# QUASI-ELLIPTIC COHOMOLOGY

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This is an attempt to understand Ganter's construction of power operations, in [Gan07]. The "quasi-elliptic cohomology" $\mathcal{E}_G$ I describe here is basically the same one she describes, except her version is defined over $\mathbb{Z}[\![q]\!]$ (and in fact, could be defined over $\mathbb{Z}[q]$), while the one I describe is only defined over $\mathbb{Z}[q^{\pm}]$. Inverting $q$ allows me to interpret some constructions more easily in terms of extensions of groups over the circle; also, it will be clear that the construction with $q$ inverted actually gives cohomology theory.

I also drop the condition that $G$ be a finite group; the only advantage to doing this is to give a direct interpretation of $\mathcal{E}_{\mathbb{T}}(\mathrm{pt})$ in terms of the Katz-Mazur group scheme. Although amusing, this is not really the correct way to handle infinite groups. For instance, things should be set up so that $\mathcal{E}_{\mathbb{T}}$ takes values in sheaves on the object $\mathbb{G}_m/\!\!/q^{\mathbb{Z}}$, which is a stacky group object defined over $\operatorname{Spec}\mathbb{Z}[q^{\pm}]$. Instead, we will only consider a construction which takes values in modules on the Katz-Mazur group scheme $T(q) := (\mathbb{G}_m/\!\!/q^{\mathbb{Z}})_{\mathrm{tors}}$ over $\operatorname{Spec}\mathbb{Z}[q^{\pm}]$.

The term "quasi-elliptic" is used because $\mathcal{E}_{\mathbb{T}}$ is not naturally attached to an elliptic curve, but after base change to $\mathbb{Z}(\!(q)\!)$ we can attach the Tate curve to this theory.

## 1. Extending groups over the circle

In what follows, we write $\mathbb{T}$ for the Lie group $\mathbb{R}/\mathbb{Z}$. Let $q\colon \mathbb{T} \to U(1)$ be the isomorphism $t \mapsto e^{2\pi i t}$. We may think of $q$ as the tautological 1-dimensional representation of $\mathbb{T}$, so that we thus fix an identification

$$K^0_{\mathbb{T}}(*) = R\mathbb{T} \approx \mathbb{Z}[q^{\pm}].$$

More generally, given an non-zero integer $n$, we write $\mathbb{T}_n$ for the Lie group $\mathbb{R}/n\mathbb{Z}$. Then we write $q^{1/n}\colon \mathbb{T}_n \to U(1)$ for the isomorphism $t \mapsto e^{2\pi i t/n}$, and thus fix an identification

$$K^0_{\mathbb{T}_n}(*) = R\mathbb{T}_n \approx \mathbb{Z}[q^{\pm 1/n}].$$

The choice is made so that the projection $\mathbb{T}_n \xrightarrow{t\mapsto t} \mathbb{T}$ induces on representation rings the evident inclusion $\mathbb{Z}[q^{\pm}] \xrightarrow{q\mapsto q} \mathbb{Z}[q^{\pm 1/n}]$. Thus the evident isomorphism $\mathbb{T}_n \xrightarrow{[t]\mapsto[t/n]} \mathbb{T}$ induces the isomorphism $\mathbb{Z}[q^{\pm}] \xrightarrow{q\mapsto q^{1/n}} \mathbb{Z}[q^{\pm 1/n}]$.

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1.1. **The groups $\widetilde{G}_\sigma$.** Let $G$ be a compact Lie group. Given an element $\sigma$ in the *center* of $G$, we define the topological group $\widetilde{G}_\sigma$ to be the quotient of $G\times\mathbb{R}$ by the subgroup generated by $(\sigma,-1)$. Thus, there is an exact sequence

$$0\to\mathbb{Z}\xrightarrow{1\mapsto(\sigma,-1)} G\times\mathbb{R}\xrightarrow{(g,t)\mapsto[g,t]}\widetilde{G}_\sigma\to 1,$$

and thus $\widetilde{G}_\sigma$ sits in an extension

$$1\to G\xrightarrow{g\mapsto[g,0]}\widetilde{G}_\sigma\xrightarrow{[g,t]\mapsto t}\mathbb{T}\to 0.$$

If $\sigma$ is an element of finite order $k$, then we get a smaller extension

$$0\to\mathbb{Z}/k\mathbb{Z}\xrightarrow{1\mapsto(\sigma,[-1])} G\times\mathbb{T}_k\xrightarrow{(g,[t])\mapsto[g,t]}\widetilde{G}_\sigma\to 1.$$

Note that there is a homomorphism $\gamma\colon\mathbb{R}\to\widetilde{G}_\sigma$, defined by $[t]\mapsto[e,t]$, whose image lies in the center of $\widetilde{G}_\sigma$. If $\sigma$ has infinite order, then $\gamma$ is injective, while if $\sigma$ has finite order $k$, it descends to an injective homomorphism $\mathbb{R}/k\mathbb{Z}\to\widetilde{G}_\sigma$, which fits in an exact sequence

$$0\to\mathbb{R}/k\mathbb{Z}\xrightarrow{t\mapsto[e,t]}\widetilde{G}_\sigma\xrightarrow{[g,t]\mapsto g} G/\langle\sigma\rangle\to 1.$$

Thus, for torsion central elements $\sigma$, the group $\widetilde{G}_\sigma$ is a $U(1)$-central extension of $G/\langle\sigma\rangle$.

1.2. **The groups $\widetilde{G}^m_\sigma$.** The above construction admits the following mild variant. Given an element $\sigma$ in the center of $G$, and $m\geq 1$ an integer, we define $\widetilde{G}^m_\sigma$ to be the quotient $G\times\mathbb{R}/(\sigma^m,-m)^{\mathbb{Z}}$. When $m=1$, this coincides with the construction described above. In general, there is an isomorphism $\widetilde{G}^m_\sigma\approx\widetilde{G}_{\sigma^m}$ given by $[g,t]\mapsto[g,t/m]$. The difference between $\widetilde{G}^m_\sigma$ and $\widetilde{G}_{\sigma^m}$ is thus purely notational, but we will have need to distinguish them carefully.

There are exact sequences

$$0\to\mathbb{Z}\xrightarrow{1\mapsto(\sigma^m,-m)} G\times\mathbb{R}\xrightarrow{(g,t)\mapsto[g,t]}\widetilde{G}^m_\sigma\to 1$$

and

$$1\to G\xrightarrow{g\mapsto[g,0]}\widetilde{G}^m_\sigma\xrightarrow{[g,t]\mapsto t}\mathbb{T}_m\to 0.$$

If $\sigma^m$ is an element of finite order $k$, there is an extension

$$0\to\mathbb{Z}/k\mathbb{Z}\xrightarrow{1\mapsto(\sigma^m,[-1])} G\times\mathbb{T}_{mk}\xrightarrow{(g,[t])\mapsto[g,t]}\widetilde{G}^m_\sigma\to 1.$$

As above, we define $\gamma\colon\mathbb{R}\to\widetilde{G}^m_\sigma$ by $[t]\mapsto[e,t]$. If $\sigma^m$ has infinite order, then $\gamma$ is injective, while if $\sigma^m$ has finite order $k$, then $\gamma$ descends to an injective homomorphism $\mathbb{R}/k\mathbb{Z}\to\widetilde{G}^m_\sigma$ which fits into an exact sequence

$$0\to\mathbb{Z}/k\mathbb{Z}\xrightarrow{t\mapsto[e,t]}\widetilde{G}^m_\sigma\xrightarrow{[g,t]\mapsto g} G/\langle\sigma^m\rangle\to 1.$$

1.3. **Representations of extensions over a circle.** We refer to the map $\pi\colon \widetilde{G}^m_\sigma \to \mathbb{T}_m$ defined by $\pi([g,t]) = [t]$ as the **tautological projection**. It induces a map of representation rings $\pi^*\colon R\mathbb{T}_m \to R\widetilde{G}^m_\sigma$. We identify $q^{1/m} \in \mathbb{R}\mathbb{T}_m$ with its image $\pi^*(q^{1/m}) \in R\widetilde{G}^m_\sigma$.

