STUFF ABOUT QUASICATEGORIES

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Contents

(Note: this is a draft, which can change daily. It includes some unorganized material which might later be incorporated into the narrative, or removed entirely. In particular, everything involving Cartesian and coCartesian fibrations needs to be completely reworked.)

1. INTRODUCTION TO ∞-CATEGORIES

I'll give a brief discussion to motivate the notion of ∞-categories.

1.1. Groupoids.

Modern mathematics is based on sets. The most basic way of constructing new sets is as sets solutions to equations. For instance, given a commutative ring R, we can consider the set $X(R)$ of tuples $(x, y, z) \in R^3$ which satisfy the equation $x^5 + y^5 = z^5$. We can express such sets as limits; for instance, $X(R)$ is the pullback of the diagram of sets

$$R \times R \xrightarrow{(x,y) \mapsto x^5 + y^5} R \xleftarrow{z^5 \leftarrow z} R.$$

Another way to construct new sets is by taking "quotients"; e.g., as sets of equivalence classes of an equivalence relation. This is in some sense much more subtle than sets of solutions to equations: mathematicians did not routinely construct sets this way until they were comfortable with the set theoretic formalism introduced by the end of the 19th century.

Some sets of equivalence classes are nothing more than that; but some have "higher" structure standing behind them, which is often encoded in the form of a *groupoid*[1]. Here are some examples.

- Given a topological space X, we can define an equivalence relation on the set of points, so $x \sim x'$ if and only if there is a continuous path connecting them. The set of equivalence classes is the set $\pi_0 X$ of path components. Standing behind this equivalence relation is the *fundamental groupoid* $\Pi_1 X$, whose objects are points of X, and whose morphisms are path-homotopy classes of paths between two points.
- Given any category C, there is an equivalence relation on the collection of objects, so that $X \sim Y$ if there exists an isomorphism between them. Equivalence classes are the isomorphism classes of objects. Standing behind this equivalence relation is the *core* of C (also called the *maximal subgroupoid*), which is a groupoid having the same objects as C, but having as morphisms only the isomorphisms in C.

Note that many interesting problems are about describing isomorphism classes; e.g., classifying finite groups of a given order, or principal G-bundles on a space. In practice, one learns that in trying to classify objects up to isomorphism, you have to have a good handle on the isomorphisms between such objects. So you need to know about the groupoid, even if it is not the primary object of interest.

You can think of a problem such as: "describe the groupoid $\mathrm{Bun}_G(M)$ of principal G-bundles on a space M" as a more sophisticated analogue of: "find the set $X(R)$ of solutions to $x^5 + y^5 = z^5$ in the ring R". (In fact, the theory of "moduli stacks" exactly develops the analogy between these two problems.) To do this, you can imagine having a "groupoid-based mathematics", generalizing the usual set-based one. Here are some observations about this.

- We regard two sets as "essentially the same" if they are *isomorphic*, i.e., if there is a bijection $f \colon X \to X'$ between them. Any such bijection has a unique inverse bijection $f^{-1} \colon X' \to X$.

   On the other hand, we regard two groupoids as "essentially the same" if they are merely *equivalent*, i.e., if there is a functor $f \colon G \to G'$ which admits an inverse *up to natural isomorphism*. It is not the case that such an inverse up to natural isomorphism is unique.

   Although any equivalence of groupoids admits some kind of inverse, the failure to be unique leads to complications. For example, for every nice space M there is an equivalence of groupoids

$$f_M \colon \mathrm{Fun}(\Pi_1 M, G) \to \mathrm{Bun}_G(M).$$

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[1] I assume familiarity with basic categorical concepts, such as in Chapter 1 of [Rie16].

Here $\mathrm{Fun}(\Pi_1 M, G)$ is a *functor groupoid*, whose objects are functors $\Pi_1 M \to G$ from the fundamental groupoid of M to the discrete group G (viewed as a groupoid with one object), and whose morphisms are natural transformations of such functors. The functor f_M can be defined by an explicit construction, and is "natural" in M. However, there is no completely "natural" way to construct inverse functors f_M^{-1}: such an inverse functor requires making an essentially arbitrary choice, for each principal bundle $\gamma\colon P \to M$, of an identification of each fiber of γ with the set G.

- We can consider "solutions to equations" in groupoids (e.g., limits). However, the naive construction of limits of groupoids may not preserve equivalences of groupoids; thus, we need to consider "weak" or "homotopy" limits.

  For example, suppose M is a space which is a union of two open sets U and V. The *weak pullback* of
  $$\mathrm{Bun}_G(U) \to \mathrm{Bun}_G(U \cap V) \leftarrow \mathrm{Bun}_G(V)$$
  is a groupoid, whose objects are triples (P, Q, α), where $P \to U$ and $Q \to U$ are G-bundles, and $\alpha\colon P|_{U \cap V} \xrightarrow{\sim} Q|_{U \cap V}$ is an isomorphism of G-bundles over $U \cap V$; the morphisms $(P, Q, \alpha) \to (P', Q', \alpha')$ are pairs $(f\colon P \to P', g\colon Q \to Q')$ are pairs of bundle maps which are compatible over $U \cap V$ with the isomorphisms α, α'. Compare this with the *strict pullback*, which consists of (P, Q) such that $P|_{U \cap V} = Q|_{U \cap V}$ as bundles; in particular, $P|_{U \cap V}$ and $Q|_{U \cap V}$ must be the *identical sets*.

  A basic result about bundles is that $\mathrm{Bun}_G(M)$ is equivalent to this weak pullback. The strict limit may fail to be equivalent to this; in fact, it is impossible to describe the strict pullback without knowing precisely what definition of G-bundle we are using, whereas the identification of weak pullback is insensitive to the precise definition of G-bundle.

These kinds of issues persist when dealing with higher groupoids and categories.

1.2. Higher groupoids.

There is a category Gpd of groupoids, whose objects are groupoids and whose morphisms are functors. However, there is even more structure here; there are *natural transformations* between functors $f, f'\colon G \to G'$ of groupoids. That is, $\mathrm{Fun}(G, G')$ forms a category. We can consider the collection consisting of (0) groupoids, (1) *equivalences* between groupoids, and (2) natural isomorphisms between equivalences; this is an example of a *2-groupoid*[2]. There is no reason to stop at 2-groupoids: there are n-groupoids, the totality of which are an example of an $(n+1)$-groupoid. (In this hierarchy, 0-groupoids are sets, and 1-groupoids are groupoids.) We might as well take the limit, and consider ∞-groupoids.

It turns out to be difficult to construct an "algebraic" definition of n-groupoid. The approach which in seems to work best in practice is to use homotopy theory. We start with the observation that every groupoid G has a *classifying space* BG. This is defined explicitly as a quotient space
$$G \mapsto BG := \left(\coprod_{x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n} \Delta^n \right)/(\sim),$$
where we put in a topological n-simplex Δ^n for each n-fold sequence of composable arrows in G, modulo certain identifications. It turns out (i) the fundamental groupoid of BG is equivalent to G, and (ii) the higher homotopy groups π_k of BG are trivial, for $k \geq 2$. We thus say that the space BG is a *1-type*. Furthermore, (iii) there is a bijection between *equivalence classes of groupoids up to equivalence* and *CW-complexes which are 1-types, up to homotopy equivalence*. (More is true, but I'll stop there for now.)

The conclusion is that groupoids and equivalences between them are modelled by 1-types and homotopy equivalences between them. This suggests that we should *define* n-groupoids as n-types (CW complexes with trivial homotopy groups in dimensions $> n$), with equivalences being homotopy

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[2]More precisely, a "quasistrict 2-groupoid".

equivalences. Removing the restriction on homotopy groups leads to modelling ∞-groupoids by CW-complexes up to homotopy equivalence.

There is a different approach, which we will follow. It uses the fact that the classiyfing space construction factors through a "combinatorial" construction, called the "nerve". That is, we have

$$(G \in \mathrm{Gpd}) \mapsto (NG \in s\mathrm{Set}) \mapsto (|NG| = BG \in \mathrm{Top}),$$

where NG is the *nerve* of the groupoid, and is an example of a *simplicial set*; $|X|$ denotes the *geometric realization* of a simplicial set X. In fact, the nerve is a particular kind of simplicial set called a *Kan complex*. It is a classical fact of homotopy theory that Kan complexes model all homotopy types. Thus, we will choose our definitions so that ∞-groupoids are precisely the Kan complexes.

1.3. ∞-categories. An ∞-category is a generalization of ∞-groupoid in which morphisms are no longer required to be invertible in any sense.

There are a number of approaches to defining ∞-categories. Here are two which build on top of the identification of ∞-groupoids with Kan complexes.

- A category C consists of a set $\mathrm{ob}\,C$ of objects, and for each pair of objects a set $\hom_C(x, y)$ of maps from x to y. If we replace the set $\hom_C(x, y)$ with a Kan complex (or more generally a simplicial set) $\mathrm{map}_C(x, y)$, we obtain a category *enriched* over Kan complexes (or simplicial sets). This leads to one model for ∞-categories: categories enriched over simplicial sets.
- The nerve construction makes sense for categories: given a category C, we have a simplicial set NC. In general, NC is not a Kan complex; however, it does land in a special class of simplicial sets, which are called quasicategories. This leads to another model for ∞-categories: *quasicategories*.

I'm going to focus on the quasicategory model.

1.4. Prerequisites. I assume only familiarity with basic concepts of category theory, such as those discussed in the first few chapters of [Rie16]. It is helpful, but not essential, to know a little algebraic topology (such as fundamental groups and groupoids, and the definition of singular homology, as described in Chs. 1–3 of Hatcher's textbook).

Some categorical prerequisites: you should be at least aware of the following notions (or know where to turn to in order to learn them):

- categories, functors, and natural transformations;
- full subcategories;
- groupoids;
- products and coproducts;
- pushouts and pullbacks;
- general colimits and limits.
- adjoint functors.

1.5. Historical remarks. Quasicategories were invented by Boardman and Vogt [BV73, §IV.2], under the name *restricted Kan complex*. They did not use them to develop a theory of ∞-categories. This development began with the work of Joyal, starting in [Joy02]. Much of the material in this course was developed first by Joyal, in published papers and unpublished manuscripts [Joy08a], [Joy08b], [JT08]. Lurie [Lur09] gives a thorough treatment of quasicategories (which he simply calls "∞-categories"), recasting and extending Joyal's work significantly.

There are significant differences between the ways that Joyal and Lurie develop the theory. In particular, they give different definitions of the notion of a "categorical equivalence" between simplicial sets, though they do in fact turn out to be equivalent [Lur09, §2.2.5]. The approach I follow here is essentially that of Joyal. However, I have tried to follow Lurie's terminology and notation in most places.

1.6. **Goal of this book.** The goal of this book is to give a reasonably approachable introduction to the subject of higher category theory. In particular, I am writing with the following ideas in mind.

- The prerequisites are merely some basic notions of category theory, as seen in a first year algebraic topology or algebraic geometry course. In particular, no advanced training in homotopy theory is assumed.
- The book is written in "lecture notes" style rather that "textbook" style. That is, I will try to avoid introducing a lot of theory in section 3 which is only to be used in section 42, even if that is the "natural" place for it. The goal is to introduce new ideas near where they are first used, so that motivations are clear.
- The structure of the exposition is organized around the following type of question: Here is a [definition we can make/theorem we can prove] for ordinary categories; how do we generalize it to quasicategories? In some cases the answer is easy. In others, it can require a significant detour.
- The exposition is largely from the bottom up, rather than from the top down. Thus, I attempt to give complete details about everything I prove, so that nothing is relegated to references. (The current document does not achieve this yet, but that is the plan; in some cases, such details will be put into appendices.)

1.7. **Things to add.** This is a place for me to remind myself of things I might add.

- A discussion of n-truncation and n-groupoids, including the equivalence of ordinary groupoids to 1-groupoids (so connecting with the introduction).
- Pointwise criterion for limits/colimits: Show that $S^{\triangleright} \to \mathrm{Fun}(D, C)$ is a colimit cone if each projection to $S^{\triangleright} \to \mathrm{Fun}(\{d\}, C) \approx C$ is one.

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Part 1. **Basic notions**

2. Simplicial sets

In the subsequent sections, we will define quasicategories as a generalization of the notion of a category. To accomplish this, we will recharacterize categories as a particular kind of *simplicial set*; relaxing this characterization will lead us to the definition of quasicategories.

Simplicial sets were introduced as a combinatorial framework for the homotopy theory of spaces. There are a number of treatments of simplicial sets from this point of view. We recommend Greg Friedman's survey [Fri12] as a starting place for learning about this viewpoint, and we will discuss this point of view later on in §**??**. Here we will focus on what we need in order to develop quasicategories.

2.1. **The simplicial indexing category Δ.** We write Δ for the category whose

- **objects** are the non-empty totally ordered sets $[n] := \{0 < 1 < \cdots < n\}$ for $n \geq 0$, and
- **morphisms** $f \colon [n] \to [m]$ are weakly monotone functions.

Note that we exclude the empty set from Δ. Morphisms in Δ are often called **simplicial operators**.

Because $[n]$ is an ordered set, you can also think of it as a category: the objects are the elements of $[n]$, and there is a morphism (necessarily unique) $i \to j$ if and only if $i \leq j$. Thus, morphisms in the category Δ are precisely the functors between the categories $[n]$. We can, and will, also think of $[n]$ as the category "freely generated" by the picture

$$\boxed{0} \to \boxed{1} \to \cdots \to \boxed{n-1} \to \boxed{n}.$$

Arbitrary non-identity morphisms in $[n]$ can be expressed uniquely as iterated composites of the arrows which are displayed in the picture.

We will often use the following notation for morphisms in Δ:

$$f = \langle f_0 \cdots f_n \rangle \colon [n] \to [m] \quad \text{with } f_0 \leq \cdots \leq f_n \quad \text{represents the function} \quad k \mapsto f_k.$$

2.2. *Remark.* There are distinguished simplicial operators called **face** and **degeneracy** operators:

$$d^i := \langle 0, \ldots, \widehat{i}, \ldots, n \rangle \colon \quad [n-1] \to [n], \quad 0 \leq i \leq n,$$

$$s^i := \langle 0 \ldots i, i, \ldots n \rangle \colon [n+1] \to [n], \quad 0 \leq i \leq n.$$

All maps in Δ can be obtained as a composition of face and degeneracy operators, and in fact Δ can be described as the category *generated* by the above symbols, subject to a set of *relations* called the "simplicial identities", which can be found in various places, e.g., [Fri12, Def. 3.2].

2.3. **Simplicial sets.** A **simplicial set** is a functor $X \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$, i.e., a contravariant functor (or "presheaf") from Δ to sets.

It is typical to write X_n for $X([n])$, and call it the set of n-**simplices** in X. Sometimes people speak informally of the set of all simplices of X, i.e., of the disjoint union of the X_ns. Furthermore, given a simplex $a \in X_n$ and a simplicial operator $f \colon [m] \to [n]$, I will write $af \in X_m$ as shorthand for $X(f)(a)$. That is, I'll think of simplicial operators as acting on simplices from the right; this is a convenient choice given that X is a *contravariant* functor. In this language, a simplicial set consists of

- a sequence of sets $X_0, X_1, X_2, \ldots,$
- functions $a \mapsto af \colon X_n \to X_m$ for each simplicial operator $f \colon [m] \to [n]$, such that
- $a\,\mathrm{id} = a$, and $(af)g = a(fg)$ for any simplex a and simplicial operators f and g whenever this makes sense.

If we need to have the simplicial operator act from the left, I'll write $f^*(a) = af$.

Sometimes it is convenient to use a subscript notation when speaking of the action of particular simplicial operators. Thus, given a simplicial operator of the form $f = \langle f_0 \cdots f_m \rangle \colon [m] \to [n]$, we can indicate the action of f on a simplex using subscripts:

$$a_{f_0 \cdots f_m} := af = a\langle f_0 \ldots f_m \rangle.$$

In particular, applying simplicial operators of the form $\langle i \rangle \colon [0] \to [n]$ gives the **vertices** $a_0, \ldots, a_n \in X_0$ of an n-simplex $a \in X_n$, while applying simplicial operators of the form $\langle ij \rangle \colon [1] \to [n]$ for $i \leq j$ gives the **edges** $a_{ij} \in X_1$ of a.

2.4. **The category of simplicial sets.** A simplicial set is a functor; therefore a **map of simplicial sets** is a natural transformation of functors. Explicitly, a map $\phi \colon X \to Y$ between simplicial sets is a collection of functions $\phi \colon X_n \to Y_n$, $n \geq 0$, which commute with simplicial operators:

$(\phi a)f = \phi(af)$ for all simplicial operators f and simplices a in X, when this makes sense.

I'll write $s\mathrm{Set}$ for the category of simplicial sets and maps between them[3].

2.5. **Discrete simplicial sets.** A simplicial set X is **discrete** if every simplicial operator f induces a bijection $f^* \colon X_n \to X_m$.

Every set S produces us a discrete simplicial set S^{disc}, defined so that $(S^{\mathrm{disc}})_n = S$, and so that each simplicial operator acts according to the identity map of S. This construction defines a functor $S \mapsto S^{\mathrm{disc}} \colon \mathrm{Set} \to s\mathrm{Set}$.

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[3]Lurie [Lur09] uses Set_Δ to denote the category of simplicial sets. **Perhaps I should try to be consistent with this?**

2.6. *Exercise.* Show that (i) every discrete simplicial set X is isomorphic to S^{disc} for some set S, (and that in fact you can take $S = X_0$), and (ii) for every pair of sets S and T, the evident function $\mathrm{Hom}_{\mathrm{Set}}(S, T) \to \mathrm{Hom}_{s\mathrm{Set}}(S^{\mathrm{disc}}, T^{\mathrm{disc}})$ is a bijection.

Let $s\mathrm{Set}^{\mathrm{disc}}$ denote the full subcategory of $s\mathrm{Set}$ spanned by discrete simplicial sets. That is, objects of $s\mathrm{Set}^{\mathrm{disc}}$ are discrete simplicial sets, and morphisms of $s\mathrm{Set}^{\mathrm{disc}}$ are all simplicial maps between them. Then the above exercise amounts to saying that *the full subcategory of discrete simplicial sets is equivalent to the category of sets.*

For this reason, it is often convenient to (at least informally) "identify" sets with their corresponding discrete simplicial sets (i.e., for a set S we also write S for the discrete simplicial set S^{disc} defined above).

2.7. Standard n-simplex. The **standard n-simplex** Δ^n is the simplicial set defined by

$$\Delta^n := \mathrm{Hom}_\Delta(-, [n]).$$

That is, the standard n-simplex is exactly the functor *represented* by the object $[n]$. Explicitly, this means that

$$(\Delta^n)_m = \mathrm{Hom}_\Delta([m], [n]) = \{\text{simplicial operators } a\colon [m] \to [n]\},$$

while the action of simplicial operators on simplices in Δ^n is given by composition: $f\colon [m'] \to [m]$ sends $a \in (\Delta^n)_m$ to $(af\colon [m'] \to [n]) \in (\Delta^n)_{m'}$.

The **generator** of Δ^n is the simplex

$$\iota_n := \langle 01 \ldots n \rangle = \mathrm{id}_{[n]} \in (\Delta^n)_n$$

corresponding to the identity map of $[n]$.

The **Yoneda lemma** (applied to the category Δ) asserts that the function

$$\mathrm{Hom}_{s\mathrm{Set}}(\Delta^n, X) \to X_n,$$

$$g \mapsto g(\iota_n),$$

is a bijection for every simplicial set X. (*Exercise:* if this fact is not familiar to you, prove it.)

The Yoneda lemma can be stated this way: for each simplex $a \in X_n$ there exists a *unique* map $f_a\colon \Delta^n \to X$ of simplicial sets which sends the generator to it, i.e., such that $f_a(\iota_n) = a$. We call the map f_a the **representing map** of the simplex a.

We will often use the bijection provided by the Yoneda lemma implicitly. In particular, instead of using notation such as f_a, we will typically abuse notation and write $a\colon \Delta^n \to X$ for the representing map of the simplex $a \in X_n$. We reiterate that the map $a\colon \Delta^n \to X$ is characterized as the *unique* map sending the generator ι_n of Δ^n to a. Thus with our notation we have $a = a(\iota_n)$, where the two appearances of "a" denote respectively the element of X_n and the representing morphism $\Delta^n \to X$.

2.8. *Exercise.* Show that the representing map $f\colon \Delta^n \to X$ of $a \in X_n$ sends $\langle f_0 \cdots f_k \rangle \in (\Delta^n)_k$ to $a\langle f_0 \cdots f_k \rangle \in X_k$.

Note that if $X = \Delta^m$ is also a standard simplex, then the Yoneda lemma gives a bijection

$$\mathrm{Hom}_{s\mathrm{Set}}(\Delta^n, \Delta^m) \xrightarrow{\sim} (\Delta^m)_n = \mathrm{Hom}_\Delta([n], [m]).$$

The inverse of this bijection sends a simplicial operator $f\colon [n] \to [m]$ to the map $\Delta^f\colon \Delta^n \to \Delta^m$ of simplicial sets defined on simplices $g \in (\Delta^n)_k = \mathrm{Hom}_\Delta([k], [n])$ by $g \mapsto fg$. (*Exercise:* prove this.)

I will commonly abuse notation, and write $f\colon \Delta^n \to \Delta^m$ instead of Δ^f for the map induced by the simplicial operator f, as it is also the representing map of the corresponding simplex $f \in (\Delta^m)_n$.

2.9. **The standard 0-simplex and the empty simplicial set.** The standard 0-simplex Δ^0 is the terminal object in sSet. Sometimes I write $*$ instead of Δ^0 for this object. Note that it is the only standard n-simplex which is discrete.

The **empty simplicial set** $\varnothing$ is the functor $\Delta^{\mathrm{op}} \to$ Set sending each $[n]$ to the empty set. It is the initial object in sSet.

2.10. *Exercise.* Show that a simplicial set X is isomorphic to the empty simplicial set if and only if X_0 is isomorphic to the empty set.

2.11. **Standard simplices on totally ordered sets.** The definition of the standard simplices Δ^n can be extended to simplicial sets "generated" by arbitrary totally ordered sets.

For instance, for *any* non-empty finite totally ordered set $S = \{s_0 < s_1 < \cdots < s_n\}$, there is a *unique* order preserving bijection $S \approx [n]$ for a unique $n \geq 0$. We write Δ^S for the simplicial set with $(\Delta^S)_k = \{\text{order preserving } [k] \to S\}$. It's not hard to see that there is a unique isomorphism $\Delta^S \approx \Delta^n$ of simplicial sets (*Exercise:* prove this). We can also apply this idea to the empty ordered set $S = \varnothing$, in which case $(\Delta^\varnothing)_k = \varnothing$ for all k, i.e., $\Delta^\varnothing$ is the empty simplicial set.

This notation is especially convenient for subsets $S \subseteq [n]$ with induced ordering, as the simplicial set Δ^S is in a natural way a *subcomplex* of Δ^n (i.e., a collection of subsets of the $(\Delta^n)_k$ closed under action of simplicial operators; we will return to the notion of subcomplex below §4.9).

Furthermore, any simplicial operator $f \colon [m] \to [n]$ factors through its image $S = f([m]) \subseteq [n]$, giving a factorization

$$[m] \xrightarrow{f_{\mathrm{surj}}} S \xrightarrow{f_{\mathrm{inj}}} [n]$$

of maps between ordered sets, and thus a factorization $\Delta^m \xrightarrow{\Delta^{f_{\mathrm{surj}}}} \Delta^S \xrightarrow{\Delta^{f_{\mathrm{inj}}}} \Delta^n$ of the induced map Δ^f of simplicial sets.

2.12. *Exercise.* Show that $\Delta^{f_{\mathrm{inj}}}$ and $\Delta^{f_{\mathrm{surj}}}$ respectively induce maps between simplicial sets which are (respectively) injective and surjective on sets of n-simplices for all n. (The case of $\Delta^{f_{\mathrm{inj}}}$ is essentially formal, but the case of $\Delta^{f_{\mathrm{surj}}}$ is not completely formal.)

2.13. **Pictures of standard simplices.** When we draw a "picture" of Δ^n, we draw a geometric n-simplex, with vertices labelled by $0, \ldots, n$. The faces of the geometric simplex correspond exactly to *injective* simplicial operators into $[n]$: these simplices are called *non-degenerate* simplices. For each non-degenerate simplex f in Δ^n, there is an infinite collection of *degenerate* simplices with the same "image" as f (when viewed as a simplicial operator with target $[n]$).

Here are some "pictures" of standard simplices, which show their non-degenerate simplices. Note that we draw the 1-simplices in Δ^n as arrows; this lets us easily see the total ordering on the vertices of Δ^n.

$$\Delta^0: \qquad \Delta^1: \qquad\qquad \Delta^2: \qquad\qquad \Delta^3:$$
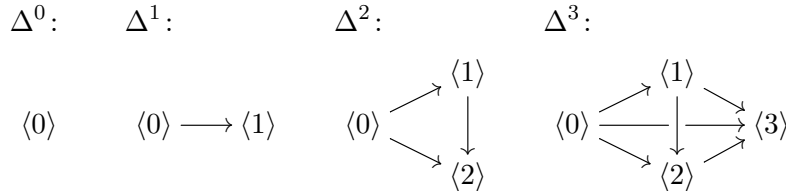
We'll extend this terminology of "degenerate" and "non-degenerate" simplices to arbitrary simplicial sets in §15.5.

3. The nerve of a category

The nerve of a category is a simplicial set which retains all the information of the original category. In fact, the nerve construction provides a full embedding of Cat, the category of categories, into sSet, which means that we are able to think of categories as just a special kind of simplicial set.

3.1. **Construction of the nerve.** Given a category C, the **nerve** of C is the simplicial set NC defined so that
$$(NC)_n := \mathrm{Hom}_{\mathrm{Cat}}([n], C),$$
the set of functors from $[n]$ to C, and so that simplicial operators $f\colon [m] \to [n]$ act by precomposition: $a \mapsto af$ for an element $a\colon [n] \to C$ in $(NC)_n$.

3.2. *Example.* There is an evident isomorphism $N[n] \approx \Delta^n$.

Given a functor $F\colon C \to D$ between categories, we obtain an evident map $N(F)\colon NC \to ND$ of simplicial sets, sending $(a\colon [n] \to C) \in (NC)_n$ to $(Fa\colon [n] \to D) \in (ND)_n$. Thus the nerve construction defines a functor $N\colon \mathrm{Cat} \to s\mathrm{Set}$.

3.3. **Structure of the nerve.** We observe the following, whose verification we leave to the reader.
- $(NC)_0$ is canonically identified with the set of objects of C.
- $(NC)_1$ is canonically identified with the set of morphisms of C.
- The operators $\langle 0 \rangle^*, \langle 1 \rangle^*\colon (NC)_1 \to (NC)_0$ assign to a morphism its source and target respectively.
- The operator $\langle 00 \rangle^*\colon (NC)_0 \to (NC)_1$ assigns to an object its identity map.
- $(NC)_2$ is in bijective correspondence with the set of pairs (f, g) of morphisms such that gf is defined, i.e., such that the target of f is the source of g. This bijection is given by sending $a \in (NC)_2$ to (a_{01}, a_{12}).
- The operator $\langle 02 \rangle^*\colon (NC)_2 \to (NC)_1$ assigns, to a simplex corresponding to such a pair (f, g) of morphisms, the composite morphism gf.

We have the following general description of n-simplices in the nerve.

3.4. **Proposition.** *Let C be a category.*
(1) *There is a bijective correspondence*
$$(NC)_n \xrightarrow{\sim} \{ (g_1, \ldots, g_n) \in (\mathrm{mor}\, C)^{\times n} \mid \mathrm{target}(g_{i-1}) = \mathrm{source}(g_i) \}$$
between $(NC)_n$ and the set of sequences $(g_1, \ldots, g_n)$ of morphisms in C such that the target of g_{i-1} is the source of g_i. This correspondence is given by the function which sends $(a\colon [n] \to C) \in (NC)_n$ to the sequence $(a\langle 0, 1 \rangle, \ldots, a\langle n-1, n \rangle)$.
(2) *With respect to the correspondence of (1), the map $f^*\colon (NC)_n \to (NC)_m$ induced by a simplicial operator $f\colon [m] \to [n]$ coincides with the function*
$$(g_1, \ldots, g_n) \mapsto (h_1, \ldots, h_m), \qquad h_k = \begin{cases} \mathrm{id} & \text{if } f(k-1) = f(k) \\ g_j g_{j-1} \cdots g_{i+1} & \text{if } f(k-1) = i < j = f(k). \end{cases}$$

Proof. For (1), one verifies that an inverse is given by the function which sends a sequence $(g_1, \ldots, g_n)$ to $(a\colon [n] \to C) \in (NC)_n$ defined on objects by $a(k) = \mathrm{target}(g_{k-1}) = \mathrm{source}(g_k)$, and on morphisms by $a(\langle ij \rangle) = g_j g_{j-1} \cdots g_{i+1}$ for $i < j$. For (2), note that for $a \in (NC)_n$ corresponding to the tuple $(g_1, \ldots, g_n)$ we can compute
$$(af)\langle k-1, k \rangle = a\langle f(k-1), f(k) \rangle = \begin{cases} \mathrm{id} & \text{if } f(k-1) = f(k), \\ g_j g_{j-1} \cdots g_{i+1} & \text{if } f(k-1) = i < j = f(k). \end{cases}$$
$\square$

In particular, you can recover the category from its nerve, up to isomorphism, since the nerve contains all information about objects, morphisms, and composition of morphisms.

3.5. *Remark.* It is clear from the above remarks that most of the information in the nerve of C is redundant: we only needed $(NC)_k$ for $k = 0, 1, 2$ and certain simplicial operators between them to recover C.

3.6. *Exercise.* Show that for any discrete simplicial set X there exists a category C and an isomorphism $NC \approx X$.

3.7. *Example.* **Nerve of a group.**

3.8. **Characterization of nerves.** This leads to the question: given a simplicial set X, how can we detect that it is isomorphic to the nerve of some category?

3.9. **Proposition.** *A simplicial set X is isomorphic to the nerve of some category if and only if for all $n \geq 2$ the function*

$$\phi_n \colon X_n \to \{\, (g_1, \ldots, g_n) \in (X_1)^{\times n} \mid g_{i-1}\langle 1 \rangle = g_i \langle 0 \rangle,\ 1 \leq i \leq n \,\}$$

which sends $a \in X_n$ to $(a_{0,1}, \ldots, a_{n-1,n})$ is a bijection.

Proof. First, suppose $X = NC$ for some category C. Then the function ϕ_n is precisely the bijection of (3.4)(1). Given this, we see that the ϕ_n are bijections even when X is merely isomorphic to some NC.

Now suppose X is a simplical set such that the ϕ_n are bijections. We *define* a category C, with

$$(\text{objects of } C) = X_0, \qquad (\text{morphisms of } C) = X_1,$$

following the discussion in (3.3). Thus, the source and target of $g \in X_1$ are g_0 and g_1 in X_0 respectively, the identity map of $x \in X_0$ is $x_{00} \in X_1$, while the composite of (g, h) such that $g_1 = h_0$ is a_{02}, where $a \in X_2$ is the unique 2-simplex with $a_{01} = g$ and $a_{12} = h$. We leave the remaining details (e.g., unit and associativity properties) to the reader, though we note that proving associativity requires consideration of elements of X_3. (Or look ahead to (5.10), where we carry out the argument explicitly in a slighlty different context.)

Next, we claim that for $a \in X_n$, and for $0 \leq i \leq j \leq k \leq n$, we have that

$$a_{i,k} = a_{j,k} a_{i,j},$$

where $a_{i,k}, a_{i,j}, a_{j,k} \in X_1$ are images of a under face operators $[1] \to [n]$, and right-hand side represents composition of two morphisms in C. To see this, note first that for $b \in X_2$, we have $b_{0,2} = b_{1,2} b_{0,1}$ by construction of C. The general case follows from this by setting $b = a_{i,j,k}$.

Now we can define maps $\psi_n \colon X_n \to (NC)_n$ by sending $a \in X_n$ to $\psi(a) \colon [n] \to C$ defined by $\psi(a)(i \leq j) = a_{i,j}$, which is a functor by the above remarks. These maps ψ_n are seen to be bijections using the bijections ϕ_n and (3.4), since $\psi_n(a)(i-1 \leq i) = a_{i-1,i}$. If $f \colon [m] \to [n]$ is a simplicial operator, then we compute

$$\psi_m(af)(i \leq j) = (af)_{i,j} = a_{f(i),f(j)} = (\psi_n(a))(f(i) \leq f(j)) = (\psi_n(a)f)(i \leq j),$$

whence ψ is a map of simplicial sets. We have thus constructed an isomorphism $\phi \colon X \to NC$ of simplicial sets, as desired.

$\square$

3.10. **A characterization of maps between nerves.** Given this, the following is a standard exercise.

3.11. **Proposition.** *The nerve functor $N \colon \mathrm{Cat} \to s\mathrm{Set}$ is fully faithful. That is, every simplicial set map $g \colon NC \to ND$ between nerves is of the form $g = N(f)$ for a unique functor $f \colon C \to D$.*

Proof sketch. We need to show that $\mathrm{Hom}_{\mathrm{Cat}}(C, D) \to \mathrm{Hom}_{s\mathrm{Set}}(NC, ND)$ is a bijection for all categories C and D. *Injectivity* is clear, as a functor f is determined by its effect on objects and morphisms, which is exactly the effect of $N(f)$ on 0- and 1-simplices of the nerves. For *surjectivity*, observe that for any map $g \colon NC \to ND$ of simplicial sets, we can define a candidate functor $f \colon C \to D$, defined on objects and morphisms by the action of g on 0- and 1- simplices. That F has the correct action on identity maps follows from the fact that g commutes with the simplicial

operator $\langle 00 \rangle \colon [1] \to [0]$. That f preserves composition uses (3.4) and the fact that g commutes with the simplicial operator $\langle 02 \rangle \colon [1] \to [2]$.

Next, note that for any simplicial maps $g, g' \colon NC \to ND$ which coincide on 0- and 1-simplices, we must have $g = g'$ (again using (3.4)), and thus in particular for a simplicial map $g \colon NC \to ND$ we have that $g = N(f)$ where f is the candidate functor constructed in the previous paragraph. $\square$

3.12. *Exercise.* Show that if C is a category and X is any simplicial set (not necessarily a nerve), then two maps $f, f' \colon X \to NC$ are equal if and only if $f_0 = f_0'$ and $f_1 = f_1'$, i.e., f and f' are equal if ond only if they coincide on 0-simplices and 1-simplices.

4. Spines

In this section we will restate our characterization of simplicial sets which are isomorphic to nerves, in terms of a certain "extension" condition. To state this condition we need the notion of a "spine" of a standard n-simplex.

4.1. **The spine of an n-simplex.** The **spine** of the n-simplex Δ^n is the simplicial set I^n defined by
$$(I^n)_k = \{ \langle a_0 \cdots a_k \rangle \in (\Delta^n)_k \mid a_k \leq a_0 + 1 \}.$$
That is, a k-simplex in I^n is a simplicial operator $a \colon [k] \to [n]$ whose image is of the form either $\{j\}$ or $\{j, j+1\}$. The action of simplicial operators on simplices in I^n is induced by their action on Δ^n. (To see that this action is well defined, observe that for $a \colon [k] \to [n]$ in $(I^n)_k$ and $f \colon [p] \to [k]$, the image of the simplicial operator af is contained in the image of a.)

There is an evident injective map $I^n \to \Delta^n$ of simplicial sets. (In fact, I^n is another example of a *subcomplex* of Δ^n, see below §4.9.) Here is a picture of I^3 in Δ^3:
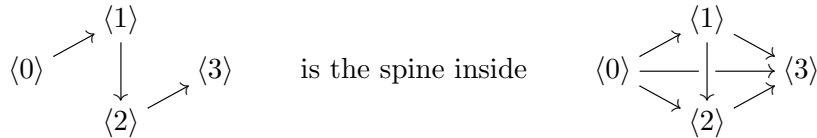
The key property of the spine is the following.

4.2. **Proposition.** *Given a simplicial set X, for every $n \geq 1$ there is a bijection*
$$\operatorname{Hom}(I^n, X) \xrightarrow{\sim} \{ (a_1, \ldots, a_n) \in (X_1)^{\times n} \mid a_i \langle 1 \rangle = a_{i+1} \langle 0 \rangle \},$$
defined by sending $f \colon I^n \to X$ to $(f(\langle 01 \rangle), f(\langle 12 \rangle), \cdots, f(\langle n-1, n \rangle))$.

We will give the proof at the end of this section, after we describe I^n as a colimit of a diagram of standard simplices; specifically, as a collection of 1-simplices "glued" together at their ends. Note also that since $I^0 = \Delta^0$, there is a bijection $\operatorname{Hom}(I^0, X) \xrightarrow{\sim} X_0$.

4.3. **Nerves are characterized by unique spine extensions.** We can now state our new characterization of nerves: they are simplicial sets such that every map $I^n \to X$ from a spine extends *uniquely* along $I^n \subseteq \Delta^n$ to a map from the standard n-simplex. That is, nerves are precisely the simplicial sets with "unique spine extensions".

4.4. **Proposition.** *A simplicial set X is isomorphic to the nerve of some category if and only if the restriction map $\operatorname{Hom}(\Delta^n, X) \to \operatorname{Hom}(I^n, X)$ is a bijection for all $n \geq 2$.*

Proof. Immediate from (4.2) and (3.9). $\square$

4.5. **Colimits of sets and simplicial sets.** Given any functor $F\colon C \to$ Set from a small category to sets, there is a "simple formula" for its colimit. First consider the coproduct (i.e., disjoint union) $\coprod_{c\in\text{ob }C} F(c)$ of the values of the functor; I'll write (c, x) for a typical element of this coproduct, with $c \in \text{ob }C$ and $x \in F(c)$. Consider the relation $\sim$ on this defined by

$$(c, x) \sim (c', x') \quad \text{if } \exists\ \alpha\colon c \to c' \text{ in } C \text{ such that } F(\alpha)(x) = x'.$$

Define

$$X := \Big(\coprod_{c\in\text{ob }C} F(c) \Big) \big/ \approx,$$

the set obtained as the quotient by the equivalence relation "$\approx$" which is *generated* by the relation "$\sim$". For each object c of C we have a function $i_c\colon F(c) \to X$ defined by $i_c(x) := [(c, x)]$. Then the data $(X, \{i_c\})$ is a colimit of the functor F: i.e., for any set S and collection of functions

$$f_c\colon F(c) \to S \quad \text{such that} \quad f_{c'} \circ F(\alpha) = f_c \quad \text{for all } \alpha\colon c \to c'$$

there exists a *unique* function $f\colon X \to S$ such that $f \circ i_c = f_c$.

4.6. *Example.* Verify that $(X, \{i_c\})$ is in fact a colimit of F.

We write $\operatorname{colim}_C F$ for a chosen colimit of F.

Note that the relation "$\sim$" is often not itself an equivalence relation, so it can be difficult to figure out what "$\approx$" actually is: the simple formula may not be so simple in practice.

4.7. *Exercise.* If C is a groupoid, then $\sim$ is always an equivalence relation.

There are cases when things are more tractable.

4.8. **Proposition.** *Let $\mathcal{A}$ be a collection of subsets of a set S, which is a partially ordered set under "$\subseteq$" and thus can be regarded as a category. Suppose $\mathcal{A}$ has the following property: for all $s \in S$, and $T_1, T_2 \in \mathcal{A}$ such that $s \in T_1 \cap T_2$, there exists $U \in \mathcal{A}$ such that $s \in U \subseteq T_1 \cap T_2$. Then the tautological map*

$$\operatorname{colim}_{T\in\mathcal{A}} T \to \bigcup_{T\in\mathcal{A}} T$$

(sending $[(T, t)] \mapsto t$) is a bijection.

Proof sketch. Show that $(T, t) \approx (T', t')$ if and only if $t = t'$. $\qquad\square$

Any functor $F\colon C \to s$Set to simplicial sets from a small category has a colimit, which is "computed objectwise". That is, we obtain a simplicial set X by taking colimits of sets in each degree: $X_n = \operatorname{colim}_C F(c)_n$, and also obtain an action of simplicial operators on the X_ns using the universal property of colimits of sets. Then one may show that the simplicial set $X = \operatorname{colim}_C F$ is a colimit of F.

4.9. **Subcomplexes.** Given a simplicial set X, a **subcomplex** is just a subfunctor of X; i.e., a collection of subsets $A_n \subseteq X_n$ which are closed under the action of simplicial operators, and thus form a simplicial set so that the inclusion $A \to X$ is a morphism of simplicial sets. We typically write $A \subseteq X$ when A is a subcomplex of X.

4.10. *Example.* Examples we have already seen include the spines $I^n \subseteq \Delta^n$ and $\Delta^S \subseteq \Delta^n$ associated to subsets $S \subseteq [n]$.

4.11. *Exercise.* For any map $f\colon X \to Y$ of simplicial sets, the image $f(X) \subseteq Y$ of f is a subcomplex of Y.

The image of any map of simplicial sets is a subcomplex. Also, for every set of simplices in a simplicial set, there is a smallest subcomplex which contains the set.

4.12. *Example.* For a vertex $x \in X_0$, we write $\{x\} \subseteq X$ for the smallest subcomplex which contains x. This subcomplex has exactly one n-simplex for each $n \geq 0$, namely $x\langle 0 \cdots 0 \rangle$, and thus is isomorphic to Δ^0.

More generally, for a collection of vertices $a, b, c, \cdots \in X_0$, we write $\{a, b, c, \dots\} \subseteq X$ for the smallest subcomplex which contains $a, b, c, \dots$. This subcomplex is a simplicial set. This choice of notation is supported by our informal identification of discrete sets with sets (2.5).

4.13. *Remark.* The result (4.8) carries over to simplicial sets, where the role of subsets is replaced by subcomplexes. Instead of stating a general result, we will note a useful consequence: if A and B are subcomplexes of a simplicial set X, then $A \cap B$ and $A \cup B$ are also subcomplexes of X; furthermore, $A \cup B$ is isomorphic to the colimit of the diagram $A \leftarrow A \cap B \rightarrow B$ of simplicial sets, where the maps are the evident inclusions. (Proof sketch: for each simplex in $A \cup B$ there is a "minimal" element of $\{A, B, A \cap B\}$ which contains it.)

4.14. **Subcomplexes of Δ^n.** For each $S \subseteq [n]$ we have a subcomplex $\Delta^S \subseteq \Delta^n$. The following says that every subcomplex of Δ^n is a union of Δ^Ss.

4.15. **Lemma.** *Let $K \subseteq \Delta^n$ be a subcomplex. If $(f \colon [m] \to [n]) \in K_m$ with $f([m]) = S$, then $f \in (\Delta^S)_m$ and $\Delta^S \subseteq K$.*

Proof. It is immediate that $f \in (\Delta^S)_m$. Consider the factorization

$$[m] \xrightarrow{f_{\mathrm{surj}}} S \xrightarrow{f_{\mathrm{inj}}} [n]$$

of f into a surjection followed by an injection. Every order preserving surjection between ordered sets admits an order preserving section, so there exists an order preserving map $s \colon S \to [m]$ such that $f_{\mathrm{surj}} s = \mathrm{id}_S$. Therefore $fs = f_{\mathrm{inj}} f_{\mathrm{surj}} s = f_{\mathrm{inj}}$. Given a simplex $(g \colon [k] \to S) \in (\Delta^S)_k$ we thus have $f_{\mathrm{inj}} g = f(sg)$, whence $f_{\mathrm{inj}} g \in K_k$ since K is a subcomplex, $f \in K_m$, and $sg \colon [k] \to [m]$ is a simplicial operator. We have proved that $\Delta^S \subseteq K$, since f^{inj} is the inclusion map of $S \subseteq [n]$. $\square$

4.16. *Remark.* Thus, a subcomplex $K \subseteq \Delta^n$ determines and is determined by a collection $\mathcal{K}$ of subsets of $[n]$ with the property that $T \subseteq S$ and $S \in \mathcal{K}$ implies $T \in \mathcal{K}$; namely, $\mathcal{K} = \{\, S \subseteq [n] \mid \Delta^S \subseteq K \,\}$.

That is, a subcomplex of Δ^n is the "same thing" as a *simplicial complex* whose vertex set is a subset of $[n]$.

We can sharpen the above: every subcomplex of Δ^n is a *colimit* of subcomplexes Δ^S.

4.17. **Proposition.** *Let $K \subseteq \Delta^n$ be a subcomplex. Let $\mathcal{A}$ be the poset of all non-empty subsets $S \subseteq [n]$ such that the inclusion map $f \colon S \to [n]$ represents a $(|S| - 1)$-simplex in K. Then the tautological map*

$$\mathrm{colim}_{S \in \mathcal{A}} \Delta^S \to K$$

is an isomorphism.

Proof. We must show that for each $m \geq 0$, the map $\mathrm{colim}_{S \in \mathcal{A}} (\Delta^S)_m \to K_m$ is a bijection. Each $(\Delta^S)_m = \{\, f \colon [m] \to [n] \mid f([m]) \subseteq S \,\}$ is a distinct subset of $K_m \subseteq (\Delta^n)_m$; i.e., $S \neq S'$ implies $(\Delta^S)_m \neq (\Delta^{S'})_m$. In view of (4.8), it suffices to show that for each $f \in K_m$ there is a minimal S in $\mathcal{A}$ such that $f \in (\Delta^S)_m$. This is immediate from (4.15), which says that $f \in (\Delta^S)_m$ and $\Delta^S \subseteq K$ where $S = f([m])$, and it is obvious that this S is minimal with this property. $\square$

4.18. **Proof of** (4.2)**.** Now we can prove our claim about maps out of a spine, using an explicit description of a spine as a colimit.

Proof of (4.2). Let $\mathcal{A}$ be the poset of all non-empty $S \subseteq [n]$ which are simplices of I^n; i.e., subsets of $[n]$ of the form $\{j\}$ or $\{j, j+1\}$. Explicitly the poset $\mathcal{A}$ has the form

$$\{0\} \to \{0, 1\} \leftarrow \{1\} \to \{1, 2\} \leftarrow \{2\} \to \cdots \leftarrow \{n-1\} \to \{n-1, n\} \leftarrow \{n\}.$$

By (4.17), $\mathrm{colim}_{S \in \mathcal{A}} \Delta^S \to I^n$ is an isomorphism. Thus $\mathrm{Hom}(I^n, X) \approx \mathrm{Hom}(\mathrm{colim}_{S \in \mathcal{A}} \Delta^S, X) \approx \lim_{S \in \mathcal{A}} \mathrm{Hom}(\Delta^S, X)$, and an elementary argument gives the result. □

5. Horns and inner horns

We now are going to give another (less obvious!) characterization of nerves, in terms of "extending inner horns", rather than "extending spines". This will allow us to weaken this new characterization to obtain the definition of a quasicategory.

5.1. **Definition of horns and inner horns.** We define a collection of subobjects of the standard simplices, called "horns". For each $n \geq 1$, these are subsimplicial sets $\Lambda_j^n \subset \Delta^n$ for each $0 \leq j \leq n$.

The jth **horn** Λ_j^n is the subcomplex of Δ^n defined by

$$(\Lambda_j^n)_k = \{\, f \colon [k] \to [n] \mid ([n] \smallsetminus \{j\}) \nsubseteq f([k]) \,\}.$$

Using the fact (4.17) that subcomplexes of Δ^n are always unions of Δ^Ss, we see that Λ_j^n is the union of "faces" $\Delta^{[n] \smallsetminus i}$ of Δ^n other than the jth face:

$$\Lambda_j^n = \bigcup_{i \neq j} \Delta^{[n] \smallsetminus i} \subset \Delta^n.$$

When $0 < j < n$ we say that $\Lambda_j^n \subset \Delta^n$ is an **inner horn**.

5.2. *Example.* The horns inside Δ^1 are just the vertices viewed as subobjects: $\Lambda_0^1 = \Delta^{\{0\}} = \{0\} \subset \Delta^1$ and $\Lambda_1^1 = \Delta^{\{1\}} = \{1\} \subset \Delta^1$. Neither is an inner horn.

5.3. *Example.* These are the three horns inside the 2-simplex.

$$
\begin{array}{ccc}
\Lambda_0^2 & \Lambda_1^2 & \Lambda_2^2
\end{array}
$$
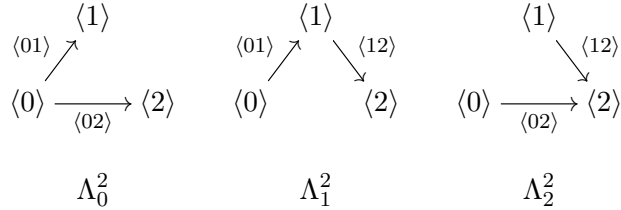
Only Λ_1^2 is an inner horn. Note that Λ_1^2 is the same as the spine I^2.

5.4. *Exercise.* Visualize the four horns inside the 3-simplex. The simplicial set Λ_j^3 actually kind of looks like a horn: you blow into the vertex $\langle j \rangle$, and sound comes out of the opposite missing face $\Delta^{[3] \smallsetminus j}$.

5.5. *Exercise.* Show that Λ_j^n is the largest subobject of Δ^n which does not contain the simplex $\langle 0 \cdots \widehat{j} \cdots n \rangle \in (\Delta^n)_{n-1}$, the "face opposite the vertex j".

We note that inner horns always contain spines: $I^n \subseteq \Lambda_j^n$ if $0 < j < n$. This is also true for non-inner horns if $n \geq 3$, but not for non-inner horns with $n = 1$ or $n = 2$.

5.6. **The inner horn extension criterion for nerves.** We can now characterize nerves as those simplicial sets which admit "unique inner horn extensions"; this is different than, but analogous to, the characterization in terms of unique spine extensions (4.4).

5.7. **Proposition.** *A simplicial set X is isomorphic to the nerve of a category, if and only if $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda_j^n, X)$ is a bijection for all $n \geq 2$, $0 < j < n$.*

The proof will take up the rest of the section.

5.8. Nerves have unique inner horn extensions. First we show that nerves have unique inner horn extensions.

5.9. Proposition. *If C is a category, then for every* inner *horn $\Lambda_j^n \subset \Delta^n$ the evident restriction map*

$$\mathrm{Hom}(\Delta^n, NC) \to \mathrm{Hom}(\Lambda_j^n, NC)$$

is a bijection.

Proof. Since inner horns contain spines, we can consider restriction along $I^n \subseteq \Lambda_j^n \subseteq \Delta^n$. The composite

$$\mathrm{Hom}(\Delta^n, NC) \to \mathrm{Hom}(\Lambda_j^n, NC) \xrightarrow{r} \mathrm{Hom}(I^n, NC)$$

of restriction maps is a bijection (4.4), so r is a surjection. Thus, it suffices to show that r is injective. This is immediate when $n = 2$, since $\Lambda_1^2 = I^2$, so we can assume $n \geq 3$.

We will show that for any inner horn Λ_j^n with $n \geq 3$ there exists a finite chain

$$I^n = F_0 \subset F_1 \subset \cdots \subset F_d = \Lambda_j^n$$

of subcomplexes, and also a list of subsets $S_1, \ldots, S_d \subset [n]$, such that (i) $F_i = F_{i-1} \cup \Delta^{S_i}$ and (ii) $I^{S_i} \subseteq F_{i-1} \cap \Delta^{S_i}$; here I^{S_i} denotes the spine of Δ^{S_i}. Given this, we see by (4.13) that F_i is isomorphic to a pushout:

$$F_i \approx \mathrm{colim}\big(F_{i-1} \leftarrow F_{i-1} \cap \Delta^{S_i} \to \Delta^{S_i}\big).$$

We then obtain a commutative diagram of sets

$$
\begin{array}{ccc}
\mathrm{Hom}(F_i, NC) & \xrightarrow{\ b\ } & \mathrm{Hom}(F_{i-1}, NC) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(\Delta^{S_i}, NC) \xrightarrow{\ a\ } \mathrm{Hom}(F_{i-1} \cap \Delta^{S_i}, NC) & \longrightarrow & \mathrm{Hom}(I^{S_i}, NC)
\end{array}
$$

where all maps are induced by restriction, in which the square is a pullback (because F_i is a pushout), and such that the horizontal composition on the bottom is a bijection. It immediately follows that a, and hence b, are injective. We can thus conclude that $\mathrm{Hom}(\Lambda_j^n, NC) \to \mathrm{Hom}(I^n, NC)$ is injective as desired, since it is a composite of injective functions such as b.

Now we prove the claim about the filtration of Λ_j^n by suitable subcomplexes F_i.

When $n = 3$, we can "attach" simplices in order explicitly:

$$\Lambda_1^3 = ((I^3 \cup \Delta^{\{0,1,2\}}) \cup \Delta^{\{1,2,3\}}) \cup \Delta^{\{0,1,3\}}, \qquad \Lambda_2^3 = ((I^3 \cup \Delta^{\{0,1,2\}}) \cup \Delta^{\{1,2,3\}}) \cup \Delta^{\{0,2,3\}}.$$

Note that, for instance, in building Λ_1^3, we must add $\Delta^{\{0,1,3\}}$ after adding $\Delta^{\{1,2,3\}}$, so that the spine $I^{\{0,1,3\}}$ of $\Delta^{\{0,1,3\}}$ is already present.
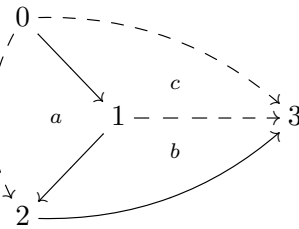
When $n \geq 4$, we have that $(\Lambda_j^n)_1 = (\Delta^n)_1$ and $(\Lambda_j^n)_2 = (\Delta^n)_2$. The procedure to "build" Λ_j^n from I^n by adding subsimplices is: (1) first attach 2-simplices one at a time, in an allowable order; then (2) attach all needed higher dimensional subsimplices. In step (2) the order doesn't matter since all 1-simplices (and hence all spines) are present in what has already been built. We leave the details of step (1) to the reader. $\qquad \square$

5.10. **Nerves are characterized by unique inner horn extension.** Let X be an arbitrary simplicial set, and suppose it has unique inner horn extensions, i.e., each $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_j, X)$ is a bijection for all $0 < j < n$ with $n \geq 2$.

Considering the unique extensions along $\Lambda^2_1 \subset \Delta^2$, we see that this defines a "composition law" on the set X_1. That is, given $f, g \in X_1$ such that $f_1 = g_0$ in X_0,[4] there is a unique map u

$$u \colon \Lambda^2_1 = \Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \xrightarrow{(f,g)} X, \qquad \langle 01 \rangle \mapsto f \in X_1, \ \langle 12 \rangle \mapsto g \in X_1.$$

Let $\widetilde{u} \colon \Delta^2 \to X$ be the *unique* extension of u along $\Lambda^2_1 \subset \Delta^2$, and define

$$g \circ f := \widetilde{u}_{02}.$$

The 2-simplex $\widetilde{u}$ is uniquely characterized by: $\widetilde{u}_{01} = f$, $\widetilde{u}_{12} = g$, $\widetilde{u}_{02} = g \circ f$.

This composition law is automatically unital. Given $x \in X_0$, write $1_x := x\langle 00 \rangle \in X_1$, so that $(1_x)_0 = x = (1_x)_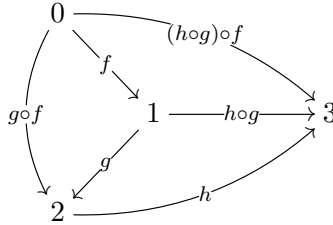1$. Then applying the composition law gives $1_x \circ f = f$ and $g \circ 1_x = g$. (Proof: consider the 2-simplices $f\langle 011 \rangle, g\langle 001 \rangle \in X_2$, and use the fact that their representing maps $\Delta^2 \to X$ are the unique extensions of their restrictions to $\Lambda^2_1 \subset \Delta^2$.)

Now consider $\Lambda^3_1 \subset \Delta^3$. Recall (4.17) that Λ^3_1 is a union (and colimit) of $\Delta^S \subseteq \Delta^3$ such that $S \not\supseteq \{0, 2, 3\}$. A map $\Lambda^3_1 \to X$ can be pictured as

so that the planar 2-cells in the picture correspond to non-degenerate 2-simplices in Δ^3 which are contained in Λ^3_1, while the edges are labelled according to their images in X, using the composition law defined above. Let $v \colon \Delta^3 \to X$ be any extension of the above picture along $\Lambda^3_1 \subset \Delta^3$, and consider the restriction $w := v\langle 023 \rangle \colon \Delta^2 \to X$ to the face $\Delta^2 \approx \Delta^{\{0,2,3\}} \subset \Delta^3$. Then $w_{01} = g \circ f$, $w_{12} = h$, and $w_{02} = (h \circ g) \circ f$, and thus the existence of w demonstrates that

$$h(g \circ f) = (h \circ g)f.$$

In other words, the *existence* of extensions along $\Lambda^3_1 \subset \Delta^3$ implies that the composition law we defined above is associative. (We could have carried out this kind of argument using $\Lambda^3_2 \subset \Delta^3$ instead.)

Thus, given an X with unique inner horn extensions, we can construct a category C, so that objects of C are elements of X_0, morphisms of C are elements of X_1, and composition is given as above.

Next we construct a map $X \to NC$ of simplicial sets. There are obvious maps $\alpha_n \colon X_n \to (NC)_n$, corresponding to restriction along spines $I^n \subseteq \Delta^n$; i.e., $\alpha(x) = (x_{01}, \ldots, x_{n-1,n})$. You can show that these maps are compatible with simplicial operators, so that they define a map $\alpha \colon X \to NC$ of simplicial sets. *Proof:* For any n-simplex $x \in X_n$, *all* of its edges are determined by edges on its spine via the composition law: $x_{ij} = x_{j-1,j} \circ x_{j-2,j-1} \circ \cdots \circ x_{i,i+1}$, for all $0 \leq i \leq j \leq n$. Thus for $f \colon [m] \to [n]$ we have $\alpha(xf) = ((xf)_{01}, \ldots, (xf)_{n-1,n}) = (x_{f_0 f_1}, \ldots, x_{f_{n-1} f_n}) = (x_{01}, \ldots, x_{n-1,n})_{f_0 \cdots f_n} = (\alpha x)f$.

Now we can prove that nerves are characterized by unique extension along inner horns.

---

[4]Recall that $f_1 = f\langle 1 \rangle$ and $g_0 = g\langle 0 \rangle$, regarded as maps $\Delta^0 \to X$ and thus as elements of X_0, using the notation discussed in §2.3.

Proof of (5.7). We have already shown (5.9) that nerves have unique extensions for inner horns. Consider a simplicial set X which has unique inner horn extension. By the discussion above, we obtain a category C and a map $\alpha\colon X \to NC$ of simplicial sets, which is clearly a bijection in degrees ≤ 2. We will show $\alpha_n\colon X_n \to (NC)_n$ is bijective by induction on n.

Fix $n \geq 3$, and consider the commutative square

$$
\begin{array}{ccc}
\mathrm{Hom}(\Delta^n, X) & \xrightarrow{\ \sim\ } & \mathrm{Hom}(\Lambda_1^n, X) \\
{\scriptstyle \alpha_{\Delta^n}}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha_{\Lambda_1^n}} \\
\mathrm{Hom}(\Delta^n, NC) & \xrightarrow[\ \sim\ ]{} & \mathrm{Hom}(\Lambda_1^n, NC)
\end{array}
$$

The horizontal maps are induced by restriction, and are bijections (top by hypothesis, bottom by (5.9)). Because Λ_1^n is a colimit of standard simplices of dimension $< n$ (4.17), the map $\alpha_{\Lambda_1^n}$ is a bijection by the induction hypothesis. Therefore so is α_{Δ^n}.   □

6. Quasicategories

We can now define the notion of a quasicategory, by removing the uniqueness part of the inner horn extension criterion for nerves.

6.1. **Identifying categories with their nerves.** From this point on, I will (at least informally) often not distinguish a category C from its nerve. In particular, I may assert something like "let C be a simplicial set which is a category", which should be read as "C is a simplicial set with is isomorphic to the nerve of some category". This should not lead to much confusion, due to the fact that the nerve functor is a fully faithful embedding of Cat into sSet (3.11).

6.2. **Definition of quasicategory.** A **quasicategory** is a simplicial set C such that for every map $f\colon \Lambda_j^n \to C$ from an inner horn, there *exists* an extension of it to $g\colon \Delta^n \to C$. That is, C is a quasicategory if the function $\mathrm{Hom}(\Delta^n, C) \to \mathrm{Hom}(\Lambda_j^n, C)$ induced by restriction along $\Lambda_j^n \subset \Delta^n$ is surjective for all $0 < j < n$, $n \geq 2$.

Any category (more precisely, the nerve of any category) is a quasicategory. In fact, by what we have shown (5.7) a category is precisely a quasicategory for which there exist *unique* extensions of inner horns.

Let C be a quasicategory. We refer to elements of C_0 as the **objects** of C, and elements of C_1 as the **morphisms** of C. Every morphism $f \in C_1$ has a **source** and **target**, namely its vertices $f_0, f_1 \in C_0$. We write $f\colon f_0 \to f_1$, just as we would for morphisms in a category. Likewise, for every object $x \in C_0$, there is a distinguished morphism $1_x\colon x \to x$, called the **identity morphism**, defined by $1_x = x_{00}$. Of course, when C is (the nerve of) a category, all the above notions coincide with the usual ones. (Note, however, that we cannot generally define *composition* of morphisms in a quasicategory.)

We now describe some basic categorical notions which admit immediate generalizations to quasicategories. Many of these generalizations apply to arbitrary simplicial sets.

6.3. **Products of quasicategories.** Simplicial sets are functors, so the product of simplicial sets X and Y is just the product of the functors. Thus, $(X \times Y)_n = X_n \times Y_n$.

6.4. **Proposition.** *The product of two quasicategories (as simplicial sets) is a quasicategory.*

Proof. Exercise, using the bijective correspondence between the sets of (i) maps $K \to X \times Y$ and (ii) pairs of maps $(K \to X, K \to Y)$.   □

6.5. *Remark.* If C and D are categories, then $N(C \times D) \approx NC \times ND$. Thus, the notion of product of quasicategories generalizes that of categories. (*Exercise:* prove this.)

6.6. **Coproducts of quasicategories.** The coproduct of simplicial sets X and Y is just the coproduct of functors, whence $(X \amalg Y)_n = X_n \amalg Y_n$, i.e., the set n-simplices of the coproduct is the disjoint union of the sets of n-simplices of X and Y. More generally, $\left(\coprod_s X_s\right)_n = \coprod_s (X_s)_n$ for an indexed collection $\{X_s\}$ of simplicial sets.

6.7. **Proposition.** *The coproduct of any indexed collection of quasicategories is a quasicategory.*

To prove this, we introduce the set of **connected components** of a simplicial set. Given a simplicial set X, we define the set of connected components by

$$\pi_0 X := \operatorname{colim}_{\Delta^{\mathrm{op}}} X,$$

the colimit of the simplicial set regarded as a functor $X \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$. Explicitly (4.5), this is the set of equivalence classes for the equivalence relation on the set $\coprod_{n \geq 0} X_n$ of all simplices, which is *generated* by the relation $\sim$, defined so that $a \sim af$ for any $a \in X_n$ and $f \colon [m] \to [n]$.

6.8. *Exercise.* Show that there is a canonical bijection

$$(X_0/\approx_1) \xrightarrow{\sim} \pi_0 X,$$

where the left-hand side denotes the set of equivalence classes in the vertex set X_0 with respect to the equivalence relation $\approx_1$ which is *generated* by the relation $\sim_1$ on X_0, defined by

$$a \sim_1 b \qquad \text{iff there exists } e \in X_1 \text{ such that } a = e_0, b = e_1.$$

We say that a simplicial set is **connected** if it has exactly one component.

6.9. *Exercise.* Show that every standard simplex Δ^n is connected, and that every horn Λ^n_j is connected.

6.10. *Exercise.* Show that a connected component S of X (regarded as a collection of subsets $S_n \subseteq X_n$ for $n \geq 0$ whose union is the equivalence class corresponding to a single element of $\pi_0 X$) is a subcomplex of X. Show that any map $K \to X$ from a connected simplicial set K factors through the inclusion of exactly one connected component $S \subseteq X$.

Proof of (6.7). If $X = \coprod_s X_s$ is a coproduct of simplicial sets, then any connected component of X must be contained in exactly one of the X_s, by an elementary argument left to the reader. The proof is now straightforward, using (6.10) and the fact that horns and standard simplices are connected (6.9). $\qquad\square$

6.11. *Exercise* (Important). Show that the evident map $\pi_0(X \times Y) \to \pi_0 X \times \pi_0 Y$ is a bijection.

6.12. **Full subquasicategories.** Given a category C and a set of objects $S \subseteq \operatorname{ob} C$, the *full subcategory spanned by S* is the subcategory $C' \subseteq C$ with $\operatorname{ob} C' = S$ and with $\operatorname{mor} C' = \{\, f \in \operatorname{mor} C \mid \operatorname{source}(f), \operatorname{target}(f) \in S \,\}$.

This has a straightforward generalization to quasicategories. Given a simplicial set C and a set $S \subseteq C_0$ of vertices, let

$$C'_n = \{\, a \in C_n \mid a_j \in S \text{ for all } j = 0, \ldots, n \,\}.$$

6.13. *Exercise.* Show that C' is a subcomplex of C, and that if C is a quasicategory then so is C'.

When C is a quasicategory, this subcomplex is called the **full subquasicategory spanned by** S.

6.14. **Opposite of a quasicategory.** Given a category C, the *opposite* category C^{op} has $\mathrm{ob}\, C^{\mathrm{op}} = \mathrm{ob}\, C$, and $\mathrm{Hom}_{C^{\mathrm{op}}}(x, y) = \mathrm{Hom}_C(y, x)$, and the sense of composition is reversed: $g \circ_{C^{\mathrm{op}}} f = f \circ_C g$.

This concept also admits a generalization to quasicategories, which we define using a non-trivial involution $\mathrm{op} \colon \Delta \to \Delta$ of the category Δ. This is the functor which on objects sends $[n] \mapsto [n]$, and on morphisms sends $\langle f_0, \ldots, f_n \rangle \colon [n] \to [m]$ to $\langle m - f_n, \ldots, m - f_0 \rangle$. (You can visualize this construction as one which "reverses the ordering" of the totally-orederd sets $[n]$.)

The **opposite** of a simplicial set $X \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ is the composite functor $X^{\mathrm{op}} := X \circ \mathrm{op}$.

We have that $(\Delta^n)^{\mathrm{op}} = \Delta^n$, while $(\Lambda^n_j)^{\mathrm{op}} = \Lambda^n_{n-j}$, so that the opposite of an inner horn is another inner horn. As a consequence, the opposite of a quasicategory is a quasicategory. It is straightforward to verify that $(NC)^{\mathrm{op}} = N(C^{\mathrm{op}})$, so the notion of opposite quasicategory generalizes the notion of opposite category. The functor $\mathrm{op} \colon \Delta \to \Delta$ satisfies $\mathrm{op} \circ \mathrm{op} = \mathrm{id}_\Delta$, so $(X^{\mathrm{op}})^{\mathrm{op}} = X$.

7. Functors and natural transformations

7.1. **Functors.** A **functor** between quasicategories is merely a map $f \colon C \to D$ between the simplicial sets.

We write QCat for the category of quasicategories and functors between them.[5] Clearly QCat $\subset$ sSet is a full subcategory. Because the nerve functor is a full embedding of Cat into QCat, any functor between ordinary categories is also a functor between quasicategories.

7.2. **Natural transformations.** Given functors $F, G \colon C \to D$ between categories, a *natural transformation* $\phi \colon F \Rightarrow G$ is a choice, for each object c of C, of a map $\phi(c) \colon F(c) \to G(c)$ in D, such that for every morphism $\alpha \colon c \to c'$ in C the square

$$
\begin{array}{ccc}
F(c) & \xrightarrow{\ \phi(c)\ } & G(c) \\
{\scriptstyle f(\alpha)}\big\downarrow & & \big\downarrow {\scriptstyle g(\alpha)} \\
F(c') & \xrightarrow[\ \phi(c')\ ]{} & G(c')
\end{array}
$$

commutes in D. There is a standard convenient reformulation of this: a natural transformation $\phi \colon F \Rightarrow G$ is the same thing as a functor

$$H \colon C \times [1] \to D,$$

so that $H | C \times \{0\} = F$, $H | C \times \{1\} = G$, and $H | \{c\} \times [1] = \alpha(c)$ for each $c \in \mathrm{ob}\, C$. (Here we make implicit use of the evident isomorphisms $C \times \{0\} \approx C \approx C \times \{1\}$.)

This admits a straightforward generalization to quasicategories. A **natural transformation** $f \colon f_0 \Rightarrow f_1$ of functors $f_0, f_1 \colon C \to D$ between quasicategories is defined to be a map

$$f \colon C \times N[1] = C \times \Delta^1 \to D$$

of simplicial sets such that $f | C \times \{i\} = f_i$ for $i = 0, 1$. For ordinary categories this coincides with the classical notion.

8. Examples of quasicategories

There are many ways to produce quasicategories, as we will see. Unfortunately, "hands-on" constructions of quasicategories are relatively rare. Here I give a few reasonably explicit examples to play with.

---

[5]Lurie [Lur09] denotes this category by Cat_Δ.

8.1. **Large vs. small.** I have been implicitly assuming that certain categories are **small**; i.e., they have *sets* of objects and morphisms. For instance, for the nerve of a category C to be a simplicial set, we need $C_0 = \mathrm{ob}\, C$ to be a set.

However, in practice many categories of interest are only **locally small**; i.e., the collection of objects is not a set but is a "proper class", although for any pair of objects $\mathrm{Hom}_C(X, Y)$ is a set. For instance, the category Set of sets is of this type: there is no set of all sets. Other examples include the categories of abelian groups, topological spaces, (small) categories, simplicial sets, etc. It is also possible to have categories which are not even locally small, e.g., the category of locally small categories. These are called **large** categories.

We would like to be able to talk about large categories in exactly the same way we talk about small categories. This is often done by positing a hierarchy of (Grothendieck) "universes". A universe U is (informally) a collection of sets which is closed under the operations of set theory. We additionally assume that for any universe U, there is a larger universe U' such that $U \in U'$. Thus, if by "set" we mean "U-set", then the category Set is a "U'-category". This idea can be implemented in the usual set theoretic foundations by postulating the existence of suitable *strongly inaccessible cardinals*.

The same distinctions occur for simplicial sets. For instance, the nerve of a small category is a small simplicial set (i.e., the simplices form a set), while the nerve of a large category is a large simplicial set.

I'm not going to be pedantic about this. I'll usually assume categories like Set, Cat, sSet, etc., are categories whose objects are "small" sets/categories/simplicial sets/whatever, i.e., are built from sets in a fixed universe U of "small sets". However, I sometimes need to consider examples of sets/categories/simplicial sets/whatever which are not small. I leave it to the reader to determine when this is the case.

