail] & C_{y/} \arrow[d, "q"] \\
& \Delta^n \star S \arrow[r] \arrow[ur, dotted] & C
\end{tikzcd}$$

(In these diagrams the maps marked x, x', x'' are all adjoints of each other.) Since y is initial, q is a trivial fibration (31.3), and therefore a lift exists since $\partial\Delta^n \star S \to \Delta^n \star S$ is a monomorphism, because joins preserve monomorphisms (32.17). We conclude that x is initial when y is.

Next we prove (1b). Suppose $y \in C_0$ is an initial object. This implies $q\colon C_{y/} \to C$ is a trivial fibration (31.3). In particular, a lift exists in

$$\begin{tikzcd}
& C_{y/} \arrow[d, "q"] \\
S \arrow[r, "p"'] \arrow[ur, dotted, "x''"] & C
\end{tikzcd}$$

By an adjunction argument (29.16), x'' corresponds to a map $x\colon \Delta^0 \to C_{/f}$ such that $\pi(x) = y$. By what we have already proved, x must be initial since $\pi(x) = y$ is initial. □

33.11. *Remark.* In fact, the converses of (1a) and (2a) in (33.10) are also true, as long as we assume that C has an initial/terminal object. The proof of these converses requires (35.12), which we have not established yet.

We can now generalize the above to arbitrary limits and colimits.

The following proposition says that colimits in $C_{/f}$ or limits in $C_{f/}$ can be "computed in the underlying quasicategory" C (if the corresponding colimit or limit in C exists).

33.12. **Proposition.** *Let $p\colon S \to C$ be a map from a simplicial set to a quasicategory.*

(1) *Let $f\colon K \to C_{/p}$ be a map such that the composite map $f_0 = \pi f\colon K \xrightarrow{f} C_{/p} \xrightarrow{\pi} C$ has a colimit cone in C. Then*
   (a) *f admits a colimit cone, and*
   (b) *if $\widetilde{f}\colon K^{\rhd} \to C_{/p}$ is such that the composite map $K^{\rhd} \xrightarrow{\widetilde{f}} C_{/p} \to C$ is a colimit cone, then $\widetilde{f}$ is a colimit cone.*

(2) *Let $f\colon K \to C_{p/}$ be a map such that the composite map $f_0 = \pi f\colon K \xrightarrow{f} C_{p/} \xrightarrow{\pi} C$ has a limit cone in C. Then*
   (a) *f admits a limit cone, and*
   (b) *if $\widetilde{f}\colon K^{\lhd} \to C_{/}p$ is such that the composite map $K^{\lhd} \xrightarrow{\widetilde{f}} C_{p/} \to C$ is a limit cone, then $\widetilde{f}$ is a limit cone.*

The proof will make use an observation sketched in the following exercise: any composite of a slice-over followed by a slice-under can be reinterpreted as a slice-under followed by a slice-over.

33.13. *Exercise* (Two-sided slice). Fix a map $p\colon A \star B \to X$ of simplicial sets. Describe a simplicial set $X_{/p/}$ which admits bijective correspondences

$$\left\{ \begin{tikzcd}
A \star B \arrow[r, "p"] \arrow[d] & X \\
A \star K \star B \arrow[ur, dotted] &
\end{tikzcd} \right\} \iff \{K \dashrightarrow X_{/p/}\},$$

natural in K. Then construct natural isomorphisms

$$(X_{p_A/})_{/\widetilde{p}_B} \approx X_{/p/} \approx (X_{/p_B})_{\widetilde{p}_A/},$$

where $p_A\colon A \to X$ and $p_B\colon B \to X$ are the evident restrictions of p to subcomplexes, and $\widetilde{p}_A\colon A \to X_{/p_B}$ and $\widetilde{p}_B\colon B \to X_{p_A/}$ are adjoints to p.

Proof of (33.12). I prove (1), as (2) is analogous. Note that $f\colon K \to C_{/p}$ is adjoint to a map $g\colon K \star S \to C$ extending p, which in turn is adjoint to a map $q\colon S \to C_{f_0/}$. Colimit cones of f_0 correspond precisely to initial objects of $C_{f_0/}$; in particular, the hypothesis of (1) asserts that $C_{f_0/}$ has an initial object. Likwise, colimit cones of f correspond exactly to initial objects of $(C_{/p})_{f/}$. As in (33.13) we have isomorphisms

$$(C_{/p})_{f/} \approx C_{/g/} \approx (C_{f_0/})_{/q}.$$

To prove (1a) here it suffices to show that $(C_{f_0/})_{/q}$ has an initial object, which follows by an application of (33.10)(1b) to the restriction functor $(C_{f_0/})_{/q} \to C_{f_0/}$ to "lift" an initial object of $C_{f_0/}$ to $(C_{f_0/})_{/q}$.

To prove (1b) here it suffices to show that the restriction functor $(C_{f_0/})_{/q} \to C_{f_0/}$ has the property that objects sent to initial objects of $C_{f_0/}$ are initial in $(C_{f_0/})_{/q}$, which is immediate from (33.10)(1a). □

33.14. **Invariance of limits and colimits.** There are some seemingly obvious facts about invariance of limits and colimits which which we cannot prove yet.

(1) Limits and colimits are invariant under categorical equivalence. For instance, if $f\colon C \to D$ is a categorical equivalence of quasicategories, and $u\colon K \to C$ is some map, then p admits a colimit in C if and only if fu admits a colimit in D, and the induced functor $C_{u/} \to D_{fu/}$ preserves colimit cones. We will prove these as (43.5) and (43.6).
(2) Limits and colimits are invariant under natural isomorphism. For instance, if $\alpha\colon f_0 \to f_1$ is a natural isomorphism of maps $f_0, f_1\colon K \to C$, then f_0 admits a colimit if and only if f_1 does, and if $\widehat{f}_0$ and $\widehat{f}_1$ are colimit cones for f_0 and f_1 respectively, there exists an isomorphism $\widehat{\alpha}\colon \widehat{f}_0 \to \widehat{f}_1$ extending α. We will prove this as (**??**).

34. The Joyal extension and lifting theorems

We are now at the point where we can state and prove Joyal's theorems about extending or lifting maps along outer horns. This will allow us to prove several facts we have stated but not yet been able to prove.

34.1. **Joyal extension theorem.** Joyal's theorem gives precise criteria for extending maps from *outer horns* into a quasicategory.

34.2. **Theorem** (Joyal extension)**.** [Joy02, Thm. 1.3] *Let C be a quasicategory, and fix a map $f\colon \Delta^1 \to C$. The following are equivalent.*

(1) *The edge represented by f is an isomorphism in C.*
(2) *Every $a\colon \Lambda^n_0 \to C$ with $n \geq 2$ such that $f = a|\Delta^{\{0,1\}}\colon \Delta^1 \to C$ admits an extension to a map $\Delta^n \to C$.*
(3) *Every $b\colon \Lambda^n_n \to C$ with $n \geq 2$ such that $f = b|\Delta^{\{n-1,n\}}\colon \Delta^1 \to C$ admits an extension to a map $\Delta^n \to C$.*

I'll call $\langle 01\rangle \in \Delta^n$ the **leading edge**, and $\langle n-1, n\rangle \in \Delta^n$ the **trailing edge**. Thus, the implications (1) ⇒ (2) and (1) ⇒ (3) say that we can always extend $\Lambda^n_0 \to C$ to an n-simplex if the *leading edge* goes to an isomorphism in C, and extend $\Lambda^n_n \to C$ to an n-simplex if the *trailing edge* goes to an isomorphism in C. **leading edge** **trailing edge**

The implications (2) ⇒ (1) and (3) ⇒ (1) are easy, and are left as an exercise.

34.3. *Exercise* (Easy part of Joyal extension). Suppose C is a quasicategory with edge $f \in C_1$, and suppose that every map $a\colon \Lambda^n_0 \to C$ with $n \in \{2,3\}$ and $f = a|\Delta^{\{0,1\}}$ admits an extension along $\Lambda^n_0 \subset \Delta^n$. Prove that f is an isomorphism.

The proof of the Joyal extension theorem will be an application of the fact that left fibrations and right fibrations are *conservative isofibrations*. **F 25 Feb**

34.4. **Conservative functors.** A functor $p\colon C \to D$ between categories is **conservative** if whenever f is a morphism in C such that $p(f)$ is an isomorphism in D, then f is an isomorphism in C. The definition of a conservative functor between *quasicategories* is precisely the same. **conservative**

34.5. *Remark.* A functor $C \to D$ of quasicategories is conservative if and only if the induced functor $hC \to hD$ on homotopy categories is conservative.

34.6. *Remark.* If $p\colon C \to D$ is an inner fibration between quasicategories which is conservative, then the fibers $p^{-1}(y) \subseteq C$ of C over any object $y \in D_0$ are necessarily quasigroupoids.

34.7. **Proposition.** *All left fibrations and right fibrations between quasicategories are conservative.*

Proof. Consider a right fibration $p\colon C \to D$, and a morphism $f\colon x \to y$ in C such that $p(f)$ is an isomorphism. We first show that f admits a preinverse.

Let $a\colon \Lambda^2_2 \to C$ such that $a_{12} = f$ and $a_{02} = 1_y$. Let $b\colon \Delta^2 \to C$ be any 2-dimensional cell exhibiting a preinverse of $p(f)$, i.e., such that $b_{12} = p(f)$ and $b_{02} = 1_{p(y)}$, so that b_{01} is a preinverse. Now we have a commutative diagram

$$\begin{tikzcd} \Lambda^2_2 \arrow[r, "a"] \arrow[d, tail] & C \arrow[d, "p"] \\ \Delta^2 \arrow[r, "b"'] \arrow[ur, dotted, "s"] & D \end{tikzcd}$$

which admits a lift since p is a right fibration. The lift s exhibits a preinverse $g := s|\Delta^{\{0,1\}}$ for f.

Because $p(f)$ was assumed to be an isomorphism in D, its preinverse $b_{01} = p(g)$ is also an isomorphism, and therefore by the above argument g admits a preinverse as well. We conclude that f is an isomorphism by (14.5). $\square$

34.8. **Isofibrations.** We say that a functor $p\colon C \to D$ of quasicategories is an **isofibration**[23] if **isofibration**

(1) p is an inner fibration, and
(2) we have "isomorphism lifting" along p. That is, for any $c \in C_0$ and *isomorphism* $g\colon p(c) \to d'$, there exists a $c' \in C_0$ and *isomorphism* $f\colon c \to c'$ such that $p(f) = g$.

Condition (2) is illustrated by the diagram

$$\begin{tikzcd} \{0\} \arrow[r, "c"] \arrow[d, tail] & C^{\mathrm{core}} \arrow[r, tail] \arrow[d, "p^{\mathrm{core}}"] & C \arrow[d, "p"] \\ \Delta^1 \arrow[r, "g"'] \arrow[ur, dotted, "f"] & D^{\mathrm{core}} \arrow[r, tail] & D \end{tikzcd}$$

Recall that if C and D are nerves of ordinary categories, then any functor $C \to D$ is an inner fibration. Thus in the case of ordinary categories, being an isofibration amounts to condition (2) only. Also, it is clear that in the case of ordinary categories condition (2) is equivalent to

(2') for any $c \in C_0$ and *isomorphism* $g'\colon d' \to p(c)$, there exists a $c' \in C_0$ and isomorphism $f'\colon c' \to c$ such that $p(f) = g'$.

[23] Joyal uses the term "quasifibration" in [Joy02]. Later in [Joy08a] this is called a "pseudofibration". Lurie uses this notion in [Lur09], but never names it. The term "isofibration" is used by Riehl and Verity [RV15].

To derive (2) from (2') for ordinary categories, just apply (2') to the (unique) inverse of g.

Note that $p\colon C \to D$ satisfies (2) if and only if $p^{\mathrm{op}}\colon C \to D$ satisfies (2'). Since $h(C^{\mathrm{op}}) = (hC)^{\mathrm{op}}$, we see that symmetry between (2) and (2') also holds for functors between quasicategories, by the following.

34.9. **Proposition.** *An inner fibration $p\colon C \to D$ between quasicategories is an isofibration if and only if $h(p)\colon h(C) \to h(D)$ is an isofibration of ordinary categories.*

Proof. ($\Longrightarrow$) Straightforward. ($\Longleftarrow$) Suppose given an isomophism $g\colon p(c) \to d'$ in D. If $h(p)\colon hC \to hD$ is an isofibration, there exists an isomorphism $f'\colon c \to c'$ in C such that $p(f') \sim_r g$. Now choose a lift in

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "a"] \arrow[d, tail] & C \arrow[d, "p"] \\ \Delta^2 \arrow[r, "b"'] \arrow[ur, dotted, "s"] & D \end{tikzcd}$$

where b exhibits $p(f') \sim_r g$ and $a(\langle 01\rangle) = f'$ and $a(\langle 12\rangle) = 1_{c'}$. The edge $f = s_{02}$ is a lift of g, and is an isomorphism since $f' \sim_r f$. $\square$

34.10. *Example.* For any quasicategory C, the tautological map $C \to \Delta^0$ to the terminal category is an isofibration.

34.11. *Exercise.* Let C be a quasicategory and suppose $C' \subseteq C$ is a subcomplex. Show that the inclusion map $C' \rightarrowtail C$ is an isofibration if and only if C' is a **replete** subcategory of C, i.e., C' is a subcategory as in (10.1) with the property that every isomorphism $f\colon x \to y$ in C with $x \in C'_0$ is itself contained in C'. **replete**

34.12. *Exercise.* (i) Let Group denote the category of groups, whose objects are pairs $G = (S, \mu)$ consisting of a set S and a function $\mu\colon S \times S \to S$ satisfying a well-known list of axioms. Show that the functor $U\colon \mathrm{Group} \to \mathrm{Set}$ which on objects sends $(S, \mu) \mapsto S$ is an isofibration between ordinary categories.

(ii) Consider the functor $U'\colon \mathrm{Group} \to \mathrm{Set}$ defined on objects by $G \mapsto \mathrm{Hom}(\mathbb{Z}, G)$. Explain why, although U' is naturally isomorphic to U, you don't know how to show whether U' is an isofibration without explicit reference to the axioms of your set theory. (Note that S and $\mathrm{Hom}(\mathbb{Z}, S)$ may be distinct sets, even though they are in bijective correpondence.) The moral is that the property of being an isofibration is not "natural isomorphism invariant".

34.13. **Left and right fibrations are isofibrations.**

34.14. **Proposition.** *All left fibrations and right fibrations between quasicategories are isofibrations.*

Proof. Suppose $p\colon C \to D$ is a right fibration (and hence an inner fibration) between quasicategories, and consider

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ \Delta^1 \arrow[r, "g"'] \arrow[ur, dotted, "f"] & D \end{tikzcd}$$

where g represents an isomorphism in D. Because p is a right fibration and $(\{1\} \subset \Delta^1) \in \mathrm{RHorn}$, there exists a lift f. Because right fibrations are conservative, f represents an isomorphism. $\square$

Note that the above proof explicitly checked isofibration condition (2') for right fibrations; thus, by symmetry we conclude that isofibration condition (2) holds for right fibrations. It seems difficult to give an elementary *direct* proof that right-fibrations satisfy (2).

34.15. **Proof of the Joyal extension theorem.**

Proof of (34.2). We prove (1) $\Rightarrow$ (2). Suppose given $a\colon \Lambda^n_0 \to C$ such that $f = a|\Delta^{\{0,1\}}$ represents an isomorphism. Observe (32.4) that $(\Lambda^n_0 \subset \Delta^n)$ is the pushout-join of a 1-horn with an $(n-2)$-cell:

$$(\Lambda^n_0 \subset \Delta^n) \approx (\Delta^{\{0\}} \subset \Delta^{\{0,1\}}) \circledast (\partial\Delta^{\{2,\dots,n\}} \subset \Delta^{\{2,\dots,n\}}),$$

since $\Lambda^n_0 \approx (\Delta^{\{0\}} \star \Delta^{\{2,\dots,n\}}) \cup (\Delta^{\{0,1\}} \star \partial\Delta^{\{2,\dots,n\}})$ inside $\Delta^n \approx \Delta^{\{0,1\}} \star \Delta^{\{2,\dots,n\}}$. Using this, we get a correspondence of lifting problems

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r, "a"] \arrow[d, tail] & C \\ & \Delta^n \arrow[ur, dotted] & \end{tikzcd} \qquad \Longleftrightarrow \qquad \begin{tikzcd} \Delta^{\{0\}} \arrow[r, "h"] \arrow[d, tail] & C_{/(a|\Delta^{\{2,\dots,n\}})} \arrow[d, "q"] \\ \Delta^{\{0,1\}} \arrow[r, "g"] \arrow[ur, dotted] \arrow[dr, "f"'] & C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \arrow[d, "p"] \\ & C \end{tikzcd}$$

where g is adjoint to $a|(\Delta^{\{0,1\}} \star \partial\Delta^{\{2,\dots,n\}})$, and h is adjoint to $a|(\Delta^{\{0\}} \star \Delta^{\{2,\dots,n\}})$. Because C is a quasicategory, and because p and q are restrictions along monomorphisms $\varnothing \subset \partial\Delta^{\{2,\dots,n\}} \subset \Delta^{\{2,\dots,n\}}$, both p and q are right fibrations (32.15), and therefore are conservative isofibrations (34.7), (34.14). Thus since f represents an isomorphism, so does g since p is conservative, and therefore a lift exists since q is an isofibration.

The proof of (2) $\Longrightarrow$ (1) is left as an exercise (34.3). The proof of (1) $\Longleftrightarrow$ (3) is similar. $\square$

34.16. **The Joyal lifting theorem.** There is a relative generalizaton, which we will have use of in the future.

34.17. **Theorem** (Joyal lifting)**.** *Let $p\colon C \to D$ be an inner fibration between quasicategories, and let $f \in C_1$ be an edge such that $p(f)$ is an isomorphism in D. The following are equivalent.*

(1) *The edge f is an isomorphism in C.*
(2) *For all $n \geq 2$, every diagram of the form*

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

admits a lift.
(3) *For all $n \geq 2$, every diagram of the form*

$$\begin{tikzcd} \Delta^{\{n-1,n\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_n \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

admits a lift.

Proof. The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are elementary, by the same argument as (34.3).

For (1) $\Rightarrow$ (2), the first step is to prove that

$$C_{/(a|\Delta^{\{2,\dots,n\}})} \xrightarrow{q} C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \times_{D_{/(pa|\partial\Delta^{\{2,\dots,n\}})}} D_{/(pa|\Delta^{\{2,\dots,n\}})} \xrightarrow{p} C$$

are both right fibrations. For instance, the map q is the pullback-slice of the inner fibration p by a monomorphism, so is a right fibration by (32.14). The map p is the composite

$$C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \times_{D_{/(pa|\partial\Delta^{\{2,\dots,n\}})}} D_{/(pa|\Delta^{\{2,\dots,n\}})} \xrightarrow{p'} C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \xrightarrow{p''} C,$$

where p' is the base change of the right fibration $D_{/(pa|\Delta^{\{2,\dots,n\}})} \to D_{/(pa|\partial\Delta^{\{2,\dots,n\}})}$, and p'' is a right fibration (in both cases by (32.15)) Then the proof of (1) $\Longrightarrow$ (2) proceeds exactly as in (34.2), since the lifting problem we want to solve is equivalent to $(\Delta^{\{0\}} \subset \Delta^{\{0,1\}}) \boxast q$. $\square$

35. Applications of the Joyal extension theorem

We can now prove a number of statements whose proofs we have deferred until now, as well as some others.

35.1. **Quasigroupoids are Kan complexes.** First we prove the identification of quasigroupoids with Kan complexes.

35.2. **Theorem** (Joyal [Joy08a]). *The quasigroupoids are precisely the Kan complexes.*

Proof. We have already noted that Kan complexes are quasigroupoids (14.14). In a quasigroupoid, the Joyal extension property (34.2) applies to all maps from Λ^n_0 and Λ^n_n with $n \geq 2$, since every edge is an isomorphism. (Recall that all simplicial sets automatically have extensions for 1-horns (14.13).) $\square$

From now on we will use terms "quasigroupoid" and "Kan complex" interchangeably.

35.3. **Invariance of slice categories.** Here is an equivalent reformulation of the Joyal extension theorem in terms of maps between slices.

35.4. **Proposition** (Reformulation of Joyal extension). *If $f\colon x \to y$ is an edge in a quasicategory C, then the following are equivalent: (1) f is an isomorphism; (2) $C_{f/} \to C_{x/}$ is a trivial fibration; (3) $C_{/f} \to C_{/y}$ is a trivial fibration.*

Proof. For all $n \geq 0$ we have a correspondence of lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, tail] & C_{f/} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & C_{x/} \end{tikzcd} \quad\Longleftrightarrow\quad \begin{tikzcd} \Delta^1 \star \varnothing \arrow[r, tail] \arrow[rr, bend left, "f"] & (\Delta^1 \star \partial\Delta^n) \cup (\{0\} \star \Delta^n) \arrow[r] \arrow[d] & C \\ & \Delta^1 \star \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

and $\big((\Delta^1 \star \partial\Delta^n) \cup (\{0\} \star \Delta^n)\big) \subset \Delta^1 \star \Delta^n) \approx (\Lambda^{1+1+n}_0 \subseteq \Delta^{1+1+n})$. The lifting problems on the right-hand side are precisely those of statement (2) of the Joyal extension theorem (34.2). $\square$

35.5. *Exercise* (Reformulation of Joyal lifting). Let $p\colon C \to D$ be an inner fibration, and $f\colon x \to y$ an edge in C such that $p(f) \in D_1$ is an isomorphism. Show that the following are equivalent: (1) f is an isomorphism in C; (2) $C_{f/} \to C_{x/} \times_{D_{p(x)/}} D_{p(f)/}$ is a trivial fibration; (3) $C_{/f} \to C_{/y} \times_{D_{/p(y)}} D_{/p(f)}$ is a trivial fibration.

35.6. **Corollary.** *If $f\colon x \to y$ is an isomorphism in a quasicategory C, then $C_{x/}$ and $C_{y/}$ are categorically equivalent, and $C_{/x}$ and $C_{/y}$ are categorically equivalent.*

Proof. Consider $C_{/x} \xleftarrow{r_0} C_{/f} \xrightarrow{r_1} C_{/y}$. We have already observed (32.15) that $r_0 \in \text{TrivFib}$, since $\{0\} \subset \Delta^0$ is left anodyne. The reformulation of Joyal extension (35.4) implies that $r_1 \in \text{TrivFib}$ when f is an isomorphism. Therefore $C_{/x}$ and $C_{/y}$ are connected by a chain of categorical equivalences. The proof for slice-under categories is analogous. $\square$

35.7. **Invariance of initial objects.** Now we prove some additional facts about initial and terminal objects. We will explicitly prove the statements about initial objects, as the case of terminal objects is similar.

35.8. **Proposition.** *Let $f\colon x \to y$ be a morphism in a quasicategory C, and let $\widetilde{f} \in (C_{x/})_0$ be the object of the slice which corresponds to $f \in C_1$. Then $\widetilde{f}$ is initial in $C_{x/}$ if and only if f is an isomorphism.*

Proof. For all $n \geq 1$ we have a correspondence of lifting problems

$$\begin{tikzcd} \{0\} \arrow[r, tail] \arrow[rr, bend left, "\widetilde{f}"] & \partial\Delta^n \arrow[r] \arrow[d, tail] & C_{x/} \\ & \Delta^n \arrow[ur, dotted] & \end{tikzcd} \iff \begin{tikzcd} \Delta^0 \star \{0\} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Delta^0 \star \partial\Delta^n \arrow[r] \arrow[d] & C \\ & \Delta^0 \star \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

and $(\Delta^0 \star \partial\Delta^n \subset \Delta^0 \star \Delta^n) \approx (\Lambda^{1+n}_0 \subseteq \Delta^{1+n})$, so a lift exists in either if and only if f is an isomorphism, by the Joyal extension theorem applied to the right-hand lifting problem.

(Alternately, we can note that $\widetilde{f}$ is initial if and only if $\pi\colon (C_{x/})_{\widetilde{f}/} \to C_{x/}$ is a trivial fibration (31.3), and that π is isomorphic to $C_{f/} \to C_{x/}$ (33.8), so the claim follows from (35.4).) ☐

35.9. *Remark.* Note that (35.8) implies that the slice $C_{x/}$ for any $x \in C_0$ in a quasicategory necessarily has an initial object, namely the vertex corresponding to the edge $1_x \in C_1$. Likewise, the slice $C_{/x}$ has a terminal object corresponding to $1_x \in C_1$.

35.10. *Remark* (Slices of quasigroupoids are quasigroupoids)*.* If C is a quasigroupoid, and $x \in C_0$ an object, then the slices $C_{/x}$ and $C_{x/}$ are quasigroupoids. This is immediate from the fact that the restriction maps $C_{/x} \to C$ and $C_{x/} \to C$ are conservative, being respectively right and left fibrations (32.15) (34.7).

35.11. *Remark* (Initial and terminal objects in quasigroupoids)*.* If C is a quasigroupoid with object $x \in C_0$, then (35.8) and its analogue for final objects, together with the fact that slices of quasigroupoids are quasigroupoids (35.10), implies that *every* object of $C_{x/}$ is initial, and *every* object of $C_{/x}$ is terminal. That is, $C_{x/} = (C_{x/})^{\mathrm{init}}$ and $C_{/x} = (C_{/x})^{\mathrm{term}}$, and they are non-empty (35.9), so both $C_{x/}$ and $C_{/x}$ are categorically equivalent to the terminal quasicategory.

Finally, we can now show that the property of being initial or terminal is isomorphism invariant. **M 28 Feb**

35.12. **Proposition.** *In a quasicategory, any object isomorphic to an initial object is also initial, and any object isomorphic to a terminal object is also terminal.*

Proof. Let x be an initial object in C, and let c be an object isomorphic to x. It is easy to see that x is initial in the homotopy category hC, and therefore c is initial in hC also. This has a useful consequence: *any* map between x and c (in either direction) must be an isomorphism in C.

We next note another fact: if x is initial, any map $f\colon S \to C$ extends along $S \subset \Delta^0 \star S$ to a map $f'\colon \Delta^0 \star S \to C$ such that $f'|\Delta^0$ represents x. This is a consequence of the fact (31.3) that $p\colon C_{x/} \to C$ is a trivial fibration, whence (25.13) there exists a map $s\colon C \to C_{x/}$ such that $ps = \mathrm{id}_C$; set f' be the adjoint to $sf\colon S \to C_{x/}$.

To show c is initial in C, we need to extend any $a\colon \partial\Delta^n \to C$ with $a_0 = c$ to a map $\widetilde{a}\colon \Delta^n\colon C$. This follows from a succession of two extension problems:

$$\begin{tikzcd}
\varnothing\star\partial\Delta^n \arrow[r, tail] \arrow[dd, tail] & (\Delta^0\star\varnothing)\amalg(\varnothing\star\partial\Delta^n) \arrow[r, "(x,a)"] \arrow[d, tail] & C \\
& \Delta^0\star\partial\Delta^n \arrow[ur, dotted, "g"] \arrow[d, tail] & \\
\varnothing\star\Delta^n \arrow[r, tail] & \Delta^0\star\Delta^n \arrow[uur, dotted, "h"'] &
\end{tikzcd}$$

The extension $g\colon \Lambda^{n+1}_0 \to C$ exists by the remarks of the previous paragraph since x is initial. The extension h exists because the leading edge of g is a map $x \to c$ in C, which is an isomorphism by the remarks of the first paragraph. The desired extension $\widetilde{a}$ is $h|(\varnothing\star\Delta^n)$. $\square$

36. Pointwise natural isomorphisms

Recall (22.4) that if C is a quasicategory then so is any function complex $\mathrm{Fun}(X, C)$ for an arbitrary simplicial set X. In this setting, say that an edge in $\mathrm{Fun}(X, C)_1$ is a **pointwise isomorphism** of maps $X \to C$ if for each for each vertex $x \in X_0$, the composite $\Delta^1 \xrightarrow{f} \mathrm{Fun}(X, C) \xrightarrow{\mathrm{res}} \mathrm{Fun}(\{x\}, C) \approx C$ represents an isomorphism in C, where f is the representing map of the edge. **pointwise isomorphism**

Note that any isomorphism in $\mathrm{Fun}(X, C)$ is automatically an pointwise isomorphism, as isomorphisms are preserved by the restriction functor (14.3). Our next goal is to prove the converse holds, so that pointwise isomorphisms in a functor category are the same as isomorphisms. We will prove this as (37.2), after some preliminary work.

36.1. **A lifting property for pointwise isomorphisms.** We establish a "lifting property" for pointwise isomorphisms.

36.2. **Proposition.** *Let $p\colon C \to D$ be an inner fibration between quasicategories, and let $i\colon S \to T$ be a monomorphism of simplicial sets such that $i_0\colon S_0 \to T_0$ is a bijection. Then a lift exists in any diagram of the form*

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d, tail] & \mathrm{Fun}(T, C) \arrow[d, "p^{\square i}"] \\
\Delta^1 \arrow[r, "v"'] \arrow[ur, dotted, "t"] & \mathrm{Fun}(S, C)\times_{\mathrm{Fun}(S,D)}\mathrm{Fun}(T, D)
\end{tikzcd}$$

where the composite $\Delta^1 \to \mathrm{Fun}(S, C)\times_{\mathrm{Fun}(S,D)}\mathrm{Fun}(T, D) \to \mathrm{Fun}(S, C)$ represents a pointwise isomorphism. Necessarily any such lift t itself represents a pointwise isomorphism since $S_0 \xrightarrow{\sim} T_0$.

In particular, when $D = \Delta^0$, this says if C is a quasicategory, $f\colon T \to C$ a map of simplicial sets, and $S \subseteq T$ a subcomplex containing all vertices of T, then any pointwise isomorphism $\overline{\alpha}\colon f|S \to \overline{g}$ of maps $S \to C$ extends to a pointwise isomorphism $\alpha\colon f \to g$ of maps $T \to C$.

36.3. *Remark.* The proposition (36.2) says that, under the hypotheses on i and p, the pullback-hom map $p^{\square i}$ satisfies a variant of the condition for being an isofibration, except that "isomorphism" is replaced by "pointwise isomorphism". In fact, it will follow from the pointwise criterion for natural isomorphisms that $p^{\square i}$ is actually an isofibration (39.2).

We need to introduce a number of ideas before we prove this. Recall (20.5) that the class of monomorphisms of simplicial sets is precisely the weak saturation $\overline{\mathrm{Cell}}$ of Cell. The same idea shows that the class of monomorphisms which are bijections on vertices is precisely the weak saturation $\overline{\mathrm{Cell}_{\geq 1}}$ of $\mathrm{Cell}_{\geq 1} := \{\, (\partial\Delta^n \subset \Delta^n) \mid n \geq 1\,\}$.

36.4. **Path category.** Next we define the **path category** of a quasicategory C. This is the full subcategory **path category**

$$\mathrm{Fun}^{\mathrm{iso}}(\Delta^1, C) \subseteq \mathrm{Fun}(\Delta^1, C)$$

spanned by objects corresponding to functors $\Delta^1 \to C$ which represent an isomorphism in C. Note that any functor $p\colon C \to D$ between quasicategories induces a functor $p_*\colon \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, C) \to \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D)$ on these subcategories, and also that restriction along $\{x\} \to \Delta^1$ for $x = 0, 1$ induces restriction functors $r_x\colon \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, C) \to \mathrm{Fun}(\{x\}, C) = C$.

36.5. **Lemma.** *Let C be a quasicategory and X a simplicial set. Then the standard bijection* $\mathrm{Hom}(X, \mathrm{Fun}(\Delta^1, C)) \approx \mathrm{Hom}(\Delta^1, \mathrm{Fun}(X, C))$ *restricts to a bijection between (i) the set of maps* $X \to \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, C)$ *and (ii) the set of maps* $\Delta^1 \to \mathrm{Fun}(X, C)$ *which represent pointwise isomorphisms in* $\mathrm{Fun}(X, C)$.

Proof. Consider $f\colon X \to \mathrm{Fun}(\Delta^1, C)$, and write $f'\colon \Delta^1 \to \mathrm{Fun}(X, C)$ and $f''\colon X \times \Delta^1 \to C$ for its adjoints. Then it is straightforward to check that f is in the set (i) iff (iii): for each $x \in X_0$ the composite $\{x\} \times \Delta^1 \rightarrowtail X \times \Delta^1 \xrightarrow{f''} C$ represents an isomorphism in C. Likewise it is similarly straightforward to check that f is in the set (ii) iff (iii) holds. $\square$

Using this, we can reformulate the statement of (36.2) as follows: given an inner fibration $p\colon C \to D$ and a monomorphism $i\colon S \to T$ of simplicial sets which induces a bijection of vertices, we need to show there exists a lift in every commutative square of the form

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail, "i"'] & \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, C) \arrow[d, "q"] \\ T \arrow[r] \arrow[ur, dotted] & \mathrm{Fun}(\{0\}, C) \times_{\mathrm{Fun}(\{0\}, D)} \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D) \end{tikzcd}$$

where q is the evident restriction of $p^{\square(\{0\}\subset\Delta^1)}$.

Proof of (36.2). Let $\mathcal{C}$ be the class of monomorphisms of simplicial sets $i\colon S \to T$ such that $i \boxslash q$ for every map $q\colon \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, C) \to \mathrm{Fun}(\{0\}, C) \times_{\mathrm{Fun}(\{0\}, D)} \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D)$ obtained by restriction from $p^{\square(\{0\}\subset\Delta^1)}$, for every inner fibration $p\colon C \to D$ between quasicategories. Is is clear from its definition that $\mathcal{C}$ is a weakly saturated class, as it is defined as a left complement to a class of maps, namely the class of q described above together with all trivial fibrations.

To prove the claim, it suffices to show that $\mathrm{Cell}_{\geq 1} \subseteq \mathcal{C}$, whence $\overline{\mathrm{Cell}}_{\geq 1} \subseteq \mathcal{C}$. That is, it suffices to show that for any $n \geq 1$, a lift exists in any commutative square of the form

$$\begin{tikzcd} (\{0\} \times \Delta^n) \cup_{\{0\}\times\partial\Delta^n} (\Delta^1 \times \partial\Delta^n) \arrow[r, "\widetilde{u}"] \arrow[d, tail] & C \arrow[d, "p"] \\ \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

where p is an inner fibration of quasicategories, and $\widetilde{u}$ is such that $\widetilde{u}|\Delta^1 \times \{y\}$ represents an isomorphism for all $y \in (\Delta^n)_0$. This follows from the following proposition (36.6) in the case of $(x, y) = (0, 0)$. $\square$

Thus we have reduced to the following proposition, which is a kind of "pushout-product" version of Joyal lifting, where we replace the horn inclusion $\Lambda^n_0 \subset \Delta^n$ with the inclusion $(\{0\} \subset \Delta^1)\square(\partial\Delta^n \subset \Delta^n)$, with the role of the "leading edge" played by $\Delta^1 \times \{0\} \subset \Delta^1 \times \Delta^n$; or alternately, replace the horn inclusion $\Lambda^n_n \subset \Delta^n$ with the inclusion $(\{1\} \subset \Delta^1)\square(\partial\Delta^n \subset \Delta^n)$, with the role of the "trailing edge" played by $\Delta^1 \times \{n\} \subset \Delta^1 \times \Delta^n$.

36.6. **Proposition** (Pushout-product Joyal lifting)**.** *Suppose $p\colon C \to D$ is an inner fibration of quasicategories, and suppose $n \geq 1$, and either $(x,y) = (0,0)$ or $(x,y) = (1,n)$. For any diagram*

$$\begin{tikzcd}
\Delta^1 \times \{y\} \arrow[r, tail] \arrow[rr, bend left, "f"] & (\{x\} \times \Delta^n) \cup_{\{x\}\times\partial\Delta^n} (\Delta^1 \times \partial\Delta^n) \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\
& \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dotted] & D
\end{tikzcd}$$

such that f represents an isomorphism in C, a lift exists.

Proof. This is a calculation, given in the appendix (78.6), which itself relies on Joyal lifting. □

36.7. *Example.* To give an idea of the proof (36.6), consider the case of $n = 1$ and $(x,y) = 0$, in which case $K = (\{0\} \times \Delta^1) \cup_{\{0\}\times\partial\Delta^1} (\Delta^1 \times \partial\Delta^1)$ can be pictured the solid-arrow part of the diagram

$$\begin{tikzcd}
(0,1) \arrow[r] & (1,1) \\
(0,0) \arrow[u] \arrow[r, "\sim", "f"'] \arrow[ur, dotted] \arrow[u, phantom, shift right=6, "a"'] & (1,0) \arrow[u, dotted] \arrow[u, phantom, shift left=6, "b"]
\end{tikzcd}$$

To lift to a map $\Delta^1 \times \Delta^1 \to C$, we first choose a lift on the 2-cell a, which is attached along an inner horn $\Lambda^2_1 \subset \Delta^2$; then we choose a lift on the 2-cell b, which is a non-inner horn $\Lambda^2_0 \subset \Delta^2$ such that $K \to C$ sends its leading edge (marked f) to an isomorphism in C, so Joyal-lifting applies.

37. Proof of the pointwise criterion for natural isomorphisms

In this section we will prove the following.

37.1. **Proposition.** *Let $j\colon K \to L$ be a monomorphism of simplicial sets such that $j_0\colon K_0 \xrightarrow{\sim} L_0$ is a bijection. Then for every quasicategory C the restriction map* $\mathrm{Fun}(j,C)\colon \mathrm{Fun}(L,C) \to \mathrm{Fun}(K,C)$ *is conservative.*

Given this, the pointwise criterion follows easily.

37.2. **Theorem** (Pointwise criterion for isomorphisms in functor categories)**.** *Let C be a quasicategory and X a simplicial set. Then an edge of* $\mathrm{Fun}(X,C)$ *is an isomorphism if and only if it is a pointwise isomorphism.*

Proof using (37.1). Consider the inclusion $j\colon \mathrm{Sk}_0 X \rightarrowtail X$ of the 0-skeleton (20.1), so that $\mathrm{Sk}_0 X = \coprod_{x\in X_0} \Delta^0$. Then $j^*\colon \mathrm{Fun}(X,C) \to \mathrm{Fun}(\mathrm{Sk}_0 X, C)$ is conservative by (37.1). So it suffices to show pointwise isomorphisms in $\mathrm{Fun}(\mathrm{Sk}_0 X, C)$ are isomorphisms. This is clear from the evident isomorphism

$$\mathrm{Fun}(\mathrm{Sk}_0 X, C) \approx \prod_{x\in X_0} C$$

which implies an isomorphism $h\,\mathrm{Fun}(\mathrm{Sk}_0 X, C) \approx \prod_{x\in X_0} hC$ (13.16), so the claim follows from the "pointwise criterion" for ordinary categories. □

First we observe that the proof of (37.1) follows from the following lemma. **W 2 Mar**

37.3. **Lemma.** *Let C be a quasicategory, and let $j\colon K \to L$ be a monomorphism of simplicial sets such that $j_0\colon K_0 \xrightarrow{\sim} L_0$ is a bijection, and write $j^*\colon$* $\mathrm{Fun}(L,C) \to \mathrm{Fun}(K,C)$ *for the restriction functor. If α is an edge in* $\mathrm{Fun}(L,C)$ *such that the edge $j^*(\alpha)$ admits a postinverse in* $\mathrm{Fun}(K,C)$*, then α admits a postinverse in* $\mathrm{Fun}(L,C)$.

Proof of (37.1) *using* (37.3). If α is a morphism in $\mathrm{Fun}(L, C)$ such that $j^*(\alpha)$ is an isomorphism in $\mathrm{Fun}(K, C)$, then in particular $j^*(\alpha)$ admits a postinverse, so by the lemma α admits a postinverse β. But then $j^*(\beta)$, being a postinverse of $j^*(\alpha)$, is also an isomorphism in $\mathrm{Fun}(K, C)$, so the same argument shows that β admits a postinverse. Therefore α is an isomorphism by (14.5). □

Finally, we prove the lemma.

Proof of (37.3). To construct a postinverse of $\alpha\colon f \to f'$ in $\mathrm{Fun}(L, C)$, we find a lift in a diagram of the form

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "u"] \arrow[d, tail] & \mathrm{Fun}(L, C) \arrow[d, "j^* = \mathrm{Fun}(j, C)"] \\ \Delta^2 \arrow[r, "v"'] \arrow[ur, dotted] & \mathrm{Fun}(K, C) \end{tikzcd}$$

where u and v may be depicted as

$$\begin{tikzcd}[row sep=small] & f' \arrow[dd, dotted, "\beta"] \\ f \arrow[ur, "\alpha"] \arrow[dr, "1_f"'] & \\ & f \end{tikzcd} \in \mathrm{Fun}(L, C) \quad \overset{j^*}{\Longrightarrow} \quad \begin{tikzcd}[row sep=small] & f'j \arrow[dd, "\overline{\beta}"] \\ fj \arrow[ur, "j^*(\alpha)"] \arrow[dr, "1_{fj}"'] & \\ & fj \end{tikzcd} \in \mathrm{Fun}(K, C)$$

so that v exhibits a postinverse $\overline{\beta}$ of αj. The lifting problem (u, v) may be regarded as a vertex of

$$\mathrm{Fun}(\Lambda^2_0, \mathrm{Fun}(L, C)) \times_{\mathrm{Fun}(\Lambda^2_0, \mathrm{Fun}(K,C))} \mathrm{Fun}(\Delta^2, \mathrm{Fun}(K, C)) = \mathrm{Fun}((\Lambda^2_0 \times L) \cup (\Delta^2 \times K), C).$$

We will construct a pointwise isomorphism $\gamma\colon (u', v') \to (u, v)$ in this quasicategory, where u' and v' are depicted as

$$\begin{tikzcd}[row sep=small] & f \arrow[dd, dotted, "1_f"] \\ f \arrow[ur, "1_f"] \arrow[dr, "1_f"'] & \\ & f \end{tikzcd} \in \mathrm{Fun}(L, C) \quad \overset{j^*}{\Longrightarrow} \quad \begin{tikzcd}[row sep=small] & fj \arrow[dd, "1_{fj}"] \\ fj \arrow[ur, "1_{fj}"] \arrow[dr, "1_{fj}"'] & \\ & fj \end{tikzcd} \in \mathrm{Fun}(K, C)$$

i.e., so that u' and v' are the composites $\Lambda^2_0 \to \{f\} \rightarrowtail \mathrm{Fun}(L, C)$ and $\Delta^2 \to \{fj\} \rightarrowtail \mathrm{Fun}(K, C)$ respectively. The lifting problem (u', v') clearly has a lift given by the the composite $s'\colon \Delta^2 \to \{f\} \rightarrowtail \mathrm{Fun}(L, C)$. Thus we have a diagram

$$\begin{tikzcd} \{0\} \arrow[r, "s'"] \arrow[d, tail] & \mathrm{Fun}(\Delta^2 \times L, C) \arrow[d] \\ \Delta^1 \arrow[r, "\gamma"'] \arrow[ur, dotted, "\delta"] & \mathrm{Fun}((\Lambda^2_0 \times L) \cup (\Delta^2 \times K), C) \end{tikzcd}$$

in which γ represents a pointwise isomorphism and j is a monomorphism which is a bijection on vertices, and therefore a lift δ exists by our result on lifting pointwise isomorphisms (36.2). The restriction $\delta|_{\{1\}}$ gives the desired lift for (u, v).

Let $r\colon \Delta^2 \times \Delta^1 \to \Delta^2$ be the map corresponding to the unique natural transformation $\langle 002\rangle \to \langle 012\rangle = \mathrm{id}_{\Delta^2}$ of functors $\Delta^2 \to \Delta^2$. That is, it is the unique map given on vertices by

$$r(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \neq (1, 0), \\ (0, 0) & \text{if } (x, y) = (1, 0). \end{cases}$$

Note that $r(\Delta^{\{0,1\}} \times \Delta^1) = \Delta^{\{0,1\}}$, and $r(\Delta^{\{0,2\}} \times \Delta^1) = \Delta^{\{0,2\}}$, so that $r(\Lambda^2_0 \times \Delta^1) = \Lambda^2_0$. We thus define $\gamma\colon (u', v') \to (u, v)$ to be the edge represented by the diagram

$$\begin{tikzcd} \Lambda^2_0 \times \Delta^1 \arrow[r, "r"] \arrow[d, tail] & \Lambda^2_0 \arrow[r, "u"] \arrow[d, tail] & \mathrm{Fun}(L, C) \arrow[d, "j^*"] \\ \Delta^2 \times \Delta^1 \arrow[r, "r"'] & \Delta^2 \arrow[r, "v"'] & \mathrm{Fun}(K, C) \end{tikzcd}$$

Since $r\colon \langle 002\rangle \to \langle 012\rangle$ it is clear that γ is a morphism $(u', v') \to (u, v)$ where u' and v' are as described above. To see that γ is a pointwise isomorphism, note that

$$r|_{\{0\}\times\Delta^1} = (0 \to 0), \qquad r|_{\{1\}\times\Delta^1} = (0 \to 1), \qquad r|_{\{2\}\times\Delta^1} = (2 \to 2).$$

Thus the restriction of $\gamma\colon \Delta^1 \to \mathrm{Fun}((\Lambda^2_0 \times L) \cup (\Delta^2 \times K), C)$ to a vertex $(x, y) \in (\Delta^2 \times L)_0$ works out to

$$\gamma|_{(0,y)} = 1_{f(y)}, \qquad \gamma|_{(1,y)} = \alpha(y), \qquad \gamma|_{(2,y)} = 1_{f(y)},$$

each of which is an isomorphism in C since α is a pointwise isomorphism. $\square$

37.4. *Exercise.* Show that if $f, g\colon C \to D$ are naturally isomorphic functors between quasicategories, then their restrictions $f^{\mathrm{core}}, g^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ to cores are also naturally isomorphic. Conclude that if $f\colon C \to D$ is a categorical equivalence between quasicategories, then the restriction $f^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ of f to cores is a categorical equivalence of quasigroupoids.

Part 6. **Isofibrations**

38. More results on isomorphisms and isofibrations

38.1. **Pullbacks of cores.** Recall that for a quasicategory C, the core $C^{\mathrm{core}} \subseteq C$ is the maximal quasigroupoid in C (14.9). The following says that maximal quasigroupoids are preserved by certain kinds of pullbacks.

38.2. **Proposition.** *Let*

$$\begin{tikzcd} C' \arrow[r, "u"] \arrow[d, "q"'] & C \arrow[d, "p"] \\ D' \arrow[r, "v"'] & D \end{tikzcd}$$

be a pullback square of simplicial sets such that the objects are quasicategories and p is an inner fibration. An edge $f \in C'_1$ is an isomorphism in C' if and only if $u(f) \in C_1$ and $q(f) \in D'_1$ are isomorphisms in C and D' respectively. Thus the induced map $(C')^{\mathrm{core}} \to C^{\mathrm{core}} \times_{D^{\mathrm{core}}} (D')^{\mathrm{core}}$ on cores is an isomorphism.

Proof. This is a straightforward application of Joyal lifting (34.17): as q is an inner fibration and $q(f)$ is an isomorphism, to show f is an isomorphism we must produce a lift in every lifting problem described by the left-hand square in

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d, tail] & C' \arrow[r, "u"] \arrow[d] & C \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dotted] \arrow[urr, dotted] & D' \arrow[r] & D \end{tikzcd}$$

Because $u(f)$ is an isomorphism, we know a lift exists in the large rectangle by Joyal lifting, and the desired lift exists because the right-hand square is a pullback. $\blacksquare$

Recall that an n-dimensional cell a of a quasicategory is in the core if and only if all of its edges a_{ij} are isomorphisms. Given this, the assertion about pullbacks of cores is immediate. □

38.3. *Exercise.* Show that the inclusion $C^{\text{core}} \to C$ of the core of a quasicategory is an isofibration.

38.4. **Kan fibrations and anodyne maps.** Let

$$\text{Horn} = \left\{ \Lambda^n_j \subset \Delta^n \;\middle|\; n \geq 1,\ 0 \leq j \leq n \right\} = \text{RHorn} \cup \text{LHorn}$$

denote the set of all horn inclusions. A map is **anodyne** if it is in $\overline{\text{Horn}}$, the weak saturation of the set of horn inclusions, and is a **Kan fibration** if it is in $\text{KanFib} := \text{Horn}^{\boxslash}$. Clearly a map is a Kan fibration if and only if it is both a left fibration and right fibration, and it is necessarily an inner fibration.

anodyne
Kan fibration

38.5. *Example.* A simplicial set X is a Kan complex iff $X \to \Delta^0$ is a Kan fibration.

38.6. **The walking isomorphism.** Let Iso be the **walking isomorphism**, i.e., the category with two objects 0 and 1, and a unique isomorphism between them. Its nerve NIso is a simplicial set, which by abuse of notation I will also denote Iso. Let $u\colon \Delta^1 \to \text{Iso}$ be the inclusion representing the unique map $0 \to 1$ in Iso.

walking isomorphism

38.7. **Proposition.** *The map* $u\colon \Delta^1 \to \text{Iso}$ *is anodyne.*

Proof. The k-dimensional cells of Iso are in one-to-one correspondence with sequences $(x_0 x_1 \cdots x_k)$ with $x_i \in \{0,1\}$. For each $k \geq 0$ there are exactly two *non-degenerate* k-dimensional cells u_k and v_k, corresponding respectively the alternating sequences $(0101\dots)$ and $(1010\dots)$ of length $k+1$. We also write $u_k, v_k\colon \Delta^k \to \text{Iso}$ for the maps representing these non-degenerate cells.

Let $F_k = u_k(\Delta^k) \subset \text{Iso}$ be the smallest subcomplex containing u_k. Observe that for a simplicial operator $f\colon [d] \to [k]$ we have $u_k f = (x_0 x_1 \cdots x_d)$ with $x_i \equiv f(i) \bmod 2$. In particular,

- $u_k\langle 1 \dots k\rangle = v_{k-1}$,
- $u_k\langle 0 \dots k-1\rangle = u_{k-1}$,
- $u_k\langle 0, 1, \dots, \widehat{i}, \dots, k-1, k\rangle$ is a degenerate cell associated to u_{k-2} if $i = 1, \dots, k-1$.

From this we can see that the only non-degenerate cells of $F_k \smallsetminus F_{k-1}$ are u_k and $v_{k-1} = u_k\langle 1 \cdots k\rangle$. Therefore $\text{Iso} = \bigcup_k F_k$, $F_1 = u(\Delta^1)$, and the commutative square

$$\begin{tikzcd} \Lambda^k_0 \arrow[r] \arrow[d, tail] & F_{k-1} \arrow[d, tail] \\ \Delta^k \arrow[r, "u_k"'] & F_k \end{tikzcd}$$

is a pushout square for all $k \geq 1$ by (20.4), since it is a pullback, and any cell in the complement of $F_{k-1} \subset F_k$ is the image of a unique cell under the map u_k. It follows that u is anodyne. □

As an immediate consequence, any map $f\colon \Delta^1 \to C$ can be extended over Iso when C is a quasigroupoid, since these are Kan complexes (35.2). We can easily refine this to give a criterion for f to represent an isomorphism in a general quasicategory.

38.8. **Proposition.** *Let C be a quasicategory, and $f\colon \Delta^1 \to C$ a map. Then there exists $f'\colon \text{Iso} \to C$ with $f'u = f$ if and only if f represents an isomorphism in C.*

Proof. ($\Longrightarrow$) Clear. ($\Longleftarrow$) If f represents an isomorphism then it factors $\Delta^1 \to C^{\text{core}} \subseteq C$ through the core, which is a Kan complex, so an extension along the anodyne map u to a map $\text{Iso} \to C^{\text{core}} \subseteq C$ exists. □

38.9. *Exercise.* Let Z be the complex of (25.6), and let $F\colon \Delta^1 \to Z$ be the map representing the edge $f \in Z_1$. Show that F is anodyne, and state and prove an analogue of (38.8) with Z in place of Iso.

38.10. *Remark.* Let Z' be the complex of (25.7), and let $F' \colon \Delta^1 \to Z'$ be the map representing the edge $f \in Z'_1$. Note that Z' is isomorphic to the subcomplex $X \subset \mathrm{Iso}$ which is the union of the images of 2-dimensional cells (010) and (101), and that under this isomorphism F' corresponds to u. It is straightforward to show that F' also satisfies an analogue of (38.8), by the same idea as the proof of (38.9).

However, it turns out that $F' \colon \Delta^1 \to Z'$ is *not* anodyne, and that $Z' \approx X \to \mathrm{Iso}$ is not a categorical equivalence. In particular, a map $X \to C$ to a quasicategory can fail to extend along $X \subset \mathrm{Iso}$.

38.11. **Isofibrations and Kan fibrations.** As we have seen, a functor $f \colon C \to D$ between quasicategories is an isofibration if (1) it is an inner fibration, and (2) every diagram

$$\begin{tikzcd} \{j\} \arrow[r] \arrow[d, tail] & C^{\mathrm{core}} \arrow[r, tail] \arrow[d, "p^{\mathrm{core}}"] & C \arrow[d, "p"] \\ \Delta^1 \arrow[r, "f"'] \arrow[ur, dotted, "g"] & D^{\mathrm{core}} \arrow[r, tail] & D \end{tikzcd}$$

with $j = 0$ admits a lift g. Furthermore, it is equivalent to require $(2')$ instead of (2), where $(2')$ is the same statement with $j = 1$.

In particular, $C \to *$ is an isofibration for any quasicategory C (because identity maps are isomorphisms). Given a functor $p \colon C \to D$ between quasicategories, we write $p^{\mathrm{core}} \colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ for its restriction to cores.

38.12. **Proposition.** *Let $p \colon C \to D$ be an inner fibration between quasicategories. Then the following are equivalent.*

1. *p is an isofibration.*
2. *p^{core} is an isofibration.*
3. *p^{core} is a Kan fibration.*

In particular, an inner fibration between Kan complexes is an isofibration if and only if it is a Kan fibration.

Proof. (1) $\iff$ (2). That p^{core} is an inner fibration is an elementary argument (17.11). It is also immediate that condition (2) holds for p iff it holds for p^{core}.

(2) $\iff$ (3). Suppose q is an inner fibration between Kan complexes, e.g., $q = p^{\mathrm{core}}$. Then Joyal lifting (34.17) implies that $(\Lambda^n_j \subset \Delta^n) \boxslash q$ for all $n \geq 2$, and all $0 \leq j \leq n$. The claim follows from the observation that q is an isofibration iff $(\Lambda^1_j \subset \Delta^1) \boxslash q$ for either of (or both of) $j = 0$ or $j = 1$. $\square$

38.13. *Exercise.* Give an example of an inner fibration between Kan complexes which is not a Kan fibration.

We have another "lifting criterion" for isofibrations involving the walking isomorphism.

38.14. **Proposition.** *A map p between quasicategories is an isofibration iff (1) it is an inner fibration, and $(2''')$ $(\{0\} \subset \mathrm{Iso}) \boxslash p$.*

Proof. ($\Longleftarrow$) Straightforward, using the fact (38.8) that every $f \colon \Delta^1 \to D$ representing an isomorphism factors through a map $\mathrm{Iso} \to D$.

($\Longrightarrow$) Solve a lifting problem ($a\colon \{0\} \to C$, $b\colon \mathrm{Iso} \to D$) of type $(\{0\} \subset \mathrm{Iso}) \boxslash p$ by solving two lifting problems in sequence

$$\begin{tikzcd} \{0\} \arrow[r, "a"] \arrow[d, tail] & C^{\mathrm{core}} \arrow[r, tail] \arrow[d, "p^{\mathrm{core}}"] & C \arrow[d, "p"] \\ \Delta^1 \arrow[r, "b'u"'] \arrow[ur, dotted, "t"] & D^{\mathrm{core}} \arrow[r, tail] & D \end{tikzcd} \qquad \begin{tikzcd} \Delta^1 \arrow[r, "t"] \arrow[d, tail, "u"'] & C^{\mathrm{core}} \arrow[r, tail] \arrow[d, "p^{\mathrm{core}}"] & C \arrow[d, "p"] \\ \mathrm{Iso} \arrow[r, "b'"'] \arrow[ur, dotted, "s"] & D^{\mathrm{core}} \arrow[r, tail] & D \end{tikzcd}$$

$b'\colon \mathrm{Iso} \to D^{\mathrm{core}}$ is the factorization of b through the core, and $u\colon \Delta^1 \to \mathrm{Iso}$ represents the morphism $0 \to 1$ in Iso. Since p is an isofibration, there exists a lift t in the left-hand square which represents an isomorphism in C. Both t and b land in the relevant cores, and so it suffices to produce a lift in the right-hand diagram, which exists because u is anodyne (38.7) and p^{core} is a Kan fibration by (38.12). $\square$

In other words, the isofibrations are precisely the maps *between quasicategories* which are contained in $\big(\mathrm{InnHorn} \cup \{\{0\} \subset \mathrm{Iso}\}\big)^{\boxslash}$. As an immediate consequence we get the following.