1.4. **Lemma.** *Suppose $\sigma$ is an element of the center of $G$, and $m \geq 1$. Then, the map $\pi^*\colon R\mathbb{T}_m \to R\widetilde{G}^m_\sigma$ exhibits $R\widetilde{G}^m_\sigma$ as a free $R\mathbb{T}_m$-module.*

*In particular, there is an $R\mathbb{T}_m$-basis of $R\widetilde{G}^m_\sigma$ given by irreducible represations $\{V_\lambda\}$, such that restriction $V_\lambda \mapsto V_\lambda|G$ to $G$ defines a bijection between $\{V_\lambda\}$ and the set $\{\lambda\}$ of irreducible representations of $G$. Furthermore, the set of all irreducible representations of $\widetilde{G}^m_\sigma$ is precisely the set of all $V_\lambda \otimes q^{k/m}$, where $k \in \mathbb{Z}$.*

*Proof.* As noted earlier, there is an isomorphism $\widetilde{G}^m_\sigma \approx \widetilde{G}_{\sigma^m}$ which is compatible via the tautological projections with the evident isomorphism $\mathbb{T}_m \approx \mathbb{T}$. Thus, without loss of generality we may reduce to the case $m = 1$.

Consider the extension

$$0 \to \mathbb{Z} \xrightarrow{1\mapsto(\sigma,-1)} G \times \mathbb{R} \xrightarrow{(g,t)\mapsto[g,t]} \widetilde{G}_\sigma \to 1.$$

We claim that there exists a bijective correspondence between

(1) isomorphism classes of irreducible $\widetilde{G}_\sigma$-representations $V$, and
(2) isomorhpism classes of pairs $(W, \chi)$, where $W$ is an irreducible $G$-representation, and $\chi\colon \mathbb{R} \to \mathbb{C}^\times$ is a character such that $\sigma \in G$ acts on $W$ via scalar multiplication by $\chi(1)$.

The correspondence sends $V$ to $(V|_G, \chi)$, where $\chi(t)$ describes the action of $[1,t]$. It is clear that any pair as in (2) arises from a unique irreducible $\widetilde{G}_\sigma$ representation.

Conversely, an arbitrary $n$-dimensional $\widetilde{G}_\sigma$ representation $V$ must have the form $\lambda \otimes \eta$, where $\lambda\colon G \to GL(n, \mathbb{C})$ is an $n$-dimensional $G$-representation, and $\eta\colon \mathbb{R} \to GL(n, \mathbb{C})$ is a homomorphism, such that $\lambda(\sigma) = \eta(1)$. If $V$ is irreducible as a $\widetilde{G}_\sigma$ representation, then $\eta(t) = \chi(t)I$ for some scalar $\chi(t) \in \mathbb{C}^*$, since $[0,t]$ is central in $\widetilde{G}_\sigma$. It follows that $\lambda$ is an irreducible $G$-representation (since a decomposition of $\lambda$ would also be a decomposition of $V$ as a $\widetilde{G}_\sigma$-representation). □

1.5. **Positive energy representations.** Recall the homomorphism $\gamma\colon \mathbb{R} \to \widetilde{G}^m_\sigma$ defined by $\gamma(t) = [e,t]$; note that $\pi\gamma\colon \mathbb{R} \to \mathbb{T}_m = \mathbb{R}/m\mathbb{Z}$ is the tautological quotient map, sending $t \mapsto [t]$.

We say that a representation $\rho\colon \widetilde{G}^m_\sigma \to GL(n, \mathbb{C})$ is a **positive energy representation** if $D_\rho := \frac{1}{2\pi i} \frac{d(\rho\circ\gamma)}{dt}|_{t=0}$ is positive semi-definite (i.e., its eigenvalues are all non-negative).

If $\rho$ is an irreducible representation of $\widetilde{G}^m_\sigma$, then $D_\rho = E(\rho)I$, and the scalar $E(\rho) \in \mathbb{R}$ is the **energy** of $\rho$. If $\sigma^m$ has finite order $k$ in $G$, then $E(\rho) \in \frac{1}{k}\mathbb{Z} \subset \mathbb{R}$. We have that $E(\rho \otimes q^{k/m}) = E(\rho) + k/m$ for $k \in \mathbb{Z}$, and more generally if $\rho$ and $\rho'$ are irreducible, then $\rho \otimes \rho'$ is a direct sum of irreducibles, each of which has energy $E(\rho)E(\rho')$. Thus, we can regard the representation ring $R\widetilde{G}^m_\sigma$ as a ring graded by energy, where for $E \in \mathbb{R}$ the $E$-homogeneous summand of $R\widetilde{G}^m_\sigma$ consists of all formal linear combinations of irreducibles of energy $E$.

Let $R_+\widetilde{G}^m_\sigma \subset R\widetilde{G}^m_\sigma$ denote the subgroup generated by positive energy representations; equivalently, it is the subgroup spanned by the irreducible positive energy representations. The representation ring $R_+\widetilde{G}^m_\sigma$ of positive energy representations is an algebra over $\mathbb{R}_+\mathbb{T}_m = \mathbb{Z}[q^{1/m}]$.

Fix $m \geq 1$. For each irreducible $G$-representation $\lambda\colon G \to GL(n, \mathbb{C})$, write $\lambda(\sigma^m) = e^{2\pi i m E} I$ for a (necessarily unique) $E \in [0, 1/m)$, and set $\chi(t) = e^{2\pi i m E t}$. Then the representation $\rho$ of $\widetilde{G}_\sigma$ determined by the pair $(\lambda, \chi)$ is irreducible with energy $E(\rho) = E$, and in fact is the irreducible of $\widetilde{G}_\sigma$ of minimal energy which restricts to $\lambda$. The collection of such minimal positive energy irreducibles gives a canonical choice of $\mathbb{Z}[q^{\pm 1/m}]$-basis of $R\widetilde{G}^m_\sigma$, which is also a $\mathbb{Z}[q^{1/m}]$-basis of $R_+\widetilde{G}^m_\sigma$.

If $\sigma$ is a central element such that $\sigma^m$ has finite order $k$, there is an injective ring homomorphism

$$R\widetilde{G}^m_\sigma \to RG[q^{\pm 1/km}],$$

obtained by restriction along the surjective homomorphism $G \times \mathbb{T}_{mk} \to \widetilde{G}_\sigma$ defined by $(g, [t]) \mapsto [g, t]$, and using our standard identification $\mathbb{R}\mathbb{T}_{mk} = \mathbb{Z}[q^{\pm 1/mk}]$. Under this map an irreducible representation $\rho$ is sent to $(\rho|G)q^{E(\rho)}$. The image of this monomorphism can be identified with the set of $V(q) = \sum V_\ell q^{\ell/mk}$ where each $V_\ell \in RG$ is a formal linear combination of irreducible representations $\lambda$ for which $\lambda(\sigma^m) = e^{2\pi i \ell/k} I$.

Even if $\sigma$ is not of finite order, we can still construct an injective ring homomorphism of the form

$$R\widetilde{G}^m_\sigma \to RG[q^c | c \in \mathbb{R}],$$

so that an irreducible $\rho$ is sent to $(\rho|G)q^{E(\rho)}$, and whose image is characterized in the same way as above.

1.6. **The extended centralizers $\Lambda(\sigma)$ and $\Lambda^m(\sigma)$.** Given an arbitrary element $\sigma$ of a compact Lie group $G$, we define $\Lambda(\sigma) = \Lambda_G(\sigma) := \widetilde{C(\sigma)}_\sigma$, where $C(\sigma) \subseteq G$ is the centralizer of $\sigma$ in $G$.