In practice, a main point of concern involves constructions limits and colimits. Many typical examples of categories $C = \mathrm{Set}, \mathrm{Cat}, s\mathrm{Set}$, etc., in which objects are built out of small sets are **small complete** and **small cocomplete**: any functor $F\colon D \to C$ from a *small* category D has a limit and a colimit in C. This is *not* true if D is not assumed to be small. In this case care about the small/large distinction is necessary.

8.2. **Morita quasicategory.** Define a simplicial set C, so that C_n is a set whose elements are data $x := (A_i, M_{ij}, f_{ijk})$, where

- for each $i \in [n]$, A_i is an associative ring,
- for each $i < j$ in $[n]$, M_{ij} is an (A_i, A_j)-bimodule,
- for each $i < j < k$ in $[n]$, $f_{ijk}\colon M_{ij} \otimes_{A_j} M_{jk} \to M_{ik}$ is an isomorphism of (A_i, A_k)-bimodules, such that
- for each $i < j < k < \ell$, the diagram

(8.3)
$$
\begin{array}{ccc}
M_{ij} \otimes M_{jk} \otimes M_{k\ell} & \xrightarrow{\mathrm{id} \otimes f_{jk\ell}} & M_{ij} \otimes M_{j\ell} \\
{\scriptstyle f_{ijk} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle f_{ij\ell}} \\
M_{ik} \otimes M_{k\ell} & \xrightarrow[f_{ik\ell}]{} & M_{i\ell}
\end{array}
$$

commutes.

For an injective map $\delta\colon [m] \to [n]$, we define $x\delta := (A_{\delta(i)}, M_{\delta(i)\delta(j)}, f_{\delta(i)\delta(j)\delta(k)})$. If δ is not injective, we must set $M_{ij} := A_{\delta(i)}$ when $\delta(i) = \delta(j)$, and set f_{ijk} to the canonical isomorphism if $\delta(i) = \delta(j)$

or $\delta(j) = \delta(k)$.

$$
\begin{array}{c}
\text{[diagram]}
\end{array}
$$
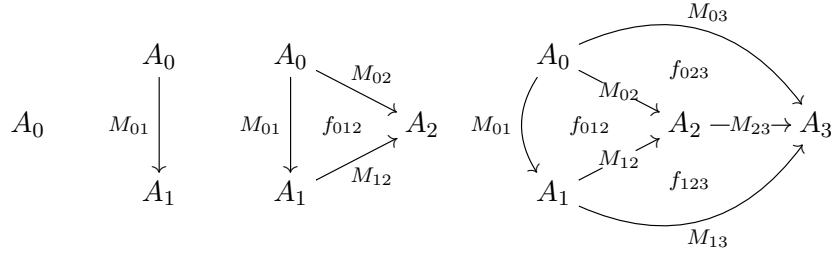
The object C is a quasicategory. Fillers for $\Lambda_1^2 \subset \Delta^2$ always exist: a map $\Lambda_1^2 \to C$ is a choice of $(A_0, M_{01}, A_1, M_{12}, A_2)$, and an extension to Δ^2 can be given by setting M_{02} to be the tensor product, and f_{012} the identity map. However, there can be more than one choice: even keeping M_{02} the same, there is a choice of isomorphism f_{012}.

Fillers for $\Lambda_1^3 \subset \Delta^3$ and $\Lambda_2^3 \subset \Delta^3$ always exist, and are unique: finding a filler amounts to choosing isomorphisms $f_{023} = f_{ik\ell}$ (for Λ_1^3) or $f_{013} = f_{ij\ell}$ (for Λ_2^3) making (8.3) commute, and such choices are unique. Similarly, all fillers in higher dimensions $\Lambda_j^n \subset \Delta^n$ with $n \geq 4$ trivially exist and are unique.

8.4. **Quasicategory of categories.** Define a simplicial set $\mathcal{C}$ so that $\mathcal{C}_n$ is a set whose elements are data $x := (C_i, F_{ij}, \phi_{ijk})$ where

- for each $i \in [n]$, C_i is a (small) category,
- for each $i < j$ in $[n]$, $F_{ij} \colon C_i \to C_j$ is a functor,
- for each $i < j < k$ in $[n]$, $\phi_{ijk} \colon F_{jk}F_{ij} \Rightarrow F_{ik}$ is a natural *isomorphism* of functors $C_i \to C_k$, such that
- for each $i < j < k < \ell$, the diagram

$$
\begin{array}{ccc}
F_{k\ell}F_{jk}F_{ij} & \xRightarrow{\phi_{jk\ell}\,\mathrm{id}_{F_{ij}}} & F_{j\ell}F_{ij} \\
{\scriptstyle \mathrm{id}_{F_{k\ell}}\,\phi_{ijk}} \big\Downarrow & & \big\Downarrow {\scriptstyle \phi_{ij\ell}} \\
F_{k\ell}F_{ik} & \xRightarrow[\phi_{ik\ell}]{} & F_{i\ell}
\end{array}
$$

commutes.

The action of simplicial operators is defined exactly as in the previous example, as is the proof that $\mathcal{C}$ is a quasicategory.

8.5. **Nerve of a crossed module.** A **crossed module** is data (G, H, ϕ, ρ), consisting of groups G and H, and homomorphisms $\phi \colon H \to G$ and $\rho \colon G \to \mathrm{Aut}\,H$, such that

$$
\phi(\rho(g)(h)) = g\phi(h)g^{-1}, \qquad \rho(\phi(h))(h') = hh'h^{-1}, \qquad \text{for all } g \in G,\ h, h' \in H.
$$

(For instance: $G = H =$ the cyclic group of order 4, with $\phi(x) = x^2$ and ρ the non-trivial action.) From this we can construct a quasicategory (in fact, a "quasigroupoid") much as in the last example: an n-simplex is data (g_{ij}, h_{ijk}) with $g_{ij} \in G$, $h_{ijk} \in H$, satisfying identities

$$
g_{ij}g_{jk} = \phi(h_{ijk})g_{ik}, \qquad h_{ijk}h_{ik\ell} = \rho(g_{ij})(h_{jk\ell})h_{ij\ell}.
$$

8.6. **Spans.** (See [Bar14, §§2–3], where this is called the *effective Burnside ∞-category*.) For each object $[n]$ of Δ, define $[n]^{\mathrm{tw}}$ to be the category with

- **objects** pairs (i, j) with $0 \leq i \leq j \leq n$, and
- a unique **morphism** $(i, j) \to (i', j')$ whenever $i' \leq i \leq j \leq j'$.

The construction $[n] \mapsto [n]^{\mathrm{tw}}$ defines a functor $\Delta \to \mathrm{Cat}$. (The category $[n]^{\mathrm{tw}}$ is called the *twisted arrow category* of $[n]$; in fact you can define a twisted arrow category C^{tw} for any category C.)

Let C be a category which has pullbacks; for an explicit example, think of the category of finite sets. Let $\mathcal{R}(C)$ be the simplicial set defined so that

$$\mathcal{R}(C)_n := \{\text{functors } ([n]^{\mathrm{tw}})^{\mathrm{op}} \to C\}.$$

Elements of $\mathcal{R}(C)_0$ are just objects of C. Elements of $\mathcal{R}(C)_1$, $\mathcal{R}(C)_2$, $\mathcal{R}(C)_3$ are respectively diagrams in C of shape
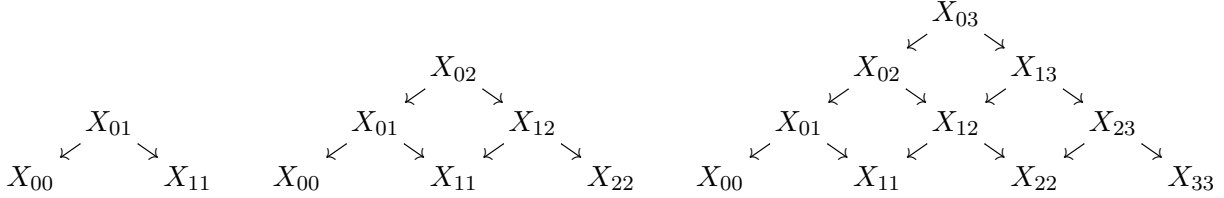
Let $\mathcal{A}(C)_n \subseteq \mathcal{R}(C)_n$ denote the subset whose n-simplices are functors $X \colon ([n]^{\mathrm{tw}})^{\mathrm{op}} \to C$ such that for every $i' \le i \le j \le j'$ the square

$$\begin{array}{ccc} X_{i'j'} & \longrightarrow & X_{ij'} \\ \downarrow & & \downarrow \\ X_{i'j} & \longrightarrow & X_{ij} \end{array}$$

is a pullback in C. Then $\mathcal{A}(C)$ is a subcomplex, and in fact is a quasicategory. This is another example in which extensions along inner horns $\Lambda^n_j \subset \Delta^n$ exist for $n \ge 2$, and are unique for $n \ge 3$.

8.7. Singular complex of a space.

The **topological n-simplex** is

$$\Delta^n_{\mathrm{top}} := \{\, (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1,\ x_i \ge 0 \,\},$$

the convex hull of the standard basis vectors. We get a functor $\Delta_{\mathrm{top}} \colon \Delta \to \mathrm{Top}$ by $\Delta_{\mathrm{top}}([n]) = \Delta^n_{\mathrm{top}}$. For a topological space T, we define $\mathrm{Sing}\, T$ to be the simplicial set $[n] \mapsto \mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{top}}, T)$.

Define **topological horns**

$$(\Lambda^n_j)_{\mathrm{top}} := \{\, x \in \Delta^n_{\mathrm{top}} \mid x_i = 0 \text{ for some } i \ne j \,\} \subset \Delta^n_{\mathrm{top}},$$

and observe that continuous maps $(\Lambda^n_j)_{\mathrm{top}} \to T$ correspond in a natural way with maps $\Lambda^n_j \to \mathrm{Sing}\, T$. (*Exercise.* This is a consequence of the fact that Λ^n_j is a colimit of all the $\Delta^S \subseteq \Lambda^n_j$, and that $(\Lambda^n_j)_{\mathrm{top}}$ is similarly a colimit of the corresponding $\Delta^S_{\mathrm{top}} \subseteq (\Lambda^n_j)_{\mathrm{top}}$.) There exists a continuous retraction $\Delta^n_{\mathrm{top}} \to (\Lambda^n_j)_{\mathrm{top}}$ (*Exercise:* describe such a retraction), and thus we see that

$$\mathrm{Hom}(\Delta^n, \mathrm{Sing}\, T) \to \mathrm{Hom}(\Lambda^n_j, \mathrm{Sing}\, T)$$

is surjective for every horn (not just inner ones).

A simplicial set X which has extensions for all horns is called a **Kan complex**. Thus, $\mathrm{Sing}\, T$ is a Kan complex, and so in particular is a quasicategory (and as we will eventually see, a "quasigroupoid").

8.8. Eilenberg-MacLane object.

Fix an abelian group A and an integer $d \ge 0$. We define a simplicial set $K = K(A, d)$, so that K_n is a set whose elements are data $a = (a_{i_0 \ldots i_d})$ consisting of

- for each $0 \le i_0 \le \cdots \le i_d \le n$, an element $a_{i_0 \ldots i_d} \in A$, such that
- $a_{i_0 \ldots i_d} = 0$ if $i_{u-1} = i_u$ for any u, and
- for each $0 \le j_0 \le \cdots \le j_{d+1} \le n$ we have $\sum_u (-1)^u a_{j_0 \ldots \widehat{j_u} \ldots j_{d+1}} = 0$.

For a map $\delta \colon [m] \to [n]$ we define

$$(a\delta)_{i_0 \ldots i_d} = a_{\delta(i_0) \ldots \delta(i_d)}.$$

The object $K(A, d)$ is a Kan complex, and hence a quasicategory (and in fact a quasigroupoid). When $d = 0$, this is just a discrete simplicial set, equal to A in each dimension.

8.9. *Exercise.* Show that $K(A, 1)$ is isomorphic to the nerve of a category, namely the nerve of the group A regarded as a category.

8.10. *Exercise.* Show that $K(A, d)$ is a Kan complex, i.e., that $\mathrm{Hom}(\Delta^n, K(A, d)) \to \mathrm{Hom}(\Lambda^n_j, K(A, d))$ is surjective for all horns $\Lambda^n_j \subset \Delta^n$. In fact, this map is *bijective* unless $n = d$. (Hint: there are four distinct cases to check, namely $n < d$, $n = d$, $n = d + 1$, and $n > d + 1$.)

8.11. *Exercise.* Given a simplicial set X, a **normalized d-cocycle** with values in A is a function $f \colon X_d \to A$ such that
  (1) $f(x_{0, \ldots i, i, \ldots d-1}) = 0$ for all $x \in X_{d-1}$ and $0 \leq i \leq d - 1$, and
  (2) $\sum (-1)^i f(x_{0, \ldots, \widehat{i}, \ldots, d+1}) = 0$ for all $x \in X_{d+1}$ and $0 \leq i \leq d + 1$. (Here "$0, \ldots, \widehat{i}, \ldots, d + 1$" is shorthand for the sequence $0, 1, \ldots, i - 1, i + 1, \ldots, d, d + 1$ of integers from 0 to $d + 1$ with i omitted.)

Show that the set $Z^d_{\mathrm{norm}}(X; A)$ of normalized d-cocycles on X is in bijective correspondence with $\mathrm{Hom}_{s\mathrm{Set}}(X, K(A, d))$. (Hint: an element $a \in K_n$ is uniquely determined by the collection of elements $a\delta \in K_d = A$, as δ ranges over injective maps $[d] \to [n]$.)

8.12. *Remark.* Eilenberg-MacLane objects are an example of a *simplicial abelian group*: the map $+ \colon K \times K \to K$ of simplicial sets defined by $(a + b)_{i_0 \ldots i_d} = a_{i_0 \ldots i_d} + b_{i_0 \ldots i_d}$ satisfies the axioms of an abelian group, reflecting the fact that $Z^d_{\mathrm{norm}}(X; A)$ is an abelian group.

9. Homotopy category of a quasicategory

Our next goal is to define the notion of an *isomorphism* in a quasicategory. This notion behaves much like that of *homotopy equivalence* in topology. Thus, we will define isomorphism by means of the *homotopy category* of a quasicategory. The homotopy category of a quasicategory is itself a special case of the notion of the *fundamental category* of a simplicial set.

9.1. **The fundamental category of a simplicial set.** A **fundamental category** for a simplicial set X consists of (i) a category hX, and (ii) a map $\alpha \colon X \to N(hX)$ of simplicial sets, such that for every category C, the map

$$\alpha^* \colon \mathrm{Hom}(N(hX), NC) \to \mathrm{Hom}(X, NC)$$

induced by restriction along α is a bijection. This is a universal property which characterizes the fundamental category up to unique isomorphism, if it exists.

9.2. **Proposition.** *Every simplicial set has a fundamental category.*

Proof sketch. Given X, we construct hX by generators and relations. First, consider the **free category** F, whose objects are the set X_0, and whose morphisms are finite "composable" sequences $[a_n, \ldots, a_1]$ of edges of X_1. Thus, morphisms in F are "words", whose "letters" are 1-simplices a_i with $(a_{i+1})_0 = (a_i)_1$, and composition is concatenation of words; the element $[a_n, \ldots, a_1]$ is then a morphism $(a_1)_0 \to (a_n)_1$. (Note: we suppose that there is an empty sequence $[]_x$ in F for each vertex $x \in X_0$; these correspond to identity maps in F.)

Then hX is defined to be the largest quotient category of F subject to the following relations:
  • $[a] \sim []_x$ for each $x \in X_0$ where $a = x_{00} \in X_1$, and
  • $[g, f] \sim [h]$ whenever there exists $a \in X_2$ such that $a_{01} = f$, $a_{12} = g$, and $a_{02} = h$.

The map $\alpha \colon X \to N(hX)$ sends $x \in X_n$ to the equivalence class of $[x_{n-1,n}, \ldots, x_{0,1}]$.
    (We will give another construction of the fundamental category in (13.17).) $\qquad\square$

As a consquence: the fundamental category construction describes a functor $h \colon s\mathrm{Set} \to \mathrm{Cat}$, which is left adjoint to the nerve functor $N \colon \mathrm{Cat} \to s\mathrm{Set}$.

In general, the fundamental category of a simplicial set is not an easy thing to get a hold of explicitly, because it is difficult to give an explicit description of a "quotient category" induced by a relation on its morphisms. When C is a quasicategory, there is a more concrete construction of hC, which in this context is called the **homotopy category** of C.

9.3. **The homotopy relation on morphisms.** Fix a quasicategory C. For $x, y \in C_0$, let $\hom_C(x, y) := \{\, f \in C_1 \mid f_0 = x, f_1 = y \,\}$ denote the set of "morphisms" in C from x to y. We write 1_x for the degenerate element $x_{00} \in \hom_C(x, x)$.
    Define relations $\sim_\ell$, $\sim_r$ on $\hom_C(x, y)$ (called **left homotopy** and **right homotopy**) by

- $f \sim_\ell g$ iff there exists $a \in C_2$ with $a_{01} = 1_x$, $a_{02} = f$, $a_{12} = g$,
- $f \sim_r g$ iff there exists $b \in C_2$ with $b_{12} = 1_y$, $b_{01} = f$, $b_{02} = g$.

Pictorally:

$$f \sim_\ell g \colon \qquad \begin{array}{c} x \xrightarrow{\;f\;} \\ {\scriptstyle 1_x}\Big\downarrow \; a \; \searrow y \\ x \xrightarrow[\;g\;]{} \end{array} \qquad\qquad f \sim_r g \colon \qquad \begin{array}{c} x \xrightarrow{\;f\;} y \\ {\scriptstyle g}\searrow \; b \; \Big\downarrow {\scriptstyle 1_y} \\ y \end{array}$$

Note that $f \sim_\ell g$ in $\hom_C(x, y)$ coincides with $g \sim_r f$ in $\hom_{C^{\mathrm{op}}}(y, x)$.

9.4. **Proposition.** *The relations $\sim_\ell$ and $\sim_r$ are identical, and are an equivalence relation on $\hom_C(x, y)$.*

Proof. We will prove

(1) $f \sim_\ell f$,
(2) $f \sim_\ell g$ and $g \sim_\ell h$ imply $f \sim_\ell h$,
(3) $f \sim_\ell g$ implies $f \sim_r g$,
(4) $f \sim_r g$ implies $g \sim_\ell f$.

These show that $\sim_\ell$ is an equivalence relation, and also that $\sim_r$ and $\sim_\ell$ coincide. The idea is to use the inner-horn extension condition for C to produce the appropriate relations.
    (1) $f \sim_\ell f$ is exhibited by $f_{001} \in C_2$.
    (2), (3), and (4) are demonstrated by the following diagrams, which present a map from an inner horn of Δ^3 to C constructed from the given data. The restriction of any extension to Δ^3 along the remaining face gives the conclusion.

$\qquad\square$
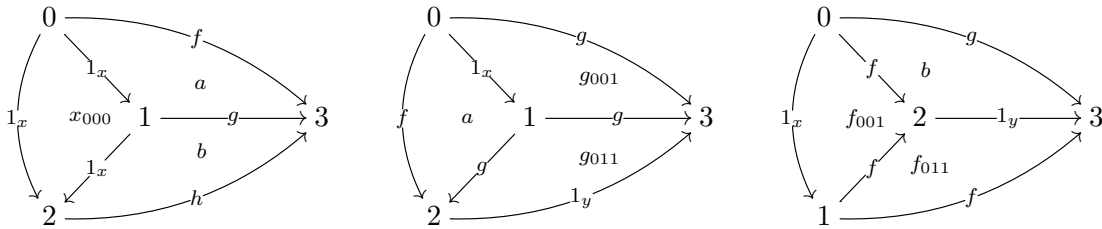
9.5. **Composition of homotopy classes of morphisms.** We now define $f \approx g$ to mean $f \sim_\ell g$ (equivalently $f \sim_r g$). We speak of **homotopy classes** $[f]$ of morphisms $f \in \hom_C(x, y)$, meaning equivalence classes under $\approx$. Next we observe that we can compose homotopy classes.

Given $f \in \hom_C(x, y)$, $g \in \hom_C(y, z)$, $h \in \hom_C(x, z)$, we say that h is a **composite** of (g, f) if there exists a 2-simplex $a \in C_2$ with $a\langle 01 \rangle = f$, $a\langle 12 \rangle = g$, $a\langle 02 \rangle = h$; thus composition is a

three-fold relation on $\hom(x,y) \times \hom(y,z) \times \hom(x,z)$. The composite relation is compatible with the homotopy relation, as shown by the following.

9.6. Lemma. *If $f \approx f'$, $g \approx g'$, h a composite of (g, f), and h' a composite of (g', f'), then $h \approx h'$.*

Proof. Since $\approx$ is an equivalence relation, it suffices prove the special cases (a) $f = f'$, and (b) $g = g'$. We prove case (b).

Let $a \in C_2$ exhibit $f \sim_\ell f'$, and let $b, b' \in C_2$ exhibit h as a composite of (g, f) and h' as a composite of (g, f') respectively. The inner horn $\Lambda^3_2 \to C$ defined by

extends to $u \colon \Delta^3 \to C$, and $u|\Delta^{\{0,1,3\}}$ exhibits $h \sim_\ell h'$.   □

Thus, composites of (g, f) live in a unique homotopy class of morphisms in C, which only depends on the homotopy classes of g and f. I will write $[g] \circ [f]$ for this class.

I'll leave the following as exercises; the proofs are much like what we have already seen.

9.7. Lemma. *Given $f \colon x \to y$, we have $[f] \circ [1_x] = [f] = [1_y] \circ [f]$.*

9.8. Lemma. *If $[g] \circ [f] = [u]$, $[h] \circ [g] = [v]$, then $[h] \circ [u] = [v] \circ [f]$.*
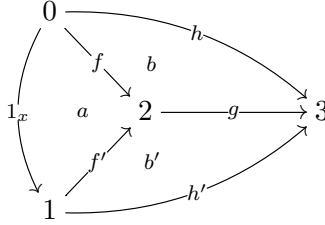
9.9. The homotopy category of a quasicategory. For any quasicategory, we define its **homotopy category** hC, so that $\mathrm{ob}(hC) := C_0$, while $\hom_{hC}(x, y) := \hom_C(x, y)/ \approx$, with composition defined by $[g] \circ [f]$. The above lemmas (9.7) and (9.8) exactly imply that hC is really a category.

We define a map $\pi \colon C \to N(hC)$ of simplicial sets as follows. On 0 simplices, π is the identity map $C_0 = N(hC)_0 = \mathrm{ob}\, hC$. On 1-simplices, the map is defined by the tautological quotient maps $\hom_C(x, y) \to \hom_C(x, y)/ \approx$. The map π sends an n-simplex $a \in C_n$ to the unique n-simplex $\pi(a) \in N(hC)_n$ such that $\pi(a)_{i-1,i} = \pi(a_{i-1,i})$. It is straightforward to check that this is well-defined: for instance, for $a \in C_2$, $\pi(a)$ is defined exactly because of the relation $[a_{12}] \circ [a_{01}] \approx [a_{02}]$.

Note that if C is an ordinary category, then $f \approx g$ if and only if $f = g$. Thus, $\pi \colon C \to N(hC)$ is an isomorphism of simplicial sets if and only if C is isomorphic to the nerve of a category.

The following says that the homotopy category of a quasicategory is its fundamental category, justifying the notation "hC".

9.10. Proposition. *Let C be a quasicategory and D a small category, and let $\phi \colon C \to D$ be a map of simplicial sets. Then there exists a unique map $\psi \colon hC \to D$ such that $\psi\pi = \phi$.*

Proof. We first show existence, by constructing a suitable extension ψ, which is necessarily a functor between categories.

On objects, ψ sends $x \in \mathrm{ob}(hC) = C_0$ to $\phi(x) \in \mathrm{ob}(D) = D_0$. On morphisms, ψ sends $[f] \in \hom_{hC}(x, y)$ to $\phi(f) \in \hom_D(\phi(x), \phi(y)) \subseteq D_1$. Observe that the function on morphisms is well-defined since if $f \sim_\ell f'$, exhibited by some $a \in C_2$, then $\phi(a) \in D_2$ exhibits the identity $\phi(f) = \phi(f')\phi(1_x) = \phi(f')$ in D. It is straightforward to show that ψ so defined is actually a functor, and that $\psi\pi = \phi$ as maps $C \to N(D)$.

The functor ψ defined above is the unique solution: the value of ψ on objects and morphisms is uniquely determined, and $\pi \colon C_k \to (hC)_k$ is bijective for $k = 0$ and surjective for $k = 1$.   □

In particular, the homotopy category construction gives adjoint functors

$$h \colon \mathrm{QCat} \rightleftarrows \mathrm{Cat} : N.$$

9.11. *Remark.* If f, g, h are any edges in a quasicategory C which fit together to give $u \colon \Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \cup \Delta^{\{0,2\}} \to C$, so that $u|\Delta^{\{01\}} = f$, $u|\Delta^{\{12\}} = g$, and $u|\Delta^{\{02\}} = h$, then there exists an extension of u to Δ^2 *if and only if* $\pi(h) = \pi(g)\pi(f)$ in hC. In particular, any edge in the $\approx$-equivalence class of h can be interpreted as a composite of g with f. To prove this, consider an extension of
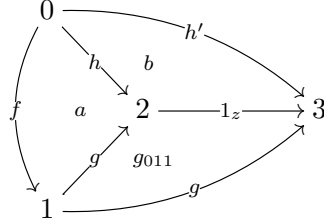
which exhibits $g \circ f \sim h'$ assuming $g \circ f \sim h$ and $h \sim_r h'$.

9.12. *Exercise.* Understand the homotopy categories of the various examples described in §8.

9.13. *Exercise.* Show that for quasicategories C and D there is an isomorphism $hC \times hD \approx h(C \times D)$.

10. Saturated classes and inner-anodyne maps

Quasicategories are defined by an "extension property": they are the simplicial sets C such that any map $K \to C$ extends over L, whenever $K \subset L$ is an inner horn inclusion $\Lambda^n_j \subset \Delta^n$. The inner horn inclusions "generate" a larger class of inclusions $K \subseteq L$ which share this extension property for maps into quasicategories; this larger class includes the "saturation" of the inner horns.

For instance, we will observe that the spine inclusions $I^n \subset \Delta^n$ are in the saturation of the set of inner horn inclusions. This will imply that quasicategories admit "spine extensions", i.e., any $I^n \to C$ extends over $I^n \subset \Delta^n$ to a map $\Delta^n \to C$.

10.1. **Saturated classes.** Consider a category (such as sSet) which has all small colimits. A **saturated class**[6] is a class $\mathcal{A}$ of morphisms in the category, which

(1) contains all isomorphisms,
(2) is closed under cobase change,
(3) is closed under composition,
(4) is closed under transfinite composition,
(5) is closed under coproducts, and
(6) is closed under retracts.

Given a class of maps S, its **saturation** $\overline{S}$ is the smallest saturated class containing S.

We need to explain some of the elements of this definition.

- "Closed under cobase change" is also called "closed under pushout": it means that if f' is the pushout of $f \colon X \to Y$ along some map $g \colon X \to Z$, then $f \in \mathcal{A}$ implies $f' \in \mathcal{A}$.
- We say that $\mathcal{A}$ is "closed under countable composition" if given a countable sequence of composable morphisms

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots$$

such that each $f_k \in \mathcal{A}$, the induced map $X_0 \to \mathrm{colim}_k X_k$ to the colimit is in $\mathcal{A}$.

---

[6]Often the term *weakly saturated* is used where we have used *saturated*, e.g., [Lur09, A.1.2.2] or [Rie14, 11.1.4]. We will use the shorter terminology for the sake of brevity. We caution that the term "saturated" is sometimes used in category theoretic contexts with a different meaning than the one we use. **I'm gonna change this to be consistent with general usage: replace "saturation" with "weak saturation" everywhere. Maybe not I dunno.**

The notion "closed under transfinite composition" is the generalization of this, where $\mathbb{N}$ is replaced by an arbitrary ordinal (i.e., a well-ordered set). This means that for an ordinal λ and a functor $X\colon \lambda \to s\mathrm{Set}$, if for every $i \in \lambda$ with $i \neq 0$ the map

$$\mathrm{colim}_{j<i} X(j) \to X(i)$$

is in $\mathcal{A}$, then the induced map $X(0) \to \mathrm{colim}_{j\in\lambda} X(j)$ is in $\mathcal{A}$.

- "Closed under coproducts" means that if $\{f_i\}$ is a set of maps in $\mathcal{A}$, then $\coprod_i f_i$ is in $\mathcal{A}$.
- We say that f is a **retract** of g if there exists a commutative diagram of the form

This is really a special case of the notion of a retract of an object in the functor category $\mathrm{Fun}([1], s\mathrm{Set})$.

10.2. *Remark.* This list of properties is not minimal: (3) is actually a special case of (4), and (5) can be deduced from (2) and (4).

10.3. *Example.* Consider the category of sets. Let $S = \{\varnothing \to \{1\}\}$ and $T = \{\{0,1\} \to \{1\}\}$. Then $\overline{S} =$ the class of injective maps, and $\overline{T} =$ the class of surjective maps.

10.4. *Example.* The class of monomorphisms of simplicial sets is a saturated class.

10.5. Inner anodyne morphisms. Let

$$\mathrm{InnHorn} := \{\, \Lambda^n_k \subset \Delta^n \mid 0 < k < n,\ n \geq 2 \,\}$$

denote the set of inner horn inclusions. The saturation $\overline{\mathrm{InnHorn}}$ is called the class of **inner anodyne**[7] morphisms. Note that inner anodyne morphisms are always monomorphisms, since monomorphisms of simplicial sets are themselves a saturated class.

10.6. Proposition. *If C is a quasicategory and $A \subseteq B$ is an inner anodyne inclusion, then any $f\colon A \to C$ admits an extension to $g\colon B \to C$ so that $g|A = f$.*

Proof. It suffices to show that the collection $\mathcal{A}$ of monomorphisms $i\colon A \to B$ such that every map from A to a quasicategory extends along i is saturated. Since $\mathrm{InnHorn} \subseteq \mathcal{A}$ we must have $\overline{\mathrm{InnHorn}} \subseteq \mathcal{A}$. This is a relatively straightforward exercise, which we leave for the reader: check that the class $\mathcal{A}$ satisfies each of the conditions (1)–(6) of a saturated class. It is highly recommended that you work through this argument this if you haven't seen it before. $\square$

10.7. *Exercise* (Easy but important). Show that every inner anodyne map induces a bijection on vertices.

10.8. Generalized horns and spines. It is useful to be able to prove that certain explicit maps are inner anodyne.

Let $S \subseteq [n]$. A **generalized horn** the subcomplex $\Lambda^n_S \subset \Delta^n$ defined by

$$\Lambda^n_S := \bigcup_{i\in S} \Delta^{[n]\smallsetminus i},$$

---

[7]Joyal calls these "mid-anodyne". The "anodyne" terminology was introduced in a related context by Gabriel and Zisman [GZ67], which we will see in §32. "Anodyne" derives from ancient Greek, meaning "without pain"; we leave it to the reader to decide whether this choice of terminology is appropiate.

i.e., the union of codimension one faces of the n-simplex indexed by elements of S. In particular, $\Lambda^n_{[n] \smallsetminus \{j\}}$ is the usual horn Λ^n_j.

An **inner generalized horn** is $\Lambda^n_S \subseteq \Delta^n$ such that there exist $s < t < s'$ with $s, s' \in S$ and $t \notin S$.

10.9. *Example.* The union $\Lambda^n_{\{0,n\}} = \Delta^{\{0,1,\dots,n-1\}} \cup \Delta^{\{1,2,\dots n\}}$ of the "first and last" faces of Δ^n is an inner generalized horn when $n \geq 2$.

10.10. **Lemma.** *Inner generalized horn inclusions are inner anodyne.*

There is a slick proof of this given by Joyal [Joy08a, Prop. 2.12], which we present in the appendix (58.1). The basic idea is simple: construct a sequence of subcomplexes

$$\Lambda^n_S = F_0 \subset F_1 \subset \cdots \subset F_d = \Delta^n$$

such that for each i, there is a subset $S_i \subseteq [n]$ such that (i) $F_i = F_{i-1} \cup \Delta^{S_i}$ and (ii) $F_{i-1} \cap \Delta^{S_i} = \Lambda^{S_i}_j \subset \Delta^{S_i}$, where this is the inclusion of an inner horn. This exhibits $\Lambda^n_S \to \Delta^n$ as a composite of maps, each of which is a cobase change of an inner horn inclusion, whence it is in the saturated class $\overline{\mathrm{InnHorn}}$.

10.11. *Remark.* The idea of this proof is reminiscent of the proof of (5.9) (that nerves of categories have unique inner horn extensions). Note that in this case, however, it is crucial that when we "attach" a simplex to an F_{i-1}, the intersection of F_{i-1} with the simplex must be *equal* to an inner horn.

Recall that every standard n-simplex contains a spine $I^n \subseteq \Delta^n$.

10.12. **Lemma.** *The spine inclusions $I^n \subset \Delta^n$ are inner anodyne for all n. Thus, for a quasicategory C, any $I^n \to C$ extends to $\Delta^n \to C$.*

This is proved in [Joy08a, Prop. 2.13]; we give the proof in the appendix (58.2).

As a consquence of (10.12), the tautological map $\pi \colon C \to hC$ from a quasicategory to (the nerve of) its homotopy category is surjective in every degree.

11. ISOMORPHISMS IN A QUASICATEGORY

Let C be a quasicategory. We say that an edge $f \in C_1$ is an **isomorphism**[8] if its image in the homotopy category hC is an isomorphism.

Explicitly, $f \colon x \to y$ is an isomorphism if and only if there exists an edge $g \colon y \to x$ such that $gf \approx 1_x$ and $fg \approx 1_y$. Here and in what follows I write "gf" to denote "any 1-simplex of C which is a composite of (g, f)", while "$\approx$" is the homotopy equivalence relation on hom-sets of C.

For instance, f is an isomorphism if there exist $g \in C_1$ and $a, b \in C_2$ such that

$$a_{01} = f = b_{12}, \qquad a_{12} = g = b_{01}, \qquad a_{02} = x_{00}, \qquad b_{02} = y_{00}.$$

so that $gf \sim_l 1_x$ and $fg \sim_r 1_y$.

---

[8]Lurie [Lur09, §1.2.4] uses the term "equivalence" for this. I prefer to go with "isomorphism" here, because they do in fact generalize the classical notion of isomorphism, and because so many other things also get to be called some kind of equivalence.

11.1. Preinverses and postinverses. Let C be a quasicategory. Given $f \colon x \to y \in C_1$, a **postinverse**[9] of f is a $g \in C_1$ such that $gf \approx 1_x$, and a **preinverse**[10] of f is an $e \in C_1$ such that $fe \approx 1_y$. An **inverse** is an $f' \in C_1$ which is both a postinverse and a preinverse. The following is trivial, but very handy.

11.2. Proposition. *Let $f \in C_1$. The following are equivalent.*

- *f is an isomorphism.*
- *f admits an inverse f'.*
- *f admits a postinverse g and a preinverse e.*
- *f admits a postinverse g and g admits a postinverse h.*
- *f admits a preinverse e and e admits a preinverse d.*

If these equivalent conditions hold, then $f \approx d \approx h$ and $f' \approx e \approx g$, and all of them are isomorphisms.

Proof. All of these are equivalent to the corresponding statements about morphisms in the homotopy category hC, where they are clearly equivalent. □

Note that inverses to a morphism in a quasicategory are generally *not unique*, though necessarily they are *unique up to homotopy*.

11.3. Quasigroupoids. A **quasigroupoid** is a quasicategory C such that hC is a groupoid, i.e., a quasicategory in which every morphism is an isomorphism.

11.4. *Exercise.* If every morphism in a quasicategory admits a preinverse, then it is a quasigroupoid. Likewise if every morphism admits a postinverse.

11.5. The core of a quasicategory. Recall that for a category A, the **core** (or **maximal subgroupoid**) of A is the subcategory $A^{\mathrm{core}} \subseteq A$ consisting of all the objects, and all the *isomorphisms* between them.

For a quasicategory C, we define $C^{\mathrm{core}} \subseteq C$ to be the subsimplicial set consisting of simplices whose edges are all isomorphisms. That is, the diagram

$$
\begin{array}{ccc}
C^{\mathrm{core}} & \longrightarrow & C \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
(hC)^{\mathrm{core}} & \longrightarrow & hC
\end{array}
$$

is a pullback of simplicial sets. Observe that $N(A^{\mathrm{core}}) = (NA)^{\mathrm{core}}$ for a category A.

11.6. Proposition. *For a quasicategory C, C^{core} is a quasigroupoid, and every subcomplex of C which is a quasigroupoid is contained in C^{core}.*

Proof. First, note that C^{core} is a subcomplex by construction: if $a \in C_n$ is such that all edges are isomorphisms, then the same is true for af for any $f \colon [m] \to [n]$.

Next, we show that C^{core} is a quasicategory. In fact, we show that given $f \colon \Delta^n \to C$ such that $f(\Lambda^n_j) \subseteq C^{\mathrm{core}}$ for some $0 < j < n$, then $f(\Delta^n) \subseteq C^{\mathrm{core}}$, so that inner-horn-filling for C implies inner-horn-filling for C^{core}. For $n = 2$ this is the fact that composites of isomorphisms are isomorphisms (true in any quasicategory because it is true in its homotopy category), while for $n \geq 3$ it is just the fact that an inner horn of Δ^n contains all the edges of Δ^n.

Thus, C^{core} is a quasicategory, and is easily seen to be a quasigroupoid, since an inverse of an isomorphism is also an isomorphism.

The final statement is clear: if $G \subseteq C$ is a subcomplex which is a quasigroupoid, then every edge in G has in inverse in G, and hence an inverse in C. □

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[9]or **left inverse**, or **retraction**,

[10]or **right inverse**, or **section**,

11.7. Kan complexes. A **Kan complex** is a simplicial set which has the extension property with respect to all horns, not just inner horns. That is, K is a Kan complex iff

$$\mathrm{Hom}(\Delta^n, K) \to \mathrm{Hom}(\Lambda_k^n, K)$$

is surjective for all $0 \le k \le n$, $n \ge 1$.

11.8. *Exercise.* Show that every simplicial set X has extensions for 1-dimensional horns; i.e., every $\Lambda_j^1 \to X$ extends over $\Lambda_j^1 \subset \Delta^1$. Thus, the stipulation that Kan complexes have extensions for 1-horns is not necessary.

11.9. Proposition. *Every Kan complex is a quasigroupoid.*

Proof. It is immediate that a Kan complex K is a quasicategory. To show K is a quasigroupoid, note that the extension condition for $\Lambda_0^2 \subset \Delta^2$ implies that every morphism in hK admits a postinverse. Explicitly, if $f\colon x \to y$ is an edge in K, let $u\colon \Lambda_0^2 \to K$ with $u_{01} = f$ and $u_{02} = f_{00} = 1_x$, so there is an extension $v\colon \Delta^2 \to K$ and $g := v_{12}$ satisfies $gf \approx 1_x$. Use (11.4). $\square$

This proposition has a converse.

A. **Deferred Proposition.** *Quasigroupoids are precisely the Kan complexes.*

This is a very important technical result, and it is not trivial; it is the main result of [Joy02]. We will give the proof in (29.2).

Recall (§8.7) that we showed that the singular complex $\mathrm{Sing}\, T$ of a topological space is a Kan complex, and therefore a quasigroupoid. It is reasonable to think of $\mathrm{Sing}\, T$ as the **fundamental quasigroupoid** of the space T.

11.10. *Exercise* (for topologists). Show that if T is a space, then $h\,\mathrm{Sing}\, T$, the homotopy category of the singular complex of T, is precisely the usual fundamental groupoid of T.

11.11. Quasigroupoids and components. Recall (6.8) that the set of connected components of a simplicial set is given by

$$\pi_0 \approx \mathrm{colim}_{\Delta^{\mathrm{op}}} X \approx X_0/\approx,$$

the equivalence classes of vertices of X under the equivalence relation generated by the relation "connected by an edge". (If T is a topological space, then elements of $\pi_0 \,\mathrm{Sing}\, T$ correspond exactly to path components of T.)

For *quasigroupoids*, π_0 recovers the set of isomorphism classes of objects.

11.12. Proposition. *If C is a quasicategory, then*

$$\pi_0\big(C^{\mathrm{core}}\big) \approx \text{*isomorphism classes of objects of C.*}$$

Proof. Straightforward, for instance using (6.8). $\square$

11.13. *Exercise.* Show that for a quasicategory C, $\pi_0(C^{\mathrm{core}}) \approx \pi_0(h(C^{\mathrm{core}})) \approx \pi_0((hC)^{\mathrm{core}})$.

12. Function complexes and the functor quasicategory

12.1. Function complexes. Given simplicial sets X and Y, we may form the **function complex** (or **mapping space**) $\mathrm{Map}(X, Y)$. This is a simplicial set with

$$\mathrm{Map}(X, Y)_n = \mathrm{Hom}(\Delta^n \times X, Y),$$

so that the simplicial operator δ^* on $\mathrm{Map}(X, Y)$ is induced by $\mathrm{Hom}(\delta \times \mathrm{id}_X, Y)\colon \mathrm{Hom}(\Delta^n \times X, Y) \to \mathrm{Hom}(\Delta^m \times X, Y)$. The vertices of $\mathrm{Map}(X, Y)$ are precisely the maps $X \to Y$.

12.2. Proposition. *The function complex construction defines a functor*

$$\mathrm{Map}\colon s\mathrm{Set}^{\mathrm{op}} \times s\mathrm{Set} \to s\mathrm{Set}.$$

Proof. Left as an exercise.   □

By construction, for each n, there is a bijective correspondence

$$\{\Delta^n \times X \to Y\} \longleftrightarrow \{\Delta^n \to \mathrm{Map}(X, Y)\}.$$

In fact, we can replace Δ^n with an arbitrary simplicial set.

12.3. Proposition. *For simplicial sets X, Y, Z, there is a bijection*

$$\mathrm{Hom}(X \times Y, Z) \xrightarrow{\sim} \mathrm{Hom}(X, \mathrm{Map}(Y, Z))$$

natural in all three variables.

Proof. The bijection sends $f\colon X \times Y \to Z$ to $\widetilde{f}\colon X \to \mathrm{Map}(Y, Z)$ defined so that for $x \in X_n$, the simplex $\widetilde{f}(x) \in \mathrm{Map}(Y, Z)_n$ is represented by the composite

$$\Delta^n \times Y \xrightarrow{x \times \mathrm{id}} X \times Y \xrightarrow{f} Z.$$

The inverse of this bijection sends $g\colon X \to \mathrm{Map}(Y, Z)$ to $\widetilde{g}\colon X \times Y \to Z$, defined so that for $(x, y) \in X_n \times Y_n$, the simplex $\widetilde{g}(x, y) \in Z_n$ is represented by

$$\Delta^n \xrightarrow{(\mathrm{id}, y)} \Delta^n \times Y \xrightarrow{g(x)} Z.$$

The proof amounts to showing that both $\widetilde{f}$ and $\widetilde{g}$ are in fact maps of simplicial sets, and that the above constructions are in fact inverse to each other. This is left as an exercise, as is the proof of naturality.   □

12.4. *Exercise.* Show, using the previous proposition, that there are natural isomorphisms

$$\mathrm{Map}(X \times Y, Z) \approx \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

of simplicial sets. This implies that the function complex construction makes sSet into a *cartesian closed category*. (Hint: show that both objects represent *isomorphic* functors $s\mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$, and apply the Yoneda lemma.)

12.5. *Remark.* The construction of the function complex is not special to simplicial sets. The construction of $\mathrm{Map}(X, Y)$ (and its properties as described above) works the same way in any category of functors $C^{\mathrm{op}} \to \mathrm{Set}$, where C is a small category (such as Δ); in the general case, the role of the standard n-simplices is played by the representable functors $\mathrm{Hom}_C(-, c)\colon C^{\mathrm{op}} \to \mathrm{Set}$.

12.6. Functor quasicategories. Given ordinary categories C, D, we have a functor category $\mathrm{Fun}(C, D)$, with *objects:* the functors $C \to D$ and *morphisms:* the natural transformations between functors. A standard fact is that if A is another category, then there is a bijective correspondence between sets of functors

$$\{A \times C \to D\} \longleftrightarrow \{A \to \mathrm{Fun}(C, D)\}.$$

For instance, a functor $f\colon A \times C \to D$ corresponds to $\widetilde{f}\colon A \to \mathrm{Fun}(C, D)$ given on objects $a \in \mathrm{ob}\, A$ by $\widetilde{f}(a)(c) = g(a, c)$ and $\widetilde{f}(a)(\gamma) = f(1_a, \gamma)$, and on morphisms $\alpha \in \mathrm{mor}\, A$ by $\widetilde{f}(\alpha)(c) = f(\alpha, 1_c)$.

12.7. *Exercise.* Show that $N\mathrm{Fun}(C, D) \approx \mathrm{Map}(NC, ND)$. (Hint: use that $N[n] = \Delta^n$, and the fact that the nerve preserves finite products.)

Thus, we expect the generalization of functor category to quasicategories to be defined by the function complex. In fact, if C and D are quasicategories, then the vertices of $\mathrm{Map}(C, D)$ are precisely the functors $C \to D$, and the edges of $\mathrm{Map}(C, D)$ are precisely the natural transformations.

It turns out that a function complex between quasicategories is again a quasicategory. In fact, we have the following.

B. Deferred Proposition. *Let K be any simplicial set and C a quasicategory. Then $\mathrm{Map}(K, C)$ is a quasicategory.*

For this reason, we will sometimes write $\mathrm{Fun}(K,C)$ for $\mathrm{Map}(K,C)$ when C is a quasicategory.

To prove (B), we need a to take a detour to develop some more technology about saturated classes of maps and lifting properties. After this, we will complete the proof in §16.

Part 2. Lifting properties

13. Lifting calculus and inner fibrations

13.1. **The lifting relation.** Given morphisms f and g in a category, a **lifting problem** for (f, g) is a pair of morphisms (u, v) such that $vf = gu$. That is, a lifting problem is any commutative square of solid arrows of the form

$$
\begin{array}{ccc}
A & \xrightarrow{u} & X \\
f \downarrow & {\scriptstyle s}\nearrow & \downarrow g \\
B & \xrightarrow{v} & Y
\end{array}
$$

A **lift** for the lifting problem is a morphism s such that $sf = u$ and $gs = v$, i.e., a dotted arrow making the diagram commute.

We write "$f \boxslash g$" if every lifting problem for (f, g) admits a lift[11]. Equivalently, $f \boxslash g$ exactly if

$$
\mathrm{Hom}(B, X) \xrightarrow{s \mapsto (sf, gs)} \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B, Y)
$$

is a surjection, where the target is the set of pairs $(u\colon A \to X, v\colon B \to Y)$ such that $gu = vf$.

When $f \boxslash g$ holds, one sometimes says f has the **left lifting property** relative to g, or that g has the **right lifting property** relative to f. Or we just say that f **lifts against** g.

We extend the notation to classes of maps, so "$\mathcal{A} \boxslash \mathcal{B}$" means: $a \boxslash b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Given a class of morphisms $\mathcal{A}$, define the **right complement** $\mathcal{A}^{\boxslash}$ and **left complement** ${}^{\boxslash}\mathcal{A}$ by

$$
\mathcal{A}^{\boxslash} = \{\, g \mid a \boxslash g \text{ for all } a \in \mathcal{A} \,\}, \qquad {}^{\boxslash}\mathcal{A} = \{\, f \mid f \boxslash a \text{ for all } a \in \mathcal{A} \,\}.
$$

13.2. **Proposition.** *For any class $\mathcal{B}$, the left complement ${}^{\boxslash}\mathcal{B}$ is a saturated class.*

13.3. *Exercise* (Important). Prove (13.2).

The above proposition (13.2) has a dual statement: any right complement $\mathcal{B}^{\boxslash}$ is **cosaturated**, i.e., satisfies dual versions of the closure properties of a saturated class, or equivalently, corresponds to a saturated class in the opposite category.

13.4. *Exercise* (Easy). Prove that if $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A}^{\boxslash} \supseteq \mathcal{B}^{\boxslash}$ and ${}^{\boxslash}\mathcal{A} \supseteq {}^{\boxslash}\mathcal{B}$. Use this to show $\mathcal{A}^{\boxslash} = ({}^{\boxslash}(\mathcal{A}^{\boxslash}))^{\boxslash}$ and ${}^{\boxslash}\mathcal{A} = {}^{\boxslash}(({}^{\boxslash}\mathcal{A})^{\boxslash})$.

13.5. *Exercise* (for those who know a little homological algebra). Fix an abelian category $\mathcal{C}$ (e.g., the category of modules over some ring R). Let $\mathcal{A}$ be the class of morphisms in $\mathcal{C}$ of the form $0 \to P$ where P is projective, and let $\mathcal{B}$ be the class of epimorphisms in $\mathcal{C}$. Show that $\mathcal{A} \boxslash \mathcal{B}$; also, show that $\mathcal{B} = \mathcal{A}^{\boxslash}$ if $\mathcal{C}$ has enough projectives.

13.6. *Exercise.* In the previous exercise, identify the class ${}^{\boxslash}\mathcal{B}$.

13.7. **Inner fibrations.** A map p of simplicial sets is an **inner fibration**[12] if $\mathrm{InnHorn} \boxslash p$. The class of inner fibrations $\mathrm{InnFib} = \mathrm{InnHorn}^{\boxslash}$ is thus the right complement of the inner horns. Note that C is a quasicategory if and only if $C \to *$ is an inner fibration.

Because InnFib is a right complement, it is cosaturated. In particular, it is closed under composition. This implies that if $p\colon C \to D$ is an inner fibration *and* D is a quasicategory, then C is also a quasicategory.

---

[11]Sometimes one sees the notation "$f \perp g$" or "$f \pitchfork g$" used instead. Our notation is taken from [Rie14, §11].

[12]Joyal calls these "mid-fibrations" [Joy02, Def. 2.1].

13.8. *Exercise.* Show that if $f: C \to D$ is any functor from a quasicategory C to a *category* D, then f is an inner fibration. In particular, all functors between categories are automatically inner fibrations. (Hint: use the fact that all inner horns mapping to a category have *unique* extensions to simplices.)

13.9. **Factorizations.** It turns out that we can always factor any map of simplicial sets into an inner anodyne map followed by an inner fibration. This is a consequence of the following general observation.

13.10. **Proposition** ("Small object argument")**.** *Let S be a* set *of morphisms in sSet. Every map f between simplicial sets admits a factorization $f = pj$ with $j \in \overline{S}$ and $p \in S^{\boxempty}$.*

The proof of this proposition is by means of what is known as the "small object argument". I'll give the proof in the next section. For now we record a consequence.

13.11. **Corollary.** *For any set S of morphisms in sSet, we have that $\overline{S} = {}^{\boxempty}(S^{\boxempty})$.*

Proof. That $\overline{S} \subseteq {}^{\boxempty}(S^{\boxempty})$ is immediate. Given f such that $f \boxempty S^{\boxempty}$, use the small object argument (13.10) to choose $f = pj$ with $j \in \overline{S}$ and $p \in S^{\boxempty}$. We have a commutative diagram of solid arrows

A map s exists making the diagram commute, because $f \boxempty p$. This exhibits f as a retract of j, whence $f \in \overline{S}$ since saturations are closed under retracts. $\square$

The proof of the corollary is called the "retract trick": given $f = pj$, $f \boxempty p$ implies that f is a retract of j, while $j \boxempty f$ implies that f is a retract of p.

In the case we are currently interested in, we have that $\overline{\mathrm{InnHorn}} = {}^{\boxempty}\mathrm{InnFib}$ and $\overline{\mathrm{InnHorn}}^{\boxempty} = \mathrm{InnFib}$, and thus any map can be factored into an inner anodyne map followed by an inner fibration.

13.12. **Weak factorization systems.** A **weak factorization system** in a category is a pair $(\mathcal{L}, \mathcal{R})$ of classes of maps such that

- every map f admits a factorization $f = r\ell$ with $r \in \mathcal{R}$ and $\ell \in \mathcal{L}$, and
- $\mathcal{L} = {}^{\boxempty}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxempty}$.

Thus, the small object argument implies that $(\overline{S}, S^{\boxempty})$ is a weak factorization in sSet for every set of maps S.

13.13. *Exercise* (for those who know some homological algebra)*.* Consider the classes $\mathcal{A} = \{0 \to \text{projective}\}$ and $\mathcal{B} = \{\text{epimorphism}\}$ of maps in an abelian category described in (13.5). Show directly that the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorization system if and only if the category has enough projectives.

13.14. **Uniqueness of liftings.** The relation $f \boxempty g$ says that lifting problems admit solutions, but not that the solutions are unique. However, we can incorporate uniqueness into the lifting calculus if our category has pushouts.

Given a map $f: A \to B$, let $f^{\vee} := (f, f): B \amalg_A B \to B$ be the evident fold map. It is straightforward to show that for a map $g: X \to Y$ we have that $\{f, f^{\vee}\} \boxempty g$ if and only if in every

commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ f\downarrow & \nearrow s & \downarrow g \\ B & \longrightarrow & Y \end{array}$$

there exists a *unique* lift s.

13.15. *Example.* Consider the category of topological spaces. Let $\mathcal{A}$ be the class of morphisms of the form $A \times \{0\} \to A \times [0,1]$, where A is an arbitrary space. Then $(\mathcal{A} \cup \mathcal{A}^\vee)^{\boxtimes}$ contains all covering maps (by the "Covering Homotopy Theorem").

A weak factorization system $(\mathcal{L}, \mathcal{R})$ in which liftings of type $\mathcal{L} \boxtimes \mathcal{R}$ are always unique is simply called a **factorization system**.

13.16. *Exercise.* Show that in a factorization system, the factorizations $f = r\ell$ are unique up to *unique isomorphism*.

13.17. *Example.* In simplicial sets, the projection map $C \to *$ is in the right complement to $S := \text{InnHorn} \cup \text{InnHorn}^\vee$ if and only if C is isomorphic to a nerve of a category (5.7). The small object argument using S, applied to a projection $X \to *$, produces a morphism $\pi\colon X \to Y$ in $\overline{S}$ with Y the nerve of a category. The uniqueness of liftings in this case implies that $\pi\colon X \to Y$ has precisely the universal property of the fundamental category of X defined in §9.1: given $f\colon X \to C$ with C a category, a unique extension of f over $X \to Y$ exists. Thus, the small object argument gives another construction of the fundamental category of an arbitrary simplicial set.

13.18. *Exercise.* Prove that if $f\colon X \to Y$ is any inner anodyne map, then the induced functor $h(f)\colon hX \to hY$ between fundamental categories is an isomorphism. (Hint: use the universal property of fundamental categories to construct an inverse to $h(f)$.)

14. The small object argument

In this section we give the proof of (13.10). For the reader: it may be helpful to first work through the special case where $Y = \Delta^0$ (the terminal object in simplicial sets).

14.1. **A factorization construction.** Fix a set of maps $S = \{s_i\colon A_i \to B_i\}$ between simplicial sets. For each $f\colon X \to Y$, we will produce a factorization

$$X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y, \qquad (Rf)(Lf) = f$$

as follows. Consider the set
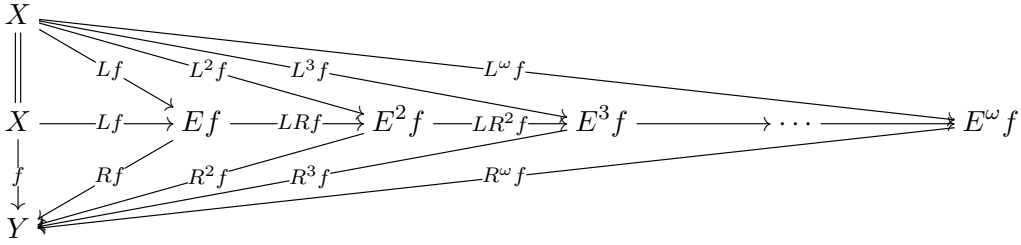
$$[S,f] := \{\, (s_i, u, v) \mid s_i \in S,\ fu = vs_i \,\} = \left\{ \begin{array}{ccc} A_i & \xrightarrow{u} & X \\ s_i\downarrow & & \downarrow f \\ B_i & \xrightarrow{v} & Y \end{array} \right\}$$

of all commutative squares which have an arrow from S on the left-hand side, and f on the right-hand side. We define Ef, Lf, and Rf using the diagram

where the the coproducts are indexed by the set $[S, f]$, and the square is a pushout. Note that $Lf \in \overline{S}$ by construction, though we do not expect to have Rf in $S^{\boxtimes}$.

We can iterate this:

here each triple $(E^\alpha f, L^\alpha f, R^\alpha f)$ is obtained by factoring the "R" map of the previous one, so that

$$(14.2) \qquad E^{\alpha+1}f := E(R^\alpha f), \quad L^{\alpha+1}f := L(R^\alpha f) \circ (L^\alpha f), \quad R^{\alpha+1}f := R(R^\alpha f).$$

Taking direct limits gives a factorization $X \xrightarrow{L^\omega f} E^\omega f \xrightarrow{R^\omega f} Y$ of f, with $E^\omega f = \mathrm{colim}_{n \to \infty} E^n f$.

We can go even further, using the magic of transfinite induction, and define compatible factorizations $(E^\lambda f, L^\lambda f, R^\lambda f)$ for each ordinal[13] λ. For successor ordinals $\alpha + 1$ use the prescription of (14.2), while for limit ordinals β take a direct limit $E^\beta f := \mathrm{colim}_{\alpha < \beta} E^\alpha f$ as in the construction of $E^\omega f$ above.

It is immediate that every $L^\alpha f \in \overline{S}$, because saturations are closed under transfinite composition. The maps $R^\alpha f$ are not generally contained in $S^{\boxtimes}$, though they do satisfy a "partial lifting property": whenever $\alpha < \beta$ there exists *by construction* a dashed arrow making

commute, for any u' and v making the square commute. Thus, we get a solution to a lifting problem of s_i against $R^\beta f$ whenever the map $A_i \to E^\beta f$ on the top of a commutative square that we want a lift for can be factored through some $E^\alpha f \to E^\beta f$ with $\alpha < \beta$. The point is that $E^{\alpha+1}f$ was obtained from $E^\alpha f$ by "formally adjoining" a solution to *every* such lifting problem.

The "small object argument" amounts to the following.

Claim. There exists an ordinal κ such that for every domain A_i of a map in S, every map $A_i \to E^\kappa f$ factors through some $E^\alpha f \to E^\kappa f$ with $\alpha < \kappa$.

Given this, it follows by the "partial lifting property" that $S \boxtimes R^\kappa f$, and so we obtain the desired factorization.

It remains to prove the claim, which we will do by choosing κ to be a *regular cardinal* which is "bigger" than all the simplicial sets A_i.

14.3. **Regular cardinals.** The **cardinality** of a set X is the smallest ordinal λ such that there exists a bijection between X and λ; we write $|X|$ for this. Ordinals which can appear this way are called **cardinals**. For instance, the first infinite ordinal ω is the countable cardinal.

Note: the class of infinite cardinals is an unbounded subclass of the ordinals, so is well-ordered and can be put into bijective correspondence with ordinals. The symbol $\aleph_\alpha$ denotes the αth infinite cardinal, e.g., $\aleph_0 = \omega$.

Say that λ is a **regular cardinal**[14] if it is an infinite cardinal, and if for every set A of ordinals such that (i) $\alpha < \lambda$ for all $\alpha \in A$, and (ii) $|A| < \lambda$, then $\sup A < \lambda$. For instance, ω is a regular

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[13]For a treatment of ordinals, see for instance the chapter on sets in [TS14].

[14]In the terminology of [TS14, §3.7], a regular cardinal is one which is equal to its own cofinality.

cardinal, since any finite collection of finite ordinals has a finite upper bound. Not every infinite cardinal is regular[15]; however, there exist arbitrarily large regular cardinals[16].

Every ordinal α defines a category, which is the poset of ordinals strictly less than α. Colimits of functors $Y \colon \kappa \to \mathrm{Set}$ with κ a regular cardinal have the following property: the map

$$(14.4) \qquad \mathrm{colim}_{\alpha < \kappa} \mathrm{Hom}(X, Y_\alpha) \to \mathrm{Hom}(X, \mathrm{colim}_{\alpha < \kappa} Y_\alpha)$$

is a bijection whenever $|X| < \kappa$. This generalizes the familiar case of $\kappa = \omega$: any map of a finite set into the colimit of a countable sequence factors through a finite stage.

14.5. *Exercise.* Prove that (14.4) is a bijection when $|X| < \kappa$.

14.6. **Small simplicial sets.** Given a regular cardinal κ, we say that a simplicial set is κ-**small** if it is isomorphic to the colimit of some functor $F \colon C \to s\mathrm{Set}$, such that (i) $|\mathrm{ob}\, C|, |\mathrm{mor}\, C| < \kappa$, and (ii) each $F(c)$ is isomorphic to a standard simplex Δ^n. Morally, we are saying that a simplicial set is κ-small if it can be "presented" with fewer than κ generators and fewer than κ relations.

Given a functor $Y \colon \kappa \to s\mathrm{Set}$ and a κ-small simplicial set X, we have a bijection as in (14.4). (This is sometimes phrased as: *κ-small simplicial sets are κ-compact.*) Thus, to prove the claim about the small object argument, we simply choose a regular cardinal κ greater than $\sup\{|A_i|\}$.

14.7. *Example.* The standard simplices Δ^n, as well as any subcomplex such as the horns Λ^n_j, are ω-small: this is a consequence of (4.17). Thus, when we carry out the small object argument for $S = \mathrm{InnHorn}$, we can take $(E^\omega f, L^\omega f, R^\omega f)$ to be the desired factorization.

14.8. **Functoriality.** The construction $f \mapsto (X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y)$ is a functor $\mathrm{Fun}([1], s\mathrm{Set}) \to \mathrm{Fun}([2], s\mathrm{Set})$, and it follows that so is $f \mapsto (X \xrightarrow{L^\alpha f} E^\alpha f \xrightarrow{R^\alpha f} Y)$ for any α. Because the choice of regular cardinal κ depends only on S, not on the map f, we see that the small object argument actually produces a *functorial factorization* of a map into a composite of an element of $\overline{S}$ with an element $S^{\boxempty}$. We will have use of this later.

15. Non-degenerate simplices and the skeletal filtration

We have noted that monomorphisms of simplicial sets form a saturated class. Here we identify an important set of maps called Cell, so that the saturation of Cell is precisely the class of monomorphisms.

15.1. **Boundary of a standard simplex.** For each $n \geq 0$, we define

$$\partial \Delta^n := \bigcup_{k \in [n]} \Delta^{[n] \smallsetminus \{k\}} \subset \Delta^n,$$

the union of all codimension-one faces of the n-simplex. Equivalently,

$$(\partial \Delta^n)_k = \{\, f \colon [k] \to [n] \mid f([k]) \neq [n] \,\}.$$

Note that $\partial \Delta^0 = \varnothing$ and $\partial \Delta^1 = \Delta^{\{0\}} \amalg \Delta^{\{1\}}$.

15.2. *Exercise.* Show that $\partial \Delta^n$ is the largest subcomplex of Δ^n which does not contain the "generator" $\langle 0 \ldots n \rangle \in (\Delta^n)_n$. In other words, $\partial \Delta^n$ is the maximal *proper* subcomplex of Δ^n.

15.3. *Exercise.* Show that if C is a category, then the evident maps $\mathrm{Hom}(\Delta^n, C) \to \mathrm{Hom}(\partial \Delta^n, C)$ defined by restriction are isomorphisms when $n \geq 3$, but not necessarily when $n \leq 2$.

---

[15]For instance, $\aleph_\omega = \sup\{\, \aleph_k \mid k < \omega \,\}$ is not regular.

[16]For instance, every successor cardinal $\aleph_{\alpha+1}$ is regular.

15.4. **Trivial fibrations and monomorphisms.** Let Cell be the set consisting of the inclusions $\partial\Delta^n \subset \Delta^n$ for $n \geq 0$. The resulting right complement is TFib := Cell$^\boxtimes$, the class of **trivial fibrations** (also sometimes called **acyclic fibrations**).

Since the elments of Cell are monomorphisms, and the class of all monomorphisms is saturated, we see that all elements of $\overline{\text{Cell}}$ are monomorphisms. We are going to prove the converse, i.e., we will show that $\overline{\text{Cell}}$ is precisely equal to the class of monomorphisms.

15.5. **Degenerate and non-degenerate simplices.** Recall $\Delta^{\text{surj}}, \Delta^{\text{inj}} \subset \Delta$, the subcategories consisting of all the objects and the *surjective* and *injective* order-preserving maps respectively.

A simplex $a \in X_n$ is said to be **degenerate** if there exists a *non-identity operator* $\sigma \in \Delta^{\text{surj}}$ and simplex b in X such that $a = b\sigma$. Equivalently, a is degenerate if there exists a *non-injective* $f \in \Delta$ and b in X such that $a = bf$; this is because any $f \in \Delta$ factors as $f = f^{\text{inj}}f^{\text{surj}}$, a surjection followed by an injection, and if f is non-injective then f^{surj} is necessarily non-identity.

Likewise, a simplex $a \in X_n$ is **non-degenerate** if whenever $a = b\sigma$ for some $\sigma \in \Delta^{\text{surj}}$ and simplex b in X, we must have that σ is an identity map. Equivalently, $a \in X_n$ is non-degenerate if $a = bf$ for some $f \in \Delta$ and b in X implies that $f \in \Delta^{\text{inj}}$.

We write $X_n^{\text{deg}}, X_n^{\text{nd}} \subset X_n$ for the sets of degenerate and non-degenerate n-simplices in a simplicial set. Note that if $f\colon A \to X$ is a map of simplcial sets, then $f(A_n^{\text{deg}}) \subseteq X_n^{\text{deg}}$, while $f^{-1}(X_n^{\text{nd}}) \subseteq A_n^{\text{nd}}$. (Neither X_n^{deg} nor X_n^{nd} define a subcomplex.)

15.6. *Exercise* (easy). For any simplicial set X, we have $X_0^{\text{nd}} = X_0$, while $X_1^{\text{nd}} \subseteq X_1$ is the complement of the image of $\langle 01 \rangle^*\colon X_0 \to X_1$.

15.7. *Exercise.* If C is an ordinary category, then a simplex $a \in N(C)_k$ of the nerve is non-degenerate if and only if it is represented by a sequence of *non-identity* maps $c_0 \to \cdots \to c_k$ in the category C.

15.8. **Proposition.** *If X is a simplicial set and $A \subseteq X$ is a subcomplex, then $A_n^{\text{nd}} = X_n^{\text{nd}} \cap A_n$.*

Proof. The inclusion $X_n^{\text{nd}} \cap A_n \subseteq A_n^{\text{nd}}$ is obvious. Conversely, let $a \in A_n^{\text{nd}}$, and suppose there exists $b \in X_k$ and $\sigma\colon [n] \to [k] \in \Delta^{\text{surj}}$ such that $b\sigma = a$. Observe that any surjection in Δ has a section (i.e., a preinverse), so there exists $\delta\colon [k] \to [n]$ such that $\sigma\delta = 1_{[k]}$. Thus $b = b\sigma\delta = a\delta \in A_k$, so b is a simplex of A. Therefore $a \in A_n^{\text{nd}}$ implies that σ is an identity map. Thus, we have proved that if a is nondegenerate in A, it must also be nondegenerate in X. $\square$

The key observation is that degenerate simplicies in a simplicial set are precisely determined by knowledge of the non-degenerate simplices.

15.9. **Proposition** (Eilenberg-Zilber lemma). *Let a be a simplex in X. Then there exists a* unique *pair (b, σ) consisting of a* non-degenerate *simplex b and a map σ in Δ^{surj} such that $a = b\sigma$.*

Proof. [GZ67, §II.3]. Given $\sigma\colon [n] \to [m]$, let $\Gamma(\sigma) = \{\, \delta\colon [m] \to [n] \mid \sigma\delta = \text{id}_{[m]} \,\}$ denote the set of sections of σ. We note the following elementary observations, whose proof is left for the reader:

- if $\sigma \in \Delta^{\text{surj}}$ then $\Gamma(\sigma)$ is non-empty, and
- if $\sigma, \sigma' \in \Delta^{\text{surj}}$ are such that $\Gamma(\sigma) = \Gamma(\sigma')$, then $\sigma = \sigma'$.

Let $a \in X_n$ be such that $a = b_i\sigma_i$ for $b_i \in X_{m_i}^{\text{nd}}$, $\sigma_i \in \Delta^{\text{surj}}([n], [m_i])$, for $i = 1, 2$. We want to show that $m_1 = m_2$, $b_1 = b_2$, and $\sigma_1 = \sigma_2$.

Pick any $\delta_1 \in \Gamma(\sigma_1)$ and $\delta_2 \in \Gamma(\sigma_2)$. Then we have

$$b_1 = b_1\sigma_1\delta_1 = a\delta_1 = b_2\sigma_2\delta_1, \qquad b_2 = b_2\sigma_2\delta_2 = a\delta_2 = b_1\sigma_1\delta_2.$$

Since b_1 and b_2 are non-degenerate, $\sigma_2\delta_1\colon [m_1] \to [m_2]$ and $\sigma_1\delta_2\colon [m_2] \to [m_1]$ must be injective. This implies $m_1 = m_2$, and since the only order-preserving injective map $[m] \to [m]$ is the identity map, we must have $\sigma_2\delta_1 = \text{id} = \sigma_1\delta_2$, from which it follows that $b_1 = b_2$. This also shows that $\delta_1 \in \Gamma(\sigma_2)$ and $\delta_2 \in \Gamma(\sigma_1)$. Since δ_1 and δ_2 were arbitrarily chosen sections, we have shown $\Gamma(\sigma_1) = \Gamma(\sigma_2)$, and therefore $\sigma_1 = \sigma_2$. $\square$

Here is an illustration of what can happen. Suppose $a \in X_2$ is a degeneracy of two simplices x and y.

- If $x, y \in X_0$, and $a = x_{000} = y_{000}$, then $x = a_0 = y$.
- If $x \in X_0$, $y \in X_1$, and $a = x_{000} = y_{001}$, then $y = a_{02} = x_{00}$, whence y is degenerate.
- If $x, y \in X_1$ and $a = x_{001} = y_{001}$, then $x = a_{01} = y$.
- If $x, y \in X_1$ and $a = x_{001} = y_{011}$, then $x = a_{12} = y_{11}$ and $y = a_{01} = x_{00}$, whence x and y are degenerate.

15.10. Corollary. *For any simplicial set X, the evident maps*

$$\coprod_{k \geq 0} X_k^{\mathrm{nd}} \times \mathrm{Hom}_{\Delta^{\mathrm{surj}}}([n], [k]) \to X_n$$

defined by $(k, x, \sigma) \mapsto x\sigma$ are bijections. Furthermore, these bijections are natural with respect to surjective simplicial operators $[n] \to [n']$.