38.15. **Proposition.** *If $p\colon C \to D$ is a trivial fibration between quasicategories, then it is an isofibration.*

Proof. All inner horn inclusions as well as $\{0\} \to \mathrm{Iso}$ are monomorphisms, so this follows from (38.14). $\square$

38.16. *Remark.* We have deliberately excluded maps between non-quasicategories from the definition of isofibration. The correct generalization of isofibration to arbitrary simplicial sets is called "categorical fibration", and will be discussed later (41).

39. Lifting properties for isofibrations

39.1. **A useful lifting result.** In view of the pointwise criterion for natural isomorphisms (37.2), the lifting property for pointwise isomorphisms (36.2) can be reforulated as follows. **F 4 Mar**

39.2. **Proposition.** *Let $p\colon C \to D$ be an inner fibration between quasicategories, and let $i\colon K \to L$ be a monomorphism of simplicial sets such that $i_0\colon K_0 \to L_0$ is a bijection. Then the induced pullback-hom map $p^{\square i}\colon \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L, D)$ is a conservative isofibration.*

Proof. By enrchied lifting (22.2) $p^{\square i}$ is an inner fibration between quasicategories. Consider a commutative square of the form

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d, tail] & \mathrm{Fun}(L, C) \arrow[d, "p^{\square i}"] \\ \Delta^1 \arrow[r, "v"'] \arrow[ur, dotted, "t"] & \mathrm{Fun}(K, C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L, D) \end{tikzcd}$$

where v represents an isomorphism. Then the composite of v with the projection to $\mathrm{Fun}(K, C)$ certainly represents a pointwise isomorphism, and thus (36.2) applies to give a lift t which is necessarily a pointwise isomorphism, and hence an isomorphism by (37.2). That the functor $p^{\square i}$ is conservative is also clear, e.g., using the pointwise criterion. $\square$

As a consequence we get the following.

39.3. **Corollary.** *Let C be a quasicategory, and $i\colon K \to L$ a monomorphism of simplical sets which induces a bijection on vertices. Then the fibers of $i^*\colon \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C)$ over any vertex of the target are quasigroupoids.*

Proof. An immediate consequence of (39.2), since i^* sends any morphism in one of its fibers to an identity morphism in the target. $\square$

39.4. **Enriched lifting for isofibrations.** Using the same ideas that prove (39.2), we can show the following variant, where we drop the condition that i be a bijection on vertices, but add the condition that p be an isofibration.

39.5. **Proposition.** *Let $p\colon C \to D$ be an isofibration between quasicategories, and $i\colon K \to L$ any monomorphism of simplicial sets. Then the induced pullback-hom map*

$$p^{\square i}\colon \operatorname{Fun}(L, C) \to \operatorname{Fun}(K, C) \times_{\operatorname{Fun}(K,D)} \operatorname{Fun}(L, D)$$

is an isofibration.

Proof. Let $\mathcal{C}$ be the class of monomorphisms i of simplicial sets such that $p^{\square i}$ is an isofibration whenever $p\colon C \to D$ is an isofibration between quasicategories. I claim that the class $\mathcal{C}$ is weakly saturated. First note that since p is an inner fibration between quasicategories, so is $p^{\square i}$ for any monomorphism i since $\overline{\mathrm{InnHorn}\square\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (22.2). Given this, $p^{\square i}$ is an isofibration iff $(\{0\} \subset \mathrm{Iso}) \boxslash p^{\square i}$ (38.14). Finally, recall that that for any map j we have $j \boxslash p^{\square i}$ iff $(i\square j) \boxslash p$ iff $i \boxslash p^{\square j}$ (21.5), and so $i \in \mathcal{C}$ iff (1) $i \in \overline{\mathrm{Cell}}$ and (2) $i \boxslash p^{\square(\{0\}\subset \mathrm{Iso})}$ for every isofibration p.

Thus, to prove the proposition it suffices to show $\mathrm{Cell} \subseteq \mathcal{C}$. We have that $(\partial\Delta^n \subset \Delta^n)$ is in $\mathcal{C}$ when $n \geq 1$ by (39.2) and the fact that isofibrations are inner fibrations by definition, while $(\partial\Delta^0 \subset \Delta^0)$ is in $\mathcal{C}$ tautologically since $p^{\square(\partial\Delta^0\subset\Delta^0)} = p$ is an isofibration by hypothesis. □

39.6. *Example.* If C is a quasicategory and $i\colon K \to L$ a monomorphism, then $i^*\colon \operatorname{Fun}(L, C) \to \operatorname{Fun}(K, C)$ is an isofibration by (39.5) and the fact that $C \to *$ is an isofibration.

39.7. **Covering homotopy extension property.** Here is a very handy consequence of enriched lifting for isofibrations (39.5). Consider maps $i\colon K \to L$ and $p\colon C \to D$ of simplicial sets, with pullback-hom map

$$p^{\square i}\colon \operatorname{Fun}(L, C) \to \operatorname{Fun}(K, C) \times_{\operatorname{Fun}(K,D)} \operatorname{Fun}(L, D).$$

A vertex (u, v) in the target of $p^{\square i}$ corresponds to a lifting problem of type $i \boxslash p$, and this lifting problem has a solution if and only if the vertex (u, v) is in the image of a vertex s in $\operatorname{Fun}(L, C)$.

An edge e in the target of $p^{\square i}$ from vertex (u_0, v_0) to vertex (u_1, v_1) corresponds to a commutative square

$$\begin{tikzcd} K \times \Delta^1 \arrow[r, "\widetilde{u}"] \arrow[d, "i\times\mathrm{id}"'] & C \arrow[d, "p"] \\ L \times \Delta^1 \arrow[r, "\widetilde{v}"'] & D \end{tikzcd}$$

such that $\widetilde{u}|K \times \{x\} = u_x$ and $\widetilde{v}|L \times \{x\} = v_x$ for $x = 0, 1$. We think of such an edge e as a "deformation" relating the two lifting problems (u_0, v_0) and (u_1, v_1). In certain circumstances, a lifting problem admits a solution if it admits a suitable deformation to a solvable lifting problem. In our setting this happens when a lifting problem is *isomorphic* to a solvable one, a principle called "covering homotopy extension".

39.8. **Proposition** (Covering homotopy extension for isofibrations)**.** *Let $i\colon K \to L$ be a monomorphism of simplicial sets, and $p\colon C \to D$ an isofibration of quasicategories. If two lifting problems (u_0, v_0) and (u_1, v_1) of type $i \boxslash p$ are represented by isomorphic objects of* $\operatorname{Fun}(K, C) \times_{\operatorname{Fun}(K,D)} \operatorname{Fun}(L, D)$*, then (u_0, v_0) admits a lift if and only if (u_1, v_1) admits a lift.*

Proof. Let e be such an isomorphism. I'll show that if (u_0, v_0) admits a lift $s\colon L \to C$, then (u_1, v_1) also admits a lift. The hypotheses on i and p, togther with the fact that $p^{\square i}$ is an isofibration (39.5), imply that a lift t exists in the commutative square

$$\begin{tikzcd} \{0\} \arrow[r, "s"] \arrow[d, tail] & \mathrm{Fun}(L,C) \arrow[d, "p^{\square i}"] \\ \Delta^1 \arrow[r, "e"'] \arrow[ur, dotted, "t"] & \mathrm{Fun}(K,C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L,D) \end{tikzcd}$$

Then the vertex $t(1) \in \mathrm{Fun}(L,C)_0$ gives the desired lift for (u_1, v_1). The proof of the reverse direction is similar, using $(\{1\} \subset \Delta^1)$ instead of $(\{0\} \subset \Delta^1)$. □

39.9. *Remark.* In the proof of (39.8), the isofibration condition is used to lift the edge e to a suitable edge t, but the lift t need not itself be an isomorphism. In fact, an analogous covering homotopy extension property can be proved in other contexts where the edge e can be shown to lift. In fact, we have already used a variant of this idea in the proof of the pointwise criterion for natural isomorphisms (37.3).

40. Isofibrations and categorical equivalences

The goal of this section is to characterize the trivial fibrations between quasicategories: they are precisely the isofibrations which are also categorical equivalences.

40.1. **Vertical categorical equivalence.** Say that a functor $p \colon C \to D$ between quasicategories is a **vertical categorical equivalence**[24] if there exists

vertical categorical equivalence

- $s \colon D \to C$ such that $ps = \mathrm{id}_D$, and
- $h \colon C \times \Delta^1 \to C$ representing a natural isomorphism $\mathrm{id}_C \to sp$, such that $ph = p\pi_C$, so that the diagram

$$\begin{tikzcd} C \times \partial\Delta^1 \arrow[rr, "(\mathrm{id}_C, sp)"] \arrow[d, tail] & & C \arrow[d, "p"] \\ C \times \Delta^1 \arrow[r, "\pi_C"'] \arrow[urr, "h"] & C \arrow[r, "p"'] & D \end{tikzcd}$$

  commutes.

Any vertical categorical equivalence is a categorical equivalence, since s is a categorical inverse to p.

40.2. *Remark.* Here is one way to think about the identity $ph = p\pi_C$: it says that the map $p_* \colon \mathrm{Fun}(C,p) \colon \mathrm{Fun}(C,C) \to \mathrm{Fun}(C,D)$ sends the isomorphism represented by h to the identity map of the object p.

40.3. *Exercise.* Show that vertical categorical equivalence can be reformulated in terms of the relative function complex of (25.10): a functor $p \colon C \to D$ is a vertical categorical equivalence iff there exists (i) an object $s \in \mathrm{Fun}_{/D}(D,C)_0$ and (ii) an isomorphism $h \colon \mathrm{id}_C \to sp$ in $\mathrm{Fun}_{/D}(C,C)_1$.

40.4. *Exercise.* Show that if $p \colon C \to D$ is a vertical categorical equivalence, then for each object $d \in D_0$ the projection $p^{-1}(d) \to \{d\}$ of a fiber to its image is a categorical equivalence.

40.5. *Exercise.* Give an example of a functor $p \colon C \to D$ between quasicategories such that $p^{-1}(d) \to \{d\}$ is a categorical equivalence for each $d \in D_0$, but p is not a vertical categorical equivalence. (Hint: there are examples where C and D are ordinary categories.)

40.6. *Exercise.* Show that any base change of a vertical categorical equivalence p along a functor from a quasicategory is also a vertical categorical equivalence.

[24]Riehl and Verity [RV22] call this a “split fiber homotopy equivalence”.

40.7. Isofibrations which are categorical equivalences are trivial fibrations.

40.8. **Proposition.** *A functor $p\colon C \to D$ of quasicategories is a trivial fibration if and only if it is both a categorical equivalence and an isofibration.*

Proof. We have already seen that trivial fibrations of quasicategories are categorical equivalences (25.11) and isofibrations (38.15). The other direction is a consquence of the two following propositions (40.9) and (40.10), which show that for an isofibration p, categorical equivalence $\Rightarrow$ vertical categorical equivalence $\Rightarrow$ trivial fibration. □

40.9. **Proposition.** *If $p\colon C \to D$ is an isofibration and a categorical equivalence, then it is a vertical categorical equivalence.*

Proof. Choose a categorical inverse $g\colon D \to C$ for p, for which there are natural isomorphisms $gp \approx \mathrm{id}_C$ and $pg \approx \mathrm{id}_D$. We will "deform" g to a functor $s\colon D \to C$ equipped with natural isomorphisms $1_{\mathrm{id}_D}\colon ps \to \mathrm{id}_D$ and $h\colon sp \to \mathrm{id}_C$ which exhibit p as a vertical categorical equivalence.

Step 1. Choose $v\colon D \times \Delta^1 \to D$ representing a natural isomorphism $pg \to \mathrm{id}_D$. Since p is an isofibration so is $\mathrm{Fun}(D,p)\colon \mathrm{Fun}(D,C) \to \mathrm{Fun}(D,D)$ (39.5), and so a lift α representing an isomorphism exists in

$$\begin{tikzcd} \{0\} \arrow[r, "g"] \arrow[d, tail] & \mathrm{Fun}(D,C) \arrow[d, "p^*"] \\ \Delta^1 \arrow[r, "\widetilde{v}"'] \arrow[ur, dotted, "\alpha"] & \mathrm{Fun}(D,D) \end{tikzcd}$$

Let $s := \alpha(1) \in \mathrm{Fun}(D,C)_0$, so that $s\colon D \to C$ is a functor such that $ps = \mathrm{id}_D$, and $\alpha\colon g \to s$ is a natural isomorphism. Thus we have natural isomorphisms $\mathrm{id}_C \approx gp \approx sp$ of functors $C \to C$, i.e., there exists a natural isomorphism $w\colon sp \to \mathrm{id}_C$, represented by an edge $w \in \mathrm{Fun}(C,C)_1$.

Step 2. We have functors $\mathrm{Fun}(C,C) \xrightarrow{p_*} \mathrm{Fun}(C,D) \xrightarrow{s_*} \mathrm{Fun}(C,C)$ induced by postcomposition with p and s. We can apply various iterations of these functors to the natural isomorphism $w\colon sp \to \mathrm{id}_C$, some of which are pictured in the following solid arrow diagram of objects and isomorphisms in $\mathrm{Fun}(C,C)$ and $\mathrm{Fun}(C,D)$:

$$\begin{tikzcd} & \mathrm{id}_C \arrow[dd, dotted, "h"] & & & & p \arrow[dd, "1_p"] & \\ sp \arrow[ur, "w"] \arrow[dr, "(sp)_*(w)"'] & & \in \mathrm{Fun}(C,C) & \overset{p_*}{\Longrightarrow} & p = psp \arrow[ur, "p_*(w)"] \arrow[dr, "p_*(w)=(psp)_*(w)"'] & & \in \mathrm{Fun}(C,D) \\ & sp & & & & p & \end{tikzcd}$$

Note that $ps = \mathrm{id}_Y$ implies that $p_*(sp) = p$ and $(sp)_*(sp) = sp$, and that $(psp)_*(w) = p_*(w)$. The right hand diagram "commutes" in $\mathrm{Fun}(C,D)$, i.e., it represents the boundary of an element $b \in \mathrm{Fun}(C,D)_2$, namely the degenerate cell $b := (p_*(w))_{011}$ associated to the edge $p_*(w)\colon psp = p \to p$.

The above picture is represented by a commutative square

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "a"] \arrow[d, tail] & \mathrm{Fun}(C,C) \arrow[d, "\mathrm{Fun}(X,p)=p_*"] \\ \Delta^2 \arrow[r, "b"'] \arrow[ur, dotted, "t"] & \mathrm{Fun}(C,D) \end{tikzcd}$$

in simplicial sets. Since p is an inner fibration and $a|\Delta^{\{0,1\}} = w$ represents an isomorphism, a lift t exists by Joyal lifting (34.17). Thus $h := t|\Delta^{\{1,2\}}\colon \Delta^1 \to \mathrm{Fun}(C,C)$ is a natural isomorphism $h\colon \mathrm{id}_C \to sp$ such that $p_*(h) = 1_p$, i.e., $ph = p\pi_C$. We have thus produced $s\colon D \to C$ and $h\colon C \times \Delta^1 \to C$ exhibiting p as a vertical categorical equivalence, as desired. □

40.10. **Proposition.** *If $p\colon C \to D$ is an isofibration and a vertical categorical equivalence, then p is a trivial fibration.*

Proof. Given such a functor p, consider a lifting problem

$$\begin{tikzcd} K \arrow[r, "u"] \arrow[d, tail, "i"'] & C \arrow[d, "p"] \\ L \arrow[r, "v"'] & D \end{tikzcd}$$

with i a monomorphism. Since p is an isofibration, the covering homotopy extension property (39.8) applies, so it suffices to show that this lifting problem is isomorphic to one we can solve. In fact, the data $(s\colon D \to C,\ h\colon C \times \Delta^1 \to C)$ of a vertical categorical equivalence provides us with such an isomorphism, via the commutative rectangle

$$\begin{tikzcd} K \times \Delta^1 \arrow[r, "u\times\mathrm{id}"] \arrow[d, tail, "i\times\mathrm{id}"] & C \times \Delta^1 \arrow[r, "h"] \arrow[d, "p\times\mathrm{id}"] & C \arrow[d, "p"] \\ L \times \Delta^1 \arrow[r, "v\times\mathrm{id}"'] & D \times \Delta^1 \arrow[r, "\pi_D"'] & D \end{tikzcd}$$

(Note that $\pi_D(p \times \mathrm{id}) = p\pi_C = ph$.) Over $\{0\} \subset \Delta^1$ this is the original lifting problem (u, v), while over $\{1\} \subset \Delta^1$ we get a lifting problem (spu, v) since $sp = h|C \times \{1\}$ and $pspu = pu = vi$. The diagram provides a morphism $e\colon (u, v) \to (spu, v)$ in $\mathrm{Fun}(K, C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L, D)$, whose projection to $\mathrm{Fun}(L, D)$ is the the identity map of v, and whose projection to $\mathrm{Fun}(K, C)$ is represented by $h(u \times \mathrm{id})$, which is an isomorphism since h is. Thus e is itself an isomorphism by (38.2).

Finally, we know a lift for (spu, v), namely $sv\colon L \to C$ (since $svi = spu$ and $psv = v$). □

The result we have just proved has many useful consequences: in many cases we can replace instances of the hypothesis "trivial fibration" with "categorical equivalence".

40.11. **Corollary.** *A quasicategory C is categorically equivalent to the terminal category Δ^0 if and only if $C \to \Delta^0$ is a trivial fibration.*

Proof. Immediate from (40.8) and the fact that $C \to \Delta^0$ is an isofibration (34.10). □

40.12. *Exercise.* Show that an object x of a quasicategory C is initial if and only if $C_{x/} \to C$ is a categorical equivalence, and is terminal if and only if $C_{/x} \to C$ is a categorical equivalence.

40.13. *Exercise.* Show that a morphism $f\colon x \to y$ of a quasicategory C is an isomorphism if and only if $C_{f/} \to C_{x/}$ is a categorical equivalence, and if and only if $C_{/f} \to C_{/y}$ is a categorical equivalence.

40.14. *Exercise.* Let C be a quasicategory and let $\pi\colon C \to hC$ be the tautological map to its homotopy category. Show that

(1) π is an isofibration, and
(2) $(\partial\Delta^n \subset \Delta^n) \boxslash \pi$ for $n = 0, 1, 2$.

Conclude that π is a categorical equivalence if and only if $(\partial\Delta^n \subset \Delta^n) \boxslash \pi$ for all $n \geq 3$.

40.15. **Monomorphisms which are categorical equivalences.** We can now give the following "lifting characterization" of monomorphisms which are categorical equivalences. **M 7 Mar**

40.16. **Proposition.** *Let $j\colon K \to L$ be a monomorphism of simplicial sets. Then j is a categorical equivalence if and only if* $\mathrm{Fun}(j, C)\colon \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C)$ *is a trivial fibration for all quasicategories C.*

Proof. Straightforward, using the fact that $\mathrm{Fun}(j, C)$ is an isofibration since j is mono (39.5), and that isofibrations which are categorical equivalences are trivial fibrations (40.8). □

40.17. *Remark.* The class $\mathrm{CatEq} \cap \overline{\mathrm{Cell}}$ of monomorphisms which are categorical equivalences is a weakly saturated class: (40.16) says it is the intersection of $\overline{\mathrm{Cell}}$ with the left complement of $\{\, p^{\square \mathrm{Cell}} \mid p\colon C \to *,\ C \in \mathrm{qCat} \,\}$.

Clearly $\overline{\mathrm{InnHorn}} \subseteq \mathrm{CatEq} \cap \overline{\mathrm{Cell}}$ by (25.17). However, $\overline{\mathrm{InnHorn}} \neq \mathrm{CatEq} \cap \overline{\mathrm{Cell}}$. For instance, every inner anodyne map is a bijection on vertices, but the monomorphism $\{0\} \to \mathrm{Iso}$ is not bijective on vertices but is a categorical equivalence. Even if we restrict to morphisms in $\mathrm{CatEq} \cap \overline{\mathrm{Cell}}$ which are bijections on vertices, we need not have an inner anodyne map, as the following example shows (40.18).

40.18. *Example* (Campbell's counterexample [Cam19]). Recall the simplicial set $H^\ell = \Delta^2/\Delta^{\{0,1\}}$ of (17.13), and write $\pi\colon \Delta^2 \to H^\ell$ for the evident quotient map. We have a commutative diagram

$$\begin{tikzcd} \Delta^1 \arrow[r, "f"] \arrow[dr, "\mathrm{id}"'] & H^\ell \arrow[d, "h"] & \Delta^1 \arrow[l, "g"'] \arrow[dl, "\mathrm{id}"] \\ & \Delta^1 & \end{tikzcd}$$

where f and g represent the edges $\pi(\langle 02 \rangle)$ and $\pi(\langle 12 \rangle)$ in H^ℓ, and h is the unique map such that $h\pi = \langle 001 \rangle$. Note that the maps f, g, h induce bijections on vertices.

The map g is actually inner anodyne: it is isomorphic to the cobase-change of the horn inclusion $\Lambda^2_1 \to \Delta^2$ along the unique map $\Lambda^2_1 \to \Delta^1$ which on vertices sends $0, 1 \mapsto 0$ and $2 \mapsto 1$.

Thus g is a categorical equivalence, whence so are h and hence f by the 2-out-of-3 property (27.11). On the other hand, we have observed that f is not an inner anodyne map (17.14). Therefore f is a categorical equivalence which is a monomorphism and a bijection on vertices, but is not an inner anodyne map.

41. Categorical fibrations

As we noted above (40.17), $\mathrm{CatEq} \cap \overline{\mathrm{Cell}}$ is weakly saturated. It is therefore natural to consider its right complement. Say that a map $p\colon X \to Y$ of simplicial sets is a **categorical fibration** (or **Joyal fibration**) if and only if $j \boxslash p$ for all j which are monomorphisms and categorical equivalences. I'll write $\mathrm{CatFib} = (\overline{\mathrm{Cell}} \cap \mathrm{CatEq})^{\boxslash}$ for the class of categorical fibrations. **categorical fibration** **Joyal fibration**

under operations such as composition, base-change, and retracts.

It turns out that the categorical equivalences *between quasicategories* are precisely the isofibrations.

41.1. **Proposition.** *A map $p\colon C \to D$ with D a quasicategory is an isofibration if and only if $j \boxslash p$ for every $j\colon K \to L$ which is both a monomorphism and a categorical equivalence.*

Proof. ($\Longleftarrow$) Immediate from the the characterization of isofibrations as maps between quasicategories in the right complement of $\mathrm{InnHorn} \cup \{\{0\} \subset \mathrm{Iso}\}$ (38.14). (Note that if p is in this right complement then in particular it is an inner fibration, so C must be a quasicategory.)

($\Longrightarrow$) Suppose p is an isofibration, and j a monomorphism and a categorical equivalence. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Fun}(L,C) \arrow[r, "p^{\square j}"'] \arrow[rr, bend left, "\mathrm{Fun}(j,C)"] \arrow[dr, bend right] & \mathrm{Fun}(K,C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L,D) \arrow[r, "q"'] \arrow[d] & \mathrm{Fun}(K,C) \arrow[d] \\ & \mathrm{Fun}(L,D) \arrow[r, "\mathrm{Fun}(j,D)"'] & \mathrm{Fun}(K,D) \end{tikzcd}$$

in which $p^{\square j}$, $\mathrm{Fun}(j, C)$, and $\mathrm{Fun}(j, D)$ are isofibrations by (39.5), and $\mathrm{Fun}(j, C)$ and $\mathrm{Fun}(j, D)$ are categorical equivalences since j is. Therefore $\mathrm{Fun}(j, C)$ and $\mathrm{Fun}(j, D)$, and hence the base-change

q, are trivial fibrations by (40.8). Therefore q is a categorical equivalence (25.11), whence $p^{\square j}$ is a categorical equivalence by 2-out-of-3 (27.11) and so a trivial fibration by (40.8). It follows that $p^{\square j}$ is surjective on vertices, i.e., $j \boxslash p$ as desired. $\square$

We have shown that isofibrations which are categorical equivalences are trivial fibrations (40.8). We can generalize this from isofibrations to categorical fibrations.

41.2. **Proposition.** *A map $p\colon X \to Y$ of simplicial sets is a trivial fibration if and only if it is a categorical fibration and a categorical equivalence.*

Proof. ($\Longrightarrow$) We know trivial fibrations are categorical equivalences (25.11), and it is clear that they are categorical fibrations by definition.

($\Longleftarrow$) If p is a categorical fibration and a categorical equivalence, factor p as $X \xrightarrow{j} Z \xrightarrow{q} Y$ with j a monomorphism and q a trivial fibration, by the small object argument applied to Cell. In particular q is a categorical equivalence (25.11), and thus so is j by 2-out-of-3 (27.11), and therefore $j \boxslash p$ since $j \in \mathrm{CatEq} \cap \overline{\mathrm{Cell}}$ and $p \in \mathrm{CatFib}$. Thus the "retract trick" (17.18) exhibits p as a retract of q, whence p is also a trivial fibration. $\square$

We have an enriched lifting property relating $\overline{\mathrm{Cell}}$ and CatFib.

41.3. **Proposition.** *If $p\colon X \to Y$ is a categorical fibration and $j\colon K \to L$ is a monomorphism, then*

$$q = p^{\square j}\colon \mathrm{Fun}(L, X) \to \mathrm{Fun}(K, X) \times_{\mathrm{Fun}(K,Y)} \mathrm{Fun}(L, Y)$$

is a categorical fibration. Furthermore, if either j or p is also a categorical equivalence, then q is a trivial fibration and hence a categorical equivalence.

Proof. To show that q is a categorical fibration, consider $i\colon A \to B$ a monomorphism which is a categorical equivalence. We have $i \boxslash q$ iff $(i\square j) \boxslash p$, so since p is a categorical fibration it suffices to show that the monomorphism $i\square j$ is a categorical equivalence, and by (40.16) it suffices to show $\mathrm{Fun}(i\square j, C)$ is a trivial fibration for every quasicategory C. This map is isomorphic to $r^{\square j}$ where $r = \mathrm{Fun}(i, C)$. Note that $r := \mathrm{Fun}(i, C)$ is an isofibration (39.5) and a categorical equivalence since i is, and therefore a trivial fibration (40.8). Thus $r^{\square j}$ is also a trivial fibration using $\overline{\mathrm{Cell}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$, and hence $\mathrm{Fun}(i\square j, C)$ is a trivial fibration as desired.

If p is also a categorical equivalence, then it is a trivial fibration by (41.2), so q is a trivial fibration by $\overline{\mathrm{Cell}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$.

If j is also a categorical equivalence, we want to shaw $i \boxslash p^{\square j}$ for any monomorphism i. But we have $i \boxslash p^{\square j}$ iff $(i\square j) \boxslash p$ iff $j \boxslash (p^{\square i})$. By what we have just proved $p^{\square i}$ is a categorical fibration, and therefore $j \boxslash p^{\square i}$ by definition. $\square$

42. Path factorization

Recall the path category of a quasicategory D, defined to be the full subcategory

$$\mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D) \subseteq \mathrm{Fun}(\Delta^1, D)$$

spanned by the objects corresponding to functors $\Delta^1 \to D$ which represent isomorphisms in D. I will sometimes write $\widehat{D} := \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D)$ as a shorthand for this. The restriction maps along $\{0\} \subset \Delta^1 \supset \{1\}$ induce functors $D \xleftarrow{r_0} \widehat{D} \xrightarrow{r_1} D$. Recall also (36.5) that functors $\widehat{H}\colon C \to \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D)$ correspond exactly to maps $H\colon C \times \Delta^1 \to D$ representing a natural isomorphism $f_0 \to f_1$ of functors $C \to D$ where $f_i = r_i\widetilde{H}_i$.

42.1. *Remark.* If D is a Kan complex (i.e., a quasigroupoid), then $\mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D) = \mathrm{Fun}(\Delta^1, D)$.

42.2. *Warning.* The path category $\mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D) \subseteq \mathrm{Fun}(\Delta^1, D)$ is not the same as the core $\mathrm{Fun}(\Delta^1, D)^{\mathrm{core}} \subseteq \mathrm{Fun}(\Delta^1, D)$, and neither of these are the same as $\mathrm{Fun}(\Delta^1, D^{\mathrm{core}})$, unless D is a quasigroupoid: the path category is always a full subcategory, whereas the core is typically not a full subcategory.

42.3. **Lemma.** *If D is a quasicategory, then the map $r = (r_0, r_1)\colon \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D) \subseteq \mathrm{Fun}(\Delta^1, D) \to D \times D$ from the path category induced by restriction along $\partial\Delta^1 \subset \Delta^1$ is an isofibration. Furthermore, each of the two functors r_0 and r_1 are trivial fibrations.*

Proof. To prove that r is an isofibration, observe that it is the composite of two maps $\mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D) \to \mathrm{Fun}(\Delta^1, D) \to \mathrm{Fun}(\partial\Delta^1, D)$ which are isofibrations: the first by an elementary argument (34.11), the second since it is a restriction along a monomorphism (39.5).

Since each of the two projections $D \times D \to D$ is an isofibration, it follows that both r_0 and r_1 are also isofibrations. To show that they are trivial fibrations, it suffices by (40.8) to show that they are categorical equivalences. In fact, the map $s\colon D \to \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D) \subseteq \mathrm{Fun}(\Delta^1, D)$ induced by restriction along projection $\Delta^1 \to *$ is a categorical inverse to either. To see this, note that $r_0 s = \mathrm{id}_D = r_1 s$, while we can easily construct natural isomorphisms $sr_0 \approx \mathrm{id}_{\mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D)} \approx sr_1$ (42.4). □

42.4. *Exercise.* Complete the proof of the lemma (42.3) by constructing natural isomorphisms $sr_0 \approx \mathrm{id} \approx sr_1$. (Hint: use suitably chosen maps $\Delta^1 \times \Delta^1 \to \Delta^1$ to define natural transformations $sr_0 \to \mathrm{id} \to sr_1$.)

42.5. **The path factorization construction.** For a functor $f\colon C \to D$ between quasicategories, we define a factorization $C \xrightarrow{j} P(f) \xrightarrow{p} D$ by means of the commutative diagram

$$\begin{tikzcd}
C \arrow[r, "j"] \arrow[dr, equal] & P(f) \arrow[r, "t"'] \arrow[d, "s_0"] \arrow[rr, bend left, "p"] & \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D) \arrow[r, "r_1"'] \arrow[d, "r_0"] & D \\
& C \arrow[r, "f"'] & D &
\end{tikzcd}$$

in which the square is a pullback square. The map j is the unique one so that $s_0 j = \mathrm{id}_C$ and $tj = \widetilde{\pi} f$, where $\widetilde{\pi}\colon D \to \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D) \subseteq \mathrm{Fun}(\Delta^1, D)$ is adjoint to the projection $D \times \Delta^1 \to D$.

42.6. *Example.* The path factorization of id_D is just $D \xrightarrow{\widetilde{\pi}} \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D) \xrightarrow{r_1} D$.

42.7. *Remark.* Note that the objects of $P(f)$ are pairs (c, α) consisting of an object $c \in C_0$ and an isomorphism $\alpha\colon f(c) \to d$ in D. The map j sends an object c to $(c, 1_{f(c)})$, while p sends (c, α) to d.

42.8. *Exercise.* Show that if $f\colon C \to D$ is a functor between ordinary categories, then $P(f)$ is also an ordinary category.

The properties of this construction are summarized by the following.

42.9. **Proposition.** *In the path factorization of f, the simplicial set $P(f)$ is a quasicategory, the map j is a categorical equivalence, and p is an isofibration. Furthermore s_0 is a trivial fibration.*

Proof. From (42.3) we know that both r_0 and r_1 are trivial fibrations. Therefore the base change s_0 of r_0 is a trivial fibration, and hence an inner fibration, which implies that $P(f)$ is a quasicategory.

Since s_0 is a trivial fibration it is a categorical equivalence (25.11), and thus j is a categorical equivalence by 2-out-of-3 (27.11).

To show that p is an isofibration, observe that there is actually a pullback square of the form

$$\begin{tikzcd} P(f) \arrow[r, "t"] \arrow[d, "s=(s_0,p)"'] & \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D) \arrow[d, "r=(r_0,r_1)"] \\ C \times D \arrow[r, "f\times \mathrm{id}_D"'] & D \times D \end{tikzcd}$$

(To see this, use patching of pullback squares where we regard $C \times D$ as a pullback of $C \xrightarrow{f} D \leftarrow D \times D$.) Since r is an isofibration (42.3), its base-change s is also an isofibration, and since the projection $\pi\colon C \times D \to D$ is an isofibration the composite $p = \pi s$ is an isofibration as desired. (I have here used several facts about isofibrations, which are consequences of the fact that isofibrations are exactly the categorical fibrations between quasicategories, and CatFib forms a weakly cosaturated class.) $\square$

43. Invariance properties of slices and limits

That isofibrations which are categorical equivalences are trivial fibrations (40.8) has a number of useful consequnece. For instance, we can reformulate the notion of limit or colimit of a functor using only the notions of slices and of categorical equivalence.

43.1. **Proposition.** *Let C be a quasicategory. Then a map $\widetilde{p}\colon K^{\triangleright} \to C$ is a colimit diagram iff the forgetful functor $C_{\widetilde{p}/} \to C_{p/}$ is a categorical equivalence, where $p := \widetilde{p}|K$. Likewise, a map $\widetilde{q}\colon K^{\triangleleft} \to C$ is a limit diagram iff the forgetful functor $C_{/\widetilde{q}} \to C_{/q}$ is a categorical equivalence, where $q := \widetilde{q}|K$.*

Proof. Immediate using the characterization of limits and colimits in terms of trivial fibrations (33.7), and the fact that the indicated forgetful functors are either left or right fibrations (32.15), and therefore are isofibrations (34.14). $\square$

43.2. **Invariance of slice categories under categorical equivalence.** We can now show that a categorical equivalence between quasicategories induces equivalences of its slice categories.

43.3. **Proposition.** *Let $f\colon C \to D$ be a categorical equivalence of quasicategories. For any map $q\colon K \to C$ of simplicial sets, the induced maps $C_{q/} \to D_{fq/}$ and $C_{/q} \to D_{/fq}$ on slice categories are also categorical equivalences.*

Proof. I'll prove the slice-under case; the slice-over case is exactly the same. Consider path factorization (42.9) of f, which gives a commutative diagram

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[rr, bend left, "f"] \arrow[dr, equal] & P(f) \arrow[r, "p"'] \arrow[d, "s_0"] & D \\ & C & \end{tikzcd}$$

where j is a categorical equivalence, p is an isofibration, and s_0 a trivial fibration. The hypothesis that f is a categorical equivalence implies that p is a categorical equivalence by 2-out-of-3 (27.11), and therefore that p is a trivial fibration by (40.8).

Recall that if f is a trivial fibration, then so is the induced map $C_{q/} \to D_{fq/}$ by $\overline{\mathrm{Cell}} \boxtimes \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$ on slices (32.13). Taking slices in the above diagram gives

$$\begin{tikzcd} C_{q/} \arrow[r, "\widetilde{j}"] \arrow[rr, bend left, "\widetilde{f}"] \arrow[dr, equal] & P(f)_{jq/} \arrow[r, "\widetilde{p}"'] \arrow[d, "\widetilde{s}_0"] & D_{fq/} \\ & C_{q/} & \end{tikzcd}$$

in which both $\widetilde{p}$ and $\widetilde{s}_0$ are trivial fibrations and thus categorical equivalences (25.11). Applying the 2-out-of-3 property shows that $\widetilde{f}$ is a categorical equivalence as desired. ☐

43.4. **Invariance of limits and colimits under categorical equivalence.** Categorical equivalences always preserve limits and colimits.

43.5. **Proposition.** *Let $f\colon C \to D$ be a categorical equivalence between quasicategories. A map $\widehat{p}\colon K^{\triangleright} \to C$ is a colimit cone in C if and only if $f\widehat{p}$ is a colimit cone in D, and a map $\widehat{q}\colon K^{\triangleleft} \to C$ is a limit cone in C if and only if $f\widehat{q}$ is a colimit cone in D,*

Proof. We prove the case of colimits. Consider the commutative diagram

$$\begin{tikzcd} C_{\widehat{p}/} \arrow[r, "f''"] \arrow[d, "\pi"'] & D_{f\widehat{p}/} \arrow[d, "\pi'"] \\ C_{p/} \arrow[r, "f'"'] & D_{fp/} \end{tikzcd}$$

where the horizontal maps are induced by f, and the vertical ones by restriction along $K \subset K^{\triangleright}$. Since f is a categorical equivalence, f' and f'' are also categorical equivalences by (43.3). Therefore by 2-out-of-3 for categorical equivalences (27.11) π is a categorical equivalence if and only if π' is, and the claim follows from (43.1). ☐

We can also show that the existence of colimit and limit cones is reflected by categorical equivalences. **W 9 Mar**

43.6. **Proposition.** *Let $f\colon C \to D$ be a categorical equivalence between quasicategories. A map $u\colon K \to C$ admits a colimit cone in C if and only if fu admits a colimit cone in D, and f admits a limit cone in C if and only if fu admits a colimit cone in D.*

Proof. We prove the case of colimits. The forward direction is immediate from (43.5), so we just prove the reverse direction. Suppose given $u\colon K \to C$ such that $w = fu$ has a colimit cone $\widehat{w}\colon K^{\triangleright} \to D$. Use a path factorization (42.9) to construct a commutative diagram of solid arrows

$$\begin{tikzcd} K \arrow[rr, tail] \arrow[d, "u"'] \arrow[dr, "v"] & & K^{\triangleright} \arrow[d, "\widehat{w}"] \arrow[dl, dotted, "\widehat{v}"'] \\ C \arrow[r, "i"] \arrow[dr, equal] & P(f) \arrow[r, "p"] \arrow[d, "s_0"] & D \\ & C & \end{tikzcd}$$

with $pi = f$, so that p and s_0 are trivial fibrations and i a categorical equivalence. Since p is a trivial fibration a lift $\widehat{v}$ exists in its square, which by (43.5) applied to the categorical equivalence p must be a colimit cone of $v := iu$. Therefore again by (43.5) applied to the categorical equivalence s_0, we see that $\widehat{u} := s_0\widehat{v}$ must be a colimit cone of $s_0 v = s_0 iu = u$, so u admits a colimit as desired. $\square$

Part 7. The fundamental theorem

44. The fundamental theorem of category theory

Recall that a functor $f\colon C \to D$ between quasicategories is said to be an *equivalence* there exists a $g\colon D \to C$ such that gf and fg are naturally isomorphic to the respective identity functors. When C and D are ordinary categories, there is a well-known criterion for the existence of such a g, namely: *f is an equivalence if and only if f is fully faithful and essentially surjective.* Here

- *fully faithful* means that $\mathrm{Hom}_C(x, y) \to \mathrm{Hom}_D(f(x), f(y))$ is a bijection of sets for every pair of objects $x, y \in \operatorname{ob} C$, and
- *essentially surjective* means that for every object $d \in \operatorname{ob} D$ there exists an object $c \in \operatorname{ob} C$ such that $f(c)$ is isomorphic to d.

I like to call this fact the *Fundamental Theorem of Category Theory.* This is non-standard and frankly pretentious terminology[25], though I am unaware of any standard abbreviated name for this result[26]. I want to give this fact a fancy name in order to signpost it, as it is quite *nonconstructive*: to prove it requires making a *choice* for each object d in D of an object c of C and an isomorphism $f(c) \approx d$ (so it in fact relies on an appropriate form of the axiom of choice).

44.1. *Exercise.* Prove the "Fundamental Theorem" for ordinary categories as follows: given $f\colon C \to D$ which is fully faithful and essentially surjective, make a choice of object $g(d) \in \operatorname{ob} C$ and isomorphism $\alpha(d)\colon f(g(d)) \to d$ for each object of d, and extend this to the data of a categorical inverse of f.

44.2. *Example.* Fix a field k. Let Mat be the category whose objects are non-negative integers $n \geq 0$, and whose morphims $A\colon n \to m$ are $(m \times n)$-matrices with entres in k, so that composition is matrix multiplication. Let Vect be the category of finite dimensional k-vector spaces and linear maps. Every basic class in linear algebra proves that the evident functor $F\colon \mathrm{Mat} \to \mathrm{Vect}$ is fully faithful and essentially surjective. Therefore F is an equivalence of categories. However, there is no *canonical* choice of an inverse functor, whose construction amounts to making an arbitrary choice of basis for each vector space.

We are going to state and then prove an analogue of this result for functors between quasicategories. This will first require an analogue of hom-sets, namely the *quasigroupoid* of maps between two objects, also called the *mapping space*.

45. Mapping spaces of a quasicategory

Given a quasicategory C and objects $x, y \in C_0$, the **mapping space** (or **mapping quasigroupoid**) from x to y is the simplicial set defined by the pullback square **mapping space** **mapping quasigroupoid**

$$\begin{tikzcd} \mathrm{map}_C(x,y) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, C) \arrow[d] \\ \{(x,y)\} \arrow[r] & C \times C \end{tikzcd}$$

That is, $\mathrm{map}_C(x, y)$ is the fiber of the restriction map $\mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\partial\Delta^1, C)$ over the point $(x, y) \in (C \times C)_0$, where we use the isomorphism $\mathrm{Fun}(\partial\Delta^1, C) \approx C \times C$ induced by the isomorphism $\partial\Delta^1 \approx \Delta^0 \amalg \Delta^0$.

[25] E.g., the Fundamental Theoroms of Arithmetic, Algebra, Calculus, etc. But if they can have a Fundamental Theorem, why can't we? (It should be mentioned that there is another candidate for a fundamental theorem, namely the Yoneda lemma.)

[26] I also don't know when it was first formulated, or who first stated it.

If $C = N(A)$ is the nerve of a category, then $\mathrm{map}_C(x, y)$ is a discrete simplicial set (2.8) corresponding to the set $\mathrm{Hom}_C(x, y)$.

45.1. **Mapping spaces are Kan complexes.** The terminology "space" is justified by the following

45.2. **Proposition.** *When C is a quasicategory, the simplicial sets $\mathrm{map}_C(x, y)$ are quasigroupoids (and hence Kan complexes by (35.2)).*

Proof. This is a consequence of (39.3), since the inclusion $i\colon \partial\Delta^1 \to \Delta^1$ induces a bijection on vertices, and so restriction along i is conservative. $\square$

We note a relative generalization, which will sometimes be useful.

45.3. **Proposition.** *If $p\colon C \to D$ is an inner fibration between quasicategories, then the induced maps $p\colon \mathrm{map}_C(x, y) \to \mathrm{map}_D(px, py)$ are Kan fibrations.*

Proof. Since $\partial\Delta^1 \subset \Delta^1$ is a bijection on vertices, the induced pullback-hom map $q = p^{\square(\partial\Delta^1 \subset \Delta^1)}\colon \mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\partial\Delta^1, C) \times_{\mathrm{Fun}(\partial\Delta^1, D)} \mathrm{Fun}(\Delta^1, D)$ is an isofibration (39.2). We have pullback squares

$$\begin{tikzcd}
\mathrm{map}_C(x,y) \arrow[r, "q'"] \arrow[d, tail] & \mathrm{Map}_D(px,py) \arrow[r] \arrow[d, tail] & \{(x,y)\} \arrow[d, tail] \\
\mathrm{Fun}(\Delta^1, C) \arrow[r, "q"'] & \mathrm{Fun}(\partial\Delta^1, C) \times_{\mathrm{Fun}(\partial\Delta^1, D)} \mathrm{Fun}(\Delta^1, D) \arrow[r] & \mathrm{Fun}(\partial\Delta^1, C)
\end{tikzcd}$$

and so the base change q' is an isofibration between Kan complexes and hence a Kan fibration. $\square$

45.4. **Mapping spaces and homotopy classes.** The set of morphisms $x \to y$ in a quasicategory C is precisely the set of objects of $\mathrm{map}_C(x, y)$. Two such are isomorphic as objects in $\mathrm{map}_C(x, y)$ if and only if they are homotopic in C.

45.5. **Proposition.** *Let C be a quasicategory. For any two maps $f, g\colon x \to y$ in C, we have that $f \approx g$ (equivalence under the relation used to define the homotopy category hC) if and only if f and g are isomorphic as objects of the quasigroupoid $\mathrm{map}_C(x, y)$. That is,*

$$\mathrm{Hom}_{hC}(x, y) \approx \pi_0 \,\mathrm{map}_C(x, y)$$

for every pair x, y of objects of C.

Proof. Suppose $f, g \in \mathrm{map}_C(x, y)_0$ are isomorphic, so that in particular there is a morphism $f \to g$ in the quasigroupoid $\mathrm{map}_C(x, y)$. This amounts to a map $\Delta^1 \times \Delta^1 \to C$ which can be represented by a diagram of cells of C of the form:

$$\begin{tikzcd}
x \arrow[r, "g"] & y \\
x \arrow[u, "1_x"] \arrow[r, "f"'] \arrow[ur, "h"] & y \arrow[u, "1_y"']
\end{tikzcd}$$

This explicitly exhibits a chain $f \sim_r h \sim_\ell g$ of homotopies, so $f \approx g$ as desired.

Conversely, if $f \approx g$, we can explicitly construct a map $H\colon f \to g$ in $\mathrm{map}_C(x, y)$: in terms of the above picture, let $h = g$, let b be an explicit choice of right-homotopy $f \sim_r g$, and let $a = g_{001}$. $\square$

45.6. **Extended mapping spaces and composition.** Given a finite list $x_0, \dots, x_n \in C_0$ of objects in a quasicategory, we have an **extended mapping space**. These are the simplicial sets defined by the pullback squares

extended mapping space

$$\begin{tikzcd} \mathrm{map}_C(x_0,\dots,x_n) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^n, C) \arrow[d] \\ \{(x_0,\dots,x_n)\} \arrow[r] & C^{\times(n+1)} \end{tikzcd}$$

where the right-hand vertical arrow is induced by restriction along $\mathrm{Sk}_0\,\Delta^n \to \Delta^n$, using the isomorphism $\mathrm{Sk}_0\,\Delta^n \approx \coprod_{n+1}\Delta^0$, whence $\mathrm{Fun}(\mathrm{Sk}_0\,\Delta^n, C) \approx C^{\times(n+1)}$. By (39.3) the extended mapping spaces are quasigroupoids.

On the other hand, we may consider the fibers of $\mathrm{Fun}(I^n, C) \to C^{\times(n+1)}$ defined by restriction along $\mathrm{Sk}_0\,\Delta^n = \mathrm{Sk}_0\, I^n \to I^n$, where $I^n \subset \Delta^n$ is the spine. The fibers of this map are isomorphic to n-fold products of mapping spaces $\mathrm{map}_C(x_{n-1}, x_n) \times \cdots \times \mathrm{map}_C(x_0, x_1)$.

45.7. **Lemma.** *The map*

$$g_n\colon \mathrm{map}_C(x_0,\dots,x_n) \to \mathrm{map}_C(x_{n-1},x_n) \times \cdots \times \mathrm{map}_C(x_0,x_1)$$

induced by restriction along the spine inclusion $I^n \subseteq \Delta^n$ is a trivial fibration. In particular, this map is a categorical equivalence between quasigroupoids.

Proof. The map g_n is a base change of $p\colon \mathrm{Fun}(\Delta^n, C) \to \mathrm{Fun}(I^n, C)$. Since $I^n \subset \Delta^n$ is inner anodyne (16.14), and C is a quasicategory, the map p is a trivial fibration by enriched lifting using $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (21.8). ☐

The inclusions $I^2 \subset \Delta^2 \supset \Delta^{\{0,2\}}$ induce restriction maps

$$\mathrm{Fun}(I^2, C) \xleftarrow{\sim} \mathrm{Fun}(\Delta^2, C) \to \mathrm{Fun}(\Delta^{\{0,2\}}, C)$$

in which the first map is a trivial fibration. As noted earlier (22.9) by choosing a categorical inverse to the first map (e.g., a section, since it is a trivial fibration) we obtain a "composition functor" $\mathrm{Fun}(I^2, C) \to \mathrm{Fun}(\Delta^1, C)$.

For any triple (x_0, x_1, x_2) of objects of C, the above maps restrict to maps between subcomplexes:

$$\mathrm{map}_C(x_1,x_2) \times \mathrm{map}_C(x_0,x_1) \xleftarrow[\sim]{g_2} \mathrm{map}_C(x_0,x_1,x_2) \to \mathrm{map}_C(x_0,x_2).$$

As g_2 is a trivial fibration (45.7), we can choose a section for it, and so by the same construction as (22.9) we obtain a "composition" functor

$$\mathrm{comp}\colon \mathrm{map}_C(x_1,x_2) \times \mathrm{map}_C(x_0,x_1) \to \mathrm{map}_C(x_0,x_2). \tag{45.8}$$

Again, this depends on a choice of categorical inverse to g_2. However, any two categorical inverses to g_2 are naturally isomorphic (24.4), and therefore comp is defined up to natural isomorphism. That is, it is a well-defined map in $h\mathrm{Kan}$, the homotopy category of Kan complexes (27.1). These noncanonical composition maps are "associative up to homotopy".

45.9. **Proposition.** *The two maps obtained by composing the sides of the square*

$$\begin{tikzcd} \mathrm{map}_C(x_2,x_3) \times \mathrm{map}_C(x_1,x_2) \times \mathrm{map}_C(x_0,x_1) \arrow[r, "\mathrm{id}\times\mathrm{comp}"] \arrow[d, "\mathrm{comp}\times\mathrm{id}"'] & \mathrm{map}_C(x_2,x_3) \times \mathrm{map}_C(x_0,x_2) \arrow[d, "\mathrm{comp}"] \\ \mathrm{map}_C(x_1,x_3) \times \mathrm{map}_C(x_0,x_1) \arrow[r, "\mathrm{comp}"'] & \mathrm{map}_C(x_0,x_3) \end{tikzcd}$$

are naturally isomorphic. That is, the diagram commutes in $h\mathrm{Kan} \subset h\mathrm{qCat}$.

Proof. Here is a diagram of Kan complexes which actually commutes "on the nose", i.e., not merely in the homotopy category, but in sSet. I use "$\langle x, y, z\rangle$" as shorthand for "$\mathrm{map}_C(x, y, z)$", etc.

$$\begin{tikzcd}
\langle x_2,x_3\rangle \times \langle x_1,x_2\rangle \times \langle x_0,x_1\rangle & \langle x_2,x_3\rangle \times \langle x_0,x_1,x_2\rangle \arrow[l, "\sim"'] \arrow[r] & \langle x_2,x_3\rangle \times \langle x_0,x_2\rangle \\
\langle x_1,x_2,x_3\rangle \times \langle x_0,x_1\rangle \arrow[u, "\sim"] \arrow[d] & \langle x_0,x_1,x_2,x_3\rangle \arrow[l, "\sim"'] \arrow[r] \arrow[u, "\sim"] \arrow[d] & \langle x_0,x_2,x_3\rangle \arrow[u, "\sim"] \arrow[d] \\
\langle x_1,x_3\rangle \times \langle x_0,x_1\rangle & \langle x_0,x_1,x_3\rangle \arrow[l, "\sim"'] \arrow[r] & \langle x_0,x_3\rangle
\end{tikzcd}$$

The maps labelled "$\xrightarrow{\sim}$" are trivial fibrations, and so are categorical equivalences. All the maps in the above diagram are obtained via restriction along inclusions in

$$\begin{tikzcd}
\Delta^{\{2,3\}} \cup \Delta^{\{1,2\}} \cup \Delta^{\{0,1\}} \arrow[r, "\sim"] \arrow[d, "\sim"] & \Delta^{\{2,3\}} \cup \Delta^{\{0,1,2\}} \arrow[d, "\sim"] & \Delta^{\{2,3\}} \cup \Delta^{\{0,2\}} \arrow[l] \arrow[d, "\sim"] \\
\Delta^{\{1,2,3\}} \cup \Delta^{\{0,1\}} \arrow[r, "\sim"] & \Delta^3 & \Delta^{\{0,2,3\}} \arrow[l] \\
\Delta^{\{1,3\}} \cup \Delta^{\{0,1\}} \arrow[r, "\sim"] \arrow[u] & \Delta^{\{0,1,3\}} \arrow[u] & \Delta^{\{0,3\}} \arrow[l] \arrow[u]
\end{tikzcd}$$

where the maps labelled "$\xrightarrow{\sim}$" are inner anodyne (being generalized inner horn inclusions (16.12)), and which therefore give rise to trivial fibrations in the previous diagram by the same argument we used to define comp. After passing from Kan $\subset s$Set to hKan the categorical equivalences become isomorphisms, and the result follows. $\square$

45.10. **Segal categories.** Thus, a quasicategory does not quite give rise in this way to a category "enriched over Kan complexes". Although we can define a composition law, it is not uniquely determined, and is only associative "up to homotopy".

What we do get is a Segal category. A **Segal category** is a functor **Segal category**

$$M\colon \Delta^{\mathrm{op}} \to s\mathrm{Set}$$

such that

(1) the simplicial set $M([0])$ is discrete, i.e., $M([0]) = \mathrm{Sk}_0\, M([0])$,
(2) for each $n \geq 1$, the simplicial set $M([n])$ is a Kan complex, and
(3) for each $n \geq 1$ the "Segal map"

$$M([n]) \xrightarrow{(\langle n-1,n\rangle^*,\dots,\langle 0,1\rangle^*)} M([1]) \times_{M([0])} \cdots \times_{M([0])} M([1])$$

is a categorical equivalence.

Given a quasicategory C, we obtain a functor $M_C\colon \Delta^{\mathrm{op}} \to s\mathrm{Set}$ by

$$\begin{aligned}
M_C([0]) &:= \mathrm{Sk}_0\, C,\\
M_C([n]) &:= \mathrm{Fun}(\Delta^n, C) \times_{\mathrm{Fun}(\mathrm{Sk}_0\,\Delta^n, C)} \mathrm{Fun}(\mathrm{Sk}_0\,\Delta^n, \mathrm{Sk}_0\, C)\\
&\approx \coprod_{x_0,\dots,x_n \in C_0} \mathrm{map}_C(x_0,\dots,x_n).
\end{aligned}$$

This object encodes all the structure we used above. For instance, the zig-zag

$$M_C([1]) \times_{M_C([0]} M_C([1]) \xleftarrow{(\langle 12\rangle^*,\langle 01\rangle^*)} M_C([2]) \xrightarrow{\langle 02\rangle^*} M_C([1])$$

is a coproduct over all triples $x_0, x_1, x_2 \in C_0$ of the zig-zag (45.8) used to define "composition".

You also get a Segal category from a "simplicially enriched" category. For instance, suppose $\mathcal{C}$ is a (small) category which is enriched over the category of Kan complexes, with object set $\mathrm{ob}\,\mathcal{C}$, and function objects $\mathcal{C}(x,x') \in \mathrm{Kan}$ for each x, x'. Then we can define $M_{\mathcal{C}} \colon \Delta^{\mathrm{op}} \to s\mathrm{Set}$ by

$$M_{\mathcal{C}}([0]) := \operatorname{ob}\mathcal{C},$$
$$M_{\mathcal{C}}([n]) := \coprod_{x_0,\dots,x_n \in \operatorname{ob}\mathcal{C}} \mathcal{C}(x_{n-1},x_n) \times \cdots \times \mathcal{C}(x_0,x_1).$$

We thus obtain functors

$$\mathrm{qCat} \to \mathrm{SeCat} \leftarrow s\mathrm{Cat}^{\mathrm{Kan}}$$

relating quasicategories, Segal categories, and Kan-complex-enriched categories. Simplicially enriched categories were proposed as a model for ∞-categories by Dwyer and Kan[27], while Segal categories were proposed as a model for ∞-categories by Hirschowitz and Simpson [HS01][28]. These models are all known to be equivalent to quasicategories in a suitable sense; see [Ber10] for more about these models and their comparison.