1.7. *Example.* Let $G = \mathbb{Z}/N$ for $N \geq 1$, and let $\sigma \in G$. Given an integer $k \in \mathbb{Z}$ which projects to $\sigma \in \mathbb{Z}/N$, let $x_k$ denote the representation of $\Lambda(\sigma)$ defined by

$$\Lambda(\sigma) = (\mathbb{Z} \times \mathbb{R})/(\mathbb{Z}(N, 0) + \mathbb{Z}(k, -1)) \xrightarrow{[a,t] \mapsto [(a+kt)/N]} \mathbb{R}/\mathbb{Z} = \mathbb{T} \underset{\sim}{\xrightarrow{q}} U(1).$$

Then $x_k$ is a 1-dimensional representation of energy $E(x_k) = k/N$, and $R\Lambda(\sigma) \approx \mathbb{Z}[q^\pm, x_k]/(x_k^N - q^k)$, where $q$ represents $\Lambda(\sigma) \xrightarrow{\pi} \mathbb{T} \xrightarrow{q} U(1)$. Observe that $x_{k+N} = q x_k$. The representations $x_k$ each restrict to the fundamental representation $\mathbb{Z}/N \xrightarrow{a \mapsto a/N} \mathbb{R}/\mathbb{Z} = \mathbb{T} \xrightarrow{q} U(1)$ of $G$. The set

$$x_k^j q^{-\lfloor jk/N \rfloor}, \qquad j = 0, \dots, N-1,$$

is the $\mathbb{Z}[q^\pm]$-basis by minimal positive energy irreducibles.

1.8. *Example.* Let $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let $\sigma \in G$. Given a number $c \in \mathbb{R}$ which projects to $\sigma$, let $z_c$ denote the representation of $\Lambda(\sigma)$ defined by

$$\Lambda(\sigma) = (\mathbb{R}\times\mathbb{R})/(\mathbb{Z}(1,0)+\mathbb{Z}(c,-1)) \xrightarrow{[x,t]\mapsto[x+ct]} \mathbb{R}/\mathbb{Z} = \mathbb{T} \underset{\sim}{\xrightarrow{q}} U(1).$$

Then $z_c$ is a 1-dimensional representation of energy $E(z_c) = c$, and $R\Lambda(\sigma) \approx \mathbb{Z}[q^{\pm}, z_c^{\pm}]$. Observe that $z_{c+1} = qz_c$. The representation $z_c$ restricts to the fundamental representation $\mathbb{T} \xrightarrow{x\mapsto x} \mathbb{T} \underset{\sim}{\xrightarrow{q}} U(1)$. The set

$$z_c^j q^{-\lfloor jc \rfloor}, \qquad j \in \mathbb{Z},$$

is the $\mathbb{Z}[q^{\pm}]$-basis by minimal positive energy irreducibles.

Under the map $\phi\colon \mathbb{Z}/N\mathbb{Z} \xrightarrow{k\mapsto k/N} \mathbb{R}/\mathbb{Z} = \mathbb{T}$, if $c = k/N$ for $k \in \mathbb{Z}$, then the restriction map $R\Lambda_{\mathbb{T}}([c]) \to R\Lambda_{\mathbb{Z}/N\mathbb{Z}}([k])$ sends $z_c$ to $x_k$.

More generally, we define $\Lambda^m(\sigma) = \Lambda^m_G(\sigma) := \widetilde{C(\sigma)}^m_\sigma$.

1.9. *Example.* Let $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\sigma \in G$, and choose $c \in \mathbb{R}$ which projects to $\sigma$. For $m \geq 1$, let $z_c$ denote the representation of $\Lambda^m(\sigma)$ defined by

$$\Lambda^m(\sigma) = (\mathbb{R}\times\mathbb{R})/(\mathbb{Z}(1,0)+\mathbb{Z}(mc,-m)) \xrightarrow{[x,t]\mapsto[x+ct]} \mathbb{R}/\mathbb{Z} = \mathbb{T} \underset{\sim}{\xrightarrow{q}} U(1).$$

Then $z_c$ is a 1-dimensional representation of energy $E(z_c) = c$, and $R\Lambda^m(\sigma) = \mathbb{Z}[q^{\pm 1/m}, z_c^{\pm}]$, and $z_{c+1} = qz_c$.

Likewise, if $G = \mathbb{Z}/N\mathbb{Z}$, and $\sigma \in G$ is the image of $k \in \mathbb{Z}$, then let $x_k$ denote the representation of $\Lambda^m(G)$ defined by

$$\Lambda^m(\sigma) = (\mathbb{Z}\times\mathbb{R})/(\mathbb{Z}(N,0)+\mathbb{Z}(mk,-m)) \xrightarrow{[a,t]\mapsto[(a+kt)/N]} \mathbb{R}/\mathbb{Z} = \mathbb{T} \underset{\sim}{\xrightarrow{q}} U(1).$$

Then $E(x_k) = k/N$, and $R\Lambda^m(\sigma) \approx \mathbb{Z}[q^{\pm 1/m}, x_k]/(x_k^N - q^k)$.

1.10. **Products of extended centralizers.** Given $\sigma \in G$ and $\tau \in H$, we may consider the fiber product $\Lambda^m_G(\sigma) \times_{\mathbb{T}_m} \Lambda^m_H(\sigma)$ of groups over the tautological projections to $\mathbb{T}_m$.

1.11. **Proposition.** *The map* $[(g,h),t] \mapsto ([g,t],[h,t])$ *defines an isomorphism of groups* $\Lambda^m_{G\times H}(\sigma,\tau) \xrightarrow{\sim} \Lambda^m_G(\sigma) \times_{\mathbb{T}_m} \Lambda^m_H(\tau)$.

We have the following result on representation rings.

1.12. **Proposition.** *The map* $R\Lambda^m_G(\sigma) \otimes_{R\mathbb{T}_m} R\Lambda^m_H(\tau) \to R\Lambda^m_{G\times H}(\sigma,\tau)$ *which sends a tensor product* $V\otimes W$ *of representations to its restriction along the inclusion* $\Lambda^m_{G\times H}(\sigma,\tau) \to \Lambda^m_G(\sigma)\times \Lambda^m_H(\tau)$ *is an isomorphism.*

*Proof.* This is a straightforward exercise using (1.4). □

1.13. **The maps $\alpha$ and $\beta$.** We describe homomorphisms

$$\Lambda^n(\sigma) \xleftarrow{\alpha} \Lambda^{mn}(\sigma) \xrightarrow{\beta} \Lambda^n(\sigma^m)$$

for all $m, n \geq 1$ as follows.