Another way to say this: the restricted functor $X|(\Delta^{\mathrm{surj}})^{\mathrm{op}} \colon (\Delta^{\mathrm{surj}})^{\mathrm{op}} \to \mathrm{Set}$ is *canonically isomorphic* to a coproduct of representable functors $\mathrm{Hom}_{\Delta^{\mathrm{surj}}}(-, [k])$ indexed by the nondegenerate simplicies of X. Or more simply: simplicial sets are *canonically free* with respect to degeneracy operators.

15.11. Example. Let $X = \Delta^k/\partial\Delta^k := \mathrm{colim}(\Delta^k \leftarrow \partial\Delta^k \to *)$. The simplicial set X has exactly two non-degenerate simplices: $* \in X_0$, and $\bar\iota \in X_k$ (the image of the generator $\iota \in (\Delta^k)_k$). The set X_n has exactly $1 + \binom{n}{k}$ elements: $1 = \left|\Delta^{\mathrm{surj}}([n], [0])\right|$ simplex associated to $*$, and $\binom{n}{k} = \left|\Delta^{\mathrm{surj}}([n], [k])\right|$ simplices associated to $\bar\iota$.

15.12. Remark. A simplicial set can be recovered up to isomorphism if you know (i) its sets of non-degenerate simplices, and (ii) the faces of the non-degenerate simplices. The proposition we proved above tells how to reconstruct the degenerate simplices; simplicial operators on degenerate simplices are computed using the fact that any simplicial operator factors into a surjection followed by an injection.

Warning. As the previous example shows, the faces of a non-degenerate simplex can be degenerate. In the case that all faces of non-degenerate simplices in X are also non-degenerate, then we do get a functor $X^{\mathrm{nd}} \colon (\Delta^{\mathrm{inj}})^{\mathrm{op}} \to \mathrm{Set}$, and the full simplicial set X can be recovered from X^{nd}. For instance, this applies to the standard simplices Δ^n, as well as any subcomplexes of such. Functors $(\Delta^{\mathrm{inj}})^{\mathrm{op}} \to \mathrm{Set}$ are the combinatorial data behind the notion of a Δ-*complex*, as seen in Hatcher's textbook on algebraic topology [Hat02, Ch. 2.1].

The following exercises give a different point of view of this principle.

15.13. Exercise. Fix an object $[n]$ in Δ, and consider the category $\Delta^{\mathrm{surj}}_{[n]/}$, which has

- **objects** the surjective morphisms $\sigma \colon [n] \to [k]$ in Δ, and
- **morphisms** commutative triangles in Δ of the form

$$[n] \begin{array}{c} \xrightarrow{\ \sigma\ } [k] \\ \searrow_{\sigma'} \ \downarrow_{\tau} \\ [k'] \end{array}$$

Show that the category $\Delta^{\mathrm{surj}}_{[n]/}$ is *isomorphic* to the poset $\mathcal{P}(\underline{n})$ of subsets of the set $\underline{n} = \{1, \ldots, n\}$. In particular, $\Delta^{\mathrm{surj}}_{[n]/}$ is a lattice (i.e., has finite products and coproducts, called *meets* and *joins* in this context).

15.14. *Exercise.* Let X be a simplicial set. Given $n \geq 0$ and $\sigma\colon [n] \to [k]$ in Δ^{surj}, let $X_n^\sigma := \sigma^*(X_k)$, the image of the operator σ^* in X_n. Show that $X_n^{\sigma \vee \sigma'} = X_n^\sigma \cap X_n^{\sigma'}$, where $\sigma \vee \sigma'$ is join in the lattice $\Delta_{[n]/}^{\mathrm{surj}}$. Conclude that for each $x \in X_n$ there exists a maximal σ such that $x \in X_n^\sigma$.

15.15. Skeleta. Given a simplicial set X, the k-**skeleton** $\mathrm{Sk}_k X \subseteq X$ is the smallest subcomplex containing all simplicies of dimensions $\leq k$. For instance, $\mathrm{Sk}_{k-1} \Delta^k = \partial \Delta^k$. Using (15.10), we see that

$$(\mathrm{Sk}_k X)_n = \bigcup_{0 \leq j \leq k} \{\, yf \mid y \in X_j,\ f\colon [n] \to [j] \in \Delta \,\} \approx \coprod_{0 \leq j \leq k} X_j^{\mathrm{nd}} \times \mathrm{Hom}_{\Delta^{\mathrm{surj}}}([n],[j]).$$

Note that $X = \mathrm{colim}_{k \to \infty} \mathrm{Sk}_k X$. The complement of $\mathrm{Sk}_{k-1} X$ in $\mathrm{Sk}_k X$ consists precisely of the *nondegenerate* k-simplcies of X (in simplicial degree k) together with their degeneracies (in simplicial degrees $> k$).

15.16. Proposition. *The evident square*

$$\begin{array}{ccc}
\coprod_{a \in X_k^{\mathrm{nd}}} \partial \Delta^k & \longrightarrow & \mathrm{Sk}_{k-1} X \\
\downarrow & & \downarrow \\
\coprod_{a \in X_k^{\mathrm{nd}}} \Delta^n & \longrightarrow & \mathrm{Sk}_k X
\end{array}$$

is a pushout of simplicial sets. More generally, for any subcomplex $A \subseteq X$, the evident square

$$\begin{array}{ccc}
\coprod_{a \in X_k^{\mathrm{nd}} \smallsetminus A_k^{\mathrm{nd}}} \partial \Delta^k & \longrightarrow & A \cup \mathrm{Sk}_{k-1} X \\
\downarrow & & \downarrow \\
\coprod_{a \in X_k^{\mathrm{nd}} \smallsetminus A_k^{\mathrm{nd}}} \Delta^k & \longrightarrow & A \cup \mathrm{Sk}_k X
\end{array}$$

is a pushout.

Proof. In each of the above squares, the complements of the vertical inclusions coincide precisely. $\square$

15.17. Corollary. $\overline{\mathrm{Cell}}$ *is precisely the class of monomorphisms.*

Proof. We know all elements of $\overline{\mathrm{Cell}}$ are monomorphisms. Any monomorphism is isomorphic to an inclusion $A \subseteq X$ of a subcomplex, so we only need show that such inclusions are contained in $\overline{\mathrm{Cell}}$. Since $X \approx \mathrm{colim}_k A \cup \mathrm{Sk}_k X$, (15.16) exhibits the inclusion as a countable composite of pushouts along coproducts of elements of Cell. $\square$

15.18. Geometric realization. Recall the the singular complex functor $\mathrm{Sing}\colon \mathrm{Top} \to s\mathrm{Set}$ (8.7). This functor has a left adjoint $|{-}|\colon s\mathrm{Set} \to \mathrm{Top}$, called **geometric realization**, constructed explicitly by

$$(15.19) \qquad |X| := \mathrm{Cok}\left[\coprod_{f\colon [m] \to [n]} X_n \times \Delta_{\mathrm{top}}^m \rightrightarrows \coprod_{[p]} X_p \times \Delta_{\mathrm{top}}^p \right];$$

that is, take a collection of topological simplices indexed by elements of X, and make identifactions according to the simplicial operators in X. (Here the symbol "Cok" represesents taking a "coequalizer", i.e., the colimit of a diagram of shape $\bullet \rightrightarrows \bullet$.)

15.20. *Exercise.* Describe the two unlabelled maps in (15.19). Then show that $|{-}|$ is in fact left adjoint to Sing.

Because geometric realization is a left adjoint, it commutes with colimits. It is straightforward to check that $|\Delta^n| \approx \Delta^n_{\text{top}}$, and that $|\partial\Delta^n| \approx \partial\Delta^n_{\text{top}}$. Applying this to the skeletal filtration, we discover that there are pushouts

$$
\begin{array}{ccc}
\coprod_{a \in X_k^{\text{nd}}} \partial\Delta^k_{\text{top}} & \longrightarrow & |\text{Sk}_{k-1}\, X| \\
\downarrow & & \downarrow \\
\coprod_{a \in X_k^{\text{nd}}} \Delta^k_{\text{top}} & \longrightarrow & |\text{Sk}_k\, X|
\end{array}
$$

of spaces, and that $|X| = \bigcup |\text{Sk}_k\, X|$ with the direct limit topology. Thus, $|X|$ is presented to us as a *CW-complex*, whose cells are in an evident bijective correspondence with the set of non-degenerate simplices of X.

16. Pushout-product and pullback-power

We are going to prove several "enriched" versions of lifting properties associated to inner anodyne maps and inner fibrations. As a consequence we'll be able to prove that function complexes of quasicategories are themselves quasicategories.

16.1. **Definition of pushout-product and pullback-power.** Given maps $f\colon A \to B$, $g\colon K \to L$ and $h\colon X \to Y$ of simplicial sets, we define the **pushout-product**[17]

$$
f\square g\colon (A \times L) \amalg_{A \times K} (B \times K) \xrightarrow{(f \times L, B \times g)} B \times L
$$

and **pullback-power**[18]

$$
h^{\square g}\colon \text{Map}(L, X) \xrightarrow{(\text{Map}(g,X), \text{Map}(L,h))} \text{Map}(K, X) \times_{\text{Map}(K,Y)} \text{Map}(L, Y).
$$

16.2. *Remark.* Typically we form $f\square g$ when f and g are monomorphisms, in which case $f\square g$ is also a monomorphism. In this case, the simplices $(b, \ell) \in B_n \times L_n$ which are *not* in the image of $f\square g$ are exactly those such that $b \in B_n \smallsetminus A_n$ *and* $\ell \in L_n \smallsetminus K_n$.

16.3. *Remark* (Important!). On 0-simplices, the pullback-power $h^{\square g}$ is just the "usual" map $\text{Hom}(L, X) \to \text{Hom}(K, X) \times_{\text{Hom}(K,Y)} \text{Hom}(L, Y)$ sending $s \mapsto (sg, hs)$. Thus, $h^{\square g}$ is surjective on 0-simplcies if and only if $g \boxslash h$.

We think of the pullback-power as encoding an "enriched" version of the lifting problem for (g, h). Thus, the target of $h^{\square g}$ is an object which "parameterizes" commutative squares involving g and h. Similarly, the source of $h^{\square g}$ "parameterizes" commutative squares together with lifts.

The product/mapping object adjunction gives rise to the following relationship between lifting problems.

16.4. **Proposition.** *We have that* $(f\square g) \boxslash h$ *if and only if* $f \boxslash (h^{\square g})$.

Proof. Compare the two lifting problems using the product/map adjunction.

$$
\begin{array}{ccc}
(A \times L) \amalg_{A \times K} (B \times K) & \xrightarrow{(u,v)} & X \\
{\scriptstyle f\square g}\downarrow \quad {\scriptstyle s}\nearrow & & \downarrow {\scriptstyle h} \\
B \times L & \xrightarrow{\quad w \quad} & Y
\end{array}
\qquad \Longleftrightarrow \qquad
\begin{array}{ccc}
A & \xrightarrow{\quad \widetilde{u} \quad} & \text{Map}(L, X) \\
{\scriptstyle f}\downarrow \quad {\scriptstyle \widetilde{s}}\nearrow & & \downarrow {\scriptstyle h^{\square g}} \\
B & \xrightarrow[(\widetilde{v},\widetilde{w})]{} & \text{Map}(K, X) \times_{\text{Map}(K,Y)} \text{Map}(L, Y)
\end{array}
$$

---

[17]This is sometimes called the **box-product**. Some also call it the **Leibniz** product, as its form is that of the Leibniz rule for boundary of a product space: $\partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$ (which is itself reminiscent of the original Leibniz rule $D(fg) = (Df)g + f(Dg)$ of calculus).

[18]Sometimes called the **box-power** or **pullback-hom**. A common alternate notation is $g \pitchfork h$. This may also be called the **Leibniz-hom**, though I don't know what rule of calculus it is related to.

The data of (u, v, w) giving a commutative square as on the left corresponds bijectively to data $(\widetilde{u}, \widetilde{v}, \widetilde{w})$ giving a commutative square as on the right. Similarly, lifts s correspond bijectively to lifts $\widetilde{s}$. $\qquad\square$

It is important to note the special cases where one or more of $A = \varnothing$, $K = \varnothing$, or $Y = *$ hold. For instance, if $K = \varnothing$ and $Y = *$, the proposition implies

$$(A \times L \xrightarrow{f \times L} B \times L) \boxtimes (X \to *) \quad \text{iff} \quad (A \xrightarrow{f} B) \boxtimes (\mathrm{Map}(L, X) \to *).$$

This is the kind of case we are interested in for proving that $\mathrm{Map}(K, C)$ is a quasicategory whenever C is. The more general statement of the proposition is a kind of "relative" version of the thing we want; it is especially handy for carrying out inductive arguments.

16.5. *Exercise* (if you like monoidal categories). Let $\mathcal{C} := \mathrm{Fun}([1], s\mathrm{Set})$, the "arrow category" of simplicial sets. Show that $\square \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ defines a symmetric monoidal structure on $\mathcal{C}$, with unit object $(\varnothing \subset \Delta^0)$. Furthermore, show that this is a *closed* monoidal structure, with $-\square g$ left adjoint to $(-)^{\square g} \colon \mathcal{C} \to \mathcal{C}$.

16.6. **Inner anodyne maps and pushout-products.** The key fact we want to prove is the following.

16.7. **Proposition.** *We have that* $\overline{\mathrm{InnHorn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$.

Proof. This will follow from (16.8) and (16.9) below. $\qquad\square$

To set up the proof we need the following.

16.8. **Proposition.** *For any sets of maps S and T, we have* $\overline{S} \square \overline{T} \subseteq \overline{S \square T}$.

Proof. Let $\mathcal{F} = (S \square T)^{\boxtimes}$. By the small object argument (13.11), $\overline{S \square T} = {}^{\boxtimes}\mathcal{F}$. First we show that $\overline{S} \square T \subseteq \overline{S \square T}$. Consider

$$\mathcal{A} := \{\, a \mid (a \square T) \boxtimes \mathcal{F} \,\}$$
$$\approx \{\, a \mid a \boxtimes (\mathcal{F}^{\square T}) \,\}$$

by the previous proposition. Thus $\mathcal{A}$ is a left complement, and so is saturated. Since $S \subseteq \mathcal{A}$ then $\overline{S} \subseteq \mathcal{A}$, i.e., $\overline{S} \square T \subseteq {}^{\boxtimes}F = \overline{S \square T}$. Now consider

$$\mathcal{B} := \{\, b \mid (\overline{S} \square b) \boxtimes \mathcal{F} \,\} \approx \{\, b \mid b \boxtimes (\mathcal{F}^{\square \overline{S}}) \,\},$$

which is likewise saturated. We have just shown that $T \subseteq \mathcal{B}$, whence $\overline{T} \subseteq \mathcal{B}$, i.e., $\overline{S} \square \overline{T} \subseteq \overline{S \square T}$. $\qquad\square$

16.9. **Lemma.** *We have* $\mathrm{InnHorn} \square \mathrm{Cell} \subseteq \overline{\mathrm{InnHorn}}$.

Proof. This is a calculation, given in [Joy08a, App. H] (and ideally the **appendix**). $\qquad\square$

Let's carry out the proof of (16.9) explicitly in one case, by showing that $(\Lambda_1^2 \subset \Delta^2) \square (\partial \Delta^1 \subset \Delta^1)$ is inner anodyne. This map is the inclusion

$$(\Lambda_1^2 \times \Delta^1) \cup_{\Lambda_1^2 \times \partial \Delta^1} (\Delta^2 \times \partial \Delta^1) \subset \Delta^2 \times \Delta^1,$$

whose target is a "prism", and whose source is a "trough". To show this is in $\overline{\mathrm{InnHorn}}$, we'll give an explicit procedure for constructing the prism from the trough by succesively attaching simplices along inner horns.

Note that $\Delta^2 \times \Delta^1 = N([2] \times [1])$, so we are working inside the nerve of a poset, whose elements (objects) are "ij" with $i \in \{0, 1, 2\}$ and $j \in \{0, 1\}$. Here is a picture of the trough, showing all the

non-degenerate simplicies.
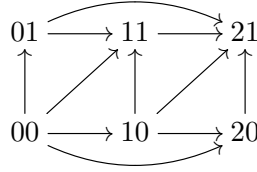
The complement of this in the prism consists of three non-degenerate 3-simplices, five non-degenerate 2-simplices (two of which form the "lid" of the trough, while the other three are in the interior of the prism), and one non-degenerate 1-simplex (separating the two 2-simplices which form the lid).

The following chart lists all non-degenerate simplices in the complement of the trough, along with their codimension one faces (in order). The "$\sqrt{}$" marks simplices which are contained in the trough.

$\langle 00,21\rangle$	$\langle 00,20,21\rangle$	$\langle 00,01,21\rangle$	$\langle 00,10,21\rangle$	$\langle 00,11,21\rangle$	$\langle 00,10,20,21\rangle$	$\langle 00,10,11,21\rangle$	$\langle 00,01,11,21\rangle$
$\sqrt{}\langle 21\rangle$	$\sqrt{}\langle 20,21\rangle$	$\sqrt{}\langle 01,21\rangle$	$\sqrt{}\langle 10,21\rangle$	$\sqrt{}\langle 11,21\rangle$	$\sqrt{}\langle 10,20,21\rangle$	$\sqrt{}\langle 10,11,21\rangle$	$\sqrt{}\langle 01,11,21\rangle$
$\sqrt{}\langle 10\rangle$	$\langle 00,21\rangle$	$\langle 00,21\rangle$	$\langle 00,21\rangle$	$\langle 00,21\rangle$	$\langle 00,20,21\rangle$	$\langle 00,11,21\rangle$	$\langle 00,11,21\rangle$
	$\sqrt{}\langle 00,20\rangle$	$\sqrt{}\langle 00,01\rangle$	$\sqrt{}\langle 00,10\rangle$	$\sqrt{}\langle 00,11\rangle$	$\langle 00,10,21\rangle$	$\langle 00,10,21\rangle$	$\langle 00,01,21\rangle$
					$\sqrt{}\langle 00,10,20\rangle$	$\sqrt{}\langle 00,10,11\rangle$	$\sqrt{}\langle 00,01,11\rangle$

Note that the simplices $\langle 00,21\rangle$, $\langle 00,10,21\rangle$, and $\langle 00,11,21\rangle$ of the complement appear multiple times as faces. We can attach simplices to the domain in the following order:

$$①\langle 00,10,21\rangle,\quad ②\langle 00,10,20,21\rangle,\quad ③\langle 00,10,11,21\rangle,\quad ④\langle 00,01,11,21\rangle.$$

In each case, the intersection of the simplex with (domain+previously attached simplices) is an inner horn. This directly exhibits $(\Lambda_1^2 \subset \Delta^2)\square(\partial\Delta^1 \subset \Delta^1)$ as an inner anodyne map.

17. Function complexes of quasicategories are quasicategories

17.1. Enriched lifting properties. We record the immediate consequences of $\overline{\text{InnHorn}}\square\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ (16.7).

17.2. Proposition.

(1) If $i\colon A \to B$ is inner anodyne and $j\colon K \to L$ a monomorphism, then
$$i\square j\colon (A\times L)\cup_{A\times K}(B\times K) \to B\times L$$
is inner anodyne.

(2) If $j\colon K \to L$ is a monomorphism and $p\colon X \to Y$ is an inner fibration, then
$$p^{\square j}\colon \text{Map}(L,X) \to \text{Map}(K,X)\times_{\text{Map}(K,Y)}\text{Map}(L,Y)$$
is an inner fibration.

(3) If $i\colon A \to B$ is inner anodyne and $p\colon X \to Y$ is an inner fibration, then
$$p^{\square i}\colon \text{Map}(B,X) \to \text{Map}(A,X)\times_{\text{Map}(A,Y)}\text{Map}(B,Y)$$
is a trivial fibration.

These can be summarized as
$$\overline{\text{InnHorn}}\square\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}},\qquad \text{InnFib}^{\square\overline{\text{Cell}}} \subseteq \text{InnFib},\qquad \text{InnFib}^{\square\overline{\text{InnHorn}}} \subseteq \text{TFib}.$$

Statement (1) is just restating (16.7). Statements (2) and (3) can be thought of as "enriched" lifting properties. In particular, $\text{InnFib}^{\square\overline{\text{InnHorn}}} \subseteq \text{TFib}$ contains $\overline{\text{InnHorn}} \boxslash \text{InnFib}$ as a special case, since trivial fibrations are necessarily surjective on vertices.

We are going to use these consequences all the time. To announce that I am using any of these, I will simply assert "$\overline{\text{InnHorn}}\square\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$" without other explanation; alternately, to indicate an application of statements (2) and (3), I will call it "enriched lifting".

There are many useful special cases, obtained by taking the domain of a monomorphism to be empty, or the target of an inner fibration to be terminal.

- If $i\colon A \to B$ is inner anodyne, so is $i \times \text{id}_L\colon A\times L \to B\times L$.

- If $p\colon X \to Y$ is an inner fibration, then so is $\mathrm{Map}(L, p)\colon \mathrm{Map}(L, X) \to \mathrm{Map}(L, Y)$.
- If $j\colon K \to L$ is a monomorphism and C a quasicategory, then $\mathrm{Map}(j, C)\colon \mathrm{Map}(L, C) \to \mathrm{Map}(K, C)$ is an inner fibration.
- If $i\colon A \to B$ is inner anodyne and C a quasicategory, then $\mathrm{Map}(i, C)\colon \mathrm{Map}(B, C) \to \mathrm{Map}(A, C)$ is a trivial fibration.
- If C is a quasicategory, so is $\mathrm{Map}(L, C)$. Thus we have proved (B).

Let's spell out the proof of (B) in a little more detail. Because $\overline{\mathrm{InnHorn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$, we have (16.7) that

$$(\Lambda^n_j \subset \Delta^n) \square (\varnothing \subseteq K) = (\Lambda^n_j \times K \to \Delta^n \times K)$$

is inner anodyne for any K and $0 < j < n$. Thus, for any diagram

$$
\begin{array}{ccc}
\Lambda^n_j \times K & \longrightarrow & C \\
\downarrow & \nearrow & \\
\Delta^n \times K & &
\end{array}
$$

with C a quasicategory, a dotted arrow exists. By adjunction, this is the same as saying we can extend $\Lambda^n_j \to \mathrm{Map}(K, C)$ along $\Lambda^n_j \subset \Delta^n$. That is, we have proved that $\mathrm{Map}(K, C)$ is a quasicategory.

17.3. Remark. Most saturated classes $\overline{S}$ that we will explictly discuss in these notes will have the property that $\overline{S} \square \overline{\mathrm{Cell}} \subseteq \overline{S}$, and thus analogues of the above remarks will hold for such classes.

17.4. Exercise (Important)**.** Show that $\overline{\mathrm{Cell}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$. (Hint: (15.17).) State the analogue of (17.2) associated to this inclusion.

17.5. Composition functors. We can use the above theory to construct "composition functors". If C is an ordinary category, the operation of composing a sequence of n maps can be upgraded to a functor:

$$\mathrm{Fun}([1], C) \times_C \cdots \times_C \mathrm{Fun}([1], C) \to \mathrm{Fun}([1], C)$$

which on *objects* describes composition of a sequence of maps. The source of this functor is the evident inverse limit in Cat; it can be identified (using simplicial set language) with $\mathrm{Fun}(I^n, C)$.

We can generalize such composition functors to quasicategories. We use the following observation: any trivial fibration admits a section, since $(\varnothing \to Y) \boxtimes (p\colon X \to Y)$ if p is a trivial fibration (17.2).

Let C be a quasicategory. Then map $r\colon \mathrm{Fun}(\Delta^n, C) \to \mathrm{Fun}(I^n, C)$ induced by restriction along $I^n \subseteq \Delta^n$ is a trivial fibration by (17.2), since the spine inclusion is inner anodyne (10.12). Therefore r admits a section s, so we get a diagram

$$
\mathrm{Fun}(I^n, C) \underset{r}{\xleftarrow{\;\;\;\;\;}} \overset{s}{\overbrace{}} \mathrm{Fun}(\Delta^n, C) \xrightarrow{\;\; r' \;\;} \mathrm{Fun}(\Delta^{\{0,n\}}, C)
$$

where r' is restriction along $\Delta^{\{0,n\}} \subset \Delta^n$. The composite $r's$ can be thought of as a kind of "n-fold composition" functor. It is not unique, since s isn't, but we'll see (**??**) that this is ok: all functors constructed this way are "naturally isomorphic" to each other.

17.6. A useful variant. The proof of (16.7) actually proves something a little stronger.

17.7. Proposition ([Joy08a, §2.3.1], [Lur09, §2.3.2])**.** *We have that* $\overline{\{\Lambda^2_1 \subset \Delta^2\} \square \overline{\mathrm{Cell}}} = \overline{\mathrm{InnHorn}}$.

Proof. We give a proof in the **appendix**. $\square$

A consequence is another characterization of quasicategories.

17.8. Corollary. *A simplicial set C is a quasicategory if and only if $f\colon \mathrm{Map}(\Delta^2, C) \to \mathrm{Map}(\Lambda^2_1, C)$ is a trivial fibration.*

Proof. Apply (17.7) together with the fact that $(\partial\Delta^k \subset \Delta^k) \boxdot f$ for all $k \geq 0$ iff $(\partial\Delta^k \subset \Delta^k)\Box(\Lambda_1^2 \subset \Delta^2) \boxdot (C \to *)$ for all $k \geq 0$. $\qquad\square$

18. Natural isomorphisms

18.1. **Natural isomorphisms of functors.** Let C and D be quasicategories. Recall that a natural transformation between functors $f_0, f_1\colon C \to D$ is a map $a\colon C \times \Delta^1 \to D$ such that $a|C \times \Delta^{\{i\}} = f_i$, or equivalently a morphism $a\colon f_0 \to f_1$ in the quasicategory $\mathrm{Fun}(C, D)$.

Say that $a\colon f_0 \to f_1$ is a **natural isomorphism** if a is an isomorphism in the quasicategory of functors $\mathrm{Fun}(C, D)$. Thus, a is a natural isomorphism iff there exists a natural transformation $b\colon f_1 \to f_0$ such that $ba \sim 1_{f_0}$ and $ab \sim 1_{f_1}$.

This notion of natural isomorphism corresponds with the usual one for ordinary categories.

Observe that "there exists a natural isomorphism $f_0 \to f_1$" is an equivalence relation on functors $C \to D$, as this relation precisely coincides with "there exists an isomorphism $f_0 \to f_1$" in the category $h\,\mathrm{Fun}(C, D)$.

Furthermore, the "naturally isomorphic" relation is compatible with composition: if f, f' are naturally isomorphic and g, g' are naturally isomorphic, then so are gf and $g'f'$. This is read off from the fact that the composition map $\mathrm{Fun}(D, E) \times \mathrm{Fun}(C, D) \to \mathrm{Fun}(C, E)$ is a functor between quasicategories, and so induces a functor

$$h\,\mathrm{Fun}(D, E) \times h\,\mathrm{Fun}(C, D) \approx h\big(\mathrm{Fun}(D, E) \times \mathrm{Fun}(C, D)\big) \to h\,\mathrm{Fun}(C, E).$$

(This uses (9.13).)

18.2. **Objectwise criterion for natural isomorphisms.** Recall that if A and B are categories, a natural transformation $\alpha\colon f_0 \to f_1$ between functors $f_0, f_1\colon A \to B$ is a natural isomorphism iff and only if α is "an isomorphism objectwise"; i.e., if for each object a of A the evident map $\alpha(a)\colon f_0(a) \to f_1(a)$ is an isomorphism in B. That natural isomorphisms have this property is immediate. The opposite implication follows from the fact that a natural transformation between functors of ordinary values can be completely recovered from its "values on objects". Thus, given α such that each $\alpha(a)\colon f_0(a) \to f_1(a)$ is an isomorphism, we may explicitly construct an inverse transformation $\beta\colon f_1 \to f_0$ by setting $\beta(a) := \alpha(a)^{-1}\colon f_1(a) \to f_0(a)$. Note that this β is in fact the *unique* inverse to α (since inverses to morphisms are unique when they exist).

One of these directions is straightforward for quasicategories.

18.3. **Proposition.** *Let C and D be quasicategories. If $\alpha\colon C \times \Delta^1 \to D$ is a natural isomorphisms between functors $f_0, f_1\colon C \to D$, then for each object c of C the induced map $\alpha(c)\colon f_0(c) \to f_1(c)$ is an isomorphism in D.*

Proof. The restriction map $\mathrm{Fun}(C, D) \to \mathrm{Fun}(\{c\}, D) = D$ is a functor between quasicategories, so it takes isomorphisms to isomorphisms. It sends α to $\alpha(c)$. $\qquad\square$

The converse to this proposition is also true.

C. **Deferred Proposition.** *A natural transformation $\alpha\colon C \times \Delta^1 \to D$ of functors between quasicategories is a natural isomorphism if and only if each of the maps $\alpha(c)$ are isomorphisms in D.*

Unfortunately, this is much more subtle to prove, as it requires using the existence of inverses to the $\alpha(c)$s to produce an inverse to α, which though it exists is not at all unique. We will prove this converse later (11.5).

18.4. *Remark.* An immediate consequence of (C) is that if D is a *quasigroupoid*, then so is $\mathrm{Fun}(C, D)$.

18.5. *Remark.* The objectwise criterion (C) can be reformulated in terms of homotopy categories. The homotopy category construction takes quasicategories to categories, and takes functors to functors. Furthermore, given a natural transformation $\alpha\colon f_0 \to f_1$ of functors $f_0, f_1\colon C \to D$ between quasicategories (i.e., a functor $\alpha\colon C \times \Delta^1 \to D$ such that $(\alpha|C \times \{j\}) = f_j$), we obtain an induced transformation $h\alpha\colon hf_0 \to hf_1$ of functors $hf_0, hf_1\colon hC \to hD$ between their homotopy categories (so that the value of $h\alpha$ at an object $c \in \mathrm{ob}\, hC = C_0$ is the homotopy class of the edge $\alpha(\{c\} \times \Delta^1) \subseteq D$). Then (C) asserts that α is a natural isomorphism if and only if $h\alpha$ is a natural isomorphism.

19. Categorical equivalence

We are now in position to define the correct generalization of the notion of "equivalence" of categories. This will be called *categorical equivalence* of quasicategories, and will be a direct generalization of the classical notion.

Given this, we use it to define a notion of categorical equivalence which applies to arbitrary maps of simplicial sets. Finally, we will show that the two definitions agree for maps between quasicategories.

19.1. **Categorical equivalences between quasicategories.** A **categorical inverse** to a functor $f\colon C \to D$ between quasicategories is a functor $g\colon D \to C$ such that gf is naturally isomorphic to 1_C and fg is naturally isomorphic to 1_D. We provisionally say that a functor f between quasicategories is a **categorical equivalence** if it admits a categorical inverse.

19.2. *Remark.* Categorical equivalence between quasicategories is a kind of "homotopy equivalence", where homotopies are natural isomorphisms between functors.

If C and D are nerves of ordinary categories, then natural isomorphisms between functors in our sense are precisely natural isomorphisms between functors in the classical sense, and that categorical equivalence between nerves of categories coincides precisely with the usual notion of equivalence of categories.

19.3. **Proposition.** *If $f\colon C \to D$ is a categorical equivalence between quasicategories, then $h(f)\colon hC \to hD$ is an equivalence of categories.*

Proof. Immediate, given that natural transformations $f \Rightarrow g\colon C \to D$ induce natural transformations $h(f) \Rightarrow h(g)\colon hC \to hD$. $\square$

Note: the converse is not at all true. For instance, there are many examples of quasicategories which are not equivalent to Δ^0, but whose homotopy categories are: e.g., $\mathrm{Sing}\, T$ for any non-contractible simply connected space T, or $K(A, d)$ for any non-trivial abelian group A and $d \geq 2$.

19.4. **General categorical equivalence.** We can extend the notion of categorical equivalence to maps between arbitrary simplicial sets. Say that a map $f\colon X \to Y$ between arbitrary simplicial sets is a **categorical equivalence** if for every quasicategory C, the induced functor $\mathrm{Fun}(f, C)\colon \mathrm{Fun}(Y, C) \to \mathrm{Fun}(X, C)$ of quasicategories admits a categorical inverse.

We claim that on maps between quasicategories this general definition of categorical equivalence coincides with the provisional notion described earlier.

19.5. **Lemma.** *For a map $f\colon C \to D$ between quasicategories, the two notions of categorical equivalence described above coincide; i.e., f admits a categorical inverse iff it is a categorical equivalence in the general sense.*

To prove this, we will need the following observation. The construction $X \mapsto \mathrm{Map}(X, E)$ is a functor $s\mathrm{Set}^{\mathrm{op}} \to s\mathrm{Set}$, and so in particular induces a natural map

$$\gamma_0\colon \mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathrm{Map}(Y, E), \mathrm{Map}(X, E))$$

of sets. The observation we need is that this construction admits an "enrichment", to a map

$$\gamma \colon \operatorname{Map}(X, Y) \to \operatorname{Map}(\operatorname{Map}(Y, E), \operatorname{Map}(X, E)),$$

which coincides with γ_0 on vertices. The map γ is defined to be adjoint to the "composition" map $\operatorname{Map}(X, Y) \times \operatorname{Map}(Y, E) \to \operatorname{Map}(X, E)$. (*Exercise:* Describe explicitly what γ does to n-simplices.) We say that the functor $\operatorname{Map}(-, E)$ is an *enriched* functor, as it gives not merely a map between hom-sets (i.e., acts on 0-simplices in function complexes), but in fact gives a map between function complexes.

Proof. ($\Longrightarrow$) When C, D, and E are quasicategories so are the function complexes between them. In this case, the above map γ takes functors $C \to D$ to functors $\operatorname{Map}(D, E) \to \operatorname{Map}(C, E)$, natural transformations of such functors to natural transformations, and natural isomorphisms of such functors to natural isomorphisms. Together with the fact that $\operatorname{Map}(-, E)$ is a functor, it follows that a categorical inverse $g \colon D \to C$ to $f \colon C \to D$ gives rise to a categorical inverse $\operatorname{Map}(g, E)$ to the induced functor $\operatorname{Map}(f, E) \colon \operatorname{Map}(D, E) \to \operatorname{Map}(C, E)$.

($\Longleftarrow$) Conversely, suppose $f \colon C \to D$ is a categorical equivalence in the general sense, so that $f^* = \operatorname{Map}(f, E)$ admits a categorical inverse for every quasicategory E, which implies that each functor

$$h(f^*) \colon h\operatorname{Fun}(D, E) \to h\operatorname{Fun}(C, E)$$

is an equivalence of ordinary categories (19.3). In particular, it follows that f^* induces a bijection of sets

$$f^* \colon \pi_0(\operatorname{Fun}(D, E)^{\mathrm{core}}) \xrightarrow{\sim} \pi_0(\operatorname{Fun}(C, E)^{\mathrm{core}});$$

recall that $\pi_0(\operatorname{Fun}(D, E)^{\mathrm{core}}) \approx \pi_0((h\operatorname{Fun}(D, E))^{\mathrm{core}})$ is precisely the set of *natural isomorphism classes* of functors $D \to E$.

Taking $E = C$, this implies that there must exist $g \in \operatorname{Fun}(D, C)_0$ such that there exists a natural isomorphism $gf \to \operatorname{id}_C$ in $\operatorname{Fun}(C, C)_1$. Taking $E = D$, we note that since

$$f^*(\operatorname{id}_D) = \operatorname{id}_D f = f \operatorname{id}_C \approx fgf = f^*(fg),$$

we must have that $\operatorname{id}_D \approx fg$, i.e., there exists a natural isomorphism $\operatorname{id}_D \to fg$ in $\operatorname{Fun}(D, D)_1$. Thus, we have shown that g is a categorical inverse of f, as desired. $\qquad\square$

19.6. *Remark.* The definition of categorical equivalence we are using here is *very* different to the definition adopted by Lurie [Lur09, §2.2.5]. It is also slightly different from the notion of "weak categorical equivalence" used by Joyal [Joy08a, 1.20]. As we will show soon (22.11), Joyal's weak categorical equivalence is equivalent to our definition of categorical equivalence. The discussion around [Lur09, 2.2.5.8] show's that Lurie's and Joyal's definitions are equivalent, and so they are both equivalent to the one we have used.

19.7. *Exercise.* Let $f \colon C \to D$ be a functor between quasicategories. Show that f is a categorical equivalence if and only if for all simplicial sets X, the induced functor $f_* \colon \operatorname{Map}(X, C) \to \operatorname{Map}(X, D)$ is a categorical equivalence.

20. Trivial fibrations and inner anodyne maps

Inner anodyne maps and trivial fibrations are particular kinds of categorical equivalences.

20.1. **Trivial fibrations to the terminal object.** Recall that a trivial fibration $p \colon X \to Y$ of simplicial sets is a map such that $(\partial \Delta^k \subset \Delta^k) \boxslash p$ for all $k \geq 0$. That is, $\mathrm{TFib} = \mathrm{Cell}^{\boxslash}$, so p is a trivial fibration if and only if $\mathrm{Cell} \boxslash p$.

20.2. *Exercise.* Consider an indexed collection of trivial fibrations $p_i \colon X_i \to Y_i$. Show that $p := \coprod p_i \colon \coprod X_i \to \coprod Y_i$ is a trivial fibration. (Hint: similar to proof of (6.7).)

20.3. Proposition. *Let X be a simplicial set and $p\colon X \to *$ be a trivial fibration whose target is the terminal simplicial set. Then X is a Kan complex (and thus a quasigroupoid) and p is a categorical equivalence.*

Proof. The hypotheses together with $\overline{\mathrm{Cell}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$ (17.4) imply that $\mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$ is a trivial fibration for all inclusions $A \subseteq B$, and thus in particular is surjective. It follows that X is a Kan complex, by taking $A \subset B$ to be a horn inclusion.

To show that p is a categorical equivalence, first note that X is non-empty, since $\mathrm{Hom}(\Delta^0, X) \to \mathrm{Hom}(\varnothing, X) = *$ is surjective. Choose any $s \in \mathrm{Hom}(\Delta^0, X)$ (2.10). Clearly $ps = \mathrm{id}_{\Delta^0}$. We will show that $sp\colon X \to X$ is naturally isomorphic to id_X. Consider

$$
\begin{array}{ccccc}
X \amalg X & = & X \times \partial\Delta^1 & \xrightarrow{(\mathrm{id}_X, sp)} & X \\
& & \downarrow & \nearrow h & \downarrow p \\
& & X \times \Delta^1 & \longrightarrow & *
\end{array}
$$

Since p is a trivial fibration, a lift h exists, which is a natural transformation $\mathrm{id}_X \to sp$; note that h represents a morphism in $\mathrm{Fun}(X, X)$. To show that h represents an isomorphism, it's enough to know that $\mathrm{Fun}(X, X)$ is actually a quasigroupoid. Observe that (since p is a trivial fibration and $\overline{\mathrm{Cell}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$), the map

$$\mathrm{Fun}(X, p)\colon \mathrm{Fun}(X, X) \to \mathrm{Fun}(X, *) = *$$

is also a trivial fibration, whence $\mathrm{Fun}(X, X)$ is a Kan complex by the argument given above. □

We will prove a partial converse to this later (33.9): if C is a quasicategory which is categorically equivalent to $*$, then $C \to *$ is a trivial fibration.

20.4. Preisomorphisms. We need a way to produce categorical equivalences between simplicial sets which are not necessarily quasicategories.

Let X be a simplicial set. Say that an edge $a \in X_1$ is a **preisomorphism** if it projects to an isomorphism under $\alpha\colon X \to hX$, the tautological map to the (nerve of the) fundamental category (9.1). If X is actually a quasicategory, the preisomorphisms are just the isomorphisms (since in that case the fundamental category is the same as the homotopy category). Note that degenerate edges are always preisomorphisms, since they go to identity maps in the fundamental category.

20.5. Proposition. *An edge $a \in X_1$ is a preisomorphism if and only if for every map $g\colon X \to C$ to a quasicategory C, the image $g(a)$ is an isomorphism in C.*

Proof. Isomorphisms in C are exactly the edges which are sent to isomorphisms under $\gamma\colon C \to hC$. Given this the proof is straightforward, using the fact that the formation of fundamental categories is functorial, and that hX is itself a category and hence a quasicategory. □

As a consequence, any map $X \to Y$ of simplicial sets takes preisomorphisms to preisomorphisms. In particular, any map *from* a quasicategory takes isomorphisms to preisomorphisms. We will use this observation below.

20.6. *Example.* Consider the subcomplex $\Lambda^3_{\{0,3\}} = \Delta^{\{0,1,2\}} \cup \Delta^{\{1,2,3\}}$ of Δ^3. Define X to be the pushout of the diagram

$$\Lambda^3_{\{0,3\}} \xleftarrow{j} \Delta^{\{0,1\}} \cup \Delta^{\{0,2\}} \cup \Delta^{\{1,3\}} \cup \Delta^{\{2,3\}} \xrightarrow{p} \Delta^{\{y<x\}},$$

where j is the evident inclusion, and p is the unique map to a 1-simplex given on vertices by $0, 2 \mapsto y$, $1, 3 \mapsto x$. The resulting complex X looks like

$$
\begin{array}{ccc}
y & \xrightarrow{\ y_{00}\ } & y \\
{\scriptstyle g}\downarrow \quad {\scriptstyle b}\ \ {\nearrow}\ {\scriptstyle f} \ {\scriptstyle a} & & \downarrow {\scriptstyle g} \\
x & \xrightarrow[\ x_{00}\ ]{} & x
\end{array}
$$

with six non-degenerate simplices x, y, f, g, a, b. The simplicial set X is not a quasicategory; however, any map $\phi\colon X \to C$ to a quasicategory sends f and g to morphisms of C which are inverse to each other, so these (and thus all) edges of X are preisomorphisms.

Say that vertices in a simplicial set X are **preisomorphic** if they can be connected by a chain of preisomorphisms (which can point in either direction). Clearly, any map $g\colon X \to C$ to a quasicategory takes preisomorphic vertices to isomorphic objects.

We can apply this to function complexes. If two maps $f_0, f_1\colon X \to Y$ are preisomorphic (viewed as vertices in $\mathrm{Map}(X, Y)$), then for any quasicategory C, the induced functors $\mathrm{Map}(f_0, C), \mathrm{Map}(f_1, C)\colon \mathrm{Map}(Y, C) \to \mathrm{Map}(X, C)$ are naturally isomorphic. To see this, consider

$$
\Delta^1 \xrightarrow{\ a\ } \mathrm{Map}(X, Y) \xrightarrow{\ b\ } \mathrm{Map}(\mathrm{Map}(Y, C), \mathrm{Map}(X, C))
$$

where b is adjoint to the composition map $\mathrm{Map}(Y, C) \times \mathrm{Map}(X, Y) \to \mathrm{Map}(X, C)$. If a represents a preisomorphism $f_0 \to f_1$ in $\mathrm{Map}(X, Y)$, then ba represents an isomorphism $\mathrm{Map}(f_0, C) \to \mathrm{Map}(f_1, C)$, since the target of b is a quasicategory. As a consequence we get the following.

20.7. Lemma. *If $p\colon X \to Y$ and $q\colon Y \to X$ are maps of simplicial sets such that qp is preisomorphic to id_X and pq is preisomorphic to id_Y, then p and q are categorical equivalences.*

20.8. Trivial fibrations are always categorical equivalences.

20.9. Proposition. *Every trivial fibration between simplicial sets is a categorical equivalence.*

Here is some notation. Given maps $f\colon A \to Y$ and $g\colon B \to Y$, we write $\mathrm{Map}_{/Y}(f, g)$ or $\mathrm{Map}_{/Y}(A, B)$ for the simplicial set defined by the pullback square

$$
\begin{array}{ccc}
\mathrm{Map}_{/Y}(A, B) & \longrightarrow & \mathrm{Map}(A, B) \\
\downarrow & & \downarrow {\scriptstyle g_* = \mathrm{Map}(A, g)} \\
\{f\} & \longrightarrow & \mathrm{Map}(A, Y)
\end{array}
$$

That is, $\mathrm{Map}_{/Y}(A, B)$ can be thought of as a simplicial set which parameterizes sections of g over f. I'll call this the **relative function complex over** Y.

Proof. Fix a trivial fibration $p\colon X \to S$. We regard both X and S as objects *over* S, via p and id_s, and consider various relative function complexes over S.

Note that since p is a trivial fibration, so are $\mathrm{Map}(X, p)$ and $\mathrm{Map}(S, p)$ since $\overline{\mathrm{Cell}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$, and therefore both

$$
\mathrm{Map}_{/S}(S, X) \to \mathrm{Map}_{/S}(S, S) = * \qquad \text{and} \qquad \mathrm{Map}_{/S}(X, X) \to \mathrm{Map}_{/S}(X, S) = *
$$

are trivial fibrations (since TFib is cosaturated and so closed under base change). It follows from (20.3) that both $\mathrm{Map}_{/S}(S, X)$ and $\mathrm{Map}_{/S}(S, Y)$ are quasigroupoids which are non-empty, and with the property that any pair of objects in each are isomorphic. These are contained in the simplicial sets $\mathrm{Map}(S, X)$ and $\mathrm{Map}(X, X)$ respectively, which however need not be quasicategories.

Pick any $s \in \mathrm{Map}_{/S}(S, X) \subseteq \mathrm{Map}(S, X)$, whence $ps = \mathrm{id}_S$, and pick any isomorphism $a\colon \mathrm{id}_X \to sp$ in $\mathrm{Map}_{/S}(X, X)$, which is hence a preisomorphism in $\mathrm{Map}(X, X)$.

Thus, we have exhibited maps p and s whose composites are preisomorphic to identity functors, and therefore they are categorical equivalences by (20.7). ☐

20.10. *Remark.* Suppose that $p\colon C \to D$ is a trivial fibration between quasicategories. As we have noted, the relative function complex $\mathrm{Map}_{/D}(C, D)$ "parameterizes sections of p". According to the proof of (20.9), this is a quasigroupoid equivalent to the terminal quasicategory. In particular, not only is p a categorical equivalence, but also

- p admits a section, which is a categorical inverse to p, and
- any two sections of p are naturally isomorphic.

We will often make use of this observation.

20.11. **Inner anodyne maps are always categorical equivalences.**

20.12. **Proposition.** *Every inner anodyne map between simplicial sets is a categorical equivalence.*

Proof. Let $j\colon X \to Y$ be a map in $\overline{\mathrm{InnHorn}}$, and let C be any quasicategory. The induced map $\mathrm{Map}(j, C)\colon \mathrm{Map}(Y, C) \to \mathrm{Map}(X, C)$ is a trivial fibration since $\overline{\mathrm{InnHorn}}\Box\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (17.2), and therefore a categorical equivalence. ☐

20.13. **Every simplicial set is categorically equivalent to a quasicategory.**

20.14. **Proposition.** *Fix a simplicial set X.*
   (1) *There exists a quasicategory C and an inner anodyne map $f\colon X \to C$, which is therefore a categorical equivalence.*
   (2) *For any two $f_i\colon X \to C_i$ as in (1), there exists a categorical equivalence $g\colon C_1 \to C_2$ such that $gf_1 = f_2$.*

Here is some more notation. Given maps $f\colon X \to A$ and $g\colon X \to B$, we write $\mathrm{Map}_{X/}(f, g)$ or $\mathrm{Map}_{X/}(A, B)$ for the simplicial set defined by the pullback square

$$
\begin{array}{ccc}
\mathrm{Map}_{X/}(A, B) & \longrightarrow & \mathrm{Map}(A, B) \\
\downarrow & & \downarrow{\scriptstyle f^*=\mathrm{Map}(f,B)} \\
\{g\} & \longrightarrow & \mathrm{Map}(X, B)
\end{array}
$$

This is the **relative function complex under X.**

Proof. (1) By the small object argument (13.10), we can factor $X \to *$ into $X \xrightarrow{j} C \xrightarrow{p} *$ where $j \in \overline{\mathrm{InnHorn}}$ and $p \in \mathrm{InnFib}$. The inner anodyne map j is the desired categorical equivalence to a quasicategory.

(2) For $i, j \in \{1, 2\}$, we have a restriction map $f_i^*\colon \mathrm{Map}(C_i, C_j) \to \mathrm{Map}(X, C_j)$, which is necessarily a trivial fibration since $\overline{\mathrm{Cell}}\Box\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$. Therefore the maps $\mathrm{Map}_{X/}(C_i, C_j) \to *$ are all trivial fibrations, i.e., each $\mathrm{Map}_{X/}(C_i, C_j)$ is a quasigroupoid with only one isomorphism class of objects (20.3). As in the proof of (20.9) we construct $g\colon C_1 \to C_2$ and $g'\colon C_2 \to C_1$ which are categorically inverse to each other; details are left to the reader. ☐

Thus, we can always "replace" a simplicial set X by a categorically equivalent quasicategory C. Although such C is not unique, it is unique up to categorical equivalence.

You can think of such a C as a quasicategory "freely generated" by the simplicial set X, an idea which is validated by the fact that $\mathrm{Fun}(j, D)\colon \mathrm{Fun}(C, D) \to \mathrm{Map}(X, D)$ is a categorical equivalence for every quasicategory D.

21. Some examples of categorical equivalences

21.1. Free monoid on one generator. Let F denote the free monoid on one generator g. This is a category with one object x, and morphism set $\{\, g^n \mid n \geq 0 \,\}$.

Associated to the generator g is a map

$$\gamma\colon S := \Delta^1/\partial\Delta^1 \to N(F)$$

sending the image of the generator $\iota \in (\Delta^1)_1$ in S to g. (We use "L/K" as a shorthand for "$L \amalg_K *$" whenever $K \subseteq L$. The object S is called the "simplicial circle".)

It is not hard to see that F is "freely generated" as a category by S, in the sense that $hS = F$ (the fundamental category of S is F). It turns out that $N(F)$ is actually freely generated as a *quasicategory* by S.

21.2. Proposition. *The map $\gamma\colon S \to N(F)$ is a categorical equivalence, and in fact is inner anodyne.*

Proof. This is an explicit calculation. Let $a_k \in N(F)_k$ denote the k-simplex corresponding to the length k chain of maps $(g, g, \dots, g)$ in F where g is the generator, and let $Y_k \subseteq N(F)$ denote the subcomplex which is the image of $a_k\colon \Delta^k \to N(F)$. We have that $\bigcup Y_k = N(F)$, since a general d-simplex in $N(F)$ corresponds to a chain of maps $(g^{m_1}, \dots, g^{m_d})$, which is evidently a face of a_m if $m \geq m_1 + \cdots + m_d$.

Clearly, $Y_1 \approx S$, while $Y_2 \approx Y_1 \cup_{\Lambda_1^2} \Delta^2$. In general, you can easily verify the following.

- A simplex x of Δ^k is such that $a_k(x)$ is in the subcomplex Y_{k-1} of Y_k if and only if x is in the subcomplex $\Lambda_{\{0,k\}}^k = \Delta^{\{0,\dots,k-1\}} \cup \Delta^{\{1,\dots,k\}}$.
- Every simplex y of Y_k not in Y_{k-1} is the image under a_k of a *unique* simplex in Δ^k.

In other words, the square

$$
\begin{array}{ccc}
\Lambda_{\{0,k\}}^k & \longrightarrow & Y_{k-1} \\
\downarrow & & \downarrow \\
\Delta^k & \xrightarrow{\;a_k\;} & Y_k
\end{array}
$$

is a pullback, and a_k induces in each degree n a bijection $(\Delta^k)_n \smallsetminus (\Lambda_{\{0,k\}}^k)_n \xrightarrow{\sim} (Y_k)_n \smallsetminus (Y_{k-1})_n$. It follows (21.3) that the square is a pushout.

The inclusion $\Lambda_{\{0,k\}}^k \subset \Delta^k$ is a *generalized inner horn*, and we have noted this is inner anodyne when $k \geq 2$ (10.10). It follows that each $Y_{k-1} \to Y_k$ is inner anodyne, whence $S \to N(F)$ is inner anodyne. $\square$

In the proof, we used the following fact, which is worth recording.

21.3. Lemma. *If*

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\
Y' & \xrightarrow{\;f\;} & Y
\end{array}
$$

is a pullback of simplicial sets such that (i) j is a monomorphism, and (ii) f induces in each degree n a bijection $Y'_n \smallsetminus i(X'_n) \xrightarrow{\sim} Y_n \smallsetminus i(X_n)$, then the square is a pushout square.

Proof. Verify the analogous statement for a pullback square of sets. $\square$

21.4. Free categories. We can generalize the above to free monoids with arbitrary sets of generators, and in fact to free categories. Let S be a **1-dimensional** simplicial set, i.e., one such that $S = \mathrm{Sk}_1 S$. These are effectively the same thing as *directed graphs* (allowed to have multiple parallel edges and loops): S_0 corresponds to the set of vertices of the directed graph, and S_1^{nd} corresponds to the set of edges of the directed graph.

Let $F := hS$. We call F the **free category** on the 1-dimensional simplicial set S. In this case, the morphisms of the fundamental category are precisely the words in the edges S_1^{nd} of the directed graph (including empty words for each vertex, corresponding to identity maps).

21.5. Proposition. *The evident map* $\gamma\colon S \to N(F)$ *is a categorical equivalence, and in fact is inner anodyne.*

Proof. This is virtually the same as the proof of (21.2). In this case, $Y_k \subseteq N(F)$ is the subcomplex generated by all $a\colon \Delta^k \to N(F)$ such that each spine-edge $a_{i-1,i}$ is in S_1^{nd}, and Y_k is obtained by attaching a generalized horn to Y_{k-1} for each such a. $\square$

As a consequence, it is "easy" to construct functors $F \to C$ from a free category to a quasicategory: start with a map $S \to C$, which amounts to specifying vertices and edges in C corresponding to elements S_0 and S_1^{nd}, and extend over $S \subseteq F$. The evident restriction map $\mathrm{Fun}(F, C) \to \mathrm{Map}(S, C)$ is a categorical equivalence, and in fact a trivial fibration.

21.6. *Exercise.* Describe the ordinary category $A := h\Lambda_0^3$ "freely generated" by Λ_0^3. Show that the tautological map $\Lambda_0^3 \to N(A)$ is inner anodyne.

21.7. Free commutative monoids. Let F be the free monoid on one generator again, with generator corresponding to simplicial circle $S = \Delta^1/\partial\Delta^1 \subset F$ Recall that $F^{\times n}$ is the free commutative monoid on n generators. Recall that the nerve functor preserves products, so $N(F^{\times n}) \approx N(F)^{\times n}$. We obtain a map

$$\delta = \gamma^{\times n}\colon S^{\times n} \to N(F^{\times n})$$

from the "simplicial n-torus".

21.8. Proposition. *The map* $\delta\colon S^{\times n} \to N(F^{\times n})$ *is a categorical equivalence, and in fact is inner anodyne.*

This is an easy consequence of the following.

21.9. Proposition. *For any simplicial set K, if $j\colon A \to B$ is inner anodyne, so is $j \times K\colon A \times K \to B \times K$.*

Proof. The map $j\square(\varnothing \subseteq K)$ is inner anodyne, since $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ by (16.7). $\square$

21.10. *Exercise.* Let $S \vee S \subseteq S^{\times 2}$ be the subcomplex obtained as the evident "one-point union" of the two "coordinate circles". Suppose given a map $\phi\colon S \vee S \to C$ to a quasicategory C, corresponding to a choice of object $x \in C_0$ together with two morphisms $f, g\colon x \to x$ in C_1. Show that there exists an extension of ϕ along $S \vee S \subset N(F^{\times 2})$ if and only if $fg \approx gf$ in hC.

21.11. *Remark.* The analogue of the above exercise for $n = 3$ isn't true. That is, given $S \vee S \vee S \to C$ corrposding to three morphisms $f, g, h\colon x \to x$ in C such that $fg \approx gh$, $gh \approx hg$, and $hf \approx fh$ in C, there need not exist an extension to $N(F^{\times 3})$. (For instance, take $C = \mathrm{Sing}\, T$, where $T \subseteq (S^1)^{\times 3}$ is the subspace of the 3-torus consisting of tuples (x_1, x_2, x_3) such that at least one x_i is the basepoint of S^1.)

21.12. Finite groups are not finite. If A is any ordinary category, then $\mathrm{Sk}_2 N(A)$ "freely generates $N(A)$ as a category", in the sense that $h(\mathrm{Sk}_2 N(A)) \approx A$, or equivalently that $\mathrm{Map}(N(A), N(B)) \to \mathrm{Map}(\mathrm{Sk}_2 N(A), N(B))$ is an isomorphism for any category B. However, it is often the case that *no finite dimensional subcomplex* "freely generates $N(A)$ as a quasicategory". In fact, this is the case for every non-trivial finite group.

21.13. *Example.* Let G be a finite group, and let $C = N(G)$. The geometric realization $BG := |N(G)|$ is the *classifying space* of G. I want to show that if G is not the trivial group, then NG is not categorically equivalent to any *finite* dimensional simplicial set K (i.e., one with no non-degenerate simplices above a certain dimension). We need to use some homotopy theory, along with a fact to be proved later: if $f \colon X \to Y$ is any categorical equivalence of simplicial sets, then the induced map $|f| \colon |X| \to |Y|$ on geometric realizations is a homotopy equivalence of spaces.

First consider $G = \mathbb{Z}/n$ with $n > 1$. A standard calculation in topology says that $H^{2k}(|N(G)|, \mathbb{Z}) \approx \mathbb{Z}/n \not\approx 0$ for all $k > 0$. This implies that $|N(G)|$ cannot be homotopy equivalent to any finite dimensional complex.

Now consider a general non-trivial finite group G, and choose a non-trivial cyclic subgroup $H \leq G$. We know the fundamental group: $\pi_1 |K| \approx \pi_1 |N(G)| = G$. Covering space theory tells us we can construct a covering map $p \colon E \to |N(G)|$ so that $\pi_1 E \to \pi_1 |N(G)|$ is the inclusion $H \to G$. In fact, E is homotopy equivalent to the classifying space BH (because $\pi_k E \approx 0$ for $k \geq 2$). But if $|N(G)|$ is finite dimensional then so is E, but this would then contracting the observation that $H^*(BH, \mathbb{Z}) \approx H^*(E, \mathbb{Z}) \approx 0$ for all $*$.

Thus, non-trivial *finite* groups are never "freely generated as a quasicategory" by finite dimensional complexes.

21.14. *Remark.* Let T be a *finite* CW-complex, and G a finite group. A theorem of Haynes Miller (the "Sullivan conjecture") implies that every functor $N(G) \to \mathrm{Sing}\, T$ is naturally isomorphic to a constant functor (i.e., one which factors through Δ^0). Thus, the singular complex of a finite CW-complex cannot "contain" any non-trivial finite group, even if its fundamental group contains a non-trivial finite subgroup.

22. The homotopy category of quasicategories

22.1. **The homotopy category of** QCat**.** The homotopy category $h\mathrm{QCat}$ of quasicategories is defined as follows. The objects of $h\mathrm{QCat}$ are the quasicategories. Morphisms $C \to D$ in $h\mathrm{QCat}$ are natural isomorphism classes of functors. That is,

$$\mathrm{Hom}_{h\mathrm{QCat}}(C, D) := \text{isomorphism classes of objects in } h\,\mathrm{Fun}(C, D) = \pi_0\big(\mathrm{Fun}(C, D)^{\mathrm{core}}\big).$$

That this defines a category results from the fact that composition of functors passes to a functor $h\,\mathrm{Fun}(D, E) \times h\,\mathrm{Fun}(C, D) \to h\,\mathrm{Fun}(C, E)$, and thus is compatible with natural isomorphism.

Note that a map $f \colon C \to D$ of quasicategories is a categorical equivalence if and only if its image in $h\mathrm{QCat}$ is an isomorphism.

22.2. *Remark.* We can similarly define a category $h\mathrm{Cat}$, whose objects are categories and whose morphisms are *isomorphism classes* of functors. The nerve functor evidently induces a full embedding $h\mathrm{Cat} \to h\mathrm{QCat}$.

22.3. *Warning.* Although we use the word "homotopy", the definition of $h\mathrm{QCat}$ given above is not an example of the notion of the homotopy category of a quasicategory defined in §9: QCat is a (large) ordinary category, so is isomorphic to its own homotopy category in that sense. Here we are using the equivalence relation on morphisms(=functors) defined by natural isomorphism.

For future reference, we note that $h\mathrm{QCat}$ has finite products, which just amount to the usual products of simplicial sets.

22.4. **Proposition.** *The terminal simplicial set Δ^0 is a terminal object in $h\mathrm{QCat}$. If C_1, C_2 are quasicategories, then the projection maps exhibit $C_1 \times C_2$ as a product in $h\mathrm{QCat}$.*

Proof. This is straightforward. The key observation for the second statement is the fact that isomorphism classes of objects in a product of quasicategories correspond to pairs of isomorphism classes in each (6.11), and the fact that $\mathrm{Map}(X, C_1 \times C_2) \xrightarrow{\sim} \mathrm{Map}(X, C_1) \times \mathrm{Map}(X, C_2)$. $\square$

22.5. The 2-out-of-3 property. A class of morphisms $\mathcal{W}$ in a category is said to satisfy the **2-out-of-3 property** if (i) $\mathcal{W}$ contains all identity maps, and (ii) whenever two maps in a triple of the form $(f, g, g \circ f)$ are in $\mathcal{W}$, so is the third.

22.6. *Example.* In any category, the class of isomorphisms satisfies 2-out-of-3, as does the class of identity maps.

22.7. *Example.* In Cat, the category of small categories, the class of equivalences satisfies 2-out-of-3.

22.8. *Exercise.* Given a functor $f \colon C \to D$ between categories, let W be the class of maps in C that f takes to isomorphisms in D. Show that W satisfies 2-out-of-3.

22.9. Proposition. *The class* CatEq *of categorical equivalences in* sSet *satisfies 2-out-of-3.*

Proof. It is immediate that the identity map of a simplicial set is a categorical equivalence.

Next, consider the special case of $h = gf$, where f, g, h are functors between *quasicategories*. In this case, we use the following fact:

- A map $f \colon C \to D$ between quasicategories is a categorical equivalence *if and only if* its image $[f]$ in hQCat is an isomorphism.

The proof of this fact amounts to showing that g is a categorical inverse of f exactly if $[g]$ is an inverse of $[f]$ in hQCat, which amounts to translating the definition of categorical inverse. Given this, 2-out-of-3 follows using the exercise (22.8) applied to the functor $h \colon \mathrm{QCat} \to \mathrm{Cat}$.

For maps $h = gf$ between arbitrary simplicial sets, we reduce to the above case by considering $\mathrm{Fun}(h, B) = \mathrm{Fun}(f, B)\,\mathrm{Fun}(g, B)$ where B is an arbitrary quasicategory.   □

22.10. Weak categorical equivalence. Joyal [Joy08a, 1.20] uses a variant of the notion of categorical equivalence, which turns out to be equivalent to what we are using. A map $f \colon X \to Y$ of simplicial sets is a **weak categorical equivalence**[19] if for every quasicategory C, the induced map $h\,\mathrm{Fun}(Y, C) \to h\,\mathrm{Fun}(X, C)$ is an equivalence of ordinary categories. Note that, like categorical equivalences, the class of weak categorical equivalences also satisfies 2-out-of-3.

22.11. Proposition. *A map is a categorical equivalence if and only if it is a weak categorical equivalence.*

Proof. ($\Longrightarrow$) Straightforward. ($\Longleftarrow$) In the case that f is a weak categorical equivalence between *quasicategories*, this is exactly what the second half of the proof of (19.5) really shows. For a general map f, use factorization to construct a commutative square

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\
X' & \xrightarrow[\ f'\ ]{} & Y'
\end{array}
$$

so that u and v are inner anodyne (and so categorical equivalences), and X' and Y' are quasicategories. Applying $h\,\mathrm{Fun}(-, C)$ to the square with C a quasicategory, we see that the vertical maps become equivalences of categories, so if f is weak categorical equivalence so is f', which is then a categorical equivalence by what we have already proved, whence f is a categorical equivalence by 2-out-of-3.   □

22.12. The homotopy 2-category of QCat. A **2-category** E is a category which is itself "enriched" over Cat. That is,

- for each pair of objects $x, y \in \mathrm{ob}\, E$, there is a category $\underline{\mathrm{Hom}}_E(x, y)$, so that
- the objects of $\underline{\mathrm{Hom}}_E(x, y)$ are precisely the set $\mathrm{Hom}_E(x, y)$ of morphisms of E, and

---

[19]This is not to be confused with "weak equivalence", which we will talk about later (32.1).

- there are "composition functors" $\mathrm{Hom}_E(y,z) \times \mathrm{Hom}_E(x,y) \to \mathrm{Hom}_E(x,z)$ for all $x,y,z \in$ ob E which on objects is just ordinary composition of morphisms in E, which
- is unital and associative in the evident sense.

One refers to the objects of $\mathrm{Hom}_E(x,y)$ as **1-morphisms** $f\colon x \to y$ of E, and the morphisms of $\mathrm{Hom}_E(x,y)$ as **2-morphisms** $\alpha\colon f \Rightarrow g$ of E. The **underlying category** of E consists of the objects and 1-morphisms only.

The standard example of a 2-category is Cat, the category of categories, with objects=categories, 1-morphisms=functors, 2-morphisms=natural transformations.

We can enlarge the category QCat of quasicategories to a **homotopy 2-category** $h_2\mathrm{QCat}$, so that
$$\mathrm{Hom}_{h_2\mathrm{QCat}}(C,D) := h\,\mathrm{Fun}(C,D).$$

That is,

- **objects** of $h_2\mathrm{QCat}$ are quasicategories,
- **1-morphisms** of $h_2\mathrm{QCat}$ are functors between quasicategories,
- **2-morphisms** of $h_2\mathrm{QCat}$ are *isomorphism classes* of natural transformations of functors.

Note that QCat sits inside $h_2\mathrm{QCat}$ as its underlying category; thus, $h_2\mathrm{QCat}$ contains all the information of QCat. On the other hand $h\mathrm{QCat}$ is obtained from $h_2\mathrm{QCat}$ by first identifying 1-morphisms (functors) which are 2-isomorphic (i.e., naturally isomorphic), and then throwing away the 2-morphisms.

Part 3. **Joins, slices, and Joyal's extension and lifting theorems**

23. JOINS

23.1. Join of categories. If A and B are ordinary categories, we can define a category $A \star B$ called the join. This has
$$\mathrm{ob}(A \star B) = \mathrm{ob}\,A \amalg \mathrm{ob}\,B, \qquad \mathrm{mor}(A \star B) = \mathrm{mor}\,A \amalg (\mathrm{ob}\,A \times \mathrm{ob}\,B) \amalg \mathrm{mor}\,B,$$

so that we put in a *unique* map from each object of A to each object of B. Explicitly,

$$\mathrm{Hom}_{A \star B}(x,y) := \begin{cases} \mathrm{Hom}_A(x,y) & \text{if } x,y \in \mathrm{ob}\,A, \\ \mathrm{Hom}_B(x,y) & \text{if } x,y \in \mathrm{ob}\,B, \\ \{*\} & \text{if } x \in \mathrm{ob}\,A,\ y \in \mathrm{ob}\,B, \\ \varnothing & \text{if } x \in \mathrm{ob}\,B,\ y \in \mathrm{ob}\,A, \end{cases}$$

with composition defined so that the evident inclusions $A \to A \star B \leftarrow B$ are functors. (Check that this really defines a category.)

23.2. *Example.* We have that $[p] \star [q] \approx [p+1+q]$.

23.3. *Exercise.* Show that functors $f\colon C \to A \star B$ are in bijective correspondence with triples of functors $(g\colon C \to [1],\ f_{\{0\}}\colon C^{\{0\}} \to A,\ f_{\{1\}}\colon C^{\{1\}} \to B)$, where $C^{\{j\}} := g^{-1}(\{j\}) \subseteq C$.

An important special case are the **left cone** and **right cone** of a category, defined by $A^{\triangleleft} := [0] \star A$ and $A^{\triangleright} := A \star [0]$. For instance, $A^{\triangleright}$ is the category obtained by adjoining one additional object v to A, as well as a unique map $x \to v$ for each object x of $A^{\triangleright}$. In this case, v becomes a terminal object for $A^{\triangleright}$, and we can say that $A \mapsto A^{\triangleright}$ *freely adjoins a terminal object* to A. (Note that a terminal object of A will not be terminal in $A^{\triangleright}$ anymore; this is why we say "freely".) Likewise, $A \mapsto A^{\triangleleft}$ *freely adjoins an initial object* to A.

Limits and colimits of functors can be characterized using cones: if $F\colon A \to B$ is a functor, a *colimit* of F is a functor $G\colon A^{\triangleright} \to B$ which is initial among functors which extend F, and likewise, a *limit* of F is a functor $G'\colon A^{\triangleleft} \to B$ which is terminal among functors which extend F.

23.4. *Remark.* It is worthwhile to spell this out in detail. Given a functor $F\colon A \to B$, to describe a functor $G\colon A^{\triangleright} \to B$ which *extends* F, it suffices to give

(1) an object $G(v)$ in B,
(2) for each object $a \in \operatorname{ob} A$ a morphism $g_a = G(a \to v)\colon F(a) = G(a) \to G(v)$ in B, such that
(3) for each morphism $\alpha\colon a \to a'$ in A we have an equality $g_{a'}F(\alpha) = g_a$ of morphisms $F(a) \to G(v)$ in B.

Given functors $G, G'\colon A^{\triangleright} \to B$, we may consider natural transformations $\phi\colon G \to G'$ which *extend the identity transformation* of F. Explicitly, such a transformation ϕ is exactly determined by

(1) a morphism $\phi_v\colon G(v) \to G'(v)$ in B such that
(2) for each object $a \in \operatorname{ob} A$ we have an equality $g_a' = \phi_v g_a$ of morphisms $F(a) \to G'(v)$ in B, where $g_a = G(a \to v)$ and $g_a' = G'(a \to v)$.

Such a G is a *colimit* of F if for every G' extending F there exists a unique map $\phi_v\colon G(v) \to G'(v)$ in B such that $g_{a'} = \phi_v g_a$ for all $a \in \operatorname{ob} A$. The object $G(v)$ is what is colloquially known as "the colimit of F", although the full data of a colimit of F is actually the functor G. We will call the functor G a *colimit cone* it what follows.

23.5. Ordered disjoint union. As noted above (23.2), the join operation on categories effectively descends to Δ. We will call this the **ordered disjoint union**. It is a functor $\sqcup\colon \Delta \times \Delta \to \Delta$, defined so that $[p] \sqcup [q] := [p + 1 + q]$, to be thought of as the disjoint union of underlying sets, ordered so that the subsets $[p]$ and $[q]$ retain their ordering, and elements of $[p]$ come before elements of $[q]$.

It is handy to extend this to the category Δ_+, the subcategory of ordered sets obtained by adjoining $[-1] := \varnothing$ to Δ. The functor $\sqcup$ extends in an evident way to $\sqcup\colon \Delta_+ \times \Delta_+ \to \Delta_+$. This extended functor makes Δ_+ into a (nonsymmetric but strict) monoidal category, with unit object $[-1]$.