45.11. **The enriched homotopy category of a quasicategory.** Given a quasicategory C we can produce a vestigial version of a category enriched over quasigroupoids, called the **enriched homotopy category** of C and denoted $\mathcal{H}C$.[29] This object will be a category *enriched over* $h\mathrm{Kan}$, where $h\mathrm{Kan}$ is the full subcategory of $h\mathrm{qCat}$ spanned by Kan complexes. The underlying category of the enriched category $\mathcal{H}C$ will just be the homotopy category hC of C. **enriched homotopy category**

We now define $\mathcal{H}C$. The objects of $\mathcal{H}C$ are just the objects of C. For any two objects $x, y \in C_0$, we have the quasigroupoid

$$\mathcal{H}C(x,y) := \operatorname{map}_C(x,y)$$

which we will regard as an object of the homotopy category $h\mathrm{Kan}$ of Kan complexes. Composition $\mathcal{H}C(x_1,x_2) \times \mathcal{H}C(x_0,x_1) \to \mathcal{H}C(x_0,x_2)$ is the composition map defined above (45.8), which is well-defined as a morphism in $h\mathrm{Kan}$. Composition is associative as shown above (45.9). (Remember that $h\mathrm{qCat}$, and thus also the full subcategory $h\mathrm{Kan}$, has finite products, which coincide with products of simplicial sets (27.4).)

The *underlying* ordinary category of $\mathcal{H}C$ is just the ordinary homotopy category hC, since

$$\operatorname{Hom}_{h\mathrm{Kan}}(\Delta^0, \operatorname{map}_C(x,y)) \approx \pi_0 \operatorname{map}_C(x,y) \approx \operatorname{Hom}_{hC}(x,y).$$

45.12. *Warning.* A quasicategory C *cannot* be recovered from its enriched homotopy category $\mathcal{H}C$, not even up to equivalence. Furthermore, there exist $h\mathrm{Kan}$-enriched categories which do not arise as $\mathcal{H}C$ for any quasicategory C. A proof is outside the scope of these notes: counterexamples may be produced (for instance) from examples of associative H-spaces which are not loop spaces, and examples of spaces which admit several inequivalent loop space structures.

45.13. *Exercise.* Let C and D be quasicategories. Show that there is an isomorphism $\mathcal{H}(C \times D) \approx \mathcal{H}C \times \mathcal{H}D$ of $h\mathrm{Kan}$-enriched categories.

46. The fundamental theorem of quasicategory theory

46.1. **Fully faithful and essentially surjective functors between quasicategories.** Note that any functor $f \colon C \to D$ of quasicategories induces functors $\operatorname{map}_C(x,y) \to \operatorname{map}_D(f(x), f(y))$ for every pair of objects x, y in C. We say that a functor $f \colon C \to D$ between quasicategories is

- **fully faithful** if for every pair $c, c' \in C_0$, the resulting map $\operatorname{map}_C(c,c') \to \operatorname{map}_D(fc, fc')$ is a categorical equivalence, and **fully faithful**
- **essentially surjective** if for every $d \in D_0$ there exists a $c \in C_0$ together with an isomorphism **essentially surjective**

[27] They called them "homotopy theories" instead of "∞-categories; see [DS95, §11.6].

[28] In fact, they generalize this to "Segal n-categories", which were the first effective model for (∞, n)-categories.

[29] Lurie usually calls this "hC" in [Lur09], though he also uses that notation for the ordinary homotopy category of C that we have already discussed. I prefer to have two separate notations.

$fc \to d$ in D; that is, if the induced functor $hf\colon hC \to hD$ of ordinary categories is essentially surjective.

46.2. *Remark.* Another way to say this is as follows: $f\colon C \to D$ is fully faithful and essentially surjective iff the induced hKan-enriched functor $\mathcal{H}f\colon \mathcal{H}C \to \mathcal{H}D$ is an equivalence of *enriched* categories. **F 11 Mar**

46.3. **Proposition.** *If $f\colon C \to D$ is a categorical equivalence between quasicategories, then f is fully faithful and essentially surjective.*

Proof. To prove essential surjectivity, choose any categorical inverse g to f and natural isomorphism $\alpha\colon fg \to \mathrm{id}_D$. Then for any $d \in D_0$ we get an object $c := g(d) \in C_0$ and an isomorphism $\alpha(d)\colon f(c) \to d$ in D.

To show that f is fully faithful, choose a categorical inverse g of f. Given $x, y \in C_0$, consider the induced diagram of quasigroupoids

$$\begin{tikzcd}
\mathrm{map}_C(x,y) \arrow[r, "f"] \arrow[rr, bend left, "gf"] & \mathrm{map}_D(fx,fy) \arrow[r, "g"] \arrow[rr, bend right, "fg"'] & \mathrm{map}_C(gfx,gfy) \arrow[r, "f"] & \mathrm{map}_D(fgfx,fgfy)
\end{tikzcd}$$

By the 2-out-of-6 property for categorical equivalences (27.11), it will suffice to show that the maps marked gf and fg are categorical equivalences between the respective mapping spaces. Since $gf\colon C \to C$ and $fg\colon D \to D$ are naturally isomorphic to the identity maps of C and D respectively, the claim follows from (46.4) which we prove below. $\square$

46.4. **Proposition.** *If $f_0, f_1\colon C \to D$ are functors which are naturally isomorphic, then f_0 is fully faithful if and only if f_1 is.*

To prove this we need to apply the following to the path category.

46.5. **Lemma.** *Any trivial fibration $p\colon C \to D$ between quasicategories is fully faithful.*

Proof. For $x, y \in C_0$ we have a diagram of pullback squares

$$\begin{tikzcd}
\mathrm{map}_C(x,y) \arrow[r] \arrow[d, "q"'] & \mathrm{Fun}(\Delta^1, C) \arrow[d, "p^{\square(\partial\Delta^1 \subset \Delta^1)}"] & \\
\mathrm{map}_D(px,py) \arrow[r] \arrow[d] & \mathrm{Fun}(\partial\Delta^1, C) \times_{\mathrm{Fun}(\partial\Delta^1, D)} \mathrm{Fun}(\Delta^1, D) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, D) \arrow[d] \\
\{(x,y)\} \arrow[r] & \mathrm{Fun}(\partial\Delta^1, C) \arrow[r] & \mathrm{Fun}(\partial\Delta^1, D)
\end{tikzcd}$$

The pullback-hom $p^{\square(\partial\Delta^1 \subset \Delta^1)}$ is a trivial fibration using $\overline{\mathrm{Cell}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$, so q is a trivial fibration and thus a categorical equivalence (25.11). $\square$

Proof of (46.4). Consider a natural isomorphism $H\colon C \times \Delta^1 \to D$ between f_0 and f_1, and write $\widetilde{H}\colon C \to \widehat{D} \subseteq \mathrm{Fun}(\Delta^1, D)$ for its adjoint, where $\widehat{D} = \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D)$. Then lemma (42.3) implies that in the commutative diagram

$$\begin{tikzcd}
 & & \mathrm{Fun}(\{0\}, D) = D \\
C \arrow[r, "\widetilde{H}"] \arrow[urr, bend left, "f_0"] \arrow[drr, bend right, "f_1"'] & \widehat{D} \arrow[ur, "r_0"'] \arrow[dr, "r_1"] & \\
 & & \mathrm{Fun}(\{1\}, D) = D
\end{tikzcd}$$

both r_0 and r_1 are trivial fibrations. Because r_0 and r_1 are trivial fibrations, for any $x, y \in C_0$ we get a commutative diagram

$$\begin{tikzcd} & & \mathrm{map}_D(f_0(x), f_0(y)) \\ \mathrm{map}_C(x,y) \arrow[r, "\widetilde{H}"] \arrow[urr, bend left, "f_0"] \arrow[drr, bend right, "f_1"'] & \mathrm{map}_{\widehat{D}}(\widetilde{H}(x), \widetilde{H}(y)) \arrow[ur, "\sim"'] \arrow[dr, "\sim"] & \\ & & \mathrm{map}_D(f_1(x), f_1(y)) \end{tikzcd}$$

in which the maps indicated by "$\sim$" are categorical equivalences by (46.5). Using the 2-out-of-3 property of categorical equivalences (27.11), we see that the map marked f_0 is a categorical equivalence if and only if the map marked $\widetilde{H}$ is, if and only if the map marked f_1 is. Thus we have shown that $f_0 \colon C \to D$ is fully faithful if and only if $f_1 \colon C \to D$ is fully faithful. □

We've finished proving the lemma we needed for the proof that categorical equivalences are fully faithful (46.3).

We note a useful fact: to check that a functor is fully faithful, it suffices to check the defining property on *representatives of isomorphism classes* of objects.

46.6. **Proposition.** *Let $f \colon C \to D$ be a functor between quasicategories, and let $S \subset C_0$ be a subset of objects which includes a representative of every isomorphism class in C. Then f is fully faithful if and only if $\mathrm{map}_C(c, c') \to \mathrm{map}_D(fc, fc')$ is a categorical equivalence for all $c, c' \in S$.*

Proof. The only-if direction is immediate from the definition of fully faithful. To prove the if direction, let $x, x' \in C_0$ and choose isomorphisms $\alpha \colon x \to c$ and $\alpha' \colon x' \to c'$ where $c, c' \in S$. We may interpret α and α' as objects of $\widehat{C} = \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, C) \subseteq \mathrm{Fun}(\Delta^1, C)$. We obtain a commutative diagram

$$\begin{tikzcd} \mathrm{map}_C(x,x') \arrow[d, "f"'] & \mathrm{map}_{\widehat{C}}(\alpha,\alpha') \arrow[l, "r_0"'] \arrow[r, "r_1"] \arrow[d, "\widehat{f}"'] & \mathrm{map}_C(c,c') \arrow[d, "f"'] \\ \mathrm{map}_D(fx,fx') & \mathrm{map}_{\widehat{D}}(\widehat{f}\alpha,\widehat{f}\alpha') \arrow[l, "r_0"] \arrow[r, "r_1"'] & \mathrm{map}_D(fc,fc') \end{tikzcd}$$

where the vertical arrows are induced by $f \colon C \to D$ and $\widehat{f} \colon \widehat{C} \to \widehat{D}$, where $\widehat{f}$ is the restriction of $\mathrm{Fun}(\Delta^1, f) \colon \mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\Delta^1, D)$ to the path categories $\widehat{C} = \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, C)$ and $\widehat{D} = \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D)$. The maps marked r_0 and r_1 are categorical equivalences by (42.3) and (46.5). Therefore the left-hand vertical arrow is a categorical equivalence using the hypothesis on f and 2-out-of-3 for categorical equivalences (27.11). □

46.7. **The fundamental theorem for quasicategories.** The converse to (46.3) is also true, whence: *A map $f \colon C \to D$ between quasicategories is a categorical equivalence if and only if it is fully faithful and essentially surjective.*

This is a non-trivial result. It gives a necessary and sufficient condition for $f \colon C \to D$ to admit a categorical inverse, but it does not spell out how to construct such an inverse. After some preliminaries, we will prove this as (48.2).

46.8. **2-out-of-6 for fully faithful essentially surjective functors.** The following result will be useful in the proof of the fundamental theorem. Recall the *2-out-of-6* and *2-out-of-3* properties of a class of morphisms (27.6), and that the class of categorical equivalences has these properties (27.11).

46.9. **Proposition.** *The class of fully faithful and essentially surjective functors between quasicategories satisfies the 2-out-of-6 property, and thus the 2-out-of-3 property.*

Proof. Any identity functor $\mathrm{id}\colon C \to C$ is manifestly fully faithful and essentially surjective.

Next note that if a functor $f\colon C \to D$ between quasicategories is fully faithful and essentially surjective, then the induced functor $hf\colon hC \to hD$ is an equivalence of ordinary categories. Conversely, if hf is an equivalence, then f is essentially surjective.

Suppose $C \xrightarrow{f} D \xrightarrow{g} E \xrightarrow{h} F$ is a sequence of functors between quasicategories such that gf and hg are fully faithful and essentially surjective. The induced sequence $hC \to hD \to hE \to hF$ of functors on homotopy categories has the same property, and thus all the functors between homotopy categories are equivalences. From this we conclude immediately that f, g, h, hgf are essentialy surjective.

Given objects $x, y \in C_0$, we have induced maps

$$\begin{tikzcd} \mathrm{map}_C(x,y) \arrow[r, "f"] \arrow[rr, bend left, "gf"] & \mathrm{map}_D(fx,fy) \arrow[r, "g"] \arrow[rr, bend right, "hg"'] & \mathrm{map}_E(gfx,gfy) \arrow[r, "h"] & \mathrm{map}_F(hgfx,hgfy) \end{tikzcd}$$

The hypothesis that gf and hg are fully faithful implies that the indicated arrows are categorical equivalences, and hence all arrows are by (27.11). Because f and gf are essentially surjective, the collections of objects $\{\, fx \mid x \in C_0 \,\} \subseteq D_0$ and $\{\, gfx \mid x \in C_0 \,\} \subseteq E_0$ include representatives of every isomorphism class of D and E respectively, and thus (46.6) implies that f, g, h, and therefore hgf, are fully faithful. $\square$

46.10. **Reduction steps.** To prove the fundamental theorem of quasicategories for a general map between quasicategories, we can reduce to the special case of isofibrations.

46.11. **Lemma.** *To prove that every fully faithful and essentially surjective functor of quasicategories is a categorical equivalence, it suffices to prove it for the special case of isofibrations.*

Proof. Let $f\colon C \to D$ be a functor which is fully faithful and essentially surjective. Consider the path factorization

$$C \xrightarrow{j} P(f) \xrightarrow{p} D$$

of f, with j a categorical equivalence and p an isofibration (42.9). Recall that the class of categorical equivalences satisfies 2-out-of-3 (27.11), as does the class of functors which are fully faithful and essentially surjective (46.9). Since every categorical equivalence (such as j) is fully faithful and essentially surjective (46.3), the claim follows. $\square$

We will prove the special case of isofibrations by showing that *if an isofibration is fully faithful and essentially surjective, then it is a trivial fibration,* i.e., that $\mathrm{Cell} \boxslash p$ for any isofibration p which is fully faithful and essentially surjective. First note the following.

46.12. **Proposition.** *An isofibration p is essentially surjective if and only if it is surjective on vertices, i.e., iff $(\partial\Delta^0 \subset \Delta^0) \boxslash p$.*

Proof. The $\Longleftarrow$ implication is obvious, while $\Longrightarrow$ is a straightforward exercise using definition of isofibration. $\square$

Thus, to complete the proof of the fundamental theorem, it suffices to show that if an isofibration p is fully faithful, then $\mathrm{Cell}_{\geq 1} \boxslash p$, which will will prove as (48.1).

47. A fiberwise criterion for trivial fibrations to quasigroupoids

We give a criterion for an isofibration to be a trivial fibration when the target is a *quasigroupoid*. This criterion is in terms of its fibers. The **fiber** of a map $p\colon X \to Y$ over a vertex $y \in Y_0$ is defined to be the pullback of p along $\{y\} \to Y$. We will write $p^{-1}(y) = \{y\} \times_Y X$ for the fiber of p over y.

fiber

Recall that a quasicategory C is categorically equivalent to the terminal category Δ^0 if and only if $C \to \Delta^0$ is a trivial fibration (40.11). We call such an object a **contractible Kan complex**. If $p\colon X \to Y$ is a trivial fibration, then since $\mathrm{TrivFib} = \mathrm{Horn}^{\boxslash}$ we see immediately that every projection $p^{-1}(y) \to *$ from a fiber is a trivial fibration; i.e., the fibers of a trivial fibration are necessarily contractible Kan complexes. The "fiberwise criterion" asserts the converse for isofibrations to Kan complexes.

contractible Kan complex

47.1. **Proposition.** *Let $p\colon C \to D$ be an isofibration in which D is a quasigroupoid. Then p is a trivial fibration if and only if every fiber of p is a contractible Kan complex.*

Proof. We have just observed $(\Longrightarrow)$, so we prove $(\Longleftarrow)$. So suppose p is an isofibration to a quasigroupoid whose fibers are contractible Kan complexes, and consider a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "a"] \arrow[d, tail] & C \arrow[d, "p"] \\ \Delta^n \arrow[r, "b"'] \arrow[ur, dotted] & D \end{tikzcd}$$

We will "deform" the lifting problem (a,b) to one of the same type which lives inside a single fiber of p. As such lifting problems have solutions by the hypothesis that the fibers of p are contractible Kan complexes, the covering homotopy extension property (39.8) implies that the original lifting problem has a solution.

Let $\gamma\colon \Delta^n \times \Delta^1 \to \Delta^n$ be the unique map which on vertices is given by $\gamma(k,0) = k$ and $\gamma(k,1) = n$, i.e., the unique natural transformation $\gamma\colon \mathrm{id}_{\Delta^n} \to \langle n \dots n\rangle$ of functors $\Delta^n \to \Delta^n$. I claim that we can construct a lift u in

$$\begin{tikzcd} \partial\Delta^n \times \{0\} \arrow[rrr, "a"] \arrow[d, tail] & & & C \arrow[d, "p"] \\ \partial\Delta^n \times \Delta^1 \arrow[r, tail] \arrow[urrr, dotted, "u"] & \Delta^n \times \Delta^1 \arrow[r, "\gamma"'] & \Delta^n \arrow[r, "b"'] & D \end{tikzcd}$$

which represents a natural isomorphism of functors $\partial\Delta^n \to C$. To see this, just note that this lifting problem is adjoint to one of the form

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d, tail] & \mathrm{Fun}(\partial\Delta^n, C) \arrow[d, "\mathrm{Fun}(\partial\Delta^n, p)"] \\ \Delta^1 \arrow[r] \arrow[ur, dotted] & \mathrm{Fun}(\partial\Delta^n, D) \end{tikzcd}$$

where the map along the bottom represents an isomorphisms since D and hence $\mathrm{Fun}(\partial\Delta^n, D)$ is a quasigroupoid, and $\mathrm{Fun}(\partial\Delta^n, p)$ is an isofibration since p is (39.5).

The lift u gives a commutative square

$$\begin{tikzcd} \partial\Delta^n \times \Delta^1 \arrow[r, "u"] \arrow[d, tail] & C \arrow[d, "p"] \\ \Delta^n \times \Delta^1 \arrow[r, "b\gamma"'] & D \end{tikzcd}$$

which represents a morphism $e = (u, b\gamma)$ in $\mathrm{Fun}(\partial\Delta^n, C) \times_{\mathrm{Fun}(\partial\Delta^n, D)} \mathrm{Fun}(\Delta^n, D)$ with vertex $e_0 = (a, b)$ the original lifting problem, and vertex $e_1 = (a', b')$ where $b' = b\gamma|\Delta^n \times \{1\}$ factors as $\Delta^n \to \{b(n)\} \rightarrowtail D$. Furthermore e is an isomorphism by (38.2) since its images u and $b\gamma$ are isomorphisms. By the covering homotopy extension property (39.8) it suffices to produce a lift for the lifting problem e_1, i.e., in the rectangle

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "a'"] \arrow[d, tail] & p^{-1}(b(n)) \arrow[r, tail] \arrow[d] & C \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] \arrow[urr, dotted] & \{b(n)\} \arrow[r, tail] & D \end{tikzcd}$$

which amounts to producing a lift in the left-hand square, which exists because $p^{-1}(b(n))$ is a contractible Kan complex. □

We often apply the fiberwise criterion in the following way.

47.2. **Corollary.** *Suppose we have a pullback square of the form*

$$\begin{tikzcd} \coprod_{\alpha \in I} C'_\alpha \arrow[r] \arrow[d, "\coprod p'_\alpha"'] & C \arrow[d, "p"] \\ \coprod_{\alpha \in I} D'_\alpha \arrow[r, "g"'] & D \end{tikzcd}$$

such that (1) D is a quasigroupoid, (2) p is an isofibration, and (3) the map g is surjective on vertices. Then p is a trivial fibration if and only if every $p'_\alpha \colon C'_\alpha \to D'_\alpha$ is a trivial fibration.

Proof. The fibers of p all appear as fibers of the p'_α by (3), so this is immediate from the fiberwise criterion (47.1), and the fact that coproducts of trivial fibrations are trivial fibrations (25.2). □

47.3. *Remark.* The proof of (47.1) actually shows something a little stronger: If $p \colon C \to D$ is an isofibration to a quasigroupoid, then for any *fixed* $n \geq 0$ we have that $(\partial\Delta^n \subset \Delta^n) \boxslash p$ if and only if $(\partial\Delta^n \subset \Delta^n) \boxslash (p^{-1}(y) \to \{y\})$ for all $y \in D_0$.

The hypothesis that the target is a quasigroupoid is necessary: there is is no "fiberwise criterion" for an arbitrary isofibration between quasicategories to be a trivial fibration.

47.4. *Exercise.* Give an example of an isofibration between quasicategories whose fibers are all categorically equivalent to Δ^0, but is not a categorical equivalence, and hence not a trivial fibration. (Hint: think small.)

47.5. **Pullback-hom criterion for fully faithful isofibrations.** Using the fiberwise criterion, we obtain a new criterion for an isofibration to be fully faithful.

47.6. **Proposition.** *Let $p \colon C \to D$ be an isofibration between quasicategories. Then p is fully faithful if and only if*

$$(p^{\square(\partial\Delta^1 \subset \Delta^1)})^{\mathrm{core}} \colon \mathrm{Fun}(\Delta^1, C)^{\mathrm{core}} \to \big((C \times C) \times_{(D \times D)} \mathrm{Fun}(\Delta^1, D)\big)^{\mathrm{core}}$$

is a trivial fibration.

Proof. We can form a commutative diagram

$$\begin{tikzcd}
\coprod_{(c,c')\in C_0\times C_0} \mathrm{map}_C(c,c') \arrow[rr, "j'"] \arrow[d, "q_{c,c'}"] & & \mathrm{Fun}(\Delta^1,C) \arrow[d, "q"] & \\
\coprod_{(c,c')\in C_0\times C_0} \mathrm{map}_D(pc,pc') \arrow[r, "j"'] \arrow[d] & (C\times C)\times_{(D\times D)} \mathrm{Fun}(\Delta^1,D) \arrow[rr] \arrow[d] & & \mathrm{Fun}(\Delta^1,D) \arrow[d] \\
(\mathrm{Sk}_0\, C)\times(\mathrm{Sk}_0\, C) \arrow[r, "i"'] & C\times C \arrow[rr, "p\times p"'] & & D\times D
\end{tikzcd}$$

in which: each square is a pullback and the map, $q = p^{\square(\partial\Delta^1\subset\Delta^1)}$ is the pullback-hom map which is an isofibration (39.5), and the horizontal maps i, j, j' are surjective on vertices. Since mapping spaces are Kan complexes (45.2), the maps i, j, and j' factor through cores, and the resulting square

$$\begin{tikzcd}
\coprod_{(c,c')\in C_0\times C_0} \mathrm{map}_C(c,c') \arrow[r] \arrow[d, "q_{c,c'}"] & \mathrm{Fun}(\Delta^1,C)^{\mathrm{core}} \arrow[d, "q^{\mathrm{core}}"] \\
\coprod_{(c,c')\in C_0\times C_0} \mathrm{map}_D(pc,pc') \arrow[r, "\widetilde{j}"'] & \big((C\times C)\times_{(D\times D)} \mathrm{Fun}(\Delta^1,D)\big)^{\mathrm{core}}
\end{tikzcd}$$

is a pullback square. The map q^{core} is an isofibration between quasigroupoids (38.12), so the fiberwise criterion (47.2) applies to show that q^{core} is a trivial fibration if and only if each $q_{c,c'}\colon \mathrm{map}_C(c,c') \to \mathrm{map}_D(pc,pc')$ is a trivial fibration, and therefore if and only if each $q_{c,c'}$ is a categorical equivalence by (40.8). The map $q_{c,c'}$ is precisely the one induced by the functor p, so the proposition is proved. □

48. Proof of the fundamental theorem

In this section, we will prove the following. **M 21 Mar**

48.1. **Proposition.** *If $p\colon C\to D$ is an isofibration which is fully faithful, then $\mathrm{Cell}_{\geq 1} \boxslash p$.*

As discussed in (46.10), this proves the following.

48.2. **Theorem** (Fundamental theorem of quasicategories)**.** *A functor $f\colon C\to D$ of quasicategories is a categorical equivalence if and only if it is fully faithful and essentially surjective.*

48.3. **The class $\mathcal{C}_p$.** Let $p\colon C\to D$ be an isofibration. We define the class

$$\mathcal{C}_p := \left\{\, i \in \overline{\mathrm{Cell}} \;\middle|\; (p^{\square i})^{\mathrm{core}} \in \mathrm{TrivFib} \,\right\}$$

of monomorphisms such that the restriction of the pullback-hom map $p^{\square i}$ to cores is a trivial fibration. Note that if $i\in\mathcal{C}_p$, then in particular $(p^{\square i})^{\mathrm{core}}$ is surjective on vertices, whence $p^{\square i}$ is also surjective on vertices and thus $i \boxslash p$. Thus to prove (48.1), it suffices to show $\mathrm{Cell}_{\geq 1}\subseteq \mathcal{C}_p$.

48.4. **The class $\mathcal{C}_p$ is weakly saturated.** First we need to show that $\mathcal{C}_p$ is weakly saturated.

48.5. **Lemma.** *Let $i\colon K\to L$ be a monomorphism of simplicial sets. Then there exists a monomorphism i' such that, for any isofibration $q\colon C\to D$ between quasicategories, we have $i\boxslash q^{\mathrm{core}}$ if and only if $i'\boxslash q$.*

Proof. Given i we construct i' as in the following diagram

$$\begin{tikzcd} K \arrow[r, tail, "u"] \arrow[d, tail, "i"'] & K' \arrow[d, tail, "j"'] \arrow[dr, tail, "i'"] & \\ L \arrow[r, tail, "v"'] & P \arrow[r, tail, "k"'] & L' \end{tikzcd}$$

where we

(1) choose an anodyne map $u\colon K \to K'$ to a Kan complex K',
(2) we form the pushout P of u along i, and
(3) choose an anodyne map $k\colon P \to L'$ to a Kan complex L',

whence $i' := kj$ is a monomorphism. The choices of u and k can be made using the small object argument (17.16) applied to the set Horn of all horn inclusions. We need to show $i \boxslash q^{\mathrm{core}}$ iff $i' \boxslash q$ for any isofibration p.

($\Longrightarrow$) Suppose $i \boxslash q^{\mathrm{core}}$, and consider a lifting problem of type $i' \boxslash q$. Since K' and L' are Kan complexes, any lifting problem of type $i' \boxslash q$ factors through cores:

$$\begin{tikzcd} K' \arrow[r] \arrow[d, "i'"'] & C^{\mathrm{core}} \arrow[r, tail] \arrow[d, "q^{\mathrm{core}}"] & C \arrow[d, "q"] \\ L' \arrow[r] \arrow[ur, dotted] & D^{\mathrm{core}} \arrow[r, tail] & D \end{tikzcd}$$

so it suffices to show $i' \boxslash q^{\mathrm{core}}$. Since j is a cobase change of i we have $j \boxslash q^{\mathrm{core}}$, while $k \boxslash q^{\mathrm{core}}$ since k is anodyne and q^{core} is a Kan fibration (41.1). Therefore $i' = kj \boxslash q^{\mathrm{core}}$ as desired.

($\Longleftarrow$) Suppose $i' \boxslash q$, and consider a lifting problem ($a\colon K \to C^{\mathrm{core}}$, $b\colon L \to D^{\mathrm{core}}$) of type $i \boxslash q^{\mathrm{core}}$. We factor this lifting problem through a diagram of the following form

$$\begin{tikzcd} K \arrow[r, tail, "u"] \arrow[d, tail, "i"'] \arrow[rrr, bend left, "a"] & K' \arrow[r, equal] \arrow[d, tail, "j"'] & K' \arrow[r, "a'"] \arrow[d, "i'"'] & C^{\mathrm{core}} \arrow[r, tail] \arrow[d] & C \arrow[d, "q"] \\ L \arrow[r, tail, "v"'] \arrow[rrr, bend right, "b"'] & P \arrow[r, tail, "k"'] & L' \arrow[r, "b''"'] \arrow[ur, dotted, "t'"] \arrow[urr, dotted, "t"] & D^{\mathrm{core}} \arrow[r, tail] & D \end{tikzcd}$$

as follows.

(1) Since u is anodyne and C^{core} is a Kan complex, we can factor $a = a'u$ for some $a'\colon K' \to C^{\mathrm{core}}$.
(2) There is a unique map $b'\colon P \to D^{\mathrm{core}}$ such that $b'v = b$ and $b'j = a'$ since P is a pushout.
(3) Since k is anodyne and D^{core} is a Kan complex, we can factor $b' = b''k$ for some $b''\colon L' \to D^{\mathrm{core}}$.

By hypothesis a lift t exists. Since L' is a Kan complex the lift t factors through $C^{\mathrm{core}} \subseteq C$, so we have a map $t'\colon L' \to C^{\mathrm{core}}$ such that $t'i' = a'$ and $q^{\mathrm{core}}t' = b''$. Then the composite $s := t'kv\colon L \to C^{\mathrm{core}}$ is the desired solution to the lifting problem (a, b). $\square$

48.6. **Lemma.** *For an isofibration p the class $\mathcal{C}_p$ is weakly saturated.*

Proof. For each $j_n\colon \partial\Delta^n \to \Delta^n$, choose a map j'_n as in (48.5) so that $j_n \boxslash q^{\mathrm{core}}$ if and only if $j'_n \boxslash q$ for some isofibration q, and apply in the case of $q = p^{\square i}$, so that $j_n \boxslash (p^{\square i})^{\mathrm{core}}$ if and only if $j'_n \boxslash p^{\square i}$ if and only if $i \boxslash p^{\square j'_n}$ Then $\mathcal{C}_p = \overline{\mathrm{Cell}} \cap {}^{\boxslash}\left\{ p^{\square j'_n} \;\middle|\; n \geq 0 \right\}$, and so is weakly saturated. $\square$

48.7. **The class $\mathcal{C}_p$ has precancellation.** We will need the following result which relates pullback-homs and composition of maps. You can think of it as an "enriched" version of the fact that $i \boxslash p$ and $j \boxslash p$ imply $ji \boxslash p$.

48.8. **Proposition** (Transitivity triangle for pullback-homs)**.** *Let $A \xrightarrow{i} B \xrightarrow{j} C$ and $p\colon X \to Y$ be maps of simplicial sets. Then there is a factorization*

$$p^{\square(j\circ i)} = q \circ p^{\square j}$$

where q is a base-change of $p^{\square i}$.

Proof. I use "$[A,X]$" as a shorthand for "$\mathrm{Fun}(A,X)$". Form the commutative diagram

$$\begin{tikzcd}
{[C,X]} \arrow[r, "p^{\square j}"] \arrow[dr, "p^{\square ji}"'] \arrow[ddr, bend right, "{[C,p]}"'] \arrow[rr, bend left, "{[j,X]}"] & {[B,X] \times_{[B,Y]} [C,Y]} \arrow[r] \arrow[d, "q"] & {[B,X]} \arrow[d, "p^{\square i}"] \arrow[dr, bend left, "{[i,X]}"] & \\
& {[A,X] \times_{[A,Y]} [C,Y]} \arrow[r] \arrow[d] & {[A,X] \times_{[A,Y]} [B,Y]} \arrow[r] \arrow[d] & {[A,X]} \arrow[d, "{[A,p]}"] \\
& {[C,Y]} \arrow[r, "{[j,Y]}"'] & {[B,Y]} \arrow[r, "{[i,Y]}"'] & {[A,Y]}
\end{tikzcd}$$

in which all three squares are pullbacks, whence in particular q is a base-change of $p^{\square i}$. The claim follows. $\square$

48.9. *Exercise.* Prove the following transitivity-triangles:

(1) $(i \circ j)\square f = k \circ (i \square f)$ where k is a cobase-change of $j \square f$.
(2) $(q \circ p)^{\square i} = r \circ p^{\square i}$ where r is a base-change of $q^{\square i}$.

Next, we show that $\mathcal{C}_p$ has the following "precancellation" property.

48.10. **Proposition.** *Let $p\colon C \to D$ be an isofibration between quasicategories. If $i\colon K \to K'$ and $j\colon K' \to K''$ are monomorphisms, then $i, ji \in \mathcal{C}_p$ implies $j \in \mathcal{C}_p$.*

Proof. By (48.8) we have $p^{\square ji} = q \circ p^{\square j}$ where q is a base-change of $p^{\square i}$. Restricting to cores gives a factorization $(p^{\square ji})^{\mathrm{core}} = q^{\mathrm{core}} \circ (p^{\square i})^{\mathrm{core}}$. Furthermore q^{core} is a base-change of $(p^{\square i})^{\mathrm{core}}$ as (38.2) applies since $p^{\square i}$ is an inner fibration between quasicategories (21.8).

We have that $(p^{\square ji})^{\mathrm{core}}$, $(p^{\square j})^{\mathrm{core}}$, $(p^{\square i})^{\mathrm{core}}$, and hence q^{core} are isofibrations (38.12). Since $ji, i \in \mathcal{C}_p$, we have that $(p^{\square ji})^{\mathrm{core}}$, $(p^{\square i})^{\mathrm{core}}$ and hence q^{core} are trivial fibrations, and therefore are categorical equivalences, whence $p^{\square j}$ is also a weak equivalence by 2-out-of-3 (27.11), and therefore $p^{\square j}$ is a trivial fibration since it is an isofibration between quasicategories (40.8). $\square$

48.11. *Exercise.* Show that the class of fully faithful functors of quasicategories has "postcancellation", in the sense that: if f, g are composable functors such that g and gf are fully faithful, then f is fully faithful. (This fact is related to the reason that "precancellation" appears in our proof of (48.1).)

48.12. **The end of the proof.** We can now prove (48.1), using the following lemma to show that if p is a fully faithful isofibration, then $\mathcal{C}_p$ contains $\mathrm{Cell}_{\geq 1}$, whence $\overline{\mathrm{Cell}}_{\geq 1} \boxslash p$ as desired. We write $j_n\colon \partial\Delta^n \to \Delta^n$ for the nth cell inclusion.

48.13. **Lemma.** *Let $\mathcal{C}$ be a weakly saturated class of monomorphism which has precancellation with respect to the class of monomorphisms, and which contains all inner horn inclusions. If $\mathcal{C}$ also contains some cell inclusion $j_n\colon \partial\Delta^n \to \Delta^n$, then $\mathrm{Cell}_{\geq n} \subseteq \mathcal{C}$.*

Proof. We show that $j_m \in \mathcal{C}$ for $m > n$ by induction on n. For any $m \geq 1$ we have a commutative diagram

$$\begin{tikzcd}
\partial\Delta^{m-1} \arrow[r] \arrow[d, tail, "j_{m-1}"'] & \Lambda^m_1 \arrow[d, tail, "i"'] \arrow[dr, "j_m i"] & \\
\Delta^{m-1} \arrow[r, "\langle 023...m\rangle"'] & \partial\Delta^m \arrow[r, tail, "j_m"'] & \Delta^m
\end{tikzcd}$$

in which the left-hand square is a pushout. By induction we have that $j_{m-1} \in \mathcal{C}$, whence $i \in \mathcal{C}$ since it is weakly saturated. We have that $j_m i \in \mathcal{C}$ since it is an inner horn inclusion. Therefore $j_m \in \mathcal{C}$ as desired by precancellation, since i and j_m are monomorphisms. $\square$

Proof of (48.1). Let p be a fully faithful isofibration. To show $\mathrm{Cell}_{\geq 1} \boxslash p$ it suffices to show $\mathrm{Cell}_{\geq 1} \subseteq \mathcal{C}_p$. We have $(\partial\Delta^1 \subset \Delta^1) \in \mathcal{C}_p$ by (47.6), and we know that $\mathcal{C}_p$ is weakly saturated (48.5) and has precancellation with respect to monomorphisms (48.10). Finally, note that $\mathcal{C}_p$ has inner horn inclusions since p is an inner fibration, so $p^{\Box i} \in \mathrm{TrivFib}$ for $i \in \overline{\mathrm{InnHorn}}$ since $\overline{\mathrm{InnHorn}}\Box\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$, and therefore $(p^{\Box i})^{\mathrm{core}} \in \mathrm{TrivFib}$ (48.14). Thus we can apply the lemma (48.13). $\square$

48.14. *Exercise.* Let $p\colon C \to D$ be a trivial fibration between quasicategories. Show that $p^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ is also a trivial fibration.

Part 8. **Model categories**

49. The Joyal model structure on simplicial sets

49.1. **Model categories.** A **model category** (in the sense of Quillen) is a category $\mathcal{M}$ with three classes of maps: W, Cof, Fib, which I will call **weak equivalences**, **cofibrations**, and **fibrations** respectively, satisfying the following axioms.

model category
weak equivalences
cofibrations
fibrations

- $\mathcal{M}$ has all small limits and colimits.
- W satisfies the 2-out-of-3 property.
- $(\mathrm{Cof} \cap \mathrm{W}, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap \mathrm{W})$ are weak factorization systems (17.20).

An object X is **cofibrant** if the map from the initial object is a cofibration, and **fibrant** if the map to the terminal object is a fibration. A map in $\mathrm{Cof} \cap W$ is called a **trivial cofibration**, and a map in $\mathrm{Fib} \cap W$ is called a **trivial fibration**.

cofibrant
fibrant
trivial cofibration
trivial fibration

49.2. *Warning.* Do not confuse the general notion of "weak equivalence" in an arbitrary model category with the specific notion of "weak homotopy equivalence of simplicial sets" defined in (52).

49.3. *Remark.* The third axiom implies that Cof, $\mathrm{Cof} \cap \mathrm{W}$, Fib, and $\mathrm{Fib} \cap \mathrm{W}$ are closed under retracts.

49.4. *Exercise.* Show that in a model category (as defined above), the class of weak equivalences is closed under retracts. *Hint:* if f is a retract of a weak equivalence g, construct a factorization of f which is itself a retract of a factorization of g[30].

[30] In many formulations of model categories, closure of weak equivalences under retracts is taken as one of the axioms. The formulation we use is described in Riehl, "A concise definition of a model category" [Rie09], which gives a solution to this exercise.

The formulation we are using is equivalent to ones used most often today, but it's worthwhile to note that there is some variation: for instance, some may weaken the definition by merely requiring existence of *finite* limits and colimits (as in Quillen's original definition in [Qui67] where it is called a "closed model category"), or strengthen it by requiring the existence of *functorial* factorization in the weak factorizations (as in Hovey's book [Hov99].)

49.5. *Exercise* (Slice model categories). Let $\mathcal{M}$ be a model category, and let X be an object of $\mathcal{M}$. Show that the slice categories $\mathcal{M}_{X/}$ and $\mathcal{M}_{/X}$ admit model category structures, in which the weak equivalences, cofibrations, and fibrations are precisely the maps whose images under $\mathcal{M}_{/X} \to \mathcal{M}$ or $\mathcal{M}_{X/} \to \mathcal{M}$ are weak equivalences, cofibrations, and fibrations in $\mathcal{M}$.

49.6. *Exercise* (Goodwillie). Classify all model category structures on the category of sets. (There are exactly nine. *Hint:* use (17.22).)

49.7. **Categorical fibrations and the small object argument.** As we have seen, the class $\mathrm{CatEq} \cap \overline{\mathrm{Cell}}$ of monomorphisms which are categorical equivalences is weakly saturated (40.17), with right complement CatFib, the class of categorical fibrations (41). In fact, the pair $(\mathrm{CatEq} \cap \overline{\mathrm{Cell}}, \mathrm{CatFib})$ is a weak factorization system, as a consequence of the small object argument (17.16) and the following.

49.8. **Proposition.** *There exists a set S of maps of simplicial sets such that $\overline{S} = \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$, whence $S^{\boxslash} = \mathrm{CatFib}$.*

Unfortunately, it's apparently not known how to write down an explicit set of maps S so that $S^{\boxslash} = \mathrm{CatFib}$. What is known is that such a set *exists*: for instance, one can take a set of representatives (up-to-isomorphism) of every morphism in $\overline{\mathrm{Cell}} \cap \mathrm{CatEq}$ in which source and target have size bounded by a large but suitably chosen cardinal. We give a proof of this fact in the appendix (80).

49.9. **The Joyal model structure.**

49.10. **Theorem** (Joyal)**.** *The category of simplicial sets admits a model structure, in which*
- $\mathrm{W} = \mathrm{CatEq}$*, the class of categorical equivalences,*
- $\mathrm{Cof} = \overline{\mathrm{Cell}}$*, the class of monomorphims,*
- $\mathrm{Fib} = \mathrm{CatFib}$*, the class of categorical fibrations.*

Furthermore, the fibrant objects are precisely the quasicategories, and the fibrations with target a fibrant object are precisely the isofibrations.

Proof. Categorical equivalences satisfy 2-out-of-3 by (27.11). We have that
- $\overline{\mathrm{Cell}} =$ monomorphisms by (20.5),
- $\mathrm{Fib} \cap \mathrm{W} = \mathrm{CatFib} \cap \mathrm{CatEq} = \mathrm{TFib} = \mathrm{Cell}^{\boxslash}$ by (41.2),
- $\mathrm{Cof} \cap \mathrm{W} = \overline{\mathrm{Cell}} \cap \mathrm{CatEq} = \overline{S}$ for some set S (49.8),
- $\mathrm{Fib} = \mathrm{CatFib} = (\mathrm{Cof} \cap \mathrm{W})^{\boxslash} = S^{\boxslash}$ by definition,

so both $(\mathrm{Cof} \cap \mathrm{W}, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap \mathrm{W})$ are weak factorization systems via the small object argument (17.16). Thus, we get a model category.

We have shown (41.1) that the categorical fibrations $p\colon C \to D$ with D a quasicategory are precisely the isofibrations. Applied when $D = *$, this implies that quasicategories are exactly the fibrant objects, and thus that fibrations with fibrant target are precisely the isofibrations. □

49.11. *Remark.* It is a fact that a model category structure is uniquely determined by its cofibrations and fibrant objects [Joy08a, Prop. E.1.10]. Thus, the Joyal model structure is the unique model structure on simplicial sets with Cof = monomorphisms and with fibrant objects the quasicategories.

49.12. **Cartesian model categories.** Recall that the category of simplicial sets is *cartesian closed*. A **cartesian model category** is a model category which is cartesian closed (and thus admits internal hom-objects, which I will write "$\mathrm{Fun}(A, X)$" as in simplicial sets), such that the terminal object is cofibrant, and with the following properties. Suppose $i\colon A \to B$ and $j\colon K \to L$ are cofibrations and $p\colon X \to Y$ is a fibration. **W 23 Mar** **cartesian model category**

- $$i \Box j \colon (A \times L) \cup_{A \times K} (B \times K) \to B \times L$$
  is a cofibration, and is in addition a weak equivalence if either i or j is also a weak equivalence, and
- $$p^{\Box j} \colon \operatorname{Fun}(L, X) \to \operatorname{Fun}(K, X) \times_{\operatorname{Fun}(K,Y)} \operatorname{Fun}(L, Y)$$
  is a fibration, and is in addition a weak equivalence if either j or p is also a weak equivalence.

In fact, we only need to specify *one* of the above two properties, as they imply each other.

49.13. **Proposition.** *The Joyal model structure is cartesian.*

Proof. This is just (41.3). $\square$

50. Model categories and homotopy colimits

We are going to exploit these model category structures now. The main purpose of model categories is to give tools for showing that a given construction preserves certain kinds of equivalence.

50.1. **Creating new model categories.** Given a model category $\mathcal{M}$, many other categories related to it can also be equipped with model category structures, such as functor categories $\operatorname{Fun}(C, \mathcal{M})$ where C is a small category. We won't consider general formulations of this here, but rather will set up some special cases.

As an example, we consider the case of $C = [1] = \{0 \xrightarrow{01} 1\}$.

50.2. **Proposition.** *There exists a model structure on $\mathcal{N} := \operatorname{Fun}([1], \mathcal{M})$ in which a map $\alpha \colon X \to X'$ is*

- *a weak equivalence if $\alpha(i) \colon X(i) \to X'(i)$ is a weak equivalence in $\mathcal{M}$ for $i = 0, 1$*
- *a cofibration if both $\alpha(0)$ and the map $(\alpha(1), X(01)) \colon X(1) \cup_{X(0)} X'(0) \to X'(1)$ are cofibrations in $\mathcal{M}$, and*
- *a fibration if $\alpha(i)$ is a fibration in $\mathcal{M}$ for $i = 0, 1$.*

Proof. It is clear that $\mathcal{N}$ has small limits and colimits, and that weak equivalences in it have the 2-out-of-3 property. It remains to show that $(\mathrm{Cof} \cap W, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap W)$ are weak factorization systems, where W, Cof, Fib are the of maps in $\mathcal{N}$ defined in the statement of the proposition.

We start with the following observation about lifting in $\mathcal{N} = \operatorname{Fun}([1], \mathcal{M})$: given maps $j \colon A \to B$ and $p \colon X \to Y$ in $\mathcal{N}$, we can solve a lifting problem (u, v) of type $j \boxslash p$ in $\mathcal{N}$ by solving a sequence of two lifting problems in $\mathcal{M}$, namely

$$\begin{tikzcd} A(0) \arrow[r, "u(0)"] \arrow[d, "j(0)"'] & X(0) \arrow[d, "p(0)"] \\ B(0) \arrow[r, "v(0)"'] \arrow[ur, dotted, "s(0)"] & Y(0) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A(1) \cup_{A(0)} B(0) \arrow[r, "{(u(1), X(01) \circ s(0))}"] \arrow[d, "{(j(1), B(01))}"'] & X(1) \arrow[d, "p(1)"] \\ B(1) \arrow[r, "v(1)"'] \arrow[ur, dotted, "s(1)"] & Y(1) \end{tikzcd}$$

where the second problem depends on the solution $s(0)$ to the first problem. Then the maps $s(0)$ and $s(1)$ fit together to give a map $s \colon B \to X$ in $\mathcal{N}$ which solve the original lifting problem.

Given this, it is not hard to prove that $\mathrm{Cof} \cap W \boxslash \mathrm{Fib}$ and $\mathrm{Cof} \boxslash \mathrm{Fib} \cap W$, using the definitions and the fact that $\mathcal{M}$ is a model category. The trickiest point is to observe that if $j \colon A \to B$ is both a cofibration and a weak equivalence in $\mathcal{N}$, then $(j(1), B(01))$ is a trivial cofibration in $\mathcal{M}$: this uses 2-out-of-3 for weak equivalences in $\mathcal{M}$ and the fact that $A(1) \to A(1) \cup_{A(0)} B(0)$ must be a trivial cofibration in $\mathcal{M}$, being a cobase-change of $j(0)$.

Next, observe that to describe a factorization of a map $f\colon X \to Y$ in $\mathcal{N}$ into $f = pj$ with $j\colon X \to U$ and $p\colon U \to Y$, it suffices to describe a sequence of two factorizations in $\mathcal{M}$, namely $f(0) = p(0) \circ j(0)$ and $h = p(1) \circ g$, as in

$$X(0) \xrightarrow{j(0)} U(0) \xrightarrow{p(0)} Y(0) \qquad \text{and} \qquad \begin{tikzcd} X(0) \arrow[r, "j(0)"] \arrow[d, "X(01)"'] & U(0) \arrow[rr, "p(0)"] \arrow[d, "\eta'"] & & Y(0) \arrow[d, "Y(01)"] \\ X(1) \arrow[r, "\eta"'] & X(1) \cup_{X(0)} U(0) \arrow[r, "g"] \arrow[rr, bend right, "h"'] & U(1) \arrow[r, "p(1)"] & Y(1) \end{tikzcd}$$

where $h = (f(1), Y(01) \circ p(0))$, so that $j(1) = g \circ \eta$ and $U(01) = g \circ \eta'$.

To factor $f = pj$ in $\mathcal{N}$ with $j \in \mathrm{Cof} \cap W$ and $p \in \mathrm{Fib}$, it suffices to successively choose factorizations of $f(0)$ and h of this type. Likewise, to factor $f = pj$ in $\mathcal{N}$ with $j \in \mathrm{Cof}$ and $p \in \mathrm{Fib} \cap W$, it suffices to successively choose factorizations of $f(0)$ and h of this type.

It remains to show that $\mathrm{Cof} \cap W = {}^{\boxslash}\mathrm{Fib}$, $\mathrm{Fib} = \mathrm{Cof} \cap W^{\boxslash}$, $\mathrm{Cof} = {}^{\boxslash}\mathrm{Fib} \cap W$, and $\mathrm{Cof}^{\boxslash} = \mathrm{Fib} \cap W$. This is an immediate consequence of the "retract trick" (17.17), together with the easily checked fact that Cof, $\mathrm{Cof} \cap W$, Fib, and $\mathrm{Fib} \cap W$ are closed under retracts, which can be proved directly using the definition and the fact that the analogous classes in $\mathcal{M}$ are closed under retracts (49.3). $\square$

The opposite of a model category is also a model category, by switching the roles of fibrations and cofibrations. Therefore, there is another model structure on $\mathrm{Fun}([1], \mathcal{M}) = (\mathrm{Fun}([1], \mathcal{M}^{\mathrm{op}}))^{\mathrm{op}}$.

50.3. **Ken Brown lemma.** The "Ken Brown lemma" gives an explicit criterion for a functor to preserve weak equivalences between large classes of objects.

50.4. **Proposition** (Ken Brown lemma). *Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor between model categories.*

1. *If F takes trivial cofibrations to weak equivalences, then F takes weak equivalences between cofibrant objects to weak equivalences.*
2. *If F takes trivial fibrations to weak equivalences, then F takes weak equivalences between fibrant objects to weak equivalences.*

Proof. I prove (1); the proof of (2) is formally dual.

Let $f\colon X \to Y$ be a weak equivalence between cofibrant objects in $\mathcal{M}$. Form the commutative diagram

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dr, "b"] \arrow[drr, bend left, "\mathrm{id}_Y"] & & \\ X \arrow[r] \arrow[rr, bend right, "a"] \arrow[rrr, bend right=40, "f"] & X \amalg Y \arrow[r, "i"] & C \arrow[r, "p"] & Y \end{tikzcd}$$

where the square is a pushout, and we have chosen a factorization of $(f, \mathrm{id}_Y)\colon X \amalg Y \to Y$ as pi, a cofibration i followed by a weak equivalence p (e.g., a trivial fibration). Because X and Y are cofibrant, the maps $X \to X \amalg Y \leftarrow Y$ are cofibrations. Using this and the 2-out-of-3 property for weak equivalences, we see that a and b are trivial cofibrations. Applying F gives

$$\begin{tikzcd} & F(Y) \arrow[d, "F(b)"'] \arrow[dr, "\mathrm{id}"] & \\ F(X) \arrow[r, "F(a)"'] & F(C) \arrow[r, "F(p)"'] & F(Y) \end{tikzcd}$$

in which $F(b)$ and $F(a)$ are weak equivalences by hypothesis, whence $F(p)$ is a weak equivalence by 2-out-of-3, and therefore $F(f) = F(p)F(a)$ is a weak equivalence, as desired. $\square$

50.5. **Quillen pairs.** Given an adjoint pair of functors $F\colon \mathcal{M} \leftrightarrows \mathcal{N} : G$ between model categories, we see from the properties of weak factorization systems that

- F preserves cofibrations if and only if G preserves trivial fibrations, and
- F preserves trivial cofibrations if and only if G preserves fibrations.

If both of these are true, we say that (F, G) is a **Quillen pair**. **Quillen pair**

Note that if (F, G) is a Quillen pair, then the Ken Brown lemma (50.4)(1) applies to F, while (50.4)(2) applies to G.

50.6. **Good colimits.** We can apply the above to certain examples of colimit functors, which we will refer to generically as "good colimits". There are three types of these: arbitrary coproducts of cofibrant objects, countable sequential colimits of cofibrant objects along cofibrations, and pushouts of cofibrant objects along a cofibration. We will show that "good colimits are weak equivalence invariant".

50.7. *Exercise.* Let S be a small discrete category (i.e., all maps are identities). Show that if $\mathcal{M}$ is a model category, then $\mathrm{Fun}(S, \mathcal{M})$ is a model category in which $\alpha\colon X \to X'$ is

- a weak equivalence, cofibration, or fibration iff each $\alpha_s\colon X_s \to X'_s$ is one in $\mathcal{M}$.

Then show that $\mathrm{colim}\colon \mathrm{Fun}(S, \mathcal{M}) \leftrightarrows \mathcal{M} : \mathrm{const}$ is a Quillen pair, and use this to prove the next proposition.

50.8. **Proposition** (Good coproducts)**.** *Given a collection $f_s\colon X_s \to X'_s$ of weak equivalences between cofibrant objects in $\mathcal{M}$, the induced map $\amalg f_s\colon \coprod X_s \to \coprod X'_s$ is a weak equivalence.*

Proof. Apply the Ken Brown lemma (50.4) to the coproduct functor $\mathrm{Fun}(S, \mathcal{M}) \to \mathcal{M}$, using the model structure of (50.7). $\square$

50.9. *Exercise.* Let ω be the category

$$0 \to 1 \to 2 \to \cdots$$

with objects indexed by natural numbers. Show that if $\mathcal{M}$ is a model category, then $\mathrm{Fun}(\omega, \mathcal{M})$ is a model category in which $\alpha\colon X \to X'$ is

- a weak equivalence if each $\alpha(i)$ is a weak equivalence in $\mathcal{M}$,
- a cofibration if (i) $\alpha(0)$ is a cofibration in $\mathcal{M}$, and $X'(i) \cup_{X(i)} X(i+1) \to X'(i+1)$ is a cofibration in $\mathcal{M}$for all $i \geq 0$, and
- a fibration if each $\alpha(i)$ is a fibration in $\mathcal{M}$.

Then show that $\mathrm{colim}\colon \mathrm{Fun}(\omega, \mathcal{M}) \leftrightarrows \mathcal{M} : \mathrm{const}$ is a Quillen pair, and use this to prove the next proposition.

50.10. **Proposition** (Good sequential colimits)**.** *Give a natural transformation $\alpha\colon X \to X'$ of functors $\omega \to \mathcal{M}$ such that all maps $\alpha(i)\colon X(i) \to X'(i)$ are weak equivalences, all objects $X(i)$ and $X(i')$ are cofibrant, and the maps $X(i) \to X(i+1)$ and $X'(i) \to X'(i+1)$ are cofibrations, the induced map $\mathrm{colim}_\omega X \to \mathrm{colim}_\omega X'$ is a weak equivalence.*

Proof. Apply the Ken Brown lemma (50.4) to the colmit functor $\mathrm{Fun}(\omega, \mathcal{M}) \to \mathcal{M}$, using the model structure of (50.9). $\square$

50.11. *Exercise.* Recall that Λ^2_0 is a category:

$$1 \xleftarrow{01} 0 \xrightarrow{12} 2.$$

Show that if $\mathcal{M}$ is a model category, then $\mathrm{Fun}(\Lambda^2_0, \mathcal{M})$ is a model category in which $\alpha\colon X \to X'$ is

- a weak equivalence if $\alpha(i)\colon X(i) \to X'(i)$ is a weak equivalence in $\mathcal{M}$ for $i = 0, 1, 2$ (i.e., an **objectwise weak equivalence**), **objectwise weak equivalence**

- a cofibration if $\alpha(0)$, $\alpha(1)$, and the evident map $X(2) \cup_{X(0)} X'(0) \to X'(2)$ are cofibrations in $\mathcal{M}$, and
- a fibration if $\alpha(1)$, $\alpha(2)$, and the evident map $X(0) \to X'(0) \times_{X'(1)} X(1)$ are fibrations in $\mathcal{M}$.

Then show that $\operatorname{colim}\colon \operatorname{Fun}(\Lambda^2_0, \mathcal{M}) \leftrightarrows \mathcal{M} : \operatorname{const}$ is a Quillen pair, and use this to prove the next proposition.