Observe that there is a pullback square of groups

$$\begin{array}{ccc} \Lambda^{mn}(\sigma) & \xrightarrow[\alpha]{[g,t]\mapsto[g,t]} & \Lambda^n(\sigma) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{T}_{mn} & \xrightarrow[{[t]\mapsto[t]}]{} & \mathbb{T}_n \end{array}$$

and thus an extension

$$0 \to \mathbb{Z}/m\mathbb{Z} \xrightarrow{1\mapsto[\sigma,-1]} \Lambda^{mn}(\sigma) \xrightarrow{\alpha} \Lambda^n(\sigma) \to 1.$$

1.14. **Proposition.** *The commutative square*

$$\begin{array}{ccc} R\Lambda^{mn}(\sigma) & \xleftarrow{R\alpha} & R\Lambda^n(\sigma) \\ {\scriptstyle R\pi}\uparrow & & \uparrow{\scriptstyle R\pi} \\ R\mathbb{T}_{mn} & \longleftarrow & R\mathbb{T}_n \end{array}$$

*is a pushout square in the category of $\lambda$-rings. In particular, there is a canonical isomorphism of $\lambda$-rings*

$$R\Lambda^n(\sigma)[q^{1/m}] \xrightarrow{\sim} R\Lambda^{mn}(\sigma),$$

*where "$A[q^{1/m}]$" is shorthand for "$\lambda$-ring pushout of $\mathbb{Z}[q^{\pm}] \to A$ along $\mathbb{Z}[q^{\pm}] \subset \mathbb{Z}[q^{\pm 1/m}]$".*

*Proof.* This is straightforward given the description of the irreducibles of $R\Lambda(\sigma)$ and $R\Lambda^m(\sigma)$. Recall that pushout in $\lambda$-rings coincides with the pushout of the underlying commutative rings. $\square$

Note that there if $\sigma^m$ has finite order $k$, there is a commutative diagram of rings

$$\begin{array}{ccc} R\Lambda^n(\sigma) & \longrightarrow & RC(\sigma)[q^{\pm 1/kn}] \\ {\scriptstyle R\alpha}\downarrow & & \downarrow \\ R\Lambda^{mn}(\sigma) & \longrightarrow & RC(\sigma)[q^{\pm 1/kmn}] \end{array}$$

where the horizontal maps are the inclusions we have defined above, and the right-hand vertical map is the evident inclusion.

The homomorphism $\beta\colon \Lambda^{mn}(\sigma) \to \Lambda^n(\sigma^m)$ is defined to be the one in the following square (which is not generally a pullback square)

$$\begin{array}{ccc} \Lambda^{mn}(\sigma) & \xrightarrow[\beta]{[g,t]\mapsto[g,t/m]} & \Lambda^n(\sigma^m) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathbb{T}_{mn} & \xrightarrow[{[t]\mapsto[t/m]}]{\sim} & \mathbb{T}_n \end{array}$$

Note that $\beta$ is an isomorphism if $C(\sigma^m) = C(\sigma)$. We observe that the induced homomorphism

$$R\Lambda^n(\sigma^m) \xrightarrow{R\beta} R\Lambda^{mn}(\sigma)$$

on representation rings is a $\lambda$-ring homomorphism which sends $q^{1/n} \mapsto q^{1/mn}$. If $\sigma$ has finite order $k$, there is a commutative diagram of rings

$$\begin{array}{ccc} R\Lambda^n(\sigma^m) & \longrightarrow & RC(\sigma^m)[q^{\pm 1/kn}] \\ {\scriptstyle R\beta}\big\downarrow & & \big\downarrow \\ R\Lambda^{mn}(\sigma) & \longrightarrow & RC(\sigma)[q^{\pm 1/kmn}] \end{array}$$

where the horizontal maps are the inclusions we have defined above, and the right-hand vertical map is defined using $q^{1/kn} \mapsto q^{1/kmn}$ and the map on representations induced by the inclusion $C(\sigma) \subseteq C(\sigma^m)$.

1.15. *Example.* Let $G = \mathbb{R}/\mathbb{T}$, and let $\sigma = [c]$ for some $c \in \mathbb{R}$. Under the identifications $\Lambda^m(\sigma) = \mathbb{Z}[q^{\pm 1/m}, z_c]$ described earlier, the maps $R\Lambda^n(\sigma) \xrightarrow{R\alpha} R\Lambda^{mn}(\sigma) \xleftarrow{R\beta} R\Lambda^n(\sigma^m)$ are described by ring homomorphisms

$$\mathbb{Z}[q^{\pm 1/n}, z_c] \xrightarrow{R\alpha} \mathbb{Z}[q^{\pm 1/mn}, z_c] \xleftarrow{R\beta} \mathbb{Z}[q^{\pm 1/n}, z_{mc}],$$

where $R\alpha$ sends $q^{1/n} \mapsto q^{1/n}$ and $z_c \mapsto z_c$, while $R\beta$ sends $q^{1/n} \mapsto q^{1/mn}$ and $z_{mc} \mapsto z_c$.

1.16. *Example.* Let $G = \mathbb{Z}/N\mathbb{Z}$, and let $\sigma = [k]$ for some $k \in \mathbb{Z}$. Under the identifications $\Lambda^m(\sigma) = \mathbb{Z}[q^{\pm 1/m}, x_k]/(x_k^N - q^k)$ described earlier, the maps $R\Lambda^n(\sigma) \xrightarrow{R\alpha} R\Lambda^{mn}(\sigma) \xleftarrow{R\beta} R\Lambda^n(\sigma^m)$ are described by ring homomorphisms

$$\mathbb{Z}[q^{\pm 1/n}, x_k]/(x_k^N - q^k) \xrightarrow{R\alpha} \mathbb{Z}[q^{\pm 1/mn}, x_k]/(x_k^N - q^k) \xleftarrow{R\beta} \mathbb{Z}[q^{\pm 1/n}, x_{mk}]/(x_{mk}^N - q^k),$$

where $R\alpha$ sends $q^{1/n} \mapsto q^{1/n}$ and $x_k \mapsto x_k$, while $R\beta$ sends $q^{1/n} \mapsto q^{1/mn}$ and $x_{mk} \mapsto x_k$.

## 2. Quasi-elliptic cohomology

Given a space $X$ with an action by $G$, and $\sigma \in G$, we let the group $\Lambda^m(\sigma)$ act on the fixed point space $X^\sigma \subseteq X$ in a way that extends the natural action of $C(\sigma)$, namebly by

$$[g,t] \cdot x := g \cdot x.$$

We note that an element $u \in G$ induces by conjugation an intertwining $\kappa_u$ of the actions $\Lambda(\sigma) \curvearrowright X^\sigma$ and $\Lambda(u\sigma u^{-1}) \curvearrowright X^{u\sigma u^{-1}}$. By this, we mean the homomorphism

$$\kappa_u \colon \Lambda(\sigma) \to \Lambda(u\sigma u^{-1}), \qquad \kappa_u([g,t]) = [ugu^{-1}, t]$$

and the map

$$\kappa_u \colon X^\sigma \to X^{u\sigma u^{-1}}, \qquad \kappa_u(x) = ux,$$

which satisfy $\kappa_u([g,t]\cdot x) = \kappa_u([g,t]) \cdot \kappa_u(x)$. Such data induces an evident isomorphism

$$\kappa_u^* \colon K^*_{\Lambda^m(u\sigma u^{-1})}(X^{u\sigma u^{-1}}) \to K^*_{\Lambda^m(\sigma)}(X^\sigma),$$

and note that $\kappa^*_{vu} = \kappa^*_u \kappa^*_v$.

2.1. **Definition of quasi-elliptic cohomology.** Let $G$ be a compact lie group. The **quasi-elliptic cohomology** of a $G$-space $X$ is defined to be

$$\mathcal{E}^*_G(X) := \Bigl(\prod_{\sigma \in G^{\mathrm{tors}}} K^*_{\Lambda(\sigma)} X^\sigma\Bigr)^G,$$

where $G^{\mathrm{tors}} \subseteq G$ is the set of torsion elements of $G$, and $G$ acts on the product as follows: an element $u \in G$ sends $x = (x_\sigma)$ to $x \cdot u$ defined by $(x\cdot u)_\sigma = \kappa^*_u(x_{u\sigma u^{-1}})$. Given a set $G^{\mathrm{tors}}_{\mathrm{conj}}$ of representatives of $G$-conjugacy classes in $G^{\mathrm{tors}}$, we can write

$$\mathcal{E}^*_G(X) \approx \prod_{\sigma \in G^{\mathrm{tors}}_{\mathrm{conj}}} K^*_{\Lambda(\sigma)} X^\sigma,$$

where $G^{\mathrm{tors}} \subseteq G$ is the set of torsion elements of $G$, and $G^{\mathrm{tors}}_{\mathrm{conj}}$ is a set of representatives of $G$-conjugacy classes in $G^{\mathrm{tors}}$. The factor $K^*_{\Lambda(\sigma)} X^\sigma$ is the usual equivariant $K$-theory of $X^\sigma$ as a $\Lambda(\sigma)$-space. The functor $X \mapsto \mathcal{E}^*_G(X)$ defines an equivariant cohomology theory on the category of $G$-CW complexes, taking values in graded commutative rings.