Note that for any map $f\colon [p] \to [q_1] \sqcup [q_2]$ in Δ_+, there is a *unique* decomposition $[p] = [p_1] \sqcup [p_2]$ such that $f = f_1 \sqcup f_2$ for some (necessarily unique) $f_i\colon [p_i] \to [q_i]$ in Δ_+. Note that we need an object $[-1]$ to be able to say this, even if $p, q_1, q_2 \geq 0$; if $f([p]) \subseteq [q_1]$ then $p_2 = -1$.

23.6. Join of simplicial sets. Let X and Y be simplicial sets. The **join** of X and Y is a simplicial set $X \star Y$ defined as follows.

The join of simplicial sets X and Y is defined by

$$(X \star Y)_n := \coprod_{[n]=[n_1] \sqcup [n_2]} X_{n_1} \times Y_{n_2},$$

where we set $X_{-1} = * = Y_{-1}$. The action of simplicial operators is defined in the evident way, using the observation of the previous paragraph: $(x, y)f = (xf_1, yf_2)$ where $f = f_1 \sqcup f_2$, $f_j\colon [m_j] \to [n_j]$.

In particular,

$$(X \star Y)_0 = X_0 \amalg Y_0,$$
$$(X \star Y)_1 = X_1 \amalg X_0 \times Y_0 \amalg Y_1,$$
$$(X \star Y)_2 = X_2 \amalg X_1 \times Y_0 \amalg X_0 \times Y_1 \amalg Y_2,$$

and so on.

The functor $\star$ defines a monoidal structure on $s\mathrm{Set}$, with unit object $\varnothing$. Note however that $\star$ is *not* symmetric monoidal, though it is true that $(Y \star X)^{\mathrm{op}} \approx X^{\mathrm{op}} \star Y^{\mathrm{op}}$. We have that

$$\Delta^p \star \Delta^q \approx \Delta^{p+1+q}.$$

An important example are the cones. The **left cone** and **right cone** of a simplicial set X are

$$X^{\triangleleft} := \Delta^0 \star X, \qquad X^{\triangleright} := X \star \Delta^0.$$

Note that outer horns are cones:
$$(\partial\Delta^n)^{\triangleleft} = \Delta^0 \star \partial\Delta^n \approx \Lambda_0^{n+1}, \qquad (\partial\Delta^n)^{\triangleright} = \partial\Delta^n \star \Delta^0 \approx \Lambda_{n+1}^{n+1}.$$

23.7. *Exercise.* Let $f\colon [m] \to [n]$ be any simplicial operator. Show that the induced map $f\colon \Delta^m \to \Delta^n$ is isomorphic to a join of maps $f_0 \star f_1 \star \cdots \star f_n$, of the form $f_j\colon \Delta^{m_j} \to \Delta^0$, where each $m_j \geq -1$ (with the convention that $\Delta^{-1} = \varnothing$).

It is straightforward to show that the nerve takes joins of categories to joins of simplicial sets: $N(A \star B) \approx N(A) \star N(B)$, and thus $N(A^{\triangleleft}) \approx (NA)^{\triangleleft}$ and $N(A^{\triangleright}) \approx (NA)^{\triangleright}$. (*Exercise:* prove this.)

23.8. **The join of quasicategories is a quasicategory.** Here is a handy rule for constructing maps into a join. Note that every join admits a canonical map $\pi\colon X \star Y \to \Delta^0 \star \Delta^0 \approx \Delta^1$.

23.9. **Lemma** ([Joy08a, Prop. 3.5]). *Maps $f\colon K \to X \star Y$ are in bijective correspondence with the set of triples*
$$\left(g\colon K \to \Delta^1, \quad f_{\{0\}}\colon K^{\{0\}} \to X, \quad f_{\{1\}}\colon K^{\{1\}} \to Y\right),$$
where $K^{\{j\}} := g^{-1}(\{j\}) \subseteq K$.

Proof. This is a straightforward exercise. In one direction, the correspondence sends f to $(\pi f, f|K^{\{0\}}, f|K^{\{1\}})$. $\qquad\square$

23.10. **Proposition.** *If C and D are quasicategories, so is $C \star D$.*

Proof. Use the previous lemma, together with the observation (which we leave as an exercise) that for any map $g\colon \Lambda_j^n \to \Delta^1$ from an *inner* horn, the preimages $g^{-1}(\{0\})$ and $g^{-1}(\{1\})$ are either inner horns or simplices, or are empty. $\qquad\square$

24. SLICES

24.1. **Slices of categories.** Given an ordinary category C, and an object $x \in \mathrm{ob}\, C$, we may form the **slice categories** $C_{x/}$ and $C_{/x}$, also called **undercategory** and **overcategory**.

For instance, the overcategory $C_{/x}$ is the category whose *objects* are maps $f\colon c \to x$ with target x, and whose *morphisms* $(f\colon c \to x) \to (f'\colon c' \to x)$ are maps $g\colon c \to c'$ such that $f'g = f$.

This can be reformulated in terms of joins. Let "T" denote the terminal category (isomorphic to $[0]$). Note that $\mathrm{ob}\, C_{/x}$ corresponds to the set of functors $f\colon [0] \star T \to C$ such that $f|T = x$, and $\mathrm{mor}\, C_{/x}$ corresponds to the set of functors $g\colon [1] \star T \to C$ such that $g|T = x$.

More generally, given a functor $p\colon A \to C$ of categories, we obtain slice categories $C_{p/}$ and $C_{/p}$ defined as follows. The category $C_{/p}$ has

- **objects:** functors $f\colon [0] \star A \to C$ such that $f|A = p$,
- **morphisms** $f \to f'$: functors $g\colon [1] \star A \to C$ such that $g|A = p$.

Likewise, the category $C_{p/}$ has

- **objects:** functors $f\colon A \star [0] \to C$ such that $f|A = p$,
- **morphisms** $f \to f'$: functors $g\colon A \star [1] \to C$ such that $g|A = p$.

24.2. *Exercise.* Describe composition of morphisms in $C_{/p}$ and $C_{p/}$.

24.3. *Exercise.* Show that $(C_{p/})^{\mathrm{op}} \approx (C^{\mathrm{op}})_{p^{\mathrm{op}}/}$ (isomorphism of categories).

24.4. *Exercise.* Fix a functor $p\colon A \to C$, and let B be a category. Construct bijections
$$\{\text{functors } f\colon B \to C_{/p}\} \leftrightarrow \{\text{functors } g\colon B \star A \to C \text{ s.t. } g|A = p\}$$
and
$$\{\text{functors } f\colon B \to C_{p/}\} \leftrightarrow \{\text{functors } g\colon A \star B \to C \text{ s.t. } g|A = p\}.$$

24.5. *Remark.* The notions of limits and colimits can be formulated very compactly in terms of the general notion of slices. Thus, given a functor $p\colon A \to C$, a *colimit* of p is the same data an initial object of $C_{p/}$, while a *limit* of p is the same data as a terminal object of $C_{/p}$. (*Exercise:* prove this; this will be the starting case for formulating notions of limits and colimits for quasicategories.)

24.6. Joins and colimits of simplicial sets. The join functor $\star\colon s\mathrm{Set} \times s\mathrm{Set} \to s\mathrm{Set}$ is in some ways analogous to the product functor $\times$, e.g., it is a monoidal functor.

The product operation $(-) \times (-)$ on simplicial sets commutes with colimits in each input, and the functors $X \times -$ and $- \times X$ admit right adjoints (in both cases, the right adjoint is $\mathrm{Map}(X, -)$). The join functor does not commute with colimits in each variable, but *almost* does so; the only obstruction is the value on the initial object

More precisely, the functors $X \star -$ and $- \star X\colon s\mathrm{Set} \to s\mathrm{Set}$ do not preserve the initial object, since $X \star \varnothing \approx X \approx \varnothing \star X$. However, (the identity map of) X is tautologically the initial object of $s\mathrm{Set}_{X/}$.

24.7. Proposition. *For every simplicial set X, the induced functors*

$$X \star -, \ - \star X\colon s\mathrm{Set} \to s\mathrm{Set}_{X/}$$

preserve colimits.

Proof. This is immediate from the degreewise formula for the join:

$$(X \star Y)_n = X_n \amalg (X_{n-1} \times Y_0) \amalg \cdots \amalg Y_n = X_n \amalg (\text{terms which are ``linear'' in } Y).$$

$\square$

24.8. *Exercise* (Trivial, but important). Show that the functors $X \star -$ and $- \star X\colon s\mathrm{Set} \to s\mathrm{Set}$ preserve *pushouts*.

24.9. Slices of simplicial sets. We have seen that the functors $S \star -\colon s\mathrm{Set} \to s\mathrm{Set}_{S/}$ and $- \star T\colon s\mathrm{Set} \to s\mathrm{Set}_{T/}$ preserve colimits, and therefore we predict that they admit right adjoints. These exist, and are called **slice** functors, denoted

$$(p\colon S \to X) \mapsto X_{p/}\colon s\mathrm{Set}_{S/} \to s\mathrm{Set}$$

and

$$(q\colon S \to X) \mapsto X_{/q}\colon s\mathrm{Set}_{S/} \to s\mathrm{Set}.$$

I will sometimes distinguish these as **slice-under** and **slice-over**, respectively. Explicitly, there are are bijective correspondences

$$(24.10) \qquad \left\{ \begin{array}{c} S \xrightarrow{\ p\ } X \\ \downarrow \quad \nearrow \\ S \star K \end{array} \right\} \iff \{K \dashrightarrow X_{p/}\}, \qquad \left\{ \begin{array}{c} T \xrightarrow{\ q\ } X \\ \downarrow \quad \nearrow \\ K \star T \end{array} \right\} \iff \{K \dashrightarrow X_{/q}\}.$$

Taking $K = \Delta^n$ we obtain the formulas

$$(X_{p/})_n = \mathrm{Hom}_{s\mathrm{Set}_{S/}}(S \star \Delta^n, X), \qquad (X_{/q})_n = \mathrm{Hom}_{s\mathrm{Set}_{T/}}(\Delta^n \star T, X),$$

which we regard as the definition of slices. (I.e., these formulas specify the n-simplices of the slices, and naturality in "Δ^n" specifies the action of simplicial operators.)

24.11. *Exercise.* Given this explicit definition of slices, verify the bijective corrrespondences (24.10).

In particular, we note the special cases associated to $x\colon \Delta^0 \to X$:

$$(X_{x/})_n = \mathrm{Hom}_{s\mathrm{Set}_{\Delta^0/}}(\Delta^0 \star \Delta^n, X) \approx \mathrm{Hom}_{s\mathrm{Set}_*}((\Delta^{n+1}, 0), (X, x)),$$

$$(X_{/x})_n = \mathrm{Hom}_{s\mathrm{Set}_{\Delta^0/}}(\Delta^n \star \Delta^0, X) \approx \mathrm{Hom}_{s\mathrm{Set}_*}((\Delta^{n+1}, n+1), (X, x)).$$

The notation (X, x) represents a pointed simplical set, the category of which is $s\mathrm{Set}_* := s\mathrm{Set}_{\Delta^0/}$. The associated adjunctions are

$$\mathrm{Hom}_{s\mathrm{Set}_*}(K^\triangleleft, (X, x)) \approx \mathrm{Hom}_{s\mathrm{Set}}(K, X_{x/}), \qquad \mathrm{Hom}_{s\mathrm{Set}_*}(K^\triangleright, (X, x)) \approx \mathrm{Hom}_{s\mathrm{Set}}(K, X_{/x}).$$

The slice construction for simplicial sets agrees with that for categories.

24.12. Proposition. *The nerve preserves slices; i.e., if $p \colon A \to C$ is a functor, then $N(C_{p/}) \approx (NC)_{Np/}$ and $N(C_{/p}) \approx (NC)_{/Np}$.*

Proof. Straightforward. $\square$

24.13. Slice as a functor. The function complex $\mathrm{Map}(-, -)$ is a functor in two variables, contravariant in the first and covariant in the second. The slice constructions also behave something like a functor of two variables, though it is a little more complicated. A more precise statement is that every diagram on the left gives rise to commutative diagrams as on the right.

$$
\begin{array}{ccc}
S & \xrightarrow{p} & X \\
{\scriptstyle j}\uparrow & & \downarrow{\scriptstyle f} \\
T & \xrightarrow{fpj} & Y
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccc}
X_{/p} & \longrightarrow & Y_{/fp} \\
\downarrow & & \downarrow \\
X_{/pj} & \longrightarrow & Y_{/fpj}
\end{array}
\qquad
\begin{array}{ccc}
X_{p/} & \longrightarrow & Y_{fp/} \\
\downarrow & & \downarrow \\
X_{pj/} & \longrightarrow & Y_{fpj/}
\end{array}
$$

There seems to be no decent notation for the maps in the right-hand squares. The whole business of joins and slices can get pretty confusing because of this.

24.14. Remark. A very precise formulation is that slice defines a functor $s\mathrm{Set}^{\mathrm{tw}} \to s\mathrm{Set}$ from the *twisted arrow category* of simplicial sets, whose objects are maps p of simplicial sets, and whose morphisms are pairs $(j, f) \colon p \to fpj$, where j and f are themselves maps of simplicial sets.

Let's spell this out. Given $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$, consider "restriction map" $X_{p/} \to Y_{fpj/}$. The composite of a map $u \colon K \to X_{p/}$ with this restriction map is described as follows. The map u corresponds to a dotted arrow in

$$
\begin{array}{ccccccc}
T & \xrightarrow{\quad j \quad} & S & \xrightarrow{\quad p \quad} & X & \xrightarrow{\ f\ } & Y \\
\downarrow & & \downarrow & & \nearrow & & \\
\downarrow & & \downarrow & {\scriptstyle \tilde{u}}\,\nearrow & & & \\
T \star K & \xrightarrow{j \star K} & S \star K & & & &
\end{array}
$$

The composite $K \xrightarrow{u} X_{p/} \to Y_{fpj/}$ corresponds to $f\tilde{u}(j \star K)$.

25. INITIAL AND TERMINAL OBJECTS

We now show that for a vertex x in a quasicategory C, the slice objects $C_{/x}$ and $C_{x/}$ are also quasicategories. Using this, we can make a definition of initial and terminal object in a quasicategory.

25.1. Left and right horns, fibrations, and anodyne maps. We introduce the following notation. The **left horns** are the set of inclusions

$$\mathrm{LHorn} := \{\, \Lambda^n_k \subset \Delta^n \mid 0 \le k < n,\ n \ge 1 \,\} = \mathrm{InnHorn} \cup \{\, \Lambda^n_0 \subset \Delta^n \mid n \ge 1 \,\}$$

and the **right horns** are set of inclusions

$$\mathrm{RHorn} := \{\, \Lambda^n_k \subset \Delta^n \mid 0 < k \le n,\ n \ge 1 \,\} = \mathrm{InnHorn} \cup \{\, \Lambda^n_n \subset \Delta^n \mid n \ge 1 \,\}.$$

The associated saturations $\overline{\mathrm{LHorn}}$ and $\overline{\mathrm{RHorn}}$ are the **left anodyne** and **right anodyne** maps. The associated right complements

$$\mathrm{LFib} := \mathrm{LHorn}^{\boxtimes}, \qquad \mathrm{RFib} := \mathrm{RHorn}^{\boxtimes}$$

are the **left fibrations** and **right fibrations**. Note that $\text{LFib} \cup \text{RFib} \subseteq \text{InnFib}$.

These classes correspond to each other under the opposite involution $(-)^{\mathrm{op}} \colon s\text{Set} \to s\text{Set}$; i.e., $\text{LHorn}^{\mathrm{op}} = \text{RHorn}$, $\text{LFib}^{\mathrm{op}} = \text{RFib}$.

25.2. Proposition. *Let C be a quasicategory and $x \in C_0$. The evident maps $C_{x/} \to C$ and $C_{/x} \to C$ which "forget x" are left fibration and right fibration respectively. In particular, $C_{x/}$ and $C_{/x}$ are also quasicategories.*

Proof. I claim that $\pi \colon C_{/x} \to C$ is a right fibration. Explicitly, this map sends the n-simplex $a \colon \Delta^n \to C_{/x}$, which corresponds to $\widetilde{a} \colon \Delta^n \star \Delta^0 \to C$ such that $\widetilde{a}|(\varnothing \star \Delta^0) = x$, to the n-simplex $\widetilde{a}|(\Delta^n \star \varnothing) \to C$. Using the join/slice adjunction, there is a bijective correspondence between lifting problems

$$
\begin{array}{ccc}
\Lambda^n_j \xrightarrow{\;f\;} C_{/x} & & \\
\Big\downarrow \qquad \nearrow \Big\downarrow{\scriptstyle \pi} & \Longleftrightarrow & \\
\Delta^n \xrightarrow{\;g\;} C & &
\end{array}
\qquad
\begin{array}{c}
\varnothing \star \Delta^0 \rightarrowtail (\Lambda^n_j \star \Delta^0) \cup_{\Lambda^n_j \star \varnothing} (\Delta^n \star \varnothing) \xrightarrow{\;(\widetilde{f},g)\;} C \\[1mm]
\quad \overset{x}{\frown} \qquad \Big\downarrow \qquad \qquad \nearrow \\
\Delta^n \star \Delta^0
\end{array}
$$

The right hand diagram is isomorphic to

$$
\begin{array}{c}
\{n+1\} \rightarrowtail \Lambda^{n+1}_j \xrightarrow{\;\;} C \\[1mm]
\overset{x}{\frown} \qquad \Big\downarrow \qquad \nearrow \\
\Delta^{n+1}
\end{array}
$$

If C is a quasicategory, then an extension exists for $0 < j \le n$.

Since right fibrations are inner fibrations, the composite $C_{/x} \to C \to *$ is an inner fibration, and thus $C_{/x}$ is a quasicategory.

The case of $C_{x/} \to C$ is similar, using the correspondence

$$
\begin{array}{ccc}
\Lambda^n_j \xrightarrow{\;\;} C_{x/} & & \\
\Big\downarrow \qquad \nearrow \Big\downarrow{\scriptstyle \pi} & \Longleftrightarrow & \\
\Delta^n \xrightarrow{\;\;} C & &
\end{array}
\qquad
\begin{array}{c}
\{0\} \rightarrowtail \Lambda^{n+1}_{j+1} \xrightarrow{\;\;} C \\[1mm]
\overset{x}{\frown} \qquad \Big\downarrow \qquad \nearrow \\
\Delta^{n+1}
\end{array}
$$

$\square$

25.3. Initial and terminal objects. An **initial object** of a quasicategory C is an $x \in C_0$ such that every $f \colon \partial\Delta^n \to C$ (for all $n \ge 1$) such that $f|\{0\} = x$, there exists an extension $f' \colon \Delta^n \to C$.

A **terminal object** of C is an initial object of C^{op}. That is, a $y \in C_0$ such that every $f \colon \partial\Delta^n \to C$ with $f|\{n\} = y$ extends to Δ^n.

$$
\begin{array}{c}
\{0\} \rightarrowtail \partial\Delta^n \xrightarrow{\;\;} C \\[1mm]
\overset{x}{\frown} \qquad \Big\downarrow \qquad \nearrow \\
\Delta^n \qquad {\scriptstyle \exists}
\end{array}
\qquad\qquad
\begin{array}{c}
\{n\} \rightarrowtail \partial\Delta^n \xrightarrow{\;\;} C \\[1mm]
\overset{y}{\frown} \qquad \Big\downarrow \qquad \nearrow \\
\Delta^n \qquad {\scriptstyle \exists}
\end{array}
$$

If C is the nerve of a category, then these definitions turn out to coincide with the usual notion of initial and terminal objects in a category. For instance, consider the definition of initial object applied to $x \in C_0$.

- The condition for $n = 1$ says that for every object y in C there exists $f \colon x \to y$,

- The condition for $n = 2$ says that for every triple of maps $f\colon x \to y$, $g\colon y \to z$, and $h\colon x \to z$, we must have $h = gf$. In particular (taking $f = 1_x$), we see there is at most one map from x to any object.
- The conditions for $n \geq 3$ are trivially satisfied.

We can characterize the property of being initial or terminal in terms of the "forgetful" functor from the slice.

25.4. Proposition. *If C is a quasicategory, then $x \in C_0$ is initial if and only if $C_{x/} \to C$ is a trivial fibration, and terminal if and only if $C_{/x} \to C$ is a trivial fibration.*

Proof. Left as an exercise. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

25.5. Remark. This implies that if x is initial, then $C_{x/} \to C$ is a categorical equivalence. Later (**??**) we'll be able to show the converse: if $C_{x/} \to C$ is a categorical equivalence, then x is initial.

A crucial fact about initial and terminal objects in a category is that they are *unique up to unique isomoprhism.* One way to formulate this is as follows: given a category C, let $C^{\text{init}} \subseteq C$ be the full subcategory spanned by the initial objects. Then one of two cases applies: either there are no initial objects, so C^{init} is empty, or there is at least one initial object, and C^{init} is equivalent to the terminal category $[0]$.

This leads to an analogous formulation for quasicategories.

25.6. Proposition. *Let C be a quasicategory. Let C^{init} and C^{term} denote respectively the full subquasicategories spanned by initial objects and terminal objects. Then (i) either C^{init} is empty or is categorically equivalent to the terminal quasicategory Δ^0, and (ii) either C^{term} is empty or is categorically equivalent to the terminal quasicategory Δ^0.*

Proof. Since $C^{\text{term}} = ((C^{\text{op}})^{\text{init}})^{\text{op}}$, we just need to consider the case of initial objects. By definition of initial object, any $f\colon \partial\Delta^n \to C^{\text{init}}$ with $n \geq 1$ can be extended to $g\colon \Delta^n \to C$, and the image of g must lie in the full subquasicategory C^{init} since all of its vertices do. If $C^{\text{init}} \neq \varnothing$, then this extension condition also holds for $n = 0$, whence $C \to \Delta^0$ is a trivial fibration, and thus C is categorically equivalent to Δ^0 by (20.1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

There are some seemingly obvious facts about initial objects that we can't prove just yet.

D. Deferred Proposition.

(1) *Given an object x in a quasicategory C, the object i_x in the slice $C_{x/}$ corresponding to the edge $1_x \in C_1$ is an initial object of $C_{x/}$.*

(2) *Any object in a quasicategory isomorphic to an initial object is also an initial object.*

Proofs will be given in (**??**).

25.7. Initial and terminal objects in functor categories. A functor whose values are all initial (or terminal) objects is itself initial (or terminal) in its functor category.

25.8. Proposition. *Consider a map $f\colon X \to C$ from a simplicial set to a category. Suppose that for every vertex $x \in X_0$ the object $f(x) \in C_0$ is initial (resp. terminal) in C. Then the functor f is initial (resp. terminal) viewed as an object of $\text{Fun}(X, C)$.*

As a consquence, if C has an initial (or terminal) object c_0, then $\text{Fun}(X, C)$ also has an initial (or terminal) object, e.g., the fubnctor which is the composite of $X \to \{c_0\} \to C$.

Proof. Assume $f(x) \in C_0$ is initial for all $x \in X_0$. Suppose given $g\colon \partial\Delta^n \to \text{Fun}(X, C)$ with $n \geq 1$ and $g|\{0\} = f$. We want to show that there exists an extension $g'\colon \Delta^n \to \text{Fun}(X, C)$ of g. We

convert this to the adjoint lifting problem:

$$\xymatrix{ \{0\} \times X \ar[r] & \partial\Delta^n \times X \ar[r]^u & C \\ & \Delta^n \times X & }$$

with f the curved arrow from $\{0\} \times X$ to C.

Consider the skeletal filtration $F_k = (\partial\Delta^n \times X) \cup \mathrm{Sk}_k(\Delta^n \times X)$, $k \geq 0$ (15.16) of the inclusion $\partial\Delta^n \times X \to \Delta^n \times X$. Note that any k-simplex $h = (a, b)\colon \Delta^k \to \Delta^n \times X$ which is not contained in in the subcomplex $\partial\Delta^n \times X$ must necessarily be such that $a\colon \Delta^k \to \Delta^n$ is surjective, whence a takes the vertex $0 \in (\Delta^k)_0$ to $0 \in (\Delta^n)_0$, and therefore $uh(0) = u(0, b(0)) = f(0, b(0))$ is an initial object in C by hypothesis. Thus we can construct the desired lift by inductively extending u to each F_k: in

$$\xymatrix{ \coprod \partial\Delta^k \ar[r]^{(h|\partial\Delta^k)} \ar@{^{(}->}[d] & F_{k-1} \ar[r]^{u|F_{k-1}} \ar@{^{(}->}[d] & C \\ \coprod \Delta^k \ar[r]_{(h)} & F_k \ar@{-->}[ur] & }$$

the square is a pushout and for each $h\colon \Delta^k \to F_k$ we have that $(hu)|\partial\Delta^k$ sends the vertex 0 to an initial object of C. $\qquad\square$

26. Joins and slices in lifting problems

Recall that for an object x in a quasicategory C, the slice objects $C_{x/}$ and $C_{/x}$ are also quasicategories. It turns out that the conclusion remains true for more general kinds of slices of quasicategories.

26.1. Proposition. *Let $p\colon S \to C$ be a map of simplicial sets, and suppose C is a quasicategory. Then both $C_{p/}$ and $C_{/p}$ are quasicategories.*

The proof is just like that of (25.2): we will show below (26.13) that $C_{p/} \to C$ is a left fibration and $C_{/p} \to C$ is a right fibration.

To set this up, we need a little technology about how joins interact with lifting problems.

26.2. Pushout-joins. We define an analogue of the box-product for the the join. Given maps $i\colon A \to B$ and $j\colon K \to L$ of simplicial sets, the **pushout-join** (or **box-join**) $i \boxast j$ is the map

$$i \boxast j\colon (A \star L) \amalg_{A\star K} (B \star K) \xrightarrow{(i\star L, B\star j)} B \star L.$$

Warning: unlike the pushout-product, the pushout-join is not symmetric: $i \boxast j \neq j \boxast i$. o

26.3. *Example.* We have already observed examples of pushout-joins in the proof of (25.2), namely

$$(\Lambda^n_j \subset \Delta^n) \boxast (\varnothing \subset \Delta^0) \approx (\Lambda^{n+1}_j \subset \Delta^{n+1}), \quad (\varnothing \subset \Delta^0) \boxast (\Lambda^n_j \subset \Delta^n) \approx (\Lambda^{1+n}_{1+j} \subset \Delta^{1+n}).$$

A straightforward calculation shows that the pushout-join of a horn with a cell is always a horn:

$$(\Lambda^n_j \subset \Delta^n) \boxast (\partial\Delta^k \subset \Delta^k) \approx (\Lambda^{n+1+k}_j \subset \Delta^{n+1+k}),$$

$$(\partial\Delta^k \subset \Delta^k) \boxast (\Lambda^n_j \subset \Delta^n) \approx (\Lambda^{k+1+n}_{k+1+j} \subset \Delta^{k+1+n}).$$

Also, the pushout-join of a cell with a cell is always a cell:

$$(\partial\Delta^n \subset \Delta^n) \boxast (\partial\Delta^k \subset \Delta^k) \approx (\partial\Delta^{n+1+k} \subset \Delta^{n+1+k}).$$

26.4. *Exercise.* Prove the isomorphisms asserted in (26.3).

26.5. *Remark.* Both pushout-product and pushout-join are special cases of a general construction: given any functor $F\colon s\mathrm{Set} \times s\mathrm{Set} \to s\mathrm{Set}$ of two variables, you get a corresponding "pushout-F" functor: $F^{\square}\colon \mathrm{Fun}([1], s\mathrm{Set}) \times \mathrm{Fun}([1], s\mathrm{Set}) \to \mathrm{Fun}([1], s\mathrm{Set})$. We will meet more examples later.

26.6. Pullback-slices. Just as the pushout-product is associated to the pullback-power, so the pushout-join is associated to two kinds of **pullback-slices** (or **box-slices**). Given maps $K \xrightarrow{j} L \xrightarrow{q} X \xrightarrow{h} Y$, we define the map

$$h^{\boxplus_q j}\colon X_{/q} \to X_{/qj} \times_{Y_{/hqj}} Y_{/hq},$$

where the components $X_{/q} \to X_{/qj}$ and $X_{/q} \to Y_{/hq}$ are the evident maps constructed by "restricting" along $p\colon K \to L$ in the first case, and "composing" with $h\colon X \to Y$ in the second case.

Similarly, given maps $A \xrightarrow{i} B \xrightarrow{p} X \xrightarrow{h} Y$ we have

$$h^{i \boxplus p}\colon X_{p/} \to X_{pi/} \times_{Y_{hpi/}} Y_{hp/}.$$

26.7. *Remark.* When $Y = *$, these pullback-slice maps are just the restriction maps $X_{/q} \to X_{/qj}$ and $X_{p/} \to X_{pi/}$.

26.8. Joins, slices, and lifting problems. The pushout-join and pullback-slice interact with lifting problems in much the same way that pushout-product and pullback-power do.

26.9. Proposition. *Given $i\colon A \to B$, $j\colon K \to L$, and $h\colon X \to Y$, the following are equivalent.*

(1) $(i \boxplus j) \boxtimes h$.
(2) $i \boxtimes (h^{\boxplus_q j})$ *for all $q\colon L \to X$.*
(3) $j \boxtimes (h^{i \boxplus p})$ *for all $p\colon B \to X$.*

Proof. A straightforward exercise. The equivalence of (1) and (2) is

$\square$

26.10. Proposition. *Let S and T be sets of maps in sSet. Then $\overline{S} \boxplus \overline{T} \subseteq \overline{S \boxplus T}$.*

Proof. This is formal and nearly identical to the proof of the saturation result for box-products (16.8). $\square$

26.11. Proposition. *We have*

$$\overline{\mathrm{Cell}} \boxplus \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}, \qquad \overline{\mathrm{RHorn}} \boxplus \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}, \qquad and \qquad \overline{\mathrm{Cell}} \boxplus \overline{\mathrm{LHorn}} \subseteq \overline{\mathrm{InnHorn}}.$$

Proof. Use (26.3). $\square$

26.12. Corollary. *Given $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$, consider the pullback-slice maps*

$$\ell\colon X_{p/} \to X_{pj/} \times_{Y_{fpj/}} Y_{fp/}, \qquad r\colon X_{/p} \to X_{/pj} \times_{Y_{/fpj}} Y_{/fp}.$$

We have the following.

(1) $j \in \overline{\mathrm{Cell}}$, $f \in \mathrm{TFib}$ *implies $\ell, r \in \mathrm{TFib}$.*
(2) $j \in \overline{\mathrm{Cell}}$, $f \in \mathrm{InnFib}$ *implies $\ell \in \mathrm{LFib}$, $r \in \mathrm{RFib}$.*
(3) $j \in \overline{\mathrm{RHorn}}$, $f \in \mathrm{InnFib}$ *implies $\ell \in \mathrm{TFib}$.*
(4) $j \in \overline{\mathrm{LHorn}}$, $f \in \mathrm{InnFib}$ *implies $r \in \mathrm{TFib}$.*

Proof. Exercise, using (26.11).   □

We are mostly interested in special cases when $X = C$ is a quasicategory, and $Y = *$.

26.13. Corollary. *Given $T \xrightarrow{j} S \xrightarrow{p} C$ with C a quasicategory and j a monomorphism, the induced map $C_{/p} \to C_{/pj}$ is a right fibration, and $C_{p/} \to C_{pj/}$ is a left fibration. In particular, $C_{/p} \to C$ is a right fibration and $C_{p/} \to C$ is a left fibration (case $T = \varnothing$).*

26.14. Corollary. *Given $T \xrightarrow{j} S \xrightarrow{p} C$ with C a quasicategory, if j is right anodyne then $C_{p/} \to C_{pj/}$ is a trivial fibration, while if j is left anodyne then $C_{/p} \to C_{/pj}$ is a trivial fibration.*

Another case we will need is when $T = \varnothing$.

26.15. Corollary. *Given $S \xrightarrow{p} X \xrightarrow{f} Y$ where f is a trivial fibration, all of the maps in*

$$X_{p/} \to X \times_Y Y_{fp/} \to Y_{fp/} \qquad and \qquad X_{/p} \to X \times_Y Y_{/fp} \to Y_{/fp}$$

are trivial fibrations.

Proof. The two pullback-slice maps are trivial fibrations by (26.12). The projections are each base changes of the trivial fibration f, and so are trivial fibrations.   □

26.16. Composition functors for slices. Here is a nice consequence of the above. Let C be a quasicategory and $f : x \to y$ a morphism in it. We obtain two functors

$$C_{/x} \xleftarrow{p} C_{/f} \xrightarrow{q} C_{/y}$$

associated to the inclusions $\{0\} \subset \Delta^1 \supset \{1\}$. The first inclusion $\{0\} \subset \Delta^1$ is a left-horn inclusion, and thus by (26.14) the restriction map p is a trivial fibration, and hence we can choose a section $s : C_{/x} \to C_{/f}$ of p.

The resulting composite $qs : C_{/x} \to C_{/y}$ can be thought of as a functor realizing the operation which sends an object $(c \xrightarrow{g} x)$ of $C_{/x}$ to "the object" $(c \xrightarrow{fg} y)$ of $C_{/y}$ defined by "composing f and g" (but remember that such composition is not uniquely defined in a quasicategory C; the choice of section s gives a collection of such choices for all g.)

26.17. *Exercise.* Show that if C is a category, then p is an isomorphism, and that qs is precisely the functor $C_{/x} \to C_{/y}$ described above.

27. Limits and colimits in quasicategories

27.1. Definition of limits and colimits. Now we can define the notion of a limit and colimit of a functor between quasicategories (and in fact of a map from a simplicial set to a quasicategory). Given a map $p : K \to C$ where C is a quasicategory, a **colimit** of p is defined to be an initial object of the slice quasicategory $C_{p/}$. Explicitly, a colimit of $p : K \to C$ is a map $\widetilde{p} : K \star \Delta^0 = K^{\triangleright} \to C$ extending p, such that for $n \geq 1$ a lift exists in every diagram of the form

Sometimes it is better to call $\widetilde{p}$ a **colimit cone**, in which case the restriction $\widetilde{p}|\varnothing \star \Delta^0$ to the cone point is an object in C which can be called a "colimit of p".

Similarly, a **limit** of p is a terminal object of $C_{/p}$; explicitly, this is a map $\widetilde{p}\colon \Delta^0 \star K = K^{\triangleleft} \to C$ extending p such that for $n \geq 1$ a lift exists in every

$$
\begin{array}{ccc}
\Delta^{\{n\}} \star K & \xrightarrow{\quad} \partial\Delta^n \star K \xrightarrow{\ \widetilde{p}\ } C & \\
\downarrow & & \nearrow \\
& \Delta^n \star K &
\end{array}
$$

27.2. Example. Consider the empty simplicial set $K = \varnothing$. Then $C_{\varnothing/} = C$, so a colimit of $p\colon \varnothing \to C$ is precisely the same as an initial object of C. Likewise, a limit of p is precisely the same as a terminal object of C.

27.3. Example. Consider $K = \Lambda_0^2$, which is the nerve of a category which we can draw as the picture $(1 \leftarrow 0 \to 2)$. Then $(\Lambda_0^2)^{\triangleright} \approx \Delta^1 \times \Delta^1$ is also an ordinary category; explicitly it has the form of a commutative square

$$
\begin{array}{ccc}
0 & \longrightarrow & 1 \\
\downarrow & \searrow & \downarrow \\
2 & \longrightarrow & v
\end{array}
$$

where v is the "cone vertex". A colimit cone $(\Lambda_0^2)^{\triangleright} \to C$ is called a **pushout diagram** in C.

Similar considerations give $(\Lambda_2^2)^{\triangleleft} \approx \Delta^1 \times \Delta^1$; a limit cone $(\Lambda_2^2)^{\triangleleft} \to C$ is called a **pullback diagram** in C.

27.4. Exercise. Let $C' \subseteq C$ be an inclusion of a full subquasicategory. Show that if $p\colon K \to C'$ has a colimit $\widetilde{p}$ in C, and if the image of $\widetilde{p}$ is contained in C', then $\widetilde{p}$ is in fact a colimit of p in C'.

27.5. Uniqueness of limits and colimits. Limits and colimits are unique if they exist.

27.6. Proposition. *Let $p\colon K \to C$ be a map to a quasicategory, and let $(C_{p/})^{\mathrm{colim}} \subseteq C_{p/}$ and $(C_{/p})^{\mathrm{lim}} \subseteq C_{/p}$ denote the full subquasicategories spanned by colimit cones and limit cones respectively. Then (i) either $(C_{p/})^{\mathrm{colim}}$ is empty or is categorically equivalent to Δ^0, and (ii) either $(C_{/p})^{\mathrm{lim}}$ is empty or is categorically equivalent to Δ^0.*

Proof. This is just the uniqueness of initial and terminal objects (25.6). $\qquad\square$

We have noted above (25.4) that an object x in a quasicategory C is initial iff $C_{x/} \to C$ is a trivial fibration, and terminal iff $C_{/x} \to C$ is a trivial fibration. There is a similar characterization of limit and colimit cones.

27.7. Proposition. *Let C be a quasicategory. Let $\widetilde{p}\colon K^{\triangleright} \to C$ be a map, and write $p := \widetilde{p}|K$. Then $\widetilde{p}$ is a colimit diagram if and only if $C_{\widetilde{p}/} \to C_{p/}$ is a trivial fibration.*

Likewise, let $\widetilde{q}\colon K^{\triangleleft} \to C$ be a map, and write $q := \widetilde{q}|K$. Then $\widetilde{q}$ is a limit diagram if and only if $C_{/\widetilde{q}} \to C_{/q}$ is a trivial fibration.

Proof. I'll just do the case of colimits.

We make an elementary observation: $(C_{p/})_{\widetilde{p}/} \approx C_{\widetilde{p}/}$. In this isomorphism, the symbol "$\widetilde{p}$" refers to both a morphism $\widetilde{p}\colon K^{\triangleright} \to C_{p/}$ and the corresponding object $\widetilde{p} \in (C_{p/})_0$. The point is that in either simplicial set, a k-simplex corresponds to a map $K \star \Delta^0 \star \Delta^k \to C$ which restricts to $\widetilde{p}$ on $K \star \Delta^0 \star \varnothing$.

Using this, the statement amounts to the special case for initial and terminal objects (25.4). $\quad\square$

27.8. Limits and colimits in slices. Given a map $p\colon S \to C$ to a quasicategory, we have "forgetful functors" $C_{/p} \to C$ and $C_{p/} \to C$ from the slices to C. The following proposition says that colimits in $C_{/p}$ or limits in $C_{p/}$ can be "computed in the underlying quasicategory" C (if the corresponding colimit or limit in C exists).

27.9. Proposition. *Let C be a quasicategory, $p\colon S \to C$ a map of simplicial sets, and K a simplicial set.*

(1) *A functor $\widetilde{f}\colon K^{\triangleright} \to C_{/p}$ is a colimit cone if and only the composite $K^{\triangleright} \xrightarrow{\widetilde{f}} C_{/p} \to C$ is a colimit cone. Furthermore, if $f\colon K \to C_{/p}$ is such that the composite $K \to C_{/p} \to C$ has a colimit in C, then f has a colimit in $C_{/p}$.*

(2) *A functor $\widetilde{f}\colon K^{\triangleleft} \to C_{p/}$ is a limit cone if and only the composite $K^{\triangleleft} \xrightarrow{\widetilde{f}} C_{p/} \to C$ is a limit cone. Furthermore, if $f\colon K \to C_{p/}$ is such that the composite $K \to C_{p/} \to C$ has a limit in C, then f has a limit in $C_{p/}$.*

(3) *A functor $\widetilde{f}\colon K^{\triangleleft} \to C_{p/}$ is a limit cone if and only if $\pi\widetilde{f}\colon K \to C$ is a limit cone. Furthermore, if $f\colon K \to C_{p/}$ is such that πf has a limit, then f has a limit.*

27.10. Exercise. Fix a map $g\colon S \star T \to C$. Write $q = g|S\colon S \to C$ and $p = g|T\colon T \to C$, and observe that g is adjoint to maps $g'\colon T \to C_{q/}$ and $g''\colon S \to C_{/p}$. Construct an isomorphism

$$\left(C_{q/}\right)_{/g'} \approx \left(C_{/p}\right)_{g''/}.$$

Proof. (See [Lur09, 1.2.13.8].) We will just prove (1). First consider $\widetilde{f}\colon K^{\triangleright} \to C_{/p}$, and let $f = \widetilde{f}|K$. Using the exercise (27.10), we see that the forgetful map $(C_{/p})_{\widetilde{f}/} \to (C_{/p})_{f/}$ is isomorphic to the map

$$(C_{\widetilde{f_0}/})_{/p'} \to (C_{f_0/})_{\pi p'/}$$

induced by the sequence $S \xrightarrow{p'} C_{\widetilde{f_0}/} \xrightarrow{\pi} C_{f_0/}$, where $\widetilde{f_0}$ and f_0 are the composites $K^{\triangleright} \xrightarrow{\widetilde{f}} C_{/p} \to C$ and $K \xrightarrow{f} C_{/p} \to C$, and p' is adjoint to $\widetilde{f}$.

We will just prove (1). First, consider $\widetilde{f}\colon K^{\triangleright} \to C_{/p}$ such that $\pi\widetilde{f}$ is a colimit cone in C. To show that $\widetilde{f}$ is a colimit cone for $f = \widetilde{f}|K$, it suffices to show produce a lift in any diagram of the form

$$
\begin{array}{ccc}
& \widetilde{f} & \\
K \star \Delta^0 \star \varnothing \longrightarrow (K \star \Delta^0 \star \partial\Delta^n) \cup (K \star \varnothing \star \Delta^n) \longrightarrow C_{/p} \\
\downarrow & \nearrow \\
K \star \Delta^0 \star \Delta^n &
\end{array}
$$

According to the definition of slice, this is equivalent to producing a lift in any diagram of the form

$$
\begin{array}{ccc}
& (\pi\widetilde{f}, p) & \\
(K \star \Delta^0 \star \varnothing \star \varnothing) \amalg (\varnothing \star \varnothing \star \varnothing \star S) \longrightarrow (K \star \Delta^0 \star \partial\Delta^n \star S) \cup (K \star \varnothing \star \Delta^n \star S) \longrightarrow C \\
\downarrow & \nearrow \\
K \star \Delta^0 \star \Delta^n \star S &
\end{array}
$$

which in turn is equivalent to producing a lift in

$$
\begin{array}{ccccc}
\varnothing \star S & \longrightarrow & \partial\Delta^n \star S & \overset{p}{\longrightarrow} & C_{\pi\widetilde{f}/} \\
& & \Big\downarrow & \nearrow & \Big\downarrow q \\
& & \Delta^n \star S & \longrightarrow & C_{\pi f/}
\end{array}
$$

Because $\pi\widetilde{f}$ is a colimit, we know q is a trivial fibration (27.7). It is clear that $\partial\Delta^n \star S \subset \Delta^n \star S$ is a monomorphism (26.11), so the desired lift exists. We have shown that if $\pi\widetilde{f}$ is a colimit then $\widetilde{f}$ is a colimit.

Next suppose given $f\colon K \to C_{p/}$ and a colimit $\widetilde{f}_0\colon K^{\triangleright} \to C$ of $\pi f\colon K \to C$. Then (27.7) $C_{\widetilde{f}_0/} \to C_{\pi f/} \cdots$

$\square$

28. The Joyal extension and lifting theorems

We are now at the point where we can state and prove Joyal's theorems about extending or lifting maps along outer horns. This will allow us to prove many of the results whose proofs we have deferred up to now.

28.1. **Joyal extension theorem.** The following gives a condition for extending maps from *outer horns* into a quasicategory.

28.2. **Theorem** (Joyal extension). [Joy02, Thm. 1.3] *Let C be a quasicategory, and fix a map $f\colon \Delta^1 \to C$. The following are equivalent.*

(1) *The edge represented by f is an isomorphism in C.*
(2) *Every $a\colon \Lambda_0^n \to C$ with $n \geq 2$ such that $f = a|\Delta^{\{0,1\}}\colon \Delta^1 \to C$ admits an extension to a map $\Delta^n \to C$.*
(3) *Every $b\colon \Lambda_n^n \to C$ with $n \geq 2$ such that $f = b|\Delta^{\{n-1,n\}}\colon \Delta^1 \to C$ admits an extension to a map $\Delta^n \to C$.*

I'll call $\langle 01 \rangle \in \Delta^n$ the **leading edge**, and $\langle n-1, n \rangle \in \Delta^n$ the **trailing edge**. Thus, the implications $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$ say that we can always extend $\Lambda_0^n \to C$ to an n-simplex if the *leading edge* goes to an isomorphism in C, and extend $\Lambda_n^n \to C$ to an n-simplex if the *trailing edge* goes to an isomorphism in C.

The implications $(2) \Rightarrow (1)$ and $(3) \Rightarrow (1)$ are easy, and are left as an exercise.

28.3. *Exercise.* Suppose C is a quasicategory with edge $f \in C_1$, and suppose that every map $a\colon \Lambda_0^n \to C$ with $n \in \{2, 3\}$ and $f = a|\Delta^{\{0,1\}}$ admits an extension along $\Lambda_0^n \subset \Delta^n$. Prove that f is an isomorphism.

The non-trivial implications of Joyal extension will lead to proofs of the deferred propositions (A), (C), and (D).

The proof of the Joyal extension theorem will be an application of the fact that left fibrations and right fibrations are *conservative isofibrations*.

28.4. **Conservative functors.** A functor $p\colon C \to D$ between categories is **conservative** if whenever f is a morphism in C such that $p(f)$ is an isomorphism in D, then f is an isomorphism in C. The definition of a conservative functor between *quasicategories* is precisely the same.

28.5. **Proposition.** *All left fibrations and right fibrations between quasicategories are conservative.*

Proof. Consider a right fibration $p\colon C \to D$, and a morphism $f\colon x \to y$ in C such that $p(f)$ is an isomorphism. We first show that f admits a preinverse.

Let $a\colon \Lambda_2^2 \to C$ such that $a_{12} = f$ and $a_{02} = 1_y$. Let $b\colon \Delta^2 \to C$ be any 2-simplex exhibiting a preinverse of $p(f)$, i.e., such that $b_{12} = p(f)$ and $b_{02} = 1_{p(y)}$, so that b_{01} is a preinverse. Now have a diagram with a lift

$$
\begin{array}{ccc}
\Lambda_2^2 & \xrightarrow{\ a\ } & C \\
\downarrow & \nearrow & \downarrow p \\
\Delta^2 & \xrightarrow{\ b\ } & D
\end{array}
$$

which exhibits a preinverse of f, which we will call g.

Because $p(f)$ was assumed invertible, its preinverse $p(g)$ is also invertible, and therefore by the above argument g admits a preinverse as well. We conclude that f is invertible by (11.2). □

28.6. Isofibrations. We say that a functor $p\colon C \to D$ of quasicategories is an **isofibration**[20] if

    (1) p is an inner fibration, and

    (2) we have "isomorphism lifting" along p. That is, for any $c \in C_0$ and *isomorphism* $g\colon p(c) \to d'$, there exists a $c' \in C_0$ and *isomorphism* $f\colon c \to c'$ such that $p(f) = g$.

Recall that if C and D are nerves of ordinary categories, then any functor $C \to D$ is an inner fibration. Thus in the case of ordinary categories, being an isofibration amounts to condition (2) only. Also, it is clear that in the case of ordinary categories we can replace (2) with the dual condition

    (2') for any $c \in C_0$ and *isomorphism* $g'\colon d' \to p(c)$, there exists a $c' \in C_0$ and isomorphism $f'\colon c' \to c$ such that $p(f) = g$.

To prove (2) from (2'), just apply condition (2') to the (unique) inverse of g.

The symmetry between (2) and (2') also holds for functors between quasicategories, by the following.

28.7. Proposition. *An inner fibration $p\colon C \to D$ between quasicategories is an isofibration if and only if $h(p)\colon h(C) \to h(D)$ is an isofibration of ordinary categories.*

Proof. ($\Longrightarrow$) Straightforward. ($\Longleftarrow$) Suppose given an isomophism $g\colon p(c) \to d'$ in D. If $h(p)$ is an isofibration, there exists an isomorphism $f'\colon c \to c'$ in C such that $p(f') \sim_r g$. Now choose a lift in

$$
\begin{array}{ccc}
\Lambda_1^2 & \xrightarrow{\ a\ } & C \\
\downarrow & \nearrow{\scriptstyle s} & \downarrow p \\
\Delta^2 & \xrightarrow{\ b\ } & D
\end{array}
$$

where b exhibits $p(f') \sim_r g$ and $a(\langle 01 \rangle) = f'$ and $a(\langle 12 \rangle) = 1_{c'}$. The edge $f = s_{02}$ is a lift of g, and is an isomorphism since $f' \sim_r f$. □

28.8. *Exercise.* (i) Let Group denote the category of groups, whose objects are pairs $G = (S, \mu)$ consisting of a set S and a function $\mu\colon S \times S \to S$ satisfying a well-known list of axioms. Show that the functor $U\colon \text{Group} \to \text{Set}$ which on objects sends $(S, \mu) \mapsto S$ is an isofibration between ordinary categories.

(ii) Consider the functor $U'\colon \text{Group} \to \text{Set}$ defined on objects by $G \mapsto \text{Hom}(\mathbb{Z}, G)$. Explain why, although U' is naturally isomorphic to U, you don't know how to show whether U' is an isofibration without explicit reference to the axioms of your set theory. The moral is that the property of being an isofibration is not "natural isomorphism invariant".

---

[20]Joyal uses the term "quasifibration" in [Joy02]. Later in [Joy08a] this is called a "pseudofibration". Lurie uses this notion, but never names it. The term "isofibration" is used by Riehl and Verity [RV15].

28.9. Left and right fibrations are isofibrations.

28.10. Proposition. *All left fibrations and right fibrations between quasicategories are isofibrations.*

Proof. Suppose $p\colon C \to D$ is a right fibration (and hence an inner fibration) between quasicategories, and consider

$$
\begin{array}{ccc}
\{1\} & \longrightarrow & C \\
\downarrow & \nearrow\!\!\!\!{}^{f} & \downarrow p \\
\Delta^1 & \xrightarrow{\ g\ } & D
\end{array}
$$

where g represents an isomorphism. Because p is a right fibration, there exists a lift f. Because right fibrations are conservative, f represents an isomorphism. $\qquad\square$

Note that the above proof checked explicitly isofibration condition (2') for right fibrations; thus, by symmetry we conclude that isofibration condition (2) holds for right fibrations. It seems difficult to give an elementary *direct* proof that right-fibrations satisfy (2).

28.11. Proof of the Joyal extension theorem.

Proof of (28.2). We prove (1) $\Rightarrow$ (2). Suppose given $a\colon \Lambda_0^n \to C$ such that $f = a|\Delta^{\{0,1\}}$ represents an isomorphism. Using the join/slice adjunction applied to

$$
(\Lambda_0^n \subset \Delta^n) \approx (\{0\} \subset \Delta^1) \boxast (\partial\Delta^{n-2} \subset \Delta^{n-2}),
$$

there is a correspondence of lifting problems

$$
\begin{array}{ccc}
& f & \\
\Delta^{\{0,1\}} \rightarrowtail \Lambda_0^n \xrightarrow{\ a\ } C & & \\
\downarrow & \nearrow & \\
\Delta^n & &
\end{array}
\qquad\Longleftrightarrow\qquad
\begin{array}{ccc}
\Delta^{\{0\}} & \xrightarrow{\ h\ } & C_{/(a|\Delta^{\{2,\dots,n\}})} \\
\downarrow & \nearrow & \downarrow q \\
\Delta^{\{0,1\}} & \xrightarrow{\ g\ } & C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \\
& {}_{f}\searrow & \downarrow p \\
& & C
\end{array}
$$

where g is adjoint to $a|(\Delta^{\{0,1\}} \star \partial\Delta^{\{2,\dots,n\}})$, and h is adjoint to $a|(\Delta^{\{0\}} \star \Delta^{\{2,\dots,n\}})$. Because C is a quasicategory, both p and q are right fibrations (26.13), and therefore are conservative isofibrations (28.5), (28.10). Thus since f represents an isomorphism, so does g since p is conservative, and therefore a lift exists since q is an isofibration.

The proof of (2) $\Longrightarrow$ (1) is left as an exercise (28.3). The proof of (1) $\Longleftrightarrow$ (3) is similar. $\qquad\square$

28.12. The Joyal lifting theorem. There is a relative generalizaton.

28.13. Theorem (Joyal lifting). *Let $p\colon C \to D$ be an inner fibration between quasicategories, and let $f \in C_1$ be an edge such that $p(f)$ is an isomorphism in D. The following are equivalent.*

(1) *The edge f is an isomorphism in C.*
(2) *For all $n \geq 2$, every diagram of the form*

$$
\begin{array}{ccc}
& f & \\
\Delta^{\{0,1\}} \rightarrowtail \Lambda_0^n \longrightarrow & C & \\
\downarrow & \nearrow & \downarrow p \\
\Delta^n & \longrightarrow & D
\end{array}
$$

admits a lift.

(3) *For all $n \geq 2$, every diagram of the form*

$$
\begin{array}{ccc}
\Delta^{\{n-1,n\}} \xrightarrow{\;\;\;} \Lambda^n_n \xrightarrow{\;\;f\;\;} C \\
\Big\downarrow \qquad \Big\downarrow \qquad \Big\downarrow p \\
\Delta^n \xrightarrow{\;\;\;} D
\end{array}
$$

admits a lift.

Proof. The implications $(2) \Rightarrow (1)$ and $(3) \Rightarrow (1)$ are elementary, as in (28.3).

For $(1) \Rightarrow (2)$, the first step is to prove that

$$
C_{/(a|\Delta^{\{2,\ldots,n\}})} \xrightarrow{q} C_{/(a|\partial\Delta^{\{2,\ldots,n\}})} \times_{D_{/(pa|\partial\Delta^{\{2,\ldots,n\}})}} D_{/(pa|\Delta^{\{2,\ldots,n\}})} \xrightarrow{p} C
$$

are both right fibrations. For instance, the map q is the slice-power of the inner fibration p by a monomorphism, so is a right fibration by (26.12). The map p is the composite

$$
C_{/(a|\partial\Delta^{\{2,\ldots,n\}})} \times_{D_{/(pa|\partial\Delta^{\{2,\ldots,n\}})}} D_{/(pa|\Delta^{\{2,\ldots,n\}})} \xrightarrow{p'} C_{/(a|\partial\Delta^{\{2,\ldots,n\}})} \xrightarrow{p''} C,
$$

where p' is the base change of the right fibration $D_{/(pa|\Delta^2\{2,\ldots,n\})} \to D_{/(pa|\partial\Delta^2\{2,\ldots,n\})}$, and p'' is a right fibration (in both cases by (26.13)) Then the proof of $(1) \implies (2)$ proceeds exactly as in (28.2). $\square$

As a consequence of Joyal lifting is a characterization of conservative inner fibrations between quasicategories in terms of a lifting property.

28.14. **Corollary.** *Let $p\colon C \to D$ be an inner fibration between quasicategories. The following are equivalent.*

(1) *The functor p is conservative.*

(2) *For all $n \geq 2$, every diagram of the form*

$$
\begin{array}{ccc}
\Delta^{\{0,1\}} \xrightarrow{\;\;\;} \Lambda^n_0 \xrightarrow{\;\;f\;\;} C \\
\Big\downarrow \qquad \qquad \Big\downarrow p \\
\Delta^n \xrightarrow{\;\;\;} D
\end{array}
$$

such that $p(f)$ represents an isomorphism in D admits a lift.

(3) *For all $n \geq 2$, every diagram of the form*

$$
\begin{array}{ccc}
\Delta^{\{n-1,n\}} \xrightarrow{\;\;\;} \Lambda^n_n \xrightarrow{\;\;f\;\;} C \\
\Big\downarrow \qquad \qquad \Big\downarrow p \\
\Delta^n \xrightarrow{\;\;\;} D
\end{array}
$$

such that $p(f)$ represents an isomorphism in D admits a lift.

29. Applications of the Joyal extension theorem

We can now prove all the statements whose proofs we have deferred until now, as well as some others.

29.1. **Quasigroupoids are Kan complexs.** First we prove (A), the identification of quasi-groupoids with Kan complexes.

29.2. **Proposition.** *Every quasigroupoid is a Kan complex.*

Proof. In a quasigroupoid, the Joyal extension property applies to all maps from Λ_0^n and Λ_n^n, since every edge is an isomorphism. $\square$

From now on we will use terms "quasigroupoid" and "Kan complex" interchangeably.

29.3. **Invariance of slice categories.** Here is an equivalent reformulation of the Joyal extension theorem in terms of maps between slices.

29.4. *Exercise* (Reformulation of Joyal extension). If $f\colon x \to y$ is an edge in a quasicategory C, then the following are equivalent: (1) f is an isomorphism; (2) $C_{f/} \to C_{x/}$ is a trivial fibration; (3) $C_{/f} \to C_{/y}$ is a trivial fibration.

29.5. **Corollary.** *If $f\colon x \to y$ is an isomorphism in a quasicategory C, then $C_{x/}$ and $C_{y/}$ are categorically equivalent, and $C_{/x}$ and $C_{/y}$ are categorically equivalent.*

Proof. Consider $C_{x/} \xleftarrow{\pi} C_{f/} \xrightarrow{\rho} C_{y/}$. We have already observed (26.14) that $\rho \in \mathrm{TFib}$, since $\{1\} \subset \Delta^1$ is right anodyne. The reformulation of Joyal extension (29.4) implies that $\pi \in \mathrm{TFib}$ when f is an isomorphism. $\square$

29.6. **Initial objects.** Now we prove (D) about initial objects.

29.7. **Proposition.** *If x is an object in a quasicategory C, then the vertex $i_x \in (C_{x/})_0$ corresponding to $1_x \in C_1$ is an initial object of $C_{x/}$.*

Proof. The map $1_x\colon x \to x$ in C is obviously an isomorphism. We have a correspondence of lifting problems

and $(\Delta^0 \star \partial\Delta^n \subset \Delta^0 \star \Delta^n) \approx (\Lambda_0^{1+n} \subseteq \Delta^{1+n})$, so a lift exists by the Joyal extension theorem. $\square$

29.8. **Proposition.** *Any object in a quasicategory isomorphic to an initial object is also initial.*

Proof. Let x be an initial object in C, and let c be an object isomorphic to x. It is easy to see that x is initial in the homotopy category hC, and therefore c is initial in hC also. This has a useful consequence: *any* map between x and c must be an isomorphism in C.

To show that c is initial in C, we want to extend $a\colon \partial\Delta^n \to C$ with $a_0 = c$ to Δ^n. Consider the left-hand diagram in

Because x is initial, p is a trivial fibration (25.4), and so a lift s exists. On vertices y of C, the map s determines a choice, of a morphism $s(y)\colon x \to y$; by what we said above, $s(c)\colon x \to c$ must be an isomorphism if c is isomorphic to x.

Now consider the right-hand diagram, in which $\widetilde{s}$ is adjoint to s. Write $b := \widetilde{s}(\Delta^0 \star a)$, a map from the horn $\Delta^0 \star \partial\Delta^n$ to C, so that so $b_0 = x$ and $b|(\varnothing \star \partial\Delta^n) = a$. The leading edge $b|\Delta^0 \star \{0\}$ is the edge $s(c)\colon x \to c$ in C, and we have observed any such is an isomorphism. Thus a lift t exists in the right-hand diagram by the Joyal lifting theorem since $(\Delta^0 \star \partial\Delta^n \subset \Delta^0 \star \Delta^n) = (\Lambda_0^{1+n} \subset \Delta^{1+n})$. The face $t|(\varnothing \star \Delta^n)$ is the desired extension of a. $\square$

29.9. **The objectwise criterion for natural isomorphisms.** Next we prove (C), the fact that natural isomorphisms are the same as natural transformations which are isomorphisms *objectwise*. We can restate this as follows. Consider the map

$$\mathrm{Fun}(j, C)\colon \mathrm{Fun}(D, C) \to \mathrm{Fun}(\mathrm{Sk}_0 D, C) \approx \prod_{d \in D_0} C$$

induced by restriction along $j\colon \mathrm{Sk}_0 D \to D$. The statement of (C) amounts to showing that this $\mathrm{Fun}(j, C)$ is conservative. This is a special case of the following proposition.

29.10. **Proposition.** *Let $j\colon K \to L$ be a monomorphism of simplicial sets such that $j\colon K_0 \xrightarrow{\sim} L_0$ is a bijection. Then for every quasicategory C the restriction map $\mathrm{Fun}(j, C)\colon \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C)$ is conservative.*

The proof we give uses ideas from [Lur09, §3.1.1].

Proof. Note that $p = \mathrm{Fun}(j, C)$ is always an inner fibration between quasicategories (since j is a monomorphism and $\overline{\mathrm{InnHorn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$). Therefore we can apply the criterion of (26.14) to show p is conservative: we show that for every $n \geq 2$ and every diagram of the form

$$
\begin{array}{ccc}
\Delta^{\{0,1\}} \xrightarrow{\quad f \quad} \Lambda_0^n \longrightarrow & \mathrm{Fun}(L, C) \\
\downarrow \qquad\qquad \nearrow{\scriptstyle a} & \downarrow{\scriptstyle p = \mathrm{Fun}(j,C)} \\
\Delta^n \longrightarrow & \mathrm{Fun}(K, C)
\end{array}
$$

such that $p(f)$ represents an isomorphism in $\mathrm{Fun}(K, C)$, a lift a exists.

We reduce this lifting problem to a different one using a retraction. For $n \geq 2$ we can define maps

$$\Delta^n \xrightarrow{s} \Delta^1 \times \Delta^n \xrightarrow{r} \Delta^n$$

uniquely characterized by their effect on vertices: $s(x) = (1, x)$, and $r(0, 1) = 0$ and $r(x, y) = y$ if $(x, y) \neq (0, 1)$. We check that

- $rs = \mathrm{id}$,
- $s(\Lambda_0^n) = \{1\} \times \Lambda_0^n \subseteq (\{0\} \times \Delta^n) \cup (\Delta^1 \times \Lambda_0^n)$,
- $r(\{0\} \times \Delta^n) = \Delta^{[n] \setminus 1} \subseteq \Lambda_0^n$, and
- $r(\Delta^1 \times \Delta^{[n] \setminus j}) = \Delta^{[n] \setminus j}$ if $j \neq 0$, whence $r(\Delta^1 \times \Lambda_0^n) = \Lambda_0^n$.

Therefore we can form the solid arrow diagram

$$
\begin{array}{ccccc}
\Lambda_0^n \xrightarrow{\ s\ } (\{0\} \times \Delta^n) \cup (\Delta^1 \times \Lambda_0^n) \xrightarrow{\ r\ } \Lambda_0^n \xrightarrow{\ u\ } & \mathrm{Fun}(L, C) \\
\downarrow \qquad\qquad\qquad \downarrow \qquad {\scriptstyle b}\nearrow \quad \downarrow \quad {\scriptstyle a}\nearrow & \downarrow{\scriptstyle p = \mathrm{Fun}(j,C)} \\
\Delta^n \xrightarrow[\ s\ ]{} \Delta^1 \times \Delta^n \xrightarrow[\ r\ ]{} \Delta^n \longrightarrow & \mathrm{Fun}(K, C)
\end{array}
$$

and observe that to produce a lift a, it suffices to produce a map b which is a lift in its rectangle: given b, take $a = bs$.

Note that r sends each edge $\Delta^1 \times \{k\}$ in $(\{0\} \times \Delta^n) \cup (\Delta^1 \times \Lambda_0^n)$ to either: the degenerate edge $\langle kk \rangle$ in Λ_0^n (if $k \neq 1$), or the leading edge $\langle 01 \rangle$ in Λ_0^n (if $k = 1$). Thus for all $k \in [n]$ the restriction of pur to $\Delta^1 \times \{k\}$ represents an isomorphism in $\mathrm{Fun}(K, C)$.

By adjunction this amounts to showing that a lift exists in

$$
\begin{array}{ccc}
\{0\} & \longrightarrow & \mathrm{Fun}(T, C) \\
\downarrow & \nearrow & \downarrow {\scriptstyle \mathrm{Fun}(i,C)} \\
\Delta^1 & \xrightarrow{\ v\ } & \mathrm{Fun}(S, C)
\end{array}
$$

where $i\colon S \to T$ is the monomorphism $(\Lambda_0^n \times L) \cup (\Delta^n \times K) \to \Delta^n \times L$, and the edge in $\mathrm{Fun}(S, C)$ represented by v is such that for every $k \in [n]$ the restriction $v|\{k\} \times K$ represents an isomorphism in $\mathrm{Fun}(\{k\} \times K)$.

Note that $i\colon S \to T$ induces a bijection on vertices, and that the hypothesis on v implies that the restriction $v|\{x\}$ of v to every vertex $x \in S_0$ represents an isomorphism in $\mathrm{Fun}(\{x\}, C) \approx C$. Thus the proposition follows from the following lemma. $\qquad \square$

29.11. Lemma. *Let C be a quasicategory, and let $i\colon S \to T$ be a monomorphism such that $i\colon S_0 \to T_0$ is a bijection. Then for every diagram*

$$
\begin{array}{ccc}
\{0\} & \longrightarrow & \mathrm{Fun}(T, C) \\
\downarrow & \overset{t}{\nearrow} & \downarrow {\scriptstyle \mathrm{Fun}(i,C)} \\
\Delta^1 & \xrightarrow{\ v\ } & \mathrm{Fun}(S, C)
\end{array}
$$

such that v represents an objectwise natural isomorphism *of functors $S \to C$ (i.e., such that the image of v under $\mathrm{Fun}(S, C) \to \mathrm{Fun}(\{x\}, C) \approx C$ represents an isomorphism in C for all vertices $x \in S_0$), a lift t exists.*

Proof. First note that in the case that a lift t does exist, it necessarily represents a natural transformation between functors $T \to C$ which is an objectwise isomorphism, since $i\colon S_0 \to T_0$ is a bijection and we have assumed v is so.

Let $\mathcal{C}$ denote the class of monomorphisms $i\colon S \to T$ such that $i\colon S_0 \to T_0$ is a bijection, and such that the conclusion of the lemma holds. A straightforward exercise shows that $\mathcal{C}$ is a saturated class. (Note that for the cases of composition and transfinite composition, this requires the observation of the previous paragraph, which is why we need to assume here that elements of $\mathcal{C}$ induce a bijection on vertices.)

Let $\mathrm{Cell}_{\geq 1} = \{\partial \Delta^n \subset \Delta^n\}_{n \geq 1}$. From the skeletal filtration (15.16) we see that $\overline{\mathrm{Cell}_{\geq 1}}$ is precisely the class of monomorphisms which are bijections on vertices. Thus the proposition follows once we show that $\mathrm{Cell}_{\geq 1} \subseteq \mathcal{C}$, which follows from the following lemma applied to the case of $D = *$ and $(i, j) = (0, 0)$. $\qquad \square$

The following lemma is a kind of "pushout-product" version of Joyal lifting.

29.12. Lemma. *Suppose $p\colon C \to D$ is an inner fibration of quasicategories, and suppose $n \geq 1$, and either $(i, j) = (0, 0)$ or $(i, j) = (1, n)$. For any diagram*

$$
\begin{array}{ccc}
\Delta^1 \times \{j\} \overset{f}{\rightarrowtail} (\{i\} \times \Delta^n) \cup_{\{i\} \times \partial \Delta^n} (\Delta^1 \times \partial \Delta^n) & \longrightarrow & C \\
\downarrow & \nearrow & \downarrow {\scriptstyle p} \\
\Delta^1 \times \Delta^n & \longrightarrow & D
\end{array}
$$

such that f represents an isomorphism in C, a lift exists.

Proof. This is a calculation, given in the appendix (59.4). □

Now that we have proved the objectwise criterion for natural isomorphisms (C), we can reinterpret the conclusion of (29.11): the functor $\mathrm{Fun}(T, C) \to \mathrm{Fun}(S, T)$ is an isofibration when $S_0 \to T_0$ is a bijection. **Perhaps show here that the bijection condition can be dropped, and do a relative version here, rather than in a later section.**

29.13. **Older proof.** I want this to go away.

Proof. Let $\mathcal{C}$ be the class of monomorphisms $j \colon K \to L$ for which the conclusion holds, i.e., such that $\mathrm{Fun}(j, C)$ is conservative for all quasicategories C. To prove the claim, it suffices to show (a) that $\mathcal{C}$ is a saturated class, and (b) that $\mathcal{C}$ contains the inclusions $\partial\Delta^n \subset \Delta^n$ for all $n \geq 1$. Given this, the result follows using the skeletal filtration (15.16).

Note that $\mathrm{Fun}(j, C)$ is always an inner fibration between quasicategories, and recall the "lifting characterization" (28.14) of conservative inner fibrations between quasicategories which follows from the Joyal lifting theorem. Using this, the proof of (a) is a straightforward exercise.

To show (b) that $\mathrm{Fun}(\Delta^n, C) \to \mathrm{Fun}(\partial\Delta^n, C)$ is conservative for $n \geq 1$, we again use (28.14), which we convert into its adjoint form. In fact, it suffices to prove that in any diagram of the form

$$
\begin{array}{ccc}
& f_0 & \\
\Delta^{\{0,1\}} \times \{0\} \rightarrowtail (\Lambda_0^m \times \Delta^n) \cup (\Delta^m \times \partial\Delta^n) \longrightarrow\!\!\!\!\!\rightarrow C \\
& \downarrow{\scriptstyle j} \qquad \nearrow & \\
& \Delta^2 \times \Delta^n &
\end{array}
$$

with $m \geq 2$ and $n \geq 1$ such that f_0 represents an isomorphism in C, a lift exists. This reduction uses that f_0 is the composite

$$
\Delta^{\{0,1\}} \xrightarrow{f} \mathrm{Fun}(\Delta^n, C) \xrightarrow{p} \mathrm{Fun}(\partial\Delta^n, C) \to \mathrm{Fun}(\{0\}, C)
$$

induced by restriction along $\{0\} \subset \partial\Delta^n \subset \Delta^n$, so that if $p(f)$ represents an isomorphism in $\mathrm{Fun}(\partial\Delta^n, C)$, then f_0 represents an isomorphism in C.

This statement is a consequence of the following lemma (29.14). □

29.14. **Proposition** (Pushout-product version of Joyal lifting). *Suppose $p \colon C \to D$ is an inner fibration between quasicategories. Suppose $m, n \geq 1$. For any diagram*

$$
\begin{array}{ccc}
& f & \\
\Delta^{\{0,1\}} \times \{0\} \rightarrowtail (\Lambda_0^m \times \Delta^n) \cup_{\Lambda_0^m \times \partial\Delta^n} (\Delta^m \times \partial\Delta^n) \longrightarrow\!\!\!\!\!\rightarrow C \\
& \downarrow{\scriptstyle j} \qquad\qquad \nearrow \qquad \downarrow{\scriptstyle p} & \\
& \Delta^m \times \Delta^n \longrightarrow D &
\end{array}
$$

such that f represents an isomorphism in C, a lift exists.