50.12. **Proposition** (Good pushouts). *Given a natural transformation $\alpha\colon X \to X'$ of functors $\Lambda^2_0 \to \mathcal{M}$, i.e., a diagram*

$$\begin{tikzcd} X(1) & X(0) \arrow[l] \arrow[r, "X(02)"] \arrow[d, "\sim"'] & X(2) \arrow[d, "\sim"'] \\ X'(1) \arrow[from=u, "\sim"'] & X'(0) \arrow[l] \arrow[r, "X'(02)"] & X'(2) \end{tikzcd}$$

in which the vertical maps are weak equivalences, all objects $X(i)$ and $X'(i)$ are cofibrant, and the maps $X(02)$ and $X'(02)$ are cofibrations, the induced map $\operatorname{colim}_{\Lambda^2_0} X \to \operatorname{colim}_{\Lambda^2_0} X'$ is a weak equivalence.

Proof. Apply the Ken Brown lemma (50.4) to the colimit functor $\operatorname{Fun}(\Lambda^2_0, \mathcal{M}) \to \mathcal{M}$, using the model structure of (50.11). □

In the Joyal model structure on $s\mathrm{Set}$, all objects are automatically cofibrant, which makes the above propositions especially handy.

We will call any colimit diagram in a model category, satisfying the hypotheses of one of (50.8), (50.12), (50.10) a **good colimit**. Thus, we see that good colimits are weak equivalence invariant. **good colimit** These "good colimits" are examples of what are called *homotopy colimits*.

Since the opposite of a model category is also a model category, all of the results of this section admit dual formulations, leading to the observation that **good limits** are homotopy invariant. **good limits**

50.13. *Exercise.* State and prove the dual versions of all the results in this section.

50.14. *Exercise.* Recall the relative function complex (25.18), which for objects $p\colon S \to K$ and $q\colon S \to C$ in $s\mathrm{Set}_{S/}$ is the simplicial set

$$\operatorname{Fun}_{S/}(K, C) = \operatorname{Fun}(K, C) \times_{\operatorname{Fun}(S,C)} \{q\}.$$

Show that if $f\colon K \to L$ is a categorical equivalence, C is a quasicategory, and both p and fp are monomorphisms, then the induced map $f^*\colon \operatorname{Fun}_{S/}(L, C) \to \operatorname{Fun}_{S/}(K, C)$ on relative function complexes is a categorical equivalence. (Hint: Both source and target of f^* can be described via good pullbacks with respect to the Joyal model structure.)

51. Homotopy pullbacks

This section is an attempt to state and prove some basic things about homotopy pullbacks (and thus homotopy pushouts). In particular, I want things like pasting. I don't actually know a reference about this that I really like. **F 25 Mar**

Let $\mathcal{M}$ be a model category, and consider $\mathcal{S} := \operatorname{Fun}(\Delta^1 \times \Delta^1, \mathcal{M})$, the category of commutative squares in $\mathcal{M}$. This contains a full subcategory $\mathcal{S}^{\mathrm{pb}} \subseteq \mathcal{S}$ spanned by the squares which are pullback squares. Given any commutative square $X\colon \Delta^1 \times \Delta^1 \to \mathcal{M}$ there exists a map $\eta\colon X \to X^{\mathrm{pb}}$, where X^{pb} is a pullback square and $\eta(i,j)\colon X(i,j) \to X^{\mathrm{pb}}(i,j)$ is the identity map when $(i,j) \neq (0,0)$.

51.1. *Exercise.* Show $\operatorname{Fun}(\Delta^1 \times \Delta^1, \mathcal{M})$ is a model category in which $\alpha\colon X \to X'$ is

- a weak equivalence if $\alpha(i,j)\colon X(i,j) \to X'(i,j)$ is a weak equivalence in $\mathcal{M}$ for all $i, j \in \{0, 1\}$,
- a cofibration if all maps $\alpha(i,j)\colon X(i,j) \to X'(i,j)$ are cofibrations in $\mathcal{M}$, and

- a fibration if the evident maps

$$X(1,1) \to X'(1,1), \quad X(1,0) \to X'(1,0) \times_{X'(1,1)} X(1,1), \quad X(0,1) \to X'(0,1) \times_{X'(1,1)} X(1,1),$$

and

$$X(0,0) \to X'(0,0) \times_{(X'(1,0) \times_{X'(1,1)} X'(0,1))} (X(1,0) \times_{X(1,1)} X(0,1))$$

are fibrations in $\mathcal{M}$

51.2. *Exercise* (Fibrant commutative squares). Show that in the model structure of the previous exercise (51.1), a commutative square X is fibrant if and only if

1. the object $X(1,1)$ is fibrant in $\mathcal{M}$, and
2. the maps $X(1,0) \to X(1,1)$ and $X(0,1) \to X(1,1)$ are fibrations in $\mathcal{M}$, and
3. the map $X(0,0) \to X(1,0) \times_{X(1,1)} X(0,1)$ is a fibration in $\mathcal{M}$.

In particular, if X is a fibrant commutative square which is a pullback, then it is a good pullback in the sense of (**??**) above.

In the following, we to a given model structure on $\mathcal{M}$, and to the above model structure on $\mathcal{S} = \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{M})$ based on it.

Say that a commutative square X is a **homotopy pullback** if there exists a weak equivalence $X \xrightarrow{\sim} Y \approx Y^{\mathrm{pb}}$ to a fibrant pullback square. **homotopy pullback**

51.3. **Proposition.** *Let X be a commutative square. The following are equivalent.*

1. *There exists a weak equivalence $X \to Y$ to a fibrant square such that $\eta_Y : Y \to Y^{\mathrm{pb}}$ is a weak equivalence.*
2. *For every weak equivalence $X \to Y$ to a fibrant square, the map $\eta_Z : Y \to Y^{\mathrm{pb}}$ is a weak equivalence.*
3. *There exists a weak equivalence $X \to Y \approx Y^{\mathrm{pb}}$ to a fibrant pullback square (i.e., X is a homotopy pullback square).*

Proof. (1) ($\Longrightarrow$) (2). Suppose given a weak equivalence $f : X \to Y$ to a fibrant square with $\eta_Y : Y \to Y^{\mathrm{pb}}$ a weak equivalence, and let $g : X \to Z$ be any weak equivalence to a fibrant square. Construct a commutative diagram as follows.

$$\begin{tikzcd}
 & Y \arrow[r, "\eta_Y"] & Y^{\mathrm{pb}} \\
X \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & W \arrow[u, "p"] \arrow[d, "s"] \arrow[r, "\eta_W"] & W^{\mathrm{pb}} \arrow[u, "p^{\mathrm{pb}}"'] \arrow[d, "s^{\mathrm{pb}}"'] \\
 & Z \arrow[r, "\eta_Z"'] & Z^{\mathrm{pb}}
\end{tikzcd}$$

First factor $g = ph$ where h is a cofibration and p is a trivial fibration in in $\mathcal{S}$. Then using the fact that Y is fibrant, use the lifiting properties of $\mathcal{S}$ to construct s such that $sh = g$. Thus each of Y, Z, W is fibrant, and by 2-out-of-3 each of the maps p and s is a weak equivalence. Therefore each of $Y^{\mathrm{pb}}, Z^{\mathrm{pb}}, W^{\mathrm{pb}}$ is a fibrant pullback square and hence a good pullback square, and both maps p^{pb} and s^{pb} give weak equivalences when evaluated at $(1,0)$, $(0,1)$, and $(1,1)$. Therefore p^{pb} and s^{pb} must be weak equivalences, and using 2-out-of-3 we see that since η_Y is a weak equivalence, so is η_Z.

(2) ($\Longrightarrow$) (3). Choose any weak equivalence $X \to Z$ to a fibrant square, which exists because $\mathcal{S}$ is a model category. By (2), η_Z is a weak equivalence, and hence so is the composite map $X \to Z^{\mathrm{pb}}$. This is the desired weak equivalence to a fibrant pullback square.

(3) ($\Longrightarrow$) (1). Immediate. $\square$

51.4. *Example.* Any commutative square in which a pair of opposite sides are weak equivalences is a homotopy pullback. For instance, suppose X is a square, with weak equivalences $X(0,0) \to X(1,0)$ and $X(0,1) \to X(1,1)$. Using the factorizations in the model category $\mathcal{S}$ of commutative squares, construct a commutative diagram

$$\begin{tikzcd} X(0,0) \arrow[r, "\sim"] \arrow[d] & X(1,0) \arrow[r, "\sim"] \arrow[d] & Y(1,0) \arrow[d] \\ X(0,1) \arrow[r, "\sim"'] & X(1,1) \arrow[r, "\sim"'] & Y(1,1) \end{tikzcd}$$

so that $X(1,1) \to Y(1,1)$ is a weak equivalence to a fibrant object, and $X(1,0) \to Y(1,0) \to Y(1,1)$ is a factorization of $X(1,0) \to Y(1,1)$ into a weak equivalence followed by a fibration. Let Y be the evident commutative square such that $Y(0,k) \to Y(1,k)$ are identity maps. Then the evident map $X \to Y$ is a weak equivalence to a fibrant pullback square.

51.5. *Example.* Any good pullback square is a homotopy pullback. For instance, suppose X is a good pullback square, so that all objects $X(i,j)$ are fibrant and with $X(1,0) \to X(1,1)$ is a fibration. Construct a commutative diagram

$$\begin{tikzcd} X(0,0) \arrow[r] \arrow[d] & Y(0,0) \arrow[r] \arrow[d] & X(1,0) \arrow[d] \\ X(0,1) \arrow[r, "\sim"'] & Y(0,1) \arrow[r] & X(1,1) \end{tikzcd}$$

where the bottom row is a factorization of $X(0,1) \to X(1,1)$ into a weak equivalence followed by a fibration, and the right-hand square is a pullback. Write Y for the right-hand square in the diagram. Thus both X and Y are good pullbacks, and the evident map $X \to Y \approx Y^{\mathrm{pb}}$ is a weak equivalence, so X is a homotopy pullback as desired.

51.6. *Remark.* Observe that when X is a fibrant square, (51.3) implies that it is a homotopy pullback if and only if η_X is a weak equivalence of squares, i.e., if and only if $X(0,0) \to X(1,0) \times_{X(1,1)} X(0,1)$ is a weak equivalence in $\mathcal{M}$.

51.7. *Remark.* Note that the notion of homotopy pullback square is invariant with respect to the evident involution of $\Delta^1 \times \Delta^1$, (i.e., switching $X(i,j)$ with $X(j,i)$) though the notion of good pullback is not invariant.

The property of being a homotopy pullback square is invariant with respect to weak equivalences.

51.8. **Proposition.** *If $f\colon X \to Y$ is a weak equivalence of commutative squares, then X is a homotopy pullback if and only if Y is a homotopy pullback.*

Proof. Choose any weak equivalence $g\colon Y \to Z$ to a fibrant commutative square. Then (51.3) applied to gf and to g implies that η_Z is a weak equivalence if and only if X is a homotopy pullback, and if and only if Y is a homotopy pullback. $\square$

51.9. **Proposition.** *Let $f\colon X \to Y$ be a map between homotopy pullback squares. Then f is a weak equivalence if and only if $f(i,j)\colon X(i,j) \to Y(i,j)$ is a weak equivalence for each of $(i,j) = (0,1), (1,0), (1,1)$.*

Proof. The only if direction is clear, so suppose f is a map which is a weak equivalence at all points except perhaps $(0,0)$. Construct a commutative diagram in $\mathcal{S}$ of the following form

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, "f"'] & X' \arrow[r, "\eta_{X'}"] \arrow[d, "f'"'] & X'^{\mathrm{pb}} \arrow[d, "f'^{\mathrm{pb}}"'] \\ Y \arrow[r, "\sim"'] & Y' \arrow[r, "\eta_{Y'}"'] & Y'^{\mathrm{pb}} \end{tikzcd}$$

where X' and Y' are fibrant squares and the indicated maps are weak equivalences of squares. Since X and Y are homotopy pullbacks, so are X' and Y' (51.8), and hence $\eta_{X'}$ and $\eta_{Y'}$ are weak equivalences (51.3). Since X'^{pb} and Y'^{pb} are fibrant pullback squares they are good pullbacks, and since good pullbacks are weak equivalence invariant we have that $f'^{\mathrm{pb}}(0,0)$ is a weak equivalence, so f'^{pb} is a weak equivalence of squares. The claim follows by 2-out-of-3. □

51.10. **Proposition** (Patching homotopy pullbacks)**.** *Suppose given a commutative diagram $X\colon \Delta^2\times\Delta^1 \to \mathcal{M}$:* **M 28 Mar**

$$\begin{tikzcd} X(0,0) \arrow[r] \arrow[d] & X(1,0) \arrow[r] \arrow[d] & X(2,0) \arrow[d] \\ X(0,1) \arrow[r] & X(1,1) \arrow[r] & X(2,1) \end{tikzcd}$$

in which the right-hand square is a homotopy pullback. Then the left-hand square is a homotopy pullback if and only if the large rectangle is a homotopy pullback.

Proof. Construct a natural transformation $\alpha\colon X \to Y$ of functors $\Delta^2\times\Delta^1 \to \mathcal{M}$ as follows:

$$\begin{tikzcd} X(0,1) \arrow[r] \arrow[d, "\alpha(0,1)"', "\sim"] & X(1,1) \arrow[r] \arrow[d, "\alpha(1,1)"', "\sim"] & X(2,1) \arrow[d, "\alpha(2,1)"', "\sim"] & X(2,0) \arrow[l] \arrow[d, "\alpha(2,0)"', "\sim"] \\ Y(0,1) \arrow[r] & Y(1,1) \arrow[r] & Y(2,1) & Y(2,0) \arrow[l] \end{tikzcd}$$

(i) choose a weak equivalence $\alpha(2,1)$ to a fibrant object, (ii) factor $X(1,1) \to Y(2,1)$ into a weak equivalence $\alpha(1,1)$ followed by a fibration, (iii) factor $X(0,1) \to Y(1,1)$ into a weak equivalence $\alpha(0,1)$ followed by a fibration, (iv) factor $X(2,0) \to Y(2,1)$ into a weak equivalence $\alpha(2,0)$ followed by a fibration, and (v) define $Y(0,0)$ and $Y(1,0)$ to be evident pullbacks, with $\alpha(0,0)$ and $\alpha(1,0)$ the unique maps making the diagram commutute.

We see that each of the three squares $Y|_{\Delta^{\{i,j\}}\times\Delta^1}$ is a fibrant pullback square. Since $X|_{\Delta^{\{1,2\}}\times\Delta^1}$ is a homotopy pullback we have that $Y|_{\Delta^{\{1,2\}}\times\Delta^1}$ is also a homotopy pullback by (51.9). Again using (51.9), we see that $\alpha(0,0)$ is a weak equivalence if and only if $X|_{\Delta^{\{0,1\}}\times\Delta^1}$ is a homotopy pullback, and also if and only if $X|_{\Delta^{\{0,2\}}\times\Delta^1}$ is a homotopy pullback. □

Part 9. Quasigroupoids and weak homotopy equivalence

52. Weak homotopy equivalence

Say that a map $f\colon X \to Y$ of simplicial sets is a **weak homotopy equivalence** if and only for every ∞-groupoid (i.e., Kan complex) C the induced functor $\mathrm{Fun}(f,C)\colon \mathrm{Fun}(Y,C) \to \mathrm{Fun}(X,C)$ is a categorical equivalence. It is immediate that every categorical equivalence is a weak homotopy equivalence, but the converse is not so. **weak homotopy equivalence**

52.1. *Exercise.* Show that either inclusion $\Delta^0 \to \Delta^1$ is a weak homotopy equivalence but not a categorical equivalence.

52.2. *Remark.* A more logical name for weak homotopy equivalence might be "groupoidal equivalence", by analogy with categorical equivalence.

52.3. *Exercise.* Show that the class of weak homotopy equivalences satisfies 2-out-of-6, and hence 2-out-of-3.

52.4. **Proposition.** *Let $i\colon X \to Y$ be a monomorphism of simplicial sets. Then the following are equivalent.*

1. *The map i is a weak homotopy equivalence.*
2. *For every isofibration $p\colon C \to D$ between quasigroupoids, the pullback-hom map $p^{\square i}$ is a trivial fibration.*
3. *For every isofibration $p\colon C \to D$ between quasigroupoids we have $i \boxslash p$.*

Then i is a weak homotopy equivalence if and only if $i \boxslash p$ for every isofibration $p\colon C \to D$ between quasigroupoids.

Proof. ($1 \implies 2$): Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(Y,C) \arrow[r, "p^{\square i}"'] \arrow[rr, bend left, "p_*"] \arrow[dr, bend right, "i^*"'] & \mathrm{Fun}(X,C) \times_{\mathrm{Fun}(X,D)} \mathrm{Fun}(Y,D) \arrow[r] \arrow[d, "q"] & \mathrm{Fun}(Y,D) \arrow[d, "i^*"] \\
& \mathrm{Fun}(X,C) \arrow[r] & \mathrm{Fun}(X,D)
\end{tikzcd}$$

in which the square is a pullback. All the maps in this diagram are isofibrations by (39.5) (in fact, they are Kan fibrations), while the maps marked i^* are categorical equivalences by hypothesis. In particular, the maps marked i^* are trivial fibrations by (40.8), and thus so is the map q obtained by basechange. Thus $p^{\square i}$ is a categorical equivalence by 2-out-of-3 and hence is a trivial fibration by (40.8).

($2 \implies 3$): The trivial fibration $p^{\square i}$ is surjective on vertices, giving $i \boxslash p$.

($3 \implies 2$): If $p\colon C \to D$ is an isofibration between quasigroupoids, then so is $p^{\square(\partial\Delta^n \subset \Delta^n)}$ for any $n \geq 0$, using (40.8). Thus (2) implies that $i \boxslash p^{\square(\partial\Delta^n \subset \Delta^n)}$, which is equivalent to $(\partial\Delta^n \subset \Delta^n) \boxslash p^{\square i}$, whence $p^{\square i}$ is a trivial fibration as desired.

($2 \implies 1$): The hypothesis implies in particular that $\mathrm{Fun}(Y,C) \to \mathrm{Fun}(X,C)$ is a trivial fibration for every quasigroupoid C, and hence a categorical equivalence (25.11). $\square$

52.5. **Corollary.** *Every anodyne map (i.e., element of $\overline{\mathrm{Horn}}$) is a weak homotopy equivalence.*

Proof. Anodyne maps i are monomorphisms such that $i \boxslash p$ for every Kan fibration. Thus statement (3) of (52.4) applies since isofibrations between quasigroupoids are Kan fibrations (38.12). $\square$

53. The Kan-Quillen model structure on simplicial sets

Say that map $p\colon X \to Y$ is a **groupoidal fibration** if $j \boxslash p$ for all j which are monomorphisms and weak equivalences. I write GpdFib for the class of groupoidal fibrations. As with the class of categorical fibrations, there a set of maps T such that $\mathrm{GpdFib} = T^{\boxslash}$; see (80). **groupoidal fibration**

53.1. The Kan-Quillen model structure.

53.2. **Theorem.** *The category of simplicial sets admits a model structure, in which*

- W = *weak homotopy equivalences (*WHEq*),*
- Cof = *monomorphims (*$\overline{\mathrm{Cell}}$*),*
- Fib = *groupoidal fibrations (*GpdFib*).*

Furthermore, the fibrant objects are precisely the Kan complexes, and the fibrations with target a fibrant object are precisely the Kan fibrations.

Proof. Weak equivalences satisfy 2-out-of-3 by (52.3). We have that

- $\mathrm{Cof} = \overline{\mathrm{Cell}}$ by definition,
- $\mathrm{Fib} \cap \mathrm{W} = \mathrm{GpdFib} \cap \mathrm{WHEq} = \mathrm{TFib} = \mathrm{Cell}^{\boxslash}$ by (41.2),
- $\mathrm{Cof} \cap \mathrm{W} = \overline{\mathrm{Cell}} \cap \mathrm{WHEq} = \overline{T}$ for some set T, as noted above.
- $\mathrm{Fib} = \mathrm{GpdFib} = (\mathrm{Cof} \cap \mathrm{W})^{\boxslash} = T^{\boxslash}$ by definition,

so both $(\mathrm{Cof} \cap \mathrm{W}, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap \mathrm{W})$ are weak factorization systems via the small object argument (17.16). Thus, we get a model category.

We have seen that Kan fibrations between Kan complexes (which are exactly the isofibrations between Kan complexes) have the lifting property of groupoidal fibrations (52.4), so the statements about fibrant objects and fibrations to fibrant objects follow just as in the categorical case. □

53.3. **Proposition.** *The Quillen model structure is cartesian.*

Proof. We must show that $p^{\square j}$ is a groupoidal fibration if j is a monomorphsm and p a groupoidal fibration, and also that it is a weak equivalence if either j or p is. This is proved by an argument nearly identical to the proof of (41.3). □

53.4. **Kan fibrations are groupoidal fibrations.** The proof of the Quillen model structure we gave above relied on (80) to produce a set T such that $\overline{T} = \mathrm{Cof} \cap \mathrm{WHEq}$. In fact, more is true. It turns out that we can take $T = \mathrm{Horn}$, so that $\mathrm{GpdFib} = \mathrm{KanFib}$. It was in this form that the model structure was first constructed by Quillen.

We will not give a proof of this here. The non-trivial part is to show that $\mathrm{KanFib} \subseteq \mathrm{GpdFib}$. This proposition is usually proved via an argument (due to Quillen) based on the theory of *minimal fibrations*. See for instance Quillen's original argument [Qui67, §II.3] or [GJ09, Ch. 1].

These arguments work by showing that KanFib is the weak cosaturation of the class of *Kan fibrations between Kan complexes*, which we know are groupoidal fibrations. In fact one can even show that every Kan fibration is a *base change* of a Kan fibration between Kan complexes, see [KLV12].

The observation that the Kan-Quillen model structure can be constructed without first showing $\mathrm{GpdFib} = \mathrm{KanFib}$, and thus (53.2) in the form I have stated it, is due to Cisinski [Cis06].

54. Groupoid completion

54.1. **Functors into the core of a quasicategory.** Given a quasicategory C and a simplicial set X, let

$$\mathrm{Fun}^{\mathrm{iso}}(X, C) \subseteq \mathrm{Fun}(X, C)$$

denote the full subcategory spanned by objects which are functors $f\colon X \to C$ with the property that $f(X) \subseteq C^{\mathrm{core}}$.

54.2. *Example.* When $X = \Delta^1$, then this is precisely the path category $\mathrm{Fun}^{\mathrm{iso}}(\Delta^1, C)$ introduced in (36.1).

Note that $\mathrm{Fun}^{\mathrm{iso}}(X, C)$ is not necessarily a quasigroupoid, unless C itself is a quasigroupoid.

We have a convenient characterization of maps into $\mathrm{Fun}^{\mathrm{iso}}(X, C)$.

54.3. **Proposition.** *For any quasicategory C and simplicial sets X and S, the evident bijection* $\mathrm{Hom}(S, \mathrm{Fun}(X, C)) \approx \mathrm{Hom}(X, \mathrm{Fun}(S, C))$ *restricts to a bijection*

$$\left\{\, S \dashrightarrow \mathrm{Fun}^{\mathrm{iso}}(X, C) \,\right\} \longleftrightarrow \left\{\, X \dashrightarrow \mathrm{Fun}(S, C)^{\mathrm{core}} \,\right\}.$$

Proof. (This is a generalization of (36.5).) Consider $f\colon S \to \mathrm{Fun}(X, C)$, and write $f'\colon X \to \mathrm{Fun}(S, C)$ and $f''\colon S \times X \to C$ for its adjoints. We have the following observations (which make use of the pointwise criterion for natural isomorphisms (37.2)).

(1) The map f factors through $\mathrm{Fun}^{\mathrm{iso}}(X, C) \subseteq \mathrm{Fun}(X, C)$ if and only if for each vertex $s \in S_0$ the induced map $f(s)\colon X \to C$ factors through $C^{\mathrm{core}} \subseteq C$. This amounts to saying that for each edge $g \in X_1$, each map $f(s)$ sends g to an isomorphism in C.
(2) The map f' factors through $\mathrm{Fun}(S, C)^{\mathrm{core}} \subseteq \mathrm{Fun}(S, C)$ if and only if for each edge $g \in X_1$ the the image $f'(g) \in \mathrm{Fun}(S, C)_1$ represents an isomorphism in $\mathrm{Fun}(S, C)$. By the objectwise criterion (37.2), this amounts to saying that $f'(g)$ sends each vertex $s \in S_0$ to an isomorphism in C.

It is thus apparent that conditions (1) and (2) are equivalent: both are amount to the requirement that $\Delta^0 \times \Delta^1 \xrightarrow{s \times g} S \times X \xrightarrow{f''} C$ represent an isomorphism in C for every $s \in S_0$ and $g \in X_1$. □

For any map $f\colon X \to Y$ of simplicial sets and any quasicategory C, the induced functor $\mathrm{Fun}(f, C)$ restricts to a functor $\mathrm{Fun}^{\mathrm{iso}}(Y, C) \to \mathrm{Fun}^{\mathrm{iso}}(X, C)$ between full subcategories.

54.4. **Proposition.** *Let $i\colon X \to Y$ be any map of simplical sets which is a monomorphism and a weak homotopy equivalence. Then for any quasicategory C, the restriction map*

$$i^*\colon \mathrm{Fun}^{\mathrm{iso}}(Y, C) \to \mathrm{Fun}^{\mathrm{iso}}(X, C)$$

is a trivial fibration, and thus in particular a categorical equivalence between quasicategories.

Proof. We need to solve lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "u"] \arrow[d, tail] & \mathrm{Fun}^{\mathrm{iso}}(Y, C) \arrow[d, "i^*"] \\ \Delta^n \arrow[r, "v"'] \arrow[ur, dotted, "s"] & \mathrm{Fun}^{\mathrm{iso}}(X, C) \end{tikzcd}$$

for all $n \geq 0$. Using (54.3) we can replace this with the adjoint lifting problem

$$\begin{tikzcd} X \arrow[r, "\widetilde{v}"] \arrow[d, tail, "i"'] & \mathrm{Fun}(\Delta^n, C)^{\mathrm{core}} \arrow[d, "p^{\mathrm{core}}"] \\ Y \arrow[r, "\widetilde{u}"'] \arrow[ur, dotted, "\widetilde{s}"] & \mathrm{Fun}(\partial\Delta^n, C)^{\mathrm{core}} \end{tikzcd}$$

where p^{core} is induced by the restriction map $p\colon \mathrm{Fun}(\Delta^n, C) \to \mathrm{Fun}(\partial\Delta^n, C)$. By (39.5) the map p is an isofibration, and thus p^{core} is an isofibration between quasigroupoids and thus a Kan fibration (38.12). Therefore a lift exists by (52.4). □

54.5. **Groupoid completion.** For any simplicial set X, we can always construct a monomorphism $i\colon X \to X'$ to a Kan complex which is a weak homotopy equivalence. For instance, factor $X \to *$ into an anodyne map followed by a Kan fibration. Any such map provides an example of a **groupoid completion** of X. **groupoid completion**

54.6. **Proposition.** *Suppose $i\colon X \to X'$ is a monomorphism of simplicial sets which is a weak homotopy equivalence, with X' a quasigroupoid. Then for any quasicategory C, restriction along i induces a trivial fibration*

$$p\colon \mathrm{Fun}(X', C) \to \mathrm{Fun}^{\mathrm{iso}}(X, C).$$

In particular, any map $f\colon X \to X^{\mathrm{core}} \subseteq C$, extends over i to a map $g\colon X' \to C^{\mathrm{core}}$, and any two such extensions are naturally isomorphic in $\mathrm{Fun}(X, C)$.

Proof. That p is a trivial fibration is immediate from (54.4) and the fact that $\mathrm{Fun}^{\mathrm{iso}}(X', C) = \mathrm{Fun}(X', C)$ since X' is a quasigroupoid. The fiber of p over a vertex representing f is thus a contractible Kan complexes, so any two objects in this fiber are isomorphic, and hence correspond to isomorphic objects of $\mathrm{Fun}(X', C)$. □

Although the groupoid completion isn't unique, it is unique up to categorical equivalence.

54.7. *Exercise.* Let $f_i\colon X \to X_i$ be groupoid completions of X, for $i = 1, 2$. Show that there exists a categorical equivalence $g\colon X_1 \to X_2$ such that $gf_1 = f_2$, and that any two such are naturally isomorphic. (Hint: proof of (25.19) and (40.16).)

We can apply this construction when X is a quasicategory, or even when X is the nerve of an ordinary category, and obtain interesting new quasigroupoids.

54.8. *Example.* It turns out that every simplicial set is weakly equivalent to the nerve of some ordinary category, and in fact to the nerve of some poset [Tho80]. Thus, for every Kan complex K, there exists an ordinary category A and a weak equivalence $NA \to K$, which therefore induces categorical equivalences $\mathrm{Fun}(K, C) \approx \mathrm{Fun}^{\mathrm{iso}}(NA, C)$ for every quasicategory C.

We note that there is also a classical groupoid completion construction, which given an ordinary category A produces an ordinary groupoid A_{Gpd} by "formally inverting all maps". We have that $h((NA)_{\mathrm{Kan}}) \approx N(A_{\mathrm{Gpd}})$, but in general $(NA)_{\mathrm{Kan}}$ is not weakly equivalent to $N(A_{\mathrm{Gpd}})$.

54.9. *Exercise.* Let A be the poset of *proper and non-empty* subsets of $\{0, 1, 2, 3\}$. Show that A_{Gpd} is equivalent to the one-object category, but that $(NA)_{\mathrm{Kan}}$ is not equivalent to the one-object category. (In the second case, you can prove non-equivalence by showing $\pi_0 \mathrm{Fun}(NA, K(\mathbb{Z}, 2)) \approx \mathbb{Z}$, using the Eilenberg-MacLane object of (12.13).)

55. Localization of quasicategories

There is a generalization of groupoid completion, which applies to a simplicial set X equipped with a subcomplex $W \subseteq X$. For a quasicategory C, let

$$\mathrm{Fun}^{W\mathrm{iso}}(X, C) \subseteq \mathrm{Fun}(X, C)$$

denote the full subcategory spanned by objects $f\colon X \to C$ such that $f(W) \subseteq C^{\mathrm{core}}$. (Note that this condition it satified if and only if f maps the edges of W to isomorphisms.) Clearly $\mathrm{Fun}^{W\mathrm{iso}}(X, C)$ is the primage of $\mathrm{Fun}^{\mathrm{iso}}(W, C)$ along the restriction map $\mathrm{Fun}(X, C) \to \mathrm{Fun}(W, C)$.

55.1. *Exercise.* Show that there is a bijective correspondence of the form

$$\left\{ S \dashrightarrow \mathrm{Fun}^{W\mathrm{iso}}(X, C) \right\} \longleftrightarrow \left\{ \begin{tikzcd} W \arrow[r, dashed] \arrow[d, tail] & \mathrm{Fun}(S, C)^{\mathrm{core}} \arrow[d, tail] \\ X \arrow[r, dashed] & \mathrm{Fun}(S, C) \end{tikzcd} \right\}.$$

Given a subcomplex $W \subseteq X$, we may define a **localization** of X with respect to W. This is any map $X \to X_{(W)}$ constructed as follows. **localization**

(1) Choose a groupoid completion $i\colon W \to W'$ of W, i.e., a monomorphsm to a Kan complex.
(2) Choose monomorphism which is a categorical equivalence from $j\colon X \cup_W W' \to X_{(W)}$ to a quasicategory $X_{(W)}$ (e.g., an inner anodyne map to a quasicategory).

If $W = X$ then $X \to X_{(X)}$ is an example of a groupoid completion of X as discussed above.

55.2. **Proposition.** *For any localization $X \to X_{(W)}$ as defined above, and any quasicategory C, the restriction map $\mathrm{Fun}(X_{(W)}, C) \to \mathrm{Fun}(X, C)$ induces a trivial fibration*

$$\mathrm{Fun}(X_{(W)}, C) \to \mathrm{Fun}^{W\mathrm{iso}}(X, C).$$

In particular, any map $f\colon X \to C$ such that $f(W) \subseteq C^{\mathrm{core}}$ extends to a functor $g\colon X_{(W)} \to C$, and any two such extensions are naturally isomorphic.

Proof. Consider

$$\begin{tikzcd} \mathrm{Fun}(X_{(W)}, C) \arrow[r, "j^*"] & \mathrm{Fun}(X \cup_W W', C) \arrow[r, "p"] \arrow[d] & \mathrm{Fun}^{W\mathrm{iso}}(X, C) \arrow[r, tail] \arrow[d] & \mathrm{Fun}(X, C) \arrow[d] \\ & \mathrm{Fun}(W', C) \arrow[r, "i^*"'] & \mathrm{Fun}^{W\mathrm{iso}}(W, C) \arrow[r, tail] & \mathrm{Fun}(W, C) \end{tikzcd}$$

in which both squares are pullbacks. The map j^* is a trivial fibration since $j \in \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$ (**??**), while i^* is a trivial fibration (54.6) since $\mathrm{Fun}^{W\mathrm{iso}}(W, C) = \mathrm{Fun}^{\mathrm{iso}}(W, C)$, whence p is a trivial fibration. □

55.3. **Quasicategories from relative categories.** A **relative category** is a pair $W \subseteq C$ consisting of an *ordinary category* C and a subcategory W containing all the objects of C. The above construction gives, for any relative category, a map **relative category**

$$C \to C_{(W)},$$

unique up to categorical equivalence. We may call $C_{(W)}$ the **localization** of C with respect to W. **localization**

It turns out that all quasicategories, up to categorical equivalence, arise as localizations of relative categories in this way [BK11].

56. Weak homotopy equivalence and homotopy groups

56.1. **Pointed simplicial sets and pointed function complexes.** Given a simplicial set X and a vertex $x \in X_0$, I'll write (X, x) for the corresponding **pointed** simplicial set, i.e., object of $s\mathrm{Set}_* := s\mathrm{Set}_{\Delta^0/}$. Given pointed simplicial sets (X, x), (Y, y), I'll write **pointed**

$$\mathrm{Fun}_*((X, x), (Y, y)) := \mathrm{Fun}_{\Delta^0/}((X, x), (Y, y))$$

for the relative function complex, and call it the pointed function complex. I'll often omit mention of the basepoints and write $\mathrm{Fun}_*(X, Y)$. This defines a functor $\mathrm{Fun}_* \colon (s\mathrm{Set}_*)^{\mathrm{op}} \times s\mathrm{Set}_* \to s\mathrm{Set}_*$, where the basepoint of $\mathrm{Fun}_*(X, Y)$ is represented by the constant map $X \to \{y\} \to Y$ to the basepoint of Y.

Say that a map of pointed simplicial sets is a weak homotopy equivalence if the underlying map of simplicial sets is a weak homotopy equivalence.

56.2. **Proposition.** *The pointed function complex is weak homotopy equivalence invariant when the target is a Kan complex. That is,*

(1) *if $f \colon A \to B$ is a weak homotopy equivalence of pointed simplicial sets, and X is a pointed Kan complex, then $f^* \colon \mathrm{Fun}_*(B, X) \to \mathrm{Fun}_*(A, X)$ is a weak homotopy equvialence, and*
(2) *if A is any pointed simplicial set and $g \colon X \to Y$ is a weak homotopy equivalence of pointed Kan complexes, then $g_* \colon \mathrm{Fun}_*(A, X) \to \mathrm{Fun}_*(B, X)$ is a weak homotopy equivalence.*

Proof. The pointed function complex $\mathrm{Fun}_*(A, X)$ is defined by a pullback $\mathrm{Fun}(A, X) \times_{\mathrm{Fun}(\{a\}, X)} \{\overline{x}\}$, which is good pullback when X is a Kan complex since $\{a\} \to A$ is always a monomorphism. The claim follows from the fact that Fun is weak homotopy invariant whenever the target X is a Kan complex. □

56.3. **Homotopy sets.** Given a Kan complex X, for each $n \geq 0$ and each vertex $x \in X_0$ we define the n**th homotopy set** to be **nth homotopy set**

$$\pi_n(X, x) := \pi_0 \mathrm{Fun}_*(\Delta^n / \partial \Delta^n, X).$$

These define functors $\pi_n \colon s\mathrm{Set}_* \to \mathrm{Set}_*$ from pointed simplicial sets to pointed sets, where the basepoint of $\pi_n(X, x)$ is the path component containing the constant map.

56.4. *Remark.* In general, $\pi_n(X,x)$ is a pointed set. In fact, when $n \geq 1$ it has a natural structure of a group, which is abelian when $n \geq 2$.

56.5. *Exercise.* Let X be a Kan complex and $x \in X_0$. Show that $\pi_1(X,x) \approx \mathrm{Hom}_{hX}(x,x)$, so that $\pi_1(X,x)$ has a group stucture defined by composition in hX, and that this group structure is natural with respect to maps between pointed Kan complexes.

We present a proof that $\pi_n(X,x)$ are groups for all $n \geq 1$ in the appendix (79).

56.6. *Remark.* If T is a topological space with basepoint $t \in T$, then the homotopy sets $\pi_n(\mathrm{Sing}(T),t)$ are in natural bijective correspondence with the "usual" homotopy sets (groups) $\pi_n(T,t)$ of the space T. This is a straightforward consequence of the observation that $\|\Delta^n/\partial\Delta^n\|$ is homeomorphic to an n-dimensional sphere.

56.7. **π_*-equivalences.** Say that a map $f\colon X \to Y$ between Kan complexes is a **π_*-equivalence** if for all $k \geq 0$ and all $x \in X_0$, the induced map $\pi_k(X,x) \to \pi_k(Y,f(x))$ is a bijection. It is clear from (56.2) that every weak equivalence of Kan complexes is a π_*-equivalence. In fact the converse is also true. **π_*-equivalence**

56.8. **Theorem.** *A map $f\colon X \to Y$ between Kan complexes is a weak homotopy equivalence if and only if it is a π_*-equivalence.*

We give a proof in an appendix (79).

57. Every quasigroupoid is equivalent to its opposite

Every ordinary groupoid C is equivalent, and in fact isomorphic, to its opposite: there is a functor $C \to C^{\mathrm{op}}$ which is the identity on objects, and which sends each morphism to its inverse. We cannot define such a functor for quasigroupoids, since inverses of morphisms in a quasigroupoid are not unique. However, it is the case that any quasigroupoid is equivalent to its opposite.

We will produce for each quasicategory C a quasigroupoid $\mathcal{S}(C)$, together with a trivial fibration $\mathcal{S}(C) \to C^{\mathrm{core}}$, with the property that $\mathcal{S}(C)$ and $\mathcal{S}(C^{\mathrm{op}})$ are isomorphic as simplicial sets.

57.1. **The functor $\mathcal{S}$.** Given a set S, let Iso^S denote the simplicial set with

$$(\mathrm{Iso}^S)_n := \mathrm{Hom}_{\mathrm{Set}}([n] = \{0,1,\dots,n\}, S),$$

with simplicial operators induced in the evident way. Observe that

$$\mathrm{Hom}_{s\mathrm{Set}}(X, \mathrm{Iso}^S) \approx \mathrm{Hom}_{\mathrm{Set}}(X_0, S),$$

so that we have a functor $\mathrm{Iso}^\bullet\colon \mathrm{Set} \to s\mathrm{Set}$ which is right adjoint to $X \mapsto X_0\colon s\mathrm{Set} \to \mathrm{Set}$. In particular, we have

$$\mathrm{Hom}_{s\mathrm{Set}}(\mathrm{Iso}^S, \mathrm{Iso}^T) \approx \mathrm{Hom}_{\mathrm{Set}}(S,T).$$

Recall that $X \mapsto X_0$ also admits a left adjoint $S \mapsto S_{\mathrm{disc}}$, sending any set to the correpsonding discrete simplcial set (2.9).

Note that the simplicial set Iso^S is the nerve of a category, with object set S and a unique morphism $x \to y$ for every pair $x,y \in S$. This is in fact a groupoid, which when S is non-empty is equivalent to the trivial groupoid. For instance, $\mathrm{Iso}^{\{0,1\}}$ is precisely the walking isomorphism Iso discussed in (38.6).

We may compose $\mathrm{Iso}^\bullet$ with the evident functor $\Delta \to \mathrm{Set}$ sending the ordered set $[n]$ to its underlying set $\{0,1,\dots,n\}$, and in this way obtain a functor $\mathrm{Iso}\colon \Delta \to s\mathrm{Set}$, so that $(\mathrm{Iso}^{[n]})_m = \mathrm{Hom}_{\mathrm{Set}}([m],[n])$. This in turn induces by restriction a functor $\mathcal{S}\colon s\mathrm{Set} \to s\mathrm{Set}$, with

$$\mathcal{S}(C)_n = \mathrm{Hom}(\mathrm{Iso}^{[n]}, C).$$

57.2. **Proposition.** *There is a natural isomorphism $\mathcal{S}(C) \to \mathcal{S}(C^{\mathrm{op}})$ of functors $s\mathrm{Set} \to s\mathrm{Set}$.*

Proof. It suffices to describe a natural isomorphism $\mathrm{Iso} \to \mathrm{Iso} \circ \mathrm{op}$ of functors $\Delta \to s\mathrm{Set}$. On each object $[n]$ in Δ this is the map $\mathrm{Iso}^r\colon \mathrm{Iso}^{[n]} \to \mathrm{Iso}^{[n]}$ induced by the order reversing bijection $r\colon [n] \to [n]$, $r(x) := n - x$. □

We write $\eta_n\colon \Delta^n \to \mathrm{Iso}^{[n]}$ for the map representing the n-cell correpsonding to the identity function $[n] \to [n]$. Then we get a natural tranformation $\epsilon\colon \mathcal{S} \to \mathrm{id}_{s\mathrm{Set}}$ of functors $s\mathrm{Set} \to s\mathrm{Set}$, which for a simplicial set C and $n \geq 0$ is given by the function $\mathrm{Hom}(\mathrm{Iso}^n, C) \to C_n$ which sends $f\colon \mathrm{Iso}^n \to C$ to to the n-cell represented by $f\eta_n\colon \Delta^n \to C$. Note that ϵ induces a bijection on vertices, since $\eta_0\colon \Delta^0 \to \mathrm{Iso}^{[0]}$ is an isomorphism.

We are going to show that $\epsilon\colon \mathcal{S}(C) \to C$ is a trivial fibration whenever C is a quasigroupoid. Together with (57.2) this gives a sequence of categorical equivalences

$$C \xleftarrow{\epsilon_C} \mathcal{S}(C) \approx \mathcal{S}(C^{\mathrm{op}}) \xrightarrow{\epsilon_{C^{\mathrm{op}}}} C^{\mathrm{op}}$$

between quasigroupoids. In particular, by choosing any section s of the trivial fibration ϵ_C we get a categorical equivalence $C \to C^{\mathrm{op}}$ which is identity on objects.

57.3. **The functor $\mathcal{R}$.** The functor $\mathcal{S}$ admits a left adjoint $\mathcal{R}$, which we can describe explicitly. We will make use of the evident identification

$$\gamma \mapsto \widetilde{\gamma}\colon \mathrm{Hom}_{s\mathrm{Set}}([n], \Delta^m) \approx \mathrm{Hom}_{s\mathrm{Set}}(\mathrm{Iso}^{[n]}, \mathrm{Iso}^{[m]}).$$

Note that $\widetilde{\iota}_n = \mathrm{id}_{\mathrm{Iso}^{[n]}}$ where $\iota_n\colon [n] \to \Delta^n$ is the evident function sending $k \mapsto \langle k\rangle \in (\Delta^n)_0$.

Define $\mathcal{R}(X)$ to be the simplicial set with n-dimensional cells

$$\mathcal{R}(X)_n := \{[n] \xrightarrow{\gamma} \Delta^m \xrightarrow{x} X\}/\sim,$$

where we quotient by the equivalence relation generated by $(xf, \gamma) \sim (x, f\gamma)$ for every simplicial operator f. The simplicial operators act on $\mathcal{R}(X)$ in the evident way: $g\colon [n'] \to [n]$ sends an equivalence class $[x, \gamma] \in \mathcal{R}(X)_n$ to $[x, \gamma g] \in \mathcal{R}(X)_{n'}$.

57.4. **Proposition.** *The functor $\mathcal{R}$ is left adjoint to $\mathcal{S}$, so that a map $f\colon X \to \mathcal{S}(C)$ corresponds to a map $g\colon \mathcal{R}(X) \to C$ which sends $[x, \gamma] \in \mathcal{R}(X)_n$ to $f(x)\widetilde{\gamma} \in C_n$.*

Proof. We just need to verify that the correspondence is well-defined, a bijection, and natural in X and C. That it is well-defined is clear, since if $[x\delta, \gamma] = [x, \delta\gamma]$ for some simplicial operator γ, we have $f(x\delta)\widetilde{\gamma} = f(x)\delta\widetilde{\gamma} = f(x)\widetilde{(\delta\gamma)}$. To see that it is a bijection, note that an inverse is given by sending $g\colon \mathcal{R}(X) \to C$ to the map $f\colon X \to \mathcal{S}(C)$ sending $x \in X_n$ to $g([x, \iota_n])$. Naturality is a straightforward verification. □

57.5. *Example.* Suppose $X = \Delta^n$. Then

$$\mathrm{Hom}(\mathcal{R}(\Delta^n), C) \approx \mathrm{Hom}(\Delta^n, \mathcal{S}(C)) \approx \mathrm{Hom}(\mathrm{Iso}^{[n]}, C)$$

so $\mathcal{R}(\Delta^n) \approx \mathrm{Iso}^{[n]}$ by Yoneda, and in fact $\mathcal{R}\colon s\mathrm{Set} \to s\mathrm{Set}$ extends $\mathrm{Iso}\colon \Delta \to s\mathrm{Set}$.

We can also see this from the explicit description of $\mathcal{R}$. Each equivalence class $[x, \gamma] \in \mathcal{R}(\Delta^n)_k$ contains a unique element of the form $([k] \xrightarrow{\iota_k} \Delta^k \to \Delta^n)$, so $\mathcal{R}(\Delta^n)_k$ is in natural bijective correspondence with the set of all functions $\gamma\colon [k] \to [n]$ (not necessarily order preserving), i.e., with the set of k-cells of $\mathrm{Iso}^{[n]}$.

In the following, it will be convenient to represent a k-cell in $\mathcal{R}(\Delta^n) \approx \mathrm{Iso}^{[n]}$ by a sequence $(a_0 \dots a_k)$ of elements $a_i \in [k]$. Note that such a k-cell is non-degenerate if and only if the sequence has no consecutive repetition, i.e., $a_{i-1} \neq a_i$ for all $i = 1, \dots, k$.

57.6. **Lemma.** *Let $K \subseteq \Delta^n$ be a subcomplex. Then the induced map $\mathcal{R}(K) \to \mathcal{R}(\Delta^n)$ is injective, whose image consists exactly of the cells represented by sequences $(a_0 \dots a_n)$ such that $\Delta^{\{a_0,\dots,a_n\}} \subseteq K$.*

Proof. If $K = \Delta^S$ for some $S \subseteq [n]$ this is immediate from the explicit description of cells of $\mathcal{R}(\Delta^n)$, in which case it is convenient to identify $\mathcal{R}(\Delta^S)$ with the corresponding subcomplex of $\mathcal{R}(\Delta^n)$. It is then clear that for $S, T \subseteq [n]$ we have $\mathcal{R}(\Delta^S) \cap \mathcal{R}(\Delta^T) = \mathcal{R}(\Delta^{S\cap T})$.

In general we can write $K \approx \operatorname{colim}_{\Delta^S \subseteq K} \Delta^S$, a colimit over a poset of some subsets $S \subseteq [n]$ (6.20). Since $\mathcal{R}$ is a left adjoint it preserves colimits, so $\mathcal{R}(K) \approx \operatorname{colim}_{\Delta^S \subseteq K} \mathcal{R}(\Delta^S)$. Then the claim follows from (6.14). □

57.7. **The proof.** The natural map $\epsilon \colon \mathcal{S}(C) \to C$ is adjoint to a natural map $\eta \colon X \to \mathcal{R}(X)$, which sends $x \in X_n$ to the element $[x, \iota_n] \in \mathcal{R}(X)_n$. When $X = \Delta^n$ this is just the tautological map $\eta_n \colon \Delta^n \to \mathcal{R}(\Delta^n) = \mathrm{Iso}^{[n]}$ described earlier.

For an arbitrary map $f \colon X \to Y$ of simplicial sets, we define maps

$$f_{\mathcal{R}} := (\mathcal{R}(f), \eta_Y) \colon \mathcal{R}(X) \cup_X Y \to \mathcal{R}(Y), \qquad g_{\mathcal{S}} := (\mathcal{S}(f), \epsilon_X) \colon \mathcal{S}(X) \to \mathcal{S}(Y) \times_Y X.$$

57.8. **Lemma.** *For any maps $i \colon K \to L$ and $p \colon C \to D$ of simplicial sets, we have that $i_{\mathcal{R}} \boxslash p$ if and only if $i \boxslash p_{\mathcal{S}}$.*

Proof. This is a straightforward verification of the equivalence of lifting problems.

$$\begin{tikzcd} \mathcal{R}(X) \cup_X Y \arrow[r, "{(u,v)}"] \arrow[d, "i_{\mathcal{R}}"'] & C \arrow[d, "p"] \\ \mathcal{R}(Y) \arrow[r, "w"'] \arrow[ur, dotted, "s"] & D \end{tikzcd} \qquad \Longleftrightarrow \qquad \begin{tikzcd} X \arrow[r, "\widetilde{u}"] \arrow[d, "i"'] & \mathcal{S}(C) \arrow[d, "p_{\mathcal{S}}"] \\ Y \arrow[r, "{(\widetilde{w},v)}"'] \arrow[ur, dotted, "\widetilde{s}"] & \mathcal{S}(D) \times_D C \end{tikzcd}$$

□

57.9. **Proposition.** *If $f \colon K \to L$ is a monomorphism, so is $f_{\mathcal{R}}$.*

Proof. Let $i_n \colon \partial\Delta^n \to \Delta^n$ be the cell inclusion. We already know that $\mathcal{R}(i_n)$ is a monomorphism by (57.6), which also explicitly describes the image of this map. Using this it is straightforward to show that $\eta_n \colon \Delta^n \to \mathcal{R}(\Delta^n)$ is also injective, and that the pullback of $\mathcal{R}(i_n)$ along η_n is precisely $\partial\Delta^n \subset \Delta^n$, from which it follows that $(i_n)_{\mathcal{R}}$ is a monomorphism.

Let $\mathcal{C}$ be the class of maps $f \colon K \to L$ such that $f_{\mathcal{R}} \in \overline{\mathrm{Cell}}$. By (57.8) and $\overline{\mathrm{Cell}} = {}^{\boxslash}\mathrm{TrivFib}$ it follows that

$$\mathcal{C} = {}^{\boxslash}(\mathrm{TrivFib}_{\mathcal{S}}) = \{\, f \mid f \boxslash p_{\mathcal{S}} \text{ for all } p \in \mathrm{TrivFib}\,\},$$

so $\mathcal{C}$ is weakly saturated. Since all $i_n \in \mathcal{C}$, we have that $\overline{\mathrm{Cell}} \subseteq \mathcal{C}$ and the claim follows. □

57.10. **Lemma.** *For each horn inclusion $j_{n,k} \colon \Lambda^n_k \subset \Delta^n$, $0 \leq k \leq n$, the induced map $(j_{n,k})_{\mathcal{R}} \colon \mathcal{R}(\Lambda^n_k) \cup_{\Lambda^n_k} \Delta^n \to \mathcal{R}(\Delta^n)$ is anodyne.*

Proof. This is an explicit calculation, which generalizes (38.7) which is the case of $(n, k) = (1, 0)$.

Let $T_m := \mathcal{R}(\Delta^n)^{\mathrm{nd}}_m \smallsetminus \mathcal{R} := \mathcal{R}(\Lambda^n_k)^{\mathrm{nd}}_m$ be the set of nondegenerate m-cells of $\mathcal{R}(\Delta^n)$ not contained in the subcomplex. In terms of representing sequences, $(a_0 \dots a_m) \in T_m$ if and only if it has no consecutive repetitions, and if $[n] \smallsetminus \{k\} \subseteq \{a_0, \dots, a_m\} \subseteq [n]$. Note in particular that $T_m = \varnothing$ if $m < n - 1$.

Partition this set as $T_m = T^1_m \amalg T^2_m$, where $(a_0 \dots a_m) \in T^1_m$ iff $a_0 = k$, and $(a_0 \dots a_m) \in T^2_m$ iff $a_0 \neq k$. Recall the notation $d^i := \langle 0 \dots \widehat{i} \dots m\rangle$ for the the simplicial face operator $[m-1] \to [m]$ (2.2). The verification of the following two statements is immediate.

1. d^0 restricts to a bijection $T^1_m \to T^2_{m-1}$.
2. For each $0 < i \leq m$, $d^i(T^1_m) \cap T^2_{m-1} = \varnothing$.

Given this, define $F_m \subseteq \mathcal{R}(\Delta^n)$ to be the smallest subcomplex containing $\mathcal{R}(\Lambda^n_k)$ and the sets T^1_i for $0 \leq i \leq m$ (and hence the sets T^2_i for $0 \leq i \leq m-1$). Then it is apparent that each inclusion $F_{m-1} \subseteq F_m$ is obtained by cobase change along a coproduct of horn inclusions $\Lambda^m_0 \subset \Delta^m$, with one copy for each element of T^1_m, and since $F_{n-1} = \mathcal{R}(\Lambda^n_k)$ and $\bigcup_m F_m = \mathcal{R}(\Delta^n)$ it follows that $\mathcal{R}(\Lambda^n_k) \to \mathcal{R}(\Delta^n)$ is anodyne.

To show that $\mathcal{R}(\Lambda^n_k) \cup_{\Lambda^n_k} \Delta^n \to \mathcal{R}(\Delta^n)$ is anodyne just note that the domain is contained[31] in F_n, and is the smallest subcomplex containing F_{n-1} and the n-cell $(01\dots n)$, which is an element of T^1_n. □

Let $\mathcal{C}$ denote the class of monomorphisms $i\colon K \to L$ of simplicial sets such that $i_\mathcal{R}$ is a weak homotopy equivalence.

57.11. **Proposition.** *The class $\mathcal{C}$ contains all monomorphisms.*

Proof. We are going to apply (48.13), so we must show that $\mathcal{C}$ contains inner horn inclusions, is weakly saturated, has precancellation with respect to monomorphisms, and contains $(\partial\Delta^0 \subset \Delta^0)$.

First note that since anodyne maps are weak homotopy equivalences, $\mathcal{C}$ contains all horn inclusions by (57.10).

Next we show that $\mathcal{C}$ is weakly saturated. Let i be a monomorphism of simplicial sets, and recall that so is $i_\mathcal{R}$ (57.9). Then $i_\mathcal{R}$ is a weak homotopy equivalence if and only if $i_\mathcal{R} \boxslash p$ for all Kan fibrations $p\colon C \to D$ between Kan complexes (52.4). Thus $i \in \mathcal{C}$ if and only if $i \boxslash p_\mathcal{S}$, so $\mathcal{C}$ is weakly saturated.

Next we show that $\mathcal{C}$ has precancellation, i.e., that $i, ji \in \mathcal{C}$ imply $j \in \mathcal{C}$. For monomorphisms $A \xrightarrow{i} B \xrightarrow{j} C$ we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{R}(A) \cup_A B \arrow[r, "i_\mathcal{R}"] \arrow[d] & \mathcal{R}(B) \arrow[d] & \\
\mathcal{R}(A) \cup_A C \arrow[r, "k"] \arrow[rr, bend right, "(ji)_\mathcal{R}"'] & \mathcal{R}(B) \cup_B C \arrow[r, "j_\mathcal{R}"] & \mathcal{R}(C)
\end{tikzcd}$$

in which the square is a pushout. If $i \in \mathcal{C}$, then $i_\mathcal{R} \in \overline{\mathrm{Cell}} \cap \mathrm{GpdEq}$, whence $k \in \overline{\mathrm{Cell}} \cap \mathrm{GpdEq}$ since this class is saturated. Since also $(ji)_\mathcal{R} \in \mathrm{GpdEq}$, we have $j_\mathcal{R}$ in GpdEq by 2-out-of-3.