Here we are mainly interested in the case when $G$ is a finite group, in which case $G^{\mathrm{tors}} = G$. When $G$ is not a finite Lie group, the product defining $\mathcal{E}_G$ will be an infinite product. It will be convenient in this case to regard $\mathcal{E}^*_G(X)$ as a pro-ring, topologized as an inverse limit of finite products. As we noted in the introduction, this is not the optimal extension of this theory to infinite compact Lie groups.

The cohomology of a point is given by

$$\mathcal{E}^*_G = \mathcal{E}^*_G(\mathrm{pt}) \approx \prod_{\sigma \in G^{\mathrm{tors}}_{\mathrm{conj}}} R\Lambda(\sigma)[U, U^{-1}].$$

The projection maps $\pi \colon \Lambda(\sigma) \to \mathbb{T}$ give ring homomorphisms $\mathbb{Z}[q^\pm] = K^0_{\mathbb{T}}(\mathrm{pt}) \to K^0_{\Lambda(\sigma)}(\mathrm{pt}) \to K^0_{\Lambda(\sigma)} X^\sigma$, and so $\mathcal{E}^*_G(X)$ is naturally a $\mathbb{Z}[q^\pm]$-algebra. By (1.4), we see that $\mathcal{E}^*_G(\mathrm{pt})$ is a flat $\mathbb{Z}[q^\pm]$-algebra; when $G$ is finite, $\mathcal{E}^*_G(\mathrm{pt})$ is a finitely generated free $\mathbb{Z}[q^\pm]$-module.

2.2. **Künneth map.** We define a Künneth map $\mathcal{E}^*_G(X)\widehat{\otimes}_{\mathcal{E}^*_G}\mathcal{E}^*_H(Y) \to \mathcal{E}^*_{G\times H}(X\times Y)$ as follows. Given $\sigma\in G$ and $\tau\in H$, recall the tautological isomorphism of groups $\Lambda_{G\times H}(\sigma,\tau)\approx \Lambda_G(\sigma)\times_{\mathbb{T}}\Lambda_H(\tau)\subset\Lambda_G(\sigma)\times\Lambda_H(\tau)$, defined by $[(g,h),t]\mapsto([g,t],[h,t])$. Thus we obtain maps

$$K^*_{\Lambda_G(\sigma)}(X^\sigma)\otimes_{K^*_{\mathbb{T}}}K^*_{\Lambda_H(\tau)}(Y^\tau)\to K^*_{\Lambda_G(\sigma)\times_{\mathbb{T}}\Lambda_H(\tau)}(X^\sigma\times Y^\tau)\approx K^*_{\Lambda_{G\times H}(\sigma,\tau)}(X\times Y)^{(\sigma,\tau)}$$

which fit together to give the desired Künneth map. Note that the symbol "$\widehat{\otimes}$" is meant to represent a suitable completed tensor product, so that

$$\mathcal{E}^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^\pm]}\mathcal{E}^*_H(Y)\overset{\text{def}}{=}\prod_{(\sigma,\tau)\in(G\times H)^{\text{tors}}}K^*_{\Lambda(\sigma)}X^\sigma\otimes_{K^*_{\mathbb{T}}}K^*_{\Lambda_H(\tau)}Y^\tau.$$

If either $G$ or $H$ is a finite group, we can omit the completion.

The Künneth map is in some cases an isomorphism, in particular, we see using (1.4) that

$$\mathcal{E}^*_G\widehat{\otimes}_{\mathcal{E}^*_e}\mathcal{E}^*_H\approx\mathcal{E}^*_{G\times H}.$$

2.3. **Change of groups.** Given a homomorphism $\phi\colon H\to G$ and an $G$-space $X$, and writing $\phi^*X$ for the $H$-space obtained by restricting the group action along $\phi$, we obtain an induced ring map

$$\phi^*\colon\mathcal{E}^*_G(X)\to\mathcal{E}^*_H(\phi^*X)$$

characterized by the commutative diagrams

$$\begin{array}{ccc}\mathcal{E}^*_G(X) & \xrightarrow{\phi^*} & \mathcal{E}^*_H(\phi^*X)\\ \downarrow{\scriptstyle\pi_{\phi(\tau)}} & & \downarrow{\scriptstyle\pi_\tau}\\ K^*_{\Lambda_G(\phi(\tau))}(X^{\phi(\tau)}) & \xrightarrow{\phi^*_\Lambda} & K^*_{\Lambda_H(\tau)}(X^{\phi(\tau)})\end{array}$$

of ring homomorphisms by means of the evident group homomorphisms $\phi_\Lambda\colon\Lambda_H(\tau)\to\Lambda_G(\phi(\tau))$, sending $[h,t]\mapsto[\phi(h),t]$.

If $H$ is a closed subgroup of $G$, and if $X$ is an $H$-space, then we obtain a change-of-group map

$$\rho^G_H\colon\mathcal{E}^*_G(G\times_H X)\to\mathcal{E}^*_H(X),$$

defined as the composite

$$\mathcal{E}^*_G(G\times_H X)\xrightarrow{\phi^*}\mathcal{E}_H(G\times_H X)\xrightarrow{i^*}\mathcal{E}_H(X),$$

where $\phi\colon H\to G$ is the inclusion homomorphism and $i\colon X\to G\times_H X$ is the $H$-equivariant map defined by $i(x):=[e,x]$.

2.4. **Proposition.** *The change of group map is an isomorphism.*

*Proof.* We will construct a ring isomorphism

$$\mathcal{E}^*_G(G\times_H X)=\prod_{\sigma\in G^{\text{tors}}_{\text{conj}}}K^*_{\Lambda_G(\sigma)}((G\times_H X)^\sigma)\underset{\sim}{\xrightarrow{\gamma}}\prod_{\tau\in H^{\text{tors}}_{\text{conj}}}K^*_{\Lambda_G(\tau)}(\Lambda_G(\tau)\times_{\Lambda_H(\tau)}X^\tau)$$

with the property that the composite

$$\mathcal{E}^*_G(G\times_H X)\xrightarrow{\gamma}\prod_{\tau\in H^{\mathrm{tors}}_{\mathrm{conj}}}K^*_{\Lambda_G(\tau)}(\Lambda_G(\tau)\times_{\Lambda_H(\tau)}X^\tau)\xrightarrow[\sim]{\prod\rho^{\Lambda_G(\tau)}_{\Lambda_H(\tau)}}\prod_{\tau\in H^{\mathrm{tors}}_{\mathrm{conj}}}K^*_{\Lambda_H(\tau)}(X^\tau)=\mathcal{E}^*_H(X)$$

of $\gamma$ with the product of $K$-theoretic change-of-group isomorphisms coincides with the change-of-group map for $\mathcal{E}$-theory.

Recall that $G^{\mathrm{tors}}_{\mathrm{conj}}$ and $H^{\mathrm{tors}}_{\mathrm{conj}}$ are sets of *representatives* of conjugacy classes. Given $\tau\in H_{\mathrm{conj}}$, there exists a unique $\sigma_\tau\in G_{\mathrm{conj}}$ such that $\tau=g_\tau\sigma_\tau g_\tau^{-1}$ for some $g_\tau\in G$. Fix a choice of $g_\tau$ for each $\tau$; any two such choices differ by right-multiplication by an element of $C_G(\sigma)$.