Note: if we take instead $m \geq 2$ and $n = 0$, the above statement becomes the Joyal lifting theorem.

Proof. Proved as [Joy08a, 5.8] (though there it is the $\Lambda_m^m \subset \Delta^m$ case that is proved). □

The idea is to produce the map j by iteratively attaching simplices along horns, which in each case are either:

(1) an inner horn $\Lambda_i^k \subset \Delta^k$ for $0 < i < k$, or
(2) a horn $\Lambda_0^k \subset \Delta^k$ with $k \geq 2$ along a map from Λ_0^k such that the restriction to its leading edge $\Delta^{\{0,1\}} \subset \Lambda_0^k$ of the map to C is identical to f.

The existence of the lifting follows, by applying the definition of inner fibration in cases of type (1), or the Joyal extension theorem in cases of type (2).

Here is a picture for the case $m = 1$, $n = 1$. The source of $(\Lambda_0^1 \subset \Delta^1) \square (\partial \Delta^1 \subset \Delta^1)$ looks like

$$
\begin{array}{ccc}
01 & \longrightarrow & 11 \\
\uparrow & & \\
00 & \underset{\sim}{\longrightarrow} & 10
\end{array}
$$

First attach the 2-simplex $\langle 00, 01, 11 \rangle$, which intersects the sounce along the inner horn Λ_1^2. Then attach the 2-simplex $\langle 00, 10, 11 \rangle$, which intersects what we have already built along the horn Λ_0^2, whose leading edge $\langle 00, 11 \rangle$ is sent to an isomorphism in C.

Here are pictures for the case $m = 2$, $n = 1$. Here is the source of $(\Lambda_0^2 \subset \Delta^2) \square (\partial \Delta^1 \subset \Delta^1)$.

$$
\begin{array}{ccccc}
11 & \longleftarrow & 01 & \longrightarrow & 21 \\
\end{array}
$$
$$
\begin{array}{ccccc}
10 & \overset{\sim}{\longleftarrow} & 00 & \longrightarrow & 20
\end{array}
$$

The following chart lists all non-degenerate simplices in the complement, with "$\checkmark$" marking those in the source.

$\langle 10, 21 \rangle$	$\langle 10, 20, 21 \rangle$	$\langle 10, 11, 21 \rangle$	$\langle 00, 10, 21 \rangle$	$\langle 00, 11, 21 \rangle$	$\langle 00, 10, 20, 21 \rangle$	$\langle 00, 10, 11, 21 \rangle$	$\langle 00, 01, 11, 21 \rangle$
$\checkmark\langle 21 \rangle$	$\checkmark\langle 20, 21 \rangle$	$\checkmark\langle 11, 21 \rangle$	$\langle 10, 21 \rangle$	$\checkmark\langle 11, 21 \rangle$	$\langle 10, 20, 21 \rangle$	$\langle 10, 11, 21 \rangle$	$\checkmark\langle 01, 11, 21 \rangle$
$\checkmark\langle 10 \rangle$	$\langle 10, 21 \rangle$	$\langle 10, 21 \rangle$	$\checkmark\langle 00, 21 \rangle$	$\langle 00, 21 \rangle$	$\checkmark\langle 00, 20, 21 \rangle$	$\langle 00, 11, 21 \rangle$	$\langle 00, 11, 21 \rangle$
	$\checkmark\langle 10, 20 \rangle$	$\checkmark\langle 10, 11 \rangle$	$\checkmark\langle 00, 10 \rangle$	$\checkmark\langle 00, 11 \rangle$	$\langle 00, 10, 21 \rangle$	$\langle 00, 10, 21 \rangle$	$\checkmark\langle 00, 01, 21 \rangle$
					$\checkmark\langle 00, 10, 20 \rangle$	$\checkmark\langle 00, 10, 11 \rangle$	$\checkmark\langle 00, 01, 11 \rangle$

Note that the simplices $\langle 10, 21 \rangle$, $\langle 00, 10, 21 \rangle$, and $\langle 00, 11, 21 \rangle$ of the complement appear multiple times as faces. We attach simplices to the domain in the following order:

$$\text{①}\langle 10, 11, 21 \rangle, \quad \text{②}\langle 00, 01, 11, 21 \rangle, \quad \text{③}\langle 00, 10, 11, 21 \rangle, \quad \text{④}\langle 00, 10, 20, 21 \rangle.$$

Only the final step involves attaching along a non-inner horn; in that case, the attaching map sends the leading edge of Λ_0^3 to $\langle 00, 10 \rangle$.

Part 4. The fundamental theorem

Recall that a functor $f \colon C \to D$ between quasicategories is said to be an *equivalence* there exists a $g \colon D \to C$ such that gf and fg are naturally isomorphic to the respective identity functors. When C and D are ordinary categories, there is a well-known criterion for the existence of such a g, namely: *f is an equivalence if and only if f is fully faithful and essentially surjective.* Here

- *fully faithful* means that $\operatorname{Hom}_C(x, y) \to \operatorname{Hom}_D(f(x), f(y))$ is a bijection of sets for every pair of objects $x, y \in \operatorname{ob} C$, and
- *essentially surjective* means that for every object $d \in \operatorname{ob} D$ there exists an object $c \in \operatorname{ob} C$ such that $f(c)$ is isomorphic to d.

I like to call this fact the *Fundamental Theorem of Category Theory*. This is non-standard and frankly pretentious terminology (I am unaware of any standard descriptive name for this result[21]). I want to give this fact a fancy name in order to signpost it, as it is quite *nonconstructive*: to prove it requires making a *choice* for each object d in D of an object c of C such that $f(c) \approx d$ (so it in fact relies on an appropriate form of the axiom of choice). (*Exercise:* prove the "Fundamental theorem" by first assuming an arbitrary choice of object $g(d) \in \operatorname{ob} C$ and isomorphism $\alpha(d) \colon f(g(d)) \to d$ for each object of d, and extending this to the data of a categorical inverse of f.)

---

[21]I also don't know when it was first formulated.

Example. Fix a field k. Let C be the category whose objects are non-negative integers $n \geq 0$, and whose morphims $A\colon n \to m$ are $(m \times n)$-matrices with entres in k, so that composition is matrix multiplication. Let D be the category of finite dimensional k-vector spaces and linear maps. Every basic class in linear algebra proves that the evident functor $F\colon C \to D$ is fully faithful and essentially surjective. Therefore F is an equivalence of categories. However, there is no *canonical* choice of an inverse functor, whose construction relies on making an arbitrary choice of basis for each vector space.

We are going to state and then prove an analogue of this result for functors between quasicategories. This will first require an analogue of hom-sets, namely the *quasigroupoid* of maps between two objects.

30. Mapping spaces of a quasicategory

Given a quasicategory C and objects $x, y \in C_0$, the **mapping space** (or **mapping quasi-groupoid**) from x to y is the simplicial set defined by the pullback square

$$
\begin{array}{ccc}
\mathrm{map}_C(x,y) & \longrightarrow & \mathrm{Fun}(\Delta^1, C) \\
\downarrow & & \downarrow \\
\{(x,y)\} & \longrightarrow & C \times C
\end{array}
$$

That is, $\mathrm{map}_C(x,y)$ is the fiber of the restriction map $\mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\partial\Delta^1, C)$ over the point (x,y), where we the isomorphism $\mathrm{Fun}(\partial\Delta^1, C) \approx C \times C$ induced by the isomorphism $\partial\Delta^1 \approx \Delta^0 \amalg \Delta^0$.

If $C = N(A)$ is the nerve of a category, then $\mathrm{map}_C(x,y)$ is a discrete simplicial set (2.5) corresponding to the set $\mathrm{Hom}_C(x,y)$.

30.1. **Mapping spaces are Kan complexes.** The terminology "space" is justified by the following

30.2. **Proposition.** *The simplicial sets $\mathrm{map}_C(x,y)$ are quasigroupoids (and hence Kan complexes by* (A)*).*

This is a special case of the following, applied to $\mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\partial\Delta^1, C)$, the restriction along $j = (\partial\Delta^1 \subset \Delta^1)$.

30.3. **Proposition.** *Let C be a quasicategory, and let $j\colon K \to L$ be a monomorphism which is a bijection on vertices. Then the fibers of the restriction map $p = \mathrm{Fun}(j, C)\colon \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C)$ are quasigroupoids.*

Proof. Consider a vertex $g \in \mathrm{Fun}(K, C)_0$, and let F denote the fiber of p over g, i.e., the pullback of p along the inclusion $\{g\} \to \mathrm{Fun}(K, C)$.

Note that $p\colon \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C)$ is a inner fibration, a case of enriched lifting using $\mathrm{InnHorn}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (16.7). Therefore $F \to \{g\}$, which is a pullback of p, is also an inner fibration, so F is a quasicategory.

To show that the quasicategory F is a quasigroupoid, it suffices to show that every $\Lambda_0^2 \to F$ extends to a 2-simplex. Thus we must consider the lifting problem

$$
\begin{array}{ccccccc}
& & & f & & & \\
\Delta^{\{0,1\}} & \rightarrowtail & \Lambda_0^2 & \longrightarrow & F & \longrightarrow & \mathrm{Fun}(L,C) \\
& & \downarrow & \nearrow & \downarrow & & \downarrow{\scriptstyle p} \\
& & \Delta^2 & \longrightarrow & \{g\} & \longrightarrow & \mathrm{Fun}(K,C)
\end{array}
$$

Since the right hand square is a pullback, it suffices to produce a lift $\Delta^2 \to \mathrm{Fun}(L, C)$. Since $p(f)$ is an identity map and hence an isomorphism in $\mathrm{Fun}(K, C)$, by the lifting criterion for conservative

inner fibrations (28.14) it suffices to show that p is conservative, which is the case by (29.10) since j is a bijection on vertices. $\square$

30.4. Mapping spaces and homotopy classes.
The set of morphisms $x \to y$ in a quasicategory is precisely the set of 0-simplices of $\mathrm{map}_C(x, y)$.

30.5. Proposition.
Let C be a quasicategory. For any two maps $f, g \colon x \to y$ in C, we have that $f \approx g$ (equivalence under the relation used to define the homotopy category hC) if and only if f and g are isomorphic as objects of the quasigroupoid $\mathrm{map}_C(x, y)$. That is,

$$\mathrm{Hom}_{hC}(x, y) \approx \pi_0 \mathrm{map}_C(x, y)$$

for every pair x, y of objects of C.

Proof. Suppose $f, g \in \mathrm{map}_C(x, y)_0$ are isomorphic, so that in particular there is a morphism $f \to g$ in the quasigroupoid $\mathrm{map}_C(x, y)$. This amounts to a map $\Delta^1 \times \Delta^1 \to C$ which can be represented by a diagram of simplices in C of the form:

$$
\begin{array}{ccc}
x & \xrightarrow{\ g\ } & y \\
{\scriptstyle 1_x}\big\downarrow & \nearrow^{a} \quad {\scriptstyle h} & \big\downarrow{\scriptstyle 1_y} \\
x & \xrightarrow[\ b\ ]{} & y \\
 & \xrightarrow[f]{} &
\end{array}
$$

This explicitly exhibits a chain $f \sim_r h \sim_\ell g$ of homotopies, so $f \approx g$ as desired.

Conversely, if $f \approx g$, we can explicitly construct a map $H \colon f \to g$ in $\mathrm{map}_C(x, y)$: in terms of the above picture, let $h = g$, let b be an explicit choice of right-homotopy $f \sim_r g$, and let $a = g_{001}$. $\square$

30.6. Extended mapping spaces and composition.
Given a finite list $x_0, \ldots, x_n \in C_0$ of objects in a quasicategory, we have an **extended mapping space**. These are the simplicial sets defined by the pullback squares

$$
\begin{array}{ccc}
\mathrm{map}_C(x_0, \ldots, x_n) & \longrightarrow & \mathrm{Fun}(\Delta^n, C) \\
\big\downarrow & & \big\downarrow \\
\{(x_0, \ldots, x_n)\} & \longrightarrow & C^{\times(n+1)}
\end{array}
$$

where the right-hand vertical arrow is induced by restriction along $\mathrm{Sk}_0 \Delta^n \to \Delta^n$, using the isomorphism $\mathrm{Sk}_0 \Delta^n \approx (\Delta^0)^{\mathrm{II}(n+1)}$. By (30.3) the extended mapping spaces are quasigroupoids.

We can compare the extended mapping spaces to the fibers of $\mathrm{Fun}(I^n, C) \to C^{\times(n+1)}$, which is seen to be an n-fold product of mapping spaces.

30.7. Lemma.
The map

$$g_n \colon \mathrm{map}_C(x_0, \ldots, x_n) \to \mathrm{map}_C(x_{n-1}, x_n) \times \cdots \times \mathrm{map}_C(x_0, x_1)$$

induced by restriction along the spine inclusion $I^n \subseteq \Delta^n$ is a trivial fibration. In particular, this map is a categorical equivalence between Kan complexes.

Proof. The map g_n is a base change of $p \colon \mathrm{Fun}(\Delta^n, C) \to \mathrm{Fun}(I^n, C)$. Since $I^n \subset \Delta^n$ is inner anodyne (10.12), and C is a quasicategory, the map p is a trivial fibration by enriched lifting using $\overline{\mathrm{InnHorn}}\square\mathrm{Cell} \subseteq \overline{\mathrm{InnHorn}}$ (16.7). $\square$

The inclusions $I^2 \subset \Delta^2 \supset \Delta^{\{0,2\}}$ induce restriction maps

$$\mathrm{Fun}(I^2, C) \leftarrow \mathrm{Fun}(\Delta^n, C) \to \mathrm{Fun}(\Delta^{\{0,2\}}, C).$$

Restricting to the fibers over a triple (x_0, x_1, x_2) of objects, we obtain a zig-zag of maps of Kan complexes

$$\mathrm{map}_C(x_1, x_2) \times \mathrm{map}_C(x_0, x_1) \xleftarrow[\sim]{g_2} \mathrm{map}_C(x_0, x_1, x_2) \to \mathrm{map}_C(x_0, x_2),$$

where the second map is induced by restriction along $\Delta^{\{0,2\}} \subset \Delta^2$, and the first map g_2 is a categorical equivalence, and in fact a trivial fibration. After choosing a categorical inverse to g_2 (perhaps a section, by (20.10)), we obtain a "composition" map

(30.8) $\qquad\qquad$ comp: $\mathrm{map}_C(x_1, x_2) \times \mathrm{map}_C(x_0, x_1) \to \mathrm{map}_C(x_0, x_2).$

This map is *not* uniquely determined, since it depends on a choice of categorical inverse to g_2. However, any two categorical equivalences to g_2 are naturally isomorphic **(proof?)**, and therefore comp is defined up to natural isomorphism. That is, it is a well-defined map in $h\mathrm{Kan}$, the homotopy category of Kan complexes (defined as a full subcategory of $h\mathrm{QCat}$).

30.9. **Proposition.** *The two maps obtained by composing the sides of the square*

$$\begin{array}{ccc}
\mathrm{map}_C(x_2, x_3) \times \mathrm{map}_C(x_2, x_1) \times \mathrm{map}_C(x_0, x_1) & \xrightarrow{\mathrm{id} \times \mathrm{comp}} & \mathrm{map}_C(x_2, x_3) \times \mathrm{map}_C(x_0, x_2) \\
{\scriptstyle \mathrm{comp} \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{comp}} \\
\mathrm{map}_C(x_1, x_3) \times \mathrm{map}_C(x_0, x_1) & \xrightarrow[\mathrm{comp}]{} & \mathrm{map}_C(x_0, x_3)
\end{array}$$

are naturally isomorphic. That is, the diagram commutes in $h\mathrm{Kan} \subset h\mathrm{QCat}$.

Proof. Here is a diagram of simplicial sets which actually commutes on the nose, i.e., not merely in the homotopy category. I use "(x, y, z)" as shorthand for "$\mathrm{map}_C(x, y, z)$", etc.

$$\begin{array}{ccccc}
(x_2, x_3) \times (x_1, x_2) \times (x_0, x_1) & \xleftarrow{\sim} & (x_2, x_3) \times (x_0, x_1, x_2) & \to & (x_2, x_3) \times (x_0, x_2) \\
\uparrow {\scriptstyle \sim} & & \uparrow {\scriptstyle \sim} & & \uparrow {\scriptstyle \sim} \\
(x_1, x_2, x_3) \times (x_0, x_1) & \xleftarrow{\ \sim\ } & (x_0, x_1, x_2, x_3) & \to & (x_0, x_2, x_3) \\
\downarrow & & \downarrow & & \downarrow \\
(x_1, x_3) \times (x_0, x_1) & \xleftarrow{\ \sim\ } & (x_0, x_1, x_3) & \to & (x_0, x_3)
\end{array}$$

The maps labelled "$\xrightarrow{\sim}$" are categorical equivalences, and in fact are trivial fibrations. All the maps in the above diagram are obtained via restriction along inclusions in

$$\begin{array}{ccccc}
\Delta^{\{2,3\}} \cup \Delta^{\{1,2\}} \cup \Delta^{\{0,1\}} & \xrightarrow{\sim} & \Delta^{\{2,3\}} \cup \Delta^{\{0,1,2\}} & \longleftarrow & \Delta^{\{2,3\}} \cup \Delta^{\{0,2\}} \\
\downarrow {\scriptstyle \sim} & & \downarrow {\scriptstyle \sim} & & \downarrow {\scriptstyle \sim} \\
\Delta^{\{1,2,3\}} \cup \Delta^{\{0,1\}} & \xrightarrow{\ \sim\ } & \Delta^3 & \longleftarrow & \Delta^{\{0,2,3\}} \\
\uparrow & & \uparrow & & \uparrow \\
\Delta^{\{1,3\}} \cup \Delta^{\{0,1\}} & \xrightarrow{\ \sim\ } & \Delta^{\{0,1,3\}} & \longleftarrow & \Delta^{\{0,3\}}
\end{array}$$

where the maps labelled "$\xrightarrow{\sim}$" are inner anodyne, and which therefore give rise to trivial fibrations in the previous diagram by the same argument we used to define comp. After passing to $h\mathrm{Kan}$ the categorical equivalences become isomorphisms, and the result follows. $\qquad\square$

30.10. **Segal categories.** Thus, a quasicategory does not quite give rise to a category "enriched over quasigroupoids". Although we can define a composition law, it is not uniquely determined, and is only associative "up to homotopy".

What we do get is a Segal category. A **Segal category** is a functor

$$M \colon \Delta^{\mathrm{op}} \to s\mathrm{Set}$$

such that

(1) the simplicial set $M([0])$ is discrete, i.e., $M([0]) = \mathrm{Sk}_0 M([0])$, and

(2) for each $n \geq 1$ the "Segal map"

$$M([n]) \xrightarrow{(\langle n-1,n\rangle^*, \ldots, \langle 0,1\rangle^*)} M([1]) \times_{M([0])} \cdots \times_{M([0])} M([1])$$

is a "weak eqvivialence" of simplicial sets.

We will define "weak equivalence" of simplicial sets below. For now, we note that a map between *Kan complexes* is a weak equivalence if and only if it is a categorical equivalence, and that if each $M([n])$ is a Kan complex, then so are the fiber products which appear in the above definition.

Given a quasicategory C, we obtain a functor $M_C \colon \Delta^{\mathrm{op}} \to s\mathrm{Set}$ by

$$M_C([0]) := \mathrm{Sk}_0\, C,$$
$$M_C([n]) := \mathrm{Fun}(\Delta^n, C) \times_{\mathrm{Fun}(\mathrm{Sk}_0\, \Delta^n, C)} \mathrm{Fun}(\mathrm{Sk}_0\, \Delta^n, \mathrm{Sk}_0\, C)$$
$$\approx \coprod_{x_0, \ldots, x_n \in C_0} \mathrm{map}_C(x_0, \ldots, x_n).$$

This object encodes all the structure we used above. For instance, the zig-zag

$$M_C([1]) \times_{M_C([0])} M_C([1]) \xleftarrow{(\langle 12\rangle^*, \langle 01\rangle^*)} M_C([2]) \xrightarrow{\langle 02\rangle^*} M_C([1])$$

is a coproduct over all triples $x_0, x_1, x_2 \in C_0$ of the zig-zag (30.8) used to define "composition".

You also get a Segal category from any "simplicially enriched" category. Suppose $\mathcal{C}$ is a (small) category which is enriched over the category of simplicial sets, with object set $\mathrm{ob}\,\mathcal{C}$, and function objects $\mathcal{C}(x, x') \in s\mathrm{Set}$ for each x, x'. Then we can define $M_{\mathcal{C}} \colon \Delta^{\mathrm{op}} \to s\mathrm{Set}$ by

$$M_{\mathcal{C}}([0]) := \mathrm{ob}\,\mathcal{C},$$
$$M_{\mathcal{C}}([n]) := \coprod_{x_0, \ldots, x_n \in \mathrm{ob}\,\mathcal{C}} \mathcal{C}(x_{n-1}, x_n) \times \cdots \times \mathcal{C}(x_0, x_1).$$

We thus obtain functors

$$\mathrm{QCat} \to \mathrm{SeCat} \leftarrow s\mathrm{Cat}$$

relating quasicategories, Segal categories, and simplicially enriched categories. Simplicially enriched categories were proposed as a model for ∞-categories by Dwyer and Kan[22], while Segal categories were proposed as a model for ∞-categories by Hirschowitz and Simpson [HS01][23]. All of these models are known to be equivalent in a suitable sense; see [Ber10] for more about these models and their comparison.

30.11. **The enriched homotopy category of a quasicategory.** Given a quasicategory C we can produce a vestigial version of a category enriched over quasigroupoids, called the **enriched homotopy category** of C and denoted $\mathcal{H}C$.[24] This object will be a category *enriched over* $h\mathrm{Kan}$, the homotopy category of Kan complexes, whose underlying category is hC.

We now define $\mathcal{H}C$. The objects of $\mathcal{H}C$ are just the objects of C. For any two objects $x, y \in C_0$, we have the quasigroupoid

$$\mathcal{H}C(x, y) := \mathrm{map}_C(x, y)$$

which we will regard as an object of the homotopy category $h\mathrm{Kan}$ of Kan complexes. Composition $\mathcal{H}C(x_1, x_2) \times \mathcal{H}C(x_0, x_1) \to \mathcal{H}C(x_0, x_2)$ is the composition map defined above. Composition is associative as shown above (30.9).

The *underlying* ordinary category of $\mathcal{H}C$ is just the ordinary homotopy category hC, since

$$\mathrm{Hom}_{h\mathrm{Kan}}(\Delta^0, \mathrm{map}_C(x, y)) \approx \pi_0 \mathrm{map}_C(x, y) \approx \mathrm{Hom}_{hC}(x, y).$$

---

[22]They called them "homotopy theories" instead of "∞-categories; see [DS95, §11.6].

[23]In fact, they generalize this to "Segal n-categories", which were the first effective model for (∞, n)-categories.

[24]Lurie usually calls this "hC", though he also uses that notation for the ordinary homotopy category of C that we have already discussed. I prefer to have two separate notations.

30.12. *Warning.* A quasicategory C *cannot* be recovered from its enriched homotopy category $\mathcal{H}C$, not even up to equivalence. In fact, there exist hKan-enriched categories which do not arise as $\mathcal{H}C$ for any quasicategory C. A proof is outside the scope of these notes; however, we note that counterexamples may be produced from associative H-spaces which are not loop spaces.

30.13. *Exercise.* Let C and D be quasicategories. Show that there is an isomorphism $\mathcal{H}(C \times D) \approx \mathcal{H}C \times \mathcal{H}D$ of hKan-enriched categories.

30.14. *Exercise.* Let C be a quasicategory. Describe how to use the enriched homotopy category $\mathcal{H}C$ to define, for each morphism $f\colon x \to y$ in C and object c in C, maps

$$f^*\colon \operatorname{map}_C(y, c) \to \operatorname{map}_C(x, c), \qquad f_*\colon \operatorname{map}_C(c, x) \to \operatorname{map}_C(c, y)$$

in hKan corresponding to pre- and post-composition with f. Show that these fit together to give a well-defined functor

$$\operatorname{map}(-, -)\colon hC^{\mathrm{op}} \times hC \to h\mathrm{Kan}.$$

31. The fundamental theorem of quasicategory theory

31.1. **Fully faithful and essentially surjective functors between quasicategories.** Note that any functor $f\colon C \to D$ of quasicategories restricts to maps $\operatorname{map}_C(x, y) \to \operatorname{map}(f(x), f(y))$ for every pair of objects x, y in C. We say that a functor $f\colon C \to D$ between quasicategories is

- **fully faithful** if for every pair $c, c' \in C_0$, the resulting map $\operatorname{map}_C(c, c') \to \operatorname{map}_D(fc, fc')$ is a categorical equivalence, and
- **essentially surjective** if the induced functor $hf\colon hC \to hD$ is essentially surjective; i.e., if for every $d \in D_0$ there exists $c \in C_0$ and an isomorphism $fc \to d$ in D_1.

Another way to say this: $f\colon C \to D$ is fully faithful and essentially surjective iff the induced hKan-enriched functor $\mathcal{H}f\colon \mathcal{H}C \to \mathcal{H}D$ is an equivalence of *enriched* categories.

31.2. **Proposition.** *If $f\colon C \to D$ is a categorical equivalence between quasicategories, then f is fully faithful and essentially surjective.*

Proof. We already know that $hf\colon hC \to hD$ is an equivalence of ordinary categories, which implies essential surjectivity.

To show that f is fully faithful, choose a categorical inverse g of f. Given $x, y \in C_0$, consider the induced diagram of quasigroupoids

$$\operatorname{map}_C(x, y) \xrightarrow{\ f\ } \operatorname{map}_D(fx, fy) \xrightarrow{\ g\ } \operatorname{map}_C(gfx, gfy) \xrightarrow{\ f\ } \operatorname{map}_D(fgfx, fgfy)$$

with arcs marked gf (over the first two) and fg (over the last two).

We will need the following lemma, which assert that being fully faithful is a natural isomorphism invariant of functors. Given this, and since $gf \approx \mathrm{id}_C$ and $fg \approx \mathrm{id}_D$, we see that the maps in the diagram marked gf and fg are categorical equivalences, i.e., they are isomorphisms in the ordinary category $h\mathrm{QCat}$. Therefore the map marked g and both maps marked f are also isomorphisms in $h\mathrm{QCat}$, i.e., are categorical equivalences. $\qquad\square$

31.3. **Lemma.** *If $f, f'\colon C \to D$ are functors which are naturally isomorphic, then f is fully faithful if and only if f' is.*

Proof. **Proof needed. Use lemma below?** $\qquad\square$

The converse is true, but not as straightforward.

E. **Deferred Proposition** (Fundamental Theorem of Quasicategory Theory). *A map $f\colon C \to D$ between quasicategories is a categorical equivalence if and only if it is fully faithful and essentially surjective.*

This is a non-trivial result. It gives a necessary and sufficient condition for $f\colon C \to D$ to admit a categorical inverse, but it does not spell out how to construct such an inverse. After many preliminaries, we will give the proof in §39.

31.4. **2-out-of-3 for fully faithful essentially surjective functors.** The following result will be useful in the proof of the fundamental theorem. Recall the *2-out-of-3* property of a class of morphisms (22.5), and that the class of categorical equivalences has this property (22.9).

31.5. **Proposition.** *The class $\mathcal{C}$ of fully faithful and essentially surjective functors between quasicategories satisfies the 2-out-of-3 property.*

Proof. Any identity functor $\mathrm{id}\colon C \to C$ is manifestly fully faithful and essentially surjective.

Next note that if a functor $f\colon C \to D$ between quasicategories is fully faithful and essentially surjective, then the induced $hf\colon hC \to hD$ is an equivalence of ordinary categories. Conversely, if hf is an equivalence, then f is essentially surjective.

Therefore if $C \xrightarrow{f} D \xrightarrow{g} E$ are functors of quasicategories such that two of $\{f, g, gf\}$ are fully faithful and essentially surjective, then the third is also essentially surjective. Thus, we only need to deal with the fully faithful property, for which there are 3 cases: showing that f, g, or gf is fully faithful assuming the other two are categorical equivalences.

Given objects $x, x' \in C_0$, we have induced maps

$$\mathrm{map}_C(x, x') \xrightarrow{f} \mathrm{map}_D(fx, fx') \xrightarrow{g} \mathrm{map}_E(gfx, gfx').$$

Proofs of two of the three cases (for f and for gf) follow immediately using (22.9).

For the case of g, note that if f and gf are categorical equivalences, it follows by the same argument that $\mathrm{map}_D(y, y') \xrightarrow{g} \mathrm{map}_E(gy, gy')$ is a categorical equivalence for any y, y' in the image of $f\colon C_0 \to D_0$. This is good enough, which follows from the following lemma. $\qquad\square$

31.6. **Lemma.** *Let $f\colon C \to D$ be a functor between quasicategories, and let $a\colon x \to y$ and $b\colon y' \to x'$ be isomorphisms in C. Then there is a commutative square*

$$
\begin{array}{ccc}
\mathrm{map}_C(x, x') & \xrightarrow{\ f\ } & \mathrm{map}_D(fx, fx') \\
{\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\
\mathrm{map}_C(y, y') & \xrightarrow[\ f\ ]{} & \mathrm{map}_D(fy, fy')
\end{array}
$$

in $h\mathrm{Kan}$, where the vertical maps are categorical equivalences (i.e., isomorphisms in $h\mathrm{Kan}$) induced by composition with a and b.

Proof. Use (30.14). $\qquad\square$

31.7. **Fully faithful and essentially surjective functors between quasigroupoids.** The special case for quasigroupoids is already interesting. Here, (E) specializes to the "Fundamental Theorem of Quasigroupoid Theory", which says that a map $f\colon X \to Y$ between Kan complexes is a categorical equivalence if and only if it

- induces categorical equivalences $\mathrm{map}_X(x_0, x_1) \to \mathrm{map}_Y(fx_0, fx_1)$ on "path spaces", and
- induces a surjection $\pi_0 X \to \pi_0 Y$.

31.8. *Remark.* A mild variant of this says that a map $f\colon X \to Y$ between Kan complexes is a categorical equivalence if and only if it

- induces categorical equivalences $\Omega_x X \to \Omega_{fx} Y$ on "loop spaces", and
- induces an isomorphism $\pi_0 X \to \pi_0 Y$.

Here $\Omega_x X := \mathrm{map}_X(x, x)$. Under the correspondence between quasigroupoids and classical homotopy theory, this turns out to be an analogue of the Whitehead theorem[25], which says that a map between CW-complexes is a homotopy equivalence iff it induces an isomorphism on all homotopy groups.

We will prove (E) after first considering the special case of quasigroupoids=Kan complexes.

32. Anodyne maps and Kan fibrations

In the next few sections, we will develop some properties related to Kan complexes. As a byproduct, we'll obtain the proof of the specialization of (E) to Kan complexes.

32.1. **Weak equivalence.** Say that a map $f\colon X \to Y$ is a **weak equivalence** of simplicial sets if and only if $\mathrm{Map}(f, G)\colon \mathrm{Map}(Y, G) \to \mathrm{Map}(X, G)$ is a categorical equivalence for every *quasigroupoid* (i.e., every *Kan complex*) G.

Every categorical equivalence is a weak equivalence, but not conversely. For maps between Kan complexes, weak equivalences and categorical equivalences coincide, as is straightforward to show via the same proof as (19.5).

32.2. **Proposition.** *Weak equivalences of simplicial sets satisfy the 2-out-of-3 property.*

Proof. Proved just as for categorical equivalences (22.9).   □

32.3. *Remark.* Given the analogy to categorical equivalence, a more sensible name might be "groupoidal equivalence". However, the term "weak equivalence" is historically well-established.

32.4. **Simplicial homotopy equivalence.** A **simplicial homotopy inverse** to a map $f\colon X \to Y$ of simplicial sets is a map $g\colon Y \to X$ such that there exists a chain of edges in $\mathrm{Map}(X, X)$ connecting id_X with gf, and a chain of edges in $\mathrm{Map}(Y, Y)$ connecting id_Y with fg. Such an f is called a **simplicial homotopy equivalence**, and of course any homotopy inverse is also a simplicial homotopy equivalence.

32.5. **Proposition.** *Any simplicial homotopy equivalence is a weak equaivalence.*

Proof. First, if $f\colon X \to Y$ is a simplicial homotopy equivalence between Kan complexes, then it is clearly a categorical equivalence, because $\mathrm{Map}(X, X)$ and $\mathrm{Map}(Y, Y)$ are quasigroupoids.

In general, suppose K is a Kan complex and consider $f^*\colon \mathrm{Map}(Y, K) \to \mathrm{Map}(X, K)$. By the same reasoning as used in the proof of (19.5), we see that f^* is a simplicial homotopy equivalence between Kan complexes, so a categorical equivalence.   □

32.6. **Anodyne maps and Kan fibrations.** Let

$$\mathrm{Horn} = \{\, \Lambda_j^n \subset \Delta^n \mid n \geq 1,\ 0 \leq j \leq n \,\} = \mathrm{RHorn} \cup \mathrm{LHorn}$$

denote the set of all horn inclusions. A map is **anodyne** if it is in $\overline{\mathrm{Horn}}$, and is a **Kan fibration** if it is in $\mathrm{KFib} := \mathrm{Horn}^{\boxslash}$.

Since Horn is a set, the small object argument (13.10) applies to it: any map can be factored $f = pj$ with $j \in \overline{\mathrm{Horn}}$ and $p \in \mathrm{KFib}$.

32.7. **Proposition.** *We have that $\overline{\mathrm{Horn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$.*

Proof. This amounts to showing $\mathrm{Horn} \square \mathrm{Cell} \subseteq \overline{\mathrm{Horn}}$. See [JT08, Theorem 3.2.2], or [GZ67], or **appendix**.   □

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[25]The "Fundamental Theorem of Classical Homotopy Theory"?

Thus, we have that

$$\mathrm{Map}(L, X) \to \mathrm{Map}(K, X) \times_{\mathrm{Map}(K,Y)} \mathrm{Map}(L, Y)$$

is a Kan fibration whenever $K \subseteq L$ and $X \to Y$ is a Kan fibration, and is a trivial fibration if $K \subseteq L$ is also anodyne.

As a special case, we learn that if X is a Kan complex and $K \subseteq L$, then $\mathrm{Fun}(L, X) \to \mathrm{Fun}(K, X)$ is a Kan fibration. Here is another consequence.

32.8. Proposition. *Every anodyne map is a weak equivalence.*

Proof. If $f \colon A \to B$ is anodyne, then $\mathrm{Map}(f, G)$ is a trivial fibration for every Kan complex G, and hence a categorical equivalence (20.9). $\qquad\square$

32.9. Exercise. Show that the inclusion $\{j\} \subseteq \Delta^n$ of any vertex into any standard n-simplex is anodyne.

32.10. Example. From (32.9) it follows that any map $f \colon \Delta^m \to \Delta^n$ between standard simplicies is a weak equivalence, using the 2-out-of-3 property (32.2). Any such f which is not an isomorphism is an example of a weak equivalence between quasicategories which is not a categorical equivalence.

32.11. Exercise. Let $f \colon X \to Y$ be any map between Kan complexes. Show that f is a Kan fibration if and only if it is an isofibration. (Hint: Joyal lifting.)

32.12. Exercise. Give an example of an inner fibration between Kan complexes which is not a Kan fibration.

32.13. The universal isomorphism. Let Iso be the "walking isomorphism", i.e., the category with two objects 0 and 1, and a unique isomorphism between them. Let $u \colon \Delta^1 \to N\mathrm{Iso}$ be the inclusion representing the unique map $0 \to 1$ in Iso.

32.14. Proposition. *The map $u \colon \Delta^1 \to N\mathrm{Iso}$ is anodyne, and hence a weak equivalence.*

Proof. The k-simplices of $N(\mathrm{Iso})$ are in one-to-one correspondence with sequences $x_0 x_1 \cdots x_k$ with $x_i \in \{0, 1\}$. For each $k \geq 0$ there are exactly two *non-degenerate* k-simplices, corresponding to the alternating sequences $0101 \ldots$ and $1010 \ldots$.

Let $u_k \colon \Delta^k \to N\mathrm{Iso}$ be the non-degenerate simplex $0101 \ldots$, and let $F_k \subset N\mathrm{Iso}$ be the smallest subcomplex containing u_k. Note that $N\mathrm{Iso} = \bigcup F_k$ and $F_1 = u(\Delta^1)$. The commutative square

$$
\begin{array}{ccc}
\Lambda_0^k & \longrightarrow & F_{k-1} \\
\downarrow & & \downarrow \\
\Delta^k & \xrightarrow{\ u_k\ } & F_k
\end{array}
$$

is a pushout square for all $k \geq 1$. This is by (21.3), since (1) it is a pullback, and (2) any simplex in the complement of $F_{k-1} \subset F_k$ is the image of a unique simplex under u_k.

It follows that u is anodyne. $\qquad\square$

We obtain as a consquence the following criterion for an edge to be an isomorphism, which we will use later.

32.15. Proposition. *Let C be a quasicategory, and $f \colon \Delta^1 \to C$ a map representing morphism in C. Then there exists $f' \colon N(\mathrm{Iso}) \to C$ such that $f'u = f$ if and only if f represents an isomorphism in C.*

Proof. ($\Longrightarrow$) Clear: consider induced maps on homotopy categories. ($\Longleftarrow$) If f represents an isomorphism then it factors through $\Delta^1 \to C^{\mathrm{core}} \subseteq C$. Since the core is a quasigroupoid, and hence a Kan complex, an extension along the anodyne map u to a map $N\mathrm{Iso} \to C^{\mathrm{core}} \subseteq C$ exists. $\qquad\square$

32.16. Remark. Let $X \subset N\mathrm{Iso}$ be the subcomplex which is the union of the images of 2-simplices 010 and 101[26]. The inclusion $v \colon \Delta^1 \to X$ representing the edge 01 has the same property described in (32.15): $f \colon \Delta^1 \to C$ represents an isomorphism if and only if it extends along v. The proof is easy: an extension of f to a map $f' \colon X \to C$ exactly encodes a choice of morphism g in C (i.e., $f'(\langle 10 \rangle)$) together with explicit homotopies $gf \sim_r 1$ and $fg \sim_\ell 1$, (i.e., $f'(\langle 010 \rangle)$ and $f'(\langle 101 \rangle)$).

However, it turns out that $\Delta^1 \to X$ is *not* a weak equivalence (and therefore that $X \to N\mathrm{Iso}$ is not a categorical equivalence). In particular, a map $X \to C$ to a quasicategry can fail to extend along $X \subset N\mathrm{Iso}$.

32.17. Exercise. Show that $X \to N\mathrm{Iso}$ is not anodyne, by constructing a map $X \to K(\mathbb{Z}, 2)$ which does not extend over $N\mathrm{Iso}$. (See (8.11).)

32.18. Covering homotopy extension property. This is the special case of $\overline{\mathrm{Horn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$ which we will use several times in the next few sections: for any inclusion $K \subseteq L$, the map

$$(K \times \Delta^1) \cup_{K \times \{j\}} (L \times \{j\}) \to L \times \Delta^1$$

with either $j = 0$ or $j = 1$ is anodyne. This amounts to saying that

has a lift whenever p is a Kan fibration. This is sometimes called the "covering homotopy extension property".

It may be helpful to think of this in the equivalent form, which asserts a lifting in

when $K \subseteq L$ and $p \colon X \to Y$ a Kan fibration. This gets used the following way: to demonstrate $(K \subseteq L) \boxtimes p$ for a given Kan fibration p and inclusion $K \subseteq L$, we "deform" a lifting problem of this type along a "path" in the space $\mathrm{Map}(K, X) \times_{\mathrm{Map}(K,Y)} \mathrm{Map}(L, Y)$ of commutative squares to a lifting problem which we know has a lift.

32.19. Fundamental theorem for Kan complexes: reduction to Kan fibrations. We are going to show the following

32.20. Theorem. *A map $f \colon X \to Y$ between Kan complexes is a weak equivalence if and only if it is fully faithful and essentially surjective.*

We will prove this by reducing to Kan fibrations.

32.21. Lemma. *To prove (32.20), it suffices to prove it for the case when f is a Kan fibration.*

Proof. ($\Longrightarrow$) Weak equivalences between Kan complexes are categorical equivalences, and we have already shown that these are fully faithful and essentially surjective (31.2).

($\Longleftarrow$) Given f between Kan complexes which is fully faithful and essentially surjective, use the small object argument (13.10) applied to Horn to factor it as

$$X \xrightarrow{j} V \xrightarrow{p} Y,$$

---

[26]This is isomorphic to the complex discussed in (20.6).

where j is anodyne and p is a Kan fibration. It follows that V is a Kan complex, j is a weak equivalence (32.8) between Kan complexes, and so is fully faithful and essentially surjective. Since the class of fully faithful and essentially surjective maps satisfy 2-out-of-3 (31.5), it follows that p also has this property. If we can use this to show p is a weak equivalence, it follows that f is a weak equivalence as desired. $\qquad\square$

We will prove the needed special case (that for Kan fibrations between Kan complexes, fully faithful and essentially surjective implies weak equivalence) in the next couple of sections, after analyzing Kan fibrations in more detail.

33. Kan fibrations between Kan complexes

In the next few sections, we are going to be considering various properties of Kan fibrations, with particular interest in Kan fibrations between Kan complexes.

In particualar, we are going to show that for a Kan fibration $p\colon X \to Y$ where X and Y are Kan complexes, all of the following are equivalent.

(1) p is a trivial fibration;
(2) p is a weak equivalence;
(3) p is a fiberwise deformation retraction;
(4) p has contractible fibers;
(5) p is fully faithful and essentially surjective.

The equivalence of (2) and (5) will complete the proof of the fundamental theorem for Kan complexs (32.20). (In fact, (1)–(4) are equivalent without the hypothesis that the objects are Kan complexes, though we will not prove that in all cases.)

33.1. Fiberwise deformation retraction. A map $p\colon X \to Y$ is said to be a **fiberwise deformation retraction** if there exists

- $s\colon Y \to X$ such that $ps = \mathrm{id}_Y$, and
- $k\colon X \times \Delta^1 \to X$ such that $k|X \times \{0\} = \mathrm{id}_X$, $k|X \times \{1\} = sp$, and $pk = p\pi$, where $\pi\colon X \times \Delta^1 \to X$ is projection; that is, the diagram

$$
\begin{array}{ccc}
X \times \{0,1\} & \xrightarrow{\;(\mathrm{id}_X,sp)\;} & X \\
\big\downarrow & {\scriptstyle k}\nearrow & \big\downarrow {\scriptstyle p} \\
X \times \Delta^1 \xrightarrow{\;\pi\;} X & \xrightarrow{\;p\;} & Y
\end{array}
$$

commutes.

Any fiberwise deformation retraction is a weak equivalence: s is a simplicial homotopy inverse to p (32.5).

33.2. *Exercise.* Show that the term "fiberwise" is justified: for each $y \in Y_0$, the projection $p^{-1}(y) \to \{y\}$ of a fiber to its image is a simplicial homotopy equivalence.

33.3. *Exercise.* Show that if $p\colon X \to Y$ is part of a fiberwise deformation retraction as above, then any base change of p is also part of a fiberwise deformation retraction.

Fiberwise deformation retractions of Kan fibrations are always trivial fibrations, as can be shown with the covering homotopy extension property.

33.4. Lemma. *Let $p\colon X \to Y$ be a Kan fibration between simplicial sets. Then p is part of a fiberwise deformation retraction if and only p is a trivial fibration.*

Proof. [JT08, Prop. 3.2.5]. ($\Longrightarrow$) We need to solve the lifting problem

$$
\begin{array}{ccc}
A & \xrightarrow{\ a\ } & X \\
{\scriptstyle i}\downarrow & {\scriptstyle u}\nearrow & \downarrow{\scriptstyle p} \\
B & \xrightarrow{\ b\ } & Y
\end{array}
$$

where i is a monomorphism. Using the data s and k of the fiberwise deformation retraction we obtain a commutative square

$$
\begin{array}{ccc}
(A \times \Delta^1) \cup (B \times \{1\}) & \xrightarrow{(k(a \times \mathrm{id}_{\Delta^1}),sb)} & X \\
{\scriptstyle j}\downarrow & {\scriptstyle t} & \downarrow{\scriptstyle p} \\
B \times \Delta^1 \xrightarrow{\ \pi\ } & B \xrightarrow{\ b\ } & Y
\end{array}
$$

Because p is a Kan fibration and j is anodyne by (32.7), a lift t exists. Then $u := t|B \times \{0\}$ is the desired lift.

($\Longleftarrow$) Left as an exercise. $\qquad\qquad\Box$

33.5. *Exercise.* Show that any trivial fibration is a fiberwise deformation retraction.

33.6. Trivial Kan fibrations between Kan complexes. We know that trivial fibrations are always categorical equivalences (20.9). We now show that any Kan fibration between Kan complexes which is also a categorical equivalence is a trivial fibration.

33.7. Proposition. *A Kan fibration $p\colon X \to Y$ between Kan complexes is a trivial fibration if and only if it is a weak equivalence.*

Proof. [JT08, Prop. 3.2.6] ($\Longrightarrow$) We have already shown that trivial fibrations between quasicategories are always categorical equivalences, which implies they are weak equivalences if between Kan complexes.

($\Longleftarrow$) On the other hand, suppose p is a Kan fibration and a weak equivalence. Being a categorical equivalence between Kan complexes, p admits a categorical inverse: there exists $f\colon Y \to X$ and maps $u\colon X \times \Delta^1 \to X$ and $v\colon Y \times \Delta^1 \to Y$ which give natural isomorphisms $u\colon fp \to \mathrm{id}_X$ and $v\colon pf \to \mathrm{id}_Y$. We will "deform" this data to a fiberwise deformation retraction.

Step 1. Since $Y \times \{0\} \subset Y \times \Delta^1$ is anodyne by (32.7), a lift α exists in

$$
\begin{array}{ccc}
Y \times \{0\} & \xrightarrow{\ f\ } & X \\
\downarrow & {\scriptstyle \alpha}\nearrow & \downarrow{\scriptstyle p} \\
Y \times \Delta^1 & \xrightarrow{\ v\ } & Y
\end{array}
$$

Let $s := \alpha|Y \times \{1\}$, so $ps = \mathrm{id}_Y$. The map α exhibits a natural isomorphism $\alpha\colon f \to s$ of functors $Y \to X$. Since fp is naturally isomorphic to id_X, we have $\mathrm{id}_X \approx fp \approx sp$, i.e., there exists a natural isomorphism $w\colon sp \to \mathrm{id}_X$.

Step 2. Consider the natural isomorphism $(sp)w\colon sp = spsp \to sp$ of functors $X \to X$. We have a commutative diagram

$$
\begin{array}{ccc}
\Lambda_0^2 & \xrightarrow{\ a\ } & \mathrm{Map}(X,X) \\
\downarrow & {\scriptstyle t}\nearrow & \downarrow{\scriptstyle \mathrm{Map}(X,p)} \\
\Delta^2 & \xrightarrow{\ b\ } & \mathrm{Map}(X,Y)
\end{array}
$$

where $a_{01} = w$, $a_{02} = spw$, and b is the degeneracy $b = (pw)_{011}$; this commutes since $pspw = pw$. Since p is a Kan fibration, so is $\mathrm{Map}(X, p)$ by (32.7), and therefore a lift t exists. Let $k = t|\Delta^{\{1,2\}}\colon \Delta^1 \to \mathrm{Map}(X, X)$. It is clear that this is a natural isomorphism $k\colon \mathrm{id}_Y \to sp$, and that this is "fiberwise", i.e., $pk = b_{12} = p\pi$ as maps $X \times \Delta^1 \to Y$.

Thus, we have exhibited p as a fiberwise deformation retraction, so p is a trivial fibration by (33.4). $\square$

33.8. Contractible Kan complexes.

33.9. Corollary. *Let X be a simplicial set. The following are equivalent.*

(1) *X is a quasicategory which is categorically equivalent to Δ^0.*
(2) *$X \to \Delta^0$ is a trivial fibration.*
(3) *Every $\partial\Delta^n \to X$ extends over $\partial\Delta^n \subset \Delta^n$.*

Such an X is necessarily a Kan complex.

Proof. We have (2) $\Leftrightarrow$ (3) by definition, and we know that (2) $\Rightarrow$ (1). Given (1), we have that X is a quasigroupoid, and hence a Kan complex (29.2), and (2) follows by the previous proposition (33.7). $\square$

We say that an X satisfying these conditions is a **contractible Kan complex**.

33.10. Monomorphisms which are weak equivalences.
We can now characterize the monomorphisms which are weak equivalences, in terms of maps into Kan complexes.

33.11. Proposition. *Let $j\colon A \to B$ be a monomorphism of simplicial sets. Then j is a weak equivalence if and only if $\mathrm{Map}(j, X)\colon \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is a trivial fibration for all Kan complexes X.*

Proof. Assume X is an arbitrary Kan complex. We know that $\mathrm{Map}(j, X)$ is always a Kan fibration between Kan complexes by (32.7). We have by definition that j is a weak equivalence iff all $\mathrm{Map}(j, X)$ are weak equivalences, which holds iff all $\mathrm{Map}(j, X)$ are trivial fibrations by (33.7). $\square$

33.12. Remark. The class $\mathrm{WkEq} \cap \overline{\mathrm{Cell}}$ of monomorphisms which are weak equivalences is a saturated class. In fact, using (33.11) it is easy to show that it is the left complement of the class of maps of the form $p^{\square j}$, where $p\colon X \to \Delta^0$ is a projection from a Kan complex X, and $j\colon \partial\Delta^n \to \Delta^n$ is a cell inclusion. Furthermore, $\mathrm{WkEq} \cap \overline{\mathrm{Cell}}$ contains the saturated class $\overline{\mathrm{Horn}}$ of anodyne maps.

It turns out that $\overline{\mathrm{Horn}} = \mathrm{WkEq} \cap \overline{\mathrm{Cell}}$, i.e., the injective weak equivalences are *precisely the same* as the anodyne maps. This is a fairly non-trivial fact, and we will address it again later. **Maybe.** As a consequence, it will follow that (33.7) and (33.15) hold without the condition that the objects be Kan complexes.

33.13. Enriched lifting for Kan fibrations between Kan complexes.

33.14. Proposition. *If $j\colon A \to B$ is a monomorphism and a weak equivalence of simplicial sets, and $p\colon X \to Y$ is a Kan fibration between Kan complexes, then*

$$p^{\square i}\colon \mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A, Y)} \mathrm{Map}(B, Y)$$

is a trivial fibration.

Proof. The map $p^{\square i}$ is a Kan fibration between Kan complexes, using $\overline{\mathrm{Horn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$ (32.7). Consider the diagram

$$\mathrm{Map}(B, X) \xrightarrow{p^{\square i}} \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y) \xrightarrow{q'} \mathrm{Map}(A, X)$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\mathrm{Map}(B, Y) \xrightarrow{q} \mathrm{Map}(A, Y)$$

in which the square is a pullback. By (33.11) the maps q and $q'(p^{\square i})$ are trivial fibrations. The pullback q' of q is also a trivial fibration, and so $p^{\square i}$ is a weak equivalence by 2-out-of-3 (32.2), and therefore a trivial fibration since it is a Kan fibration between Kan complexes (33.7). $\qquad\square$

We also obtain another characterization of Kan fibrations between Kan complexes.

33.15. Corollary. *A map $p\colon X \to Y$ between Kan complexes is a Kan fibration if and only if $j \boxempty p$ for all j which are monomorphisms and weak equivalences.*

Proof. ($\Longleftarrow$) Straightforward, since inner horn inclusions are monomorphisms and weak equivalences. ($\Longrightarrow$) Immediate from the previous proposition. $\qquad\square$

34. The fiberwise criterion for trivial fibrations

We give another criterion for Kan fibration to be a trivial fibration, in terms of its fibers.

34.1. Fiberwise criterion for trivial fibrations. The **fiber** $p^{-1}(y)$ of a map $p\colon X \to Y$ over a vertex $y \in Y_0$ is the pullback of p along $\{y\} \to Y$.

If $p\colon X \to Y$ is a trivial fibration, then since $\mathrm{TFib} = \mathrm{Horn}^{\boxempty}$ we see immediately that every projection $p^{-1}(y) \to *$ from a fiber is a trivial fibration; i.e., the fibers of a trivial fibration are necessarily contractible Kan complexes.

34.2. Proposition. *Let $p\colon X \to Y$ be a Kan fibration. Then p is a trivial fibration if and only if every fiber of p is a contractible Kan complex.*

Proof. We have just observed ($\Longrightarrow$) . So suppose p is a Kan fibration whose fibers are contractible Kan complexes, and consider a lifting problem

$$
\begin{array}{ccc}
\partial\Delta^n & \xrightarrow{\;a\;} & X \\
\downarrow & \nearrow^{t} & \downarrow{p} \\
\Delta^n & \xrightarrow[b]{} & Y
\end{array}
$$

We will construct a lift t by using the covering homotopy extension property to "deform" this to a lifting problem involving a single fiber of p, which admits a solution by the hypothesis on the fibers of p.

Let $\gamma\colon \Delta^n \times \Delta^1 \to \Delta^n$ be the unique map given on vertices by $\gamma(i,0) = i$ and $\gamma(i,1) = n$. Thus $\gamma|\Delta^n \times \{0\} = \mathrm{id}$, while $\gamma|\Delta^n \times \{1\}$ factors through the vertex $\{n\} \subseteq \Delta^n$. Since $(\partial\Delta^n \times \{0\} \subset \partial\Delta^n \times \Delta^1)$ is anodyne by $\overline{\mathrm{Horn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$, a lift exists in

$$
\begin{array}{ccc}
\partial\Delta^n \times \{0\} & \xrightarrow{\qquad\qquad a \qquad\qquad} & X \\
\downarrow & \quad\nearrow^{c} & \downarrow{p} \\
\partial\Delta^n \times \Delta^1 \rightarrowtail \Delta^n \times \Delta^1 \xrightarrow{\gamma} \Delta^n \xrightarrow{b} & & Y
\end{array}
$$

Since $b\gamma|\Delta^n \times \{1\}$ factors through a vertex $y \in Y_0$, we have a lift d in

$$
\begin{array}{ccc}
\partial\Delta^n \times \{1\} \xrightarrow{c|\partial\Delta^n\times\{1\}} p^{-1}(y) \rightarrowtail^{\;j\;} & & X \\
\downarrow \quad\nearrow^{d} \qquad\quad \downarrow & & \downarrow{p} \\
\Delta^n \times \{1\} \xrightarrow{\qquad} \{y\} \xrightarrow{y} & & Y
\end{array}
$$

since by hypothesis $p^{-1}(y)$ is a contractible Kan complex. Putting this together we obtain a commutative square

$$
\begin{array}{ccc}
(\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\}) & \xrightarrow{(c,jd)} & X \\
\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle p} \\
\Delta^n \times \Delta^1 & \xrightarrow{b\gamma} & Y
\end{array}
$$

which admits a lift s since $(\partial\Delta^n \subset \Delta^n)\square(\{1\} \subset \Delta^1)$ is anodyne. The restriction $t := s|\Delta^n \times \{0\}$ provides a solution to the original lifting problem, since $t|\partial\Delta^n \times \{0\} = c|\partial\Delta^n \times \{0\} = a$ and $pt = (b\gamma)|\Delta^n \times \{0\} = b$. $\qquad\square$

We often apply the above result in the following way.

34.3. **Corollary.** *Suppose*

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
Y' & \xrightarrow{g} & Y
\end{array}
$$

is a pullback square such that (1) p is a Kan fibration and (2) $g_0 \colon Y_0' \to Y_0$ is surjective. Then p is a trivial fibration if and only if p' is a trivial fibration. Furthermore, if all objects are Kan complexes, then p is a weak equivalence if and only if p' is a weak equivalence.

Proof. The fibers of p all appear as fibers of p' by (2). Use the fiberwise criterion (34.2) and (33.7). $\qquad\square$

34.4. **Criterion for fully faithful map between Kan complexes.** As an application of the fiberwise criterion, we have the following criterion for a Kan fibration between Kan complexes to be fully faithful.

34.5. **Proposition.** *A Kan fibration $p\colon X \to Y$ between Kan complexes is fully faithful iff the induced map*

$$
q = p^{\square(\partial\Delta^1 \subset \Delta^1)} \colon \operatorname{Map}(\Delta^1, X) \to \operatorname{Map}(\Delta^1, Y) \times_{\operatorname{Map}(\partial\Delta^1, Y)} \operatorname{Map}(\partial\Delta^1, X)
$$

is a trivial fibration

Proof. Consider the pullback square

$$
\begin{array}{ccc}
\coprod_{x,x'} \operatorname{map}_X(x, x') & \longrightarrow & \operatorname{Map}(\Delta^1, X) \\
\downarrow & & \downarrow{\scriptstyle q} \\
\coprod_{x,x'} \operatorname{map}_Y(fx, fx') & \longrightarrow & \operatorname{Map}(\Delta^1, Y) \times_{\operatorname{Map}(\partial\Delta^1, Y)} \operatorname{Map}(\partial\Delta^1, X)
\end{array}
$$

where the products are over all pairs of elements of X_0. The map along the bottom is a bijection on vertices. The map q is a Kan fibration by enriched lifting (32.7), and all the objects are Kan complexes, again using (32.7). The result follows using (34.3). $\qquad\square$

35. Fundamental theorem for Kan complexes

In this section, we will prove quasigroupoid version of the fundamental theorem (32.20), i.e., that fully faithful and essentially surjective maps between quasigroupoids (=Kan complexes) are weak equivalences. We note that we have already reduced (32.21) to the case of Kan fibrations.

35.1. Proposition. *A Kan fibration $p\colon X \to Y$ between Kan complexes which is fully faithful and essentially surjective is a weak equivalence.*

Proof. Consider

$$X \xrightarrow{j=(ip,\,\mathrm{id}_X)} \mathrm{Map}(\Delta^1, Y) \times_Y X \xrightarrow{\pi_1} \mathrm{Map}(\Delta^1, Y) \xrightarrow{\langle 1 \rangle^*} Y$$

where the fiber product is constructed from $\mathrm{Map}(\Delta^1, Y) \xrightarrow{\langle 0 \rangle^*} Y \xleftarrow{p} X$, and $i\colon Y \to \mathrm{Map}(\Delta^1, Y)$ is adjoint to the projection $Y \times \Delta^1 \to \Delta^1$. Since $\langle 1 \rangle^* i = \mathrm{id}_Y$ we see that $\langle 1 \rangle^* \pi_1 j = p$.

A straightforward argument shows that the map $\langle 0 \rangle^*\colon \mathrm{Map}(\Delta^1, Y) \to Y$ is the projection part of a fiberwise deformation retraction, with section i; the homotopy is the adjoint to restriction along a suitable map $\Delta^1 \times \Delta^1 \to \Delta^1$. The projection map $\pi_2\colon \mathrm{Map}(\Delta^1, Y) \times_Y X \to X$ is the base change of $\langle 0 \rangle^*$ along p, and so is also a fiberwise deformation retraction (33.3), and so is a simplicial homotopy equivalence, and hence a weak equivalence.

Since $\pi_2 j = \mathrm{id}_X$, it follows that j is a weak equivalence by 2-out-of-3 (32.2). Again using 2-out-of-3, we see that to show p is a weak equivalence it suffices to show that $r_1 := \langle 1 \rangle^* \pi_1$ is a weak equivalence.

Now consider the diagram

$$
\begin{array}{ccccc}
\mathrm{Map}(\Delta^1, Y) \times_{Y \times Y} (X \times X) & \xrightarrow{\;\;\;\;\;\;} & X \times X & \xrightarrow{\;\;\;\;} & X \\[2pt]
{\scriptstyle \mathrm{id} \times (\mathrm{id} \times p)} \big\downarrow & & {\scriptstyle \pi_2'} \quad \quad {\scriptstyle \pi_2'} & & \big\downarrow {\scriptstyle p} \\[2pt]
\mathrm{Map}(\Delta^1, Y) \times_{Y \times Y} (X \times Y) & \xrightarrow{\pi_2} & X \times Y & \xrightarrow{\pi_2} & Y
\end{array}
$$

in which the composite along the bottom is also r_1, and the rectangle is a pullback. We complete the proof using the fiberwise criterion (34.3) to show that r_1 (which is a Kan fibration) is a trivial fibration; i.e., we need that p is surjective on 0-simplices and that r_1' is a trivial fibration.

First, note that since p is an essentially surjective Kan fibration, p must actually be surjective on 0-simplices, by a straightforward argument using the fact that $(\{0\} \subset \Delta^1) \boxslash p$.

Both projections on the top row are Kan fibrations, since $X \to \Delta^0$ and $\mathrm{Map}(\Delta^1, Y) \to \mathrm{Map}(\partial \Delta^1, Y)$ are Kan fibrations. Thus r_1' is a Kan fibration between Kan complexes, so we only need to show that r_1' is a weak equivalence, since then (33.7) will give that r_1' is a trivial fibration.

Now consider

$$\mathrm{Map}(\Delta^1, X) \xrightarrow{q} \mathrm{Map}(\Delta^1, Y) \times_{\mathrm{Map}(\partial \Delta^1, Y)} \mathrm{Map}(\partial \Delta^1, X) \xrightarrow{r_1'} X$$

where q is the evident box-power map. The composite $r_1' q$ is equal to the evident restriction map along $\{1\} \subset \Delta^1$. Since $\{1\} \subset \Delta^1$ is anodyne it follows that $r_1' q$ is a trivial fibration. Finally, since p is fully faithful, (34.5) gives that q is a trivial fibration, and hence it follows that r_1' is a weak equivalence by 2-out-of-3. The proof is complete. $\square$

36. Isofibrations

In this section, we return to isofibrations, which were defined in (28.6). The moral is that isofibrations between quasicategories play a role analogous to Kan fibrations between Kan complexes.

36.1. **Characterizations of isofibrations.** Recall that a functor $f\colon C \to D$ between quasicategories is an isofibration if (1) it is an inner fibration, and (2) every

$$
\begin{array}{ccc}
\{j\} & \longrightarrow & C \\
\downarrow & \nearrow g \nearrow & \downarrow p \\
\Delta^1 & \xrightarrow{\ f\ } & D
\end{array}
$$

with $j = 0$ such that f represents an isomorphism admits a lift g which is also represents isomorphism. Furthermore, it is equivalent to require (2') instead of (2), where (2') is the same statement with $j = 1$.

Note that $C \to *$ is an isofibration for any quasicategory C (because identity maps are isomorphisms).

We have the following "lifting criterion" for isofibrations.

36.2. **Proposition.** *An map p between quasicategories is an isofibration iff (1) it is an inner fibration and (2) $(\{0\} \subset N(\mathrm{Iso})) \boxtimes p$.*

Proof. ($\Longleftarrow$) Straightforward, using the fact (32.15) that every $f\colon \Delta^1 \to D$ representing an isomorphism factors through a map $N(\mathrm{Iso}) \to D$.

($\Longrightarrow$) Let p be an isofibration. To solve the lifting problem

$$
\begin{array}{ccc}
\{0\} & \longrightarrow & C \\
\downarrow & \nearrow s \nearrow & \downarrow p \\
N(\mathrm{Iso}) & \longrightarrow & D
\end{array}
$$

recall from the proof of (32.14) that $N(\mathrm{Iso}) = \bigcup F_k$ where F_k is obtained from F_{k-1} by gluing along $\Lambda_0^k \subset \Delta^k$; we construct lifts $s_k\colon F_k \to C$ inductively. A lift $s_1\colon F_1 = \Delta^1 \to C$ exists by the definition of isofibration, and we may assume that s_1 is an isomorphism in C. Then the Joyal lifting theorem (28.13) provides lifts s_k for $k \geq 2$. $\qquad\square$

In other words, the isofibrations are precisely the maps *between quasicategories* which are contained in $\big(\mathrm{InnHorn} \cup \{\{0\} \subset N(\mathrm{Iso})\}\big)^{\boxtimes}$. In particular, the pullback of an isofibration along a map from a quasicategory is also a quasicategory.

36.3. *Remark.* We have deliberately excluded maps between non-quasicategories from the definition of isofibration. The correct generalization of isofibration to arbitrary simplicial sets is called "categorical fibration", and will be discussed later.

Here is another characterization of isofibrations in terms of cores (=maximal sub-quasigroupoids, defined in (11.5)). Remember that any functor $p\colon C \to D$ restricts to a functor $p^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ between Kan complexes.

36.4. **Proposition.** *A map $p\colon C \to D$ between quasicategories is a isofibration if and only if (1) it is an inner fibration, and (2) $p^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ is a Kan fibration.*

Proof. ($\Longrightarrow$) Let p be an isofibration. Then p^{core} is also an isofibration, by an elementary argument. (The point is that the relevant lifting problems for p^{core} clearly have lifts s with target C, since p is an isofibration; an easy argument shows that the image of such lifts s must actually land in C^{core}.)

Thus have reduced to showing that any isofibration between Kan complexes is a Kan fibration, which is a straightforward exercise using Joyal lifting (32.11).

($\Longleftarrow$) If p^{core} is a Kan fibration, then it is immediate that property (2) of an isofibration holds. $\quad\square$

In particular, isofibrations between Kan complexes are precisely Kan fibrations. (This can be proved directly using Joyal lifting, as in (32.11).)

36.5. **Lifting properties for isofibrations.** We are now ready to prove the following proposition, which will be the key tool in what follows.

36.6. **Proposition.** *Let $p\colon C \to D$ be an isofibration between quasicategories, and $i\colon K \to L$ any monomorphism of simplicial sets. Then the induced pullback-power map*

$$p^{\square i}\colon \operatorname{Fun}(L, C) \to \operatorname{Fun}(K, C) \times_{\operatorname{Fun}(K, D)} \operatorname{Fun}(L, D)$$

is an isofibration.

Proof. Fix a map $p\colon C \to D$ between quasicategories. We first note that the class maps $i\colon K \to L$ such that $p^{\square i}$ is an isofibration is saturated. In fact, let $S := \operatorname{InnHorn} \cup \{\{0\} \subset N(\operatorname{Iso})\}$. Two applications of (16.4) show that $S \boxslash (p^{\square i})$ iff $(S \square i) \boxslash p$ iff $i \boxslash (p^{\square S})$. Therefore, (36.2) implies that the class $\mathcal{C} := \{\, i \mid p^{\square i} \in \operatorname{IsoFib} \,\}$ is the left complement of $p^{\square S}$, and thus saturated.

Therefore, to show that $\mathcal{C}$ contains all monomorphisms, it suffices show that it contains $i = (\partial \Delta^n \subset \Delta^n)$ for $n \geq 0$.

Note that we will certainly have that $p^{\square i}$ is an inner fibration, using using $\overline{\operatorname{InnHorn}} \square \overline{\operatorname{Cell}} \subseteq \overline{\operatorname{InnHorn}}$ (16.7).

If $n = 0$, then $p^{\square i} = p$ so the claim is trivial[27].

Now assume $n \geq 1$. Since $p^{\square i}$ is an inner fibration between quasicategories, it suffices to solve the lifting problem

$$
\begin{array}{ccc}
\{0\} & \longrightarrow & \operatorname{Fun}(\Delta^n, C) \\
\downarrow & \nearrow & \downarrow{\scriptstyle p^{\square i}} \\
\Delta^1 & \xrightarrow{\ f\ } & \operatorname{Fun}(\partial\Delta^n, C) \times_{\operatorname{Fun}(\partial\Delta^n, D)} \operatorname{Fun}(\Delta^n, D)
\end{array}
\qquad \Longleftrightarrow \qquad
\begin{array}{ccc}
(\{0\} \times \Delta^n) \cup (\Delta^1 \times \partial\Delta^n) & \xrightarrow{\ g\ } & C \\
\downarrow & \nearrow & \downarrow{\scriptstyle p} \\
\Delta^1 \times \Delta^n & \longrightarrow & D
\end{array}
$$

where f represents an isomorphism in the target.

The edge $f' := (g|\Delta^1 \times \Delta^{\{0\}})$ is the same as the composite

$$\Delta^1 \xrightarrow{\ f\ } \operatorname{Fun}(\partial\Delta^n, C) \times_{\operatorname{Fun}(\partial\Delta^n, D)} \operatorname{Fun}(\Delta^n, C) \to \operatorname{Fun}(\partial\Delta^n, C) \to \operatorname{Fun}(\{0\}, C).$$

Since f is an isomorphism in the fiber product, it follows that f' is an isomorphism in C. Therefore a lift exists by the pushout-product version of Joyal lifting (29.12), since $n \geq 1$. $\qquad\square$

One consequence of the above is that if p is an isofibration and i is a monomorphisms, then

$$(p^{\square i})^{\operatorname{core}}\colon \operatorname{Fun}(L, C)^{\operatorname{core}} \to \big(\operatorname{Fun}(K, C) \times_{\operatorname{Fun}(K, D)} \operatorname{Fun}(L, D)\big)^{\operatorname{core}}$$

is a Kan fibration (36.4). It turns out we can replace the target (the core of a pullback) with a pullback of cores.

36.7. **Corollary.** *If $p\colon C \to D$ is any functor between quasicategories and $K \subseteq L$, then*

$$(36.8) \qquad \big(\operatorname{Fun}(K, C) \times_{\operatorname{Fun}(K, D)} \operatorname{Fun}(L, D)\big)^{\operatorname{core}} = \operatorname{Fun}(K, C)^{\operatorname{core}} \times_{\operatorname{Fun}(K, D)^{\operatorname{core}}} \operatorname{Fun}(L, D)^{\operatorname{core}}.$$

Proof. Both sides of (36.8) can be regarded as subobjects of $\operatorname{Fun}(K, C) \times_{\operatorname{Fun}(K, D)} \operatorname{Fun}(L, D)$, which we note is a quasicategory since $r\colon \operatorname{Fun}(L, D) \to \operatorname{Fun}(K, D)$ is an inner fibration and $\operatorname{Fun}(K, C)$ is a quasicategory. The left-hand side is clearly contained in the right-hand side, since any functor between quasicategories takes isomorphisms to isomorphisms. By (36.6), $r\colon \operatorname{Fun}(L, D) \to \operatorname{Fun}(K, D)$ is actually a isofibration, and so r^{core} is a Kan fibration (36.4). Thus the right-hand side of (36.8) is a pullback of Kan complexes along a Kan fibration, and thus is a Kan complex, which is necesarily contained in the left-hand side of (36.8). $\qquad\square$

---

[27]This step is the only place in the proof where we actually use the fact that p is an isofibration, and not merely an inner fibration! In fact, if p is merely an inner fibration, but $K_0 = L_0$, then $p^{\square i}$ is a isofibration. This proof is closely related to that of (29.10).

36.9. **Trivial fibrations between quasicategories.** Now we can prove a generalization of (33.7), which identified trivial fibrations between Kan complexes as the Kan fibrations which are weak equivalences.