Finally note that $j := (\partial\Delta^0 \subset \Delta^0) \in \mathcal{C}$, since in fact $j_\mathcal{R}$ is an isomorphism.

□

57.12. **Proposition.** *For every Kan fibration $p\colon C \to D$ between Kan complexes, we have that $p_\mathcal{S}\colon \mathcal{S}(C) \to \mathcal{S}(D) \times_D C$ is a trivial fibration. In particular, $\mathcal{S}(C) \to C$ is a trivial fibration for every quasigroupoid C.*

Proof. It suffices to show that $i \boxslash p_\mathcal{S}$ for every monomorphism i, or equivalently that $i_\mathcal{R} \boxslash p$. That this is so is because $i_\mathcal{R}$ is a weak homotopy equivalence by (57.11), so $i_\mathcal{R} \boxslash p$ by (52.4). □

Note that if $i\colon \varnothing \to X$, then $i_\mathcal{R} = \eta_X\colon X \to \mathcal{R}(X)$, and this map is a weak homotopy equivalence by (57.11). Thus for any simplicial set X, we get a diagram

$$X \xleftarrow{\eta_X} \mathcal{R}(X) \approx \mathcal{R}(X^{\mathrm{op}}) \xrightarrow{\eta_{X^{\mathrm{op}}}} X^{\mathrm{op}}$$

in which the maps are weak homotopy equivalences. Thus we learn that every simplicial set is weakly homotopy equivalent to its opposite.

Part 10. **Understanding join and slice**

58. The alternate slice

Given a quasicategory C and an object $x \in C_0$, we have constructed the slice quasicategories $C_{x/}$ **W 30 Mar**

[31]Owen Barrett points out this is not true when $k = 0$.

and $C_{/x}$, which come equipped with forgetful functors $C_{x/} \to C$ and $C_{/x} \to C$. When C is an ordinary category, the slices and can also be described as pullbacks. For instance, for an ordinary category C, the slice $C_{/x}$ is isomorphic to the fiber of the restriction functor $\mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\{1\}, C)$ over the vertex representing $x \in C_0$.

58.1. *Exercise.* Prove that if C is an ordinary category, this fiber is isomorphic to $C_{/x}$.

58.2. **Alternate slices over and under an object.** For a general simplicial set, we take this as the definition of the **alternate slice**. Thus, given a simplicial set C and $x \in C_0$, we define simplicial sets $C^{x/}$ and $C^{/x}$ together with maps $p\colon C^{x/} \to C$ and $q\colon C^{/x} \to C$ via the pullback squares **alternate slice**

$$\begin{tikzcd} C^{x/} \arrow[r] \arrow[d, "p"'] & \mathrm{Fun}(\Delta^1, C) \arrow[d] & C^{/x} \arrow[l] \arrow[d, "q"] \\ \{x\} \times C \arrow[r, "j_0"'] & \mathrm{Fun}(\partial\Delta^1, C) & \{x\} \times C \arrow[l, "j_1"] \end{tikzcd}$$

where the maps j_k, $k = 0, 1$, are induced by the inclusions $\{x\} \to \mathrm{Fun}(\{k\}, C) = C$.

These alternate slices are not generally isomorphic to the slice we have already defined, but note that there are evident bijections $(C_{x/})_0 \approx (C^{x/})_0$ and $(C_{/x})_0 \approx (C^{/x})_0$ on sets of vertices. We will eventually show that if C is a quasicategory, then there are categorical equivalences $C_{x/} \to C^{x/}$ and $C_{/x} \to C^{/x}$.

A key feature of alternate slices is that, unlike ordinary slices, it is straightforward to relate them to the mapping spaces of a quasicategory.

58.3. **Proposition.** *Given a quasicategory C and objects $x, y \in C_0$, there are pullback squares*

$$\begin{tikzcd} \mathrm{map}_C(x,y) \arrow[r] \arrow[d] & C^{x/} \arrow[d, "p"] \\ \{y\} \arrow[r] & C \end{tikzcd} \qquad \begin{tikzcd} \mathrm{map}_C(x,y) \arrow[r] \arrow[d] & C^{/y} \arrow[d, "q"] \\ \{x\} \arrow[r] & C \end{tikzcd}$$

That is, the fibers of the forgetful functors of alternate slices of C are mapping spaces of C.

Proof. Immediate from the definitions of mapping spaces and alternate slices. □

58.4. *Remark.* The fibers of the forgetful functors $C_{x/} \to C$ over y and $C_{/y} \to C$ over x for the usual slices are called the **right and left mapping spaces** respectively, and are denoted $\mathrm{map}^R_C(x, y)$ and $\mathrm{map}^L_C(x, y)$. For a quasicategory C they are categorically equivalent to the usual mapping space $\mathrm{map}_C(x, y)$, as a consequence of the equivalence slices and alternate slices which we will prove. **right and left mapping spaces**

Given a map $p\colon C \to D$ and $x \in C_0$ we have evident restriction maps $p'\colon C^{x/} \to C \times_D D^{px/}$ and $p''\colon C^{/x} \to C \times_D D^{/px}$. This "pullback-alternate-slice" map is closely related to the pullback-hom map.

58.5. **Lemma.** *For any map $p\colon C \to D$ and $x \in C_0$, there are pullback squares of the form*

$$\begin{tikzcd} C^{x/} \arrow[r] \arrow[d, "p'"'] & \mathrm{Fun}(\Delta^1, C) \arrow[d, "p^{\square(\partial\Delta^1 \subset \Delta^1)}"] & C^{/x} \arrow[l] \arrow[d, "p''"] \\ C \times_D D^{px/} \arrow[r] & \mathrm{Fun}(\partial\Delta^1, C) \times_{\mathrm{Fun}(\partial\Delta^1, D)} \mathrm{Fun}(\Delta^1, D) & C \times_D D^{/px} \arrow[l] \end{tikzcd}$$

Proof. This is a straightforward exercise. For instance, $C \times_D D^{px/}$ is seen to be the pullback of $\mathrm{Fun}(\partial\Delta^1, C) \times_{\mathrm{Fun}(\partial\Delta^1, D)} \mathrm{Fun}(\Delta^1, D) \to (C \times C) \times_{D \times D} (D \times D) \leftarrow (\{x\} \times C) \times_{\{px\} \times D} (\{px\} \times D)$.

Precomposing the left arrow with $p^{\square(\partial\Delta^1\subset\Delta^1)}$ and using the obvious isomorphisms gives the pullback square defining $C^{x/}$. $\square$

58.6. **Alternate slice for arbitrary maps.** We can generalize the alternate slice construction to arbitrary maps of simplicial sets. Thus, suppose given a map $f\colon K\to C$, which corresponds to a vertex $\widetilde{f}\in \mathrm{Fun}(K,C)_0$. We define the general **alternate slices** $C^{f/}$ and $C^{/f}$ via the pullback squares **alternate slices**

$$\begin{tikzcd}
C^{f/} \arrow[r] \arrow[d] & \mathrm{Fun}(K,C)^{\widetilde{f}/} \arrow[r] \arrow[d] & \mathrm{Fun}(K\times\Delta^1,C) \arrow[d] & \mathrm{Fun}(K,C)^{/\widetilde{f}} \arrow[l] \arrow[d] & C^{/f} \arrow[l] \arrow[d] \\
C \arrow[r, "\widetilde{\pi}_K"'] & \{\widetilde{f}\}\times\mathrm{Fun}(K,C) \arrow[r, "j_0"'] & \mathrm{Fun}(K\times\partial\Delta^1,C) & \mathrm{Fun}(K,C)\times\{\widetilde{f}\} \arrow[l, "j_1"] & C \arrow[l, "\widetilde{\pi}_K"]
\end{tikzcd}$$

where the maps j_k, $k=0,1$ are induced by the inclusions $\{\widetilde{f}\}\to\mathrm{Fun}(K\times\{k\},C)$, and $\widetilde{\pi}_K$ is adjoint to the projection $\pi\colon C\times K\to C$. Thus, the general alternate slices are obtained by base-change from the special case for functors from Δ^0.

The fibers of general alternate slices can also be described as mapping spaces, namely as spaces of natural transformations to or from a constant functor.

58.7. **Proposition.** *Given a quasicategory C, a map $f\colon K\to C$ of simplicial sets, and $x,y\in C_0$, there are pullback squares*

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{Fun}(K,C)}(f,\widetilde{\pi}_K y) \arrow[r] \arrow[d] & C^{f/} \arrow[d] \\
\{y\} \arrow[r] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{map}_{\mathrm{Fun}(K,C)}(\widetilde{\pi}_K x,f) \arrow[r] \arrow[d] & C^{/f} \arrow[d] \\
\{x\} \arrow[r] & C
\end{tikzcd}$$

where $\widetilde{\pi}_K x\colon K\to C$ represents the constant map with value x, i.e., the composite $K\to\{x\}\to C$.

Given a sequence of maps $K\xrightarrow{i}L\xrightarrow{f}C\xrightarrow{p}D$ of simplicial sets, we have evident restriction maps $C^{f/}\to C^{fi/}\times_{D^{pfi/}}D^{pf/}$ and $C^{/f}\to C^{/fi}\times_{D^{/pfi}}D^{/pf}$. These are alternate analogs of the pullback-slice maps of (32.14). Applied to a sequence of the form $\varnothing\to\{x\}\to C\xrightarrow{p}D$ this gives the special case already discussed. It turns out that the general case can be obtained via base-change from the special case.

58.8. **Proposition.** *There are pullback squares of the form*

$$\begin{tikzcd}
C^{f/} \arrow[r] \arrow[d] & U^{\widetilde{f}/} \arrow[d] \\
C^{fi/}\times_{D^{pfi/}}D^{pf/} \arrow[r] & U\times_V U^{(\widetilde{fi},\widetilde{pf})/}
\end{tikzcd}
\qquad
\begin{tikzcd}
C^{/f} \arrow[r] \arrow[d] & U^{/\widetilde{f}} \arrow[d] \\
C^{/fi}\times_{D^{/pfi}}D^{/pf} \arrow[r] & U\times_V V^{/(\widetilde{fi},\widetilde{pf})}
\end{tikzcd}$$

where $U\to V$ is the pullback-hom map $q=p^{\square i}\colon \mathrm{Fun}(L,C)\to\mathrm{Fun}(K,C)\times_{\mathrm{Fun}(K,D)}\mathrm{Fun}(L,D)$, and the vertical maps in each square are alternate pullback-slice maps associated to the sequences $K\xrightarrow{i}L\xrightarrow{f}C\xrightarrow{p}D$ and $\varnothing\to\{\widetilde{f}\}\to U\xrightarrow{q}V$ respectively.

Proof. This is a difficult to visualize but ultimately straightforward argument. A key observation is that the four alternate-slice objects associated to f, pf, fi, and pfi are each described by a pullback square $[1]^{\times 2}\to s\mathrm{Set}$, which therefore fit together to give a functor $[1]^{\times 4}\to s\mathrm{Set}$, which is a limit cone. Decomposing this 4-dimensional cartesian cube in a different way gives the desired pullback description. It may be helpful to note that the lower horizontal map in the left square is really a map of the form

$$C \times_{\left(C\times_D D\right)} \left(C^{fi/} \times_{D^{pfi/}} D^{pf/}\right) \to$$
$$\mathrm{Fun}(L,C) \times_{\left(\mathrm{Fun}(K,C)\times_{\mathrm{Fun}(K,D)}\mathrm{Fun}(L,D)\right)} \left(\mathrm{Fun}(K,C)^{\widetilde{fi}/} \times_{\mathrm{Fun}(K,D)^{\widetilde{pfi}/}} \mathrm{Fun}(L,D)^{\widetilde{pf}/}\right),$$

involving 14 of the 16 vertices of the 4-dimensional cube. □

59. The alternate join

Just as the usual slices are adjoint to a join construction, the alternate slices are adjoint to an alternate join construction.

59.1. **Definition of alternate join.** Given simplicial sets X and Y, we define the **alternate join** by the pushout diagram **alternate join**

$$\begin{tikzcd} (X\times\{0\}\times Y) \amalg (X\times\{1\}\times Y) \arrow[r] \arrow[d, tail] & (X\times\{0\}\times\Delta^0) \amalg (\Delta^0\times\{1\}\times Y) \arrow[d] \\ X\times\Delta^1\times Y \arrow[r, tail] & X\diamond Y \end{tikzcd}$$

where the map on the left is induced by inclusion $\partial\Delta^1 \subset \Delta^1$, and the map on the top by the projections $X \to \Delta^0$ and $Y \to \Delta^0$.

59.2. *Example.* We have isomorphsims $X \diamond \varnothing \approx \varnothing \approx \varnothing \diamond X$.

59.3. *Example.* We have isomorphisms

$$X \diamond \Delta^0 \approx (X\times\Delta^1)/(X\times\{1\}), \qquad \Delta^0 \diamond Y \approx (\Delta^1\times Y)/(\{0\}\times Y).$$

59.4. *Remark.* Unlike the join, the alternate join is not monoidal: $(X\diamond Y)\diamond Z \not\approx X\diamond(Y\diamond Z)$ in general. Also, the alternate join of two quasicategories is not usually a quasicategory.

Viewed as a functor of either variable, the alternate join gives functors

$$S\diamond -\colon s\mathrm{Set} \to s\mathrm{Set}_{S/}, \qquad -\diamond T\colon s\mathrm{Set} \to s\mathrm{Set}_{T/}$$

from simplicial sets to the evident slice categories.

59.5. **Proposition.** *The alternate join is left adjoint to the alternate slices, in the sense that* $S\diamond -\colon s\mathrm{Set} \to s\mathrm{Set}_{S/}$ *and* $-\diamond -T\colon s\mathrm{Set} \to s\mathrm{Set}_{T/}$ *are left adjoint to*

$$(f\colon S\to C) \mapsto C^{f/}\colon s\mathrm{Set}_{S/} \qquad \textit{and} \qquad (g\colon T\to C) \mapsto C^{/g}\colon s\mathrm{Set}_{T/} \to s\mathrm{Set}$$

respectively. Thus, we have natural bijections

$$\mathrm{Hom}(K, C^{f/}) \approx \mathrm{Hom}_{S/}(S\diamond K, C), \qquad \mathrm{Hom}(K, C^{/g}) \approx \mathrm{Hom}_{T/}(K\diamond T, C).$$

Proof. Straightforward. □

59.6. **Enriched adjunction for alternate join/slice.** In fact, we can do a little better: the alternate join and slices participate in **enriched adjunctions** involving the relative function complex (25.18), which for objects $p\colon S\to K$ and $f\colon S\to C$ in $s\mathrm{Set}_{S/}$ is a simplicial set **enriched adjunctions**

$$\mathrm{Fun}_{S/}(K,C) = \mathrm{Fun}(K,C) \times_{\mathrm{Fun}(S,C)} \{f\}.$$

First, note that for any simplicial sets S, K, and X, we have pushout squares of the form

$$\begin{tikzcd} X\times S \arrow[r] \arrow[d] & S \arrow[d] \\ X\times(S\diamond K) \arrow[r, "\ell"'] & S\diamond(X\times K) \end{tikzcd} \qquad \begin{tikzcd} X\times S \arrow[r] \arrow[d] & S \arrow[d] \\ X\times(K\diamond S) \arrow[r, "r"'] & (X\times K)\diamond S \end{tikzcd}$$

where the top horizontal maps are projections, the vertical maps are the evident inclusions, and the bottom horizontal maps are the evident ones induced by the fact that the objects are quotients of the same product $X\times S\times K\times\Delta^1$ (up to permuting the factors). Using this, for any $f\colon K\to C$ we obtain isomorphisms

$$\operatorname{Fun}(K, C^{f/}) \xrightarrow{\sim} \operatorname{Fun}_{S/}(S\diamond K, C), \qquad \operatorname{Fun}(K, C^{/f}) \xrightarrow{\sim} \operatorname{Fun}_{S/}(K\diamond S, C).$$

In the left-hand case, we define the map by describing a natural transformation

$$\left\{\; X \dashrightarrow \operatorname{Fun}(K, C^{f/}) \;\right\} \Longrightarrow \left\{\; X \dashrightarrow \operatorname{Fun}_{S/}(S\diamond K, C) \;\right\}$$

sending

$$\left\{ \begin{tikzcd} S \arrow[d] \arrow[dr, "f"] & \\ S\diamond(X\times K) \arrow[r, dashed] & C \end{tikzcd} \right\} \Longrightarrow \left\{ \begin{tikzcd} X\times S \arrow[r, "\text{proj}"] \arrow[d] & S \arrow[d, "f"] \\ X\times(S\diamond K) \arrow[r, dashed] & C \end{tikzcd} \right\}$$

by precomposing with ℓ defined above. The transformation is an isomorphism exactly because the square defining ℓ is a pushout square.

We note that this provides us with another description of the alternate slice.

59.7. **Proposition.** *For any map $f\colon S\to C$ of simplicial sets, we have isomorphisms*

$$C^{f/} \approx \operatorname{Fun}_{S/}(S\diamond\Delta^0, C), \qquad C^{/f} \approx \operatorname{Fun}_{S/}(\Delta^0\diamond S, C).$$

Proof. These are the enriched adjunction maps for alternate join/slice for $K=\Delta^0$. □

59.8. **Enriched adjunction for join/slice.** In an analogous way we can define enriched adjunction maps for the usual join and slices, using commutative squares

$$\begin{tikzcd} X\times S \arrow[r] \arrow[d] & S \arrow[d] \\ X\times(S\star K) \arrow[r, "\ell'"'] & S\star(X\times K) \end{tikzcd} \qquad \begin{tikzcd} X\times S \arrow[r] \arrow[d] & S \arrow[d] \\ X\times(K\star S) \arrow[r, "r'"'] & (X\times K)\star S \end{tikzcd}$$

which however are not pushouts in general (though they are trivially pushouts when $X=\Delta^0$). For instance the map ℓ' is the one correspoing by (28.16) to the triple of maps

$$\pi\colon X\times(S\star K)\to X\times\Delta^1\to\Delta^1, \qquad \pi^{-1}(\{0\}) = X\times S \xrightarrow{\text{proj}} S, \qquad \pi^{-1}(\{1\}) = X\times K \xrightarrow{\text{id}} X\times K.$$

Following the same recipe as for alternate join and slices, given $f\colon K\to C$ we have natural maps

$$\operatorname{Fun}(K, C_{f/}) \to \operatorname{Fun}_{S/}(S\star K, C), \qquad \operatorname{Fun}(K, C_{/f}) \to \operatorname{Fun}_{S/}(K\star S, C),$$

which we call **enriched adjunction maps** for join and slices. These are not in general isomorphisms, but they are bijections on vertices. Although they are not isomorphisms, we will later show that these enriched adjunction maps are categorical equivalences when C is a quasicategory. For now, we show that when C is a quasicategory they induce bijections on isomorphism classes of objects.

enriched adjunction maps

M 4 Apr

59.9. **Proposition.** *For K a simplicial set and $p\colon S\to C$ a map to a quasicategory C, the enriched adjunction map for join/slice induces bijections*

$$\pi_0\big(\operatorname{Fun}(K, C_{p/})^{\text{core}}\big) \xrightarrow{\sim} \pi_0\big(\operatorname{Fun}_{S/}(S\star K, C)^{\text{core}}\big), \quad \pi_0\big(\operatorname{Fun}(X, C_{/p})^{\text{core}}\big) \xrightarrow{\sim} \pi_0\big(\operatorname{Fun}_{S/}(K\star S, C)^{\text{core}}\big),$$

Proof. We give the proof in the slice-under case. Since the enriched adjunction map gives a bijection on objects, it suffices to prove injectivity on sets of isomorphism classes. Thus, we suppose given $f_0, f_1 \colon X \to C_{p/}$ representing objects of $\mathrm{Fun}(X, C_{p/})$ such that the corresponding objects $\widetilde{f}_0, \widetilde{f}_1 \colon S \star X \to C$ represent isomorphic objects of $\mathrm{Fun}_{S/}(S \star X, C)$, and show that there is a natural isomorphism $f_0 \to f_1$.

Let $C \xrightarrow{i} \widehat{C} \xrightarrow{r=(r_0,r_1)} C \times C$ be the standard path category for C (36.4), with $\widehat{C} = \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, C)$. An isomorphism $\widetilde{f}_0 \to \widetilde{f}_1$ in $\mathrm{Fun}_{S/}(S \star K, C)$ amounts to a choice of lift $\widetilde{f}$ in

$$\begin{tikzcd}
S \arrow[r, "p"] \arrow[d, tail] & C \arrow[r, "i"] & \widehat{C} \arrow[d, "r=(r_0,r_1)"] \\
S \star X \arrow[rr, "(\widetilde{f}_0,\widetilde{f}_1)"'] \arrow[urr, "\widetilde{f}"] & & C \times C
\end{tikzcd}$$

Applying the join/slice adjunction to the lower triangle gives a diagram

$$\begin{tikzcd}
C_{/p} \arrow[r, "i'"] & \widehat{C}_{pi/} \arrow[d, "r'=(r'_0,r'_1)"] \\
X \arrow[r, "(f_0,f_1)"'] \arrow[ur, "f"] & C_{p/} \times C_{/p}
\end{tikzcd}$$

Since $r_0, r_1 \colon \widehat{C} \to C$ are trivial fibrations (42.3), so are the induced maps $r'_0, r'_1 \colon \widehat{C}_{pi/} \to C_{/p}$ on slices (32.16). Thus every functor in

$$\begin{tikzcd}
 & & \mathrm{Fun}(X, C_{p/}) \\
\mathrm{Fun}(X, C_{p/}) \arrow[r, "i'_*"] \arrow[urr, bend left, "\mathrm{id}"] \arrow[drr, bend right, "\mathrm{id}"'] & \mathrm{Fun}(X, \widehat{C}_{pi/}) \arrow[ur, "(r'_0)_*"'] \arrow[dr, "(r'_1)_*"] & \\
 & & \mathrm{Fun}(X, C_{p/})
\end{tikzcd}$$

is a categorical equivalence, whence r'_0 and r'_1 induces the same bijection on isomorphism classes of objects. we see that every arrow in this diagram is a categorical equivalence, and therefore both π_0 and π_1 induce *the same bijection* on isomorphism classes on objects, and thus $f_0 = (r'_0)_*(f)$ and $f_1 = (r'_1)_*(f)$ are isomorphic as desired.

□

59.10. *Exercise.* Construct a natural "distributivity" map $X \times (S \star T) \to (X \times S) \star (X \times T)$.

60. Equivalence of join and alternate join

The proof that slice and alternate slice are equivalent will rely on an equivalence between join and alternate join.

60.1. **Comparison map from alternate join to join.** There is a canonical comparison map $X \diamond Y \to X \star Y$, natural in both variables, which by (28.16) corresponds to the triple of maps

$$\pi \colon X \diamond Y \to \Delta^0 \diamond \Delta^0 \approx \Delta^1, \qquad \pi^{-1}(\{0\}) = X \xrightarrow{\mathrm{id}} X, \qquad \pi^{-1}(\{1\}) = Y \xrightarrow{\mathrm{id}} Y.$$

60.2. **Proposition.** *The canonical comparison map $X \diamond Y \to X \star Y$ is a categorical equivalence for all simplicial sets X and Y.*

We will give the proof at the end of this section.

60.3. **Categorical invariance of joins.** First we note that the alternate join is a "categorically invariant" construction.

60.4. **Proposition.** *The alternate join $\diamond$ preserves categorical equivalences in either variable. That is, if $Y \to Y'$ is a categorical equivalence, then so are $X \diamond Y \to X \diamond Y'$ and $Y \diamond Z \to Y' \diamond Z$.*

Proof. The $\diamond$ functor is constructed (59.1) using finite products and a "good" pushout, i.e., a pushout along a cofibration (=monomorphism). The result follows because both finite products (27.5) and good pushouts (50.12) preserve categorical equivalences. □

Once we prove equivalence of join and alternate join, this will imply the categorical invariance of the usual join.

60.5. **Corollary.** *The join $\star$ preserves categorical equivalences in either variable. That is, if $Y \to Y'$ is a categorical equivalence, then so are $X \star Y \to X \star Y'$ and $Y \star Z \to Y' \star Z$.*

Proof. Immediate using (60.2), the invariance of the alternate join under categorical equivalence (60.4), and the 2-out-of-3 property of categorical equivalences (27.11). □

60.6. **Skeletal induction.** To prove (60.2) we will use the following strategy.

60.7. **Proposition** (Skeletal induction)**.** *Let $\mathcal{C}$ be a class of simplicial sets with the following properties.*

- *Every $\Delta^n \in \mathcal{C}$.*
- *The class $\mathcal{C}$ is closed under* good colimits. *That is:*
  (a) *any coproduct of objects of $\mathcal{C}$ is in $\mathcal{C}$;*
  (b) *any pushout of a diagram $X_0 \leftarrow X_1 \to X_2$ of objects in $\mathcal{C}$ along a monomorphism $X_1 \to X_2$ is in $\mathcal{C}$;*
  (c) *any colimit of a countable sequence $X_0 \to X_1 \to X_2 \to \cdots$ of objects in $\mathcal{C}$, such that each $X_k \to X_{k+1}$ is a monomorphism, is in $\mathcal{C}$.*

Then $\mathcal{C}$ is the class of all simplicial sets.

Proof. This is a straightforward consequence of the skeletal filtration (20.3). To show $X \in \mathcal{C}$, it suffices to show each $\operatorname{Sk}_n X \in \mathcal{C}$ by (c). So we show that all n-skeleta are in $\mathcal{C}$ by induction on n, with base case $n = -1$ (the empty simplicial set), which is really a special case of (a). Since $\operatorname{Sk}_{n-1} X \subseteq \operatorname{Sk}_n X$ is a pushout along a coproduct of maps $\partial\Delta^n = \operatorname{Sk}_{n-1} \Delta^n \to \Delta^n$, this follows using (a), (b), the fact that all $\Delta^n \in \mathcal{C}$, and the inductive hypothesis, which tells us that $\partial\Delta^n \in \mathcal{C}$. □

We will use skeletal induction to show that certain natural transformations from simplicial sets to a model category take values in weak equivalences.

60.8. **Proposition.** *Let $\alpha\colon F \to F'$ be a natural transformation between functors $s\mathrm{Set} \to \mathcal{M}$, where $\mathcal{M}$ is some model category. If*

(1) *F and F' preserve colimits,*
(2) *F and F' take monomorphisms to cofibrations,*
(3) *F and F' take inner anodyne maps to to weak equivalences in $\mathcal{M}$, and*
(4) *$\alpha(\Delta^1)\colon F(\Delta^1) \to F'(\Delta^1)$ is a weak equivalence in $\mathcal{M}$,*

then $\alpha(X)\colon F(X) \to F'(X)$ is a weak equivalence in $\mathcal{M}$ for all simplicial sets X.

Proof. [Lur09, 4.2.1.2] Consider the class of simplicial sets $\mathcal{C} := \{\, X \mid \alpha(X) \text{ is a weak equivalence} \,\}$. We use skeletal induction (60.7) to show that $\mathcal{C}$ contains all simplicial sets.

Because F and F' preserve colimits (1) and cofibrations (2), they take good colimit diagrams in $s\mathrm{Set}$ to good colimit diagrams in $\mathcal{M}$. Since good colimits are weak equivalence invariant (50.8), (50.12), (50.10), we see that $\mathcal{C}$ is closed under forming good colimits. It remains to show that $\Delta^n \in \mathcal{C}$ for all n.

We have $\Delta^1 \in \mathcal{C}$ by (4). Since Δ^0 is a retract of Δ^1, we get that $\Delta^0 \in \mathcal{C}$ since weak equivalences in $\mathcal{M}$ are closed under retracts (49.4).

The spines I^n can be built from Δ^0 and Δ^1 by a sequence of good pushouts (glue on one 1-simplex at a time), so the $I^n \in \mathcal{C}$. The inclusions $I^n \subset \Delta^n$ are inner anodyne (16.14), so by (3) and the 2-out-of-3 property of weak equivalences in $\mathcal{M}$ it follows that $\Delta^n \in \mathcal{C}$. $\square$

60.9. **Proof of the equivalence.** We will apply this idea to functors $s\mathrm{Set} \to s\mathrm{Set}_{X/}$, where the slice category $s\mathrm{Set}_{X/}$ inherits its model structure from the Joyal model structure on $s\mathrm{Set}$ (49.5).

Proof of (60.2). The functors $X \diamond (-), X \star (-), (-) \diamond X, (-) \star X \colon s\mathrm{Set} \to s\mathrm{Set}_{X/}$ satisfy the first three properties required of the functors in the previous proposition (60.8). That is, they (1) preserve colimits, (2) take monomorphisms to monomorphisms, and hence to cofibrations, and (3) take inner anodyne maps to categorical equivalences. Condition (3) for $\diamond$ follows from (60.4), while condition (3) for $\star$ follows from $\overline{\mathrm{Cell}} \circledast \overline{\mathrm{LHorn}} \subseteq \overline{\mathrm{InnHorn}}$ and $\overline{\mathrm{RHorn}} \circledast \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (32.13) and the fact that $\overline{\mathrm{InnHorn}} \subseteq \overline{\mathrm{LHorn}} \cap \overline{\mathrm{RHorn}}$.

Thus, to show $X \diamond Y \to X \star Y$ is a categorical equivalence for a fixed X and arbitrary Y, it suffices by the previous proposition to show (4), i.e., that $X \diamond \Delta^1 \to X \star \Delta^1$ is a categorical equivalence. The same argument in the other variable lets us reduce to the case when $X = \Delta^1$, i.e., to showing that a single map $\overline{f} \colon \Delta^1 \diamond \Delta^1 \to \Delta^1 \star \Delta^1$ is a categorical equivalence, which is the following lemma (60.10). $\square$

60.10. **Lemma.** *The canonical comparison map $\overline{f} \colon \Delta^1 \diamond \Delta^1 \to \Delta^1 \star \Delta^1$ is a categorical equivalence.* **W 6 Apr**

Proof. We will show $\overline{f}$ is a categorical equivalence by producing a map $\overline{g} \colon \Delta^1 \star \Delta^1 \to \Delta^1 \diamond \Delta^1$ such that $\overline{f}\overline{g} = \mathrm{id}_{\Delta^1 \star \Delta^1}$ and $\overline{g}\overline{f}$ is preisomorphic to the identity map of $\Delta^1 \diamond \Delta^1$, via (25.9).

Recall that $\Delta^1 \diamond \Delta^1$ is a quotient of a cube, so that it is isomorphic to the pushout of

$$\Delta^1 \times \{0\} \times * \ \amalg \ * \times \{1\} \times \Delta^1 * \leftarrow \Delta^1 \times \{0\} \times \Delta^1 \ \amalg \ \Delta^1 \times \{1\} \times \Delta^1 \rightarrowtail \Delta^1 \times \Delta^1 \times \Delta^1.$$

We write vertices in $(\Delta^1)^{\times 3}$ as sequences (xty) where $x, t, y \in \{0, 1\}$. Note that $\Delta^1 \diamond \Delta^1$ has exactly four vertices, corresponding to equivalence classes $\{(000), (001)\}, \{(100), (101)\}, \{(010), (110)\}, \{(011), (111)\}$ of vertices in the cube, and that the only edges in $\Delta^1 \diamond \Delta^1$ which connect a vertex to itself are degenerate.

Let

$$f \colon (\Delta^1)^{\times 3} \to \Delta^1 \star \Delta^1 = \Delta^3$$

be the map which on vertices sends

$$(xty) \mapsto (1-t)x + t(2+y) = \begin{cases} x & \text{if } t = 0, \\ 2 + y & \text{if } t = 1. \end{cases}$$

$$\begin{tikzcd}
& & 010 \arrow[dll] \arrow[drr] & & & & & & 2 \arrow[dll] \arrow[drr] & & \\
110 \arrow[drr] & & 000 \arrow[u] \arrow[dll] \arrow[drr] & & 011 \arrow[dll] & & 2 \arrow[drr] & & 0 \arrow[u] \arrow[dll] \arrow[drr] & & 3 \arrow[dll] \\
100 \arrow[u] \arrow[drr] & & 111 & & 001 \arrow[u] \arrow[dll] & \xmapsto{f} & 1 \arrow[u] \arrow[drr] & & 3 & & 0 \arrow[u] \arrow[dll] \\
& & 101 \arrow[u] & & & & & & 1 \arrow[u] & & \\
& & (\Delta^1)^{\times 3} & & & & & & \Delta^3 & &
\end{tikzcd}$$

On passage to quotients this gives the comparison map $\overline{f} \colon \Delta^1 \diamond \Delta^1 \to \Delta^1 \star \Delta^1$ we want. Note that $\overline{f}$ is a bijection on vertices.

Let $g \colon \Delta^3 \to (\Delta^1)^{\times 3}$ be the map classifying the cell $\langle (000), (100), (110), (111) \rangle$, and let $\overline{g} \colon \Delta^3 \to \Delta^1 \diamond \Delta^1$ be the composite with the quotient map. We have $fg = \mathrm{id}_{\Delta^3} = \overline{f}\overline{g}$.

Let $h \in \mathrm{Fun}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})_0$ and $a, b \in \mathrm{Fun}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})_1$ be the unique elements whose action on vertices is as indicated in the following picture.

$$\begin{tikzcd}[column sep=small]
& 010 \arrow[dl] \arrow[dr] & & & & 010 \arrow[dl] \arrow[dr] & & & & 110 \arrow[dl] \arrow[dr] & \\
110 \arrow[dr] & 000 \arrow[u] \arrow[dl] \arrow[dr] & 011 \arrow[dl] & & 110 \arrow[dr] & 000 \arrow[u] \arrow[dl] \arrow[dr] & 011 \arrow[dl] & & 110 \arrow[dr] & 000 \arrow[u] \arrow[dl] \arrow[dr] & 111 \arrow[dl] \\
100 \arrow[u] \arrow[dr] & 111 & 001 \arrow[u] \arrow[dl] & \xLeftarrow{a} & 100 \arrow[u] \arrow[dr] & 111 & 000 \arrow[u] \arrow[dl] & \xRightarrow{b} & 100 \arrow[u] \arrow[dr] & 111 & 000 \arrow[u] \arrow[dl] \\
& 101 \arrow[u] & & & & 100 \arrow[u] & & & & 100 \arrow[u] & \\
& \mathrm{id} & & & & h & & & & gf &
\end{tikzcd}$$

These pass to 1-cells $\overline{h}$, $\overline{a}$, $\overline{b}$ in $\mathrm{Fun}(\Delta^1 \diamond \Delta^1, \Delta^1 \diamond \Delta^1)$. The edges $\overline{a}$ and $\overline{b}$ are preisomorphisms, as one sees that for each vertex $v \in (\Delta^1 \diamond \Delta^1)$, the induced maps $\Delta^1 \times \{v\} \subset \Delta^1 \times (\Delta^1 \diamond \Delta^1) \xrightarrow{\overline{a} \text{ or } \overline{b}} \Delta^1 \diamond \Delta^1$ represent degenerate edges of $\Delta^1 \diamond \Delta^1$. Thus $\overline{f}\overline{g}$ and $\overline{g}\overline{f}$ are preisomorphic to identity maps, and hence $\overline{f}$ is a categorical equivalence as desired. □

61. Equivalence of slice and alternate slice

61.1. **Comparison map from slice to alternate slice.** The natural comparison map $X \diamond Y \to X \star Y$ from alternate join and join induces adjoint comparison maps from slice to alternate slice. Thus, given $f : S \to C$, we have maps

$$C_{f/} \to C^{f/} \qquad \text{and} \qquad C_{/f} \to C^{/f}$$

which can be described using the adjuntion as follows: given $g : K \to C_{f/}$, the composite $h : K \to C^{f/}$ with the comparison map is adjoint to $\widetilde{g}\alpha$ in

$$\begin{tikzcd}
S \arrow[r, equal] \arrow[d, tail] & S \arrow[r, "f"] \arrow[d, tail] & C \\
S \diamond K \arrow[r, "\alpha"'] & S \star K \arrow[ur, "\widetilde{g}"'] &
\end{tikzcd}$$

and similarly for slice-over.

To compare slices with alternate slices, we will use the following commutative diagram of function complexes.

61.2. **Proposition.** *For all simplicial sets K and all maps $f : S \to C$ of simplicial sets, we have commutative squares*

$$\begin{tikzcd}
\mathrm{Fun}(K, C_{f/}) \arrow[r] \arrow[d] & \mathrm{Fun}_{S/}(S \star K, C) \arrow[d] & \mathrm{Fun}(K, C_{/f}) \arrow[r] \arrow[d] & \mathrm{Fun}_{S/}(K \star S, C) \arrow[d] \\
\mathrm{Fun}(K, C^{f/}) \arrow[r, "\approx"'] & \mathrm{Fun}_{S/}(S \diamond K, C) & \mathrm{Fun}(K, C^{/f}) \arrow[r, "\approx"'] & \mathrm{Fun}_{S/}(K \diamond S, C)
\end{tikzcd}$$

in which the horizontal maps are the respective enriched adjunction maps, the left vertical map in each square is induced by the slice comparison maps $C_{f/} \to C^{f/}$ or $C_{/f} \to C^{/f}$, and the right vertical map in each square is induced by the join comparison maps $S \diamond K \to S \star K$ or $K \diamond S \to K \star S$.

Proof. This is straightforward from the definitions of the enriched adjunction maps and the comparison maps for joins and slices. Explicitly, consider a map $\gamma : X \to \mathrm{Fun}(K, C_{f/})$ from some test object X, and follow it through the diagram. We are led to consider a diagram

$$\begin{tikzcd}
X \times S \arrow[r, "\mathrm{proj}"] \arrow[d, tail] & S \arrow[d, tail] \arrow[ddr, bend left, "f"] & \\
X \times (S \diamond K) \arrow[r] \arrow[d, "\mathrm{id}_X \times \alpha"'] & S \diamond (X \times K) \arrow[d, "\alpha \times \mathrm{id}_{X \star K}"] & \\
X \times (S \star K) \arrow[r] & S \times (X \star K) \arrow[r, "\widetilde{\gamma}"'] & C
\end{tikzcd}$$

where maps labelled α are comparison maps from alternate join to join (60.1), the top square is the pushout used to define enriched adjunction for alternate join/slice (59.6), and the rectangle is the pushout used to define enriched adjunction for join/slice (59.8). The result follows exactly because the lower square commutes. $\square$

61.3. **Equivalence of slice and alternate slice.**

61.4. **Proposition.** *For any quasicategory C and map $f\colon S \to C$, the comparison maps $C_{f/} \to C^{f/}$ and $C_{/f} \to C^{/f}$ are categorical equivalences.*

Proof. [Lur09, 4.2.1.5] We do the first case. First recall that if $f\colon A \to B$ is a functor between quasicategories, then f is a categorical equivalence if and only if the induced maps $\pi_0\big(\mathrm{Fun}(X, A)^{\mathrm{core}}\big) \to \pi_0\big(\mathrm{Fun}(X, B)^{\mathrm{core}}\big)$ are bijections for all simplicial sets X (27.14)(4).

We refer to the left-hand commutative square of (61.2). We know that the bottom horizontal map is an isomorphism (59.6). By (59.9) the top map is a bijection on isomorphism classes of objects. By (60.2) $\alpha S \diamond K \to S \star K$ is a categorical equivalence, and therefore the right-hand vertical map in the commutative square is a categorical equivalence using (50.14).

Therefore, both of the horizontal maps and the right-hand vertical map induce bijections on isomorphism classes of objects, and hence so does the left-hand vertical map. Since this holds for every simplicial set K, it follows that $C_{f/} \to C^{f/}$ is a categorical equivalence by the criterion of (27.14)(4). The proposition is proved. $\square$

61.5. **Corollary.** *For any quasicategory C map $f\colon S \to C$, and simplicial set X, the enriched adjunction maps $\mathrm{Fun}(X, C_{f/}) \to \mathrm{Fun}_{S/}(S \star X, C)$ and $\mathrm{Fun}(X, C_{/f}) \to \mathrm{Fun}_{S/}(X \star S, C)$ for join/slice are categorical equivalences.*

Proof. Immediate from the proof of (61.4) $\square$

61.6. **Slices as fibers of cone-restriction.** As a consequence, we obtain another variant of the slice construction. Let C be a quasicategory and S a simplicial set, and consider the forgetful functors

$$p\colon \mathrm{Fun}(S^{\rhd}, C) \to \mathrm{Fun}(S, C), \qquad q\colon \mathrm{Fun}(S^{\lhd}, C) \to \mathrm{Fun}(S, C).$$

For a given map $f\colon S \to C$, the fibers of p and q over the vertex of $\mathrm{Fun}(S, C)$ corresponding to f are precisely the relative function complexes $\mathrm{Fun}_{S/}(S^{\rhd}, (C, f))$ and $\mathrm{Fun}_{S/}(S^{\lhd}, (C, f))$. These fibers are in fact equivalent to the evident slice categories.

61.7. **Corollary.** *For any map $f\colon S \to C$ to a quasicategory, we have commutative squares*

$$\begin{tikzcd}
C_{f/} \arrow[r] \arrow[d] & \mathrm{Fun}_{S/}(S^{\rhd}, (C, f)) \arrow[d] \\
C^{f/} \arrow[r, "\approx"'] & \mathrm{Fun}_{S/}(S \diamond \Delta^0, (C, f))
\end{tikzcd} \qquad
\begin{tikzcd}
C_{/f} \arrow[r] \arrow[d] & \mathrm{Fun}_{S/}(S^{\lhd}, (C, f)) \arrow[d] \\
C^{/f} \arrow[r, "\approx"'] & \mathrm{Fun}_{S/}(\Delta^0 \diamond S, (C, f))
\end{tikzcd}$$

in which the lower horizontal maps are isomorphisms, the top horizontal maps are bijections on sets of objects, and every map is a categorical equivalence.

Proof. These are just the commutative squares of (61.2) with $K = \Delta^0$, together with (61.4) and (61.5). □

62. Properties of the alternate slice

Recall that given a sequence of maps $K \xrightarrow{i} L \xrightarrow{f} C \xrightarrow{p} D$ where i is a monomorphism and p an inner fibration, the induced pullback-slice maps $C_{f/} \to C_{fi/} \times_{D_{pfi/}} D_{pf/}$ and $C_{/f} \to C_{/fi} \times_{D_{/pfi}} D_{/pf}$ are left fibration and right fibration respectively (32.14), and therefore in particular are conservative isofibrations (34.7) (34.14). Furthermore, they are trivial fibrations if either p is a trivial fibration or if i is right or left anodyne respectively (32.14). We will show that the alternate pullback-slice maps share these properties, at least when C and D are quasicategories.

62.1. **Proposition.** *Let $p\colon C \to D$ be an inner fibration between quasicategories. Then for any object $x \in C_0$, the evident induced map $p'\colon C^{x/} \to C \times_D D^{px/}$ is a left fibration, and $p''\colon C^{/x} \to C \times_D D^{/px}$ is a right fibration.*

Proof. We deal with the case of p', as the case of p'' is similar. Since p' is a base-change of $p^{\square(\partial\Delta^1 \subset \Delta^1)}$, it is an inner fibration. To produce a lift in the left-hand square of

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r] \arrow[d, tail] & C^{x/} \arrow[r] \arrow[d, "p'"] & \mathrm{Fun}(\Delta^1, C) \arrow[d, "p^{\square(\partial\Delta^1\subset\Delta^1)}"] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & C \times_D D^{px/} \arrow[r] & \mathrm{Fun}(\partial\Delta^1, C) \times_{\mathrm{Fun}(\partial\Delta^1, D)} \mathrm{Fun}(\Delta^1, D)
\end{tikzcd}$$

for $n \geq 1$, it suffices to produce a lift in the large rectangle. This lifting problem is equivalent to one of the form

$$\begin{tikzcd}
\{0\} \times \Delta^{\{0,1\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & (\partial\Delta^1 \times \Delta^n) \times_{\partial\Delta^1 \times \Lambda^n_0} (\Delta^1 \times \Lambda^n_0) \arrow[r, "t"] \arrow[d, tail] & C \arrow[d, "p"] \\
& \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dotted] & D
\end{tikzcd}$$

in which the map t sends $\{0\} \times \Delta^n$ to $\{x\} \subseteq C$, so that in particular f represents 1_x in C. The claim follows from the following (62.2). □

62.2. **Lemma** (Another pushout-product version of Joyal lifting)**.** *Let $p\colon C \to D$ be an inner fibration between quasicategories. Then if $n \geq 0$, and for either: (i) $(x, y) = (0, 0)$, $\{a, b\} = \{0, 1\}$, or (ii) $(x, y) = (1, n)$, $\{a, b\} = \{n - 1, n\}$, a lift exists in any diagram of the form*

$$\begin{tikzcd}
\{x\} \times \Delta^{\{a,b\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & (\partial\Delta^1 \times \Delta^n) \times_{\partial\Delta^1 \times \Lambda^n_y} (\Delta^1 \times \Lambda^n_y) \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\
& \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dotted] & D
\end{tikzcd}$$

such that f represents an isomorphism in C.

Proof. We give a proof in the appendix as another application of Joyal lifting (78.8). □

As a special case (i.e., if $D = \Delta^0$), we learn that for a quasicategory C we get a left fibration $C^{x/} \to C$ and a right fibration $C^{/x} \to C$, which are therefore both conservative isofibrations.

In fact, we have the same property for the general alternate pullback slice map.

62.3. **Corollary.** *Consider a sequence $K \xrightarrow{i} L \xrightarrow{f} C \xrightarrow{p} D$ where i is a monomorphism and p is an inner fibration between quasicategories. Then the alternate pullback slice map $p' \colon C^{f/} \to C^{fi/} \times_{D^{pfi/}} D^{pf/}$ is a left fibration and the alternate pullback slice map $p'' \colon C^{/f} \to C^{/fi} \times_{D^{/pfi}} D^{/pf}$ is a right fibration. In particular, both p' and p'' are conservative isofibrations.*

Proof. We consider the slice-under case. Consider the pullback-hom map $q := p^{\square i} \colon U \to V$. This is an inner fibration by (22.2). Thus for any object $u \in U_0$ the induced map $q' \colon U^{u/} \to U \times_V V^{qu/}$ is a left fibration (62.1). The claim follows because p' is a basechange of q' (58.8). ☐

We note the following special case.

62.4. **Corollary.** *For a sequence $K \xrightarrow{i} L \xrightarrow{f} C$ where i is a monomorphism and C is a quasicategory, the restriction functor $C^{f/} \to C^{fi/}$ is a left fibration and $C^{/f} \to C^{/fi}$ is a right fibrations, hence both are conservative isofibrations.*

Proof. Immediate from (62.3) (where $D = \Delta^0$), and (34.7) and (34.14). ☐

This means that for any sequence $K \xrightarrow{i} L \xrightarrow{f} C \xrightarrow{p} D$ as in (62.3), the targets of the alternate pullback slice maps p' and p'' are not merely pullbacks, but good pullbacks in the Joyal model strucutre. Thus the equivalence between slice and alternate slice extends to the target of the pullback-slice maps.

62.5. **Proposition.** *Consider a sequence $K \xrightarrow{i} L \xrightarrow{f} C \xrightarrow{p} D$ where i is a monomorphism and p is an inner fibration between quasicategories. Then the horizontal maps in the commutative squares*

$$\begin{tikzcd}
C_{f/} \arrow[r] \arrow[d] & C^{f/} \arrow[d] \\
C_{fi/} \times_{D_{pfi/}} D_{pf/} \arrow[r] & C^{fi/} \times_{D^{pfi/}} D^{pf/}
\end{tikzcd}
\qquad
\begin{tikzcd}
C_{/f} \arrow[r] \arrow[d] & C^{/f} \arrow[d] \\
C_{/fi} \times_{D_{/pfi}} D_{/pf} \arrow[r] & C^{/fi} \times_{D^{/pfi}} D^{/pf}
\end{tikzcd}$$

induced by the comparison between slice and alternate slice are all categorical equivalences. In particular, under these hypotheses the slice-pullback map is a categorical equivalence (and hence a trivial fibration) if and only if the corresponding alternate slice-pulllback map is a categorical equivalence (and hence a trivial fibration).

Proof. The first statement is immediate from the categorical equivalence of slice and alternate slice (61.4) once we see that the pullbacks along the bottom row are all good pullbacks with respect to the Joyal model structure, and hence homotopy pullbacks, so that we may apply (51.9). The final statement is a consequence of 2-out-of-3 (27.11), and the fact that the pullback-slice and alternate pullback-slice maps are isofibrations and so are categorical equivalence if and only if they are trivial fibrations (40.8). ☐

62.6. **Corollary.** *Consider a sequence $K \xrightarrow{i} L \xrightarrow{f} C \xrightarrow{p} D$ where i is a monomorphism and $p \colon C \to D$ is an inner fibration between quasicategories. If i is right anodyne then the alternate pullback-slice map $C^{f/} \to C^{fi/} \times_{D^{pfi/}} D^{pf/}$ is a trivial fibration, and if i is left anodyne then the alternate pullback-slice map $C^{/f} \to C^{/fi} \times_{D^{/pfi}} D^{/pf}$ is a trivial fibration.* **F 8 Apr**

Proof. Immediate from (62.5) and the corresponding facts for the corresponding pullback-slice maps (32.14). ☐

We will often combine the above with the following.

62.7. **Proposition.** *For any monomorphism $j\colon K \to L$ of simplicial sets, the map $\Delta^0 \star j\colon K^{\lhd} \to L^{\lhd}$ is left anodyne, and the map $j \star \Delta^0\colon K^{\rhd} \to L^{\rhd}$ is right anodyne.*

Proof. We prove the first case. Note that for any map $p\colon C \to D$ of simplicial sets, we have that $(\Delta^0 \star j) \boxslash p$ if and only if $j \boxslash (C_{x/} \to D_{px/})$ for all vertices $x \in C_0$. Since $(\overline{\text{LHorn}}, \text{LFib})$ is a factorization system, we see that that the class of maps j such that $j \star \Delta^0$ is weakly saturated, so it suffices to show that the cell inclusions have this property. But we know that $\Delta^0 \star (\partial\Delta^n \subset \Delta^n) = (\Lambda^{n+1}_0 \subset \Delta^{n+1})$. □

In particular, inclusions $\{v\} \subseteq L^{\lhd}$ and $\{v\} \subseteq L^{\rhd}$ are left anodyne and right anodyne respectively.

As a consequence we get the following, which says that slices *under a left cone* or *over a right cone* are equivalent to the corresponding slices under or over the "cone point".

62.8. **Corollary.** *For any map $\widehat{f}\colon S^{\rhd} \to C$ to a quasicategory, the restriction maps $C_{\widehat{f}/} \to C_{\widehat{f}(v)/}$ and $C^{\widehat{f}/} \to C_{\widehat{f}(v)/}$ induced by restriction along the inclusion of the cone point of $S^{\rhd}$ are trivial fibrations. Likewise, for any map $\widehat{g}\colon S^{\lhd} \to C$ to a quasicategory, the restriction maps $C_{/\widehat{g}} \to C_{/\widehat{g}(v)}$ and $C^{/\widehat{g}} \to C^{/\widehat{g}(v)}$ induced by restriction along the inclusion of the cone point of $S^{\rhd}$ are trivial fibrations.*

As a consequence, for each $c \in C_0$ the induced maps

$$\operatorname{map}_{\operatorname{Fun}(S^{\rhd},C)}(\widehat{f}, \widetilde{\pi}_{S^{\rhd}}c) \to \operatorname{map}_C(\widehat{f}(v), c) \qquad \textit{and} \qquad \operatorname{map}_{\operatorname{Fun}(S^{\lhd},C)}(\widetilde{\pi}_{S^{\lhd}}c, \widehat{g}) \to \operatorname{map}_C(c, \widehat{g}(v))$$

are equivalences.

Proof. We do the case of $\widehat{f}$. By (62.7) the inclusion $\{v\} \to S^{\rhd}$ is right anodyne. Hence the restriction map $C_{\widehat{f}/} \to C_{\widehat{f}(v)/}$ is a trivial fibration by (32.14), and $C^{\widehat{f}/} \to C^{\widehat{f}(v)/}$ is a trivial fibration by (62.6). The equivalence of mapping spaces is immediate from (58.7) and the fact that all restriction maps to C are isofibrations. □

Part 11. More on limits and colimits

63. Limits, colimits, and mapping spaces

63.1. **Pushout products and right and left anodyne maps.** Recall that $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ (21.8) and $\overline{\text{Horn}} \square \overline{\text{Cell}} \subseteq \overline{\text{Horn}}$ (**??**). We have an analogous fact for left or right anodyne maps, which in fact implies both prior results.

63.2. **Proposition.** *We have that $\overline{\text{LHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{LHorn}}$ and $\overline{\text{RHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{RHorn}}$.*

Proof. This is a calculation. See the appendix (78). □

Therefore we have versions of enriched lifting which apply in this case.

63.3. **Corollary.**

(1) *If $p\colon X \to Y$ is a left fibration and $j\colon K \to L$ is a monomorphism, then $p^{\square j}$ is a left fibration. If furthermore j is left anodyne, then $p^{\square j}$ is a trivial fibration.*
(2) *If $p\colon X \to Y$ is a right fibration and $j\colon K \to L$ is a monomorphism, then $p^{\square j}$ is a right fibration. If furthermore j is right anodyne, then $p^{\square j}$ is a trivial fibration.*

63.4. **Fiberwise criterion for trivial fibrations, revisited.** We note the following "fiberwise" criterion for a left or right fibration to be a trivial fibration (and hence a categorical equivalence), analogous to the one we proved for isofibrations to quasigroupoids (47.1). Recall that the fibers of any left or right fibration between simplicial sets are Kan complexes.

63.5. **Proposition.** *Let $p\colon X \to Y$ be either a left or right fibration of simplicial sets. Then p is a trivial fibration if and only if it has contractible fibers.*

Proof. [Lur09, 2.1.3.4]. We consider the case of a left fibration, and note that the direction ($\Longrightarrow$) is imediate.

Suppose given a left fibration p with contractible fibers. We will show that $(\partial\Delta^n \subset \Delta^n) \boxslash p$ for all $n \geq 0$, by a variant of the covering homotopy extension technique we used to prove (47.1), to "deform" a given lifting problem to one which lives in a single fiber.

As in the proof of (47.1) we consider a lifting problem of type $(\partial\Delta^n \subset \Delta^n) \boxslash p$, i.e., a vertex $(a, b) \in \mathrm{Fun}(\partial\Delta^n, X) \times_{\mathrm{Fun}(\partial\Delta^n, Y)} \mathrm{Fun}(\Delta^n, Y)$. Let $\gamma\colon \Delta^n \times \Delta^1 \to \Delta^n$ be the unique map given on vertices by $\gamma(k, 0) = k$ and $\gamma(k, 1) = n$, so that γ represents a natural transformation $\mathrm{id}_{\Delta^n} \to \langle n \cdots n\rangle$ of functors $\Delta^n \to \Delta^n$. Then there exists a lift u in

$$\begin{tikzcd}
\partial\Delta^n \times \{0\} \arrow[rrr, "a"] \arrow[d, tail] & & & X \arrow[d, "p"] \\
\partial\Delta^n \times \Delta^1 \arrow[r, tail] \arrow[urrr, dotted, "u"] & \Delta^n \times \Delta^1 \arrow[r, "\gamma"'] & \Delta^n \arrow[r, "b"'] & Y
\end{tikzcd}$$

since $\overline{\mathrm{LHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{LHorn}}$ (63.2) so $\partial\Delta^n\{0\} \subset \partial\Delta^n \times \Delta^n$ is left anodyne. The lower right triangle represents an edge e in $\mathrm{Fun}(\partial\Delta^n, X) \times_{\mathrm{Fun}(\partial\Delta^n, Y)} \mathrm{Fun}(\Delta^n, Y)$ connecting the vertex $e_0 = (a, b)$ to a vertex $e_1 = (a', b')$, where $b' = b\gamma|\Delta^n \times \{1\}$ factor as $\Delta^n \to \{b(n)\} \to Y$. As the lifting problem (a', b') lives in a single fiber, by hypothesis it admits a solution $t\colon \Delta^n \to X$.

Thus we have a solid arrow commutative diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\}) \arrow[r, "{(u,t)}"] \arrow[d, tail] & X \arrow[d, "p"] \\
\Delta^n \times \Delta^1 \arrow[r, "b\gamma"'] \arrow[ur, dotted, "s"] & Y
\end{tikzcd}$$

If we can produce a lift s, then the restriction $s|\Delta^n \times \{0\}$ is the desired solution to the original lifting problem (a, b).