We have maps

$$\Lambda_G(\tau)\times_{\Lambda_H(\tau)}X^\tau\xrightarrow{[[a,t],x]\mapsto[a,x]}(G\times_H X)^\tau\xrightarrow[\sim]{[u,x]\mapsto[g_\tau^{-1}u,x]}(G\times_H X)^\sigma.$$

The first map is $\Lambda_G(\tau)$ equivariant, and the second map is equivariant with respect to the homomorphism $c_{g_\tau}\colon\Lambda_G(\sigma)\to\Lambda_H(\tau)$ sending $[u,t]\mapsto[g_\tau u g_\tau^{-1},t]$. Taking a coproduct over the set of $H$-conjugacy classes in $H$ which are $G$-conjugate to $\sigma$, we obtain an isomorphism

$$\gamma_\sigma\colon\coprod_{\{\,\tau\in H_{\mathrm{conj}}\mid\sigma_\tau=\sigma\,\}}\Lambda_G(\tau)\times_{\Lambda_H(\tau)}X^\tau\xrightarrow{\sim}(G\times_H X)^\sigma$$

which is equivariant with respect to $c_{g_\tau}$. We thus define $\gamma$ as the composite of the map

$$\prod_{\sigma\in G^{\mathrm{tors}}_{\mathrm{conj}}}K^*_{\Lambda_G(\sigma)}((G\times_H X)^\sigma)\xrightarrow[\sim]{(\gamma_\sigma)}\prod_{\sigma\in G^{\mathrm{tors}}_{\mathrm{conj}}}K_{\Lambda_H(\tau)}\left(\coprod_{\{\,\tau\in H^{\mathrm{tors}}_{\mathrm{conj}}\mid\sigma_\tau=\sigma\,\}}\Lambda_G(\tau)\times_{\Lambda_H(\tau)}X^\tau\right)$$

defined by the $\gamma_\sigma$, followed by the product isomorphism on cohomology. It is straightforward to check that the composite of $\gamma$ with the change-of-group isomorphisms for $K$-theory recovers our definition of the change-of-group map for $\mathcal{E}$. ☐

## 3. Quasi-elliptic $\lambda$-ring structure

**3.1. The variant $\mathcal{E}^*_{G,m}(X)$.** We define

$$\mathcal{E}^*_{G,m}(X):=\left(\prod_{\sigma\in G^{\mathrm{tors}}}K^*_{\Lambda^m(\sigma)}X^\sigma\right)\approx\prod_{\sigma\in G^{\mathrm{tors}}_{\mathrm{conj}}}K^*_{\Lambda_m(\sigma)}X^\sigma.$$

It is clear that $\mathcal{E}^*_{G,m}$ is a multiplicative cohomology theory, taking values in $\mathbb{Z}[q^{\pm 1/m}]$-algebras, that it is equipped with natural Künneth maps and change-of-group isomorphisms.

Recall from (1.14) the natural isomorphisms of $\Lambda$-rings

$$R\Lambda^n(\sigma)[q^{1/m}]=R\Lambda^n(\sigma)\otimes_{\mathbb{Z}[q^{\pm 1/n}]}\mathbb{Z}[q^{\pm 1/mm}]\approx R\Lambda^{mn}(\sigma).$$

It follows that for an arbitrary $G/\langle\sigma\rangle$-space $Y$, the induced map

$$(K^*_{\Lambda^n(\sigma)}Y)[q^{1/m}]\xrightarrow{\sim}K^*_{\Lambda^{mn}(\sigma)}Y$$

is an isomorphism of $\Lambda$-rings; both sides are cohomology theories on the category of $G/\langle\sigma\rangle$-CW complexes. Furthermore, the map is an in isomorphism of $\mathbb{Z}[q^{\pm 1/mn}]$-algebras. We thus have an natural isomorphism

$$\mathcal{E}^*_G(X)[q^{1/m}] = \mathcal{E}^*_G(X) \otimes_{R\mathbb{T}} R\mathbb{T}_m \xrightarrow{\sim} \mathcal{E}^*_{G,m}(X).$$

3.2. **The operations $\mu^m$.** Since $\mathcal{E}^0_G(X)$ is a product of conventional equivariant $K$-theory rings, it evidently comes with the structure of a $\lambda$-ring. We now produce $\lambda$-ring homomorphisms $\mu^m\colon \mathcal{E}^*_G(X) \to \mathcal{E}^*_G(X)[q^{1/m}]$.

The map $\mu^m\colon \mathcal{E}^*_G(X) \to \mathcal{E}^*_G(X)[q^{1/m}]$ will be a map

$$\prod_{\sigma \in G^{\mathrm{tors}}_{\mathrm{conj}}} K^*_{\Lambda(\sigma)} X^\sigma \to \prod_{\sigma \in G^{\mathrm{tors}}_{\mathrm{conj}}} K^*_{\Lambda_m(\sigma)} X^\sigma.$$

We thus define $\mu^m$ to be the map whose projection to the $\sigma$-coordinate of the target is given by

$$\begin{array}{ccccc}
\prod_\sigma K^*_{\Lambda(\sigma)} X^\sigma & & \longrightarrow & & \prod_\sigma K^*_{\Lambda^m(\sigma)} X^\sigma \\
\downarrow & & & & \downarrow \\
K^*_{\Lambda(\sigma^m)} X^{\sigma^m} & \xrightarrow[\beta^*]{} & K^*_{\Lambda^m(\sigma)} X^{\sigma^m} & \longrightarrow & K^*_{\Lambda^m(\sigma)} X^\sigma
\end{array}$$

where the map $\beta^*$ is induced by the homomorphism $\beta\colon \Lambda_n(\sigma) \to \Lambda(\sigma^n)$ defined by $\beta([g,t]) = [g,t]$, and the second maps on the bottom is restriction along $X^\sigma \subseteq X^{\sigma^m}$.

It is immediate that with this definition, $\mu^m$ is a map of $\Lambda$-rings, and that $\mu^m(q) = q^{1/m}$.

3.3. **Quasi-elliptic $\lambda$-rings.** Recall that if $A$ is a $\mathbb{Z}[q^\pm]$-algebra, we set $A[q^{1/m}] := A \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}[q^{\pm 1/m}]$. If $A$ is a $\lambda$-ring and $\mathbb{Z}[q^\pm] \to A$ is a $\lambda$-ring homomorphism, then $A[q^{1/m}]$ acquires an evident $\lambda$-ring structure. We can regard the construction $A \mapsto A[q^{1/m}]$ as a functor from $\lambda$-rings under $\mathbb{Z}[q^\pm]$ to $\lambda$-rings under $\mathbb{Z}[q^{\pm 1/m}]$. Furthermore, there are evident natural isomorphisms $A[q^{1/m}][q^{1/n}] \approx A[q^{1/mn}]$ of $\lambda$-rings under $\mathbb{Z}[q^{\pm 1/mn}]$.

A **quasi-elliptic $\lambda$-ring** is a $\lambda$-ring $A$ equipped with a $\lambda$-ring homomorphism $\mathbb{Z}[q^\pm] \to A$, together with $\lambda$-ring homomorphsms

$$\mu^m\colon A \to A[q^{1/m}], \qquad m \geq 1,$$

with $\mu^m(q) = q^{1/m}$, such that $\mu^1 = \mathrm{id}_A$, and such that for all $m, n \geq 1$ the composite

$$A \xrightarrow{\mu^m} A[q^{1/m}] \xrightarrow{\mu^n[q^{1/m}]} A[q^{1/n}][q^{1/m}] \approx A[q^{1/mn}]$$

is equal to $\mu^{mn}$.