36.10. **Proposition.** *Let $p\colon C \to D$ be a map between quasicategories. Then p is a trivial fibration if and only if it is an isofibration and a categorical equivalence.*

Proof. [Joy08a, Theorem 5.15]. ($\Longrightarrow$) If p is a trivial fibration, it is an inner fibration and $(\{0\} \subset N(\mathrm{Iso})) \boxdot p$, so it is an isofibration (36.2). We have already shown that p is a categorical equivalence (20.9).

($\Longleftarrow$) Conversely, suppose p is isofibration and categorical equivalence. It suffices to show that for any inclusion $i\colon K \subseteq L$, the map

$$g := (p^{\Box i})^{\mathrm{core}}\colon \mathrm{Fun}(L,C)^{\mathrm{core}} \to \mathrm{Fun}(K,C)^{\mathrm{core}} \times_{\mathrm{Fun}(K,D)^{\mathrm{core}}} \mathrm{Fun}(L,D)^{\mathrm{core}}$$

is surjective on 0-simplicies. In fact, we will show that g is a trivial fibration, which implies what we want. By (36.6), the map $p^{\Box i}\colon \mathrm{Fun}(L,C) \to \mathrm{Fun}(K,C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L,D)$ is an isofibration; by (36.4), the restriction $(p^{\Box i})^{\mathrm{core}}$ of this map to cores is a Kan fibration; by (36.7), this restriction is precisely g. Thus g is a Kan fibration between Kan complexes. It will therefore suffice by (33.7) to show that g is also a categorical equivalence.

The maps

$$\mathrm{Fun}(L,C) \to \mathrm{Fun}(L,D), \qquad \mathrm{Fun}(K,C) \to \mathrm{Fun}(K,D)$$

are categorical equivalences since p is, and so induce categorical equivalences on cores. These maps are also isofibrations by (36.6), and therefore the restrictions to cores are Kan fibrations. Thus $\mathrm{Fun}(L,C)^{\mathrm{core}} \to \mathrm{Fun}(L,D)^{\mathrm{core}}$ and $\mathrm{Fun}(K,C)^{\mathrm{core}} \to \mathrm{Fun}(K,D)^{\mathrm{core}}$ are Kan fibrations and weak equivalences between Kan complexes, and thus are trivial fibrations (33.7). Thus in

$$\mathrm{Fun}(L,C)^{\mathrm{core}} \xrightarrow{g} \mathrm{Fun}(K,C)^{\mathrm{core}} \times_{\mathrm{Fun}(K,D)^{\mathrm{core}}} \mathrm{Fun}(L,D)^{\mathrm{core}} \to \mathrm{Fun}(L,D)^{\mathrm{core}}$$

the second map is a trivial fibration being a pullback of $\mathrm{Fun}(K,p)^{\mathrm{core}}$, and the composite is a weak equivalence. It follows that g is a weak equivalence by the 2-out-of-3 property for weak equivalences (32.2), and the result is proved. $\square$

This proof made essential use of our characterization of trivial fibrations between Kan complexes, which is why we had to prove that special case first.

Now we show that isofibrations have liftings with respect to all monomorphisms which are categorical equivalences.

36.11. **Proposition.** *A map $p\colon C \to D$ with D a quasicategory is an isofibration if and only if $j \boxdot p$ for every $j\colon K \to L$ which is both a monomorphism and a categorical equivalence.*

Proof. ($\Longleftarrow$) Immediate from the the characterization of isofibrations as maps between quasicategories in the right complement of $\mathrm{InnHorn} \cup \{\{0\} \subset N\mathrm{Iso}\}$ (36.2).

($\Longrightarrow$) Suppose p is an isofibration. It suffices to show that for $j \in \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$,

$$q\colon \mathrm{Fun}(K,C) \to \mathrm{Fun}(L,C) \times_{\mathrm{Fun}(L,D)} \mathrm{Fun}(K,D)$$

is a trivial fibration, whence it is surjective on 0-simplices and thus $j \boxdot p$. Since j is a monomorphism, (36.6) says q is an isofibration between quasicategories, so by the criterion for an isofibration to be a trivial fibration (36.10) it suffices to show that q is a categorical equivalence. Since j is a categorical equivalence both $\mathrm{Fun}(j,C)$ and $\mathrm{Fun}(j,D)$ are categorical equivalences. The map $\mathrm{Fun}(j,D)$ is also an isofibration by (36.6), so it is a trivial fibration by (36.10). Therefore the basechange $\mathrm{Fun}(L,C) \times_{\mathrm{Fun}(L,D)} \mathrm{Fun}(K,D) \to \mathrm{Fun}(L,C)$ of $\mathrm{Fun}(j,D)$ is a trivial fibration, and the result follows using the 2-out-of-3 property of categorical equivalences (22.9). $\square$

36.12. **Monomorphisms which are categorical equivalences.** We can now prove a generalization of (33.11), which characterized the injective weak equivalences.

36.13. **Proposition.** *Let $j\colon K \to L$ be a monomorphism of simplicial sets. Then j is a categorical equivalence if and only if* $\mathrm{Map}(j, C)\colon \mathrm{Map}(L, C) \to \mathrm{Map}(K, C)$ *is a trivial fibration for all quasicategories C.*

Proof. Straightforward using the fact that $\mathrm{Map}(j, C)$ is an isofibration for any inclusion (36.6), and that isofibrations which are categorical equivalences are trivial fibrations (36.10). $\square$

36.14. *Remark.* As a consequence of (36.13), we see that the class $\overline{\mathrm{Cell}} \cap \mathrm{CatEq}$ of injective categorical equivalences between simplicial sets is a saturated class: it is the left complement of the class of maps of the form $p^{\square i}$ where $p\colon C \to \Delta^0$ is a projection from a quasicategory, and i is a cell inclusion. Furthermore, this class contains the class of inner anodyne maps.

This class is *not the same* as $\overline{\mathrm{InnHorn}}$. For instance, every inner anodyne map is a bijection on vertices, but $\{0\} \to N\mathrm{Iso}$ which is not bijective on vertices is an injective categorical equivalence. Neither is it the same as the saturation of $\mathrm{InnHorn} \cup \{\{0\} \subset N\mathrm{Iso}\}$. **(Proof?)** This is a significant way in which the theory of quasicategories is not entirely parallel with the theory of Kan complexes.

37. LOCALIZATION OF QUASICATEGORIES

37.1. **Quasigroupoidification.** Let C be a quasicategory, and X a simplicial set. Let

$$\mathrm{Fun}^{(X)}(X, C) \subseteq \mathrm{Fun}(X, C)$$

denote the full subquasicategory spanned by objects which are functors $f\colon X \to C$ with the property that $f(X) \subseteq C^{\mathrm{core}}$.

Note that $\mathrm{Fun}^{(X)}(X, C)$ is a quasicategory, but not necessarily a quasigroupoid: for instance, morphisms are functors $f\colon X \times \Delta^1 \to C$ such that $f(X \times \{0\}) \subseteq C^{\mathrm{core}}$ and $f(X \times \{1\}) \subseteq C^{\mathrm{core}}$, but need not satisfy $f(X \times \Delta^1) \subseteq C^{\mathrm{core}}$.

37.2. **Lemma.** *Consider $\Delta^1 \subset N(\mathrm{Iso})$ representing the map $0 \to 1$ in Iso. The restriction map* $\mathrm{Fun}(N(\mathrm{Iso}), C) \to \mathrm{Fun}(\Delta^1, C)$ *factors through a trivial fibration*

$$\mathrm{Fun}(N(\mathrm{Iso}), C) \to \mathrm{Fun}^{(\Delta^1)}(\Delta^1, C).$$

Proof. We've already proved this is surjective on zero-simplices (32.15). In that proof, we noted that $N(\mathrm{Iso}) = \bigcup F_k$ with $F_k = F_{k-1} \cup_{\Lambda_0^k} \Delta^k$ and $F_1 = \Delta^1$, and the result was immediate using the Joyal extension theorem.

To prove this lemma, it suffices to show that each $\mathrm{Fun}^{(\Delta^1)}(F_k, C) \to \mathrm{Fun}^{(\Delta^1)}(F_{k-1}, C)$ is a trivial fibration for $k \geq 2$. There are pullback squares

$$
\begin{array}{ccc}
\mathrm{Fun}^{(\Delta^1)}(F_k, C) & \longrightarrow & \mathrm{Fun}^{(\Delta^{\{0,1\}})}(\Delta^k, C) \\
\downarrow & & \downarrow \\
\mathrm{Fun}^{(\Delta^1)}(F_{k-1}, C) & \longrightarrow & \mathrm{Fun}^{(\Delta^{\{0,1\}})}(\Lambda_0^k, C)
\end{array}
$$

so it suffices to solve the lifting problems $(\partial \Delta^n \subset \Delta^n) \boxtimes (\mathrm{Fun}^{(\Delta^{\{0,1\}})}(\Delta^k, C) \to \mathrm{Fun}^{(\Delta^{\{0,1\}})}(\Lambda_0^k, C))$ when $n \geq 0$ and $k \geq 0$. This adjoints to the lifting problem

$$
\begin{array}{ccc}
\Delta^{\{0,1\}} \rightarrowtail \Lambda_0^k & \xrightarrow{\quad f \quad} & \mathrm{Fun}(\Delta^n, C) \\
\downarrow & \nearrow & \downarrow \\
\Delta^k & \longrightarrow & \mathrm{Fun}(\partial \Delta^n, C)
\end{array}
$$

where f is such that for each $j \in [n]$ the composite $\Delta^{\{0,1\}} \to \operatorname{Fun}(\Delta^n, C) \to \operatorname{Fun}(\{j\}, C) \approx C$ represents an isomorphism in C. By the objectwise criterion for natural isomorphisms (C) we see that f itself represents an isomorphism in $\operatorname{Fun}(\Delta^n, C)$, so the desired lift exists by Joyal lifting (28.13). $\square$

37.3. Proposition. *Let $i\colon X \to X'$ be any anodyne map to a Kan complex X'. Then the restriction map $\operatorname{Fun}(i, C)\colon \operatorname{Fun}(X', C) \to \operatorname{Fun}(X, C)$ factors through a trivial fibration*

$$q\colon \operatorname{Fun}(X', C) \to \operatorname{Fun}^{(X)}(X, C).$$

Proof. Every edge in X maps to an isomorphism in X'. Therefore $\operatorname{Fun}(i, C)$ must factor through a map q into the subcomplex $\operatorname{Fun}^{(X)}(X, C) \subseteq \operatorname{Fun}(X, C)$. We also know that $\operatorname{Fun}(i, C)$ is an isofibration by (36.6), and therefore the map q is also an isofibration. Therefore by (36.10) it is enough to show that for each such i the map q is a categorical equivalence.

Given any two anodyne maps $i\colon X \to X'$ and $i'\colon X \to X''$ to Kan complexes, there exists a categorical equivalence $f\colon X' \to X''$ such that $fi = i'$ (because $\operatorname{Fun}(X', X'') \to \operatorname{Fun}(X, X'')$ is a trivial fibration by (32.7)). Therefore in the commutative diagram

$$\operatorname{Fun}(X', C) \xrightarrow{\operatorname{Fun}(f,C)} \operatorname{Fun}(X'', C)$$

$$q \searrow \qquad \swarrow q'$$

$$\operatorname{Fun}^{(X)}(X, C)$$

the map $\operatorname{Fun}(f, C)$ is a categorical equivalence, so q is a categorical equivalence iff q' is.

Thus, given a simplicial set X, to show that *any* anodyne $i\colon X \to X'$ to a Kan complex induces a categorical equivalence $q\colon \operatorname{Fun}(X', C) \to \operatorname{Fun}^{(X)}(X, C)$, it suffices to do so for a *single choice* of such an i.

Now given any simplicial sets X, construct maps f and g of the form

$$X \xrightarrow{f} Y := X \cup_{\coprod \Delta^1} \coprod N(\mathrm{Iso}) \xrightarrow{g} X',$$

where the coproducts defining Y are taken over all maps $\Delta^1 \to X$, and g is any inner anodyne map to a quasicategory. By construction f and g are anodyne, and therefore $i = gf$ is anodyne.

I claim that the quasicategory X' is actually a Kan complex, i.e., a quasigroupoid, i.e., a quasicategory such that $h(X')$ is a groupoid. To see this, note that every edge in Y factors through a map $\mathrm{Iso} \to Y$ by construction; from this, we see that every edge in the simplicial set Y must be a preisomorphism, whence its fundamental category hY is a groupoid. Since g is inner anodyne, $hY \to hX'$ is an equivalence, so hX' is a groupoid as desired.

Now $\operatorname{Fun}(i, C)$ is the composite of

$$\operatorname{Fun}(X', C) \xrightarrow{\operatorname{Fun}(g,C)} \operatorname{Fun}(Y, C) \xrightarrow{p} \operatorname{Fun}^{(X)}(X, C) \subseteq \operatorname{Fun}(X, C).$$

Since g is inner anodyne, $\operatorname{Fun}(g, C)$ is a trivial fibration by (36.6). The map p is a pullback of maps $\operatorname{Fun}(N(\mathrm{Iso}), C) \to \operatorname{Fun}^{(\Delta^1)}(\Delta^1, C)$, which are also trivial fibrations by (37.2). The result is proved. $\square$

We write $X \to X_{\mathrm{Kan}}$ for any choice of anodyne map to a Kan complex, and call it a **quasi-groupoidification** of X. What we have shown is that for any quasicategory C we get a categorical equivalence

$$\operatorname{Fun}(X_{\mathrm{Kan}}, C) \approx \operatorname{Fun}^{(X)}(X, C).$$

We can apply this construction when X is a quasicategory, or even when X is the nerve of an ordinary category, and obtain interesting new Kan complexes.

37.4. *Example.* It turns out that every simplicial set is weakly equivalent to the nerve of some ordinary category, and in fact the nerve of some poset [Tho80]. Thus, for every Kan complex K, there exists an ordinary category A and a weak equivalence $NA \to K$, and hence a categorical equivalence $(NA)_{\mathrm{Kan}} \to K$ between Kan complexes.

We note that there is also a classical groupoidification construction, which given an ordinary category A produces an ordinary groupoid A_{Gpd} by "formally inverting all maps". We have that $h((NA)_{\mathrm{Kan}}) \approx N(A_{\mathrm{Gpd}})$, but in general $(NA)_{\mathrm{Kan}}$ is not weakly equivalent to $N(A_{\mathrm{Gpd}})$.

37.5. *Exercise.* Let A be the poset of *proper and non-empty* subsets of $\{0, 1, 2, 3\}$. Show that A_{Gpd} is equivalent to the one-object category, but that NA_{Kan} is not equivalent to the one-object category. (In the second case, the idea is that the geometric realization of NA is a 2-sphere. Explicitly, you can prove non-equivalence by showing $\pi_0 \operatorname{Fun}(NA, K(\mathbb{Z}, 2)) \approx \mathbb{Z}$, using the Eilenberg-MacLane object of §8.8.)

37.6. Localization of quasicategories. There is a more general construction, which applies to a simplicial set X equipped with a subcomplex $W \subseteq X$. Let

$$\operatorname{Fun}^{(W)}(X, C) \subseteq \operatorname{Fun}(X, C)$$

denote the full subcategory spanned by objects $f \colon X \to C$ such that $f(W) \subseteq C^{\mathrm{core}}$. Note that this really only depends on the 1-simplices in W.

37.7. Proposition. *Given an inclusion $W \subseteq X$, choose an anodyne map $W \to W_{\mathrm{Kan}}$ to a Kan complex, and then choose an inner anodyne map $g \colon Y := X \cup_W W_{\mathrm{Kan}} \to X'$ to a quasicategory. Then for any quasicategory C, the restriction map $\operatorname{Fun}(X', C) \to \operatorname{Fun}(X, C)$ factors through a trivial fibration*

$$\operatorname{Fun}(X', C) \to \operatorname{Fun}^{(W)}(X, C).$$

Proof. We have

$$
\begin{array}{ccccccc}
\operatorname{Fun}(X', C) & \xrightarrow{\operatorname{Fun}(g, C)} & \operatorname{Fun}(Y, C) & \longrightarrow & \operatorname{Fun}^{(W)}(X, C) & \rightarrowtail & \operatorname{Fun}(X, C) \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \operatorname{Fun}(W_{\mathrm{Kan}}, C) & \xrightarrow[p]{} & \operatorname{Fun}^{(W)}(W, C) & \rightarrowtail & \operatorname{Fun}(W, C)
\end{array}
$$

in which both squares are pullbacks. The map $\operatorname{Fun}(g, C)$ is a trivial fibration since g is inner anodyne, while p is a trivial fibration as we have shown (37.3). $\qquad\square$

We sometimes write $X \to X_{(W)}$ for any map $X \to X'$ obtained as in the proposition. Note that any $X \to X_{(X)}$ is an example of $X \to X_{\mathrm{Kan}}$. The observation is that for any quasicategory C, we have a categorical equivalence

$$\operatorname{Fun}^{(W)}(X, C) \approx \operatorname{Fun}(X_{(W)}, C).$$

37.8. Quasicategories from relative categories. A **relative category** is a pair $W \subseteq C$ consisting of an *ordinary category* C and a subcategory W containing all the objects of C. The above construction gives, for any relative category, a map

$$C \to C_{(W)},$$

unique up to categorical equivalence. We may call $C_{(W)}$ the **localization** of C with respect to W.

It turns out that many quasicategories of interest arise as such localizations.

38. The path fibration

To prove the fundamental theorem of quasicategories for a general map between quasicategories, we will reduce to the special case of isofibrations. We do this by means of the "path fibration" construction, which provides a factorization of a map into a categorical fibration followed by an isofibration.

38.1. **The path fibration for quasicategories.** Let $f\colon C \to D$ be a functor between quasicategories. We define a factorization $C \xrightarrow{j} P(f) \xrightarrow{p} D$ by means of the following diagram.

$$
\begin{array}{ccccc}
& & & \overset{p}{\overgroup{}} & \\
C & \xrightarrow{\ j\ } & P(f) & \longrightarrow \mathrm{Fun}^{(\Delta^1)}(\Delta^1, D) \xrightarrow[r_1]{} & D \\
& \Vert & \ \downarrow{\scriptstyle s_0} & \ \ \downarrow{\scriptstyle r_0} & \\
& & C & \xrightarrow[\ f\ ]{} & D
\end{array}
$$

Here the the square is a pullback square. The map j is the unique one so that $s_0 j = \mathrm{id}_C$, and tj is induced by $C \xrightarrow{f} D \xrightarrow{\widetilde{\pi}} \mathrm{Fun}(\Delta^1, D)$ where $\widetilde{\pi}$ is adjoint to the projection $D \times \Delta^1 \to D$. The maps $r_i\colon \mathrm{Fun}^{(\Delta^1)}(\Delta^1, D) \subseteq \mathrm{Fun}(\Delta^1, D) \to D$ are induced by restriction along $\{i\} \subset \Delta^1$.

In particular, the objects of $P(f)$ are pairs (c, α) consisting of an object $c \in C_0$ and an isomorphism $\alpha\colon f(c) \to d$ in D. The map j sends an object c to $(c, 1_{f(c)})$, while p sends (c, α) to d.

The factorization $C \xrightarrow{j} P(f) \xrightarrow{p} D$ is called the **path fibration** of f, because of the following.

38.2. **Lemma.** *The map j is a categorical equivalence and p is an isofibration.*

Proof. For $i \in \{0, 1\}$ the inclusion $\{i\} \subset N(\mathrm{Iso})$ is an equivalence of ordinary categories, hence a categorical equivalence. Therefore the composite map

$$
\mathrm{Fun}(N(\mathrm{Iso}), D) \xrightarrow{q} \mathrm{Fun}^{(\Delta^1)}(\Delta^1, D) \xrightarrow{r_i} D
$$

induced by restiction along $\{i\} \subset N(\mathrm{Iso})$ is a categorical equivalence. We have shown (37.2) that q is a trivial fibration, and therefore the r_i are categorical equivalences by 2-out-of-3 (22.9).

The restriction map $\mathrm{Fun}(\Delta^1, D) \to \mathrm{Fun}(\partial\Delta^1, D) = D \times D$ is an isofibration (36.6). We claim that $r\colon \mathrm{Fun}^{(\Delta^1)}(\Delta^1, D) \to D \times D$ is also an isofibration. It easily seen that r is an inner fibration, since $\mathrm{Fun}^{(\Delta^1)}(\Delta^1, D) \subseteq \mathrm{Fun}(\Delta^1, D)$ is full. To show that r is also isofibration is then straightforward (e.g., if $\alpha\colon \alpha_0 \to \alpha_1$ is a natural isomorphism of functors $\Delta^1 \to D$, and $\alpha_0(\Delta^1) \subseteq D^{\mathrm{core}}$, then $\alpha_1(\Delta^1) \subseteq D^{\mathrm{core}}$).

Since the projections $D \times D \to D$ are isofibrations, it follows that the r_i are isofibrations, and hence trivial fibrations since they are categorical equivalences (36.10).

Therefore s_0, being a pullback of r_0, is a trivial fibration. Therefore $P(f)$ is a quasicategory and j is a categorical equivalence.

In the commutative diagram

$$
\begin{array}{ccc}
& P(f) \longrightarrow \mathrm{Fun}^{(\Delta^1)}(\Delta^1, D) \\
{\scriptstyle p}\swarrow \ \ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle r=(r_0, r_1)} \\
D \xleftarrow[\pi]{} C \times D \xrightarrow[f \times \mathrm{id}_D]{} D \times D
\end{array}
$$

the square is a pullback. We have shown that r is an isofibration, and hence so is its pullback s. Therefore $p = \pi s$ is an isofibration, as desired. $\qquad\square$

38.3. **The path fibration for Kan complexes.** If $f: C \to D$ is a functor between quasicategories, and D is a Kan complex, then $\mathrm{Fun}^{(\Delta^1)}(\Delta^1, D) = \mathrm{Fun}(\Delta^1, D)$, and the diagram defining the path fibration takes the form

$$
\begin{array}{ccccc}
C & \xrightarrow{\ j\ } & P(f) & \xrightarrow{\hspace{1.5em}} & \mathrm{Fun}(\Delta^1, D) \xrightarrow{\ r_1\ } D \\
& \| & \ \downarrow{\scriptstyle s_0} & & \ \downarrow{\scriptstyle r_0} \\
& & C & \xrightarrow{\ f\ } & D
\end{array}
$$

38.4. **Proposition.** *If $f: C \to D$ is a functor between Kan complexes, then j is a weak equivalence and p is a Kan fibration.*

Proof. Immediate. $\hspace{1em}\square$

39. PROOF OF THE FUNDAMENTAL THEOREM

We are ready to finish the proof of (E), The Fundamental Theorem of Quasicategories.

39.1. **Proposition.** *If $f: C \to D$ is a fully faithful and essentially surjective functor between quasicategories, then f is a categorical equivalence.*

We will prove this below. First note that the quasigroupoid version of this result (which we proved earlier) gives us the following.

39.2. **Lemma.** *If $f: C \to D$ is a fully faithful and essentially surjective functor between quasicategories, then $f: C^{\mathrm{core}} \to D^{\mathrm{core}}$ is a categorical equivalence (in fact, a weak equivalence).*

Proof. Note that if f is fully faithful and essentially surjective, so is f^{core}. Apply (32.20). $\hspace{1em}\square$

Note that the path fibration construction gives us a factorization of f into $C \xrightarrow{j} P(f) \xrightarrow{p} D$, where j is a categorical equivalence and p is an isofibration. Because both categorical equivalences and (fully faithful + essentially surjective) are classes which satisfy 2-out-of-3 (22.9), (31.5), proving the fundamental theorem reduces to the case that f is itself an isofibration.

To prove the isofibration case of (39.1), we will deduce it from the following.

39.3. **Proposition.** *If $p: C \to D$ is an isofibration which is fully fathiful and essentially surjective, then $q = (p^{\square i})^{\mathrm{core}}: \mathrm{Fun}(L, C)^{\mathrm{core}} \to \mathrm{Fun}(K, C)^{\mathrm{core}} \times_{\mathrm{Fun}(K,D)^{\mathrm{core}}} \mathrm{Fun}(L, D)^{\mathrm{core}}$ is a trivial fibration for every monomorphism $i: K \to L$.*

Proof that (39.3) implies (39.1). As noted above, it is enough to consider isofibrations $p: C \to D$ which are fully faithful and essentially surjective. By (39.3), for any monomorphism $i: K \to L$ the map $q = (p^{\square i})^{\mathrm{core}}$ is a trivial fibration, and therefore surjective on vertices. The core of a quasicategory has all its objects, and thus the box power map $p^{\square i}: \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}(L, D)$ is surjective on vertices. Thuse, we have $i \boxempty p$ for every monomorphism i, whence p is a trivial fibration, and thus a categorical equivalence. $\hspace{1em}\square$

39.4. **Proof of** (39.3)**.** We start with the following lemma, which says that isofibrations between quasicategories which are trivial fibrations on cores are characterized by a lifting property.

39.5. **Lemma.** *There exists a set of maps S such that for any isofibration $q: C \to D$ between quasicategories, we have $S \boxempty q$ iff $q^{\mathrm{core}} \in \mathrm{TFib}$.*

Proof. Given an inclusion $K \subseteq L$ of simplicial sets, we can use two applications of the small object argument (13.10) to construct

$$
\begin{array}{ccc}
K & \longrightarrow & K_{\mathrm{Kan}} \\
\downarrow & & \downarrow \qquad \searrow \\
L & \longrightarrow L' & \longrightarrow L_{\mathrm{Kan}}
\end{array}
$$

in which the square is pushout, the horizontal maps are anodyne, and the objects K_{Kan} and L_{Kan} are Kan complexes. (So $K \to K_{\mathrm{Kan}}$ and $L \to L_{\mathrm{Kan}}$ are examples of quasigroupoidification as in (37.1).

We will show that $(K \subseteq L) \boxslash q^{\mathrm{core}}$ iff $(K_{\mathrm{Kan}} \subseteq L_{\mathrm{Kan}}) \boxslash q$. The lemma will follow immediately by taking $S = \{ (\partial \Delta^n)_{\mathrm{Kan}} \subset (\Delta^n)_{\mathrm{Kan}} \mid n \geq 0 \}$.

($\Longrightarrow$) Suppose $(K \subseteq L) \boxslash q^{\mathrm{core}}$. Since K_{Kan} and L_{Kan} are Kan complexes, any maps from them to quasicategories must factor through cores. Thus it suffices to find a lift in the right-hand square of

$$
\begin{array}{ccccccc}
K & \longrightarrow & K_{\mathrm{Kan}} & = & K_{\mathrm{Kan}} & \longrightarrow & C^{\mathrm{core}} \\
\downarrow & & \downarrow & {}^{s} & \downarrow & {}^{s''} & \downarrow {}^{q^{\mathrm{core}}} \\
L & \longrightarrow & L' & \longrightarrow & L_{\mathrm{Kan}} & \longrightarrow & D^{\mathrm{core}}
\end{array}
$$

By hypothesis, a lift s exists, and therefore a lift s' since the left-hand square is a pushout. Because $L' \to L_{\mathrm{Kan}}$ is anodyne and q^{core} is a Kan fibration (36.4), an extension to a lift s'' exists.

($\Longleftarrow$) Suppose $(K_{\mathrm{Kan}} \subseteq L_{\mathrm{Kan}}) \boxslash q$. Consider a lifting problem

$$
\begin{array}{ccc}
K & \xrightarrow{a} & C^{\mathrm{core}} \\
\downarrow & & \downarrow {}^{q^{\mathrm{core}}} \\
L & \longrightarrow & D^{\mathrm{core}}
\end{array}
$$

Because C^{core} is a Kan complex and $K \to K_{\mathrm{Kan}}$ is anodyne, the map a factors through some $a' \colon K_{\mathrm{Kan}} \to C^{\mathrm{core}}$, and there is a unique compatible map $b' \colon L' \to D^{\mathrm{core}}$ from the pushout along $K \subset L$. Again, b' factors through $b'' \colon L_{\mathrm{Kan}} \to D^{\mathrm{core}}$. Thus we have extended the original square to a diagram

$$
\begin{array}{ccccccccc}
K & \longrightarrow & K_{\mathrm{Kan}} & = & K_{\mathrm{Kan}} & \xrightarrow{a'} & C^{\mathrm{core}} & \rightarrowtail & C \\
\downarrow & & \downarrow & & \downarrow & {}^{t'} & \downarrow {}^{t} & & \downarrow {}^{q} \\
L & \longrightarrow & L' & \longrightarrow & L_{\mathrm{Kan}} & \xrightarrow{b''} & D^{\mathrm{core}} & \rightarrowtail & D
\end{array}
$$

A lift t exists by hypothesis, and since L_{Kan} is a Kan complex it factors through a unique lift t' (using that $C^{\mathrm{core}} \to C$ and $D^{\mathrm{core}} \to D$ are monomorphisms). The composite $L \to L_{\mathrm{Kan}} \to C^{\mathrm{core}}$ is the desired lift. $\qquad \square$

Fix an isofibration $p \colon C \to D$ between quasicategories which is fully faithful and essentially surjective. Consider the class

$$
\mathcal{C}_p := \{ i \mid i \text{ is a monomorphism and } (p^{\square i})^{\mathrm{core}} \in \mathrm{TFib} \}.
$$

The statement of (39.3) amounts to showing that $\mathcal{C}_p$ contains every monomorphism.

39.6. Lemma. *The class $\mathcal{C}_p$ is saturated.*

Proof. First note that for any monomorphism i, the map $p^{\square i}$ is an isofibration since p is (36.6). Using the set of maps S provided by the previous lemma (39.5), for a monomorphism i we have that $(p^{\square i})^{\mathrm{core}} \subseteq \mathrm{TFib}$ iff $S \boxslash (p^{\square i})$ iff $i \boxslash (p^{\square S})$. Thus $\mathcal{C}_p$ is the intersection of ${}^{\boxslash}(p^{\square S})$ with $\overline{\mathrm{Cell}}$, and so is saturated. $\qquad \square$

39.7. Lemma. *Let $p\colon C \to D$ be an isofibration between quasicategories. If $K \subseteq L$, and if $(\varnothing \subset K)$ and $(\varnothing \subset L)$ are elements of $\mathcal{C}_p$, then $(K \subseteq L) \in \mathcal{C}_p$.*

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
& \xrightarrow{\quad \mathrm{Map}(L,p)^{\mathrm{core}} \quad} & \\
\mathrm{Map}(L,C)^{\mathrm{core}} \xrightarrow[(p^{\square i})^{\mathrm{core}}]{} \big(\mathrm{Map}(K,C) \times_{\mathrm{Map}(K,D)} \mathrm{Map}(L,D)\big)^{\mathrm{core}} \xrightarrow[q]{} & \mathrm{Map}(L,D)^{\mathrm{core}} \\
& \downarrow & \downarrow \\
& \mathrm{Map}(K,C)^{\mathrm{core}} \xrightarrow[\mathrm{Map}(K,p)^{\mathrm{core}}]{} & \mathrm{Map}(K,D)^{\mathrm{core}}
\end{array}
$$

where the square is a pullback (36.7). By hypothesis, both $\mathrm{Map}(K,p)^{\mathrm{core}}$ and $\mathrm{Map}(L,p)^{\mathrm{core}}$ are trivial fibrations, whence q is also a trivial fibration. By 2-out-of-3, we have that $(p^{\square i})^{\mathrm{core}}$ is a categorical equivalence. Since p is an isofibration, we have that $(p^{\square i})^{\mathrm{core}}$ is a Kan fibration (36.6), (36.7) between Kan complexes, and therefore is a trivial fibration (33.7) as desired. $\qquad\square$

Next we observe that $(\varnothing \subseteq \Delta^n) \in \mathcal{C}_p$ if p is fully faithful and essentially surjective.

39.8. Proposition. *If $p\colon C \to D$ is an isofibration which is fully faithful and essentially surjective, then $\mathrm{Fun}(\Delta^n, C)^{\mathrm{core}} \to \mathrm{Fun}(\Delta^n, D)^{\mathrm{core}}$ is a trivial fibration for all $n \geq 0$.*

Proof. In the case $n = 0$, this means showing that $p^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ is a trivial fibration, which we have already observed (39.2).

For $n \geq 1$, consider the diagram

$$
\begin{array}{ccc}
\displaystyle\coprod_{c_0,\ldots,c_n} \mathrm{map}_C(c_0,\ldots,c_n) \rightarrowtail & \mathrm{Fun}(\Delta^n, C)^{\mathrm{core}} & \\
{\scriptstyle \coprod q_{c_0,\ldots,c_n}} \downarrow \qquad\qquad & \downarrow {\scriptstyle q} & \searrow {\scriptstyle \mathrm{Fun}(\Delta^n,p)^{\mathrm{core}}} \\
\displaystyle\coprod_{c_0,\ldots,c_n} \mathrm{map}_D(pc_0,\ldots,pc_n) \xrightarrow{j} & (C^{\times n+1})^{\mathrm{core}} \times_{(D^{\times n+1})^{\mathrm{core}}} \mathrm{Fun}(\Delta^n,D)^{\mathrm{core}} \xrightarrow{r} & \mathrm{Fun}(\Delta^n,D)^{\mathrm{core}} \\
\downarrow \qquad\qquad & \downarrow & \downarrow \\
C_0^{\times n+1} \rightarrowtail & (C^{\times n+1})^{\mathrm{core}} \xrightarrow{(p^{\times n+1})^{\mathrm{core}}} (D^{\times n+1})^{\mathrm{core}} &
\end{array}
$$

in which all the squares are pullbacks. The map r is a base change of $(p^{\times n+1})^{\mathrm{core}}\colon (C^{\times n+1})^{\mathrm{core}} \to (D^{\times n+1})^{\mathrm{core}}$, which is isomorphic to the $(n+1)$-fold product of $p^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$, which we have just noted is a trivial fibration. Thus, r is a trivial fibration, so it will suffice to show that q is a trivial fibration, for which we will use the fiberwise criterion (34.3).

We know that $p^{\square(\mathrm{Sk}_0\,\Delta^n \subset \Delta^n)}$ is an isofibration (36.6), and thus $q = (p^{\square(\mathrm{Sk}_0\,\Delta^n \subset \Delta^n)})^{\mathrm{core}}$ is a Kan fibration (36.4). Because p is an essentially surjective isofibration, p^{core} is surjective on vertices, and thus j is surjective on 0-simplices. Thus by the fiberwise criterion (34.3) we need to show that $\coprod q_{c_0,\ldots,c_n}$ is a trivial fibration. Since coproducts of trivial fibrations are trivial fibrations (20.2) we thus reduce to showing that each $q_{c_0,\ldots,c_n}$ (which is a Kan fibration between Kan complexes being a pullback of q) is a weak equivalence, and hence a trivial fibration (33.7). This is immediate from the fact that p is fully faithful and induces maps compatible with the weak equivalences $\mathrm{map}_C(c_0,\ldots,c_n) \to \mathrm{map}_C(c_0,c_1) \times \cdots \times \mathrm{map}_C(c_{n-1},c_n)$ and $\mathrm{map}_D(pc_0,\ldots,pc_n) \to \mathrm{map}_D(pc_0,pc_1) \times \cdots \mathrm{map}_D(pc_{n-1},pc_n)$. $\qquad\square$

Proof of (39.3). As we have already noted it suffices to show that

$$
\overline{\mathrm{Cell}} \subseteq \mathcal{C}_p = \{\, i \mid i \in \overline{\mathrm{Cell}},\ (p^{\square i})^{\mathrm{core}} \in \mathrm{TFib} \,\}
$$

for any fully faithful and essentially surjective isofibration $p\colon C \to D$. As $\mathcal{C}_p$ is saturated (39.6), it is enough to show that $(\partial\Delta^n \subset \Delta^n) \in \mathcal{C}_p$ for all $n \geq 0$. We will do this by induction on n.

For the case $n = 0$, this is immediate from (39.8). For $n \geq 1$, suppose we have $(\partial\Delta^k \subset \Delta^k) \in \mathcal{C}_p$ for $k < n$. Since $\partial\Delta^n$ is equal to its own $(n-1)$-skeleton, skeletal filtration (15.16) gives that $(\varnothing \subset \partial\Delta^n) \in \mathcal{C}_p$. Then use (39.7) and (39.8) to conclude that $(\partial\Delta^n \subset \Delta^n) \in \mathcal{C}_p$ as desired. $\qquad\square$

Part 5. Model categories

40. Categorical fibrations

A map $p\colon X \to Y$ of simplicial sets is a **categorical fibration** if and only if $j \boxslash p$ for all j which are monomorphisms and categorical equivalences. I'll write CatFib for the class of categorical fibrations.

Categorical fibrations generalize isofibrations. In fact, a map $p\colon C \to D$ with D a quasicategory is a categorical fibration if and only if it is an isofibration, as we proved in (36.11).

40.1. Proposition. *A map $p\colon X \to Y$ of simplicial sets is a trivial fibration if and only if it is a categorical fibration and a categorical equivalence.*

Proof. ($\Longrightarrow$) We have already proved this: (**??**) and (**??**). ($\Longleftarrow$) If p is a categorical fibration and a categorical equivalence, factor p as $X \xrightarrow{j} Z \xrightarrow{q} Y$ with j a monomorphism and q a trivial fibration. Then the usual argument shows that p is a retract of q, using the fact that $j \boxslash p$ since j is a categorical equivalence by 2-of-3. $\qquad\square$

40.2. Proposition. *If $p\colon X \to Y$ is a categorical fibration and $j\colon K \to L$ is a monomorphism, then*
$$q\colon \operatorname{Map}(L, X) \to \operatorname{Map}(K, X) \times_{\operatorname{Map}(K,Y)} \operatorname{Map}(L, Y)$$
is a categorical fibration. Furthermore, if either j or p is also a categorical equivalence, then so is q.

Proof. For the first, let $i\colon A \to B$ be a monomorphism which is a categorical equivalence. We have $i \boxslash q$ iff $(i \square j) \boxslash p$. By definition of categorical fibration, it suffices to show that $i \square j$ is a categorical equivalence, i.e., to show $\operatorname{Map}(i \square j, C)$ is a categorical equivalence for every quasicategory C. In fact, $\operatorname{Map}(i, C)$ is an isofibration and a categorical equivalence, hence a trivial fibration, and therefore $j \square \operatorname{Map}(i, C)$.

If p is also a categorical equivalence, then it is a trivial fibration, and the result follows.

If j is also a categorical equivalence, then for any monomorphism i, we have $i \boxslash q$ iff $(i \square j) \boxslash p$ iff $j \boxslash (p^{\square i})$. But $p^{\square i}$ is a categorical fibration by what we have just proved, so the result holds. $\qquad\square$

40.3. Categorical fibrations and the small object argument. Clearly, $\text{CatFib} = \left(\overline{\text{Cell}} \cap \text{CatEq}\right)^{\boxslash}$ is a right complement to a class of maps. We would like to know that CatFib is the right complement to a *set* of maps; then we could use the small object argument to factor any map into an injective categorical equivalence followed by a categorical fibration.

Unfortunately, it's apparently not known how to write down an explicit set of maps S so that $S^{\boxslash} = \text{CatFib}$. What is known is that such a set *exists*.

40.4. Proposition. *There exists a set S of maps of simplicial sets such that $\overline{S} = \overline{\text{Cell}} \cap \text{CatEq}$, whence $S^{\boxslash} = \text{CatFib}$.*

In the rest of this section we will sketch a proof. The idea is to show that CatFib is the right complement of the class of all injective categorical equivalences $K \to L$ for which the number of simplices in K and L is bounded by some explicit cardinal κ. We obtain S by choosing one representative for each isomorphism class in this class; then S is a set because of the cardinality bound.

We will define a **detection functor** $F\colon \operatorname{Fun}([1], s\text{Set}) \to \operatorname{Fun}([1], \text{Set})$ on categories of morphisms. This will have the following properties:

- For each map $f\colon X \to Y$, the map $F(f)$ is a monomorphism of sets.

- A map $f\colon X \to Y$ is a categorical equivalence if and only if $F(f)$ is a bijection.
- The functor F commutes with κ-filtered colimits for some regular cardinal κ.
- The functor F takes κ-small simplicial sets to κ-small sets.

We define F as the composite of several intermediate steps.

Step 1: Recall that the small object argument gives a *functorial* way to factor a map f as $f = pi$, with $i \in \overline{S}$ and $p \in S^{\boxtimes}$. "Funtorial factorization" means that we get a section of the functor $\mathrm{Fun}([2], s\mathrm{Set}) \to \mathrm{Fun}([1], s\mathrm{Set})$ defining composition.

We can apply this using $S = \mathrm{InnHorn}$. Thus, given any simplicial set, we *functorially* obtain an inner anodyne map $X \to X_{\mathrm{QCat}}$ to a quasicategory X_{QCat}. As a result, we have a functor $f \mapsto f_{\mathrm{QCat}}\colon \mathrm{Fun}([1], s\mathrm{Set}) \to \mathrm{Fun}([1], s\mathrm{Set})$, with the property that f is a categorical equivalence if and only if f_{QCat} is, and both source and target of f_{QCat} are quasicategories.

Step 2: Form the path fibration $Q(f)\colon P(f_{\mathrm{QCat}}) \to Y_{\mathrm{QCat}}$ of f_{QCat}. The map $Q(f)$ is thus an isofibration between quasicategories, and is a trivial fibration if and only if f is a categorical equivalence.

Step 3: Write $Q(f)\colon X' \to Y'$. Define $E(f)$ to be the map of sets

$$E(f)\colon \coprod_n \mathrm{Hom}(\Delta^n, X') \to \coprod_n \mathrm{Hom}(\partial\Delta^n, X') \times_{\mathrm{Hom}(\partial\Delta^n, Y')} \mathrm{Hom}(\Delta^n, Y').$$

Thus, f is a categorical equivalence if and only if $E(f)$ is *surjective*.

Step 4: Write $E(f)\colon E_0(f) \to E_1(f)$, and define $F(f)$ by

$$F(f)\colon \mathrm{colim}\big[E_0(f) \times_{E_1(f)} E_0(f) \rightrightarrows E_0(f)\big] \to E_1(f).$$

In other words, $F(f)$ is the map from the image of $E(f)$ to $E_1(f)$. Thus, $F(f)$ is always a monomorphism, and f is a categorical equivalence if and only if $F(f)$ is a bijection.

There exists a regular cardinal κ such that F commutes with κ-filtered colimits, and takes κ-small simplicial sets to κ-small sets. (In fact, we can take $\kappa = \omega^+$, the sucessor to the countable cardinal).

Using the detection functor, we can prove the following key lemma.

40.5. **Lemma.** *Let $f\colon X \subseteq Y$ be an inclusion which is a categorical equivalence. Every κ-small subcomplex $A \subseteq Y$ is contained in a κ-small subcomplex $B \subseteq Y$ with the property that $B \cap X \subseteq B$ is a categorical equivalence.*

Proof. For a subcomplex $A \subseteq Y$ let f_A denote the inclusion $A \cap X \subseteq A$. The collection of all κ-small subcomplexes of Y is κ-filtered. Thus

$$\mathrm{colim}_{\kappa\text{-small } A \subseteq Y} F(f_A) = F(f),$$

which we have assumed is an isomorphism. Thus for any κ-small $A \subseteq$ there must exist a κ-small $A' \supset A$ such that a lift exists in

$$
\begin{array}{ccc}
F_0(f_A) & \longrightarrow & F_0(f_{A'}) \\
\downarrow & \nearrow & \downarrow \\
F_1(f_A) & \longrightarrow & F_1(f_{A'})
\end{array}
$$

This is because $F_1(f_A)$ is a κ-small set, so any lift $F_1(f_A) \to F_0(F)$ factors through some stage of the κ-filtered colimit.

We use transfinite induction to obtain a sequence $\{A_i\}$ indexed by $i < \kappa$, where at limit ordinals we take a colimit. Set $B := \mathrm{colim}\, A_i$. Because κ is regular $|B| < \kappa$, and we have that $F(f_B)$ is an isomorphism by construction. $\square$

Consider the collection of monomorphisms $i\colon A \to B$ such that i is a categorical equivalence and $|B| < \kappa$. Choose a set S of such spanning all isomorphism classes of such maps; this is a set because of the cardinality bound. Clearly $S \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$.

40.6. **Proposition.** *We have* $\overline{S} = \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$.

Proof. [Joy08a, D.2.16]. Given an injective categorical equivalence $X \subseteq Y$, we consider the following poset $\mathcal{P}$. The objects of $\mathcal{P}$ are subobjects $P \subseteq Y$ such that $X \subseteq P$ so that the inclusion $X \to P$ is contained in $\overline{S}$. The morphisms of $\mathcal{P}$ are inclusions $P \to Q$ of subobjects of Y which are contained in $\overline{S}$. Because $\overline{S}$ is saturated, the hypotheses of Zorn's lemma apply to give a maximal element M of $\mathcal{P}$. Since $X \subseteq Y$ is assumed to be a categorical equivalence, 2-out-of-3 gives that $M \subseteq Y$ is a categorical equivalence.

If $M = Y$ we are done, so suppose $M \neq Y$. Then there exists a κ-small $A \subseteq Y$ not contained in M, which by the above lemma can be chosen so that $A \cap M \subseteq A$ is a categorical equivalence, and thus an element of S. The pushout $M \subseteq A \cup M$ of this map is thus in $\overline{S}$ contradicting the maximality of M. $\qquad\square$

In particular, we learn that every map can be factored into an injective categorical equivalence followed by a categorical fibration.

41. The Joyal model structure on simplicial sets

41.1. **Model categories.** A **model category** (in the sense of Quillen) is a category $\mathcal{M}$ with three classes of maps: W, Cof, Fib, conventionally called *weak equivalences*, *cofibrations*, and *fibrations*, satisfying the following axioms.

- $\mathcal{M}$ has all small limits and colimits.
- W satisfies the 2-out-of-3 property.
- $(\mathrm{Cof} \cap \mathrm{W}, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap \mathrm{W})$ are weak factorization systems (13.12).

Conventionally, an object X is **cofibrant** if the map from the initial object is a cofibration, and **fibrant** if the map to the terminal object is a fibration.

41.2. *Remark.* The third axiom implies that Cof, Cof $\cap$ W, Fib, and Fib $\cap$ W are closed under retracts.

41.3. *Exercise.* Show that in a model category (as defined above), the class of weak equivalences is closed under retracts. *Hint:* construct a factorization of f which is itself a retract of a factorization of g[28].

41.4. *Exercise* (Slice model categories). Let $\mathcal{M}$ be a model category, and let X be an object of $\mathcal{M}$. Show that the slice categories $\mathcal{M}_{X/}$ and $\mathcal{M}_{/X}$ admit model category structures, in which the weak equivalences, cofibrations, and fibrations are precisely the maps whose images under $\mathcal{M}_{/X} \to \mathcal{M}$ or $\mathcal{M}_{X/} \to \mathcal{M}$ are weak equivalences, cofibrations, and fibrations in $\mathcal{M}$.

41.5. **The Joyal model category.**

41.6. **Theorem** (Joyal)**.** *The category of simplicial sets admits a model structure, in which*

- W = *categorical equivalences (*CatEq*),*
- Cof = *monomorphims (*$\overline{\mathrm{Cell}}$*),*
- Fib = *categorical fibrations (*CatFib*).*

Furthermore, the fibrant objects are precisely the quasicategories, and the fibrations with target a fibrant object are precisely the isofibrations.

Proof. Categorical equivalences satisfy 2-out-of-3 by (22.9). We have that

- Cof = $\overline{\mathrm{Cell}}$ by definition,
- Fib $\cap$ W = TFib = $\mathrm{Cell}^{\boxslash}$ by (40.1),

---

[28]In many formulations of model categories, weak equivalences being closed under retracts is taken as an axiom. The formulation we use is described in Riehl, "A concise definition of a model category" [Rie09], which gives a solution to this exercise.

- Cof ∩ W = $\overline{S}$ for some set S (40.4),
- Fib = CatFib = $(\mathrm{Cof} \cap \mathrm{W})^{\boxtimes} = S^{\boxtimes}$ by definition,

so both $(\mathrm{Cof} \cap \mathrm{W}, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap \mathrm{W})$ are weak factorization systems via the small object argument (13.10). Thus, we get a model category.

We have shown (36.11) that the categorical fibrations $p\colon C \to D$ with D a quasicategory are precisely the isofibrations. Applied when $D = *$, this implies that quasicategories are exactly the fibrant objects, and thus that fibrations with fibrant target are precisely the isofibrations.   □

41.7. *Remark.* It is a standard fact that a model category structure is uniquely determined by its cofibrations and fibrant objects. Thus, the Joyal model structure is the unique model structure on simplicial sets with Cof = monomorphisms and with fibrant objects the quasicategories.

41.8. Cartesian model categories. Recall that the category of simplicial sets is *cartesian closed*. A **cartesian model category** is a model category which is cartesian closed, with the following properties. Suppose $i\colon A \to B$ and $j\colon K \to L$ are cofibrations and $p\colon X \to Y$ is a fibration. Then

-
$$i\square j\colon (A \times L) \cup_{A \times K} (B \times K) \to B \times L$$

  is a cofibration, and is in addition a weak equivalence if either i or j is also a weak equivalence, and

-
$$p^{\square j}\colon \mathrm{Map}(L, X) \to \mathrm{Map}(K, X) \times_{\mathrm{Map}(K,Y)} \mathrm{Map}(L, Y)$$

  is a fibration, and is in addition a weak equivalence if either j or p is also a weak equivalence.

In fact, we only need to specify *one* of the above two properties, as they imply each other.

41.9. Proposition. *The Joyal model structure is cartesian.*

Proof. This is just (40.2).   □

42. The Kan-Quillen model structure on simplicial sets

A map $p\colon X \to Y$ is a **groupoidal fibration** if and only if $j \boxtimes p$ for all j which are monomorphisms and weak equivalences. I write GpdFib for the class of categorical fibrations.

42.1. The Kan-Quillen model structure.

42.2. Theorem (Cisinski)**.** *The category of simplicial sets admits a model structure, in which*

- W = *weak equivalences,*
- Cof = *monomorphims,*
- Fib = *groupoidal fibrations.*

Furthermore, the fibrant objects are precisely the Kan complexes, and the fibrations with target a fibrant object are precisely the Kan fibrations.

Proof. This goes very much the same way as the Joyal model structure, and I won't spell it out in detail. First build a detecting functor F so that a map f is a weak equivalence iff $F(f)$ is a bijection; this is just as in the categorical equivalence case, except that we make use of functorial replacement $X \mapsto X_{\mathrm{Kan}}$ by Kan complexes, rather than by quasicategories. Using this, we can show that $\overline{\mathrm{Cell}} \cap \mathrm{WkEq} = \overline{S}$ and $\mathrm{GpdFib} = S^{\boxtimes}$ for some set S, giving the factorization system $(\mathrm{Cof} \cap \mathrm{W}, \mathrm{Fib})$.

We know that trivial fibrations are weak equivalences and are certainly groupoidal fibrations. The converse is proved just as in the categorical fibration case (40.1). This gives the other factorization system $(\mathrm{Cof}, \mathrm{Fib} \cap \mathrm{W})$.

We have already proved that Kan fibrations between Kan complexes have the lifting property of groupoidal fibrations (33.15), so the statements about fibrant objects and fibrations to fibrant objects follow just as in the categorical case.   □

42.3. **Proposition.** *The Quillen model structure is cartesian.*

Proof. We must show that $p^{\square j}$ is a groupoidal fibration if j is a monomorphsm and p a groupoidal fibration, and also that it is a weak equivalence if either j or p is. This is proved by an argument nearly identical to the proof of (40.2). $\square$

42.4. **Kan fibrations are groupoidal fibrations.** The above model structure was actually first produced by Quillen. In Quillen's formulation, the fibrations were taken to be the *Kan fibrations*. In fact, this is the same model structure, by

42.5. **Proposition** (Quillen). KFib = GpdFib.

We will not give a proof of this here. The non-trivial part is to show that KFib $\subseteq$ GpdFib; note that we already know that a Kan fibration between Kan complexes is a groupoidal fibration by (33.15). This is usually done via an argument (due to Quillen) based on the theory of *minimal fibrations*. See for instance Quillen's original argument [Qui67, §II.3] or [GJ09, Ch. 1].

These arguments work by showing that KFib is the cosaturation of the class of *Kan fibrations between Kan complexes*. In fact one can even show that every Kan fibration is a *base change* of a Kan fibration between Kan complexes, see [KLV12].

43. Model categories and homotopy colimits

We are going to exploit these model category structures now. The main purpose of model categories is to give tools for showing that a given construction preserves certain kinds of weak equivalences.

43.1. **Creating new model categories.** Given a model category $\mathcal{M}$, many other categories related to it can also be equipped with model category structures, such as functor categories $\text{Fun}(C, \mathcal{M})$ where C is a small category. We won't consider general formulations of this here, but we will consider some special cases.

As an example, we consider the case of $C = [1] = \{0 \xrightarrow{01} 1\}$.

43.2. **Proposition.** *There exists a model structure on $\mathcal{N} := \text{Fun}([1], \mathcal{M})$ in which a map $\alpha\colon X \to X'$ is*

- *a weak equivalence if $\alpha(i)\colon X(i) \to X'(i)$ is a weak equivalence in $\mathcal{M}$ for $i = 0, 1$*
- *a cofibration if $\alpha(i)$ is a cofibration in $\mathcal{M}$ for $i = 0, 1$, and*
- *a fibration in $\mathcal{M}$ if both $\alpha(1)$ and the map $(X(01), \alpha(0))\colon X(0) \to X(1) \times_{X'(1)} X'(0)$ are fibrations in $\mathcal{M}$.*

Proof. It is clear that $\mathcal{N}$ has small limits and colimits, and that weak equivalences in it have the 2-out-of-3 property. It remains to show that $(\text{Cof} \cap W, \text{Fib})$ and $(\text{Cof}, \text{Fib} \cap W)$ are weak factorization systems.

Let $\alpha\colon X \to X'$ be a map in $\mathcal{N}$. Choose

- a factorization $X(1) \to Y(1) \to X'(1)$ of $\alpha(1)$ of the form $\text{Fib} \circ \text{Cof} \cap W$, and
- a factorization $X(0) \to Y(0) \to X'(0) \times_{X'(1)} Y(1) \dots$

$\square$

43.3. **Reedy lemma.**

43.4. **Proposition** (Reedy lemma). *Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor between model categories which takes trivial cofibrations to weak equivalences. Then F takes weak equivalences between cofibrant objects to weak equivalences.*

Proof. Let $f\colon X \to Y$ be a weak equivalence between cofibrant objects in $\mathcal{M}$. Form the commutative diagram

$$\begin{array}{c}
\varnothing \longrightarrow Y \\
\downarrow \quad\quad \downarrow \quad \overset{\mathrm{id}_Y}{\searrow} \\
X \longrightarrow X \amalg Y \overset{b}{\longrightarrow} C \overset{p}{\to} Y \\
\underset{a}{} \quad \underset{f}{}
\end{array}$$

where the square is a pushout, and we have chosen a factorization $X \amalg Y \to C \to Y$ into a cofibration followed by a weak equivalence (e.g., a trivial fibration). Because X and Y are cofibrant, the maps $X \to X \amalg Y \leftarrow Y$ are cofibrations. Using this and the 2-of-3 property, we see that a and b are trivial cofibrations. Applying F gives

$$\begin{array}{c}
F(Y) \\
{\scriptstyle F(b)}\downarrow \quad \overset{\mathrm{id}}{\searrow} \\
F(X) \underset{F(a)}{\longrightarrow} F(C) \underset{F(p)}{\longrightarrow} F(Y)
\end{array}$$

in which $F(b)$ and $F(a)$ are weak equivalences by hypothesis, whence $F(b)$ is a weak equivalence by 2-of-3, and therefore $F(f) = F(p)F(b)$ is a weak equivalence, as desired. $\qquad\square$

The opposite of a model categry is also a model category, by switching the roles of fibrations and cofibrations. Thus, there is a dual formulation of the Reedy lemma.

43.5. Proposition (Reedy lemma, dual form)**.** *Let $G\colon \mathcal{N} \to \mathcal{M}$ be a functor between model categories which takes trivial fibrations to weak equivalences. Then G takes weak equivalences between fibrant objects to weak equivalences.*

43.6. Quillen pairs. Given an adjoint pair of functors $F\colon \mathcal{M} \leftrightarrows \mathcal{N} \colon G$ between model categories, we see from the properties of weak factorization systems that

- F preserves cofibrations if and only if G preserves trivial fibrations, and
- F preserves trivial cofibrations if and only if G preserves fibrations.

If both of these are true, we say that (F, G) is a **Quillen pair**.

Note that if (F, G) is a Quillen pair, then the Reedy lemma (43.4) applies to F, and the dual form of the Reedy lemma (43.5) applies to G.

43.7. Good colimits. We can apply the above to certain examples of colimit functors.

43.8. *Exercise.* Let S be a small discrete category (i.e., all maps are identities). Show that if $\mathcal{M}$ is a model category, then $\mathrm{Fun}(S, \mathcal{M})$ is a model category in which $\alpha\colon X \to X'$ is

- a weak equivalence, cofibration, or fibration iff each $\alpha_s\colon X_s \to X'_s$ is one in $\mathcal{M}$.

Thes show that $\mathrm{colim}\colon \mathrm{Fun}(S, \mathcal{M}) \leftrightarrows \mathcal{M} \colon \mathrm{const}$ is a Quillen pair, and use this to prove the next proposition.

43.9. Proposition. *Given a collection $f_s\colon X_s \to X'_s$ of weak equivalences between cofibrant objects in $\mathcal{M}$, the induced map $\amalg f_s\colon \coprod X_s \to \coprod X'_s$ is a weak equivalence.*

43.10. *Exercise.* Let V be the category

$$0 \overset{10}{\longleftarrow} 1 \overset{12}{\longrightarrow} 2.$$

Show that if $\mathcal{M}$ is a model category, then $\mathrm{Fun}(V, \mathcal{M})$ is a model category in which $\alpha\colon X \to X'$ is

- a weak equivalence if $\alpha(i)\colon X(i) \to X'(i)$ is a weak equivalence for $i = 0, 1, 2$ (i.e., an **objectwise weak equivalence**), and is

- a cofibration if $\alpha(0)$ and $\alpha(1)$ are cofibrations, and the evident map $X(2) \cup_{X(1)} X'(1) \to X'(2)$ is a cofibration.

Then show that colim$\colon \mathrm{Fun}(V, \mathcal{M}) \leftrightarrows \mathcal{M} \colon$ const is a Quillen pair, and use this to prove the next proposition. (Hint: determine what the fibrations in $\mathrm{Fun}(V, \mathcal{M})$ must be according to the lifting property.)

43.11. Proposition. *Given a natural transformation $\alpha \colon X \to X'$ of functors $V \to \mathcal{M}$, i.e., a diagram*

$$
\begin{array}{ccccc}
X(0) & \longleftarrow & X(1) & \xrightarrow{X(12)} & X(2) \\
\sim \downarrow & & \sim \downarrow & & \sim \downarrow \\
X'(0) & \longleftarrow & X'(1) & \xrightarrow{X'(12)} & X'(2)
\end{array}
$$

in which the vertical maps are weak equivalences, all objects $X(i)$ and $X'(i)$ are cofibrant, and the maps $X(12)$ and $X'(12)$ are cofibrations, the induced map $\mathrm{colim}_V X \to \mathrm{colim}_V X'$ is a weak equivalence.

43.12. Exercise. Let ω be the category

$$0 \to 1 \to 2 \to \cdots$$

with objects indexed by natural numbers. Show that if $\mathcal{M}$ is a model category, then $\mathrm{Fun}(\omega, \mathcal{M})$ is a model category in which $\alpha \colon X \to X'$ is

- a weak equivalence if each $\alpha(i)$ is a weak equivalence,
- a cofibration if (i) $\alpha(0)$ is a cofibration, and $X'(i) \cup_{X(i)} X(i+1) \to X'(i+1)$ is a cofibration for all $i \geq 0$.

Then show that colim$\colon \mathrm{Fun}(\omega, \mathcal{M}) \leftrightarrows \mathcal{M} \colon$ const is a Quillen pair, and use this to prove the next proposition.

43.13. Proposition. *Give a natural transformation $\alpha \colon X \to X'$ of functors $\omega \to \mathcal{M}$ such that all maps $\alpha(i) \colon X(i) \to X'(i)$ are weak equivalences, all objects $X(i)$ and $X(i')$ are cofibrant, and the maps $X(i) \to X(i+1)$ and $X'(i) \to X'(i+1)$ are cofibrations, the induced map $\mathrm{colim}_\omega X \to \mathrm{colim}_\omega X'$ is a weak equivalence.*

In the Joyal and Quillen model structures, all objects are automatically cofibrant, which makes the above propositions especially handy.

We will call any colimit diagram in a model category, satisfying the hypotheses of one of (43.9), (43.11), (43.13) a **good colimit**. Thus, we see that good colimits are homotopy invariant. These "good colimits" are examples of what are called *homotopy colimits*.

Since the opposite of a model category is also a model category, all of the results of this section admit dual formulations, leading to the observation that **good limits** are homotopy invariant.

43.14. Exercise. State and prove the dual versions of all the results in this section.

44. A SIMPLICIAL SET IS WEAKLY EQUIVALENT TO ITS OPPOSITE

Recall the opposite construction $X \mapsto X^{\mathrm{op}}$ on general simplicial sets. We will now prove the following, which is a kind of a generalization of the fact that every groupoid is isomorphic to its own opposite.

44.1. Proposition. *Every simplicial set X is weakly equivalent to its opposite, in the sense that there exists a zig-zag of weak equivalences connecting X and X^{op}.*

Note that for any ordinary groupoid G, it is straightforward to construct an isomorphism $G \to G^{\mathrm{op}}$, by sending each map to its inverse. Unfortunately, this proof cannot be replicated in our setting, even if we assume that X is a quasigroupoid.

44.2. *Remark.* Here is one possible proof (in some sense, the most natural proof). Note that there is a *homeomorphism* of geometric realizations $|X| \approx |X^{\mathrm{op}}|$. Then use the fact that geometric realization induces an equivalence $h(s\mathrm{Set}, \mathrm{WkEq}) \approx h(\mathrm{Top}, \mathrm{WkEq})$. Of course, we haven't actually proved this fact about homotopy categories yet.

I'll give some different proofs which are internal to simplicial sets. I do this in part because it's interesting to see how this is done, but also because it allows me to set up some technology which will be useful later.

44.3. General "singular" and "realization" functors. Consider a functor $C \colon \Delta \to \mathcal{A}$ into some category $\mathcal{A}$, i.e., a **cosimplicial object** in $\mathcal{A}$. Often we write $C^n \in \mathcal{A}$ instead of $C([n])$ for the values of this functor, and so write $C^\bullet$ for the whole functor. In most of our examples, we will actually have $\mathcal{A} = s\mathrm{Set}$, so $C^\bullet$ will be a cosimplicial simplicial set.

Given such a $C^\bullet$, **realization** and **singular** functors are an adjoint pair

$$\mathrm{Re} = \mathrm{Re}_{C^\bullet} \colon s\mathrm{Set} \rightleftarrows \mathcal{A} \colon \mathrm{Si}_{C^\bullet} = \mathrm{Si}$$

associated to $C^\bullet$. The singular functor Si is always defined, and is defined by

$$(\mathrm{Si}\, X)_n := \mathrm{Hom}_{s\mathrm{Set}}(C([n]), X),$$

with simplicial operators induced by the fact that C is a functor on Δ. The left adjoint Re is defined if $\mathcal{A}$ is cocomplete[29].

44.4. *Exercise.* Show that if $\mathcal{A}$ is cocomplete, and $X \in s\mathrm{Set}$, then

$$\mathrm{Re}\, X \approx \mathrm{Cok}\left[\coprod_{f \colon [m] \to [n]} \coprod_{x \in X_n} C^m \rightrightarrows \coprod_{[p]} \coprod_{x \in X_p} C^p \right].$$

(Part of the exercise is to figure out what the two maps are.)

44.5. *Example.* For the cosimplicial space $\Delta^\bullet_{\mathrm{top}} \colon \Delta \to \mathrm{Top}$ taking $[n]$ to the topological n-simplex, $\mathrm{Re}_{\Delta^\bullet_{\mathrm{top}}}$ and $\mathrm{Si}_{\Delta^\bullet_{\mathrm{top}}}$ are just the usual geometric realization and singular complex functors.

44.6. *Example.* For the tautological functor $\Delta^\bullet \colon \Delta \to s\mathrm{Set}$ sending $[n] \mapsto \Delta^n$, the functors $\mathrm{Re}_{\Delta^\bullet}$ and $\mathrm{Si}_{\Delta^\bullet}$ are isomorphic to the identity functor on simplicial sets.

44.7. *Example.* For the functor $(\Delta^\bullet)^{\mathrm{op}} = \Delta^\bullet \circ \mathrm{op} \colon \Delta \to s\mathrm{Set}$, the functors $\mathrm{Re}_{\Delta^\bullet \circ \mathrm{op}}$ and $\mathrm{Si}_{\Delta^\bullet \circ \mathrm{op}}$ are both isomorphic to the *opposite* functor $X \mapsto X^{\mathrm{op}}$.

44.8. The cosimplicial object $\mathrm{Iso}^\bullet$. Consider $\mathrm{Iso}^\bullet \colon \Delta \to s\mathrm{Set}$, where Iso^n is the (nerve of the) groupoid with objects $\{0, 1, \ldots, n\}$ and unique isomophisms between every object. This has corresponding realization and singular functors

$$\mathrm{Re} = \mathrm{Re}_{\mathrm{Iso}^\bullet} \colon s\mathrm{Set} \to s\mathrm{Set} \colon \mathrm{Si}_{\mathrm{Iso}^\bullet} = \mathrm{Si}.$$

There is an evident natural transformation $\alpha \colon \Delta^\bullet \to \mathrm{Iso}^\bullet$, by the unique functors $[k] \to \mathrm{Iso}^k$ sending object j to j. This gives an adjoint-related pair of natural transformations

$$\eta_X \colon X \to \mathrm{Re}_{\mathrm{Iso}^\bullet}\, X, \qquad \epsilon_X \colon \mathrm{Si}_{\mathrm{Iso}^\bullet}\, X \to X.$$

If we precompose with op, we obtain a cosimplicial object $\mathrm{Iso}^\bullet \circ \mathrm{op} = (\mathrm{Iso}^\bullet)^{\mathrm{op}}$. One sees that there are natural isomorphisms

$$(44.9) \qquad\qquad \mathrm{Re}_{\mathrm{Iso}^\bullet \circ \mathrm{op}} \approx (\mathrm{Re}_{\mathrm{Iso}^\bullet})^{\mathrm{op}}, \qquad \mathrm{Si}_{\mathrm{Iso}^\bullet \circ \mathrm{op}} \approx (\mathrm{Si}_{\mathrm{Iso}^\bullet})^{\mathrm{op}},$$

and that the evident natural transformation $\alpha \circ \mathrm{op} \colon \Delta^\bullet \circ \mathrm{op} \to \mathrm{Iso}^\bullet \circ \mathrm{op}$ gives rise to an adjoint-related pair of natural transformations

$$\eta'_X \colon X^{\mathrm{op}} \to \mathrm{Re}_{\mathrm{Iso}^\bullet \circ \mathrm{op}}\, X, \qquad \epsilon'_X \colon \mathrm{Si}_{\mathrm{Iso}^\bullet \circ \mathrm{op}}\, X \to X^{\mathrm{op}}.$$

---

[29]This means that $\mathcal{A}$ has colimits for all functors $F \colon B \to \mathcal{A}$ from small categories B.

Furthermore, under the isomorphisms of (44.9) the maps η'_X and ϵ'_X are identified with $\eta_{X^{\mathrm{op}}}$ and $\epsilon_{X^{\mathrm{op}}}$.

44.10. *Exercise.* Prove the statements of the previous paragraph.

Although $\Delta^\bullet$ and $\Delta^\bullet \circ \mathrm{op}$ are not isomorphic as functors $\Delta \to s\mathrm{Set}$, it is the case that $\mathrm{Iso}^\bullet \approx \mathrm{Iso}^\bullet \circ \mathrm{op}$, using the isomorphisms of categories $\mathrm{Iso}^n \to \mathrm{Iso}^n$ given on objects by $x \mapsto n - x$. Putting all this together, we obtain natural transformations

$$X \xrightarrow{\eta_X} \mathrm{Re}_{\mathrm{Iso}^\bullet} X \approx \mathrm{Re}_{\mathrm{Iso}^\bullet} X^{\mathrm{op}} \xleftarrow{\eta_{X^{\mathrm{op}}}} X^{\mathrm{op}}, \qquad X \xleftarrow{\epsilon_X} \mathrm{Si}_{\mathrm{Iso}^\bullet} X \approx \mathrm{Si}_{\mathrm{Iso}^\bullet} X^{\mathrm{op}} \xrightarrow{\epsilon_{X^{\mathrm{op}}}} X^{\mathrm{op}}.$$

We'll show that that η_X, and hence $\eta_{X^{\mathrm{op}}}$, are always weak equivalences, and that ϵ_X, and hence $\epsilon_{X^{\mathrm{op}}}$, are weak equivalences whenever X is a Kan complex. In the following, $\mathrm{Re} = \mathrm{Re}_{\mathrm{Iso}^\bullet}$ and $\mathrm{Si} = \mathrm{Si}_{\mathrm{Iso}^\bullet}$.

44.11. **Lemma.** *For each monomorphism $K \to L$, the induced map $(\mathrm{Re}\, K) \amalg_K L \to \mathrm{Re}\, L$ is a monomorphism. In particular,*

- Re *preserves monomorphisms and* Si *preserves trivial fibrations, and*
- $\eta_L \colon L \to \mathrm{Re}\, L$ *is always a monomorphism.*

Proof. Formally, it is enough to check the case of $\partial\Delta^n \subset \Delta^n$. To see this, check that the lifting problems

$$
\begin{array}{ccc}
(\mathrm{Re}\, K) \cup_K L \longrightarrow X & & K \longrightarrow \mathrm{Si}\, X \\
\downarrow \quad\quad\quad\quad \nearrow \quad \downarrow & \Longleftrightarrow & \downarrow \quad\quad \nearrow \quad\quad \downarrow \\
\mathrm{Re}\, L \longrightarrow Y & & L \longrightarrow (\mathrm{Si}\, Y) \times_Y X
\end{array}
$$

are equivalent. This means we need to show that all monomorphisms are contained in the saturated class $^{\boxtimes}\mathcal{C}$, where $\mathcal{C}$ is the class of all the maps $(\mathrm{Si}\, p, \epsilon_X) \colon \mathrm{Si}\, X \to (\mathrm{Si}\, Y) \times_Y X$ such that $p \in \mathrm{TFib}$, which means we only need to show that Cell is contained in it.