Form the diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\}) \arrow[r, "g"] \arrow[d, tail] \arrow[rr, bend left, "{(u,t)}"] & C \arrow[r, "h"'] \arrow[d, "p'"'] & X \arrow[d, "p"] \\
\Delta^n \times \Delta^1 \arrow[r, "\gamma"'] \arrow[ur, dotted, "s'"] & \Delta^n \arrow[r, "b"'] & Y
\end{tikzcd}$$

where the right-hand square is a pullback. Observe that (i) p' is a left fibration, and hence an inner fibration, between quasicategories, and that (ii) γ sends the edge $\{n\} \times \Delta^1$ to the degenerate edge $\langle nn\rangle$ in Δ^n. Therefore g sends the edge $\{n\} \times \Delta^1$ into the fiber of p' over $n \in (\Delta^n)_0$, which is isomorphic to the fiber of p over $b(n)$, which is by hypothesis a contractible Kan complex. Thus, $g|\{n\} \times \Delta^1$ represents an isomorphism in the quasicategory C. Therefore the pushout-product version of Joyal lifting (36.6) gives a lift s', and so $s := hs'$ is the desired lift. $\square$

63.6. **Initial and terminal objects via mapping spaces.** We can apply this fiberwise criterion to pullback-slice maps or their alternate analogs, since these are often either left or right fibrations. Using this we get a natural criterion for an object to be initial or terminal, in terms of spaces of maps from or to the object.

63.7. **Proposition.** *Let C be a quasicategory and $x \in C_0$ an object of C.*

(1) *The object x is initial in C if and only if for every object $c \in C_0$ the mapping space* $\mathrm{map}_C(x,c)$ *is contractible.*
(2) *The object x is terminal in C if and only if for every object $c \in C_0$ the mapping space* $\mathrm{map}_C(c,x)$ *is contractible.*

Proof. I'll prove case (1). Consider the commutative diagram

$$\begin{tikzcd} C_{x/} \arrow[rr, "\gamma"] \arrow[dr, "p"'] & & C^{x/} \arrow[dl, "q"] \\ & C & \end{tikzcd}$$

where p and q are the evident forgetful functors and γ is the comparison map. The object x is initial if and only if p is a categorical equivalence (43.1), hence by 2-out-of-3 (27.11) if and only if q is a categorical equivalence. Since q is a left fibration and thus an isofibration (34.14), this is so if and only if q is a trivial fibration (40.8), and this is the case if and only if the fibers of q are contractible (63.5). The claim follows because the fibers of q are precisely the mapping spaces $\mathrm{map}_C(x,c)$ (58.3). □

63.8. **Limits and colimits via mapping spaces.** We have a similar result for general limits and colimits, so that whether a cone is a colimit or limit is characterized in terms of the spaces of natural transformations from or to *constant* functors.

63.9. **Proposition.**

(1) *For any map $\widehat{f}\colon S^{\rhd} \to C$ to a quasicategory with $\widehat{f}|S = f$, the following are equivalent.*
  (a) *The slice restriction functor $C_{\widehat{f}/} \to C_{f/}$ is a trivial fibration, i.e., $\widehat{f}$ is a colimit cone.*
  (b) *The slice restriction functor $C_{\widehat{f}/} \to C_{f/}$ is a categorical equivalence.*
  (c) *The alternate slice restriction functor $C^{\widehat{f}/} \to C^{f/}$ is a trivial fibration.*
  (d) *The alternate slice restriction functor $C^{\widehat{f}/} \to C^{f/}$ is a categorical equivalence.*
  (e) *For each object $c \in C_0$, the restriction map*
$$\mathrm{map}_{\mathrm{Fun}(S^{\rhd},C)}(\widehat{f}, \widetilde{\pi}_{S^{\rhd}}c) \to \mathrm{map}_{\mathrm{Fun}(S,C)}(f, \widetilde{\pi}_S c)$$
  is an equivalence, where $\widetilde{\pi}_{S^{\rhd}}\colon C \to \mathrm{Fun}(S^{\rhd}, C)$ and $\widetilde{\pi}_S\colon C \to \mathrm{Fun}(S,C)$ are adjoints to projection.

  Furthermore, if any of these hold, restriction along $\{v\} \subseteq S^{\rhd}$ gives an equivalence $\mathrm{map}_{\mathrm{Fun}(S,C)}(f, \widetilde{\pi}_S c) \xrightarrow{\sim} \mathrm{map}_C(\widehat{f}(v), c)$.

(2) *For any map $\widehat{g}\colon S^{\lhd} \to C$ to a quasicategory with $\widehat{g}|S = f$, the following are equivalent.*
  (a) *The slice restriction functor $C_{/\widehat{g}} \to C_{/g}$ is a trivial fibration, i.e., $\widehat{g}$ is a limit cone.*
  (b) *The slice restriction functor $C_{/\widehat{g}} \to C_{/g}$ is a categorical equivalence.*
  (c) *The alternate slice restriction functor $C^{/\widehat{g}} \to C^{/g}$ is a trivial fibration.*
  (d) *The alternate slice restriction functor $C^{/\widehat{g}} \to C^{/g}$ is a categorical equivalence.*
  (e) *For each object $c \in C_0$, the restriction map*
$$\mathrm{map}_{\mathrm{Fun}(S^{\lhd},C)}(\widetilde{\pi}_{S^{\lhd}}c, \widehat{g}) \to \mathrm{map}_{\mathrm{Fun}(S,C)}(\widetilde{\pi}_S c, g)$$
  is an equivalence, where $\widetilde{\pi}_{S^{\lhd}}\colon C \to \mathrm{Fun}(S^{\lhd}, C)$ and $\widetilde{\pi}_S\colon C \to \mathrm{Fun}(S,C)$ are adjoints to projection.

  Furthermore, if any of these hold, restriction along $\{v\} \subseteq S^{\lhd}$ gives an equivalence $\mathrm{map}_{\mathrm{Fun}(S,C)}(\widetilde{\pi}_S c, g) \xrightarrow{\sim} \mathrm{map}_C(c, \widehat{g}(v))$.

Proof. We prove case (1), following the same strategy as the proof of (63.7). The equivalence of (a)–(d) is straightforward using (62.5) to compare pullback-slices with alternate pullback-slices, and the fact that each of the maps is an isofibration. For the equivalence with (e) we refer to the diagram of pullback squares (58.7)

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{Fun}(S^\triangleright,C)}(\widehat{f},\widetilde{\pi}_{S^\triangleright}c) \arrow[r, "p_c"] \arrow[d, tail] & \mathrm{map}_{\mathrm{Fun}(S,C)}(f,\widetilde{\pi}_S c) \arrow[r] \arrow[d, tail] & \{c\} \arrow[d, tail] \\
C^{\widehat{f}/} \arrow[r, "p"'] & C^{f/} \arrow[r] & C
\end{tikzcd}$$

Any vertex of $C^{f/}$ is contained in are contained in $\mathrm{map}_{\mathrm{Fun}(S,C)}(f,\widetilde{\pi}_S c)$ for some $c \in C_0$, so we deduce from the fiberwise criterion (63.5) that p is a trivial fibration if and only if each p_c is a trivial fibration. This is the case if and only if each p_c is a categorical equivalence, since each p_c is a pullback of the left fibration p (62.4) and so is an isofibration. The final claim that $\mathrm{map}_{\mathrm{Fun}(S,C)}(f,\widetilde{\pi}_S c) \approx \mathrm{map}_C(\widehat{f}(v), c)$ if $\widehat{f}$ is a colimit cone is immediate from (62.8). $\square$

63.10. **Limits and colimits in quasicategories vs. homotopy limits and colimits in simplicial sets.** We can sometimes use the mapping-space criterion for limits/colimits to convert the property of being a limit/colimits in a quasicategory to a question about *homotopy limits/colimits* in simplicial sets.

For instance, let S be a set, corresponding to a discrete simplicial set which I also call S. For each $s \in S = S_0$, write $i_s \colon \Delta^1 \approx \{s\}^\triangleright \to S^\triangleright$ for the evident inclusion, so that we get commutative diagrams

$$\begin{tikzcd}
\mathrm{Fun}(\{v\},C) \arrow[d, "\approx"] & \mathrm{Fun}(S^\triangleright,C) \arrow[l] \arrow[r] \arrow[d, "i_s^*"] & \mathrm{Fun}(S,C) \arrow[d, "\{s\}^*"] \\
C & \mathrm{Fun}(\Delta^1,C) \arrow[l, "\langle 0\rangle^*"] \arrow[r, "\langle 1\rangle^*"'] & C
\end{tikzcd}$$

where all the maps are induced by restriction, and note that $\mathrm{Fun}(S,C) \approx \prod_{s\in S} C$.

Given a functor $\widehat{f} \colon S^\triangleright \to C$ with restriction $f = \widehat{f}|_S \colon S \to C$, and an object $c \in C_0$ with corresponding constant functor $\widetilde{\pi}_{S^\triangleright} c \colon S^\triangleright \to C$, evaluating the above diagram on mapping spaces gives us

$$\begin{tikzcd}
\mathrm{map}_C(\widehat{f}(v),c) \arrow[d, equal] & \mathrm{map}_{\mathrm{Fun}(S^\triangleright,C)}(\widehat{f},\widetilde{\pi}_{S^\triangleright}c) \arrow[l, "\sim"'] \arrow[r, "\alpha"] \arrow[d, "\sim"'] & \prod_{s\in S}\mathrm{map}_C(f(s),c) \arrow[d] \\
\mathrm{map}_C(\widehat{f}(v),c) & \mathrm{map}_{\mathrm{Fun}(\Delta^1,C)}(\widehat{f}i_s,\widetilde{\pi}_{\Delta^1}c) \arrow[l, "\sim"] \arrow[r, "\beta_s"'] & \mathrm{map}_C(f(s),c)
\end{tikzcd}$$

where the maps indicated "$\sim$" are equivalences by (**??**) and 2-out-of-3, since $\{v\} \subset S^\triangleright$ and $\{0\} \subset \Delta^1$ are left anodyne.

The mapping space criterion for colimits says that $\widehat{f}$ is a colimit cone (i.e., a coproduct in C) if and only if the map α is an equivalence, ...

64. Functoriality of limits and colimits

Let S be a simplicial set and C a quasicategory, and consider the functor $p \colon \mathrm{Fun}(S^\triangleright, C) \to \mathrm{Fun}(S,C)$ induced by restriction along $S \subset S^\triangleright$. Let us consider the following diagram. **M 11 Apr**

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{colim}}(S^\triangleright,C) \arrow[r, tail] \arrow[d, "p'"'] & \mathrm{Fun}(S^\triangleright,C) \arrow[r, "e"] \arrow[d, "p"] & C \\
\mathrm{Fun}^{\exists\,\mathrm{colim}}(S,C) \arrow[r, tail] & \mathrm{Fun}(S,C) &
\end{tikzcd}$$

Here $\mathrm{Fun}^{\mathrm{colim}}(S^{\triangleright}, C)$ denotes the full subcategory of $\mathrm{Fun}(S^{\triangleright}, C)$ spanned by $\widehat{f}\colon S^{\triangleright} \to C$ which are colimit cones, and $\mathrm{Fun}^{\exists\,\mathrm{colim}}(S, C)$ denotes the full subcategory of $\mathrm{Fun}(S, C)$ spanned by $f\colon S \to C$ which admit an extension to a colimit cone. The functor p' is the evident restriction of p, and is necessarily surjective on objects. The functor e is defined by evaluation at the cone point of $S^{\triangleright}$.

By a **colimit cone functor**, we mean a section s of the the functor p', i.e., a *functorial* assignment of a colimit cone $\widehat{f}$ extending each functor f which admits a colimit. If such a colimit functor exists, then we can compose it with restriction to the cone point: the resulting composite will be a functor $\mathrm{Fun}^{\exists\,\mathrm{colim}}(S, C) \to C$, which we will call a **colimit functor**. In particular, in the happy situation that *every* $f\colon S \to C$ admits a colimit cone, so that $\mathrm{Fun}^{\exists\,\mathrm{colim}}(S, C) = \mathrm{Fun}(S, C)$, we get a functor **colimit cone functor** **colimit functor**

$$\mathrm{colim}_S\colon \mathrm{Fun}(S, C) \to C.$$

The colimit cone functor always exists, and is essentially unique, because of the following fact, which we will prove as (69.13): the restriction functor $p'\colon \mathrm{Fun}^{\mathrm{colim}}(S^{\triangleright}, C) \to \mathrm{Fun}^{\exists\,\mathrm{colim}}(S, C)$ is a trivial fibration. As a consequence, p' has a contractible Kan complex of sections, and any of these sections is a colimit cone functor.

Of course, there is an analogous series of definitions and results involving limits, whose statement we leave for the reader.

64.1. *Exercise.* Show that if C is an ordinary category, the functor p' is fully faithful, and hence is a trivial fibration since it is surjective on objects.

We might want to prove our result by showing that p' is fully faithful when C is any quasicategory. The mapping space criterion for colimits tells us only that $\mathrm{map}_{\mathrm{Fun}(S^{\triangleright},C)}(\widehat{f}, \widehat{g}) \to \mathrm{map}_{\mathrm{Fun}(S,C)}(\widehat{f}|_S, \widehat{g}|_S)$ is an equivalence for a colimit cone $\widehat{f}$ when $\widehat{g}$ is a constant functor $\widetilde{\pi}_{S^{\triangleright}} c$. So the idea of the proof is to show that having equivalences for constant $\widehat{g}$ gives equivalences for general $\widehat{g}$.

To set up the proof, we first need to understand the fibers of the restriction $p\colon \mathrm{Fun}(S^{\triangleright}, C) \to \mathrm{Fun}(S, C)$. Given $f\colon S \to C$, we have the following diagram, in which both squares are pullbacks.

$$\begin{tikzcd}
C_{f/} \arrow[r, "\alpha"] & p^{-1}(f) \arrow[r, "\beta"] \arrow[d, tail] & C^{f/} \arrow[r] \arrow[d, tail] & \{f\} \arrow[d, tail] \\
& \mathrm{Fun}(S^{\triangleright}, C) \arrow[r, "r"] \arrow[rr, bend right, "p"'] & \mathrm{Fun}(S \diamond \Delta^0, C) \arrow[r, "q"] & \mathrm{Fun}(S, C)
\end{tikzcd}$$

Here $p = qr$, q, and r are the evident maps induced by restriction along $S \to S \diamond \Delta^0 \to S \star \Delta^0$. Since both p and q are induced by restriction along an isomorphism they are isofibrations (39.6). Since $S \diamond \Delta^0 \to S \star \Delta^0$ is a categorical equivalence (60.2) so is r. Furthermore, both maps α and β are categorical equivalences: note that $p^{-1}(f) = \mathrm{Fun}_{S/}\, S^{\triangleright}, (C, f)$ as in (61.7), and that α is a bijection on objects.

64.2. **Lemma.** *A map $\widehat{f}\colon S^{\triangleright} \to C$ is a colimit cone if and only if it is an initial object of the fiber $p^{-1}(f)$ of $p\colon \mathrm{Fun}(S^{\triangleright}, C) \to \mathrm{Fun}(S, C)$, where $f = \widehat{f}|S$.*

Proof. Apply the fact that initial objects are invariant under equivalences (43.5) to β. □

65. Relative initial and terminal objects

Let $p\colon C \to D$ be an inner fibration of quasicategories. We say that an object $x \in C_0$ is p**-initial** **p-initial** if a lift exists in every commutative diagram of the form

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r, "u"] \arrow[d, tail] & C \arrow[d, "p"] \\ & \Delta^n \arrow[r, "v"'] \arrow[ur, dotted] & D \end{tikzcd}$$

with $n \geq 1$, and with $u(0) = x$. Similarly, the object is p**-terminal** if a lift exists in every commutative diagram of the form **p-terminal**

$$\begin{tikzcd} \{n\} \arrow[r] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r, "u"] \arrow[d, tail] & C \arrow[d, "p"] \\ & \Delta^n \arrow[r, "v"'] \arrow[ur, dotted] & D \end{tikzcd}$$

with $n \geq 1$, and with $u(n) = x$.

65.1. *Example.* For the projection $p\colon C \to \Delta^0$, being p-initial or p-terminal is the same as being initial or terminal in C.

The property of an object being relatively initial or terminal is closed under basechange.

65.2. **Lemma.** *Consider a pullback square of quasicategories*

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r] & B \end{tikzcd}$$

in which p is an inner fibration, and let $x \in E'_0$. If $f(x)$ is p-initial then x' is p'-initial, and if $f(x)$ is p-terminal then x' is p'-terminal.

In particular, if $y \in E_0$ is p-initial/p-terminal, then it is initial/terminal in the fiber quasicategory $p^{-1}(py) = \{y\} \times_B E$.

Proof. The first statement is immediate from the definition, and the second statement is a special case of the first. □

65.3. *Example* (Initial in fiber does not imply relative initial). The map $p\colon \partial\Delta^1 \to \Delta^1$ is an inner fibration of categories, such that the object 0 is initial in its fiber but is not p-initial.

65.4. **Proposition.** *Let $p\colon C \to D$ be an inner fibration of quasicategories, and let $x \in C_0$. Then the following are equivalent.*

(1) *The object x is p-initial.*
(2) *The pullback-slice map $C_{x/} \to C \times_D D_{px/}$ is a trivial fibration.*
(3) *The alternate pullback-slice map $C^{x/} \to C \times_D D^{px/}$ is a trivial fibration.*
(4) *For each $c \in C_0$, the map $\operatorname{map}_C(x, c) \to \operatorname{map}_D(px, pc)$ induced by p is an equivalence.*

Proof. The equivalence of (1) and (2) is an immediate consequence of the join/slice adjunction, as in (31.3). The equivalence of (2) and (3) is immediate from (62.5). The equivalence of (3) and (4) is a consequence of the fiberwise criterion for trivial fibrations (63.5) (because the alternate pullback-slice map $q\colon C^{x/} \to C \times_D D^{px/}$ is a left fibration (62.3)), and the fact that the induced map $\operatorname{map}_C(x, c) \to \operatorname{map}_D(px, py)$ is precisely the map induced by q on fibers over $c \in C_0$ (58.3). □

Given an inner fibration $p\colon C \to D$ between quasicategories, we consider the following commutative diagram

$$\begin{tikzcd} C^{p\,\mathrm{init}} \arrow[r, tail] \arrow[d, "p'"'] & C \arrow[d, "p"] \\ D^{\exists p\,\mathrm{init}} \arrow[r, tail] & D \end{tikzcd}$$

where $C^{p\,\mathrm{init}} \subseteq C$ is the full subcategory spanned by p-initial objects, and $D^{\exists p\,\mathrm{init}} \subseteq D$ is the full subcategory spanned by objects which are the image of some p-initial object in C. The functor p' is the evident restriction of p, which by construction is surjective on objects.

65.5. **Proposition.** *If $p\colon C \to D$ is an inner fibration between quasicategories, the restricted functor $p'\colon C^{p\,\mathrm{init}} \to D^{\exists p\,\mathrm{init}}$ is a trivial fibration.*

Proof. We want $(\partial\Delta^n \subset \Delta^n) \boxslash p'$ for all $n \geq 0$. When $n = 0$ this is by definition, while if $n \geq 1$ is is an immediate consequence of the fact that all objects in $C^{p\,\mathrm{init}}$ are p-initial. □

We will apply this to the restriction functor $p\colon \mathrm{Fun}(S^{\triangleright}, C) \to \mathrm{Fun}(S, C)$. Thus, to prove the existence of a colimit cone functor, it will suffice to prove that every object in $\mathrm{Fun}(S^{\triangleright}, C)$ which is initial in its fiber over p (i.e., is a colimit cone) is in fact p-initial, after which we can apply (65.5).

66. Cartesian and coCartesian morphisms

Consider a map $p\colon E \to B$ between simplicial sets. An edge $f\colon x \to y$ in E is said to be **p-Cartesian edge** (or a **p-Cartesian morphism** if E and B are quasicategories) if a lift exists in every square of the form

p-Cartesian edge
p-Cartesian morphism

$$\begin{tikzcd} \Delta^{\{n-1,n\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_n \arrow[r] \arrow[d, tail] & E \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

with $n \geq 1$, i.e., if we can lift every right-horn whose trailing edge maps to f. Likewise, f is said to be a **p-coCartesian edge** (or a **p-coCartesian morphism** if E and B are quasicategories) if a lift exists in every square of the form

p-coCartesian edge
p-coCartesian morphism

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d, tail] & E \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

with $n \geq 1$, i.e., if we can lift every left-horn whose leading edge maps to f.

66.1. *Remark.* In [Lur09, 2.4.1], Lurie makes it a precondition for an edge to be Cartesian or coCartesian that the map p be an inner fibration; however, he drops this condition in [Lur21, Definition 01T5]. In practice, it is rarely useful to consider this notion when p is not at least an inner fibration.

66.2. *Example.* If $p\colon E \to B$ is an inner fibration, then it is a left fibration if and only if every edge in E is p-coCartesian, Likewise, it is a right fibration if and only if every edge in E is p-Cartesian.

The Joyal lifting theorem describes the Cartesian and coCartesian morphisms which project to isomorphisms.

66.3. **Proposition.** *Let $p\colon C \to D$ be an inner fibration between quasicategories, and suppose $f \in C_1$ is a morphism in C such that $p(f)$ is an isomorphism in D. Then f is a p-Cartesian morphism if and only if f is an isomorphism if and only if f is a p-coCartesian morphism.*

Proof. Immediate from the statement of Joyal lifting (34.17). □

66.4. **Lemma.** *Let $C \xrightarrow{p} D \xrightarrow{q} E$ be a sequence of maps of simplicial sets, and let $f \in C_1$ be an edge of C. If (i) f is p-Cartesian, and (ii) $p(f)$ is q-Cartesian, then f is qp-Cartesian. Likewise, if (i') f is p-coCartesian, and (ii') $p(f)$ is q-coCartesian, then f is qp-Cartesian.*

Proof. Straightforward from the defintions. □

66.5. **Lemma.** *Let*

$$\begin{tikzcd} C' \arrow[r, "u"] \arrow[d, "p'"'] & C \arrow[d, "p"] \\ D' \arrow[r] & D \end{tikzcd}$$

be a pullback square of simplicial sets, and let $f \in C'_1$ be an edge in C'. If $u(f)$ is p-Cartesian then f is p'-Cartesian, and if $u(f)$ is p-coCartesian then f is p'-coCartesian.

Proof. Straightforward from the definitions. □

66.6. **Characterizations of Cartesian and coCartesian edges.** We have a number of characterizations of Cartesian and coCartesian edges, which I'll state in the Cartesian case.

66.7. **Proposition.** *Let $p\colon E \to B$ be a map of simplicial sets, and $f \in E_1$ an edge with $x = f_0, y = f_1 \in E_0$. Then f is a p-Cartesian edge if and only if the pullback-slice map $E_{/f} \to E_{/y} \times_{B_{/py}} B_{/pf}$ is a trivial fibration.*

Proof. This amounts to the the equivalence of lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, tail] & E_{f/} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & E_{x/} \times_{B_{px/}} B_{pf/} \end{tikzcd} \iff \begin{tikzcd} \varnothing \star \Delta^1 \arrow[r, tail] \arrow[rr, bend left, "f"] & (\partial\Delta^n \star \Delta^1) \cup (\Delta^n \star \{1\}) \arrow[r] \arrow[d, tail, "j"'] & E \arrow[d, "p"] \\ & \Delta^n \star \Delta^1 \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

where j is isomorphic to $\Lambda^{n+2}_{n+2} \subset \Delta^{n+2}$. □

66.8. **Proposition.** *Let $p\colon E \to B$ be an inner fibration of quasicategories, and $f \in E_1$ an edge with $x = f_0, y = f_1 \in E_0$. Then f is a p-Cartesian edge if and only if the alternate pullback-slice map $E^{/f} \to E^{/y} \times_{B^{/py}} B^{/pf}$ is a trivial fibration.*

Proof. Deduce from the pullback-slice case (66.7) using (62.5) to compare the two types of pullback-slice map. □

66.9. **Proposition.** *Let $p\colon E \to B$ be an inner fibration of quasicategories. Then a morphism $f\colon x \to y$ in E is a p-Cartesian morphism if and only if it corresponds to a q-terminal object for the slice restriction functor $q\colon E_{/y} \to B_{/py}$, if and only if it corresponds to a q'-terminal object for the alternate slice restriction functor $q'\colon E^{/y} \to B^{/py}$.*

Proof. The first part is a consquence of the equivalence of lifting problems

$$\begin{tikzcd} \{n\} \arrow[r, tail] \arrow[rr, bend left, "f"] & \partial\Delta^n \arrow[r] \arrow[d, tail] & E_{y/} \arrow[d, "q"] \\ & \Delta^n \arrow[r] \arrow[ur, dotted] & B_{py/} \end{tikzcd} \iff \begin{tikzcd} \varnothing \star \Delta^0 \arrow[r, tail] \arrow[rr, bend left, "f"] & \partial\Delta^n \star \Delta^0 \arrow[r] \arrow[d, tail, "j"'] & E \arrow[d, "p"] \\ & \Delta^n \star \Delta^0 \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

where j is isomorphic to $\Lambda^{n+1}_{n+1} \subset \Delta^{n+1}$. Deduce the second part from the first using (62.5). $\square$

66.10. **Mapping space criterion for (co)Cartesian morphisms.**

66.11. **Lemma.** *Let C be a quasicategory, $f\colon x \to y$ a morphism in C, and $c \in C_0$ an object. Consider the diagrams* **W 13 Apr**

$$\operatorname{map}_C(c,x) \xleftarrow{r} \operatorname{map}_{\operatorname{Fun}(\Delta^1,C)}(1_c, f) \xrightarrow{r'} \operatorname{map}_C(c,y)$$

and

$$\operatorname{map}_C(x,c) \xleftarrow{r'} \operatorname{map}_{\operatorname{Fun}(\Delta^1,C)}(f, 1_c) \xrightarrow{r} \operatorname{map}_C(y,c)$$

in which $1_c, f \in C_1$ are regarded in an evident way as objects of $\operatorname{Fun}(\Delta^1, C)$*, and the maps are induced by the evident restrictions to subcomplexes of Δ^1. Then the maps marked r' are Kan fibrations, and the maps marked r are trivial fibrations.*

Proof. I'll prove the claims for the first diagram. We have a commutative square of alternate slices

$$\begin{tikzcd} C^{/f} \arrow[r, "q'"] \arrow[d, "q"'] & C^{/y} \arrow[d] \\ C^{/x} \arrow[r] & C \end{tikzcd}$$

where each map is induced by restriction to the evident subcomplex of Δ^1. Furthermore, the map q' is a right fibration (62.4) and the map q is a trivial fibration (66.7). Taking the fiber over $c \in C_0$ of each map to C in this diagram gives the zig-zag in the statement of the lemma, and the claim follows since being a trivial fibration or right right fibration is preserved under base change, and right fibrations between Kan complexes are Kan fibrations. $\square$

Given a functor $p\colon C \to D$ of quasicategories, we get an induced commutative diagram of quasigroupoids

$$\begin{tikzcd} \operatorname{map}_C(c,x) \arrow[d] & \operatorname{map}_{\operatorname{Fun}(\Delta^1,C)}(1_c, f) \arrow[l, "\sim"'] \arrow[r] \arrow[d] & \operatorname{map}_C(c,y) \arrow[d] \\ \operatorname{map}_D(pc,px) & \operatorname{map}_{\operatorname{Fun}(\Delta^1,D)}(1_{pc}, pf) \arrow[l, "\sim"] \arrow[r] & \operatorname{map}_D(pc,py) \end{tikzcd}$$

in which the left-pointing arrows are equivalences. Furthermore, if p is an inner fibration then the vertical arrows are Kan fibrations.

66.12. *Exercise.* Let $p\colon C \to D$ be a functor of ordinary categories, and let $f\colon x \to y$ be a morphism in C. Show that f is p-Cartesian if and only if for every object c of C, the square

$$\begin{tikzcd} \hom_C(c,x) \arrow[r, "f\circ-"] \arrow[d, "p"'] & \hom_C(c,y) \arrow[d, "p"] \\ \hom_D(pc,px) \arrow[r, "pf\circ-"'] & \hom_D(pc,py) \end{tikzcd}$$

is a pullback of sets.

66.13. **Proposition.** *Let $p\colon C \to D$ be an inner fibration between quasicategories, and let $f\colon x \to y$ be a morphism in C.*

The morphism f is p-Cartesian if and only if for every $c \in C_0$ the right-hand square in the above diagram is a homotopy pullback, i.e., if $\operatorname{map}_{\operatorname{Fun}(\Delta^1,C)}(1_c, f) \to \operatorname{map}_C(c,y) \times_{\operatorname{map}_D(pc,py)} \operatorname{map}_{\operatorname{Fun}(\Delta^1,D)}(1_{pc}, pf)$ *is an equivalence.*

Proof. (Based on [Lur09, 2.4.4.3].) Recall that f is p-Cartesian if and only if the alternate pullback slice map $q\colon C^{/f} \to C^{/y} \times_{D^{/py}} D^{/pf}$ is a trivial fibration (66.8). Note that since p is an inner fibration of quasicategories the map q is a right fibration (62.3), and therefore is a trivial fibration if and only if it has contractbile fibers (63.5).

For each $c \in C_0$ both squares in

$$\begin{tikzcd}
\{c\} \times_C C^{/f} \arrow[r, "q_c"] \arrow[d, tail] & (\{c\} \times_C C^{/y}) \times_{\{pc\}\times_D D^{/py}} (\{py\} \times_D D^{/pf}) \arrow[r] \arrow[d, tail] & \{c\} \arrow[d, tail] \\
C^{/f} \arrow[r, "q"'] & C^{/y} \times_{D^{/py}} D^{/pf} \arrow[r] & C
\end{tikzcd}$$

are pullbacks. Thus to show that q has contractible fibers it suffices to show that each q_c (which is also a right fibration) has contractible fibres. But q_c is isomorphic to the map

$$\mathrm{map}_{\mathrm{Fun}(\Delta^1,C)}(1_c, f) \to \mathrm{map}_C(c, y) \times_{\mathrm{map}_D(pc,py)} \mathrm{map}_{\mathrm{Fun}(\Delta^1,D)}(1_{pc}, pf),$$

which is we wanted to prove is an equivalence. $\square$

67. Cartesian and coCartesian fibrations

A map $p\colon C \to D$ is a **Cartesian fibration** if it is an inner fibration and if for every diagram of the form **Cartesian fibration**

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\
\Delta^1 \arrow[r] \arrow[ur, dotted, "f"] & D
\end{tikzcd}$$

a lift exists such that f represents a p-Cartesian edge. Likewise p is a **coCartesian fibration** if it is an inner fibration and if for every diagram of the form **coCartesian fibration**

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\
\Delta^1 \arrow[r] \arrow[ur, dotted, "f"] & D
\end{tikzcd}$$

a lift exists such that f represents a p-coCartesian edge.

67.1. *Example.* Every left fibration between simplicial sets is a coCartesian fibration, and every right fibration between simplicial sets is a Cartesian fibration.

67.2. *Example.* Every Cartesian fibration and coCartesian fibration between quasicategories is an isofibration. The point is that if $p\colon C \to D$ is an inner fibration of quasicategories, then the p-Cartesian and p-coCartesian morphisms of C which map to isomorphisms to D are precisely the isomorphisms in C (66.3).

In particular, if $p\colon C \to D$ is a map to a Kan complex D, then p is a Cartesian fibration if and only if it is an isofibration (and thus also if and only if it is a coCartesian fibration).

Our first application of these notions is the following, which will be the key to our proof of functoriality of limits and colimits.

67.3. **Proposition.** *Suppose $p\colon C \to D$ is a Cartesian fibration between quasicategories. Then an object $c \in C_0$ is p-initial if and only if c is initial in its fiber $p^{-1}(pc)$.*

Proof. We have already shown that if c is p-initial then it is initial in its fiber (65.2), so we only need to prove the converse. Given c which is initial in its fiber, we want to show that for any $y \in C_0$, the map $p'\colon \mathrm{map}_C(c, y) \to \mathrm{map}_D(pc, py)$ induced by p is an equivalence. Since p is an inner fibration the map p' is a Kan fibration (45.3), so by the fiberwise criterion (47.1) it suffices to show that the fiber of p' over any $g\colon pc \to py$ is contractible.

Since p is a Cartesian fibration, we can choose a Cartesian morphism $f\colon x \to y$ such that $pf = g$. Now consider the diagram

$$\begin{tikzcd} \mathrm{map}_C(c,x) \arrow[d, "q_0"'] & \mathrm{map}_{\mathrm{Fun}(\Delta^1,C)}(1_c,f) \arrow[l, "r_c"'] \arrow[r, "r'_c"] \arrow[d, "q"'] & \mathrm{map}_C(c,y) \arrow[d, "p'"'] \\ \mathrm{map}_D(pc,pc) & \mathrm{map}_{\mathrm{Fun}(\Delta^1,D)}(1_{pc},g) \arrow[l, "r_{pc}"] \arrow[r, "r'_{pc}"'] & \mathrm{map}_D(pc,py) \end{tikzcd}$$

in which both squares are homotopy pullbacks by (66.13), and all three vertical maps are Kan fibrations (45.3) since they are induced by inner fibrations between quasicategories. Suppose there exists an object h of $\mathrm{map}_{\mathrm{Fun}(\Delta^1,D)}(1_{pc},g)$ such that $r_{pc}(h) = 1_{pc}$ and $r'_{pc}(h) = g$. Then we get induced maps on fibers

$$q_0^{-1}(1_{pc}) \leftarrow q^{-1}(h) \to p'^{-1}(g)$$

which are weak equivalences. Furthermore, the object $q_0^{-1}(1_{pc})$ is exactly the mapping space $\mathrm{map}_{p^{-1}(pc)}(c,x)$ in the fiber category, which is contractible by the hypothesis that c is initial in its fiber. Thus $p'^{-1}(g)$, the fiber of p' over g, is also contractible as desired.

It is not hard to produce such an h, which will correspond to a map $\Delta^1 \times \Delta^1 \to D$ which can be pictured as

$$\begin{tikzcd} pc \arrow[r, "1_{pc}"] \arrow[d, "1_{pc}"'] \arrow[dr] & pc \arrow[d, "g"] \\ pc \arrow[r, "g"'] & py \end{tikzcd}$$

so that the labels on edges and vertices correspond to the constraints on h. For instance, we can build such a map by sending both of the two pictured 2-cells to the degenerate cell $g_{001} \in D_2$.

$\square$

68. Relative initial and terminal objects for pullback-hom maps

We need to be able to produce examples of Cartesian and coCartesian morphisms in certain cases. Since such morphisms can be characterized in tems of relative terminal or inital objects, we start with a method for producing these. **F 15 Apr**

68.1. **Relative terminal objects for pullback-hom maps.** We will find examples of relative terminal objects using the following.

68.2. **Lemma.** *Let $p\colon E \to B$ be an inner fibration of quasicategories. In any commutative diagram*

$$\begin{tikzcd} \Delta^m \times \partial\Delta^n \cup \partial\Delta^m \times \Delta^n \arrow[r, "r"] \arrow[d, tail, "i"'] & E \arrow[d, "p"] \\ \Delta^m \times \Delta^n \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

with $m \geq 0$ and $n \geq 1$, if $r(m,n) \in B_0$ is a p-terminal object, then a lift exists.

Proof. Consider the skeletal filtration of i. The non-degenerate k-cells of $\Delta^m \times \Delta^n$ not in the image of i are the pairs (a,b) where $a\colon [k] \to [m]$ and $b\colon [k] \to [n]$ are both surjective functions. In particular $k \geq 1$ since $n \geq 1$. The restriction

$$\partial\Delta^k \xrightarrow{(a,b)|\partial\Delta^k} \Delta^m \times \partial\Delta^n \cup \partial\Delta^m \times \Delta^n \xrightarrow{u} C$$

to the boundary of Δ^k sends the vertex k to the object $r(m,n) \in E_0$, which is p-terminal by hypothesis. Thus we can inductively construct a lift cell-by-cell along the skeletal filtration of i. $\square$

68.3. *Remark.* Consider the pullback-hom map

$$q = p^{\square(\partial\Delta^m \subset \Delta^m)} \colon \operatorname{Fun}(\Delta^m, E) \to \operatorname{Fun}(\partial\Delta^m, E) \times_{\operatorname{Fun}(\partial\Delta^m, B)} \operatorname{Fun}(\Delta^m, B).$$

The lifting problem described in (68.2) amounts to asserting that a lift exists in

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "r'"] \arrow[d, tail] & \operatorname{Fun}(\Delta^m, E) \arrow[d, "q"] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & \operatorname{Fun}(\partial\Delta^m, E) \times_{\operatorname{Fun}(\partial\Delta^m, B)} \operatorname{Fun}(\Delta^m, B)
\end{tikzcd}$$

whenever $n \geq 1$ and $r'(n) \in E_0$ is p-terminal. Thus, it implies that if $f \colon \Delta^m \to E$ is such that $f(m) \in E_0$ is p-terminal, then f corresponds to a q-terminal object of $\operatorname{Fun}(\Delta^m, E)$.

Also note that if $m \geq 1$, any object (u, v) in the target of q with the property that $u(m) \in E_0$ is p-terminal is the image of some p-terminal object f in the domain of q, as $u(m)$ being p-terminal is precisely what is needed to find a solution f to the lifting problem (u, v), and such a solution necessarily satisfies $f(m) = u(m)$ and so is q-terminal by the above remarks.

We will apply the above observation to produce relative terminal objects for more general kinds of pullback-hom maps. Let $i \colon K \to L$ be a monomorphism of simplicial sets. Say that a vertex in L is ***i*-right critical** if it is of the form a_n for some $n \geq 0$ and some $a \in L_n^{\mathrm{nd}} \smallsetminus i(K_n^{\mathrm{nd}})$. Likewise, a vertex is ***i*-left critical** if it is of the form a_0 for some $n \geq 0$ and some $a \in L_n^{\mathrm{nd}} \smallsetminus i(K_n^{\mathrm{nd}})$. Note that the sets of i-right critical and i-left critical vertices include every vertex of the complement $L_0 \smallsetminus i(K_0)$. The point of the definition is that, because of the skeletal filtration of i, we obtain L from K by succesively attaching cells along maps $\partial\Delta^k \to L$ such that the final vertex k is always sent to an i-right critical vertex, and the intial vertex 0 is always sent to an i-left critical vertex.

***i*-right critical**
***i*-left critical**

68.4. *Example.* Consider the inclusions $K \xrightarrow{i} K \star L \xleftarrow{j} L$ into the factors of the join. Then:

- all vertices are i-left critical and j-right critical,
- the i-right critical vertices are precisely those of L, and
- the j-left critical vertices are precisely those of K.

If instead we consider the inclusion $\gamma \colon K \amalg L \to K \star L$, then the γ-left critical vertices are precisely those in K, and the γ-right critical vertices are precisely those in L.

The following gives a condition for detecting relative terminal objects in pullback-hom maps, via "evaluation at i-right critical vertices".

68.5. **Proposition.** *Let $p \colon E \to B$ be an inner fibration of quasicategories, $i \colon K \to L$ a monomorphism of simplicial sets, and write $q := p^{\square i} \colon \operatorname{Fun}(L, E) \to \operatorname{Fun}(K, E) \times_{\operatorname{Fun}(K,B)} \operatorname{Fun}(L, B)$ for the pullback-hom map.*

Consider a map $f \colon L \to E$, regarded as an object of $\operatorname{Fun}(L, E)$. If for each i-right critical vertex $s \in L_0$ the object $f(s) \in E_0$ is p-terminal, then f is q-terminal.

Proof. To show that f is q-terminal, we need to produce a lift in every commutative square of the form

$$\begin{tikzcd}
\{n\} \arrow[r, tail] \arrow[rr, bend left, "f"] & \partial\Delta^n \arrow[r, "u"] \arrow[d, tail] & \operatorname{Fun}(L, E) \arrow[d, "q"] \\
& \Delta^n \arrow[r, "(v,w)"'] \arrow[ur, dotted] & \operatorname{Fun}(K, E) \times_{\operatorname{Fun}(K,B)} \operatorname{Fun}(L, B)
\end{tikzcd}$$

with $n \geq 1$ such that $u(n) = f$, using the hypothesis on f which says that $u(n)(s) = f(s) \in E_0$ is p-terminal for each i-right critical vertex $s \in L_0$. Such a lifting problem is equivalent to one of the form

$$\begin{tikzcd}
K \arrow[r, "\widetilde{v}"] \arrow[d, tail, "i"'] & \mathrm{Fun}(\Delta^n, E) \arrow[d, "q'"] \\
L \arrow[r, "(\widetilde{u},\widetilde{w})"'] \arrow[ur, dotted] & \mathrm{Fun}(\partial\Delta^n, E) \times_{\mathrm{Fun}(\partial\Delta^n, B)} \mathrm{Fun}(\Delta^n, B)
\end{tikzcd}$$

with the property that for each i-right critical vertex $s \in L_0$, the object $\widetilde{u}(s)(n) \in E_0$ is p-terminal. We construct such a lift inductively along the skeletal filtration of i (20.3), and thus reduce to the special case when i is the inclusion $\partial\Delta^m \to \Delta^m$ for some $m \geq 0$, and we have that $\widetilde{u}(m)(n) \in E_0$ is p-terminal, exactly because for any $\sigma \in L_m^{\mathrm{nd}} \setminus i(K_m^{\mathrm{nd}})$ we have that $\sigma(m) \in L_0$ is i-right critical. This special case is in turn equivalent to the lifting problem solved by the lemma (68.2). $\square$

The following uses the same idea to produce examples of relative terminal objects in pullback-hom maps.

68.6. **Proposition.** *Let $p\colon E \to B$ be an inner fibration of quasicategories, $i\colon K \to L$ a monomorphism of simplicial sets, and write $q := p^{\square i}\colon \mathrm{Fun}(L, E) \to \mathrm{Fun}(K, E) \times_{\mathrm{Fun}(K,B)} \mathrm{Fun}(L, B)$ for the pullback-hom map.*

Consider an object (u, v) of $\mathrm{Fun}(K, E) \times_{\mathrm{Fun}(K,B)} \mathrm{Fun}(L, B)$. If

(a) *u takes every i-right critical vertex in $i(K)$ to a p-terminal object of E, and*
(b) *v takes every vertex in $L \setminus i(K)$ to an object of B which is the image of some p-terminal object of E,*

then there exists a q-terminal object f in $\mathrm{Fun}(L, E)$ such that $q(f) = (u, v)$.

Proof. We will construct a lift in

$$\begin{tikzcd}
K \arrow[r, "u"] \arrow[d, tail, "i"'] & E \arrow[d, "p"] \\
L \arrow[r, "v"'] \arrow[ur, dotted, "f"] & B
\end{tikzcd}$$

with the property that $f(s) \in E_0$ is p-terminal for each right i-critical vertex $s \in L_0$. By (68.5) this lift wil correspond to the desired q-terminal object of $\mathrm{Fun}(L, E)$.

We construct such a lift inductively along the skeletal filtration of $i\colon K \to L$ (20.3). Thus, we can reduce to the case when i is a cell inclusion $\partial\Delta^m \to \Delta^m$. When $m = 0$, the existence of such a lift taking value at a p-terminal object is just hypothesis (b), while when $m > 0$, the existence of such a lift sending the vertex $m \in (\Delta^m)_0$ to a p-terminal object is given by (a). $\square$

68.7. *Example.* Let C be a quasicatategory, and conside the projection functor $q\colon C \times C \to C$ to the second factor. If (x, y) is an object of $C \times C$, then (x, y) is q-terminal if and only if x is terminal in C. The if direction follows from the proposition (68.5) applied to $p\colon C \to *$ and $i\colon \{1\} \to \{0, 1\}$, since in this case $q = p^{\square i}$, while the only if direction follows because q-terminal objects are always terminal in their fiber (65.2).

68.8. *Example.* Let C be a quasicategory, let $i\colon \partial\Delta^1 \to \Delta^1$, and consider the restriction functor $q = i^*\colon \mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\partial\Delta^1, C) = C \times C$. Suppose $f\colon x \to y$ is a morphism of C, considered as an object of $\mathrm{Fun}(\Delta^1, C)$.

Then if y is terminal in C then f is q-terminal, and if x is initial in C then f is q-initial.

Note that the converse is not true. For instance, if C is a poset, then you can show that every object of $\mathrm{Fun}(\Delta^1, C)$ is q-terminal.

69. Cartesian and coCartesian morphisms for pullback-hom maps

69.1. **Pullback-hom-map criterion for (co)Cartesian morphisms.** As we have seen, coCartesian/Cartesian morphisms correspond to relative intial/terminal objects for slice restrictions (66.9). In fact, they also correspond to relative initial/terminal objects for pullback-hom maps.

69.2. **Proposition.** *Let $p\colon E \to B$ be an inner fibration of quasicategories, and let $f\colon x \to y$ be a morphism in E. Then*

1. *f is p-coCartesian if and only if f corresponds to a q-initial object in $q\colon \mathrm{Fun}(\Delta^1, E) \to \mathrm{Fun}(\{0\}, E) \times_{\mathrm{Fun}(\{0\},B)} \mathrm{Fun}(\Delta^1, B)$, where $q = p^{\square(\{0\}\subset\Delta^1)}$ is the pullback-hom map, and*
2. *f is p-Cartesian if and only if f corresponds to a q'-terminal object in $q'\colon \mathrm{Fun}(\Delta^1, E) \to \mathrm{Fun}(\{1\}, E) \times_{\mathrm{Fun}(\{1\},B)} \mathrm{Fun}(\Delta^1, B)$, where $q' = p^{\square(\{1\}\subset\Delta^1)}$ is the pullback-hom map.*

Proof. For the if direction we consider case (2), where we have a commutative diagram

$$\begin{tikzcd}
E^{/y} \arrow[r, "\overline{q}"] \arrow[d, tail] & B^{/py} \arrow[r] \arrow[d, tail] & \{y\} \arrow[d, tail] \\
\mathrm{Fun}(\Delta^1, E) \arrow[r, "q"'] & \mathrm{Fun}(\{1\}, E) \times_{\mathrm{Fun}(\{1\},B)} \mathrm{Fun}(\Delta^1, B) \arrow[r] & E
\end{tikzcd}$$

in which both squares are pullbacks. The morphism f corresponds to an object $\widetilde{f}$ of $E^{/y}$ which maps to the corresponding vertex $\widetilde{f}'$ of $\mathrm{Fun}(\Delta^1, E)$. By (66.9), it suffices to show that $\widetilde{f}$ is $\overline{q}$-terminal, and this is immediate from the fact that $\overline{q}$ is a basechange of q (65.2).

For either (1) or (2), the only-if direction is immediate from (69.3) below. □

69.3. **Proposition.** *Let $p\colon E \to B$ be an inner fibration of simplicial sets, and f an edge in E. We consider commutative diagrams of the form*

$$\begin{tikzcd}
\Delta^1 \times \{y\} \arrow[r, tail] \arrow[rr, bend left, "f"] & (\{x\} \times \Delta^n) \cup_{\{x\}\times\partial\Delta^n} (\Delta^1 \times \partial\Delta^n) \arrow[r] \arrow[d, tail] & E \arrow[d, "p"] \\
& \Delta^1 \times \Delta^n \arrow[r] & B
\end{tikzcd}$$

1. *If f represents a p-coCartesian edge, then a lift exists in every such diagram with $n \geq 0$ and $(x, y) = (0, 0)$.*
2. *If f represents a p-Cartesian edge, then a lift exists in every such diagram with $n \geq 0$ and $(x, y) = (1, n)$.*

Proof. This is a calculation, given in the appendix (78.7). In fact, the proof is an immediate generalization of the proof of the pushout-product form of Joyal lifting (36.6). □

To show that an inner fibration $q\colon E \to B$ is a Cartesian fibration, we need to be able to solve lifting problems of the form

$$\begin{tikzcd}
\{1\} \arrow[r, "u"] \arrow[d, tail] & E \arrow[d, "q"] \\
\Delta^1 \arrow[r, "v"'] \arrow[ur, dotted, "s"] & B
\end{tikzcd}$$

so that the lift s represents a q-Cartesian edge. This admits a nice reformulation in terms of relative initial or terminal objects.

69.4. **Lemma.** *Let $q\colon E \to B$ be an inner fibration of quasicategories. In a lifting problem (u, v) as above* (69.1)*, lifts s representing a q-Cartesian edge are in bijective correspondence with r-terminal objects which map to (u, v), where*

$$r := q^{\square(\{1\}\subset\Delta^1)}\colon \operatorname{Fun}(\Delta^1, E) \to \operatorname{Fun}(\{1\}, E) \times_{\operatorname{Fun}(\{1\},B)} \operatorname{Fun}(\Delta^1, B).$$

In particular, q is a Cartesian fibration if and only if every vertex of the target of r is the image of some r-terminal object in the source.

Proof. Straightforward: a vertex s such that $r(s) = (u, v)$ is precisely a lift for the lifting problem, and we have seen that s is r-terminal if and only if it corresponds to a q-Cartesian morphism (69.2) □

69.5. **Sufficient criteria for Cartesian morphisms in pullback-hom maps.** We can now give the following sufficient criterion for a morphism to be Cartesian for a pullback-hom map, in terms of "restriction to right-critical vertices".

69.6. **Proposition.** *Let $p\colon C \to D$ be an inner fibration of quasicategories, and let $i\colon K \to L$ be a monomorphism of simplicial sets. Write $q := p^{\square i}\colon \operatorname{Fun}(L, C) \to \operatorname{Fun}(K, C) \times_{\operatorname{Fun}(K,D)} \operatorname{Fun}(L, D)$ for the pullback-hom map.*

If $\alpha\colon f \to f'$ is a morphism in $\operatorname{Fun}(L, C)$ such that for each i-right critical vertex $s \in L_0$, the restriction $\alpha(s)\colon f(s) \to f'(s)$ is a p-Cartesian morphism in C, then α is a q-Cartesian morphism.

Proof. Write $q' := p^{\square(\{1\}\subset\Delta^1)}\colon \operatorname{Fun}(\Delta^1, C) \to \operatorname{Fun}(\{1\}, C) \times_{\operatorname{Fun}(\{1\},D)} \operatorname{Fun}(\Delta^1, D)$. Observe that since $q = p^{\square i}$ is itself a pullback-hom map, we have isomorphisms of maps

$$q^{\square(\{1\}\subset\Delta^1)} \approx p^{i\square(\{1\}\subset\Delta^1)} \approx q'^{\square i}.$$

We are going to make several uses of (69.2), which says that q-Cartesian morphisms correspond to $q^{\square(\{1\}\subset\Delta^1)}$-terminal objects, where q is a general inner fibration of quasicategories.

For instance, we see that our α is q-Cartesian if and only if it corresponds to a relative terminal object for $r := q^{\square(\{1\}\subset\Delta^1)}$, which is isomorphic to $r' := q'^{\square i}$. Thus it suffices to apply the criterion of (68.5) to the object $\widetilde{\alpha}$ of $\operatorname{Fun}(L, \operatorname{Fun}(\Delta^1, C))$ (the domain of r') which corresponds to the object α of $\operatorname{Fun}(\Delta^1, \operatorname{Fun}(L, C))$. This criterion asserts that $\widetilde{\alpha}$ is r'-terminal if for every i-right critical vertex $s \in L_0$, the object $\widetilde{\alpha}(s)$ of $\operatorname{Fun}(\Delta^1, C)$ is q'-terminal. Or in other words, again (69.2), we need to show that the edge of C represented by the map $\widetilde{\alpha}(s)\colon \Delta^1 \to C$ is p-Cartesian. This is precisely the hypothesis we are given. □

The following gives a sufficient criterion for lifting morphisms to Cartesian morphisms for pullback-hom maps. **M 18 Apr**

69.7. **Proposition.** *Let $p\colon C \to D$ be an inner fibration of quasicategories, and let $i\colon K \to L$ be a monomorphism of simplicial sets. Write $q := p^{\square i}\colon \operatorname{Fun}(L, C) \to \operatorname{Fun}(K, C) \times_{\operatorname{Fun}(K,D)} \operatorname{Fun}(L, D)$ for the pullback-hom map.*

Consider a commutative square

$$\begin{tikzcd} \{1\} \arrow[r, "u"] \arrow[d, tail] & \operatorname{Fun}(L, C) \arrow[d, "q"] \\ \Delta^1 \arrow[r, "{(v,w)}"'] \arrow[ur, dotted, "t"] & \operatorname{Fun}(K, C) \times_{\operatorname{Fun}(K,D)} \operatorname{Fun}(L, D) \end{tikzcd}$$

Suppose

(a) *$v(s) \in C_1$ is p-Cartesian for each i-right critical vertex $s \in K_0$, and*

(b) *for each vertex s in $L \smallsetminus i(K)$, there exists a lift $t(s)$ in*

$$\begin{tikzcd} \{1\} \arrow[r, "u(s)"] \arrow[d, tail] & C \arrow[d, "p"] \\ \Delta^1 \arrow[r, "w(s)"'] \arrow[ur, dotted, "t(s)"] & D \end{tikzcd}$$

which represents a p-Cartesian morphism.

Then there exists a lift t in the diagram above which represents a q-Cartesian morphism.

Proof. As in the proof of (69.7), we refer to an adjunction of lifting problems. A lift t in the square of the proposition is equivalent to a lift $\widetilde{t}$ in the square

$$\begin{tikzcd} K \arrow[r, "\widetilde{v}"] \arrow[d, tail] & \mathrm{Fun}(\Delta^1, C) \arrow[d, "q'"] \\ L \arrow[r, "(\widetilde{u},\widetilde{w})"'] \arrow[ur, dotted, "\widetilde{t}"] & \mathrm{Fun}(\{1\}, C) \times_{\mathrm{Fun}(\{1\},D)} \mathrm{Fun}(\Delta^1, D) \end{tikzcd}$$

where $q' = p^{\square(\{1\}\subset\Delta^1)}$. Furthermore, t represents a q-Cartesian morphism if and only if $\widetilde{t}$, regarded as a vertex of $\mathrm{Fun}(L, \mathrm{Fun}(\Delta^1, C))$, is an r'-terminal vertex for $r' = q'^{\square i}$. According to the criterion of (68.6), to show that the desired lift $\widetilde{t}$ exists, we should show

(a) $\widetilde{v}$ takes each i-right critical vertex in $i(K)$ to a q'-terminal object of $\mathrm{Fun}(\Delta^1, C)$, and

(b) $(\widetilde{u}, \widetilde{w})$ takes every vertex in $L \smallsetminus i(K)$ to an object of $\mathrm{Fun}(\{1\}, C) \times_{\mathrm{Fun}(\{1\},D)} \mathrm{Fun}(\Delta^1, D)$ which is the image of some q'-terminal object of $\mathrm{Fun}(\Delta^1, C)$.

Undwinding this we precisely recover the hypotheses of the proposition. □

This criterion is easiest to state when there are no right critical vertices in the subcomplex.

69.8. **Corollary.** *Let $p\colon C \to D$ be a Cartesian fibration of quasicategories, and let $i\colon K \to L$ be a monomorphism of simplicial sets such that K contains no i-right critical vertices. Then $q = p^{\square i}\colon \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L, D)$ is a Cartesian fibration. Furthermore, any morphism $\alpha \in \mathrm{Fun}(L, C)_1$ such that $\alpha(s)$ is p-Cartesian for all $s \in L_0 \smallsetminus i(K_0)$ is q-Cartesian.*

69.9. **Restriction functors which are (co)Cartesian fibrations.** For the functor $p\colon C \to *$, the p-Cartesian morphisms are precisely the isomorphisms in C, and such a p is always a Cartesian fibration. So we obtain the following.

69.10. **Proposition.** *Let C be a quasicategory and $i\colon K \to L$ a monomorphism of simplicial sets, and write*

$$q = i^*\colon \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C)$$

for the restriction functor. If $\alpha \in \mathrm{Fun}(L, C)_1$ is a morphism such that $\alpha(s)$ is an isomorphism in C for any i-right critical vertex $s \in K_0$, then u is q-Cartesian. Furthermore, if K contains no i-right critical vertices, then q is a Cartesian fibration.