3.4. **Proposition.** *The operations $\mu^m\colon \mathcal{E}^*_G(X) \to \mathcal{E}^*_G(X)[q^{1/m}]$ defined above give the sturcture of a quasi-elliptic $\lambda$-ring on $\mathcal{E}^*_G(X)$.*

*Proof.* This is largely straightforward. To prove the identity $\mu^n[q^{1/m}] \circ \mu^m = \mu^{mn}$ ... ☐

3.5. $\mathcal{E}^0_{U(1)}(\text{pt})$ **as a quasi-elliptic $\lambda$-ring.** We use the evident identification $U(1)^{\text{tors}} = U(1)^{\text{tors}}_{\text{conj}} \approx \mathbb{Q}/\mathbb{Z}$. Thus, we have ring isomorphisms

$$\mathcal{E}^0_{U(1)} \approx \left(\prod_{c\in\mathbb{Q}} R\Lambda(e^{2\pi ic})\right)^S \approx \left(\prod_{c\in\mathbb{Q}} \mathbb{Z}[q^{\pm}, z_c^{\pm}]\right)^S.$$

The two terms on the right are the invariant subrings under a certain endomorphism $S$. For the middle term, $S$ is the ring endomorphism defined on $x = (x_c) \in \prod R\Lambda(e^{2\pi ic})$ by $(Sx)_c := S(x_{c+1})$ (since $e^{2\pi i(c+1)} = e^{2\pi ic}$ in $U(1)$). In terms of the right-hand expression, in which $z_c\colon \Lambda(e^{2\pi ic}) \to U(1)$ is the representation defined by $z_c([u,t]) = e^{2\pi ict}u$, the map $S$ is given by

$$(Sf)_c(q, z_c) = f_{c+1}(q, z_{c+1}) = f_c(q, qz_c),$$

where we write $f = (f_c(q, z_c))$, where the $f_c$ are Laurent polynomials in two variables.

The $\lambda$-ring structure is characterized by

$$(\psi^n f)_c(q, z_c) = f_c(q^n, z_c^n).$$

The $\mu^m$ operations are given by

$$(\mu^m f)_c(q, z_c) = f_{mc}(q^{1/m}, z_c).$$

3.6. $E^0_{\mathbb{Z}/N\mathbb{Z}}(\text{pt})$ **as a quasi-elliptic $\Lambda$-ring.**

## 4. Quasi-elliptic cohomology for orbifolds

Given a space $X$ with an action by a compact Lie group $G$, the **torsion inertia groupoid** $I^{\text{tors}}(X/\!\!/G)$ is a groupoid in spaces, with

- objects are the space $\coprod_{\sigma\in G^{\text{tors}}} X^\sigma$,
- morphisms the space $\coprod_{\sigma,\sigma'\in G^{\text{tors}}} C(\sigma,\sigma') \times X^\sigma$, where $C(\sigma,\sigma')$ is the subspace $\{\, g\in G \mid g\sigma = \sigma' g \,\} \subseteq G$.

We write $(\sigma, x)$ for a typical object. A point $(g, x) \in C(\sigma,\sigma') \times X^\sigma$ is viewed as a morphism $(\sigma, x) \to (\sigma', gx)$.

Observe that a complete set of representatives for isomorphism classes of objects in $I^{\text{tors}}(X/\!\!/G)$ is given by $\coprod_{[\sigma]} X^\sigma$, where $[\sigma]$ ranges over conjugacy classes of $G^{\text{tors}}$, and that $C(\sigma,\sigma)$ is the centralizer of $\sigma$ in $G$.

For each pair $\sigma, \sigma' \in G^{\text{tors}}$, we define the space $\Lambda_G(\sigma,\sigma') = \Lambda(\sigma,\sigma')$ to be the quotient of $C(\sigma,\sigma') \times \mathbb{R}$ under the action of $\mathbb{Z}$, where the action of the generator of $\mathbb{Z}$ is given by

$$(g, t) \mapsto (g\sigma, t+1) = (\sigma' g, t+1).$$

We write $[g, t] \in \Lambda(\sigma,\sigma')$ for the orbit of the pair $(g, t)$.

We thus let $\Lambda(X/\!\!/G)$ be the groupoid with the same objects as $I^{\text{tors}}(X/\!\!/G)$, and with morphisms $\coprod_{\sigma,\sigma'\in G^{\text{tors}}} \Lambda(\sigma,\sigma') \times X^\sigma$. A point $([g,t], x) \in \Lambda(\sigma,\sigma') \times X^\sigma$ is viewed as a morphism from $(\sigma, x) \to (\sigma', gx)$. Composition is defined by the rule $[g_1, t_1]\cdot[g_2, t_2] = [g_1g_2, t_1+t_2]$.

Let $\pi\colon \Lambda(X/\!\!/G) \to \mathbb{T}$ be the functor which sends the morphism $[g, t]$ to $[t] \in \mathbb{T}$.

We can then define

$$\mathcal{E}^*_G(X) \stackrel{\text{def}}{=} K^*_{\text{orb}}(\Lambda(X/\!\!/G)).$$

If we choose representatives for conjugacy classes of elements in $G^{\text{tors}}$, then we have

$$\mathcal{E}^*_G(X) \approx \prod_{[\sigma]} K^*_{\Lambda(\sigma)} X^\sigma.$$

The theory described above for abelian Lie groups carries over in this case, in most cases without change. To show that $\mathcal{E}^*_G(X)$ is a quasi-elliptic $\Lambda$-ring, it is convenient to argue a little differently. First, for each pair $\sigma, \sigma' \in G^{\text{tors}}$, let $\Lambda_n(\sigma, \sigma')$ denote the quotient of $C(\sigma, \sigma') \times \mathbb{R}$ under the action of $\mathbb{Z}$, where the action of the generator of $\mathbb{Z}$ is given by

$$(g, t) \mapsto (g\sigma^n, t+1) = (\sigma'^n g, t+1).$$

Then define for $n \geq 1$ a groupoid $\Lambda_n(X/\!\!/G)$ with

- objects the space $\coprod_{\sigma \in G^{\text{tors}}} X^\sigma$,
- morphisms the space $\coprod_{\sigma, \sigma' \in G^{\text{tors}}} \Lambda_n(\sigma, \sigma') \times X^\sigma$.

A point $([g,t], x) \in \Lambda_n(\sigma, \sigma') \times X^\sigma$ is viewed as a morphism $(\sigma, x) \to (\sigma', gx)$. As before, there is a functor $\pi\colon \Lambda_n(X/\!\!/G) \to \mathbb{T}$ sending $[g,t]$ to $[t]$.

We define two functors $\alpha, \beta\colon \Lambda_n(X/\!\!/G) \to \Lambda(X/\!\!/G)$. The functor $\alpha$ is given on objects and morphisms by

$$(\sigma, x) \mapsto (\sigma, x), \qquad [g,t] \mapsto [g, nt].$$

The functor $\beta$ is given on objects and morphisms by

$$(\sigma, x) \mapsto (\sigma^n, x), \qquad [g,t] \mapsto [g,t].$$

Observe that

$$\begin{array}{ccc} \Lambda_n(X/\!\!/G) & \xrightarrow{\alpha} & \Lambda(X/\!\!/G) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{T} & \xrightarrow[{[t] \mapsto [nt]}]{} & \mathbb{T} \end{array}$$

is a pullback square in groupoids. Then show using (1.4) that

$$\begin{array}{ccc} K_{\text{orb}}(\Lambda(X/\!\!/G)) & \xrightarrow{\alpha^*} & K_{\text{orb}}(\Lambda_n(X/\!\!/G)) \\ \uparrow & & \uparrow \\ K_{\text{orb}}(*/\!\!/\mathbb{T}) & \longrightarrow & K_{\text{orb}}(*/\!\!/\mathbb{T}) \end{array}$$

is a pushout square in $\Lambda$-rings, and so induces a natural isomorphism

$$K^*_{\text{orb}}(\Lambda(X/\!\!/G))[q^{1/n}] \xrightarrow{\sim} K^*_{\text{orb}}(\Lambda_n(X/\!\!/G)).$$