This is a calculation: $\mathrm{Re}(\partial\Delta^n) \to \mathrm{Re}(\Delta^n) = N(\mathrm{Iso}^n)$ is isomorphic to inclusion of the subcomplex $K \subseteq N(\mathrm{Iso}^n)$ whose k-simplices are sequences $x_0 \to \cdots \to x_k$ such that $\{x_0, \ldots, x_k\} \neq \{0, \ldots, n\}$. To show this, use the fact that Δ^n is a colimit of its $(n-1)$-dimensional faces along their intersections, and that Re preserves colimits. The image of the simplex $(0, 1, \ldots, n)$ in $\mathrm{Re}(\Delta^n)$ intersects K exactly in its boundary, so $(\mathrm{Re}\, \partial\Delta^n) \cup_{\partial\Delta^n} \Delta^n \to \mathrm{Re}\, \Delta^n$ is a monomorphism as desired. $\square$

44.12. *Remark.* Given any natural transformation $\lambda \colon F \to G$ of functors, and map $f \colon X \to Y$, we get induced maps

$$F(Y) \cup_{F(X)} G(X) \to G(Y), \qquad F(X) \to F(Y) \times_{G(Y)} G(X).$$

These can be thought of as a variant of the "box" construction we've considered elsewhere (26.5), but associated to the "evaluation pairing" $\mathrm{Fun}(s\mathrm{Set}, s\mathrm{Set}) \times s\mathrm{Set} \to s\mathrm{Set}$ rather than a functor $s\mathrm{Set} \times s\mathrm{Set} \to s\mathrm{Set}$.

44.13. **Skeletal induction.** The next step is to show that $K \to \mathrm{Re}\, K$ is a weak equivalence for every simplicial set K. To do this, we will use the following bit of machinery.

44.14. **Proposition** (Skeletal induction)**.** *Let $\mathcal{C}$ be a class of simplicial sets with the following properties.*

(1) *If $X \in \mathcal{C}$, then every object isomorphic to X is in $\mathcal{C}$.*
(2) *Every $\Delta^n \in \mathcal{C}$.*
(3) *The class $\mathcal{C}$ is closed under* good colimits. *That is:*
  (a) *any coproduct of objects of $\mathcal{C}$ is in $\mathcal{C}$;*
  (b) *any pushout of a diagram $X_0 \leftarrow X_1 \to X_2$ of objects in $\mathcal{C}$ along a monomorphism $X_1 \to X_2$ is in $\mathcal{C}$;*

(c) *any colimit of a countable sequence $X_0 \to X_1 \to X_2 \to \cdots$ of objects in $\mathcal{C}$, such that each $X_k \to X_{k+1}$ is a monomorphism, is in $\mathcal{C}$.*

Then $\mathcal{C}$ is the class of all simplicial sets.

Proof. This is a straightforward consequence of the skeletal filtration (15.16). To show $X \in \mathcal{C}$, it suffices to show each $\operatorname{Sk}_n X \in \mathcal{C}$ by (3c). So we show that all n-skeleta are in $\mathcal{C}$ by induction on n, with base case $n = -1$ (the empty simplicial set), which is really a special case of (3a). Since $\operatorname{Sk}_{n-1} X \subseteq \operatorname{Sk}_n X$ is a pushout along a coproduct of maps $\partial \Delta^n = \operatorname{Sk}_{n-1} \Delta^n \to \Delta^n$, this follows using (2), (3a), (3b), and the inductive hypothesis. $\qquad\square$

44.15. Proposition. *For every simplicial set X, the map $X \to \operatorname{Re} X$ is a weak equivalence.*

Proof. Let $\mathcal{C}$ be the class of X such that $\eta\colon X \to \operatorname{Re} X$ is a weak equivalence. We verify the hypotheses of the above proposition. Property (1) is obvious.

To prove property (2) recall that $\eta_{\Delta^1}\colon \Delta^1 \to \operatorname{Iso}^1$ is anodyne (32.14). We can identify η_{Δ^n} as a retract of $(\eta_{\Delta^1})^{\times n}\colon (\Delta^1)^{\times n} \to (\operatorname{Iso}^1)^{\times n}$, which is necessarily anodyne since a product of an anodyne map with any identity map is anodyne (similar to (21.9)). The retractions $\Delta^n \xrightarrow{f} (\Delta^1)^{\times n} \xrightarrow{g} \Delta^n$ and $\operatorname{Iso}^n \xrightarrow{f} (\operatorname{Iso}^1)^{\times n} \xrightarrow{g} \operatorname{Iso}^n$ are maps which are given on vertices by

$$f(k) = (\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{n-k}), \qquad g(k_1, \ldots, k_n) = \max\{\, j \mid k_j = 1 \,\}.$$

Property (3) involves colimits, which in every case are *good colimits*. In each case, we need to show that a map $\operatorname{colim} X_i \to \operatorname{Re} \operatorname{colim} X_i$ is a weak equivalence when each $X_i \to \operatorname{Re} X_i$ is. The functor Re preserves colimits and monomorphisms (44.11), so in every case we are comparing good colimits, so the result follows from (43.9), (43.11), (43.13). $\qquad\square$

We thus obtain the desired result.

44.16. Corollary. *Every simplicial set is weakly equivalent to its opposite X^{op}.*

Proof. Both maps in $X \xrightarrow{\eta_X} \operatorname{Re} X \approx \operatorname{Re} X^{\mathrm{op}} \xleftarrow{\eta_{X^{\mathrm{op}}}} X^{\mathrm{op}}$ are weak equivalences (44.15). $\qquad\square$

44.17. Proposition. *For each monomorphism $K \to L$, the induced map $(\operatorname{Re} K) \amalg_K L \to \operatorname{Re} L$ is a monomorphism and a weak equivalence.*

Proof. Both squares

$$
\begin{array}{ccc}
K & \xrightarrow{\;\eta\;} & \operatorname{Re} K \\
\downarrow & & \downarrow \\
L & \xrightarrow{\;\eta\;} & (\operatorname{Re} K)\amalg_K L
\end{array}
\qquad\qquad
\begin{array}{ccc}
\operatorname{Re} K & \xrightarrow{\;\mathrm{id}\;} & \operatorname{Re} K \\
\downarrow & & \downarrow \\
\operatorname{Re} L & \xrightarrow{\;\mathrm{id}\;} & \operatorname{Re} L
\end{array}
$$

are good pushouts, using (44.11). The evident map from the left square to the right square is a weak equivalence at the upper left, upper right, and lower left corners (44.15), so the result follows from the invariance of good pushouts (43.11). $\qquad\square$

44.18. Corollary. *If $p\colon X \to Y$ is a Kan fibration, then $\operatorname{Si} X \to \operatorname{Si} Y \times_Y X$ is a trivial fibration. In particular, if X is a Kan complex, then $\operatorname{Si} X \to X$ is a trivial fibration.*

In particular, for any Kan complex X, both maps in $X \xleftarrow{\epsilon_X} \operatorname{Si} X \approx \operatorname{Si} X^{\mathrm{op}} \xrightarrow{\epsilon_{X^{\mathrm{op}}}} X^{\mathrm{op}}$ are trivial fibrations.

Proof. Straightforward, using (44.17). $\qquad\square$

44.19. Remark. The object $\operatorname{Re} X$ is not generally categorically equivalent to X.

It can be shown that if C is a quasicategory, then $\operatorname{Si} C$ is categorically equivalent to C^{core}.

45. Initial and terminal objects, revisited

Recall the definition of initial and terminal objects in a quasicategory. One characterization was: x is an initial object of C iff the left fibration $p\colon C_{x/} \to C$ is a trivial fibration, and a terminal object iff the right fibration $p'\colon C_{/x} \to C$ is a trivial fibration.

When C is the nerve of an ordinary category, these reduce to the usual definitions of initial and terminal object. In this case, there is an equivalent characterization: x is initial if and only if $\mathrm{Hom}_C(x,y)$ is a singleton set for all objects y of C, and terminal if and only if $Hom_C(y,x)$ is a singleton set for all y.

We would like to generalize this to the case of quasicategories.

F. **Deferred Proposition.** *An object x of a quasicategory is initial if and only if* $\mathrm{map}_C(x,c)$ *is contractible for all objects c of C, and terminal if and only if* $\mathrm{map}_C(c,x)$ *is contractible for all objects c of C.*

To prove this, you need to be able to relate mapping spaces of a quasicategory to the join/slice constructions that we used to define initial and terminal. We will establish such a relation in the next few sections.

45.1. **Right and left mapping spaces.** Let x, y be objects of a quasicategory C. We define the **right mapping space** $\mathrm{map}_C^R(x,y)$ and **left mapping space** $\mathrm{map}_C^L(x,y)$ by pullback diagrams

$$
\begin{array}{ccc}
\mathrm{map}_C^R(x,y) \longrightarrow C_{x/} & \qquad & \mathrm{map}_C^L(x,y) \longrightarrow C_{/y} \\
\downarrow \qquad\qquad \downarrow{\scriptstyle \pi} & & \downarrow \qquad\qquad \downarrow{\scriptstyle \pi} \\
\Delta^0 \xrightarrow{\quad y \quad} C & & \Delta^0 \xrightarrow{\quad x \quad} C
\end{array}
$$

where the maps labelled π are the evident forgetful maps.

For instance, an n-simplex of $\mathrm{map}_C^R(x,y)$ is precisely a map $a\colon \Delta^{n+1} \to C$ such that $a|\Delta^{\{0,\dots,n\}}$ is the constant simplex at x, and $a|\Delta^{\{n+1\}} = y$. In particular, a 0-simplex in $\mathrm{map}_C^R(x,y)$ is a morphism $x \to y$ in C, while a 1-simplex in $\mathrm{map}_C^R(x,y)$ is a 2-simplex in C exhibiting the $\sim_r$ relation between two maps, which we used to define the homotopy category in §9.

Recall (26.14) that when C is a quasicategory, the maps $C_{x/} \to C$ and $C_{/y} \to C$ are left fibrations and right fibrations respectively. Thus both $\mathrm{map}_C^R(x,y)$ and $\mathrm{map}_C^L(x,y)$ are Kan complexes, by the following.

45.2. *Exercise.* Show that if $X \to \Delta^0$ is a left fibration or a right fibration, then X is a Kan complex. (Hint: Joyal lifting.)

Furthermore, by the above remarks relating 1-simplices in the right and left mapping spaces to the homotopy relation, we have that $\pi_0 \mathrm{map}_C^R(x,y) \approx \pi_0 \mathrm{map}_C^L(x,y) \approx \mathrm{Hom}_{hC}(x,y)$.

We will show below that both $\mathrm{map}_C^R(x,y)$ and $\mathrm{map}_C^L(x,y)$ are actually *weakly equivalent* to the standard mapping space $\mathrm{map}_C(x,y)$.

45.3. **Box products and right and left anodyne maps.** Recall that $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (16.7) and $\overline{\mathrm{Horn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$. We have an analogous fact for left or right anodyne maps.

45.4. **Proposition.** *We have that* $\overline{\mathrm{LHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{LHorn}}$ *and* $\overline{\mathrm{RHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{RHorn}}$.

Proof. See **appendix**. □

45.5. Fiberwise criterion for trivial fibrations, revisited. Recall the fiberwise criterion for trivial fibrations (34.2): a Kan fibration p is a trivial fibration if and only if the fibers of p are contractible Kan complexes. In fact, this still holds if we only know p is a left or right fibration.

45.6. Proposition. *Suppose $p\colon X \to Y$ is a right fibration or left fibration of simplicial sets. Then p is a trivial fibration if and only if it has contractible fibers.*

Proof. [Lur09, 2.1.3.4]. Let's consider the case of $p\colon X \to Y$ a left fibration. The direction ($\Longrightarrow$) is immediate, so we only need to prove ($\Longleftarrow$).

We attempt to carry out the argument of the proof of (34.2), and show that $(\partial\Delta^n \subset \Delta^n) \boxempty p$ for all $n \geq 0$. The case of $n = 0$ is immediate, since the fibers of p must be non-empty, since they are contractible, so we can assume $n \geq 1$.

Examining that proof of (34.2), we see that we used only the hypothesis that p is a Kan fibration in order to have that

$$(\partial\Delta^n \times \{0\} \subset \partial\Delta^n \times \Delta^1) \boxempty p, \qquad \big((\partial\Delta^n \subset \Delta^n)\square(\{1\} \subset \Delta^1)\big) \boxempty p.$$

In the first case, the inclusion $(\partial\Delta^n \times \{0\} \subset \partial\Delta^n \times \Delta^1)$ is left anodyne by (45.4), so the lifting problem stil has a solution when p is only a left fibration.

In the second case, we need to argue a little differently. In the proof of (34.2) this lifting problem appears when producing a lifting (for $n \geq 1$) in a diagram of the form

$$
\begin{array}{ccc}
(\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\}) & \xrightarrow{\ (c,jd)\ } & X \\
\downarrow & \nearrow{\scriptstyle s} & \downarrow{\scriptstyle p} \\
\Delta^n \times \Delta^1 & \xrightarrow[\ b\gamma\ ]{} & Y
\end{array}
$$

Pulling back along the factorization of the bottom map $b\gamma$, we obtain a diagram

$$
\begin{array}{ccccc}
(\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\}) & \longrightarrow & C & \longrightarrow & X \\
\downarrow & \nearrow{\scriptstyle s'} & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
\Delta^n \times \Delta^1 & \xrightarrow[\ \gamma\ ]{} & \Delta^n & \xrightarrow[\ b\ ]{} & Y
\end{array}
$$

where the right-hand square is a pullback. Observe that (i) p' is a left fibration, and hence an inner fibration, between quasicategories, and that (ii) the map γ (as defined in the proof of (34.2)) sends the edge $\{n\} \times \Delta^1$ to the degenerate edge $\langle nn \rangle$ in Δ^n. Therefore we can apply the pushout-product version of Joyal lifting (29.12) to produce a lift s'. $\square$

45.7. Corollary. *An object x of a quasicategory C is initial if and only if $\operatorname{map}_C^R(x,c)$ is contractible for all objects c of C, and is final if and only if $\operatorname{map}_C^L(c,x)$ is contractible for all objects c of C.*

Proof. The fibers of the left fibration $C_{x/} \to C$ are precisely the right mapping spaces $\operatorname{map}_C^R(x,c)$. By what we just proved (45.6) these fibers are all contractible if and only if $C_{x/} \to C$ is a trivial fibration, which we have noted (25.4) is equivalent to x being initial in C. $\square$

46. The alternate join and slice

We now want to compare the right and left mapping spaces, which are fibers of the projections $C_{x/} \to C$ and $C_{/x} \to C$, to the ordinary mapping spaces, which are fibers of $\operatorname{Fun}(\Delta^1, C) \to \operatorname{Fun}(\partial\Delta^1, C)$. We do this using constructions called the "alternate join" and "alternate slice" [Lur09, §4.2.1].

Given an object x in C, consider the map

$$q\colon \operatorname{Fun}(\Delta^1, C) \times_{\operatorname{Fun}(\{0\},C)} \{x\} \to \operatorname{Fun}(\{1\}, C) = C$$

induced by restriction along $\{0\} \subset \Delta^1$. Note that the fiber of q over some object c of C is precisely the quasigroupoid $\mathrm{map}_C(x, c)$. The domain of q is an example of what we will call the "alternate slice" construction, for which we will use the (unmemorable) notation $C^{x/}$.

46.1. *Exercise.* Show that if C is an ordinary category, then $C^{x/}$ is *isomorphic* to the usual slice category $C_{x/}$, and q is *isomorphic* to the usual projection $p\colon C_{x/} \to C$.

For a general quasicategory, q is *not* isomorphic p. What is true is that there is a commutative diagram

$$C_{x/} \xrightarrow{\ \ f\ \ } C^{x/} = \mathrm{Fun}(\Delta^1, C) \times_{\mathrm{Fun}(\{0\}, C)} \{x\}$$

$$p \searrow \qquad \swarrow q$$

$$C$$

The map f sends a simplex $a\colon \Delta^k \to C_{/x}$, which corresponds to $\widetilde{a}\colon \Delta^{k+1} \to C$ such that $\widetilde{a}_0 = x$, to a simplex in $C^{x/}$ corresponding to $\widetilde{a}r\colon \Delta^k \times \Delta^1 \to C$, where $r\colon \Delta^k \times \Delta^1 \to \Delta^{k+1}$ is the unique map given on vertices by $r(i, 0) = 0$, $r(i, 1) = i + 1$.

The characterization (F) of initial objects in terms of contractible mapping spaces thus amounts to the claim that p is a trivial fibration if and only if q has contractible fibers. In fact, we'll prove that

- both p and q are left fibrations,
- f is a categorical equivalence.

Because p and q are left fibrations, they are trivial fibrations iff their fibers are contractible (45.6). Because f is a categorical equivalence, p is a categorical equivalence if and only if q is by 2-out-of-3 (22.9). The result follows because p and q are in particular isofibrations (28.10), and an isofibration is a trivial fibration if and only if it is a categorical equivalence (36.10).

In other words, we can regard $C^{x/}$ as an alternate version of the slice construction, so we call it the "alternate slice". It is related to an alternate version of the join, denoted $X \diamond Y$ and called the "alternate join", which we define first.

46.2. **The alternate join.** Given simplicial sets X and Y, define the **alternate join** by the pushout diagram

$$(X \times \{0\} \times Y) \amalg (X \times \{1\} \times Y) \rightarrowtail X \times \Delta^1 \times Y$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$(X \times \{0\} \times \Delta^0) \amalg (\Delta^0 \times \{1\} \times Y) \rightarrowtail X \diamond Y$$

where the maps on top and left are induced by the evident inclusion and projection maps.

The alternate join comes with a natural comparison map

$$X \diamond Y \to X \star Y,$$

defined as follows. Using the recipe of (23.9) for constructing maps to a join, we get a map $X \times \Delta^1 \times Y \to X \star Y$ corresponding to the triple $(g, f^{(0)}, f^{(1)})$, where $g\colon X \times \Delta^1 \times Y \to \Delta^1$, $f^{(0)}\colon X \times \{0\} \times Y \to X$, and $f^{(1)}\colon X \times \{1\} \times Y \to Y$ are the evident projections. A similar procedure produces compatible maps to $X \star Y$ from the other vertices of the pushout square defining $X \diamond Y$. Note that the comparison map induces a bijection on vertices.

46.3. *Example.* We have

$$X \diamond \Delta^0 \approx (X \times \Delta^1)/(X \times \{1\}), \qquad \Delta^0 \diamond Y \approx (\Delta^1 \times Y)/(\{0\} \times Y),$$

simplicial sets obtained by collapsing subcomplexes to a single point. These come with evident maps $X \diamond \Delta^0 \to X^{\triangleright}$ and $\Delta^0 \diamond Y \to Y^{\triangleleft}$.

Like the true join, $X \diamond \varnothing \approx X \approx \varnothing \diamond X$, and the functors $X \diamond - \colon s\mathrm{Set} \to s\mathrm{Set}_{X/}$ and $- \diamond Y \colon s\mathrm{Set} \to s\mathrm{Set}_{Y/}$ commute with colimits.

Warning: when X and Y are non-empty, $X \times \Delta^1 \times Y \to X \diamond Y$ is surjective, but this is not the case when either X or Y are empty.

Unlike the true join, the alternate join is not monoidal: $(X \diamond Y) \diamond Z \not\approx X \diamond (Y \diamond Z)$ in general. Also, the alternate join of two quasicategories is not usually a quasicategory.

The alternate join is a categorically invariant construction.

46.4. Proposition. *The alternate join $\diamond$ preserves categorical equivalences in either variable. That is, if $Y \to Y'$ is a categorical equivalence, then so are $X \diamond Y \to X \diamond Y'$ and $Y \diamond Z \to Y' \diamond Z$.*

Proof. The $\diamond$ product is constructed using a "good" pushout, i.e., a pushout along a cofibration. The result follows because both products and good pushouts preserve categorical equivalences (43.11). $\qquad\square$

46.5. Alternate slice. Given $p \colon S \to X$ and $q \colon T \to X$, we define the **alternate slices** $X^{p/}$ and $X^{/q}$ via the bijective correspondences

$$\left\{ \begin{array}{c} S \diamond \varnothing \\ \downarrow \qquad \searrow^{p} \\ S \diamond K \;-\!-\!\dashrightarrow\; X \end{array} \right\} \Longleftrightarrow \{K \dashrightarrow X^{p/}\}, \qquad \left\{ \begin{array}{c} \varnothing \diamond T \\ \downarrow \qquad \searrow^{q} \\ K \diamond T \;-\!-\!\dashrightarrow\; X \end{array} \right\} \Longleftrightarrow \{K \dashrightarrow X^{/q}\}.$$

just as we defined ordinary slices using joins. These constructions give right adjoints to the alternate join functors:

$$S \diamond (-) \colon s\mathrm{Set} \rightleftarrows s\mathrm{Set}_{S/} : (p \mapsto X^{p/}), \qquad (-) \diamond T \colon s\mathrm{Set} \rightleftarrows s\mathrm{Set}_{T/} : (q \mapsto X^{/q}).$$

Alternate slices are "functorial" in exactly the sense that ordinary slices are (24.13): a sequence of maps $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$ induces $X^{p/} \to Y^{fpj/}$ and $X^{/p} \to Y^{/fpj}$.

46.6. Exercise. Show that there are pullback squares of the form

$$
\begin{array}{ccc}
X^{p/} & \longrightarrow & \mathrm{Map}(S \times \Delta^1, X) \\
\downarrow & & \downarrow \\
X & \xrightarrow{(\widetilde{p},c)} & \mathrm{Map}(S \times \{0\}, X) \times \mathrm{Map}(S \times \{1\}, X)
\end{array}
\qquad
\begin{array}{ccc}
X^{/p} & \longrightarrow & \mathrm{Map}(\Delta^1 \times S, X) \\
\downarrow & & \downarrow \\
X & \xrightarrow{(c,\widetilde{p})} & \mathrm{Map}(\{0\} \times S, X) \times \mathrm{Map}(\{1\} \times S, X)
\end{array}
$$

where $\widetilde{p} \colon X \to \mathrm{Map}(S, X)$ is adjoint to $X \times S \xrightarrow{\mathrm{proj}} S \xrightarrow{p} X$, and $c \colon X \to \mathrm{Map}(S, X)$ is adjoint to $X \times S \xrightarrow{\mathrm{proj}} X \xrightarrow{\mathrm{id}} X$.

Using the adjunction relation between joins and slices, and alternate joins and slices, the natural comparison map $X \diamond Y \to X \star Y$ induces natural comparison maps on alternate slices. That is, given $p \colon S \to X$ and $q \colon T \to Y$ we have natural comparison maps

$$X_{p/} \to X^{p/} \qquad \text{and} \qquad Y_{/q} \to Y^{/q}.$$

46.7. Joins, slices, and function complexes. Recall the function complex $\mathrm{Map}(X, Y) \in s\mathrm{Set}$, defined for any pair of simplicial sets X, Y. Recall also (20.13) the relative function complex under S, which for objects $p \colon S \to X$ and $q \colon S \to Y$ in $s\mathrm{Set}_{S/}$ is a simplicial set

$$\mathrm{Map}_{S/}(X, Y) := \mathrm{Map}(X, Y) \times_{\mathrm{Map}(S,Y)} \{q\}$$

with bijective correspondences

$$\left\{ K \dashrightarrow \mathrm{Map}_{S/}(X, Y) \right\} \Longleftrightarrow \left\{ \begin{array}{c} K \times S \xrightarrow{\mathrm{proj}} S \\ \scriptstyle \mathrm{id} \times p \downarrow \qquad \downarrow \scriptstyle q \\ K \times X \;-\!-\!\dashrightarrow\; Y \end{array} \right\}$$

natural in the simplicial set K. The set of vertices of $\mathrm{Map}_{S/}(X, Y)$ is precisely the set $\mathrm{Hom}_{s\mathrm{Set}_{S/}}(X, Y)$ of morphisms in the category $s\mathrm{Set}_{S/}$.

Given X and $p\colon S \to Y$, we have join/slice adjunctions

$$\mathrm{Hom}_{s\mathrm{Set}}(X, Y_{p/}) \approx \mathrm{Hom}_{s\mathrm{Set}_{S/}}(S \star X, Y), \qquad \mathrm{Hom}_{s\mathrm{Set}}(X, Y_{/p}) \approx \mathrm{Hom}_{s\mathrm{Set}_{S/}}(X \star S, Y).$$

We now construct maps

$$\mathrm{Map}(X, Y_{p/}) \to \mathrm{Map}_{S/}(S \star X, Y), \qquad \mathrm{Map}(X, Y_{/p}) \to \mathrm{Map}_{S/}(X \star S, Y).$$

which are natural in both X and p, and which on vertices are exactly the join/slice adjunctions. We will call these the **enriched adjunction maps** for join/slice; they are not isomorphisms in general.

I write this out in the case of slice-over, by constructing a transformation

$$\left\{\, K \dashrightarrow \mathrm{Map}(X, Y_{p/}) \,\right\} \Longrightarrow \left\{\, K \dashrightarrow \mathrm{Map}_{S/}(S \star X, Y) \,\right\}$$

natural in the simplicial set K. Applying the product/function complex adjunction, and the join/slice adjunction, this amounts to defining natural maps

$$\left\{ \begin{array}{c} S \star \varnothing \\ \downarrow \quad \searrow^{p} \\ S \star (K \times X) \dashrightarrow Y \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} K \times (S \star \varnothing) \xrightarrow{\mathrm{proj}} S \\ \downarrow \qquad\qquad \downarrow^{p} \\ K \times (S \star X) \dashrightarrow Y \end{array} \right\}$$

Thus it suffices to produce natural maps

$$K \times (S \star X) \to S \star (K \times X)$$

which in the case that $X = \varnothing$ reduce to the projection map $K \times S \to S$. We take this to be the map corresponding by (23.9) to the triple $(g, f_{\{0\}}, f\{1\})$ so that g is the composite

$$K \times (S \star X) \to K \times (\Delta^0 \star \Delta^0) \to \Delta^0 \star \Delta^0 = \Delta^1,$$

and

$$f^{(0)} = \mathrm{proj}\colon K \times (S \star \varnothing) \to S, \qquad f^{(1)} = \mathrm{id}\colon K \times (\varnothing \star X) \to K \times X.$$

It is now straightforward to derive explicit formulas for the desired transformation (by specializing to $K = \Delta^n$), and to show that is is natural.

46.8. *Exercise.* Construct a natural "distributivity" map $K \times (X \star Y) \to (K \times X) \star (K \times Y)$.

46.9. **Alternate joins, alternate slices, and function complexes.** We can carry out the same procedure for alternate joins and slices, to obtain maps

$$\mathrm{Map}(X, Y^{p/}) \to \mathrm{Map}_{S/}(S \diamond X, Y), \qquad \mathrm{Map}(X, Y^{/p}) \to \mathrm{Map}_{S/}(X \diamond S, Y)$$

which are natural in both X and p, and which on vertices are exactly the alternate join/slice adjunctions. We will call these the **enriched adjunction maps** for alternate join/slice.

Tracing through the same steps as in the previous section, we see that (in the first case) we need natural maps

$$K \times (S \diamond X) \to S \diamond (K \times X)$$

which when $X = \varnothing$ reduce to the projection map $K \times S \to S$. In this case it is entirely straightforward to construct such a map, since both objects are naturally quotients of the product $K \times S \times \Delta^1 \times X \approx S \times \Delta^1 \times K \times X$. In fact, examination of the constructions shows that the evident diagram

$$\begin{array}{ccc} K \times S & \xrightarrow{\ \mathrm{proj}\ } & S \\ \downarrow & & \downarrow \\ K \times (S \diamond X) & \longrightarrow & S \diamond (K \times X) \end{array}$$

is a pushout square. (*Exercise:* prove this.) Given this consideration, we see that we have actually defined natural *isomorphisms*

$$\mathrm{Map}(X, Y^{p/}) \xrightarrow{\approx} \mathrm{Map}_{S/}(S \diamond X, Y), \qquad \mathrm{Map}(X, Y^{/p}) \xrightarrow{\approx} \mathrm{Map}_{S/}(X \diamond S, Y).$$

Furthermore, these natural isomorphisms are compatible with the transformations for join/slice.

46.10. **Proposition.** *The evident diagrams*

$$
\begin{array}{ccc}
\mathrm{Map}(X, Y_{p/}) & \longrightarrow & \mathrm{Map}_{S/}(S \star X, Y) \\
\downarrow & & \downarrow \\
\mathrm{Map}(X, Y^{p/}) & \xrightarrow{\approx} & \mathrm{Map}_{S/}(S \diamond X, Y)
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Map}(X, Y_{/p}) & \longrightarrow & \mathrm{Map}_{S/}(X \star S, Y) \\
\downarrow & & \downarrow \\
\mathrm{Map}(X, Y^{/p}) & \xrightarrow{\approx} & \mathrm{Map}_{S/}(X \diamond S, Y)
\end{array}
$$

commute.

Proof. This amounts to showing that the evident diagram

$$
\begin{array}{ccc}
K \times (S \diamond X) & \longrightarrow & S \diamond (K \times X) \\
\downarrow & & \downarrow \\
K \times (S \star X) & \longrightarrow & S \star (K \times X)
\end{array}
$$

commutes, which we leave to the reader. $\square$

Below we will show that if $Y = C$ is a quasicategory, then all of the maps in these diagrams are categorical equivalences. As a consequence, we will obtain categorical equivalences $C_{p/} \to C^{p/}$ and $C_{/p} \to C^{/p}$.

47. Equivalence of the two join and slice constructions

47.1. **The enriched adjunction map for joins/slices preserves isomorphism classes of objects.** We now consider the natural maps

$$\mathrm{Map}(X, C_{p/}) \to \mathrm{Map}_{S/}(S \star X, C), \qquad \mathrm{Map}(X, C_{/p}) \to \mathrm{Map}_{S/}(X \star S, C)$$

in the case when $p\colon S \to C$ is a map to a quasicategory C. In this case both sources and targets of the natural maps in question are themselves quasicategories, and both induce bijections on sets of objects. Eventually we will show that these functors are categorical equivalencs. Right now we will just prove that these functors induce bijections on *isomorphism classes of objects*.

47.2. **Proposition.** *For X a simplicial set and $p\colon S \to C$ a map to a quasicategory, the enriched adjunction map for join/slice induces bijections*

$$\pi_0\big(\mathrm{Map}(X, C_{p/})^{\mathrm{core}}\big) \xrightarrow{\sim} \pi_0\big(\mathrm{Map}_{S/}(S \star X, C)^{\mathrm{core}}\big), \quad \pi_0\big(\mathrm{Map}(X, C_{/p})^{\mathrm{core}}\big) \xrightarrow{\sim} \pi_0\big(\mathrm{Map}_{S/}(X \star S, C)^{\mathrm{core}}\big),$$

Proof. We give the proof in the slice-over case. Since the enriched adjunction map gives a bijection on objects, it suffices to prove injectivity on sets of isomorphism classes.

Let $f_0, f_1\colon X \to C_{p/}$ be objects of $\mathrm{Map}(X, C_{p/})$, which correspond to objects $\widetilde{f}_0, \widetilde{f}_1\colon S \star X \to C$ of $\mathrm{Map}_{S/}(S \star X, C)$, with $\widetilde{f}_j|S = p$. If $\widetilde{f}_0$ and $\widetilde{f}_1$ are isomorphic objects, then there exists a map $N\mathrm{Iso} \to \mathrm{Map}_{S/}(S \star X, C)$ representing such an isomorphism (32.15). The data of such a map amounts to a an arrow $\widetilde{f}$ fitting in the commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\ p\ } C \longrightarrow & \mathrm{Map}(N\mathrm{Iso}, C) \\
\downarrow & \nearrow{\scriptstyle \widetilde{f}} & \downarrow \\
S \star X & \xrightarrow[(\widetilde{f}_0, \widetilde{f}_1)]{} & C \times C
\end{array}
$$

where $C \to \mathrm{Map}(N\mathrm{Iso}, C) \to C \times C$ are induced by restriction along $* \leftarrow N\mathrm{Iso} \leftarrow \{0, 1\}$. Write $D = \mathrm{Map}(N\mathrm{Iso}, C)$ and $\bar{p}\colon S \to D$ for the map along the top of the rectangle. Applying the join/slice adjunction, we see that we have a diagram

$$
\begin{array}{ccc}
 & & D_{\bar{p}/} \\
 & \nearrow^{f} & \downarrow \\
X & \xrightarrow[(f_0, f_1)]{} & C_{p/} \times C_{/p}
\end{array}
$$

That is, we have produced an object f in $\mathrm{Map}(X, D_{\bar{p}/})$ which under the two evident projections $\pi_0, \pi_1\colon \mathrm{Map}(X, D_{\bar{p}/}) \to \mathrm{Map}(X, C)$ is sent to f_0 and f_1 respectively.

We have that both projections $D \to C$ are trivial fibrations, whence so are both projections $D_{\bar{p}/} \to C_{p/}$ **(this needs a proof)**, and hence both projections π_0 and π_1. Considering the induced commutative diagram

$$
\begin{array}{ccc}
 & \xrightarrow{\mathrm{id}} & \mathrm{Map}(X, C_{p/}) \\
 & \nearrow & \\
\mathrm{Map}(X, C_{p/}) \longrightarrow \mathrm{Map}(X, D_{\bar{p}/}) & \xrightarrow{\pi_0} & \\
 & \searrow_{\pi_1} & \\
 & \xrightarrow{\mathrm{id}} & \mathrm{Map}(X, C_{p/})
\end{array}
$$

we see that every arrow in this diagram is a categorical equivalence, and therefore both π_0 and π_1 induce *the same bijection* on isomorphism classes on objects. Therefore f_0 and f_1 are isomorphic objects, as desired. $\square$

47.3. Equivalence of join and alternate join. The key result of this section is the following.

47.4. Proposition. *The canonical comparison map $X \diamond Y \to X \star Y$ is a categorical equivalence for all simplicial sets X and Y.*

What we have proved implies the categorical invariance of the usual join.

47.5. Corollary. *The join $\star$ preserves categorical equivalences in either variable. That is, if $Y \to Y'$ is a categorical equivalence, then so are $X \star Y \to X \star Y'$ and $Y \star Z \to Y' \star Z$.*

Proof. Immediate using (47.4), the invariance of the alternate join under categorical equivalence (46.4), and the 2-out-of-3 property of categorical equivalences (22.9). $\square$

The proof is based on the following general strategy.

47.6. Proposition. *Let $\alpha\colon F \to F'$ be a natural transformation between functors $s\mathrm{Set} \to \mathcal{M}$, where $\mathcal{M}$ is some model category. If*

(1) *F and F' preserve colimits,*
(2) *F and F' take monomorphisms to cofibrations,*
(3) *F and F' take inner anodyne maps to to weak equivalences in $\mathcal{M}$, and*
(4) *$\alpha(\Delta^1)\colon F(\Delta^1) \to F'(\Delta^1)$ is a weak equivalence in $\mathcal{M}$,*

then $\alpha(X)\colon F(X) \to F'(X)$ is a weak equivalence in $\mathcal{M}$ for all simplicial sets X.

Proof. [Lur09, 4.2.1.2] Consider the class of simplicial sets $\mathcal{C} := \{\, X \mid \alpha(X) \text{ is a weak equivalence} \,\}$. We use skeletal induction (44.14) to show that $\mathcal{C}$ contains all simplicial sets.

It is clear that $\mathcal{C}$ is closed under isomorphic objects. Because F and F' preserve colimits (1) and cofibrations (2), they take good colimit diagrams in $s\mathrm{Set}$ to good colimit diagrams in $\mathcal{M}$. Since good colimits are weak equivalence invariant (43.9), (43.11), (43.13), we see that $\mathcal{C}$ is closed under forming good colimits. It remains to show that $\Delta^n \in \mathcal{C}$ for all n.

We have $\Delta^1 \in \mathcal{C}$ by (4). Since Δ^0 is a retract of Δ^1, we get that $\Delta^0 \in \mathcal{C}$ since weak equivalences in $\mathcal{M}$ are closed under retracts (41.3).

The spines I^n can be built from Δ^0 and Δ^1 by a sequence of good pushouts (glue on one 1-simplex at a time), so the $I^n \in \mathcal{C}$. The inclusions $I^n \subset \Delta^n$ are inner anodyne (10.12), so by (3) and the 2-out-of-3 property of weak equivalences in $\mathcal{M}$ it follows that $\Delta^n \in \mathcal{C}$. $\qquad\square$

We will apply this idea to functors $s\mathrm{Set} \to s\mathrm{Set}_{X/}$, where the slice category $s\mathrm{Set}_{X/}$ inherits its model structure from the Joyal model structure on $s\mathrm{Set}$ (41.4).

Proof of (47.4). The functors $X \diamond (-), X \star (-), (-) \diamond X, (-) \star X \colon s\mathrm{Set} \to s\mathrm{Set}_{X/}$ satisfy the first three properties required of the functors in the previous proposition (47.6). That is, they (1) preserve colimits, (2) take monomorphisms to monomorphisms, and (3) take inner anodyne maps to categorical equivalences. Condition (3) for $\diamond$ follows from (46.4), while condition (3) for $\star$ this follows from (26.11) since $\overline{\mathrm{InnHorn}} \subseteq \overline{\mathrm{LHorn}} \cap \overline{\mathrm{RHorn}}$.

Thus, to show $X \diamond Y \to X \star Y$ is a categorical equivalence for a fixed X and arbitrary Y, it suffices by the previous proposition to show that $X \diamond \Delta^1 \to X \star \Delta^1$ is a categorical equivalence. The same argument lets us reduce to the case when $X = \Delta^1$, i.e., to showing that a single map $\overline{f} \colon \Delta^1 \diamond \Delta^1 \to \Delta^1 \star \Delta^1$ is a categorical equivalence.

We will show $\overline{f}$ is a categorical equivalence by producing a map $\overline{g} \colon \Delta^1 \star \Delta^1 \to \Delta^1 \diamond \Delta^1$ such that $\overline{f}\overline{g} = \mathrm{id}_{\Delta^1 \star \Delta^1}$ and $\overline{g}\overline{f}$ is preisomorphic to the identity map of $\Delta^1 \diamond \Delta^1$, via (20.7) .

Since $\Delta^1 \diamond \Delta^1$ is a quotient of a cube, we start with maps involving the cube. I will write vertices in $(\Delta^1)^\times$ as sequences $(a_1 a_2 a_3)$ where $a_i \in \{0,1\}$. Let

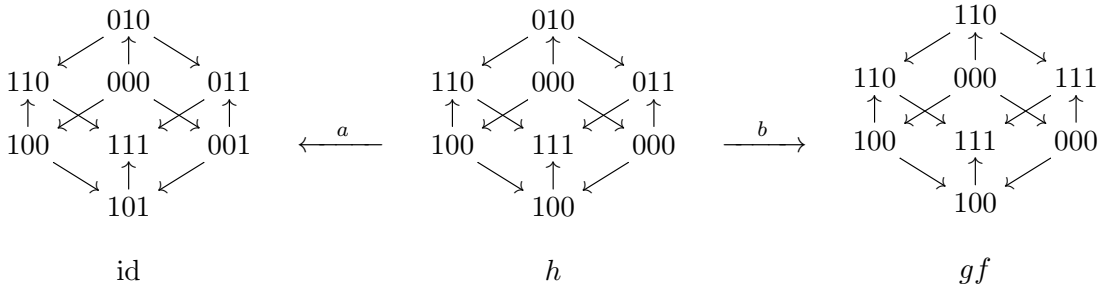$$f \colon (\Delta^1)^{\times 3} \to \Delta^3 = \Delta^1 \star \Delta^1$$

be the map which on vertices sends

$$(a_1 a_2 a_3) \mapsto \sup \{\, k \mid a_k = 1 \,\}.$$

On passage to quotients this gives the comparison map $\overline{f} \colon \Delta^1 \diamond \Delta^1 \to \Delta^1 \star \Delta^1$ of the proposition.

Let $g \colon \Delta^3 \to (\Delta^1)^{\times 3}$ be the map classifying the simplex $\langle (000), (100), (110), (111) \rangle$, and let $\overline{g} \colon \Delta^3 \to \Delta^1 \diamond \Delta^1$ be the composite with the quotient map. We have $fg = \mathrm{id}_{\Delta^3} = \overline{f}\overline{g}$.

Let $h \in \mathrm{Map}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})_0$ and $a, b \in \mathrm{Map}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})_1$ be as indicated in the following picture.

These pass to simplices $\overline{h}, \overline{a}, \overline{b}$ in $\mathrm{Map}(\Delta^1 \diamond \Delta^1, \Delta^1 \diamond \Delta^1)$. The edges $\overline{a}$ and $\overline{b}$ are preisomorphisms, as one sees that for each vertex $v \in (\Delta^1 \diamond \Delta^1)$, the induced maps $\Delta^1 \times \{v\} \subset \Delta^1 \times (\Delta^1 \diamond \Delta^1) \xrightarrow{\overline{a} \text{ or } \overline{b}} \Delta^1 \diamond \Delta^1$ represent degenerate edges. Thus $\overline{f}\overline{g}$ and $\overline{g}\overline{f}$ are preisomorphic to identity maps, and hence $\overline{f}$ is a categorical equivalence as desired. $\qquad\square$

47.7. Equivalence of slice and alternate slice.

47.8. **Proposition.** *For any quasicategory C and map $p \colon S \to C$, the comparison maps $C_{p/} \to C^{p/}$ and $C_{/p} \to C^{/p}$ are categorical equivalences.*

Proof. We do the first case. We use the following fact: if $f\colon A \to B$ is a functor between quasicategories, then f is a categorical equivalence if and only if the induced maps $\pi_0\big(\mathrm{Fun}(X,A)^{\mathrm{core}}\big) \to \big(\mathrm{Fun}(X,B)^{\mathrm{core}}\big)$ are bijections for all simplicial sets X. **I probably did this before somewhere.**
Recall (46.10) that we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Map}(X, C_{p/}) & \longrightarrow & \mathrm{Map}_{S/}(S \star X, C) \\
\downarrow & & \downarrow \\
\mathrm{Map}(X, C^{p/}) & \xrightarrow[\approx]{} & \mathrm{Map}_{S/}(S \diamond X, C)
\end{array}
$$

in which the bottom map is an isomorphism. By (47.2) the top map is a bijection on isomorphism classes of objects By (47.4) $S \diamond X \to S \star X$ is a categorical equivalence, and thus the right-hand map is a categorical equivalence, and hence a bijection on isomorphism classes of objects. It follows that the left-hand map is a bijection on isomorphism classes of objects, and the proposition is proved. $\square$

47.9. Corollary. *For any quasicategory C and map $p\colon S \to C$, the enriched adjunction maps* $\mathrm{Fun}(X, C_{p/}) \to \mathrm{Map}_{S/}(S \star X, C)$ *and* $\mathrm{Fun}(X, C_{/p}) \to \mathrm{Map}_{S/}(X \star S, C)$ *are categorical equivalences.*

47.10. Alternate pushout-join. Just as we defined the "pushout-join" $⊞$, we can define the "alternate pushout-join" $⧄$: given $f\colon A \to B$ and $g\colon K \to L$, we obtain

$$
f ⧄ g \colon (B \diamond K) \cup_{A \diamond K} (A \diamond L) \to B \diamond L.
$$

47.11. Proposition. *We have that*

$$
\overline{\mathrm{RHorn}} ⧄ \overline{\mathrm{Cell}} \ \cup \ \overline{\mathrm{Cell}} ⧄ \overline{\mathrm{LHorn}} \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}.
$$

Proof. We'll show that $\overline{\mathrm{RHorn}}⧄\overline{\mathrm{Cell}} \subseteq \mathrm{Cell}\cap\mathrm{CatEq}$. It is straightforward to show that the $⧄$-product of two monomorphisms is a monomorphism. Thus, it suffices to show that for $f\colon A \to B$ right anodyne and any inclusion $g\colon K \to L$, the map $f ⧄ g$ is a categorical equivalence. We know that $\overline{\mathrm{RHorn}} ⊞ \mathrm{Cell} \subseteq \overline{\mathrm{InnHorn}} \subseteq \mathrm{CatEq}$ (26.11), so $f ⊞ g$ is a categorical equivalence. Furthermore, in

$$
\begin{array}{ccc}
(B \diamond K) \cup_{A \diamond K} (A \diamond L) & \longrightarrow & B \diamond L \\
\downarrow & & \downarrow \\
(B \star K) \cup_{A \star K} (A \star L) & \longrightarrow & B \star L
\end{array}
$$

the vertical maps are categorical equivalences; this uses the result proved above (47.4), as well as the fact that since f and g are monomorphisms, the domains of $f ⧄ g$ and $f ⊞ g$ are constructed from good pushouts. $\square$

Question: is $\overline{\mathrm{LHorn}} ⧄ \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$?

47.12. Proposition. *Given $K \xrightarrow{j} L \xrightarrow{p} C$, if C is a quasicategory and j is a monomorphism, then $C^{p/} \to C^{pj/}$ is a left fibration, and $C^{/p} \to C^{/pj}$ is a right fibration.*

Proof. Follows from $\overline{\mathrm{LHorn}} ⧄ \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$ and $\overline{\mathrm{Cell}} ⧄ \overline{\mathrm{RHorn}} \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$. $\square$

47.13. Equivalence of various mapping spaces. Finally we can prove our original goal.

47.14. Proposition. *For any quasicategory C and object $x \in C_0$, the natural comparison maps* $\mathrm{map}_C^R(x,y) \to \mathrm{map}_C(x,y) \leftarrow \mathrm{map}_C^L(x,y)$ *are weak equivalences.*

Proof. In

$$
\begin{array}{ccc}
C_{x/} & \xrightarrow{\ f\ } & C^{x/} \\
 & {}_{p}\searrow \quad \swarrow_{q} & \\
 & C &
\end{array}
$$

the map f is a categorical equivalence (47.8) and p and q are left fibrations by (**??**) and (**??**) respectively, and hence are categorical fibrations (**??**). It follows that the induced maps on fibers $\operatorname{map}_C^R(x,c) \to \operatorname{map}_C(x,c)$ are categorical equivalences and hence weak equivalences, since the pullbacks describing the pullbacks are good pullbacks (with respect to the Joyal model structure). □

47.15. **Slices as fibers. Rewrite this in terms of the enriched adjunction maps.**

The alternate slice $C^{f/}$ has another convenient characterization: it is the fiber over f of a map between functor categories.

47.16. **Proposition.** *For a map $f \colon S \to X$ of simplicial sets, the alternate slice $X^{f/}$ is isomorphic to the fiber of the restriction map*

$$\operatorname{Map}(S \diamond \Delta^0, X) \to \operatorname{Map}(S, X).$$

over f.

Proof. Let F be the fiber of the restriction map. There is an evident correspondence

$$
\begin{array}{ccc}
& & S \times K \xrightarrow{\ \pi\ } S \\
K \dashrightarrow F & \Longleftrightarrow & \downarrow \qquad\qquad \downarrow f \\
& & (S \diamond \Delta^0) \times K \dashrightarrow C
\end{array}
$$

The claim follows by showing that the evident quotient map $S \times \Delta^1 \times K \to (S \diamond \Delta^0) \times K$ extends to an isomorphism

$$S \diamond K \xrightarrow{\ \sim\ } ((S \diamond \Delta^0) \times K) \cup_{S \times K} S$$

compatible with the standard inclusions of S. □

We can also consider the fiber of the inclusion $S \subset S \star \Delta^0$ into the standard cone. This gives yet another version of the slice.

47.17. **Corollary.** *Let C be a quasicategory, and let $F(f) :=$ the fiber of $\operatorname{Fun}(S^{\triangleright}, C) \to \operatorname{Fun}(S, C)$ over f. Then there is a chain of categorical equivalences*

$$F(f) \to C^{f/} \leftarrow C_{f/}.$$

Furthermore, $F(f)$ and $C_{f/}$ have the same set of 0-simplices, and both arrows above coincide on 0-simplices.

Proof. The second equivalence is just (47.8). For the first equivalence, note that

$$
\begin{array}{ccc}
\operatorname{Fun}(S \star \Delta^0, C) & \longrightarrow & \operatorname{Fun}(S \diamond \Delta^0, C) \\
\downarrow & & \downarrow \\
\operatorname{Fun}(S, C) & =\!=\!=\!= & \operatorname{Fun}(S, C)
\end{array}
$$

the top horizontal map is a categorical equivalence using (47.4), while the vertical maps are both categorical fibrations. Therefore the induced map on fibers over f is a categorical equivalence, since the pullback squares in question are good.

The 0-simplices of $F(f)$ and $C_{f/}$ are exactly the set $\{S^{\triangleright} \to C\}$. Both inclusions $F(f)_0 \to (C^{f/})_0 \leftarrow (C_{f/})_0$ are induced by restriction along the standard comparison map $S \diamond \Delta^0 \to S \star \Delta^0$. □

Part 6. **Cartesian fibrations**

Part 7. **Old stuff**

Note. From this point forward, these notes are not an organize narrative, but rather a collection of bits and pieces that might be worked into something useful at some point.

48. Quasicategories as a quasicategory

48.1. **Functors to sets and categories.** In ordinary category theory, the category Set of sets plays a distinguished role. Notably, for every category[30] C and every object $x \in \operatorname{ob} C$, we have the **representable** and **corepresentable** functors

$$\rho_x = \operatorname{Hom}_C(-, x) \colon C^{\mathrm{op}} \to \operatorname{Set}, \qquad \rho^x = \operatorname{Hom}_C(x, -) \colon C \to \operatorname{Set}.$$

These are found everywhere, and are the subject of the *Yoneda lemma*: natural transformations $\rho_x \to F$ are in natural bijective correspondence with elements of $F(x)$.

In many situation we have examples of functors $C \to \operatorname{Cat}$ to the category of categories.

48.2. *Example.* Let C be the category of associative rings. We have a functor

$$\operatorname{Mod} \colon C^{\mathrm{op}} \to \operatorname{Cat},$$

which sends R to Mod_R, the category of left R-modules, and sends a homomorphism $f \colon R \to S$ to the restriction functor $f^* \colon \operatorname{Mod}_S \to \operatorname{Mod}_R$, so f^*M has underlying abelian group M and R-module structure given by $r \cdot m := f(r)m$.

Sometimes we have something that looks like a functor $C \to \operatorname{Cat}$, but isn't quite: these are **pseudofunctors**.

48.3. *Example.* Let C be associative rings as in the last example. We have an operation

$$\operatorname{Mod}' \colon C \to \operatorname{Cat},$$

which sends R to Mod_R, but sends a homomorphism $f \colon R \to S$ to the "extension of scalars functor", given on objects by

$$f_*M := S \otimes_R M.$$

Given $R \xrightarrow{f} S \xrightarrow{g} T$, we only have a natural isomorphism $T \otimes_S (S \otimes_R M) \approx T \otimes_R M$, so $g_* f_* \neq (gf)_*$, and thus Mod' is not really a functor. If we incorporate the data of suitable natural isomorphisms $g_* f_* \approx (gf)_*$ and $\operatorname{id}_* \approx \operatorname{id}$ into the definition, we get the notion of a pseudofunctor. **Give reference here.**

48.4. **Functors to ∞-groupoids and ∞-categories?** Given a quasicategory C and an object x, we would like to consider the functor represented by x. Roughly, this associates to an object c in C the object $\operatorname{map}_C(c, x)$, which is a Kan complex. Thus, we might imagine that we get a functor

$$\operatorname{map}_C(-, x) \colon C^{\mathrm{op}} \to \operatorname{Kan}.$$

Of course, $\operatorname{map}_C(-, x)$ is not a functor: given a morphism $f \colon c \to c'$ in C, we only know how to produce a zig-zag $\operatorname{map}_C(c', x) \xleftarrow{\sim} \bullet \to \operatorname{map}_C(c, x)$.

48.5. **Three models for an ∞-category.** We have touched on several different models for an ∞-category:

(1) quasicategories,
(2) categories enriched over Kan complexes.

Although QCat and Kan are not "naturally" quasicategories, they can be regarded naturally as of type (2).

Thus, we have a simplicially enriched category $\operatorname{QCat}_{s,\mathrm{core}}$, whose underlying objects are quasicategories, and whose mapping spaces are

$$\operatorname{Fun}(C, D)^{\mathrm{core}}.$$

---

[30]locally small category: the hom-sets of C must actually be objects of Set

We can extract an actual quasicategory from this by using some kind of machine. For instance, Lurie defines

$$\mathrm{Cat}_\infty := \mathcal{N}(\mathrm{QCat}_{s,\mathrm{core}})$$

where $\mathcal{N}$ denotes the "simplicial nerve" construction.

If we are interested in functors from C^{op} to ∞-categories, we can model this by the simplicially enriched category

$$(s\mathrm{Set}^{\mathrm{Cart}}_{/C})_{s,\mathrm{core}}.$$

Here, $s\mathrm{Set}_{/C}$ is the slice category, which has a simplicial enrichment with function objects

$$\mathrm{Map}_{/C}(p,p') \subseteq \mathrm{Map}(E,E')$$

for $p\colon E \to C$ and $p'\colon E' \to C$. We can consider a full subcategory consisting of $p\colon E \to C$ which are "Cartesian fibrations" (to be defined below). Cartesian fibrations are a special kind of inner fibration, so $\mathrm{Map}_{/C}(p,p')$ is a quasicategory in this case. We obtain a category enriched over Kan complexes, with function objects

$$\mathrm{Map}_{/C}(p,p')^{\mathrm{core}}.$$

This is the candidate for $\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Cat}_\infty)$.

49. Coherent nerve

49.1. The coherent nerve. The coherent nerve $\mathcal{N}$ is a construction which turns a simplicially enriched category into a simplicial set, and in particular turns a Kan-enriched category into a quasicategory. It was invented by Cordier [Cor82]. The coherent nerve is constructed as right adjoint of a "realization/singular" pair

$$\mathfrak{C}\colon s\mathrm{Set} \rightleftarrows s\mathrm{Cat} :\mathcal{N}.$$

Given a finite totally ordered set S, define

$$\mathcal{P}(S) := \{\, A \subseteq S \mid \{\min,\max\} \subseteq A \subseteq S \,\}.$$

This is a poset, ordered by set containment; here $\min,\max$ denote the least and greatest elements of S (possibly the same). If S is empty, so is $P(S)$.

Let $\mathfrak{C}(\Delta^n)$ denote the simplicially enriched category defined as follows.

- The objects are elements of $[n] = \{0,\ldots,n\}$.
- For objects $x,y \in [n]$, the function complex is

$$\mathrm{Map}_{\mathfrak{C}(\Delta^n)}(x,y) := N\mathcal{P}([x,y]), \qquad [x,y] := \{\, t \in [n] \mid x \le t \le y \,\},$$

  which is set to be empty if $x > y$.
- Composition is induced by the union operation on subsets:

$$(T,S) \mapsto T \cup S\colon \mathcal{P}([y,z]) \times \mathcal{P}([x,y]) \to \mathcal{P}([x,z]).$$

Every $f\colon [m] \to [n]$ in Δ gives rise to an enriched functor $\mathfrak{C}(f)\colon \mathfrak{C}(\Delta^m) \to \mathfrak{C}(\Delta^n)$, which on objects operates as f does on elements of the ordered sets, and is induced on morphisms by

$$S \mapsto f(S)\colon \mathcal{P}([x,y]) \to \mathcal{P}([fx,fy])$$

We obtain (after identifying Δ with its image in $s\mathrm{Set}$) a functor $\mathfrak{C}\colon \Delta \to s\mathrm{Cat}$.

Given a simplicially enriched category C, its **coherent nerve** (or **simplicial nerve**) is the simplicial set $\mathcal{N}C$ defined by

$$(\mathcal{N}C)_n = \mathrm{Hom}_{s\mathrm{Cat}}(\mathfrak{C}(\Delta^n),C).$$

...

49.2. **Quasicategories from simplicial nerves.**

49.3. Proposition. *If $\mathcal{C}$ is a category enriched over Kan complexes, then $\mathcal{N}(\mathcal{C})$ is a quasicategory.*

Proof. $\square$

50. CORRESPONDENCES

A **correspondence** is defined to be an inner fibration $p\colon M \to \Delta^1$. A map of correspondences is a morphism in the slice category $s\mathrm{Set}_{/\Delta^1}$.

50.1. **Correspondences of ordinary categories.** If M is an ordinary category, then any functor $p\colon M \to \Delta^1$ is an inner fibration. Given such a functor, we can identify the following data:

- categories $C := p^{-1}(\{0\})$ and $D := p^{-1}(\{1\})$, the preimages of the vertices, and
- for each pair of objects $c \in \mathrm{ob}\, C$, $d \in \mathrm{ob}\, D$, a set
$$\mathcal{M}(c,d) := \mathrm{Hom}_M(c,d),$$
  which
- fit together to give a functor
$$\mathcal{M}\colon C^{\mathrm{op}} \times D \to \mathrm{Set}.$$

Conversely, given the data of categories C and D, and a functor $\mathcal{M}\colon C^{\mathrm{op}} \times D \to \mathrm{Set}$, we can construct a category M with functor $p\colon M \to \Delta^1$ in the evident way, with

$$\mathrm{ob}\, M := \mathrm{ob}\, C \amalg \mathrm{ob}\, D, \qquad \mathrm{mor}\, M := \mathrm{mor}\, C \amalg \left(\coprod_{c,d} \mathcal{M}(c,d) \right) \amalg \mathrm{mor}\, D.$$

Under the above, maps $f\colon M \to M'$ between correspondences which are categories are sent to data consisting of: functors $u\colon C \to C'$ and $v\colon D \to D'$, and natural transformations

$$\mathcal{M} \to \mathcal{M}' \circ (u \times v) \quad \text{of functors} \quad C^{\mathrm{op}} \times D \to \mathrm{Set}.$$

50.2. *Example.* If C and D are categories, then the functor $C \star D \to \Delta^0 \star \Delta^0 \approx \Delta^1$ is an example of a correspondence. The corresponding functor $\mathcal{M}\colon C^{\mathrm{op}} \times D \to \mathrm{Set}$ is the one with $\mathcal{M}(c,d) = \{*\}$ for all objects.

50.3. *Example.* Let $F\colon C \to D$ be a functor between categories. Then we get a functor $\mathcal{M}\colon C^{\mathrm{op}} \times D \to \mathrm{Set}$ defined by
$$\mathcal{M}(c,d) := \mathrm{Hom}_D(F(c), D),$$
and thus an associated correspondence $p\colon M \to \Delta^1$.

Similarly, let $G\colon D \to C$ be a functor between categories. Then we get a functor $\mathcal{M}'\colon C^{\mathrm{op}} \times D \to \mathrm{Set}$ defined by
$$\mathcal{M}'(c,d) := \mathrm{Hom}_C(c, G(d)),$$
and thus an associated correspondence $p'\colon M' \to \Delta^1$.

50.4. *Example.* Suppose $F\colon C \rightleftarrows D \colon G$ is an adjoint pair of functors. If we form $\mathcal{M}$ and $\mathcal{M}'$ as in the previous example, we see that the adjunction gives a natural isomorphism $\mathcal{M} \approx \mathcal{M}'$ of functors $C^{\mathrm{op}} \times D \to \mathrm{Set}$. The associated correspondences $M \to \Delta^1$ and $M' \to \Delta^1$ are isomorphic.

51. Cartesian and cocartesian morphisms

In the following, we fix an inner fibration $p\colon M \to S$. We will often assume that S (and thus M) is a quasicategory.

Consider an edge $f\colon x \to y$ in M. We say that the edge represented by $f\colon \Delta^1 \to M$ is p-**cartesian** if a lift exists in every diagram of the form

$$
\begin{array}{ccc}
\Delta^{\{n-1,n\}} & \overset{f}{\rightarrowtail} \Lambda^n_n \longrightarrow & M \\
\downarrow & \nearrow & \downarrow p \\
\Delta^n & \longrightarrow & S
\end{array}
$$

for all $n \geq 2$.

There is a dual notion of a p-**cocartesian edge**, where Λ^n_n is replaced by Λ^n_0, and we use the leading edge of the simplex instead of the trailing edge.

We have already seen examples of this property.

- Let $p\colon C \to *$ where C is a quasicategory By the Joyal extension theorem (28.2), we have that an edge in C is p-cartesian if and only if it is p-cocartesian if and only if it is an isomorphism.
- Let $p\colon M \to S$ be an inner fibration between quasicategories, and suppose $f \in M_1$ is an edge such that $p(f)$ is an isomorphism in S. By the Joyal lifting theorem (28.13), f is p-cartesian if and only if it is p-cocartesian if and only if f is an isomorphism in M.
- If $p\colon M \to S$ is a right fibration, then every edge in M is p-cartesian. Likewise, if p is a left fibration, then every edge in M is p-cocartesian.

Thus, Joyal's theorem completely describe cartesian/cocartesian edges over an *isomorphism* in a quasicategory.

We have an equivalent formulation: f is p-cartesian if and only if

$$
M_{/f} \to M_{/y} \times_{S_{/p(y)}} S_{/pf}
$$

is a trivial fibration.

51.1. **Cartesian edges and correspondence.** Let $p\colon M \to \Delta^1$ be a correspondence, with M an ordinary category. We write $C := p^{-1}(\{0\})$, $D := p^{-1}(\{1\})$, and $\mathcal{M}\colon C^{\mathrm{op}} \times D \to \mathrm{Set}$ for the associated functor.

Suppose $f\colon c \to d$ is an edge such that $p(f) = \langle 01 \rangle$.

51.2. **Lemma.** *The edge f is p-catesian if and only if, for each $u\colon x \to d$ with $p(u) = \langle 01 \rangle$, there exists a unique $v\colon x \to c$ such that $fv = u$.*

In particular, if f is p-cartesian, then composition

$$
f_*\colon \mathrm{Hom}_M(x, c) \to \mathrm{Hom}_M(x, d)
$$

is a bijection for all $x \in \mathrm{ob}\, C$. Equivalently, the map

$$
\mathrm{Hom}_C(x, c) \to \mathcal{M}(x, d), \qquad v \mapsto fv
$$

is a bijection, so $\mathcal{M}(-, d)\colon C^{\mathrm{op}} \to \mathrm{Set}$ is represented by c.

51.3. Box criterion for cartesian edges.

51.4. Proposition. [Lur09, 2.4.1.8] *Let $p\colon M \to S$ be an inner fibration, and $f \in M_1$ an edge. Then f is p-cartesian if and only if a lift exists in every diagram of the form*

$$
\begin{array}{ccc}
\Delta^1 \times \{n\} \overset{f}{\longrightarrow} (\{1\} \times \Delta^n) \cup_{\{1\} \times \partial \Delta^n} (\Delta^1 \times \partial \Delta^n) \longrightarrow & M \\
\downarrow & & \downarrow p \\
\Delta^1 \times \Delta^n \longrightarrow & S
\end{array}
$$

for all $n \geq 1$.

Proof. The if part is just like the proof of the box version of Joyal lifting.   □

We reformulate this criterion. Consider the box power map

$$
q := p^{\Box(\{1\} \subset \Delta^1)}\colon \ \mathrm{Map}(\Delta^1, M) \to \mathrm{Map}(\Delta^1, S) \times_{\mathrm{Map}(\{1\}, S)} \mathrm{Map}(\{1\}, M).
$$

Then the above proposition says that f is p-cartesian iff a lift exists in every diagram

$$
\begin{array}{ccc}
\partial \Delta^n \overset{a}{\longrightarrow} & \mathrm{Map}(\Delta^1, M) \\
\downarrow & & \downarrow q \\
\Delta^n \overset{b}{\longrightarrow} & \mathrm{Map}(\Delta^1, S) \times_{\mathrm{Map}(\{1\}, S)} \mathrm{Map}(\{1\}, M)
\end{array}
$$

such that $n \geq 1$ and $a(n) = f \in \mathrm{Map}(\Delta^1, M)_0$.

51.5. Uniqueness of lifts to Cartesian edges.
Let $U \subseteq \mathrm{Map}(\Delta^1, M)$ be the full subsimplicial set spanned by the vertices which represent p-cartesian edges. Likewise, let $V \subseteq \mathrm{Map}(\Delta^1, S) \times_{\mathrm{Map}(\{1\}, S)} \mathrm{Map}(\{1\}, M)$ denote the essential image of U under q, i.e., the full subsimplicial set spanned by the vertices $q(U_0)$. Obviously, the map q restricts to a map $q'\colon U \to V$.

Note in particular that V_0 is the subset of $\{ (g, y) \in S_1 \times M_0 \mid g_1 = p(y) \}$ such that there *exists* a Cartesian edge $f \in M_1$ with $f_1 = y$ and $p(f) = g$, and the preimage of (g, y) under $q'\colon U \to V$ is the set of all choices of lifts. The following in particular asserts a kind of uniqueness for choices of lifts.

51.6. Proposition. *The map $q'\colon U \to V$ is a trivial fibration.*

Proof. Consider

$$
\begin{array}{ccccc}
\partial \Delta^n \overset{a}{\longrightarrow} & U \,\rightarrowtail & \mathrm{Map}(\Delta^1, M) \\
\downarrow & & \downarrow q|U & & \downarrow q \\
\Delta^n \longrightarrow & V \,\rightarrowtail & \mathrm{Map}(\Delta^1, S) \times_{\mathrm{Map}(\{1\}, S)} \mathrm{Map}(\{1\}, M)
\end{array}
$$

If $n \geq 1$, then a lift $s\colon \Delta^n \to \mathrm{Map}(\Delta^1, M)$ exists by the previous proposition, since $a(n) \in U_0 \subseteq \mathrm{Map}(\Delta^1, M)_0$ represents a p-cartesian edge. Because $(\partial \Delta^n)_0 = (\Delta^n)_0$ when $n \geq 1$, we see that s maps into the full subcomplex U.

If $n = 0$, this amounts to $U_0 \to V_0$ being surjective, which holds by definition.   □

51.7. Cartesian fibration.
A **cartesian fibration** is a map $p\colon M \to S$ which is an inner fibration, and is such that for all $(g, y) \in S_1 \times M_0$ with $g_1 = p(y)$, there exists a p-cartesian edge f with $p(f) = g$ and $f_1 = y$.

51.8. *Example.* Every left or right fibration is a cartesian fibration, since all edges are cartesian.

By the above, we see that an inner fibration $p\colon M \to S$ is a cartesian fibration if and only if $V = \mathrm{Map}(\Delta^1, S) \times_{\mathrm{Map}(\{1\}, S)} \mathrm{Map}(\{1\}, M)$.

51.9. Cartesian correspondences. Given a map $p\colon M \to S$, for any simplex $a \in S_k$ write

$$M_a := \mathrm{Map}_{/S}(\Delta^k, M) = \mathrm{Map}_{/S}(a, p).$$

Note that if $a = bf$ for some simplicial operator $f\colon [k] \to [l]$, we obtain an induced restriction map

$$f^*\colon M_b \to M_a.$$

Given a correspondence $p\colon M \to \Delta^1$, we obtain

$$C = M_{\langle 0 \rangle} \xleftarrow{\langle 0 \rangle^*} M_{\langle 01 \rangle} \xrightarrow{\langle 1 \rangle^*} M_{\langle 1 \rangle} = D.$$

Note that these are all quasicategories. The objects of $M_{\langle 01 \rangle}$ are precisely the edges in M lying over $\langle 01 \rangle$.

51.10. Proposition. *Let $p\colon M \to S$ be a cartesian fibration, and let $M_{\langle 01 \rangle}^{\mathrm{cart}} \subseteq M_{\langle 01 \rangle}$ be the full subcategory spanned by elements corresponding to cartesian edges. Then $M_{\langle 01 \rangle}^{\mathrm{cart}} \to M_{\langle 1 \rangle}$ is a trivial fibration.*

Proof. Every square in

is a pullback. The result follows because $qi = q'$ is a trivial fibration. $\square$

More generally, given an inner fibration $p\colon M \to S$ and a simplex $a \in S_k$, the objects of the quasicategory M_a correspond to k-simplices $b \in M_k$ such that $p(b) = a$. Let $M_a^{\mathrm{cart}} \subseteq M_a$ denote the full subcategory spanned by objects corresponding to $b \in M_k$ such that all edges of b are p-cartesian.

51.11. Proposition. *Let $p\colon M \to S$ be an inner fibration, and $f \in M_1$ an edge. Consider*

where $\Lambda_j^k \subset \Delta^k$ is a right horn inclusion, and f represents a p-cartesian edge. Then a lift exists whenever $n, k \geq 1$, and also when $k \geq 2$, $n = 0$.

Proof. This should also be like the box version of Joyal lifting. Note that if $k = 0$, we recover the definition of p-cartesian edge. $\square$

This admits a reformulation: if f is p-cartesian, then if $0 < j \leq k$ and $a(n) = f$ is p-cartesian there is a lift in

$$
\begin{array}{ccc}
\partial\Delta^n & \xrightarrow{\ a\ } & \mathrm{Map}(\Delta^k, M) \\
\downarrow & \nearrow & \downarrow q \\
\Delta^n & \xrightarrow{\ b\ } & \mathrm{Map}(\Delta^k, S) \times_{\mathrm{Map}(\Lambda_j^k, S)} \mathrm{Map}(\Lambda_j^k, M)
\end{array}
$$

when $k \geq 1$ and $n \geq 1$, or for all $n \geq 0$ if $k \geq 2$.

51.12. Cartesian fibrations and right fibrations.

51.13. Proposition. [Lur09, 2.4.2.4] *A map $p\colon M \to S$ is a right fibration iff it is a cartesian fibration whose fibers are Kan complexes.*

Proof. We have already seen that a right fibration is a cartesian fibration, and has Kan complexes as fibers.

Now suppose p is cartesian fibration with Kan complex fibers. Let $f\colon x \to y$ be an edge in M. Since p is cartesian, there exists a p-cartesian edgee $f'\colon x' \to y$ over $p(f)$. Since p is cartesian fibration and f' a cartesian edge, there exists $a \in M_2$ with $a_{02} = f$ and $a_{12} = f'$ and $p(a) = (p(f))_{001}$. Thus $g := a_{01}$ is an edge in the fiber over $(p(f))_0$, so is an isomorphism in that fiber. $\square$

51.14. Mapping space criterion for cartesian edges.

51.15. Proposition. [Lur09, 2.4.4.3] *Let $p\colon C \to D$ be an inner fibration between quasicategories, and $f\colon x \to y$ a morphism in C. The following are equivalent.*

  (1) *f is p-cartesian.*
  (2) *For every $c \in C_0$, the diagram*

$$
\begin{array}{ccc}
\mathrm{map}_C(c, x) & \xrightarrow{\ f_*\ } & \mathrm{map}_C(c, y) \\
\downarrow & & \downarrow \\
\mathrm{map}_D(p(c), p(x)) & \xrightarrow[p(f)_*]{} & \mathrm{map}_D(p(c), p(y))
\end{array}
$$

  is a homotopy pullback.

52. Limits and colimits as functors

Suppose J and C are categories. We say that C has all J-colimits if every functor $F\colon J \to C$ has a colimit in J. It is a standard observation that if F is such a functor, then we can assemble a functor

$$\mathrm{colim}_J\colon \mathrm{Fun}(J, C) \to C.$$

In fact, we can regard this functor as a composite of functors

$$\mathrm{Fun}(J, C) \xrightarrow{s} \mathrm{Fun}(J^\triangleright, C) \xrightarrow{\text{eval. at } v} C,$$

where s is some section of the restriction functor $\mathrm{Fun}(J^\triangleright, C) \to \mathrm{Fun}(J, C)$ which takes values in colimit cones.