69.11. *Example.* Consider $q = i^*\colon \mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\{0\}, C)$, restriction along the inclusion $\{0\} \subset \Delta^1$. The only i-right critical vertex is 1, which is not in the image of i. Thus q is a Cartesian fibration, and any morphism $f \in \mathrm{Fun}(\Delta^1, C)_1$ such that $f(1)$ is an isomorphism is q-Cartesian.

69.12. **Proof of functoriality of colimits.** We can now complete the proof of functoriality of colimits.

69.13. **Theorem.** *Let C be a quasicategory and S a simplicial set. Consider the commutative square*

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{colim}}(S^{\triangleright}, C) \arrow[r, tail] \arrow[d, "p'"'] & \mathrm{Fun}(S^{\triangleright}, C) \arrow[d, "p"] \\ \mathrm{Fun}^{\exists\,\mathrm{colim}}(S, C) \arrow[r, tail] & \mathrm{Fun}(S, C) \end{tikzcd}$$

where the quasicategories on the left-hand side are the full subcategories spanned respectively by colimit cones and by functors which admit colimit cones. Then p' is a trivial fibration, and thus there exists a colimit cone functor.

Proof. The only right-critical vertex of the inclusion $S \subset S^{\triangleright}$ is the cone point. Therefore p is a Cartesian fibration by (69.10). The colimit cones are precisely the vertices of $\mathrm{Fun}(S^{\triangleright}, C)$ which are initial in their fiber along p (64.2). By (67.3), these are precisely the p-initial vertices. Then (65.5) applies to show that p' is a trivial fibration, as desired. $\square$

70. Cartesian and coCartesian path fibrations

Given a functor $f\colon C \to D$ (not necessarily an inner fibration) between quasicategories, we obtain functors

$$q\colon\ \mathrm{Fun}(\{0\}, C) \times_{\mathrm{Fun}(\{0\},D)} \mathrm{Fun}(\Delta^1, D) \to \mathrm{Fun}(\{1\}, D)$$

and

$$q'\colon\ \mathrm{Fun}(\{1\}, C) \times_{\mathrm{Fun}(\{1\},D)} \mathrm{Fun}(\Delta^1, D) \to \mathrm{Fun}(\{0\}, D),$$

where all maps are induced in the evident way by f and by restriction to subcomplexes of Δ^1. I will call these the **coCartesian path fibration** and **Cartesian path fibration** of f respectively. This terminology will be justified soon. **coCartesian path fibration** **Cartesian path fibration**

70.1. *Remark.* Note that the fibers of q and q' over an object $d \in D_0$ are isomorphic to the pullbacks $C \times_D D^{/d}$ and $C \times_D D^{d/}$ of the alternate slices along f.

70.2. **Lemma.** *Let $f\colon C \to D$ be a functor between quasicategories. Then the coCartesian path fibration and Cartesian path fibration associated to f are inner fibrations.*

Proof. We prove the case of the coCartesian path fibration q. Consider the commutative diagram

$$\begin{tikzcd} (C \times D) \times_{D\times D} \mathrm{Fun}(\Delta^1, D) \arrow[r, "r'"] \arrow[d] & C \times D \arrow[r, "\pi"] \arrow[d, "f\times\mathrm{id}_D"] & D \\ \mathrm{Fun}(\Delta^1, D) \arrow[r, "r"'] & D \times D & \end{tikzcd}$$

in which r is restriction along $\{0,1\} = \partial\Delta^1 \subset \Delta^1$, π is the evident projection, and the square is a pullback. Then both π and r, and hence r', are inner fibrations. The claim follows because q is isomorphic to the map $\pi r'$. $\square$

70.3. **Lemma.** *Let $f\colon C \to D$ be a functor between quasicategories, and consider the coCartesian path fibration*

$$q\colon\ \mathrm{Fun}(\{0\}, C) \times_{\mathrm{Fun}(\{0\},D)} \mathrm{Fun}(\Delta^1, D) \to \mathrm{Fun}(\{1\}, D) = D.$$

If $\alpha\colon \Delta^1 \to \mathrm{Fun}(\{0\}, C) \times_{\mathrm{Fun}(\{0\},D)} \mathrm{Fun}(\Delta^1, D)$ is such that the composite

$$\Delta^1 \xrightarrow{\alpha} \mathrm{Fun}(\{0\}, C) \times_{\mathrm{Fun}(\{0\},D)} \mathrm{Fun}(\Delta^1, D) \xrightarrow{\pi'} \mathrm{Fun}(\{0\}, C) = C$$

represents an isomorphism in C, then α is a q-coCartesian morphism.

Proof. Here is a picture of α:

$$\begin{tikzcd} x_0 \arrow[d, "u"'] \\ y_0 \end{tikzcd} \quad \text{in } C, \qquad \begin{tikzcd} fx_0 \arrow[r] \arrow[d, "fu"'] & y_0 \arrow[d, "v"] \\ fx_1 \arrow[r] & y_1 \end{tikzcd} \quad \text{in } D.$$

The hypothesis on α is that u is an isomorphism in C.

We again refer to the commutative diagram

$$\begin{tikzcd} (C \times D) \times_{D\times D} \mathrm{Fun}(\Delta^1, D) \arrow[r, "r'"] \arrow[d] & C \times D \arrow[r, "\pi"] \arrow[d, "f\times \mathrm{id}_D"] & D \\ \mathrm{Fun}(\Delta^1, D) \arrow[r, "r"'] & D \times D & \end{tikzcd}$$

in which $\pi r'$ is isomorphic to q. To show that α is q-coCartesian, by (66.4) it suffices to show that (i) $r'(\alpha) = (f, g)$ is π-coCartesian, and (ii) α is r'-coCartesian. Since f is an isomorphism, (i) is immediate from (66.3). For (ii), since r' is a basechange of r, it suffices by (66.5) to show that the edge represented by $fu\colon \Delta^1 \to D$ is r-coCartesian. (66.5), which is the following lemma (70.4). $\square$

70.4. **Lemma.** *Let D be a quasicategory, and consider the restriction functor $r\colon \mathrm{Fun}(\Delta^1, D) \to \mathrm{Fun}(\partial\Delta^1, D) \approx D \times D$. If $\beta\colon f \to g$ is a morphism in $\mathrm{Fun}(\Delta^1, D)$ such that f is an isomorphism in D, then β is r-coCartesian, while if g is an isomorphism in D, then β is r-Cartesian.*

Proof. We use the criterion of (69.10). For instance, since 1 is the only i-right critical vertex of $i\colon \partial\Delta^1 \to \Delta^1$, we have that β is r-coCartesian if and only if g is π-coCartesian for $\pi\colon D \to *$, i.e., if and only if f is an isomorphism. $\square$

70.5. **Proposition.** *Let $p\colon C \to D$ be a functor between quasicategories. Then the coCartesian path fibration $q\colon \mathrm{Fun}(\{0\}, C) \times_{\mathrm{Fun}(\{0\},D)} \mathrm{Fun}(\Delta^1, D) \to \mathrm{Fun}(\{1\}, D) = D$ of p is a coCartesian fibration, and the Cartesian path fibration $q'\colon \mathrm{Fun}(\{1\}, C) \times_{\mathrm{Fun}(\{1\},D)} \mathrm{Fun}(\Delta^1, D) \to \mathrm{Fun}(\{0\}, D) = D$ is a Cartesian fibration.*

Proof. The map q is an inner fibration (70.2). To show that q is a coCartesian fibration, suppose given $(x_0 \in C_0, \alpha_0\colon fx_0 \to y_0 \in D_1)$ (i.e., an object of the domain of q), and a map $(v\colon y_0 \to y_1) \in D_1$. We want to lift v along q to a q-coCartesian morphism in C with source x_0. By (70.3), it suffices to fill in the dotted part of the picture

$$\begin{tikzcd} x_0 \arrow[d, dotted, "u"'] \\ y_0 \end{tikzcd} \quad \text{in } C, \qquad \begin{tikzcd} fx_0 \arrow[r, "\alpha_0"] \arrow[d, dotted, "fu"'] \arrow[dr, dotted] & y_0 \arrow[d, "v"] \\ fx_1 \arrow[r, dotted] & y_1 \end{tikzcd} \quad \text{in } D.$$

where u is some isomorphism in C. We can do this easily: for instance, set $u = \mathrm{id}_{x_0}$, and extend the induced map $(\partial\Delta^1 \times \Delta^1) \cup (\Delta^1 \times \{0\}) \to D$ to the two non-degenerate 2-cells in the square using extension along the inner horn $\Delta^2_1 \subset \Delta^2$ followed by extenison along $\Delta^2_0 \subset \Delta^2$, which exists by Joyal extension (34.2). $\square$

Part 12. Adjoint functors

In ordinary category theory, *adjunction* is often regarded as a relation between pairs of functors $f\colon C \leftrightarrows D : g$. In fact, an adjunction involves a choice of additional structure, namely a choice of isomorphisms $\alpha_{c,d}\colon \hom_C(c, g(d)) \xrightarrow{\sim} \hom_D(f(c), d)$ for all pairs of objects, which must fit together to give a natural isomorphism of functors $C^{\mathrm{op}} \times D \to \mathrm{Set}$.

Another point of view is that being an adjoint is a *property* of a functor. For instance, one may say that $f\colon C \to D$ is a *left adjoint* if for every object d of D, the functor

$$C^{\mathrm{op}} \to \mathrm{Set}, \qquad c \mapsto \hom_C(f(c), d)$$

is representable by some object of C. One shows that if f has this property, then you can construct a functor g and natural isomorphism α as above: in fact, once you pick for each object d in D an object $g(d)$ of C representing $\hom_C(f(-), d)$, all other choices are forced.

We cannot replicate the definition of adjoint pair of functors in quasicategories using the theory we have so far, as this requires a notion of mapping space which is *fully functorial*, in the sense that for any quasicategory C we get a functor $C^{\mathrm{op}} \times C \to \mathcal{S}$, where $\mathcal{S}$ is a suitable quasicategory of ∞-groupoids. This can be done, but it requires a lot more work.

However, we can replicate the property of being of adjoint, by means of a trick: it turns out that the functor $\hom_C(f(-), d)\colon C^{\mathrm{op}} \to \mathrm{Set}$ is representable if and only if the projection map $C \times_D D_{/d} \to C$ is a right fibration, and this are notions which we can make sense for in qusiacategories.

70.6. *Exercise.* Let $f\colon C \to D$ be a functor of 1-categories, d an object of D, and let $q\colon C \times_D D_{/d} \to C$ be the evident projection map, where the fiber product is the pullback of slice-restriction $D_{/d} \to D$ along f. Show that $\hom_C(f(-), d)$ is representable by some object c in C if and only if d is isomorphic (as an object of $s\mathrm{Set}_{/D}$) to the slice-restriction $C_{/c} \to C$.

71. Representable fibrations

A **representable fibration** is a right fibration $p\colon E \to C$ between quasicategories such that E has a terminal object. Likewise, a **corepresentable fibration** is a left fibration $p\colon E \to C$ between quasicategories such that E has an initial object.

representable fibration

corepresentable fibration

71.1. *Example.* The slice restrictions $C_{/c} \to C$ and $C^{/c} \to C$ are representable fibrations for any object c of a quasicategory C, since the vertex corresponding to 1_c is respectively terminal or initial in the slices (35.9).

Maps between representable fibrations are equivalences exactly when they preserve the terminal object.

71.2. **Proposition.** *Consider a diagram*

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & C & \end{tikzcd}$$

where C is a quasicategory, p is a representable fibrations, and p' is a right fibration. The functor f is an equivalence if and only if it takes some terminal object of E to a terminal object of E'. When this is the case, p' is also a representable fibration.

Proof. One direction is clear: any equivalence preserves terminal objects (43.5). In particular, if f is an equivalence then p' is also a representable fibration.

It remains to show that if f sends a terminal object $t \in E_0$ to a terminal object $t' = f(t) \in E'_0$, then f is a categorical equivalence. I will show that f is essentially surjective and fully faithful, so that the "fundamental theorem" applies (48.2).

First we show f is essentially surjective. Suppose $x' \in E'_0$ is any object. Since t' is terminal in E' there exists a morphism $\alpha'\colon x' \to t'$ in E'. Since p is a right fibration we have $(\{1\} \subset \Delta^1) \boxslash p$, so we can find $\alpha\colon x \to t$ with $p(\alpha) = p'(\alpha')$. From this we obtain a commutative diagram

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "u"] \arrow[d, tail] & E' \arrow[d, "p"] \\ \Delta^2 \arrow[r, "v"'] \arrow[ur, dotted, "s"] & C \end{tikzcd}$$

so that $v = p(\alpha'_{001})$ and u represents the solid arrows of the diagram

$$\begin{tikzcd} & x' \arrow[dr, "\alpha'"] & \\ f(x) \arrow[ur, dotted, "\beta"] \arrow[rr, "f(\alpha)"'] & & t' \end{tikzcd}$$

in E. Since p is a right fibration a lift s exists. Then $\beta := s_{01}\colon f(x) \to x'$ is a morphism contained in a fiber of the right fibration p' over the vertex $p(x) = p(x')$, so is an isomorphism since the fiber is a Kan complex. Thus we have shown that f is essentially surjective.

To show that f is fully faithful, consider any $x, y \in E_0$, and choose any map $\alpha\colon y \to t$, which exists because t is terminal. Then we have a commutative diagram

$$\begin{tikzcd} \mathrm{map}_E(x,y) \arrow[d, "f"] & \mathrm{map}_{\mathrm{Fun}(\Delta^1,E)}(1_x,\alpha) \arrow[l, "\sim"'] \arrow[r] \arrow[d, "f"] & \mathrm{map}_E(x,t) \arrow[d, "f", "\sim"'] \\ \mathrm{map}_{E'}(fx,fy) \arrow[d, "p'"] & \mathrm{map}_{\mathrm{Fun}(\Delta^1,E')}(1_{fx},f\alpha) \arrow[l, "\sim"'] \arrow[r] \arrow[d, "p'"] & \mathrm{map}_{E'}(fx,t') \arrow[d, "p'"] \\ \mathrm{map}_C(px,py) & \mathrm{map}_{\mathrm{Fun}(\Delta^1,C)}(1_{px},p\alpha) \arrow[l, "\sim"'] \arrow[r] & \mathrm{map}_C(px,pt) \end{tikzcd}$$

in which:

- The horizontal maps in the left-hand column are equivalences (66.11).
- The lower right square is a homotopy pullback, since the morphism $f\alpha$ is p'-Cartesian (since p' is a right fibration) (66.13).
- Likewise, the right-hand vertical rectangle is a homotopy pullback, since the morphism α is p-Cartesian.
- It follows that the upper right square is a homotopy pullback, by patching of homotopy pullbacks (51.10).
- The map $\mathrm{map}_{E'}(x,t) \to \mathrm{map}_E(fx,t')$ induced by f is an equivalence since both spaces are contractible, since t and $t' = f(t)$ are terminal objects.

Since the upper squares are homotopy pullbacks, we can conclude that $\mathrm{map}_E(x,y) \to \mathrm{map}_{E'}(fx,fy)$ induced by f is an equivalence, so f is fully faithful as desired. ☐

72. Slice restrictions as representable fibrations

Next we will show that all representable fibrations are equivalent to slice restrictions. The key is the following lemma, which we will prove below.

72.1. **Lemma.** *Let $p\colon E \to C$ be a right fibration of simplicial sets. For any vertex $e \in E$, there exists a commutative diagram*

$$\begin{tikzcd} C_{/c} \arrow[rr, "g"] \arrow[dr, "\pi"'] & & E \arrow[dl, "p"] \\ & C & \end{tikzcd}$$

such that $g(1_c) = e$, where π is the slice projection.

72.2. **Proposition.** *Let $p\colon E \to C$ be a right fibration to a quasicategory. The following are equivalent:*

1. *The map p is a representable fibration (i.e., E has a terminal object).*
2. *There exists an object $c \in C_0$ and a commutative diagram*

$$\begin{tikzcd} C_{/c} \arrow[rr, "g"] \arrow[dr, "\pi"'] & & E \arrow[dl, "p"] \\ & C & \end{tikzcd}$$

such that g is an equivalence, where π is the slice restriction.

3. *There exists an object $c \in C_0$ and a commutative diagram*

$$\begin{tikzcd} C^{/c} \arrow[rr, "g"] \arrow[dr, "\pi'"'] & & E \arrow[dl, "p"] \\ & C & \end{tikzcd}$$

such that g is an equivalence, where π' is the slice restriction.

Proof. To show that (1) implies (2), note that by (72.1) we can construct a map g such that $pg = \pi$, and such that $g(1_c)$ is any chosen terminal object of E. Then g is an equivalence by (71.2). To show that (2) implies (1), note that since g is an equivalence it preserves terminal objects (43.5), and thus $g(1_c)$ must be terminal in E since 1_c is terminal in $C_{/c}$ (35.9).

That (2) and (3) are equivalent is immediate from the existence of a commutative diagram

$$\begin{tikzcd} C_{/c} \arrow[r, "g"] \arrow[dr, "\pi"] & C^{/c} \arrow[r, "h"] \arrow[d, "\pi'"] & C_{/c} \arrow[dl, "\pi"] \\ & C & \end{tikzcd}$$

in which both g and h are equivalences. The equivalence g is just the standard equivalence between slice and alternate slice (61.4), while h can be chosen to be a categorical inverse of g which is compatible with the projections π and π', since these are both isofibrations and thus fibrations in the Joyal model structure (**??**). $\square$

Now we turn to the proof of (72.1). It is based on the following.

72.3. **Lemma.** *Let C be any simplicial set, and $c \in C_0$ a vertex. Then the inclusion $\{1_c\} \subset C_{/c}$ is right anodyne, and the inclusion $\{1_c\} \subset C_{c/}$ is left anodyne.*

Proof of (72.1), *using* (72.3). We want a lift in the commutative square

$$\begin{tikzcd} \{1_c\} \arrow[r, "e"] \arrow[d, tail] & E \arrow[d, "p"] \\ C_{/c} \arrow[r, "\pi"'] \arrow[ur, dotted, "g"] & C \end{tikzcd}$$

whose existence is immediate from the fact that i is right anodyne and p is a right fibration. $\square$

72.4. **Contractions of simplicial sets.** We now turn to the proof of (72.3), which relies on the notion of a "contraction" of a simplicial set. Let Δ_R denote the category whose

- objects are totally ordered sets $[n]^R := [n] \sqcup \{R\} = \{0 < 1 < \cdots < n < R\}$ for $n \geq -1$, and
- morphisms are order preserving functions which take the "right basepoint" R to R.

There is an evident "inclusion" functor $\Delta \rightarrowtail \Delta_R$ sending $[n] \mapsto [n]_R$, and extending a simplicial operator $f\colon [m] \to [n]$ by the rule $f(R) := R$. I'll silently identify Δ with the corresponding subcategory of Δ_R.

A **right contraction** of a simplicial set X is a choice of extension of $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ to a functor $X\colon (\Delta_R)^{\mathrm{op}} \to \mathrm{Set}$. There is an evident dually defined category Δ_L and corresponding notion of **left contraction**. **right contraction** **left contraction**

Note that a right contraction of X in particular provides **contraction operators** $Q^*\colon X_{n-1} \to X_n$ for all $n \geq 0$, induced by the surjective function $Q := \langle 0, 1, \dots, n-1, R, R\rangle\colon [n]^R \to [n-1]^R$ (which is not in the subcategory Δ). The functions Q admit sections in Δ_R, and thus the contraction operators are always injective. In particular, we get a monomorphism $Q^*\colon X_{-1} \to X_0$. **contraction operators**

72.5. *Example* (Slices have contractions). The functor $\Delta \to s\mathrm{Set}_*$ defined by $[n] \mapsto (\Delta^n)^{\triangleright}$ manifestly extends to a functor $\Delta_R \to s\mathrm{Set}_*$, given on objects by the rule $[n]^R \mapsto \Delta^{[n]^R} = (\Delta^n)^{\triangleright}$, which is compatible with all order preserving maps in Δ_R. Note that for all $f\colon [m]_R \to [n]_R$ in Δ_R, the induced map $(\Delta^m)^{\triangleright} \to (\Delta^n)^{\triangleright}$ sends the cone point to the cone point.

Given a simplicial set C with vertex $c \in C_0$, the slice $C_{/c}$ is a simplicial set with n-cells corresponding to maps $(\Delta^n)^{\triangleright} \to C$ sending the cone point to c. Thus any such slice admits a canonical right contraction, as the functor $C\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ extends to a functor $\Delta_R^{\mathrm{op}} \to \mathrm{Set}$.

72.6. **Proposition.** *Let X be a simplicial set equipped with a right contraction. Consider the evident inclusion $f\colon S \to X$, where S is the discrete simpicial set with underlying set X_{-1}. Then f is right anodyne. (In fact, f is in the weak saturation of $\{\, \Lambda^n_n \subset \Delta^n \mid n \geq 1\,\}$.)*

Proof. We will make use of the standard notation for simplicial operators:

$$s^i\colon [n] \to [n-1], \quad i = 0, \dots, n-1, \qquad s^i(x) = \begin{cases} x & \text{if } x \leq i, \\ x-1 & \text{if } x > i. \end{cases}$$

$$d^j\colon [n-1] \to [n], \quad j = 0, \dots, n, \qquad d^j(x) = \begin{cases} x & \text{if } x < j, \\ x+1 & \text{if } x \geq j. \end{cases}$$

These extend to give morphisms in Δ_R, so that $s^i(R) = R = d^j(R)$.

Consider the collection X^{nd} of non-degenerate cells of X not contained in S. Partition X^{nd} into disjoint subsets $X^I \amalg X^{II}$, where $X^I = (X^{\mathrm{nd}} \smallsetminus S^{\mathrm{nd}}) \cap XQ$, the set of nondegenerate cells which are in the image of the contraction operators. Note that there is a bijection $Q^*\colon S \to X^I_0$.

I claim that the contraction operators Q restrict to bijections $\phi\colon X^{II}_{n-1} \to X^I_n$ for $n \geq 1$, with inverses given by $(d^n)^*\colon X^I_n \to X^{II}_{n-1}$, and that furthermore $X^I_n d^i \subseteq X^I_{n-1} \cup X^{\mathrm{deg}}_{n-1}$ when $0 \leq i < n$. We make the following observations for $x \in X_{n-1}$.

- If $xQ = ys^i \in X_n$ with $i \in \{0, \dots, n-2\}$, then $x = xQd^n = ys^id^n = yd^ns^i$. Thus, x is degenerate in this case.
- If $xQ = ys^{n-1} \in X_n$, then $x = xQd^n = ys^{n-1}d^n = y$ and $xd^{n-1}Q = xQd^{n-1} = ys^{n-1}d^{n-1} = y$. Thus, x is in the image of Q in this case.
- Taken together, the last two statements imply that Q applied to an element of X^{II} (i.e., a non-degenerate cell not in the image of Q) must produce a non-degenerate cell. Thus $Q(X^{II}) \subseteq X^I$, so ϕ is well-defined.
- We have $xQd^n = x$ for $x \in X_{n-1}$, so ϕ is injective.
- If $x = ys^i \in X_{n-1}$ with $i \in \{0, \dots, n-2\}$, then $xQ = ys^iQ = yQs^i$, i.e., x degenerate implies xQ degenerate.

- If $x = yQ \in X_{n-1}$, then $xQ = yQQ = yQs_{n-1}$, i.e., Q sends cells in the image of Q to degenerate cells.
- Taken together, the last two statements imply that every non-degenerate cell in the image of Q is the image of a non-degenerate cell not in the image of Q. That is, ϕ is surjective, with inverse given by d^n.
- We have $xQd^i = xd^iQ \in X^I_{n-1} \cup X^{\mathrm{deg}}_{n-1}$ if $i \in \{0, \dots, n-1\}$.

Now we can filter X by subcomplexes E_n, where $E_{-1} = S$, while for $n \geq 0$ we let E_n be the smallest subcomplex containing $\mathrm{Sk}_{n-1} X$ and X^I_n. Thus the non-degenerate cells in the complement $E^{\mathrm{nd}}_n \smallsetminus E^{\mathrm{nd}}_{n-1}$ are precisely those in X^I_n and $X^I_n d_n = X^{II}_{n-1}$, while $X^I_n d^i \subseteq (E_{n-1})_{n-1}$ when $0 \leq i < n$. Thus each inclusion $E_{n-1} \subseteq E_n$ is obtained by attaching each of the cells in X^I_n along $\Lambda^n_n \subset \Delta^n$. $\square$

73. Equivalences of right fibrations

Say that a functor $f\colon C \to D$ between quasicategories is a **weak right fibration** if the square

weak right fibration

$$\begin{tikzcd} \mathrm{Fun}(L, C) \arrow[r, "f_*"] \arrow[d, "j^*"'] & \mathrm{Fun}(L, D) \arrow[d, "j^*"] \\ \mathrm{Fun}(K, C) \arrow[r, "f_*"'] & \mathrm{Fun}(K, D) \end{tikzcd}$$

is a homotopy pullback in the Joyal model structure for all right anodyne maps $j\colon K \to L$. Since both restriction maps j^* are isofibrations (39.6), this is equivalent to saying that

$$f^{\square j}\colon \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L, D)$$

is a categorical equivalence for all $j \in \overline{\mathrm{RHorn}}$. There is an analogous notion of **weak left fibration**.

weak left fibration

For isofibrations, weak right fibrations are the same as right fibrations.

73.1. **Proposition.** *A right fibration $f\colon C \to D$ of quasicategories is always a weak right fibration. The converse holds whenever f is an isofibration.*

Proof. If f is a right fibration then $f^{\square j}$ is a trivial fibration for every $j \in \overline{\mathrm{RHorn}}$ (63.3). For an isofibration f and monomorphism j the pullback-hom map $f^{\square j}$ is also an isofibration (39.5). If f is also a weak right fibration then $f^{\square j}$ is also a categorical equivalence, and thus is a trivial fibration (40.8). Therefore $f^{\square j}$ is surjective on vertices, whence $j \boxslash f$. $\square$

The class of weak right fibration has properties not shared in general by the class of right fibrations.

73.2. **Proposition.** *Consider a commutative square*

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, "f"'] & C' \arrow[d, "f'"] \\ D \arrow[r, "v"'] & D' \end{tikzcd}$$

where C, C', D, D' are quasicategories, and u and v are categorical equivalences. Then f is a weak right fibration if and only if f' is a weak right fibration.

Proof. The categorical equivalences induce a weak equivalence between the commutative squares $\mathrm{Fun}(j, f)$ and $\mathrm{Fun}(j, f')$, so that one square is a homotopy pullback if and only if the other is (51.8). The claim follows. $\square$

73.3. **Proposition.** *Let $C \xrightarrow{f} D \xrightarrow{g} E$ be functors between quasicategories, and suppose g is a weak right fibration. Then f is a weak right fibration if and only if gf is a weak right fibration.*

Proof. Let $j\colon K \to L$ be any right anodyne map, and consider the commutative diagram

$$\begin{tikzcd} \mathrm{Fun}(L,C) \arrow[r, "f_*"] \arrow[d, "j^*"'] & \mathrm{Fun}(L,D) \arrow[r, "g^*"] \arrow[d, "j^*"'] & \mathrm{Fun}(L,E) \arrow[d, "j^*"'] \\ \mathrm{Fun}(K,C) \arrow[r, "f_*"'] & \mathrm{Fun}(K,D) \arrow[r, "g_*"'] & \mathrm{Fun}(K,E) \end{tikzcd}$$

The claim is immediate from patching of homotopy pullbacks (51.10). □

73.4. **Lemma.** *Let $E \xrightarrow{f} F \xrightarrow{q} C$ be functors of quasicategories, and let $p = qf$. If p and q are right fibrations, then there exists a factorization of the form $f = rj$ where j is a categorical equivalence and r is a right fibration.*

Proof. Choose any factorization $f = rj$ into a categorical equivalence j followed by an isofibration r (42.9), and consider the resulting commutative diagram

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[dr, "p"'] & E' \arrow[r, "r"] \arrow[d, "p'"] & F \arrow[dl, "q"] \\ & C & \end{tikzcd}$$

Note that since p and q are right fibrations, they are in particular weak right fibrations (73.1). Since j is an equivalence p' is also a weak right fibration by (73.2), and therefore r is a weak right fibration by (73.3). But r is an isofibration, and thus is also a right fibration (73.1). □

73.5. **Proposition.** *Consider a commutative square*

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "p"'] & F \arrow[d, "q"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

of quasicategories in which p and q are right fibrations. Then the square is a homotopy pullback in the Joyal model structure if and only if it induces equivalences on all fibers, i.e., if and only if $p^{-1}(c) \to q^{-1}(fc)$ is an equivalence of quasigroupoids for all $c \in C_0$.

Proof. The only if direction is immediate. In fact, if p and q are merely isofibrations, then taking fibers over a vertex is a good pullback and thus an instance of a homotopy pullback (**??**).

For the other direction, let $g'\colon E \to C \times_D F$ be the evident map to the pullback, so that we want to show that g is a categorical equivalence. Choose a factorization $g' = jr$ into a categorical equivalence j followed by a right fibration (73.4). Thus by 2-out-of-3, it suffices to show that r is a categorical equivalence. Since r is a right fibration, it suffices by the fiberwise criterion (**??**) to show that all its fibers are contractible.

Write $q' := qg'\colon C \times_D F \to C$ and $p' := q'r\colon E' \to C$. Given any vertex $c \in C_0$ we have induced maps on fibers

$$p^{-1}(c) \xrightarrow{j_c} (p')^{-1}(c) \xrightarrow{r_c} (q')^{-1}(c) \approx q^{-1}(f(c)).$$

Any fiber of r is a fiber of r_c for some c. We know that both j_c and $r_c j_c$ are equivalences, whence r_c is an equivalence. The map r_c is a right fibration since it is the basechange of one, so since it is an equivalence its fibers will be contractible. But every fiber of r appears as the fiber of some r_c, so we are done. □

74. Adjoint functors

A functor $f\colon C \to D$ is a **left adjoint** if for every pullback square **left adjoint**

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

in which p is a representable fibration, then p' is also a representable fibration. Likewise, f is a **right adjoint** if for every pullback square as above in which p is a corepresentable fibration, then p' is also a corepresentable fibration. **right adjoint**

Several standard properties of adjoint functors are easy.

74.1. **Proposition.** *Every categorical equivalence between quasicategories is both a a left adjoint and a right adjoint. Any composite of two left adjoints is a left adjoint, and any composite of two right adjoints is a right adjoint.*

Proof. The first statement is immediate from (71.2) and the fact that a pullback of a categorical equivalence along an isofibration is a categorical equivalence. The second statement is immediate. □

Since representable functors are always equivalent to slice restrictions, we can refomulate the definition of adjoint using only these.

74.2. **Proposition.** *Let $f\colon C \to D$ be a functor. The following are equivalent.*

(1) *The functor f is a left adjoint (i.e., pullback along f preserves representable fibrations).*
(2) *For every object $d \in D_0$ there exists an object $c \in C_0$ and a homotopy pullback square of the form*

$$\begin{tikzcd} C^{/c} \arrow[r, "f'"] \arrow[d, "\pi"'] & D^{/d} \arrow[d, "\pi'"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

where the maps π and π' are slice restrictions.
(3) *For every object $d \in D_0$ there exists an object $c \in C_0$ and a commutative square as in (2) such that for each object $x \in C_0$ the map*

$$\mathrm{map}_C(x, c) \to \mathrm{map}_D(fx, d)$$

induced by f' is an equivalence.

Proof. (1) ($\Longrightarrow$) (2). Since f is a left adjoint, any basechange $\widetilde{\pi}'\colon C \times_D D^{/d} \to C$ of π' along f is a representable fibration. Thus by (72.2) we can find a $c \in C_0$ and an equivalence $C^{/c} \to C \times_D D^{/d}$ compatible with the projections π and $\widetilde{\pi}'$. Since both these projections are right fibrations and thus isofibrations, we get the desired homotopy pullback square.

(2) ($\Longrightarrow$) (1). By (72.2) a right fibration $E \to D$ is representable if and only if it is weakly equivalent (in the Joyal model structure on $s\mathrm{Set}_{/D}$) to one of the form $\pi'\colon D^{/d} \to D$. Pullback along f defines a functor $f^*\colon s\mathrm{Set}_{/D} \to s\mathrm{Set}_{/C}$ which preserves categorical fibrations, and thus preserves categorical equivalences between fibrant objects by the Ken Brown lemma (50.4). Given this the claim is straightforward.

(2) ($\Longleftrightarrow$) (3). Since π and π' are right fibrations whose fibers are the indicated mapping spaces, we see from (73.5) that the square is a homotopy pullback if and only if the induced map on the mapping spaces are equivalences. □

75. Adjoints and limits and colimits

Next we show that adjoint functors preserve colimits or limits as the case may be.

75.1. **Lemma.** *Let K be a simplicial set, C a quasicategory, and $c \in C_0$ an object. There are isomorphisms of simplicial sets*

$$\mathrm{Fun}(K, C^{c/}) \xrightarrow{\sim} \mathrm{Fun}(K, C)^{\widetilde{\pi}_K c/}, \qquad \mathrm{Fun}(K, C^{/c}) \xrightarrow{\sim} \mathrm{Fun}(K, C)^{/\widetilde{\pi}_K c},$$

where $\widetilde{\pi}_K \colon C \to \mathrm{Fun}(K, C)$ is adjoint to projection. Furthermore, these isomorphisms are natural in K, and are compatible with the evident projections to $\mathrm{Fun}(K, C)$.

Proof. Apply $\mathrm{Fun}(K, -)$ to the pullback square defining the alternate slice $C^{/c}$, giving a pullback square

$$\begin{tikzcd} \mathrm{Fun}(K, C^{/c}) \arrow[r] \arrow[d] & \mathrm{Fun}(K, \mathrm{Fun}(\Delta^1, C)) \arrow[d] \\ \mathrm{Fun}(K, C \times \{c\}) \arrow[r] & \mathrm{Fun}(K, C \times C) \end{tikzcd}$$

Product-hom adjunction gives an isomorphism between this square and the pullback square

$$\begin{tikzcd} \mathrm{Fun}(K, C)^{\widetilde{\pi}_K c/} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathrm{Fun}(K, C)) \arrow[d] \\ \mathrm{Fun}(K, C) \times \{\widetilde{\pi}_K c\} \arrow[r] & \mathrm{Fun}(K, C) \times \mathrm{Fun}(K, C) \end{tikzcd}$$

defining the alternate slice $\mathrm{Fun}(K, C)^{\widetilde{\pi}_K c/}$. $\square$

75.2. **Proposition.** *Left adjoints preserve all colimit cones, and right adjoints preserve all limit cones.*

Proof. We prove the case of left adjoints and colimits. Let $f \colon C \to D$ be a left adjoint, let S be a simplicial set, and suppose $u \colon S^{\triangleright} \to C$ is a colimit cone. We want to show that the composite $fu \colon S^{\triangleright} \to D$ is a colimit cone. We apply the criterion of (63.9), and show that for every object d in D the restriction map $\mathrm{map}_{\mathrm{Fun}(S^{\triangleright}, D)}(fu, \widetilde{\pi}_{S^{\triangleright}} d) \to \mathrm{map}_{\mathrm{Fun}(S, D)}(fu|_S, \widetilde{\pi}_S d)$ is an equivalence.

Since f is a left adjoint, by (74.2) we can construct for every object d of D an object c of C homotopy pullback diagram of the form

$$\begin{tikzcd} C^{/c} \arrow[r, "f'"] \arrow[d] & D^{/d} \arrow[d] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

where the vertical maps are the evident restriction functors. For any simplicial set K, applying $\mathrm{Fun}(K, -)$ to the above diagram gives another homotopy pullback square of quasicategories (**??**), which by the previous proposition (75.1) we can write as

$$\begin{tikzcd} \mathrm{Fun}(K, C)^{/\widetilde{\pi}_K c} \arrow[r] \arrow[d] & \mathrm{Fun}(K, D)^{/\widetilde{\pi}_K d} \arrow[d] \\ \mathrm{Fun}(K, C) \arrow[r] & \mathrm{Fun}(K, D) \end{tikzcd}$$

where the vertical maps are the evident restriction functors. Taking fibers of the vertical maps over vertices u in $\mathrm{Fun}(K, C)$ and fu and $\mathrm{Fun}(K, D)$ respectively gives an equivalence

$$\mathrm{map}_{\mathrm{Fun}(K,C)}(u, \widetilde{\pi}_K c) \to \mathrm{map}_{\mathrm{Fun}(K,D)}(fu, \widetilde{\pi}_K d).$$

Taking either $K = S$ or $K = S^{\rhd}$, and using the naturality of the construction, thus gives a commutative square

$$\begin{tikzcd} \mathrm{map}_{\mathrm{Fun}(S^{\rhd},C)}(u, \widetilde{\pi}_{S^{\rhd}} c) \arrow[r] \arrow[d, "\sim"'] & \mathrm{map}_{\mathrm{Fun}(S,C)}(u|_S, \widetilde{\pi}_S c) \arrow[d, "\sim"] \\ \mathrm{map}_{\mathrm{Fun}(S^{\rhd},D)}(fu, \widetilde{\pi}_{S^{\rhd}} d) \arrow[r] & \mathrm{map}_{\mathrm{Fun}(S,D)}(fu|_S, \widetilde{\pi}_S d) \end{tikzcd}$$

in which the vertical maps are equivalences. Since u is a colimit cone, the top horizontal map is also an equivalence, and hence the bottom horizontal map is also by 2-out-of-3. Thus fu is a colimit cone as desired. $\square$

76. Adjoints of adjoints

Suppose $p\colon C \to D$ is an inner fibration of categories. We have observed (65.5) that the restricted functor $p'\colon C^{p\,\mathrm{init}} \to D^{\exists p\,\mathrm{init}}$ is a trivial fibration. Thus if *every* object of D is the image of some p-initial object of C, we can always produce a section $s\colon D \to C^{p\,\mathrm{init}} \subseteq C$ of p taking values in p-initial objects. Note that any such section s is fully faithful, as it restricts to an equivalence $D \to C^{p\,\mathrm{init}} \subseteq C$ to a full subcategory.

It turns out in this situation the functor s is always a left adjoint, and p is always a right adjoint.

76.1. **Proposition.** *Let $p\colon C \to D$ be an inner fibration of quasicategories, and let $s\colon D \to C$ be a functor such that $ps = 1_D$, and such that for all objects $d \in D_0$ the object $s(d) \in C_0$ is p-initial. Then p is a right adjoint and s is a left adjoint.*

Proof. First we show that p is a right adjoint. Given $y \in D_0$, consider the commutative square

$$\begin{tikzcd} C^{sy/} \arrow[r, "p'"] \arrow[d] & D^{y/} \arrow[d] \\ C \arrow[r, "p"'] & D \end{tikzcd}$$

where p' is the functor induced by p (which is defined since $psy = y$), and the vertical arrows are slice restrictions, which are right fibrations. Far any $c \in C_0$, the map p' induces a map on fibers of the form

$$\mathrm{map}_C(sy, c) \to \mathrm{map}_D(psy, pc) = \mathrm{map}_D(y, pc)$$

which is identical to the map induced by p, and thus is an equivalence since sy is p-initial (65.4). Therefore the square is a homotopy pullback (**??**), and thus p is a right adjoint (**??**).

Next we show that s is a left adjoint. We construct the following commutative diagram

$$\begin{tikzcd} D^{/px} \arrow[r, "s'"] \arrow[d] & C^{/x} \arrow[r, "p'"] \arrow[d] & D^{/px} \arrow[d] \\ D \arrow[r, "s"'] & C \arrow[r, "p"'] & D \end{tikzcd}$$

in which the vertical maps are slice restrictions and thus right fibrations (**??**), the map p' is induced by p, and the map s' is any map making the diagram commute and sending 1_{px} to 1_x, which exists by (71.2) and (**??**). The composite $p's'$ sends 1_{px} to itself, and thus is an equivalence (**??**). For any $d \in D_0$ we have induced maps on fibers

$$\mathrm{map}_D(d, px) \xrightarrow{s''} \mathrm{map}_C(sd, x) \xrightarrow{p''} \mathrm{map}_D(psd, px) = \mathrm{map}_D(d, px).$$

Since $p's'$ is an equivalence, so is $p''s''$, Since sd is a p-initial object, the map p'' is also an equivalence, and therefore s'' is an equivalence by 2-out-of-3. Thus s is a left adjoint (**??**). $\square$

76.2. *Example* (Restriction to a cone vertex). Let C be a quasicategory and S a simplicial set. Consider the functor $q\colon \operatorname{Fun}(S^{\triangleright}, C) \to \operatorname{Fun}(\{v\}, C) = C$ induced by restriction along $\{v\} \subset S^{\triangleright}$. Then every object $c \in C_0$ lifts along q to a q-terminal object. In fact, the constant functor $\widetilde{\pi}_{S^{\triangleright}}(c)\colon S^{\triangleright} \to C$ is q-terminal, since as we have seen the restriction map $\operatorname{map}_{\operatorname{Fun}(S^{\triangleright},C)}(\widehat{f}, \widetilde{\pi}_{S^{\triangleright}}(c)) \to \operatorname{map}_C(\widehat{f}(v), c)$ is always an equivalence since the slice restriction $C^{\widehat{f}/} \to C^{\widehat{f}(v)/}$ is always a trivial fibration (62.8). Note that we have a convenient choice of section of q, namely the functor $t\colon C \to \operatorname{Fun}(S^{\triangleright}, C)$ induced by restriction along $S^{\triangleright} \to \Delta^0$. Thus we obtain a pair of functors

$$\begin{tikzcd} \operatorname{Fun}(S^{\triangleright}, C) \arrow[r, shift left, "q"] & C \arrow[l, shift left, "t"] \end{tikzcd}$$

in which q is a left adjoint and t is a right adjoint, and $qt = \mathrm{id}_C$.

76.3. *Example* (Colimit as an adjoint). Let C be a quasicategory and S a simplicial set, and suppose C has colimits for all functors $S \to C$. Let $p\colon \operatorname{Fun}(S^{\triangleright}, C) \to \operatorname{Fun}(S, C)$ be induced by restriction along $S \subset S^{\triangleright}$. By hypothesis, every object of $\operatorname{Fun}(S, C)$ lifts to a a colimit cone, and hence to some p-initial object (**??**). Thus if we take any choice of section $\operatorname{Fun}(S, C) \to \operatorname{Fun}^{\mathrm{colim}}(S^{\triangleright}, C) \subseteq \operatorname{Fun}(S^{\triangleright}, C)$ (**??**) we obtain a pair of functors

$$\begin{tikzcd} \operatorname{Fun}(S, C) \arrow[r, shift left, "s"] & \operatorname{Fun}(S^{\triangleright}, C) \arrow[l, shift left, "p"] \end{tikzcd}$$

in which s is a left adjoint and p is a right adjoint, and $ps = \mathrm{id}_{\operatorname{Fun}(S,C)}$.

We can combine this with the previous example, so we have a diagram

$$\begin{tikzcd} \operatorname{Fun}(S, C) \arrow[r, shift left, "s"] & \operatorname{Fun}(S^{\triangleright}, C) \arrow[l, shift left, "p"] \arrow[r, shift left, "q"] & C \arrow[l, shift left, "t"] \end{tikzcd}$$

in which qs is a left adjoint and pt is a right adjoint. The functor $pt = \widetilde{\pi}_S$ is the "constant-functor functor", and the functor qs is a model for the colimit functor $\operatorname{colim}_S\colon \operatorname{Fun}(S, C) \to C$.

76.4. *Example.* Let $f\colon C \to D$ be a functor between quasicategories. Consider the coCartesian path fibration diagram:

$$\begin{tikzcd} C \arrow[r, shift left, "s"] & C \times_D \operatorname{Fun}(\Delta^1, D) \arrow[l, shift left, "p"] \arrow[r, shift left, "q"] & D \arrow[l, shift left, dotted, "t"] \end{tikzcd}$$

where the fiber product is defined using f and restriction along $\{0\} \subset \Delta^1$, the functor q is induced by restriction along $\{1\} \subset \Delta^1$, the functor p is the evident projection, and the functor s is induced by restriction along $\Delta^1 \to *$.

The functor s sends each object $c \in C_0$ to the object $(c, 1_{fc})$, which is a p-initial object (76.5). Thus s is a left adjoint and p is a right adjoint.

Recall that since q is a coCartesian fibration, an object of its domain is q-terminal if and only if it is terminal in its fiber. The fibers of q are exactly

$$q^{-1}(d) = C \times_D D^{/d},$$

the pullbacks of the slices $D^{/d} \to D$ along f. Thus, f is a left adjoint if and only if each object $d \in D_0$ admits a lift along q to a q-terminal object $(gd, \epsilon_d\colon fgd \to d)$. Thus, there exists a section $t = (g, \epsilon)$ of q taking values in q-terminal objects. In particular, q is a left adjoint and t is a right adjoint.

The composite $qs = f$ is the functor we started with. The above discussion shows that if f is a left adjoint, then the associated functor $g = pt$ is a right adjoint.

76.5. **Lemma.** *Let $f\colon C \to D$ be a functor between quasicategories, and consider the projection map*

$$p\colon \operatorname{Fun}(\{0\}, C) \times_{\operatorname{Fun}(\{0\},D)} \operatorname{Fun}(\Delta^1, D) \to \operatorname{Fun}(\{0\}, C),$$

and an object (c, α) in the domain of p, with $c \in C_0$ and $\alpha \in D_1$. If α is an isomorphism in D then (c, α) is a p-terminal object.

Proof. The map p is the base-change of the restriction map $p'\colon \operatorname{Fun}(\Delta^1, D) \to \operatorname{Fun}(\{0\}, D)$, so it suffices by (65.2) to show that α is p'-terminal. But $p' = \pi^{\square(\{0\}\subset\Delta^1)}$ where $\pi\colon D \to *$, so by the pullback-hom criterion for Cartesian morphisms (69.2) we see that α is a p'-terminal object in $\operatorname{Fun}(\Delta^1, D)$ iff and only if it corresponds to a π-Cartesian edge in D. But we know that the π-Cartesian edges are exactly the isomorphisms (66.3). $\square$

Part 13. **Appendices**

77. Appendix: Generalized horns

A **generalized horn**[32] is a subcomplex $\Lambda^n_S \subset \Delta^n$ of the standard n-simplex, where $S \subseteq [n]$ and **generalized horn**

$$(\Lambda^n_S)_k := \{\, f\colon [k] \to [n] \;\mid\; S \not\subseteq f([k]) \,\}.$$

In other words, a generalized horn is a union of some codimension 1 faces of the n-simplex:

$$\Lambda^n_S = \bigcup_{s\in S} \Delta^{[n]\smallsetminus s}.$$

In particular,

$$\Lambda^n_{[n]} = \partial\Delta^n, \quad \Lambda^n_{[n]\smallsetminus j} = \Lambda^n_j, \quad \Lambda^n_{\{j\}} = \Delta^{[n]\smallsetminus j}, \quad \Lambda^n_{\varnothing} = \varnothing.$$

In general $S \subseteq T$ implies $\Lambda^n_S \subseteq \Lambda^n_T$.

77.1. **Proposition** (Joyal [Joy08a, Prop. 2.12])**.** *Let $S \subsetneq [n]$ be a* proper *subset.*

(1) $(\Lambda^n_S \subset \Delta^n) \in \overline{\text{Horn}}$ *if $S \neq \varnothing$.*
(2) $(\Lambda^n_S \subset \Delta^n) \in \overline{\text{LHorn}}$ *if $n \in S$.*
(3) $(\Lambda^n_S \subset \Delta^n) \in \overline{\text{RHorn}}$ *if $0 \in S$.*
(4) $(\Lambda^n_S \subset \Delta^n) \in \overline{\text{InnHorn}}$ *if S is not an "interval"; i.e., if there exist $a < b < c$ with $a, c \in S$ and $b \notin S$.*

Proof. We start with an observation. Consider $S \subsetneq [n]$ and $t \in [n] \smallsetminus S$. Observe the diagram

$$\begin{tikzcd}
\Delta^{[n]\smallsetminus t} \cap \Lambda^n_S \arrow[r, tail] \arrow[d, tail] & \Delta^{[n]\smallsetminus t} \arrow[d, tail] & \\
\Lambda^n_S \arrow[r, tail] & \Lambda^n_{S\cup t} \arrow[r, tail] & \Delta^n
\end{tikzcd}$$

in which the square is a pushout, and the top arrow is isomorphic to the generalized horn $\Lambda^{[n]\smallsetminus t}_S \subset \Delta^{[n]\smallsetminus t}$. Thus, $(\Lambda^n_S \subset \Delta^n)$ is contained in the weak saturation of any set contaning the two inclusions

$$\Lambda^{[n]\smallsetminus t}_S \subset \Delta^{[n]\smallsetminus t} \quad \text{and} \quad \Lambda^n_{S\cup t} \subset \Delta^n.$$

Each of the statements of the proposition is proved by an evident induction on the size of $[n] \smallsetminus S$, using the above observation. I'll do case (4), as the other cases are similar. If $S \subset [n]$ is not an interval, there exists some $s < u < s'$ with $s, s' \in S$ and $u \notin S$. If $[n] \smallsetminus S = \{u\}$ then we already have an inner horn. If not, then choose $t \in [n] \smallsetminus (S \cup \{u\})$, in which case $S \cup t$ is not an interval in

[32] This notion is from [Joy08a, §2.2.1]. However, I have changed the sense of the notation: our Λ^n_S is Joyal's $\Lambda^{[n]\smallsetminus S}$. I find my notation easier to follow, but note that it does conflict with the standard notation for horns. Maybe I should use something like $\Lambda^{n,S}$?

$[n]$, and S is not an interval in $[n] \smallsetminus t$. Therefore both $\Lambda^{[n]\smallsetminus t}_S \subset \Delta^{[n]\smallsetminus t}$ and $\Lambda^n_{S\cup t} \subset \Delta^n$ are inner anodyne by the inductive hypothesis. The proofs of the other cases are similar. □

77.2. **Proposition** (Joyal [Joy08a, Prop. 2.13])**.** *For all $n \geq 2$, we have that $(I^n \subset \Delta^n) \in \overline{\text{InnHorn}}$.*

Proof. We can factor the spine inclusion as $h_n = g_n f_n$:

$$I^n \xrightarrow{f_n} \Delta^{\{1,\dots,n\}} \cup I^n \xrightarrow{g_n} \Delta^n.$$

We show by induction on n that $f_n, g_n, h_n \in \overline{\text{InnHorn}}$, noting that the case $n = 2$ is immediate.

To show that $f_n \in \overline{\text{InnHorn}}$, consider the pushout square

$$\begin{tikzcd} I^{\{1,\dots,n\}} \arrow[r, tail] \arrow[d, tail] & \Delta^{\{1,\dots,n\}} \arrow[d, tail] \\ I^n \arrow[r, tail, "f_n"'] & \Delta^{\{1,\dots,n\}} \cup I^n \end{tikzcd}$$

in which the top arrow is isomorphic to h_{n-1}, which is in $\overline{\text{InnHorn}}$ by induction.

To show that $g_n \in \overline{\text{InnHorn}}$, consider the diagram

$$\begin{tikzcd} \Delta^{\{1,\dots,n-1\}} \cup I^{\{0\dots,n-1\}} \arrow[r, tail, "g_{n-1}"] \arrow[d, tail] & \Delta^{\{0,\dots,n-1\}} \arrow[d, tail] & \\ \Delta^{\{1,\dots,n\}} \cup I^n \arrow[r, tail] & \Delta^{\{1,\dots,n\}} \cup \Delta^{\{0,\dots,n-1\}} \arrow[r, tail] & \Delta^n \end{tikzcd}$$

in which the square is a pushout, the top horizontal arrow is isomorphic to g_{n-1}, an element of $\overline{\text{InnHorn}}$ by induction, and the bottom right horizontal arrow is equal to $\Lambda^n_{\{0,n\}} \subset \Delta^n$, which is in $\overline{\text{InnHorn}}$ by (77.1)(4).

□

78. Appendix: Box product lemmas

Here is where I'l prove various statements mentioned in the text.

- $\text{LHorn}\square\text{Cell} \subseteq \overline{\text{LHorn}}$ (63.2), proved in (78.1) below.
- $\text{RHorn}\square\text{Cell} \subseteq \overline{\text{RHorn}}$ (63.2), proved in (78.1) below.
- $\text{Horn}\square\text{Cell} \subseteq \overline{\text{Horn}}$, is a consequence of the above, since $\text{Horn} = \text{LHorn} \cup \text{RHorn}$ and $\overline{\text{LHorn}} \cup \overline{\text{RHorn}} \subseteq \overline{\text{Horn}}$.
- $\text{InnHorn}\square\text{Cell} \subseteq \overline{\text{InnHorn}}$ (21.10), proved in (78.3) below.

78.1. **Left and right horns.** We prove the case of $\text{LHorn}\square\text{Cell} \subseteq \overline{\text{LHorn}}$ here. Given this $\text{RHorn}\square\text{Cell} \subseteq \overline{\text{RHorn}}$ follows since op: $s\text{Set} \to s\text{Set}$ carries LHorn to RHorn and preserves Cell.

Joyal [Joy08a, 2.25][33] observes that $(\Lambda^n_k \subset \Delta^n)$ is a retract of $(\Lambda^n_k \subset \Delta^n)\square(\{0\} \subset \Delta^1)$ when $0 \leq k < n$. The retraction is

$$\Delta^n \xrightarrow{s} \Delta^n \times \Delta^1 \xrightarrow{r} \Delta^n$$

defined by $s(x) = (x, 1)$ and

$$r(x,0) = \begin{cases} x & \text{if } x \leq k, \\ k & \text{if } x \geq k, \end{cases} \qquad r(x,1) = x.$$

Note that $r(\Delta^{[n]\smallsetminus j} \times \Delta^1) = \Delta^{[n]\smallsetminus j}$ if $j \neq k$, and $r(\Delta^n \times \{0\}) = \Delta^{\{0,\dots,k\}} \subseteq \Delta^{[n]\smallsetminus (k+1)}$, so this gives the desired retraction.

[33] Lurie [Lur09, 2.1.2.6] states this incorrectly in my edition.

The existence of the retraction reduces showing LHorn$\square$Cell $\subseteq \overline{\text{LHorn}}$ to proving

$$(\{0\} \subset \Delta^1)\square\text{Cell} \subseteq \overline{\text{LHorn}},$$

since $(\Lambda^n_k \subset \Delta^n) \in \overline{\text{Cell}}$ and thus $(\Lambda^n_k \subset \Delta^n)\square\text{Cell} \subseteq \overline{\text{Cell}}$.

78.2. **Lemma.** *We have that* $(\{0\} \subset \Delta^1)\square\text{Cell} \subseteq \overline{\text{LHorn}}$.

Proof. ...Let $K = (\{0\} \times \Delta^n) \cup (\Delta^1 \times \partial\Delta^n)$, so that $(\{0\} \subset \Delta^1)\square(\partial\Delta^n \subset \Delta^n)$ is the inclusion $K \to \Delta^1 \times \Delta^n$. We will show that we can build $\Delta^1 \times \Delta^n$ from K by an explicit sequence of steps, where in each case we attach an $(n+1)$-sequence along a left horn.

For each $0 \leq a \leq n$ let τ_a be the $(n+1)$-dimensional cell of $\Delta^1 \times \Delta^n$ defined by

$$\tau_a = \langle (0,0), \dots, (0,a),\ (1,a), \dots, (1,n) \rangle.$$

We obtain an ascending filtration of $\Delta^1 \times \Delta^n$ by starting with K and attaching simplices in the following order:

$$\tau_n, \tau_{n-1}, \dots, \tau_1, \tau_0.$$

The τs range through all non-degenerate $(n+1)$-dimensional cells of $\Delta^1 \times \Delta^n$, so $K \cup \bigcup \tau_a = \Delta^1 \times \Delta^n$. (Here I am using the same notation for elements $\tau_a \in (\Delta^1 \times \Delta^n)_{n+1}$ and for the corresponding subcomplex of $\Delta^1 \times \Delta^n$ which is isomorphic to Δ^{n+1}.)