Then $\mu^n$ is defined to be the map

$$K^*_{\text{orb}}(\Lambda(X/\!\!/G)) \xrightarrow{\beta^*} K^*_{\text{orb}}(\Lambda_n(X/\!\!/G)) \approx K^*_{\text{orb}}(\Lambda(X/\!\!/G))[q^{1/n}].$$

## 5. Norm construction

Let $p\colon X \to Y$ be a finite covering map between $G$-spaces, where $G$ is a compact Lie group. I want to produce a norm map $N\colon \mathcal{E}^0_G X \to \mathcal{E}^0_G Y$. This will work as follows. Given $V \in \mathcal{E}^0_G X$ we want to produce $NV = W \in \mathcal{E}^0_G Y$. Thus, we have

- for each $\sigma \in G$ and $x \in X^\sigma$, a vector space $V^\sigma_x$, and
- for each $[g,t] \in \Lambda(\sigma,\sigma')$, a map $[g,t]\colon V^\sigma_x \to V^{g\sigma g^{-1}}_{gx}$,

and using this we must produce similar data $W^\sigma_y$ and $[g,t]\colon W^\sigma_y \to W^{g\sigma g^{-1}}_{gy}$.

We first make an observation. If $\sigma \in G$, and if $x \in X$ is such that the $\sigma$-orbit of $x$ has size $n$, then we can *canonically* identify the vector spaces $V^{\sigma^n}_{x'}$, as $x'$ ranges over the $\sigma$-orbit of $x$. That is, whenever $x' = \sigma^r x$ for some $r \in \mathbb{Z}$, we have an isomorphism

$$[\sigma^r, \tfrac{r}{n}]\colon V^{\sigma^n}_x \to V^{\sigma^r\sigma^n\sigma^{-r}}_{\sigma^r x} = V^{\sigma^n}_{x'};$$

when $r \equiv 0 \mod n$ this isomorphism is precisely the identity map of $V^{\sigma^n}_x$, and these isomorphisms are compatible, in the sense that the composite of canonical isomorphisms $V^{\sigma^n}_x \to V^{\sigma^n}_{x'} \to V^{\sigma^n}_{x''}$ is the canonical isomorphism identifying the spaces for $x$ and $x''$.

Let $\sigma \in G$ and $y \in Y^\sigma$. The fiber $p^{-1}(y)$ is preserved by the action of the element $\sigma$. Choose a list $x_1, \dots, x_d$ of representatives of the $\sigma$-orbits in $p^{-1}(y)$, and let $n_i = |\langle\sigma\rangle x_i|$ be the size of each orbit. Set

$$W^\sigma_y \overset{\text{def}}{=} \bigotimes_{i=1}^d V^{\sigma^{n_i}}_{x_i}.$$

By the above remarks, this does not truly depend on the choice of representatives. That is, if we choose representatives $x'_i = \sigma^{r_i} x_i$, then we have a *canonical* choice of isomorphism

$$\bigotimes_{i=1}^d V^{\sigma^{n_i}}_{x_i} \xrightarrow{\otimes[\sigma^{r_i}, \frac{r_i}{n_i}]} \bigotimes_{i=1}^d V^{\sigma^{n_i}}_{x'_i}.$$

Now suppose $[g,t] \in \Lambda(\sigma,\sigma')$; we must define $[g,t]\colon W^\sigma_y \to W^{\sigma'}_{gy}$, where $\sigma' = g\sigma g^{-1}$. Note that action by $g$ gives a bijection $p^{-1}(y) \to p^{-1}(gy)$, and thus a bijection $\langle\sigma\rangle\backslash p^{-1}(y) \to \langle\sigma'\rangle\backslash p^{-1}(gy)$. Therefore $x'_1 = gx_1, \dots, x'_d = gx_d$ are representatives of the $\sigma'$-orbits in $p^{-1}(gy)$, and $n_i = |\langle\sigma'\rangle x'_i|$.

The map $[g,t]\colon W^\sigma_y \to W^{\sigma'}_{gy}$ is then defined by

$$\bigotimes_{i=1}^d [g, t/n_i]\colon \bigotimes_{i=1}^d V^{\sigma^{n_i}}_{x_i} \to \bigotimes_{i=1}^d V^{g\sigma^{n_i}g^{-1}}_{gx'_i} = \bigotimes_{i=1}^d V^{\sigma'^{n_i}}_{'x'_i}.$$

We must check that this is well-defined. In particular, we must check: that it does not depend on the integers $r_i$, but only on the residue mod $n_i$ of $r_i$; that it does not depend on the representative of the element of $\Lambda(\sigma,\sigma')$; and that it does not depend on the choice of representatives of $\sigma$-orbits in $p^{-1}(y)$ and $p^{-1}(gy)$.

5.1. **Another description.** Given $p\colon X \to Y$ a finite covering map between $G$-spaces, define groupoids $A(p)$ and $B(p)$ as follows.

Let $A(p)$ be the groupoid with the following data.

- The objects $A(p)$ are tuples $(\sigma, y, f, s)$ where $\sigma \in G^{\mathrm{tors}}$, $y \in Y^\sigma$, $f : S \to p^{-1}(y)$ is a function from a finite set $S$ such that $f(S) \subseteq p^{-1}(y)$ contains exactly one element in each $\sigma$-orbit of $p^{-1}(y)$, and $s \in S$.
- The morphisms $(\sigma, y, f, s) \to (\sigma', y', f', s')$ are elements $[g, t] \in \Lambda(\sigma, \sigma')$ such that $\sigma' = g\sigma g^{-1}$, $y' = gy$, and $gf(s) = \sigma'^r f'(s')$ for some $r \in \mathbb{Z}$.

Then $B(p)$ is the groupoid with the following data.

- The objects $A(p)$ are tuples $(\sigma, y, f)$ where $\sigma \in G^{\mathrm{tors}}$, $y \in Y^\sigma$, and $f : S \to p^{-1}(y)$ is a function from a finite set $S$ such that $f(S) \subseteq p^{-1}(y)$ contains exactly one element in each $\sigma$-orbit of $p^{-1}(y)$.
- The morphisms $(\sigma, y, f) \to (\sigma', y', f')$ are elements $[g, t] \in \Lambda(\sigma, \sigma')$ such that $\sigma' = g\sigma g^{-1}$, $y' = gy$.

There is an evident functor $\pi\colon A(p) \to B(p)$. We also define a functor $i\colon A(p) \to \Lambda(X/\!\!/G)$, which on objects sends

$$(\sigma, y, f, s) \mapsto (\sigma^n, y),$$

where $n = |\langle\sigma\rangle s|$, and on morphisms sends $[g, t]\colon (\sigma, y, f, s) \to (\sigma', y', f', s')$ to $[\sigma'^r g, (t + r)/n] = [g\sigma^r, (t + r)/n]$, where $f'(s') = \sigma'^r f(s)$.

We define a functor $j\colon B(p) \to \Lambda(Y/\!\!/G)$, which on objects sends $(\sigma, y, f) \mapsto (\sigma, y)$, and an morphisms sends $[g, t]\colon (\sigma, y, f) \to (\sigma', y', f')$ to $[g, t]\colon (\sigma, y) \to (\sigma', y')$.

Now we can define a norm map by

$$K^0_{\mathrm{orb}}(\Lambda(X/\!\!/G)) \xrightarrow{i^*} K^0_{\mathrm{orb}}(A(p)) \xrightarrow{N_\pi} K^0_{\mathrm{orb}}(B(p)) \underset{\sim}{\xleftarrow{j^*}} K^0_{\mathrm{orb}}(\Lambda(Y/\!\!/G)).$$

## References

[Gan07] Nora Ganter, *Stringy power operations in Tate K-theory* (2007), available at `arXiv:math/0701565`.

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