The claim is that each attachment is along a specified horn inclusion. More precisely, for $a \in [n]$ the simplex τ_a is attached to $K \cup \bigcup_{k>a} \tau_k$ along the horn at the vertex $(0,a)$ in τ_a, i.e., via a $\Lambda^{n+1}_a \subset \Delta^n$ horn inclusion. Note that if when $a > 0$ this is an inner horn, while when $a = 0$ this is the inclusion $\Lambda^{n+1}_0 \subset \Delta^n$; in either case, it is a left horn. Given the claim, it follows that $(\{0\} \subset \Delta^1)\square(\partial\Delta^n \subset \Delta^n) \in \overline{\text{LHorn}}$ as desired.

The proof of the claim amounts to the following list of elementary observations about τ_a:

- Every codimension-one face is contained in $\Delta^1 \times \partial\Delta^n$ except: the face opposite vertex $(0,a)$, and the face opposite vertex $(1,a)$.
- The face opposite vertex $(1,a)$ is contained in $\{0\} \times \Delta^n$ if $a = n$, or is a face of τ_{a+1} if $a < n$.
- The face opposite vertex $(0,a)$ is not contained in $\Delta^1 \times \partial\Delta^n$, nor in $\{0\} \times \Delta^n$. Nor is it contained in any τ_i with $i > a$ (beacuse the vertex $(1,a)$ is in this face but not in τ_i with $i > a$).

Taken together these show that $\tau_a \cap (K \cup \bigcup_{k>a} \tau_k)$ is the ath horn in the $(n+1)$-simplex τ_a.

$\square$

78.3. **Inner horns.** Here is an argument for the key case for inner horns, following the proof of [Lur09, 2.3.2.1]. I will show the following.

78.4. **Lemma.** *We have* $\overline{\text{InnHorn}} = \overline{(\Lambda^2_1 \subset \Delta^2)\square\text{Cell}} = \overline{(\Lambda^2_1 \subset \Delta^2)\square\overline{\text{Cell}}}$.

Given this, to show InnHorn$\square$Cell $\subseteq \overline{\text{InnHorn}}$ it suffices to note that $(\Lambda^2_1 \subset \Delta^2)\square\text{Cell}\square\text{Cell} \subseteq (\Lambda^2_1 \subseteq \Delta^2)\square\overline{\text{Cell}}$, using $\overline{\text{Cell}}\square\overline{\text{Cell}} \subseteq \overline{\text{Cell}}$ (22.5).

Proof. We prove this by showing that for each term in the sequence

$$\text{InnHorn}, \qquad (\Lambda^2_1 \subset \Delta^2)\square\overline{\text{Cell}}, \qquad (\Lambda^2_1 \subset \Delta^2)\square\text{Cell}, \qquad \text{InnHorn},$$

the weak saturation of each term is contained in the weak saturation of the next one in the list.

Case 1: $(\Lambda^2_1 \subset \Delta^2)\square\overline{\text{Cell}} \subseteq \overline{(\Lambda^2_1 \subset \Delta)\square\text{Cell}}$. This is immediate from (21.9), which implies that $S\square\overline{T} \subseteq \overline{S}\square\overline{T} \subseteq \overline{S\square T}$ for any sets of maps S and T.

Case 2: InnHorn $\subseteq \overline{(\Lambda^2_1 \subset \Delta^2)\square\overline{\text{Cell}}}$. This is proved by exhibiting inner horn inclusions as retracts. Consider $\Delta^n \xrightarrow{s} \Delta^2 \times \Delta^n \xrightarrow{r} \Delta^n$, the unique maps which are given on vertices by

$$s(y) = \begin{cases} (0,y) & \text{if } y < j, \\ (1,y) & \text{if } y = j, \\ (2,y) & \text{if } y > j, \end{cases} \qquad r(x,y) = \begin{cases} y & \text{if } x = 0 \text{ and } y < j, \\ y & \text{if } x = 2 \text{ and } y > j, \\ j & \text{otherwise.} \end{cases}$$

These explicitly exhibit $(\Lambda^n_j \subset \Delta^n)$ as a retract of $(\Lambda^2_1 \subset \Delta^2)\square(\Lambda^n_j \subset \Delta^n)$, so

$$\text{InnHorn} \subseteq \{\Lambda^2_1 \subset \Delta^2\}\square\overline{\text{Cell}}.$$

Case 3: $(\Lambda^2_1 \subset \Delta^2)\square\text{Cell} \subseteq \overline{\text{InnHorn}}$. This is proved as the following lemma (78.5). $\square$

78.5. **Lemma.** *For all $n \geq 0$ we have that $(\Lambda^2_1 \subset \Delta^2)\square(\partial\Delta^n \subset \Delta^n) \in \overline{\text{InnHorn}}$.*

Proof. For each $0 \leq a \leq b < n$, let σ_{ab} be the $(n+1)$-simplex of $\Delta^2 \times \Delta^n$ defined by

$$\sigma_{ab} = \langle (0,0), \dots, (0,a),\ (1,a), \dots, (1,b),\ (2,b+1), \dots, (2,n) \rangle.$$

For each $0 \leq a \leq b \leq n$, let τ_{ab} be the $(n+2)$-cell of $\Delta^2 \times \Delta^n$ defined by

$$\tau_{ab} = \langle (0,0), \dots, (0,a),\ (1,a), \dots, (1,b),\ (2,b), \dots, (2,n) \rangle.$$

The set $\{\tau_{ab}\}$ consists of all the non-degenerate $(n+2)$-dimensional cells. Note that σ_{ab} is a face of τ_{ab} and of $\tau_{a,b+1}$, but not a face of any other τ.

We attach simplices to $K := (\Lambda^2_1 \times \Delta^n) \cup (\Delta^2 \times \partial\Delta^n)$ in the following order:

$$\sigma_{00},\quad \sigma_{01}, \sigma_{11},\quad \sigma_{02}, \sigma_{12}, \sigma_{22},\quad \dots\quad \sigma_{0,n-1}, \dots, \sigma_{n-1,n-1},$$

followed by

$$\tau_{00},\quad \tau_{01}, \tau_{11},\quad \tau_{02}, \tau_{12}, \tau_{22},\quad \dots\quad \tau_{0,n}, \dots, \tau_{n,n}.$$

The τs range through all the non-degenerate $(n+2)$-dimensional cells of $\Delta^2 \times \Delta^n$, so that $K \cup \bigcup \sigma_{a,b} \cup \bigcup \tau_{a,b} = \Delta^2 \times \Delta^n$.

The claim is that each attachment is along an inner horn inclusion. More precisely, each σ_{ab} gets attached along the horn at the vertex $(1,a)$ in σ_{ab}, i.e., via a $\Lambda^{n+1}_{a+1} \subset \Delta^{n+1}$ horn inclusion, which is always inner since $a \leq b < n$. Likewise, each τ_{ab} gets attached along the horn at vertex $(1,a)$ in τ_{ab}, i.e., via a $\Lambda^{n+2}_{a+1} \subset \Delta^{n+2}$ horn inclusion, which is always inner since $a \leq b \leq n$.

The proof of the claim amounts to the following lists of elementary observations.

For $\sigma_{a,b}$:

- Every codimension-one face is contained in $\Delta^2 \times \partial\Delta^n$, except the following: the face opposite vertex $(0,a)$, and the face opposite vertex $(1,a)$.
- The face opposite vertex $(0,a)$ is either contained in $\Lambda^2_0 \times \Delta^n$ if $a = 0$, or a face of $\sigma_{a-1,b}$ if $a > 0$.
- The face of $\sigma_{a,b}$ opposite vertex $(1,a)$ is not contained in $\Delta^2 \times \partial\Delta^n$, nor in $\Lambda^2_0 \times \Delta^n$, nor in any $\sigma_{i,b}$ with $i < a$ (because of the vertex $(0,a)$), nor in any $\sigma_{i,j}$ with $i \leq j < b$ (because of the vertex $(1,b)$ if $a < b$, or the vertex $(0,a)$ if $a = b$).

For $\tau_{a,b}$ when $a < b$:

- Every codimension-one face is contained in $\Delta^2 \times \partial\Delta^n$ except the following: the face opposite vertex $(0,a)$, the face opposite vertex $(1,a)$, the face opposite vertex $(1,b)$, and the face opposite vertex $(2,b)$.
- The face opposite vertex $(2,b)$ is $\sigma_{a,b}$, while the face opposite vertex $(1,b)$ is $\sigma_{a,b-1}$.
- The face opposite vertex $(0,a)$ is either contained in $\Lambda^2_1 \times \Delta^n$ if $a = 0$, or is a face of $\tau_{a-1,b}$ if $a > 0$.

- The face opposite vertex $(1,a)$ is not contained in $\Delta^2\times\partial\Delta^n$, nor in $\Lambda^2_1\times\Delta^n$, nor in any $\sigma_{i,j}$ (because of the vertices $(1,b)$ and $(2,b)$), nor in any $\tau_{i,b}$ with $i<b$ (because of the vertex $(0,a)$), nor in any $\tau_{i,j}$ with $i\leq j<b$ (because of the vertex $(1,b)$).

For $\tau_{a,b}$ when $a=b$:

- Every codimension-one face is contained in $\Delta^2\times\partial\Delta^n$ except the following: the face opposite vertex $(0,a)$, the face opposite vertex $(1,a)=(1,b)$, and the face opposite vertex $(2,b)$.
- The face opposite vertex $(2,b)$ is $\sigma_{a,b}$.
- The face opposite vertex $(0,a)$ is contained in $\Lambda^2_1\times\Delta^n$ if $a=0$, or is a face of $\tau_{a-1,b}$ if $a>0$.
- The face opposite vertex $(1,a)=(1,b)$ is not contained in $\Delta^2\times\partial\Delta^n$, nor in $\Lambda^2_1\times\Delta^n$, nor in any $\sigma_{i,j}$ (because of the vertices $(0,a)$ and $(2,b)$), nor in any $\tau_{i,b}$ with $i<b$ (because of the vertex $(0,a)$), nor in any $\tau_{i,j}$ with $i\leq j<b$ (because of the vertex $(0,a)$).

□

78.6. **A pushout-product version of Joyal lifting.** We now give a proof of (36.6): we will prove the case of $(x,y)=(0,0)$, i.e., given $p\colon C\to D$ an inner fibration of quasicategories, $n\geq 1$, and

$$\begin{tikzcd}
\Delta^1\times\{0\} \arrow[r, tail] \arrow[rr, bend left, "f"] & (\{0\}\times\Delta^n)\cup_{\{0\}\times\partial\Delta^n}(\Delta^1\times\partial\Delta^n) \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\
& \Delta^1\times\Delta^n \arrow[r] \arrow[ur, dotted] & D
\end{tikzcd}$$

such that f represents an isomorphism in C, we will construct a lift. (Note that if $n=0$ such a lift does not generally exist.)

We refer to the proof of (78.2), where we observed that we can build $\Delta^1\times\Delta^n$ from $K=(\{0\}\times\Delta^n)\cup(\Delta^1\times\partial\Delta^n)$ by successively attaching a sequence $\tau_n,\dots,\tau_0$ of $(n+1)$-cells along horns; in particular, τ_a is attached to $K\cup\bigcup_{k>a}\tau_k$ along a horn inclusion isomorphic to $\Lambda^{n+1}_a\subset\Delta^{n+1}$.

Given this, we thus construct the desired lift by inductively choosing a lift defined on each τ_a relative to the given lift on its Λ^{n+1}_a-horn. When $a>0$ such a lift exists because p is an inner fibration and τ_a is attached along an inner horn, while when $a=0$ a lift exists by Joyal lifting (34.17), as $\Delta^1\times\{0\}$ is the leading edge of τ_0.

78.7. **A pushout-product characterization of (co)Cartesian edges.** We now give a proof of (69.3): we will prove the case of $(x,y)=(0,0)$. In fact, the proof is exactly as in (78.6): construct a lift inductively by choosing a lift defined on each $(n+1)$-cell τ_a, where a lift exists for $a>0$ because p is an inner fibration, and a lift exists for $a=0$ because the edge f is coCartesian.

78.8. **Another pushout-product version of Joyal lifting.** We show that given $p\colon C\to D$ an inner fibration of categories, $n\geq 1$, and

$$\begin{tikzcd}
\{0\}\times\Delta^{\{0,1\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & (\partial\Delta^1\times\Delta^n)\cup_{\partial\Delta^1\times\Lambda^n_0}(\Delta^1\times\Lambda^n_0) \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\
& \Delta^1\times\Delta^n \arrow[r] \arrow[ur, dotted] & D
\end{tikzcd}$$

such that f represents an isomorphisms in C, we can construct a lift.

We refer to the notation of the proof of (78.2), so that $\tau_0,\dots,\tau_n$ are the nondegenerate $(n+1)$-cells of $\Delta^1\times\Delta^n$. I claim that $\Delta^1\times\Delta^n$ can be built from $K=(\partial\Delta^1\times\Delta^n)\cup(\Delta^1\times\Lambda^n_0)$ by successively attaching the sequence $\tau_0,\dots,\tau_n$ along *generalized horns*, so that

- τ_0 is attached along a horn inclusion isomorphic to Λ^{n+1}_1,

- τ_a, $0 < a < n$, is attached along a generalized horn inclusion isomorphic to $\Lambda^{n+1}_{[n+1]\smallsetminus 0,k+1}$, and
- τ_n is attached along a horn inclusion isomorphic to Λ^{n+1}_0.

In each except the last case the inclusion is a generalized *inner* horn, while the leading edge of τ_n is precisely $\Delta^1 \times \{0\}$.

79. Appendix: Weak equivalences and homotopy groups

In this appendix we give a proof of (56.8), that the weak homotopy equivalences between Kan complexes are precisely the π_*-eqivalences.

79.1. **Models for the simplicial sphere.** We first note that we can replace $\Delta^n/\partial\Delta^n$ in the definition of $\pi_n(X,x) := \pi_0 \operatorname{Fun}_*(\Delta^n/\partial\Delta^n, X)$ with any other "model" of a a simplicial n-sphere.

79.2. **Proposition.** *Let (S,s) be a pointed simplicial set which is weakly homotopy equivalent to $(\Delta^n/\partial\Delta^n, *)$ in $s\mathrm{Set}_*$, i.e., such that there exists a zig-zag of basepoint preserving weakly homotopy equivalences*

$$(S,s) \leftarrow (S_1,s_1) \rightarrow (S_2,s_2) \leftarrow \cdots \leftarrow (\Delta^n/\partial\Delta^n, *).$$

Then there exists a bijection $\pi_n(X,x) \approx \pi_0 \operatorname{Fun}_*(S,X)$, functorial in X.

Proof. Immediate using (56.2)(1). □

For instance, the boundary of an $(n+1)$-simplex is a simplicial n-sphere.

79.3. **Proposition.** *There is a weak homotopy equivalence $(\partial\Delta^{n+1}, \{0\}) \to (\Delta^n/\partial\Delta^n, *)$ of pointed simplicial sets, and thus natural isomorphisms $\pi_n(X,x) \approx \pi_0 \operatorname{Fun}_*((\partial\Delta^{n+1}, \{0\})), (X,x)$ for Kan complexes X.*

Proof. The inclusion $\{0\} \subseteq \Lambda^{n+1}_0$ is anodyne (79.4) and thus a weak homotopy equivalence (52.5). Therefore by application of good pushouts (50.12) the induced map $\partial\Delta^{n+1} = \partial\Delta^{n+1}/\{0\} \to \partial\Delta^{n+1}/\Lambda^{n+1}_0$ is a weak homotopy equivalence. The claim follows because we have an isomorphism $\partial\Delta^{n+1}/\Lambda^{n+1}_0 \approx \Delta^n/\partial\Delta^n$, induced by $\langle 1,\dots,n+1\rangle \colon \Delta^n \to \partial\Delta^{n+1}$. The description of π_n is immediate from (56.2)(1). □

79.4. *Exercise.* Define subcomplexes $F_k \subseteq \Delta^n$ for $0 \le k \le n$, so that F_k is the union of all $\Delta^S \subseteq \Delta^n$ such that (i) $0 \in S \subseteq [n]$ and (ii) $|S| \le k+1$. Show that each inclusion $\Delta^{\{0\}} = F_0 \subset F_1 \subset \cdots \subset F_n = \Lambda^n_0$ is anodyne, whence $\{0\} \subseteq \Lambda^n_0$ is anodyne.

79.5. **π_*-equivalences.**

79.6. **Proposition.** *The class of π_*-equivalences between Kan complexes satisfies 2-out-of-6, and thus satisfies 2-out-of-3.*

Proof. This is much like the proof that functors which are essentially surjective and fully faithful share this property. One ingredient is to prove that if $f_0, f_1 \colon X \to Y$ are functors which are naturally isomorphic, then f_0 is a π_*-equivalence if and only if f_1 is. Another ingredient is the observation that to check that f is a π_*-equivalence, it suffices to check $\pi_k(X,x) \to \pi_k(Y,fx)$ for $x \in S$ where $S \subseteq X_0$ is a set of representatives of $\pi_0 X$. □

Since every weak homotopy equivalence $f \colon X \to Y$ between Kan complexes is a π_*-equivalence, to show (56.8) we can reduce to the case when f is a Kan fibration. In fact, we will show that f is a trivial fibration, using the following.

79.7. **Proposition.** *Let $p \colon X \to Y$ be a Kan fibration between Kan complexes, and consider $n \ge 0$. Then $\mathrm{Cell}_{\le n} \boxslash p$ if and only if, for all $0 \le k < n$ and all $x \in X_0$, the induced map*

$$\pi_k(X,x) \to \pi_k(Y,p(x))$$

is a bijection, and is a surjection for $k = n$.

Proof. This will be immediate from (**??**) and (79.9) below. $\square$

In the following, given a vertex $x \in X_0$, we write $\overline{x}\colon K \to X$ for any constant map with value x, i.e., any map of the form $K \to \{x\} \to X$.

79.8. **Lemma.** *Let $p\colon X \to Y$ be a Kan fibration between Kan complexes. Let $n \geq 0$ be such that $p_*\colon \pi_n(X,x) \to \pi_n(Y,px)$ is surjective for all $x \in X_0$. Then $(* \subset \Delta^n/\partial\Delta^n) \boxslash p$.*

Proof. Suppose given a lifting problem (u,v) of type $(* \subset \Delta^n/\partial\Delta^n) \boxslash p$. Note that $u = \overline{x}$ for some $x \in X$. We will show that we can deform this lifting problem to one of the form $(\overline{x}, \overline{y})$ where $y = f(x) \in Y_0$, which tautologically admits a lift (i.e., the constant map $\overline{x}\colon \Delta^n/\partial\Delta^n \to X$), so that the claim follows by covering homotopy extenison (39.8).

The hypothesis that p_* is surjective says exactly that there exists a map $\widetilde{e}\colon (\Delta^n/\partial\Delta^n) \times \Delta^1 \to Y$ such that (i) $\widetilde{e}|(\Delta^n/\partial\Delta^n) \times \{0\} = v$, (ii) $\widetilde{e}|(\Delta^n/\partial\Delta^n) \times \{1\} = \overline{y}$, where $\overline{y}$ is the unique map factoring through $\{y = px\} \to Y$, and (ii) $\widetilde{e}| * \times \Delta^1 = \overline{y}$ is also a constant map. By adjunction, we see that this exactly gives the data of an edge e in $\mathrm{Fun}(*, X) \times_{\mathrm{Fun}(*,Y)} \mathrm{Fun}(\Delta^n/\partial\Delta^n)$ with $e_0 = (\overline{x}, v)$, and $e_1 = (\overline{x}, \overline{y})$, as desired. $\square$

79.9. **Lemma.** *Let $p\colon X \to Y$ be a Kan fibration between Kan complexes. Let $n \geq 0$ be such that $\pi_n(X,x) \to \pi_n(Y,px)$ is injective for all $x \in X_0$. Then $(* \subset \Delta^{n+1}/\partial\Delta^{n+1}) \boxslash p$ implies $(\partial\Delta^{n+1} \subset \Delta^{n+1}) \boxslash p$.*

Proof. Suppose given a lifting problem (u,v) of type $(\partial\Delta^{n+1} \subset \Delta^{n+1}) \boxslash p$. We show that this can be deformed to a lifting problem (u', v') such that $u' = \overline{x}$ for some $x \in X$, i.e., so that the lifting problem (u', v') factors as

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, tail] & * \arrow[r, "\overline{x}"] \arrow[d, tail] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] & \Delta^n/\partial\Delta^n \arrow[r, "v''"'] \arrow[ur, dotted] & Y
\end{tikzcd}$$

Then by hypothesis the lifting problem $(\overline{x}, v'')$ admits a solution, so the claim follows using covering homotopy extenison (39.8).

Let $x := u(\langle 0 \rangle) \in X_0$ and $y = p(x)$. Recall that $\partial\Delta^{n+1}$ is weakly homotopy equivalent to $\Delta^n/\partial\Delta^n$ (79.3), so that we have natural isomorphism $\pi_0 \mathrm{Fun}_*((\partial\Delta^{n+1}, \{0\}), (T,t)) = \pi_n(T,t)$ (79.2). In particular, $u\colon \partial\Delta^{n+1} \to X$ represents an element of $\pi_n(X,x)$. Consider the composite

$$\partial\Delta^n \times \Delta^1 \to \Delta^n \times \Delta^1 \xrightarrow{\gamma} \Delta^n \xrightarrow{v} Y$$

where γ represents the natural transformation $\mathrm{id}_{\Delta^n} \to \langle n \dots n \rangle$. This gives an edge in $\mathrm{Fun}_*((\partial\Delta^n, \{0\}), (Y,y))$ connecting pu with $\overline{y}$, i.e., pu represents the trivial element of $\pi_n(Y,y)$. By hypothesis we conclude that u represents the trivial element of $\pi_n(X,x)$, so there exists $h\colon \partial\Delta^{n+1} \times \Delta^1 \to X$ so that $h|\partial\Delta^{n+1} \times \{0\} = u$, $h|\partial\Delta^{n+1} \times \{1\} = \overline{x}$, and $h|\{0\} \times \Delta^1 = \overline{x}$. Now consider

$$\begin{tikzcd}
(\partial\Delta^{n+1} \times \Delta^1) \cup_{\partial\Delta^{n+1} \times \{0\}} \Delta^{n+1} \times \{0\} \arrow[r, "{(ph,v)}"] \arrow[d, tail, "i"'] & Y \\
\Delta^{n+1} \times \Delta^1 \arrow[ur, dotted, "k"'] &
\end{tikzcd}$$

which is well-defined since $ph|\partial\Delta^{n+1} \times \{0\} = pu = v|\partial\Delta^{n+1} \times \{0\}$. Not that that since i is the pushout product $(\partial\Delta^{n+1} \subset \Delta^{n+1}) \square (\{0\} \subset \Delta^1)$ which is in $\mathrm{Cell}\square\mathrm{Horn} \subseteq \overline{\mathrm{Horn}}$ (**??**), and Y is a Kan complex, a lift k exists. Therefore the pair (h,k) represents an edge in $\mathrm{Fun}(\partial\Delta^{n+1}, X) \times_{\mathrm{Fun}(\partial\Delta^{n+1},Y)} \mathrm{Fun}(\Delta^{n+1}, Y)$ connecting (u,v) with $(\overline{x}, v')$ where $v' = k|\Delta^{n+1} \times \{1\}$. This is what we needed. $\square$

79.10. **Group structures.** We give some exercises which lead to a proof that $\pi_n(X,x)$ is a group when $n \geq 1$, and abelian when $n \geq 2$. We have already noted that $\pi_1(X,x) = \mathrm{Hom}_{hX}(x,x)$, so has a natural group structure (56.5).

Write $\partial(\Delta^p \times \Delta^q) := (\Delta^p \times \partial\Delta^q) \cup (\partial\Delta^p \times \Delta^q)$.

79.11. *Exercise.* Show that $\partial(\Delta^n \times \Delta^1)$, with any choice of basepoint, is weakly homotopy equivalent to $\Delta^n/\partial\Delta^n$ in $s\mathrm{Set}_*$. (Hint: let $S := (\Delta^n \times \{0\}) \cup (\partial\Delta^n \times \Delta^1)$, and note that S is weak homotopy equivalent to Δ^0, and that $\partial(\Delta^n \times \Delta^1)/S \approx \Delta^n/\partial\Delta^n$.)

79.12. *Exercise.* Show that $(\Delta^n \times \Delta^1)/\partial(\Delta^n \times \Delta^1)$ is weakly homotopy equivalent to $\Delta^{n+1}/\partial\Delta^{n+1}$ in $s\mathrm{Set}_*$. (Hint: use (79.3).)

79.13. *Exercise.* Show that $\pi_n(X,x) \approx \pi_{n-1}(\mathrm{map}_X(x,x), 1_X)$ for all $n \geq 1$. Conclude that $\pi_n(X,x)$ is a group if $n \geq 1$.

80. Appendix: Sets generating weakly saturated classes

We show that the weakly saturated classes $\mathrm{CatEq} \cap \overline{\mathrm{Cell}}$ and $\mathrm{GpdEq} \cap \overline{\mathrm{Cell}}$ are each generated by some set S, and so in particular are parts of weak factorization systems

$$(\mathrm{CatEq} \cap \overline{\mathrm{Cell}},\ \mathrm{CatFib}) \qquad \text{and} \qquad (\mathrm{GpdEq} \cap \overline{\mathrm{Cell}},\ \mathrm{GpdFib}).$$

In either case, we will show that the weakly saturated class is generated by the class of injective maps $K \rightarrowtail L$ in the class for which the number of cells in K and L is bounded by some explicit regular cardinal. We obtain S by choosing one representative for each isomorphism class in this class; then S is a set because of the cardinality bound.

80.1. **Lemma.** *Let U be any weakly saturated class of maps of simplicial sets. Suppose Y is a simplicial set with subcomplex $X \subseteq Y$. Then there exists a subcomplex $X' \subseteq Y$ which is maximal with respect to the properties that (i) $X \subseteq X' \subseteq Y$ and (ii) $(X \to X') \in U$.*

Proof. Let $\mathcal{P}$ be the set of all subcomplexes Z of Y such that $X \subseteq Z$ and $(X \to Z) \in U$. Say that $Z \leq Z'$ for $Z, Z' \in \mathcal{P}$ if $Z \subseteq Z'$ and $(Z \to Z') \in U$. Then $\mathcal{P}$ is a partially ordered set since U is closed under composition (but note that $Z \subseteq Z'$ need not imply $Z \leq Z'$). Furthermore, $\mathcal{P}$ is non-empty since $X \in \mathcal{P}$.

I claim that $\mathcal{P}$ satisfies the hypothesis of Zorn's lemma. In fact, suppose $\mathcal{C} \subseteq \mathcal{P}$ is a non-empty chain. Using the axiom of choice we can choose an ordinal λ and a cofinal map $f\colon \lambda \to \mathcal{C}$ (i.e., one such that for all $Z \in \mathcal{C}$ there exists $\alpha \in \lambda$ with $Z \leq f(\alpha)$). Then $B := \mathrm{colim}_{\alpha<\lambda} f(\alpha) = \bigcup_{\alpha<\lambda} f(\alpha)$ is such that $(f(0) \to B) \in U$ since U is closed under transfinite composition, and thus $B \in \mathcal{P}$. Since B is clearly an upper bound for $\mathcal{C}$, Zorn's lemma applies, and $\mathcal{P}$ has a maximal element X'. $\square$

80.2. **Lemma.** *Let T be a weakly saturated class of monomorphisms. Suppose that there exists a regular cardinal κ with the following property: for any inclusion $X \subseteq Y$ of simplicial sets with $(X \to Y) \in T$, Y is equal to the union of the collection of all subcomplexes $B \subseteq Y$ such that (i) B is κ-small, and (ii) $(B \cap X \to B) \in T$. Then $T = \overline{S}$ where $S \subseteq T$ is the subclass of maps whose codomains are κ-small simplicial sets.*

Proof. Let S be the class decribed in the statement. We want to show that any element of T is in $\overline{S}$. Since the maps in T are monomorphisms, it suffices to consider inclusions of subcomplexes $K \subseteq Y$ which are in T, and to show that these are in $\overline{S}$.

Apply (80.1) with the class $\overline{S}$ to obtain $K \subseteq K' \subseteq Y$ maximal with respect to the property that $(K \to K') \in \overline{S}$. If $K' = Y$ we are done. If not, then by hypothesis applied to $K' \subseteq Y$ there exists a κ-small subcomplex $B \subseteq Y$ with $B \not\subseteq K'$ and $(B \cap K' \to B) \in S \subseteq T$. This implies $(K' \to B \cup K') \in \overline{S}$ since this is a cobase change of $B \cap K' \to B$, and therefore $(K \to B \cup K') \in \overline{S}$. This contradicts the maximality of K'. $\square$

80.3. **Detection functors.** Let $\mathcal{C}$ be a class of morphisms in sSet. A **detection functor** for $\mathcal{C}$ is a functor $F\colon \mathrm{Fun}([1],\mathrm{Set}) \to \mathrm{Fun}([1], s\mathrm{Set})$ on arrow categories such that there exists a regular cardinal κ, with the following properties: **detection functor**

1. For any map f of simplicial sets (i.e., object of $\mathrm{Fun}([1], s\mathrm{Set})$) we have $f \in \mathcal{C}$ if and only if $F(f)\colon F_0(f) \to F_1(f)$ is a bijection of sets.
2. The functor F commutes with κ-filtered colimits.
3. The functor F takes maps between κ-small simplicial sets to maps between κ-small sets.

80.4. **Proposition.** *Let F be a detection functor for $\mathcal{C}$, with associated infinite regular cardinal κ. Suppose $f\colon X \subseteq Y$ is an inclusion of a subcomplex which is an element of $\mathcal{C}$. Then Y is a union of all subcomplexes $B \subseteq Y$ such that (i) B is κ^+-small and (ii) $(B \cap X \to B) \in \mathcal{C}$. (Here κ^+ is the successor cardinal to κ, which is also regular.)*

Proof. (Adapted from [Joy08a, D.2.16].) Let $\mathcal{P}$ be the poset of subcomplexes of Y, so that the detection functor gives a composite functor

$$\mathcal{P} \to \mathrm{Fun}([1],\mathrm{Set}), \qquad A \mapsto F(f_A), \qquad f_A\colon A \cap X \rightarrowtail A,$$

which commutes with κ-filtered colimits since F does. For any cardinal α let $\mathcal{P}_\alpha \subseteq \mathcal{P}$ be the subset consisting of subcomplexes $A \subseteq Y$ with $|A| < \alpha$. We will show that that every $A \in \mathcal{P}_\kappa$ is contained in some $B \in \mathcal{P}_{\kappa+}$ such that $F(f_B)$ is a bijection. Since $Y = \bigcup_{A\in\mathcal{P}_\kappa} A = \mathrm{colim}_{A\in\mathcal{P}_\kappa} A$ this proves the claim.

Suppose given $A \in \mathcal{P}_\kappa$. Note that $\mathcal{P}_\kappa$ is κ-filtered and tthus

$$\mathrm{colim}_{A\in\mathcal{P}}\, F(f_A) \approx F(f)$$

since F preserves κ-filtered colimits. Furthermore since $f \in \mathcal{C}$ we have that $F(f)\colon F_0(f) \to F_1(f)$ is a bijection of sets. Therefore, for any $A \in \mathcal{P}_\kappa$ we can choose $A^+ \in \mathcal{P}_\kappa$ with $A \subseteq A^+$ such that a lift exists in

$$\begin{tikzcd} F_0(f_A) \arrow[r] \arrow[d] & F_0(f_{A^+}) \arrow[d] \\ F_1(f_A) \arrow[r] \arrow[ur, dotted, "s"] & F_1(f_{A^+}) \end{tikzcd}$$

Now define a functor $A_\bullet\colon \kappa \to \mathcal{P}_\kappa$ by transfinite induction, so that

- $A_0 := A$,
- $A_{\lambda+1} := A^+$,
- $A_\lambda := \mathrm{colim}_{i<\lambda} A_i$ if $\lambda \leq \kappa$ is a limit ordinal.

Let $B := \mathrm{colim}_{i<\kappa} A_i = \bigcup_{i<\kappa} A_i$, which will be an element of $\mathcal{P}_{\kappa^+}$. Since F preserves κ-filtered colimits, we have $F(f_B) = \mathrm{colim}_{i<\kappa} F(f_{A_i})$, which is seen to be a bijection. Thus we have proved that $A \subseteq B$ with $|B| < \kappa^+$ and $(B \cap X \to X) \in \mathcal{C}$ as desired. $\square$

80.5. **Corollary.** *Suppose $\mathcal{C}$ is a class of maps in* sSet *for which there exists a detection functor, and suppose $T := \mathcal{C} \cap \overline{\mathrm{Cell}}$ is weakly saturated. Then $T = \overline{S}$ for some set S.*

80.6. **Construction of detection functors.** It remains to construct detection functors for the classes CatEq and GpdEq. We obtain the detection functor as a composite of several intermediate steps, so $F := F^{(4)}F^{(3)}F^{(2)}F^{(1)}$.

Step 1: Recall (18.9) that the small object argument gives a *functorial* way to factor a map f as $f = pi$, with $i \in \overline{S}$ and $p \in S^{\boxslash}$ for some set S.

We can apply this with with $S = \mathrm{InnHorn}$, so that we obtain a functor $s\mathrm{Set} \to \mathrm{Fun}([1], s\mathrm{Set})$ sending X to $i_X\colon X \to X'$, where i_X is a categorical equivalence and X' is a quasicategory. Alternately we can apply this with $S = \mathrm{Horn}$, so that i_X is a weak homotopy equivalence and X' a Kan complex.

In either case, for any map of simplicial sets $f\colon X \to Y$ we obtain a commutative square

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "i_Y"'] & Y' \end{tikzcd}$$

so that we get a functor $F^{(1)}\colon \mathrm{Fun}([1], s\mathrm{Set}) \to \mathrm{Fun}([1], \mathrm{qCat}) \subseteq \mathrm{Fun}([1], s\mathrm{Set})$, which on objects sends $f \mapsto f'$. If $S = \mathrm{InnHorn}$ we have that f' is a categorical equivalence if and only if f is, while if $S = \mathrm{Horn}$ we have that f' is a categorical equivalene if and only if f is a weak homotopy equivalence.

Step 2: Define a functor $F^{(2)}\colon \mathrm{Fun}([1], \mathrm{qCat}) \to \mathrm{Fun}([1], \mathrm{qCat})$ which on objects sends $f\colon C \to D$ to the path factorization $p = F^{(2)}(f)\colon \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, C) \times_C D \to D$. We have that p is an isofibration, and is a categorical equivalence if and only if f is. Therefore, f is a categorical equivalence if and only if p is a trivial fibration (40.8).

Step 3: Define a functor $F^{(3)}\colon \mathrm{Fun}([1], \mathrm{qCat}) \to \mathrm{Fun}([1], \mathrm{Set})$ sending $f\colon X \to Y$ to the map of sets

$$F^{(3)}(f)\colon \coprod_{n\geq 0} \mathrm{Hom}(\Delta^n, X) \to \coprod_{n\geq 0} \mathrm{Hom}(\partial\Delta^n, X) \times_{\mathrm{Hom}(\partial\Delta^n, Y)} \mathrm{Hom}(\Delta^n, Y).$$

Thus, f is a trivial fibration if and only if $F^{(3)}(f)$ is surjective.

Step 4: Define a functor $F^{(4)}\colon \mathrm{Fun}([1], \mathrm{Set}) \to \mathrm{Fun}([1], \mathrm{Set})$ sending $f\colon X \to Y$ to

$$F^{(4)}(f)\colon \operatorname{colim}\big[X \times_Y X \rightrightarrows X\big] \to Y.$$

Thus, f is a surjection if and only if $F^{(4)}(f)$ is a bijection.

It is clear that the composite functor F is such that $F(f)$ is a bijection if and only if f is a categorical equivalence (or weak homotopy equivalence). We can choose an infinite regular cardinal such that each of $F^{(i)}$ preserves κ-filtered colimits and takes κ-small simplicial sets to κ-small sets or simplicial sets as the case may be. In fact, any infinite regular cardinal $> \omega$ satisfies when $i = 2, 3, 4$, while for for $i = 1$ we choose κ greater than the size of the domains and codomains of objects in S.

81. Stuff

Just dumping some stuff here that may make it into a future new section.

81.1. **Proposition.** *Let C be a quasicategory, and let i be the inclusion*

$$i\colon A = \partial\Delta^m \times \Delta^n \times \Delta^1 \,\cup\, \Delta^m \times \partial\Delta^n \times \Delta^1 \,\cup\, \Delta^m \times \Delta^n \times \{1\} \to \Delta^m \times \Delta^n \times \Delta^1,$$

with $m, n \geq 1$. Then any $f\colon A \to C$ which sends the edge $\{(m,n)\} \times \Delta^1$ to an isomorphism in C extends over i.

Proof. We have already proved the case of $n = 0$. Prove the general case by building $\Delta^m \times \Delta^n$ from its boundary by attaching cells. □

Here's a better statement of the previous.

81.2. **Proposition.** *Let C be a quasicategory and $i\colon K \to L$ a monomorphism which is a bijection on vertices. Let $S \subseteq L_0$ be the subset of vertices of the form a_k, for all $a \in L_k^{\mathrm{nd}} \smallsetminus K_k^{\mathrm{nd}}$. Then in any diagram*

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, tail] & \mathrm{Fun}(\Delta^1, C) \arrow[d] \\ L \arrow[r] \arrow[ur, dotted] & \mathrm{Fun}(\{1\}, C) \end{tikzcd}$$

if f sends each vertex in S to a vertex in $\mathrm{Fun}(\Delta^1, C)$ *corresponding to an isomorphism in C, then a lift exists.*

81.3. **Corollary.** *Let C be a quasicategory, and let $p\colon \mathrm{Fun}(\Delta^n, C) \to \mathrm{Fun}(\partial\Delta^n, C)$ be the restriction map, with $n \geq 1$. Then an edge α of $\mathrm{Fun}(\Delta^n, C)$ is p-Cartesian iff its corresponding to natural transformation $\alpha\colon f \Rightarrow f'$ of functors $\Delta^n \to C$ is such that $\alpha(n)\colon f(n) \to f'(n)$ is an isomorphism in C.*

81.4. **Lemma.** *Let $p\colon C \to D$ be an inner fibration of quasicategories. In any commutative diagram*

$$\begin{tikzcd} \Delta^m \times \partial\Delta^n \cup \partial\Delta^m \times \Delta^n \arrow[r, "u"] \arrow[d, tail, "i"'] & C \arrow[d, "p"] \\ \Delta^m \times \Delta^n \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

with $m \geq 0$ and $n \geq 1$, if $u(m,n) \in C_0$ is a p-terminal object, then a lift exists.

Proof. Consider the skeletal filtration of i. The non-degenerate k-cells of $\Delta^m \times \Delta^n$ not in the image of i are the pairs (a,b) where $a\colon [k] \to [m]$ and $b\colon [k] \to [n]$ are both surjective functions. In particular $k \geq 1$ since $n \geq 1$. The restriciton

$$\partial\Delta^k \xrightarrow{(a,b)|\partial\Delta^k} \Delta^m \times \partial\Delta^n \cup \partial\Delta^m \times \Delta^n \xrightarrow{u} C$$

to the boundary of Δ^k sends the vertex k to the object $u(m,n) \in E_0$, which is p-terminal by hypothesis. Thus we can inductively construct a lift cell-by-cell along the skeletal filtration of i. $\square$

Let $i\colon K \to L$ be a monomorphism of simplicial sets. Say that a vertex in L is **i-right critical** if it is of the form a_n for some $n \geq 1$ and some $a \in L_n^{\mathrm{nd}} \smallsetminus i(K_n^{\mathrm{nd}})$. Likewise, a vertex is **i-left critical** if it is of the form a_0 for some $n \geq 0$ and some $a \in L_n^{\mathrm{nd}} \smallsetminus i(K_n^{\mathrm{nd}})$. **$i$-right critical** **$i$-left critical**

81.5. **Proposition.** *Let $p\colon E \to B$ be an inner fibration of quasicategories, and $i\colon K \subseteq L$ an inclusion of simplicial sets. Write $q := p^{\square i}\colon \mathrm{Fun}(L, E) \to \mathrm{Fun}(K, E) \times_{\mathrm{Fun}(K,B)} \mathrm{Fun}(L, B)$ for the pullback-hom map*

1. *Consider a map $f\colon L \to E$, regarded as an object of $\mathrm{Fun}(L, E)$. If for each i-right critical vertex $s \in L_0$ the object $f(s) \in E_0$ is p-terminal, then f is q-terminal.*
2. *Consider an object (u, v) of $\mathrm{Fun}(K, E) \times_{\mathrm{Fun}(K,B)} \mathrm{Fun}(L, B)$. If (i) u takes every i-right critical vertex in K to a p-terminal object of E, and (ii) v takes every i-right critical vertex in $L \smallsetminus K$ to an object of B which is the image of some p-terminal object of E, then there exist a q-terminal object s in $\mathrm{Fun}(L, E)$ such that $q(s) = (u, v)$.*

Proof. We need to produce a lift in every commutative square of the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "u"] \arrow[d, tail] & \mathrm{Fun}(L, E) \arrow[d, "q"] \\ \Delta^n \arrow[r, "(v,w)"'] \arrow[ur, dotted] & \mathrm{Fun}(K, E) \times_{\mathrm{Fun}(K,B)} \mathrm{Fun}(L, B) \end{tikzcd}$$

with $n \geq 1$ such that $u(n) = f$, which by hypothesis implies that $u(n)(s) \in E_0$ is p-terminal for each i-right critical vertex $s \in L_0$. Such a lifting problem is equivalent to one of the form

$$\begin{tikzcd} K \arrow[r, "\widetilde{v}"] \arrow[d, tail, "i"'] & \mathrm{Fun}(\Delta^n, E) \arrow[d, "q'"] \\ L \arrow[r, "(\widetilde{u},\widetilde{w})"'] \arrow[ur, dotted] & \mathrm{Fun}(\partial\Delta^n, E) \times_{\mathrm{Fun}(\partial\Delta^n,B)} \mathrm{Fun}(\Delta^n, B) \end{tikzcd}$$

with the property that for each i-right critical vertex $s \in L_0$, the object $\widetilde{u}(s)(n) \in E_0$ is p-terminal. We can construct such a lift inductively using the skeletal filtration of i, and thus reduce to the special case when i is the inclusion $\partial\Delta^m \to \Delta^m$ for some $m \geq 0$. This special case is equivalent to the one whose solution is provided by the lemma. □

81.6. *Example.* If C is a quasicategory with terminal object x, and if $q\colon C \times C \to C$ is projection to the second factor, then any object of the form (x, y) is q-terminal. This is just the proposition applied to $p\colon C \to *$ and $i\colon \{1\} \to \Delta^1$.

81.7. **Proposition.** *Let $p\colon C \to D$ be an inner fibration of quasicategories, and $i\colon K \subseteq L$ an inclusion of simplicial sets which is a bijection on vertices. Let $S \subseteq L_0$ denote the set of vertices of L which have the form a_n for some $n \geq 1$ and some $a \in L_n^{\mathrm{nd}} \smallsetminus K_n^{\mathrm{nd}}$. Write $q = p^{\square i}\colon \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L, D)$ for the pullback-hom map.*

If $\alpha\colon F \to f'$ is a morphism in $\mathrm{Fun}(L, C)$ such that for each $s \in S$, the restriction $\alpha(s)\colon f(s) \to f'(s)$ is a p-Cartesian morphism in C, then α is a q-Cartesian morphism.

Proof. We need to solve a □

81.8. **Proposition.** *Let C be a quasicategory and K a simplicial set, and consider the restriction functor*

$$p\colon \mathrm{Fun}(K^{\rhd}, C) \to \mathrm{Fun}(K \cup \{v\}, C),$$

where v is the cone point of $K^{\rhd}$. Then for any edge $\alpha \in \mathrm{Fun}(K^{\rhd}, C)_1$, if $\alpha(v)$ is an isomorphism in C, then α is p-Cartesian.

81.9. **Corollary.** *If C is a quasicategory and K a simplicial set, the restriction functor*

$$q\colon \mathrm{Fun}(K^{\rhd}, C) \to \mathrm{Fun}(K, C)$$

is a Cartesian fibration.

81.10. **Corollary.** *If C is a quasicategory and K a simplicial set, and $q\colon \mathrm{Fun}(K^{\rhd}, C) \to \mathrm{Fun}(K, C)$ is restriction, then an object in $\mathrm{Fun}(K^{\rhd}, C)$ relatively q-initial iff it is an initial object in its fiber.*

Given a functor $p\colon C \to D$ between quasicategories, we obtain functors

$$q\colon \mathrm{Fun}(\{0\}, C) \times_{\mathrm{Fun}(\{0\},D)} \mathrm{Fun}(\Delta^1, D) \to \mathrm{Fun}(\{1\}, D)$$

and

$$q'\colon \mathrm{Fun}(\{1\}, C) \times_{\mathrm{Fun}(\{1\},D)} \mathrm{Fun}(\Delta^1, D) \to \mathrm{Fun}(\{0\}, D),$$

where all maps are induced in the evident way by p and by restriction to subcomplexes of Δ^1. I will call these the **coCartesian path fibration** and **Cartesian path fibration** respectively. This terminology will be justified soon. **coCartesian path fibration** **Cartesian path fibration**

81.11. **Lemma.** *Let $p\colon C \to D$ be a functor between quasicategories. Then the coCartesian path fibration and Cartesian path fibration associated to p are inner fibrations.*

Proof. We prove the case of the coCartesian path fibration q. Consider the commutative diagram

$$\begin{tikzcd}
(C \times D) \times_{D \times D} \mathrm{Fun}(\Delta^1, D) \arrow[r, "r'"] \arrow[d] & C \times D \arrow[r, "\pi"] \arrow[d] & D \\
\mathrm{Fun}(\Delta^1, D) \arrow[r, "r"'] & D \times D &
\end{tikzcd}$$

in which r is restriction along $\partial\Delta^1 \subset \Delta^1$ and π is projection to the second factor, and the square is a pullback. Then both π and r, and hence r', are inner fibrations (**??**). The claim follows because q is isomorphic to $\pi r'$. □

81.12. **Lemma.** *Let $p\colon C \to D$ be a functor between quasicategories, and consider the coCartesian path fibration*

$$q\colon\ \mathrm{Fun}(\{0\}, C) \times_{\mathrm{Fun}(\{0\},D)} \mathrm{Fun}(\Delta^1, D) \to \mathrm{Fun}(\{1\}, D).$$

If $f\colon \Delta^1 \to \mathrm{Fun}(\{0\}, C)\times_{\mathrm{Fun}(\{0\},D)}\mathrm{Fun}(\Delta^1, D)$ is such that the composite $\Delta^1 \to \mathrm{Fun}(\{0\}, C)\times_{\mathrm{Fun}(\{0\},D)}$ $\mathrm{Fun}(\Delta^1, D) \to \mathrm{Fun}(\{0\}, C)$ represents an isomorphism in C, then f is a q-coCartesian morphism.

Proof. We again refer to the commutative diagram

$$\begin{tikzcd}
(C \times D) \times_{D\times D} \mathrm{Fun}(\Delta^1, D) \arrow[r, "r'"] \arrow[d] & C \times D \arrow[r, "\pi"] \arrow[d] & D \\
\mathrm{Fun}(\Delta^1, D) \arrow[r, "r"'] & D \times D &
\end{tikzcd}$$

in which $\pi r' = q$. To show that f is q-coCartesian, by (**??**) it suffices to show that (i) $r'(f)$ is π-coCartesian, and (ii) f is r'-coCartesian. Since f is an isomorphism, (i) is immediate from (**??**). For (ii), since r' is a basechange of r, it suffices to show that the edge represented by $pf\colon \Delta^1 \to D$ is r-coCartesian. This is immediate from (**??**). ☐

81.13. **Proposition.** *Let $p\colon C \to D$ be a functor between quasicategories. Then the coCartesian path fibration is a coCartesian fibration.*

82. Mapping spaces in alternate slices

Let C be a quasicategory with morphisms $g\colon b \to x$ and $f\colon a \to x$ to a common object x. The morphisms f and g also correspond to objects of the alternate slice $C^{/x}$, which we will also denote by f and g. The goal of this section is to identitfy the mapping space $\mathrm{map}_{C^{/x}}(g, f)$ as the "homotopy fiber" over g of a "map"

$$f_*\colon\ \mathrm{map}_C(a, b) \to \mathrm{map}_C(a, x)$$

which corresponds to "postcomposition with f". In fact, we construct f_* not as a single map but as a zig-zag of maps in which the backwards map is an equivalence.

Consider the subcomplex

$$K := \{(0, 0)\} \ \cup\ \{1\} \times \Delta^1 \ \cup\ \Delta^1 \times \{1\},$$

define $\mathrm{map}'_C(a, b)$ to be the pullback in

$$\begin{tikzcd}
\mathrm{map}'_C(a, b) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1 \times \Delta^1, C) \arrow[d] \\
\{f'\} \arrow[r] & \mathrm{Fun}(K, C)
\end{tikzcd}$$

where $f'\colon K \to C$ is the map with $f'|\{(0, 0)\} = a$, $f'|\{1\} \times \Delta^1 = f$, and $f'|\Delta^1 \times \{1\} = 1_x$.

82.1. **Lemma.** *Restriction along $\Delta^1 \times \{0\} \subset \Delta^1 \times \Delta^1$ induces an equivalence $\mathrm{map}'_C(a, b) \to \mathrm{map}_C(a, b)$.*

Proof. We are comparing the fibers of the vertical maps in

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1 \times \Delta^1, C) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1 \times \{0\}, C) \\
\mathrm{Fun}(K, C) & \mathrm{Fun}(\partial\Delta^1 \times \{0\}, C)
\end{tikzcd}$$

☐

We can construct the following commutative diagram

$$\begin{tikzcd}
\mathrm{map}_{C/x}(g,f) \arrow[r] \arrow[d] & \mathrm{map}'_C(a,b) \arrow[r] \arrow[d] & \mathrm{map}_C(a,b) \\
\{g\} \arrow[r] & \mathrm{map}_C(a,x) &
\end{tikzcd}$$

Let $K \subseteq \Delta^1 \times \Delta^1$ be the subcomplex

$$K := \{(0,0)\}\amalg$$

where $\mathrm{map}'_C(a,b)$ is defined to be the pullback in

$$\begin{tikzcd}
\mathrm{map}'_C(a,b) \arrow[r] & \mathrm{Fun}(\Delta^1 \times \Delta^1, C) \\
\{(a,1_b,f)\} & \mathrm{Fun}(K,C)
\end{tikzcd}$$

83. Additional stuff

83.1. **Large vs. small.** I have been implicitly assuming that certain categories are **small**; i.e., they have *sets* of objects and morphisms. For instance, for the nerve of a category C to be a simplicial set, we need $C_0 = \mathrm{ob}\, C$ to be a set.

small

However, in practice many categories of interest are only **locally small**; i.e., the collection of objects is not a set but is a "proper class", although for any pair of objects $\mathrm{Hom}_C(X,Y)$ is a set. For instance, the category Set of sets is of this type: there is no set of all sets. Other examples include the categories of abelian groups, topological spaces, (small) categories, simplicial sets, etc. It is also possible to have categories which are not even locally small, e.g., the category of locally small categories. These may be called **large** categories.

locally small

large

We would like to be able to talk about large categories in exactly the same way we talk about small categories. This is often done by positing a hierarchy of (Grothendieck) "universes". A universe U is (informally) a collection of sets which is closed under the operations of set theory. We additionally assume that for any universe U, there is a larger universe U' such that $U \in U'$. Thus, if by "set" we mean "U-set", then the category Set is a "U'-category". This idea can be implemented in the usual set theoretic foundations by postulating the existence of suitable *strongly inaccessible cardinals*.

The same distinctions occur for simplicial sets. For instance, the nerve of a small category is a small simplicial set (i.e., the elements form a set), while the nerve of a large category is a large simplicial set.

I'm not going to be pedantic about this. I'll usually assume categories like Set, Cat, sSet, etc., are categories whose objects are "small" sets/categories/simplicial sets/whatever, i.e., are built from sets in a fixed universe U of "small sets". However, I sometimes need to consider examples of sets/categories/simplicial sets/whatever which are not small. I leave it to the reader to determine when this is the case.

In practice, a main point of concern involves constructions such as limits and colimits. Many typical examples of categories $C = \mathrm{Set}, \mathrm{Cat}, s\mathrm{Set}$, etc., in which objects are built out of small sets are **small complete** and **small cocomplete**: any functor $F\colon D \to C$ from a *small* category D has a limit and a colimit in C. This is *not* true if D is not assumed to be small. In this case care about the small/large distinction is necessary.

small complete
small cocomplete

References

[BK11] Clark Barwick and D. M. Kan, *A Thomason-like Quillen equivalence between quasi-categories and relative categories* (2011), available at `arXiv:1101.0772`.

[Ber10] Julia E. Bergner, *A survey of* $(\infty, 1)$*-categories*, Towards higher categories, IMA Vol. Math. Appl., vol. 152, Springer, New York, 2010, pp. 69–83.

[BV73] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin-New York, 1973.

[Cam19] Alexander Campbell, *A counterexample in quasicategory theory* (2019), available at `arXiv:1904.04965`.

[Cis06] Denis-Charles Cisinski, *Les préfaisceaux comme modèles des types d'homotopie*, Astérisque **308** (2006), xxiv+390 (French, with English and French summaries).

[DS95] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126.

[Fri12] Greg Friedman, *Survey article: an elementary illustrated introduction to simplicial sets*, Rocky Mountain J. Math. **42** (2012), no. 2, 353–423, available at `arXiv:0809.4221`.

[GZ67] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York, 1967.

[GJ09] Paul G. Goerss and John F. Jardine, *Simplicial homotopy theory*, Modern Birkhäuser Classics, Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition [MR1711612].

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[HS01] André Hischowitz and Carlos Simpson, *Descente pour les* n*-champs* (2001), available at `arXiv:math/9807049`.

[Hov99] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR1650134 (99h:55031)

[Joy02] A. Joyal, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra **175** (2002), no. 1-3, 207–222. Special volume celebrating the 70th birthday of Professor Max Kelly.

[Joy08a] André Joyal, *The theory of quasi-categories and its applications*, 2008, Notes for course at CRM, Barcelona, February 2008. At `http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf`.

[Joy08b] ———, *Notes on quasi-categories*, 2008. At `http://www.math.uchicago.edu/~may/IMA/Joyal.pdf`.

[JT08] André Joyal and Myles Tierney, *Notes on simplicial homotopy theory*, 2008, Notes for course at CRM, Barcelona, February 2008. At `http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern47.pdf`.

[KLV12] Chris Kapulkin, Peter LeFanu Lumsdaine, and Vladimir Voevodsky, *Univalence in simplicial sets* (2012), available at `arXiv:1203.2553`.

[Lei14] Tom Leinster, *Basic category theory*, Cambridge Studies in Advanced Mathematics, vol. 143, Cambridge University Press, Cambridge, 2014.

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Lur12] ———, *Higher algebra* (2012), available at `http://www.math.harvard.edu/~lurie/`.

[Lur21] ———, *Kerodon* (2021), `https://kerodon.net`.

[Qui67] Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967.

[Rie16] Emily Riehl, *Category theory in context*, Dover Publications, 2016. Available at `http://www.math.jhu.edu/~eriehl/context.pdf`.

[Rie14] ———, *Categorical homotopy theory*, New Mathematical Monographs, vol. 24, Cambridge University Press, Cambridge, 2014.

[Rie09] ———, *A concise definition of a model category* (2009), preprint. Available at `http://www.math.jhu.edu/~eriehl/modelcat.pdf`.

[RV15] Emily Riehl and Dominic Verity, *The 2-category theory of quasi-categories*, Adv. Math. **280** (2015), 549–642.

[RV22] ———, *Elements of* ∞*-category theory*, Cambridge Studies in Advanced Mathematics, vol. 194, Cambridge University Press, Cambridge, 2022.

[TS14] The Stacks Project Authors, *Stacks Project*, 2014. At `http://stacks.math.columbia.edu`.

[Tho80] R. W. Thomason, *Cat as a closed model category*, Cahiers Topologie Géom. Différentielle **21** (1980), no. 3, 305–324.

Department of Mathematics, University of Illinois, Urbana, IL
Email address: `rezk@illinois.edu`