Even when C does not have all J-colimits, we can assert the following. Consider the diagram

$$
\begin{array}{ccc}
\mathrm{Fun}^{\text{colim cone}}(J^\triangleright, C) & \rightarrowtail & \mathrm{Fun}(J^\triangleright, C) \\
p \downarrow & & \downarrow \\
\mathrm{Fun}^{\exists\, \text{colim}}(J, C) & \rightarrowtail & \mathrm{Fun}(J, C)
\end{array}
$$

in which the objects on the left are the evident full subcategories of the corresponding objects on the right, i.e., the ones consisting of colimit cones, and of functors which admit colimits. Then p is an equivalence of categories, and in fact is a trivial fibration. Therefore there is a contractible groupoid of sections of p, and any section s gives rise to a colimit functor

$$\mathrm{Fun}^{\exists\, \mathrm{colim}}(J, C) \xrightarrow{s} \mathrm{Fun}^{\mathrm{colim\ cone}}(J^{\triangleright}, C) \xrightarrow{\mathrm{eval.\ at}\ v} C.$$

We want to prove the analogous statement for quasicategories. Thus, given a quasicategory C and a simplicial set S, let $\mathrm{Fun}^{\mathrm{colim\ cone}}(S^{\triangleright}, C) \subseteq \mathrm{Fun}(S^{\triangleright}, C)$ denote the full subquasicategory spanned by $S^{\triangleright} \to C$ which are colimit cones, and let $\mathrm{Fun}^{\exists\, \mathrm{colim}}(S, C) \subseteq \mathrm{Fun}(S, C)$ denote the full subquasicategory spanned by $S \to C$ for which a colimit exists.

52.1. Proposition. *The induced projection* $q\colon \mathrm{Fun}^{colim\ cone}(S^{\triangleright}, C) \to \mathrm{Fun}^{\exists\, colim}(S, C)$ *is a trivial fibration.*

We refer to this as the *functoriality of colimits*. We will prove it below.

The strategy is to show (1) that q is an isofibration, and (2) that q is fully faithful and essentially surjective. Then (39.1) applies to show that q is a categorical equivalence, and so a trivial fibration by (40.1).

Parts of this are already clear. For instance, q is certainly an inner fibration, since $p\colon \mathrm{Fun}(S^{\triangleright}, C) \to \mathrm{Fun}(S, C)$ is one, and q is the restriction of p to full subquasicategories. Likewise, q is manifestly essentially surjective.

52.2. Conical maps. In what follows, C will be a quasicategory and S a simplicial set, and we write

$$V = V(S) := \mathrm{Fun}(S^{\triangleright}, C), \qquad U = U(S) := \mathrm{Fun}(S, C).$$

Let $p\colon V \to U$ be the evident restriction map.

Let's say that a morphism $\widehat{\alpha}\colon \widehat{f} \to \widehat{g}$ in V is **conical** if its evalutation $\widehat{\alpha}(v)\colon \widehat{f}(v) \to \widehat{g}(v)$ at the cone point of $S^{\triangleright}$ is an isomorphism in C.

What follows are two propositions involving conical maps. We will prove them soon. The first says that any morphism in U can be lifted to a *conical* morphism in V with prescribed target.

52.3. Proposition. *Fix a quasicategory C and a simplicial set S. Suppose given*
- *a functor $\widehat{g}\colon S^{\triangleright} \to C$, and*
- *a natural transformation $\alpha\colon f \Rightarrow g$ of functors $S \to C$ such that $g = \widehat{g}|S$.*

Then there exists a conical morphism $\widehat{\alpha}\colon \widehat{f} \to \widehat{g}$ in V such that $\widehat{\alpha}|S = \alpha$.

$$
\begin{array}{ccc}
\{1\} & \xrightarrow{\ \widehat{g}\ } & \mathrm{Fun}(S^{\triangleright}, C) = V \\
\downarrow & {\widehat{\alpha}}\ \nearrow & \downarrow \\
& \nearrow\ \mathrm{conical} & \\
\Delta^1 & \xrightarrow[\ \alpha\ ]{} & \mathrm{Fun}(S, C) = U
\end{array}
$$

The second says that morphisms in V can be "transported" along conical maps.

52.4. Proposition. *Fix a quasicategory C, simplicial set S, and a map $\widehat{\alpha}\colon \widehat{f} \to \widehat{g}$ in V, and let $\alpha\colon f \to g$ denote $\widehat{\alpha}|S$. For any object $\widehat{h}$ of V, consider the square*

$$
\begin{array}{ccc}
\mathrm{map}_V(\widehat{h}, \widehat{f}) & \xrightarrow{\ \widehat{\alpha}\circ\ } & \mathrm{map}_V(\widehat{h}, \widehat{g}) \\
\downarrow & & \downarrow \\
\mathrm{map}_U(h, f) & \xrightarrow[\ \alpha\circ\ ]{} & \mathrm{map}_U(h, g)
\end{array}
$$

where $h = \widehat{h}|S$, and the horizontal maps are induced by postcomposition with $\widehat{\alpha}$ and α respectively. If $\widehat{\alpha}$ is conical, then the above square is a homotopy pullback square.

We will explain and prove these two propositions soon. For the time being, you should convince yourself that if C is the nerve of an ordinary category, then both propositions are entirely straightforward to prove.

52.5. **Proof of functoriality of colimits, using properties of conical maps.** Recall that $\widehat{f}\colon S^{\triangleright} \to C$ extending $f\colon S \to C$ is a colimit cone if and only if it corresponds to an initial object of $C_{f/}$. Using the categorical equivalences

$$F(f) \to C^{f/} \leftarrow C_{f/}$$

where $F(f) \subseteq V$ is the fiber of $p\colon V \to U$ over f, we see that it is equivalent to say that $\widehat{f}$ is initial in $F(f)$.

The following gives a criterion for being a colimit cone in terms of the whole functor category $V = \mathrm{Fun}(S^{\triangleright}, C)$, rather than just in terms of the fiber over some f.

52.6. **Proposition.** *A functor $\widehat{f}\colon S^{\triangleright} \to C$ is a colimit cone if and only if*

$$p'\colon \mathrm{map}_V(\widehat{f}, \widehat{g}) \to \mathrm{map}_U(f, g)$$

is a weak equivalence for every $\widehat{g}\colon S^{\triangleright} \to C$, $g = \widehat{g}|S = p(\widehat{g})$.

Proof. Since $p\colon V \to U$ is a categorical fibration, the induced maps p' on mapping spaces are Kan fibrations. Thus, p' is a weak equivalence if and only if its fibers are contractible.

($\Longleftarrow$) Suppose every p' is a weak equivalence. Then in particular p' is a weak equivalence for any $\widehat{g}\colon S^{\triangleright} \to C$ such that $\widehat{g}|S = f$. In this case, the fiber of p' over $1_f \in \mathrm{map}_U(f, f)$ is precisely the mapping space $\mathrm{map}_{F(f)}(\widehat{f}, \widehat{g})$ in the fiber quasicategory $F(f) \subseteq \mathrm{Fun}(S^{\triangleright}, C)$, and this fiber is contractible. Therefore, $\widehat{f}$ is an initial object of $F(f)$, and therefore $\widehat{f}$ is initial in $C_{f/}$ by the above discussion. We have shown that $\widehat{f}$ is a colimit cone.

($\Longrightarrow$) Suppose $\widehat{f}$ is a colimit cone. Therefore for $\widehat{f'}$ such that $\widehat{f'}|S = f$ the fiber of $\mathrm{map}_V(\widehat{f}, \widehat{f'}) \to \mathrm{map}_U(f, f)$ over 1_f is contractible. We need to show that the fiber of $p'\colon \mathrm{map}_V(\widehat{f}, \widehat{g}) \to \mathrm{map}_U(f, g)$ over a general $\alpha \in \mathrm{map}_U(f, g)$ is contractible.

Given such an α, choose a conical map $\widehat{\alpha}\colon \widehat{f'} \to \widehat{g}$ with $\widehat{\alpha}|S = \alpha$ (52.3), and consider the resulting square

$$
\begin{array}{ccc}
\mathrm{map}_V(\widehat{f}, \widehat{f'}) & \xrightarrow{\widehat{\alpha}\circ} & \mathrm{map}_V(\widehat{f}, \widehat{g}) \\
{\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p''} \\
\mathrm{map}_U(f, f) & \xrightarrow[\alpha\circ]{} & \mathrm{map}_U(f, g)
\end{array}
$$

$$1_f \longmapsto \alpha$$

Since $\widehat{\alpha}$ is conical, the square is a homotopy pullback square (52.4). Therefore, the fiber of p'' over α is weakly equivalent to the fiber of p' over 1_f, which is contractible since $\widehat{f}$ is a colimit cone. $\quad\square$

Proof of (52.1). First we show that $q\colon \mathrm{Fun}^{\mathrm{colim\,cone}}(S^{\triangleright}, C) \to \mathrm{Fun}^{\exists\,\mathrm{colim}}(S, C)$ is an isofibration; we have already observed that it is an inner fibration. Given an isomorphism $\alpha\colon f \to g$ between objects in $\mathrm{Fun}^{\exists\,\mathrm{colim}}(S, C) \subseteq U$ and a choice of colimit cone $\widehat{g}$ over g, chose a conical lift $\widehat{\alpha}\colon \widehat{f} \to \widehat{g}$. The arrow $\widehat{\alpha}\colon S^{\triangleright} \times \Delta^1 \to C$ restricts to an isomorphism at each vertex of $S^{\triangleright}$, and so is a natural isomorphism by the objectwise criterion for natural isomorphisms. Thus $\widehat{f}$ is also a colimit cone by (52.6), so $\widehat{\alpha}$ is an isomorphism in $\mathrm{Fun}^{\mathrm{colim\,cone}}(S^{\triangleright}, C)$.

We have already observed that q is essentially surjective (in fact, it is surjective on 0-simplices). That q is fully faithful is immediate from (52.6). □

52.7. Proof of properties of conical maps.

Proof of (52.3). Recall the situation: we are given a natural transformation $\alpha \colon f \Rightarrow g$ of functors $S \to C$, and a lift $\widehat{g} \colon S^{\triangleright} \to C$ of the target to the cone, and we want to find a *conical* lift of α:

$$
\begin{array}{ccc}
\{1\} & \xrightarrow{\ \widehat{g}\ } & \mathrm{Fun}(S^{\triangleright}, C) \\
\Big\downarrow & \nearrow^{\widehat{\alpha}} & \Big\downarrow \\
& \text{conical} & \\
\Delta^1 & \xrightarrow[\ \alpha\ ]{} & \mathrm{Fun}(S, C)
\end{array}
$$

We make use of a natural map

$$\kappa \colon S^{\triangleright} \times K \to (S \times K)^{\triangleright}.$$

Note that this map sends $\{v\} \times K$ to the cone point $\{v\}$. Consider the composite

$$\lambda \colon (S \times \Delta^1) \cup_{S \times \{1\}} (S^{\triangleright} \times \{1\}) \to S^{\triangleright} \times \Delta^1 \xrightarrow{\kappa} (S \times \Delta^1)^{\triangleright}$$

where the first map is the box-product $(S \subset S^{\triangleright}) \square (\{1\} \subset \Delta^1)$. By inspection, we see that the composite map can be identified with the box-join

$$(S \times \{1\} \subseteq S \times \Delta^1) \boxast (\varnothing \subseteq \Delta^0).$$

Since $\mathrm{RHorn} \square \mathrm{Cell} \subseteq \overline{\mathrm{RHorn}}$ (45.4) we have that $(S \times \{1\} \subseteq S \times \Delta^1)$ is right anodyne. Likewise, since $\mathrm{RHorn} \boxast \mathrm{Cell} \subseteq \overline{\mathrm{InnHorn}}$ (26.11), we conclude that λ is inner anodyne. Therefore, an extension $\overline{\alpha}$ exists in

$$
\begin{array}{ccc}
(S \times \Delta^1) \cup_{S \times \{1\}} (S^{\triangleright} \times \{1\}) & \xrightarrow{(\alpha, \widehat{g})} & C \\
\Big\downarrow & & \\
S^{\triangleright} \times \Delta^1 & & \\
\kappa \Big\downarrow & \nearrow_{\overline{\alpha}} & \\
(S \times \Delta^1)^{\triangleright} & &
\end{array}
$$

We set $\widetilde{\alpha} := \overline{\alpha} \circ \kappa$. It is clear that $\widetilde{\alpha}$ is conical: $\widehat{\alpha}(v)$ is the identity map of $\overline{\alpha}(v)$. □

For the proof of (52.4), let's first note that, as stated, it actually doesn't make sense! This proposition asserts that for conical $\widehat{\alpha}$, the diagram

$$
\begin{array}{ccc}
\mathrm{map}_V(\widehat{h}, \widehat{f}) & \xrightarrow{\ \widehat{\alpha}\circ\ } & \mathrm{map}_V(\widehat{h}, \widehat{g}) \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{map}_U(h, f) & \xrightarrow[\ \alpha\circ\ ]{} & \mathrm{map}_U(h, g)
\end{array}
$$

is a homotopy pullback. However, the horizontal maps ("postcomposition" with α and $\widehat{\alpha}$) are only defined as a homotopy class of maps in $h\mathrm{Kan}$. For instance, "$\alpha\circ$" is the homotopy class defined by the zig-zag around the left and top of the diagram

$$
\begin{array}{ccc}
\mathrm{map}_U(h, f, g)_\alpha & \longrightarrow \mathrm{map}_U(h, f, g) \xrightarrow{\ \mathrm{comp}\ } & \mathrm{map}_U(h, g) \\
\sim \Big\downarrow & \Big\downarrow \sim & \\
\mathrm{map}_U(h, f) \times \{\alpha\} & \longrightarrow \mathrm{map}_U(h, f) \times \mathrm{map}(f, g) &
\end{array}
$$

where the left-hand square is a pullback. The correct statement of (52.4) is that in

$$\mathrm{map}_V(\widehat{h},\widehat{f}) \xleftarrow{\ \sim\ } \mathrm{map}_V(\widehat{h},\widehat{f},\widehat{g})_{\widehat{\alpha}} \longrightarrow \mathrm{map}_V(\widehat{h},\widehat{g})$$
$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$
$$\mathrm{map}_U(h,f) \xleftarrow{\ \sim\ } \mathrm{map}_U(h,f,g)_{\alpha} \longrightarrow \mathrm{map}_U(h,g)$$

the right-hand square is a homotopy pullback.

We can refine this a little further. Fix a map $e\colon \Delta^{\{1,2\}} \to C$. For a simplicial set S, let $K \subseteq S^{\triangleright} \times \Delta^2$ be a subcomplex containing the edge $\{v\} \times \Delta^{\{1,2\}}$, and define $\mathrm{Map}(K,C)_e$ by the pullback square

$$\mathrm{Map}(K,C)_e \rightarrowtail \mathrm{Map}(K,C)$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$\{e\} \rightarrowtail \mathrm{Map}(\{v\} \times \Delta^{\{1,2\}}, C)$$

To prove our proposition, it suffices to show that for every isomorphism e in C, the map

$$\mathrm{Map}(S^{\triangleright} \times \Delta^2, C)_e \to \mathrm{Map}((S^{\triangleright} \times \Lambda_2^2) \cup_{S \times \Lambda_2^2} (S \times \Delta^2), C)_e$$

is a trivial fibration. Equivalently, we must produce a lift in each diagram of the form

$$(S^{\triangleright} \times \partial\Delta^m) \cup_{S \times \partial\Delta^m} (S \times \Delta^m) \longrightarrow \mathrm{Map}(\Delta^2, C)$$

$$\{v\} \times \Delta^{\{1,2\}} \rightarrowtail S^{\triangleright} \times \Delta^m \longrightarrow \mathrm{Map}(\Lambda_2^2, C) \longrightarrow \mathrm{Map}(\Delta^{\{1,2\}}, C)$$

$$\{e\}$$

We reduce to producing a lift in

$$\{n\} \times \Delta^{\{1,2\}} \rightarrowtail (\Delta^n \times \Lambda_2^2) \cup_{\partial\Delta^n \times \Lambda_2^2} (\partial\Delta^n \times \Delta^2) \xrightarrow{\ e\ } C$$

$$\Delta^n \times \Delta^2$$

where e is an isomorphism in C. This is precisely the box-version of Joyal extension.

53. MORE STUFF

Recall that the join constructions $K \star -$ and $- \star K$ are colimit preserving functors $s\mathrm{Set} \to s\mathrm{Set}_{K/}$ to the category of simplicial sets *under* K. In particular, viewed as functors $s\mathrm{Set} \to s\mathrm{Set}$ to plain simplicial sets, they preserve pushouts, and transfinite compositions.

53.1. Proposition. *If $\mathcal{A}$ is a class of maps in sSet, then $K \star \overline{\mathcal{A}} \subseteq \overline{K \star \mathcal{A}}$ and $\overline{\mathcal{A}} \star K \subseteq \overline{\mathcal{A} \star K}$.*

Proof. Check that $K \star -\colon s\mathrm{Set} \to s\mathrm{Set}$ preserves isomorphisms, transfinite composition, pushouts, and retracts. $\qquad\square$

53.2. Remark. Given $f\colon X \to Y$ and K, we have a factorization of $K \star f$ as

$$K \star X \to (K \star X) \amalg_{\varnothing \star X} (\varnothing \star Y) \xrightarrow{(\varnothing \subseteq K) \boxtimes f} K \star Y.$$

53.3. Proposition. *We have* $\Delta^0 \star \overline{\text{Cell}} \subseteq \overline{\text{LHorn}}$ *and* $\overline{\text{Cell}} \star \Delta^0 \subseteq \overline{\text{RHorn}}$.

53.4. Proposition. *Let C be a quasicategory and x an object of C. Then x is an initial object iff $\{x\} \to C$ is left anodyne, and x is a terminal object iff $\{x\} \to C$ is right anodyne.*

Proof. ($\Longrightarrow$) Let x be terminal, and consider $j \colon \{x\} \to C$. Since $j^{\triangleright}$ is right anodyne, it suffices to show that j is a retract of $j^{\triangleright}$. To do this, we construct a map r fitting into

$$
\begin{array}{ccccc}
 & \text{id} & & & \\
\{x\} & \rightarrowtail \{x\}^{\triangleright} & \longrightarrow & \{x\} \\
\downarrow{\scriptstyle j} & \downarrow{\scriptstyle j^{\triangleright}} & & \downarrow{\scriptstyle j} \\
C & \rightarrowtail C^{\triangleright} & \dashrightarrow_{r} & C \\
 & \text{id} & & &
\end{array}
$$

This amounts to solving the lifting problem

$$
\begin{array}{ccc}
C \cup \{x\}^{\triangleright} & \xrightarrow{(\text{id}, 1_x)} & C \\
\downarrow & \nearrow & \\
C^{\triangleright} & &
\end{array}
\qquad \Longleftrightarrow \qquad
\begin{array}{ccc}
\{x\} & \xrightarrow{1_x} & C_{x/} \\
\downarrow & \nearrow & \downarrow \\
C & =\!\!=\!\!= & C
\end{array}
$$

Since x is terminal, $C_{x/} \to C$ is a trivial fibration (**??**), so a lift exists.

($\Longleftarrow$) Suppose $j \colon \{x\} \to C$ is right anodyne. Since $C_{/x} \to C$ is a right fibration, a lift exists in

$$
\begin{array}{ccc}
\{x\} & \xrightarrow{1_x} & C_{x/} \\
\downarrow & \nearrow & \downarrow \\
C & =\!\!=\!\!= & C
\end{array}
$$

which is equivalent to x being terminal. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

53.5. Corollary. *Let $p \colon D \to C$ be a right fibration between quasicategories, and let x be an object of C. Then the induced map*

$$\text{Map}(C_{/x}, D) \to \text{Map}(\{1_x\}, D) \times_{\text{Map}(\{1_x\}, C)} \text{Map}(C_{/x}, C)$$

is a trivial fibration. In particular, the map

$$\text{Map}_C(C_{/x}, D) \to \text{Map}_C(\{1_x\}, D)$$

induced by restriction over the projection map $(C_{/x} \to C) \in \text{Map}(C_{/x}, C)$ is a trivial fibration between Kan complexes.

54. Introduction: the cover/functor correspondence

Consider the following classes of maps between simplicial sets, called respectively **covers**, **left covers** and **right covers**.

$$\text{Cover} := (\text{Horn II Horn}^{\vee})^{\boxtimes},$$

$$\text{LCover} := (\text{LHorn II LHorn}^{\vee})^{\boxtimes}, \qquad \text{RCover} := (\text{RHorn II RHorn}^{\vee})^{\boxtimes}.$$

Thus, Cover is the subclass of Kan fibrations, which admit *unique* lifting for every lifting problem with respect to horns, and therefore with respect to all anodyne maps. Similar statements hold for LCover and RCover.

We obtain for each simplicial set X full subcategories $s\text{Set}_{/X}^{\text{Cover}}$, $s\text{Set}_{/X}^{\text{LCover}}$, $s\text{Set}_{/X}^{\text{RCover}}$, of the slice category $s\text{Set}_{/X}$. We will see that each of these categories is equivalent to a certain category of functors from hX to sets.

54.1. **Simplicial covering maps.** Covers $p\colon E \to X$ are precisely the simplicial analogues of covering maps of spaces. To see this, will first note that covers are "locally trivial".

54.2. **Lemma.** *Cover* $= (S \amalg S^\vee)^\boxtimes$, *where S is the set of all maps $\Delta^m \to \Delta^n$ between standard simplices. Thus, $p\colon E \to X$ is a cover if and only if for each simplex $a \in X_n$ and each simplicial operator $\delta\colon [m] \to [n]$, the function $x \mapsto x\delta\colon p^{-1}(a) \to p^{-1}(a\delta)$ is a bijection of sets.*

Proof. Given a class $\mathcal{B}$ of maps, let $^{\boxtimes!}\mathcal{B}$ denote the class of maps f which have *unique* lifting with respect to maps in $\mathcal{B}$; i.e., such that $\{f, f^\vee\} \boxtimes \mathcal{B}$. An easy argument shows that $^{\boxtimes!}\mathcal{B}$ is saturated, and also has the property that if $f, g \circ f \in {}^{\boxtimes!}\mathcal{B}$, then $g \in {}^{\boxtimes!}\mathcal{B}$.

Because any inclusion $\Delta^0 \to \Delta^n$ of a vertex is anodyne and thus in $^{\boxtimes!}$Cover, applying the observation of the previous paragraph to a composite $\Delta^0 \to \Delta^m \xrightarrow{\delta} \Delta^n$ shows that every such $\delta \in {}^{\boxtimes!}$Cover. Hence Cover $\subseteq (S \amalg S^\vee)^\boxtimes$.

Conversely, let $\mathcal{B} := (S \amalg S^\vee)^\boxtimes$, so that $S \subseteq {}^{\boxtimes!}\mathcal{B}$. Since $^{\boxtimes!}\mathcal{B}$ is saturated, we can show that Horn $\subseteq {}^{\boxtimes!}\mathcal{B}$, by building horn inclusions out of (injective) maps between simplices. Thus $(S \amalg S^\vee)^\boxtimes \subseteq$ Cover. $\qquad\square$

54.3. **Corollary.** *A map $p\colon E \to X$ is a cover if and only if for each map $f\colon \Delta^n \to X$ there is a pullback square of the form*

$$
\begin{array}{ccc}
\coprod \Delta^n & \longrightarrow & E \\
{\scriptstyle proj}\downarrow & & \downarrow{\scriptstyle p} \\
\Delta^n & \xrightarrow{\ f\ } & X
\end{array}
$$

Proof. Left as an exercise, using the lemma. $\qquad\square$

Given a cover $p\colon E \to X$, we define a functor $F\colon hX \to \mathrm{Set}^{\mathrm{core}}$ as follows. For each object $x \in X_0$, we let $F(x) := p^{-1}(x)$. For each edge $f \in X_1$, we define a bijection $F(f)\colon F(f_0) \to F(f_1)$ as the composite

$$
p^{-1}(f_0) \xleftarrow{\langle 0 \rangle} p^{-1}(f) \xrightarrow{\langle 1 \rangle} p^{-1}(f_1),
$$

where both maps are bijections by the lemma. For each 2-simplex $a \in X_2$, we have a commutative diagram of bijections which shows that $F(a_{12})F(a_{01}) = F(a_{02})$. By the universal property of fundamental category, this extends uniquely to a functor $F\colon hX \to \mathrm{Set}^{\mathrm{core}}$.

Conversely, given a functor $F\colon hX \to \mathrm{Set}^{\mathrm{core}}$, we define a cover $p\colon E \to X$ as follows. Set

$$
E_n := \coprod_{x \in X_n} F(x_0),
$$

and for each simplicial operator $\delta\colon [m] \to [n]$, define $\delta^*\colon E_n \to E_m$ by

$$
\delta^*(x, t) := \big(x\delta,\, F(x_{0,\delta(0)})(t)\big).
$$

54.4. **Proposition.** *The above constructions give an adjoint equivalence of categories*

$$
s\mathrm{Set}_{/X}^{\mathrm{Cover}} \approx \mathrm{Fun}(hX, \mathrm{Set}^{\mathrm{core}}).
$$

Proof. Left as an exercise, including showing that $\mathrm{Un}(F)$ is in fact a cover. $\qquad\square$

Let $\Pi_1 X := (hX)_{\mathrm{Gpd}}$, the groupoid obtained by formally inverting all morphisms in hX. The above passes to an equivalence of categories $s\mathrm{Set}_{/X}^{\mathrm{Cover}} \approx \mathrm{Fun}(\Pi_1 X, \mathrm{Set})$. This is the simplicial set analogue of the classification of covering spaces.

54.5. Left and right covers correspond to functors to sets. I'll state the analogues of the above results for left and right covers. Proofs are left for the reader.

54.6. Lemma. $\mathrm{LCover} = (S\amalg S^\vee)^\boxtimes$ and $\mathrm{RCover} = (T\amalg T^\vee)^\boxtimes$, where $S = \{\, \delta\colon \Delta^m \to \Delta^n \mid \delta(0) = 0 \,\}$ and $T = \{\, \delta\colon \Delta^m \to \Delta^n \mid \delta(m) = n \,\}$.

Given a left cover $p\colon E \to X$, we define a functor $F\colon hX \to \mathrm{Set}$ much as we did above, so that $F(x) := p^{-1}(x)$, while $F(f)$ is the map obtained as the composite

$$p^{-1}(f_0) \xleftarrow[\sim]{\langle 0 \rangle} p^{-1}(f) \xrightarrow{\langle 1 \rangle} p^{-1}(f_1),$$

where the first map is a bijection by the lemma. We write $\mathrm{St}(p) := F$ and call it the **straightening** of p.

Conversely, given a functor $F\colon hX \to \mathrm{Set}$, we define a left cover $p\colon E \to X$ by

$$E_n := \coprod_{x \in X_n} F(x_0), \qquad \delta^*(x, t) := \big(x\delta, F(x_{0,\delta(0)})(t)\big).$$

The constructions for right covers are similar, except that the corresponding functors are contravariant. Thus, given a right cover $p\colon E \to X$ we obtain a straightening $\mathrm{St}(p) = F\colon hX^{\mathrm{op}} \to \mathrm{Set}$ given by $F(x) = p^{-1}(x)$ and $F(f)$ the composite

$$p^{-1}(f_0) \xleftarrow{\langle 0 \rangle} p^{-1}(f) \xrightarrow[\sim]{\langle 1 \rangle} p^{-1}(f_1),$$

while for a functor $F\colon hX^{\mathrm{op}} \to \mathrm{Set}$, we obtain a right cover $p\colon E \to X$ by

$$E_n := \coprod_{x \in X_n} F(x_n), \qquad \delta^*(x, t) := \big(x\delta, F(x_{\delta(m),m})(t)\big).$$

54.7. Proposition. *The above constructions give adjoint equivalences of categories*

$$\mathrm{St}\colon s\mathrm{Set}_{/X}^{\mathrm{LCover}} \rightleftarrows \mathrm{Fun}(hX, \mathrm{Set}) : \mathrm{Un}, \qquad \mathrm{St}\colon s\mathrm{Set}_{/X}^{\mathrm{RCover}} \rightleftarrows \mathrm{Fun}(hX^{\mathrm{op}}, \mathrm{Set}) : \mathrm{Un}\,.$$

Note also that left covers of X correspond to right covers of X^{op}.

54.8. Straightening and unstraightening over a category. Suppose $X = hX = C$ is itself a category and $F\colon C \to \mathrm{Set}$ a functor, and consider its unstraightening $p\colon E \to C$. It is straightforward to show that E is the nerve of a category, namely the **category of elements** of F, which has

- **objects:** pairs (c, x), $c \in C_0$ and $x \in F(x)$, and
- **morphisms** $(c, x) \to (c', x')$**:** morphisms $\alpha\colon c \to c'$ in C such that $F(\alpha)(x) = x'$.

The left cover map p is then just the evident forgetful functor. An analogous statement holds for contravariant functors.

Given any object x in C, we have the slice projections $C_{x/} \to C$ and $C_{/x} \to C$.

54.9. Proposition. *The projections $C_{x/} \to C$ and $C_{/x} \to C$ are left and right covers respectively. Their straightenings are representable functors:* $\mathrm{St}(C_{x/} \to C) = \mathrm{Hom}_C(x, -)$ *and* $\mathrm{St}(C_{/x} \to C) = \mathrm{Hom}_C(-, x)$.

Proof. Exercise.   □

54.10. Fiber product corresponds to tensor product. Let C be a category. Given functors $F\colon C^{\mathrm{op}} \to \mathrm{Set}$ and $G\colon C \to \mathrm{Set}$, the **tensor product** is the set defined by

$$F \otimes_C G := \mathrm{colim}\left[\coprod_{c_0 \to c_1 \in C_1} F(c_1) \times G(c_0) \rightrightarrows \coprod_{c \in C} F(c) \times G(c) \right].$$

54.11. *Exercise.* Given $x \in C_0$, let $\rho_x := \mathrm{Hom}_C(-, x)\colon C^{\mathrm{op}} \to \mathrm{Set}$ and $\rho^x := \mathrm{Hom}_C(x, -)\colon C \to \mathrm{Set}$ denote the representable functors. Show that $\rho_x \otimes_C G \approx G(x)$ and $F \otimes_C \rho^x \approx F(x)$.

If $A \to C$ and $B \to C$ are the unstraightenings of F and G respectively, then

$$F \otimes_C G \approx \pi_0(A \times_C B).$$

54.12. The universal left cover. What happens if we unstraighten the identity functor of Set? We get the category of elements of Set, which is precisely $\mathrm{Set}_* := \mathrm{Set}_{*/}$, the category of based sets.

55. STRAIGHTENING AND UNSTRAIGHTENING

Let $\mathfrak{D}_{\Delta^n} \colon \mathfrak{C}(\Delta^n)^{\mathrm{op}} \to s\mathrm{Set}$ be the simplicialy enriched functor defined as follows.
- For each object $x \in \{0, \ldots, n\}$, set

$$\mathfrak{D}_{\Delta^n}(x) := N\mathcal{P}_\ell(x),$$

  the nerve of the poset

$$\mathcal{P}_\ell(x) := \bigl\{ S \mid \{x\} \subseteq S \subseteq [x, n] \bigr\}$$

  of subsets of the interval $[x, n] = \{x, \ldots, n\}$ which contain the left endpoint.
- The structure of enriched functor is induced by the union operation on subsets:

$$(T, S) \mapsto T \cup S \colon \mathcal{P}(x, y) \times \mathcal{P}_\ell(y) \to \mathcal{P}_\ell(x).$$

- For each map $\delta \colon \Delta^m \to \Delta^n$, we define a natural transformation

$$\mathfrak{D}_\delta \colon \mathfrak{D}_{\Delta^m} \to \mathfrak{D}_{\Delta^n} \circ \mathfrak{C}(\delta)^{\mathrm{op}}$$

  of simplicially enriched functors $\mathfrak{C}(\Delta^m)^{\mathrm{op}} \to s\mathrm{Set}$, which at each object x of $\mathfrak{C}(\Delta^m)^{\mathrm{op}}$ is a map $\mathfrak{D}_{\Delta^m}(x) \to \mathfrak{D}_{\Delta^n}(\delta x)$ induced by the map of posets

$$S \mapsto \delta(S) \colon \mathcal{P}_\ell(x) \to \mathcal{P}_\ell(\delta x).$$

55.1. Remark. The functor $\mathcal{D}_{\Delta^n} \colon \mathfrak{C}(\Delta^n)^{\mathrm{op}} \to s\mathrm{Set}$ is isomorphic to the representable functor $\mathrm{Map}_{\mathfrak{C}((\Delta^n)^{\triangleright})}(-, v)$, where v represents the cone point of $(\Delta^n)^{\triangleright}$. Likewise, the natural transformation $\mathfrak{D}_\delta \colon \mathfrak{D}_{\Delta^m} \to \mathfrak{D}_{\Delta^n} \circ \mathfrak{C}(\delta)^{\mathrm{op}}$ coincides with the transformation

$$\mathrm{Map}_{\mathfrak{C}((\Delta^m)^{\triangleright})}(-, v) \xrightarrow{\mathfrak{C}(\delta^{\triangleright})} \mathrm{Map}_{\mathfrak{C}((\Delta^n)^{\triangleright})}(\delta(-), v)$$

induced by $\delta^{\triangleright} \colon (\Delta^m)^{\triangleright} \to (\Delta^n)^{\triangleright}$.

Fix a simplical set S, and consider a simplicially enriched functor $F \colon \mathfrak{C}(S)^{\mathrm{op}} \to s\mathrm{Set}$. We define a morphism

$$\mathrm{Un}_S(F) \colon X \to S$$

of simplicial sets, called the **unstraightening** of F over S, as follows.
- An n-simplex of $\mathrm{Un}_S F$ is a pair

$$f \colon \Delta^n \to S, \qquad t \colon \mathfrak{D}_{\Delta^n} \to F \circ \mathfrak{C}(f),$$

  where f is a map of simplicial sets, and t is a map of simplicially enriched functors $\mathfrak{C}(\Delta^n)^{\mathrm{op}} \to s\mathrm{Set}$.
- To a map $\delta \colon \Delta^m \to \Delta^n$ we have an induced map $(\mathrm{Un}_S F)_n \to (\mathrm{Un}_S F)_m$, which sends an n-simplex (f, t) to the pair

$$\Delta^m \xrightarrow{\delta} \Delta^n \xrightarrow{f} S, \qquad \mathfrak{D}_{\Delta^m} \xrightarrow{\mathfrak{D}_\delta} \mathfrak{D}_{\Delta^n} \circ \mathfrak{C}(\delta)^{\mathrm{op}} \xrightarrow{t \circ \mathfrak{C}(\delta)^{\mathrm{op}}} F \circ \mathfrak{C}(f) \circ \mathfrak{C}(\delta)^{\mathrm{op}}.$$

56. Pullback of right anodyne map along left fibration

The arguments here are based on the appendix to [Mos15].

56.1. Lemma. *Consider a pullback square*

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow{\scriptstyle p} \\
\Lambda^n_k & \longrightarrow & \Delta^n
\end{array}
$$

If p is a left fibration and $k \geq 1$, then $A \to B$ is right anodyne.

56.2. Contraction maps. Let Δ_R denote the category whose
- objects are totally ordered sets $[n]^R := [n] \sqcup \{R\} = \{0 < 1 < \cdots < n < R\}$ for $n \geq -1$, and
- morphisms are order preserving functions which take the "right basepoint" R to R.

There is an evident "inclusion" functor $\Delta \to \Delta_R$ sending $[n] \mapsto [n]_R$.

A **right contraction** of a simplicial set X is a choice of extension of $X \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ to a functor $X \colon (\Delta_R)^{\mathrm{op}} \to \mathrm{Set}$. There is an evident dually defined category Δ_L and corresponding notion of **left contraction**.

A right contraction of a simplicial set X is completely determined by the inclusion $X_{-1} \to X_0$ together with its **contraction operators**, which are the maps $X_{n-1} \to X_n$ for $n \geq 0$ induced by the surjective map $Q \colon [n]_R \to [n-1]_R$ which sends $n \mapsto R$.

Let $\Delta_R^{\mathrm{surj}} \subset \Delta_R$ denote the subcategory consisting of all objects, and all *surjective* maps. Given a right-contracted simplicial set $X \colon (\Delta^R)^{\mathrm{op}} \to \mathrm{Set}$, say that $a \in X_n$ is **contracted** if there exists a *non-identity* $\sigma \colon [n]_R \to [k]_R$ in Δ_R^{surj} and $b \in X_k$ such that $a = b\sigma$. That is, a contracted simplex is one which is *either* degenerate or in the image of the contraction operators Q. Say $a \in X_n$ is **non-contracted** if $a = b\sigma$ for $\sigma \in \Delta_R^{\mathrm{surj}}$ we have $\sigma = \mathrm{id}$.

We have an anologue of the Eilenberg-Zilber lemma (15.9) for right contracted simplicial sets, which is proved in the same way.

56.3. Proposition. *For a in X, there exists a* unique *pair (b, σ) consisting of a map $\sigma \in \Delta_R^{\mathrm{surj}}$ and a non-contracted simplex $b \in X$ such that $a = b\sigma$.*

56.4. Corollary. *For any right contracted simplicial set, the evident maps*

$$
\coprod_{k \geq -1} \coprod_{b \in X_k^{\mathrm{nc}}} \mathrm{Hom}_{\Delta_R^{\mathrm{surj}}}([n]_R, [k]_R) \to X([n]_R)
$$

defined by $(b, \sigma) \mapsto b\sigma$ are bijections. Furthermore, these bijections are natural with respect to surjective morphisms $[n]_R \to [n']_R$.

There is an analogue of the skeletal filtration: let $F_n X \subseteq X$ denote the smallest subobject containing all simplices of dimensions $\leq n$. Then the simplices of $F_n X$ consist of those which are degeracies or contractions of noncontracted simplices of dimension $\leq n$.

Comparing this with the skeletal filtration, we discover the following.

56.5. Lemma. *For all $n \geq 0$, the contraction operators $Q \colon X_{n-1} \to X_n$ restrict to bijections $X_{n-1}^{\mathrm{nc}} \to (X_n^{\mathrm{nd}} \smallsetminus X_n^{\mathrm{nc}})$.*

56.6. Proposition. *Let X be a simplicial set equipped with a right contraction. Consider the evident inclusion $f \colon S \to X$, where S is the discrete simpicial set with underlying set X_{-1}. Then f is right anodyne; in fact, $f \in \overline{\{\, \Lambda^n_n \subset \Delta^n \mid n \geq 1 \,\}}$.*

We note that for $x \in X_{n-1}$,

- $d_i Q(x) = Q d_i(x)$ if $i \in \{0, \ldots, n-1\}$, i.e., the contraction operators commute with most face maps;
- we have $d_n Q(x) = x$;
- $s_i Q(x) = Q s_i(x)$ if $i \in \{0, \ldots, n-1\}$, i.e., the contraction operators commute with most degeneracy maps;
- we have $s_n Q(x) = QQ(x)$.

In fact, a right contraction for a simplicial set X is equivalent to choosing data $(X_{-1}, d_0 \colon X_0 \to X_{-1}, \{Q \colon X_{n-1} \to X_n\})$, where the Q satisfy the above identities.

There is an evident complementary notion of **left contraction**.

If X admits a right contraction, then it is "right contractible" to a discrete simplicial set.

56.7. Proposition. *Let X be a simplicial set equipped with a right contraction. Consider the evident inclusion $f \colon S \to X$, where S is the discrete simpicial set with underlying set X_{-1}. Then f is right anodyne; in fact, $f \in \overline{\{\, \Lambda_n^n \subset \Delta^n \mid n \geq 1 \,\}}$.*

Proof. Consider the collection X^{nd} of non-degenerate simplices of X not contained in S. Partition X^{nd} into disjoint subsets $X^I \amalg X^{II}$, where $X^I = (X^{\mathrm{nd}} \smallsetminus S^{\mathrm{nd}}) \cap Q(X)$, the set of nondegenerate simplices which are in the image of the contraction operators.

The claim is that the contraction operators define bijections $\phi \colon X_{n-1}^{II} \to X_n^I$ for $n \geq 1$. To see this, we note the following.

- If $Qx = s_i y \in X_n$ with $i \in \{0, \ldots, n-2\}$, then $x = d_n Qx = d_n s_i y = s_i d_n y$. Thus, x is degenerate.
- If $Qx = s_{n-1} y \in X_n$, then $x = d_n Qx = d_n s_{n-1} y = y$ and $Q d_{n-1} x = d_{n-1} Qx = d_{n-1} s_{n-1} y = y$. Thus, x is in the image of Q.
- Taken together, the last two statements imply that Q applied to an element of X^{II} is non-degenerate. Thus, ϕ is a well-defined map.
- If $x = s_i y \in X_{n-1}$ with $i \in \{0, \ldots, n-2\}$, then $Q s_i y = s_i Q y$, i.e., Qx is degenerate whenever x is degenerate. Thus, ϕ is surjective.
- We have $d_n Qx = x$ for $x \in X_{n-1}$. Therefore, ϕ is injective.

Now we can filter X by subcomplexes E_n, where $E_{-1} = S$, while E_n is the smallest subcomplex containing $\mathrm{Sk}_{n-1} X$ and X_n^I. For each $Qx \in X_n^I$ we have $d_i Qx = Q d_i x \in E_{n-1}$ when $i \in \{0, \ldots, n-1\}$, while $d_n Qx = x \notin E_{n-1}$. Thus each inclusion $E_{n-1} \subseteq E_n$ is obtained by attaching the collection X_n^I of n-simplices along $\Lambda_n^n \subset \Delta^n$. $\square$

56.8. Proposition. *Let X be a simplicial set and $x \in X_0$. Then the inclusion $\{1_x\} \to X_{/x}$ is right anodyne, and the inclusion $\{1_x\} \to X_{x/}$ is left anodyne.*

Proof. The functor $\Delta \to s\mathrm{Set}_*$ to pointed simplicial sets defined by $[n] \mapsto (\Delta^n)^{\triangleright}$ manifestly extends to a functor $\Delta^R \to s\mathrm{Set}_*$. From this we obtain a canonical contraction on $X_{/x}$. $\square$

56.9. Proposition. *Suppose X is a contractible Kan complex. Then for any choice of vertex $* \in X_0$, the inclusion $\{*\} \to X$ admits a right contraction.*

Proof. We define a right contraction by inductively constructing contraction operators. Set $X_{-1} := \{*\}$, and let $Q \colon X_{-1} \to X_0$ denote the evident inclusion.

Suppose we have already defined contraction operators $Q \colon X_{k-1} \to X_k$ for $k < n$, which satisfy all the identities which make sense. That is, we have a presheaf of sets on the subcategory of Δ^R generated by Δ and by the contraction operators $[k]_R \to [k-1]_R$ for $k < n$. We will construct $Q \colon X_{n-1} \to X_n$ so that the following hold.

(1) If $x = s_i y$, then $Qx = s_i Q y$.
(2) If $x = Qy$, then $Qx = s_{n-1} Q y$.
(3) We have that $d_n Qx = x$ and $d_i Qx = Q d_i x$ for $i \in \{0, \ldots, n-1\}$.

Doing so precisely gives an extension of the given presheaf to the subcategory of Δ^R generated by Δ and contraction operators $[k]_R \to [k-1]_R$ for $k \leq n$.

The idea is to use the above as a prescription for $Q\colon X_{n-1} \to X_n$: that is,

(1) if $x = s_i y$, set $Qx := s_i Qy$;
(2) if $x = Qy$, set $Qx := s_{n-1} Qy$;
(3) otherwise, choose any Qx such that $d_n Qx = x$ and $d_i Qx = Qd_i x$ for $i \in \{0, \dots, n-1\}$.

We need to check that this makes sense: that cases (1) and (2) agree when they overlap, and that a choice in (3) is always possible. We consider the various cases, for $x \in X_{n-1}$.

- Suppose $x = s_i y = s_j z$ for some $i, j \in \{0, \dots, n-2\}$. Thus, both y and z are common degeneracies of some non-degenerate simplex, i.e., $y = \sigma u$ and $z = \tau u$ for some $u \in X_d$ with $d \leq n-3$, while $\sigma, \tau\colon [n-2] \to [d]$ are surjective operators with $\sigma s_i = \tau s_j$. Then $yQs_i = u\sigma Qs_i = uQ\sigma s_i = uQ\tau s_j = u\tau Qs_j = zQs_j$.
- Suppose $x = s_i y = Qz$ for some $i \in \{0, \dots, n-2\}$. Then there exists

That a choice in (3) exists follows from the fact that the collection of $Qd_0 x, \dots, Qd_{n-1} x, x$ fit together to give a map $\partial\Delta^n \to X$. $\qquad\square$

Fix a vertex k of Δ^n. We define the **excess** of a simplex $\delta\colon \Delta^m \to \Delta^n$ (relative to k) as follows.

- If some vertex of Δ^n other than k is *not* in the image of δ, the excess of δ is -1.
- Otherwise, the excess of δ is

$$|\{\, j \in (\Delta^m)_0 \mid \delta(j) \neq k \,\}| - n.$$

Note that the simplices of excess $= -1$ form the subcomplex Λ_k^n.

Given a map $p\colon B \to \Delta^n$, define the excess of a simplex $\Delta^m \to B$ (relative to k and p) to be the excess of $px\colon \Delta^m \to \Delta^n$. Observe that the collection of simplices B with excess $\leq d$ form a subcomplex $E_d \subseteq B$, and that E_{-1} is the pullback of B over Λ_k^n.

56.10. Lemma. *If $k \in \{1, \dots, n\}$ and p is a left fibration, then each map $E_{d-1} \to E_d$ is right anodyne.*

Here is a formalism for "simplicial operators"[31]. Given a finite totally ordered set I, we can define $\Delta^I := N(I)$. We have that Δ^I is isomorphic to a *unique* standard simplex Δ^n (or to the empty simplicial set if I is empty) by a *unique* isomorphism. The *I-simplices* X_I of a simplicial set X are the set of maps $\Delta^I \to X$; we have a canonical identification $X_I \approx X_n$ where $|I| = n+1$.

We define the following maps between totally ordered sets.

- $d^a\colon I \smallsetminus \{a\} \to I$ for $a \in I$ is the evident inclusion.
- $s_{b,c}^a\colon I \to I \cup_{\{b,c\}} \{a\}$ for consecutive elements $b < c$ in I is a projection map; the target is the totally ordered set obtained by identifying b and c and relabelling the new element as "a" (which is assumed not to be an element of $I \smallsetminus \{b,c\}$, though we are allowed to consider $s_{a,b}^a$ or $s_{a,b}^b$).
- $s_b^a\colon I \to I \cup_{\{b\}} \{a\}$ is an isomorphism, where the target is obtained by removing b and replacing it with a.

The resulting operators on a simplicial set are $d_a\colon X_i \to X_{I \smallsetminus \{a\}}$ (face operators), $s_a^{b,c}\colon X_{I \cup_{\{b,c\}}\{a\}} \to X_I$ (degeneracy operators), and $s_a^b\colon X_{I \cup_{\{b\}}\{a\}} \to X_I$. Then we have the following simplicial identities:

- $d_a d_b = d_b d_a$,
- $s_a^{b,c} s_d^{e,f} = s_d^{e,f} s_a^{b,c}$ if a, b, c, d, e, f are pairwise distinct,
- $s_c^{d,c} s_a^{b,c} = s_b^{b,d} s_a^{b,c}$,
- $d_a s_b^{c,d} = s_b^{c,d} d_a$ if a, b, c, d pairwise distinct,
- $d_b s_a^{b,c} = s_a^c$, $d_c s_a^{b,c} = s_a^b$.

---

[31]Adapted from the appendix to [Mos15].

In other words, these identities say that simplicial operators generally commute, except when we compose $f s_a^{b,c}$ where f is an operator involving b or c.

Fix a map $p\colon B \to \Delta^n$ and a vertex k of Δ^n. For each simplex $u\colon \Delta^d \to B$ such that pu factors through the face $\Delta^{[n]\smallsetminus\{k\}} \subset \Delta^n$ opposite k, we obtain a coaugmented simplicial set $B(u)\colon (\Delta^+)^{\mathrm{op}} \to$ Set. Here $B(u)_j$ is the the set of fillers in

$$
\begin{array}{ccc}
\Delta^d \approx \Delta^{d_L} \star \varnothing \star \Delta^{d_R} & \xrightarrow{\ \ u\ \ } & B \\
\big\downarrow & \nearrow\ \ \ & \big\downarrow {\scriptstyle p} \\
\Delta^{d_L} \star \Delta^j \star \Delta^{d_R} & \longrightarrow \Delta^{k-1} \star \{k\} \star \Delta^{n-k-1} \approx \Delta^n
\end{array}
$$

Note that any simplicial operator $\delta\colon \Delta^{d'} \to \Delta^d$ induces a map $B(\delta)\colon B(u) \to B(u\delta)$ of coaugmented simplicial sets.

56.11. Lemma. *Suppose there exist, for each simplex $u\colon \Delta^d \to B$ such that the image of pu is equal to $\Delta^{[n]\smallsetminus k}$, a right contraction Q of $B(u)$, and that these contraction maps are compatible with degeneracy operators, in the sense that for surjective $\sigma\colon \Delta^{d'} \to \Delta^d$ we have $Q \circ B(\sigma) = B(\sigma) \circ Q$. Then $A \to B$ is right anodyne.*

Proof. Observe that the set of simplices of B in the complement of A is precisely the disjoint union of the $B(u)$, where u ranges over simplices such that pu surjects to the face opposite k.

Let $S_n := B_n^{\mathrm{nd}} \smallsetminus A_n^{\mathrm{nd}}$, the set of non-degenerate n-simplices in the complement of A. Let $S_n' =$ the set of such which are in the image of the degeneracy operators, and let S_n'' be its complement in S_n. We claim that the Qs induce a bijection

$$\phi\colon S_{n-1}'' \to S_n'.$$

To see this, we observe the following for simplices in the complement of A.

- If $Q(x)$ is a degenerate simplex, then x is also degenerate. Therefore Q induces a function $\phi\colon S_{n-1}'' \to S_n'$.
- If x is a degenerate simplex, then so is $Q(x)$. Therefore ϕ is surjective.
- We have that $d_z(Q(x)) = x$, where $z = \sup V_x$. Therefore ϕ is injective.

Furthermore, we have for $x \in S_{n-1}''$ that

- For $a \in V_{Q(x)}$ but $a \neq \sup V_{Q(x)}$, we have that $d_a(Q(x)) = Q(d_a(x))$. Therefore these faces $d_a(Q(x))$ are either degenerate or are in the image of Q.
- For $a \in [n] \smallsetminus V_{Q(x)}$, we have that $d_a(Q(x))$ has strictly smaller excess than x and $Q(x)$.
- $z = \sup V_{Q(x)}$ is not equal to 0 (since $k > 0$.).

We can now construct B from A by attaching right horns. Let $E_{r,n} \subseteq E_r$ denote the smallest subcomplex of E_r containing S_d' for $d \leq n$; let $E_{r,-1} = E_{r-1}$. Then each inclusion $E_{r,n-1} \to E_{r,n}$ is obtained by attaching right horns of dimension n, according to the elements of S_n' which have excess r. $\qquad\square$

56.12. Lemma. *Suppose $B \to \Delta^n$ is a left fibration. Then contraction maps as in the previous lemma exist.*

Proof. For non-degenerate simplices in the complement of A, we define Q inductively, as follows.

(1) If $x = s_a^{b,c} y$, set $Q_z x := s_a^{b,c} Q_z y$.
(2) If $x = Q_w y$, set $Q_z x := s_w^{w,z} Q_w y$.
(3) Otherwise, choose $Q_z x$ to be any n-simplex such that $d_z Q_z x = x$ and $d_a Q_z x = Q_z d_a x$ for $a \in V_x$.

$\qquad\square$

57. Cartesian fibrations

Let $p\colon C \to D$ a functor between ordinary categories. A morphism $f\colon x' \to x$ in C is called p-**Cartesian** if for every object c of C the evident commutative square

$$\begin{array}{ccc}
\mathrm{Hom}_C(c, x') & \xrightarrow{\ f\circ\ } & \mathrm{Hom}_C(c, x) \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\
\mathrm{Hom}_D(p(c), p(x')) & \xrightarrow[p(f)\circ]{} & \mathrm{Hom}_D(p(c), p(x))
\end{array}$$

is a pullback square of sets.

Given an object $x \in \mathrm{ob}\, C$ and a morphism $g\colon y' \to p(x)$ in D, a **Cartesian lift of g at x** is a p-Cartesian morphism $f\colon x' \to x$ such that $p(f) = g$.

We say that $p\colon C \to D$ is a **Cartesian fibration** of categories if every pair $(x \in \mathrm{ob}\, C, g\colon y' \to p(x) \in D)$ admits a Cartesian lift.

Here are some observations, whose verification we leave to the reader. Fix a functor $p\colon C \to D$.

- Every isomorphism in C is p-Cartesian.
- Every Cartesian lift of an isomorphism in D is itself an isomorphism.
- If $f\colon x' \to x$ is p-Cartesian, then for any $g\colon x'' \to x'$ in C, we have that g is p-Cartesian if and only if gf is p-Cartesian.
- Any two Cartesian lifts of g at x are "canonically isomorphic".

  Explicitly, fix $g\colon y' \to y$ in D and an object x in C such that $y = p(x)$. If $f_1\colon x_1' \to x$ and $f_2\colon x_2' \to x$ are any two Cartesian lifts of g, then there exists a unique map $u\colon x_1' \to x_2'$ such that $p(u) = 1_{y'}$ and $f_2 u = f_1$; the map u is necessarily an isomorphism.
- The map p is a right fibration if and only if it is a Cartesian fibration and every morphism in C is p-Cartesian.

Now suppose that $p\colon C \to D$ is a Cartesian fibration. For an object y of D, we write $C_y := p^{-1}(y)$ for the fiber of C over y.

- The map p is an isofibration.
- For each morphism $g\colon y' \to y$ in D and object x in C with $p(x) = y$, fix a choice of Cartesian lift $\widetilde{g}_x$ of g at x. Using this data, we obtain functors

$$g^!\colon C_y \to C_y'$$

  so that for morphism $\alpha\colon x_1 \to x_2$ in C_y, the map $g^!(\alpha)$ in $C_{y'}$ is the unique one fitting into

$$\begin{array}{ccc}
x_1' & \xrightarrow{\ \widetilde{g}_{x_1}\ } & x_1 \\
{\scriptstyle g^!(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha} \\
x_2' & \xrightarrow[\widetilde{g}_{x_2}]{} & x_2
\end{array}$$

  The functor $g^!$ depends on the choices of Cartesian lifts of g. Any two set of choices of lifts give rise to isomorphic functors.
- For each pair of morphisms $y'' \xrightarrow{h} y' \xrightarrow{g} y$, we obtain a natural isomorphism of functors

$$\gamma\colon h^! \circ g^! \overset{\cong}{\Rightarrow} (hg)^!\colon C_y \to C_{y''}.$$

This natural transformation is given by the unique maps γ_x in $C_{y''}$ fitting into

$$
\begin{array}{ccc}
x'' & \xrightarrow{\ \widetilde{h}_{x'}\ } & x' \\
{\scriptstyle \gamma_x}\downarrow & \quad {\scriptstyle \widetilde{g}_x} & \downarrow \\
x''' & \xrightarrow[\ \widetilde{(gh)}_x\ ]{} & x
\end{array}
$$

Similarly, there is a natural isomorphism $\mathrm{id} \overset{\sim}{\Rightarrow} (1_y)_! \colon C_y \to C_y$. The data of the functors $g^!$ together with these natural isomorphisms define a **pseudofunctor** $D^{\mathrm{op}} \to \mathrm{Cat}$, which on objects sends $y \mapsto C_y$.

- We can produce an actual functor $F \colon D^{\mathrm{op}} \to \mathrm{Cat}$ with $F(y)$ equivalent to C_y as follows. Given functors $p' \colon C' \to D$ and $p \colon C \to D$, let $\mathrm{Fun}_D(C', C)$ denote the category of fiberwise functors and natural transformations; i.e., the fiber of $p\circ \colon \mathrm{Fun}(C', C) \to \mathrm{Fun}(C', D)$ over q. Let $\mathrm{Fun}_D^+(C', C) \subseteq \mathrm{Fun}_D(C', C)$ denote the full subcategory of functors $f \colon C' \to C$ which take p'-Cartesian morphisms to p-Cartesian morphims.

  We obtain a functor $F \colon D^{\mathrm{op}} \to \mathrm{Cat}$, given on objects by

  $$
  F(y) := \mathrm{Fun}_D^+(D_{/y}, C).
  $$

  One can show that restriction to $\{1_y\} \subseteq D_{/y}$ defines an equivalence of categories $F(y) \to C_y$.

- Given D, there is a 2-category $\mathcal{F}_D$, whose objects are Cartesian fibrations $p \colon C \to D$; for any two objects $p \colon C \to D$ and $p' \colon C' \to D$ we take $\mathrm{Fun}_D^+(C', C)$ as the category of morphisms from p' to p. One can show that $\mathcal{F}_D$ is 2-equivalent to the 2-category $\mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Cat})$.

Part 8. Appendices

58. Appendix: Generalized horns

A **generalized horn**[32] is a subcomplex $\Lambda_S^n \subset \Delta^n$ of the standard n-simplex, where $S \subseteq [n]$ and

$$
(\Lambda_S^n)_k := \{\, f \colon [k] \to [n] \mid S \not\subseteq f([k]) \,\}.
$$

In other words, a generalized horn is a union of some codimension 1 faces of the n-simplex:

$$
\Lambda_S^n = \bigcup_{s \in S} \Delta^{[n] \smallsetminus s}.
$$

In particular,

$$
\Lambda_{[n]}^n = \partial\Delta^n, \quad \Lambda_{[n] \smallsetminus j}^n = \Lambda_j^n, \quad \Lambda_{\{j\}}^n = \Delta^{[n] \smallsetminus j}, \quad \Lambda_\varnothing^n = \varnothing.
$$

In general $S \subseteq T$ implies $\Lambda_S^n \subseteq \Lambda_T^n$.

58.1. Proposition (Joyal [Joy08a, Prop. 2.12])**.** *Let $S \subsetneq [n]$ be a* proper *subset.*

(1) $(\Lambda_S^n \subset \Delta^n) \in \overline{\mathrm{Horn}}$ *if $S \neq \varnothing$.*
(2) $(\Lambda_S^n \subset \Delta^n) \in \overline{\mathrm{LHorn}}$ *if $n \in S$.*
(3) $(\Lambda_S^n \subset \Delta^n) \in \overline{\mathrm{RHorn}}$ *if $0 \in S$.*
(4) $(\Lambda_S^n \subset \Delta^n) \in \overline{\mathrm{InnHorn}}$ *if S is not an interval; i.e., if there exist $a < b < c$ with $a, c \in S$ and $b \notin S$.*

---

[32]This notion is from [Joy08a, §2.2.1]. However, I have changed the sense of the notation: our Λ_S^n is Joyal's $\Lambda^{[n] \smallsetminus S}$. I find my notation easier to follow, but note that it does conflict with the standard notation for horns. Maybe I should use something like $\Lambda^{n,S}$?

Proof. Consider $S \subsetneq [n]$ and $t \in [n] \smallsetminus S$. Observe the diagram

$$
\begin{array}{ccccc}
\Delta^{[n]\smallsetminus t} \cap \Lambda^n_S & \longrightarrow & \Delta^{[n]\smallsetminus t} & & \\
\downarrow & & \downarrow & & \\
\Lambda^n_S & \longrightarrow & \Lambda^n_{S\cup t} & \longrightarrow & \Delta^n
\end{array}
$$

in which the square is a pushout, and the top arrow is isomorphic to the generalized horn $\Lambda^{[n]\smallsetminus t}_S \subset \Delta^{[n]\smallsetminus t}$. Thus, $(\Lambda^n_S \subset \Delta^n)$ is contained in the saturation of the set contaning the two inclusions

$$
\Lambda^{[n]\smallsetminus t}_S \subset \Delta^{[n]\smallsetminus t} \quad \text{and} \quad \Lambda^n_{S\cup t} \subset \Delta^n.
$$

Each of the statements of the proposition follows by an evident induction on the size of $[n] \smallsetminus S$. I'll do case (4). If $S \subset [n]$ is not an interval, there exists some $s < u < s'$ with $s, s' \in S$ and $u \notin S$. If $[n] \smallsetminus S = \{u\}$ then we already have an inner horn. If not, then choose $t \in [n] \smallsetminus (S \cup \{u\})$, in which case $S \cup t$ is not an interval in $[n]$, and S is not an interval in $[n] \smallsetminus t$. The proofs of the other cases are similar. $\qquad\square$

58.2. **Proposition** (Joyal [Joy08a, Prop. 2.13]). *For all $n \geq 2$, we have that $(I^n \subset \Delta^n) \in \overline{\mathrm{InnHorn}}$.*

Proof. We can factor the inclusion spine inclusion $h_n = g_n f_n$ as

$$
I^n \xrightarrow{f_n} \Delta^{\{1,\dots,n\}} \cup I^n \xrightarrow{g_n} \Delta^n.
$$

We show by induction on n that $f_n, g_n \in \overline{\mathrm{InnHorn}}$, noting that the case $n = 2$ is immediate.

To show that $f_n \in \overline{\mathrm{InnHorn}}$, consider the pushout square

$$
\begin{array}{ccc}
I^{\{1,\dots,n\}} & \longrightarrow & \Delta^{\{1,\dots,n\}} \\
\downarrow & & \downarrow \\
I^n & \xrightarrow{f_n} & \Delta^{\{1,\dots,n\}} \cup I^n
\end{array}
$$

in which the top arrow is equal to $h_{n-1} = g_{n-1}f_{n-1}$, which is in $\overline{\mathrm{InnHorn}}$ by induction.

To show that $g_n \in \overline{\mathrm{InnHorn}}$, consider the diagram

$$
\begin{array}{ccccc}
\Delta^{\{1,\dots,n-1\}} \cup I^{\{0\dots,n-1\}} & \xrightarrow{g_{n-1}} & \Delta^{\{0,\dots,n-1\}} & & \\
\downarrow & & \downarrow & & \\
\Delta^{\{1,\dots,n\}} \cup I^n & \longrightarrow & \Delta^{\{1,\dots,n\}} \cup \Delta^{\{0,\dots,n-1\}} & \longrightarrow & \Delta^n
\end{array}
$$

in which the square is a pushout, the top horizontal arrow is equal to g_{n-1}, an element of $\overline{\mathrm{InnHorn}}$ by induction, and the bottom right horizontal arrow is equal to $\Lambda^n_{\{0,n\}} \subset \Delta^n$, which is in $\overline{\mathrm{InnHorn}}$ by (58.1)(4).

$\qquad\square$

59. Appendix: Box product lemmas

Here is where I'l prove various statements mentioned in the text.

- InnHorn$\square$Cell $\subseteq \overline{\mathrm{InnHorn}}$ (16.9).
- LHorn$\square$Cell $\subseteq \overline{\mathrm{LHorn}}$ (45.4).
- RHorn$\square$Cell $\subseteq \overline{\mathrm{RHorn}}$ (45.4).
- Horn$\square$Cell $\subseteq \overline{\mathrm{Horn}}$.

59.1. **Left and right horns.** Joyal [Joy08a, 2.25][33] observes that $(\Lambda_k^n \subset \Delta^n)$ is a retract of $(\Lambda_k^n \subset \Delta^n) \square (\{0\} \subset \Delta^1)$ when $0 \le k < n$. This reduces showing LHorn$\square$Cell $\subseteq \overline{\text{LHorn}}$ to $\{(\{0\} \subset \Delta^1)\}\square$Cell $\subseteq \overline{\text{LHorn}}$. The retraction is

$$\Delta^n \xrightarrow{s} \Delta^n \times \Delta^1 \xrightarrow{r} \Delta^n$$

defined by $s(x) = (x, 1)$ and

$$r(x, 0) = \begin{cases} x & \text{if } x \le k, \\ k & \text{if } x \ge k, \end{cases} \qquad r(x, 1) = x.$$

Note that $r(\Delta^{[n]\smallsetminus j} \times \Delta^1) = \Delta^{[n]\smallsetminus j}$ if $j \ne k$, and $r(\Delta^n \times \{0\}) = \Delta^{\{0,\dots,k\}} \subseteq \Delta^{[n]\smallsetminus (k+1)}$, so this gives the desired retraction.

59.2. **Inner horns.** Here is an argument for the key case for inner horns.

Consider $\Delta^n \xrightarrow{s} \Delta^2 \times \Delta^n \xrightarrow{r} \Delta^n$, the unique maps which are given on vertices by

$$s(y) = \begin{cases} (0, y) & \text{if } y < j, \\ (1, y) & \text{if } y = j, \\ (2, y) & \text{if } y > j, \end{cases} \qquad r(x, y) = \begin{cases} y & \text{if } x = 0 \text{ and } y < j, \\ y & \text{if } x = 2 \text{ and } y > j, \\ j & \text{otherwise.} \end{cases}$$

These explicitly exhibit $(\Lambda_j^n \subset \Delta^n)$ as a retract of $(\Lambda_1^2 \subset \Delta^2)\square(\Lambda_j^n \subset \Delta^n)$, so

$$\text{InnHorn} \subseteq \{\Lambda_1^2 \subset \Delta^2\}\square\overline{\text{Cell}}.$$

We have (17.4) that $\overline{\text{Cell}}\square\overline{\text{Cell}} \subseteq \overline{\text{Cell}}$, so the above implies that InnHorn$\square\overline{\text{Cell}} \subseteq \{\Lambda_1^2 \subset \Delta^2\}\square\overline{\text{Cell}}$. Thus the assertions "InnHorn$\square\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$" and "$\{\Lambda_0^2 \subset \Delta^2\}\square\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$" are equivalent. Thus both assertions follow from the following.

59.3. **Lemma.** *For all $n \ge 0$ we have that $(\Lambda_1^2 \subset \Delta^2)\square(\partial\Delta^n \subset \Delta^n) \in \overline{\text{InnHorn}}$.*

Proof. [Lur09, 2.3.2.1].

For each $0 \le a \le b < n$, let σ_{ab} be the $(n+1)$-simplex of $\Delta^2 \times \Delta^n$ defined by

$$\sigma_{ab} = \langle (0,0), \dots, (0,a), (1,a), \dots, (1,b), (2,b+1), \dots, (2,n)\rangle.$$

For each $0 \le a \le b \le n$, let τ_{ab} be the $(n+2)$-simplex of $\Delta^2 \times \Delta^n$ defined by

$$\tau_{ab} = \langle (0,0), \dots, (0,a), (1,a), \dots, (1,b), (2,b), \dots, (2,n)\rangle.$$

The set $\{\tau_{ab}\}$ consists of all the non-degenerate $(n+2)$-simplices. Note that σ_{ab} is a face of τ_{ab} and of $\tau_{a,b+1}$, but not a face of any other τ.

We attach simplices to $K := (\Lambda_1^2 \times \Delta^n) \cup (\Delta^2 \times \partial\Delta^n)$ in the following order:

$$\sigma_{00}, \quad \sigma_{01}, \sigma_{11}, \quad \sigma_{02}, \sigma_{12}, \sigma_{22}, \quad \dots \quad \sigma_{0,n-1}, \dots, \sigma_{n-1,n-1},$$

followed by

$$\tau_{00}, \quad \tau_{01}, \tau_{11}, \quad \tau_{02}, \tau_{12}, \tau_{22}, \quad \dots \quad \tau_{0,n}, \dots, \tau_{n,n}.$$

The τs range through all the non-degenerate $(n+2)$-simplices of $\Delta^2 \times \Delta^n$, so that $K \cup \bigcup \sigma_{a,b} \cup \bigcup \tau_{a,b} = \Delta^2 \times \Delta^n$.

The claim is that each attachment is along an inner horn inclusion. More precisely, each σ_{ab} gets attached along the horn at the vertex $(1, a)$ in σ_{ab}, i.e., via a $\Lambda_{a+1}^{n+1} \subset \Delta^{n+1}$ horn inclusion, which is always inner since $a \le b < n$. Likewise, each τ_{ab} gets attached along the horn at vertex $(1, a)$ in τ_{ab}, i.e., via a $\Lambda_{a+1}^{n+2} \subset \Delta^{n+2}$ horn inclusion, which is always inner since $a \le b \le n$.

The proof of the claim amounts to the following lists of elementary observations.

For $\sigma_{a,b}$:

---

[33]Lurie [Lur09, 2.1.2.6] states this incorrectly.

- Every codimension-one face is contained in $\Delta^2 \times \partial\Delta^n$, except the following: the face opposite vertex $(0, a)$, and the face opposite vertex $(1, a)$.
- The face opposite vertex $(0, a)$ is either contained in $\Lambda_0^2 \times \Delta^n$ if $a = 0$, or a face of $\sigma_{a-1,b}$ if $a > 0$.
- The face of $\sigma_{a,b}$ opposite vertex $(1, a)$ is not contained in $\Delta^2 \times \partial\Delta^n$, nor in $\Lambda_0^2 \times \Delta^n$, nor in any $\sigma_{i,b}$ with $i < a$ (because of the vertex $(0, a)$), nor in any $\sigma_{i,j}$ with $i \leq j < b$ (because of the vertex $(1, b)$ if $a < b$, or the vertex $(0, a)$ if $a = b$).

For $\tau_{a,b}$ when $a < b$:

- Every codimension-one face is contained in $\Delta^2 \times \partial\Delta^n$ except the following: the face opposite vertex $(0, a)$, the face opposite vertex $(1, a)$, the face opposite vertex $(1, b)$, and the face opposite vertex $(2, b)$.
- The face opposite vertex $(2, b)$ is $\sigma_{a,b}$, while the face opposite vertex $(1, b)$ is $\sigma_{a,b-1}$.
- The face opposite vertex $(0, a)$ is either contained in $\Lambda_1^2 \times \Delta^n$ if $a = 0$, or is a face of $\tau_{a-1,b}$ if $a > 0$.
- The face opposite vertex $(1, a)$ is not contained in $\Delta^2 \times \partial\Delta^n$, nor in $\Lambda_1^2 \times \Delta^n$, nor in any $\sigma_{i,j}$ (because of the vertices $(1, b)$ and $(2, b)$), nor in any $\tau_{i,b}$ with $i < b$ (because of the vertex $(0, a)$), nor in any $\tau_{i,j}$ with $i \leq j < b$ (because of the vertex $(1, b)$).

For $\tau_{a,b}$ when $a = b$:

- Every codimension-one face is contained in $\Delta^2 \times \partial\Delta^n$ except the following: the face opposite vertex $(0, a)$, the face opposite vertex $(1, a) = (1, b)$, and the face opposite vertex $(2, b)$.
- The face opposite vertex $(2, b)$ is $\sigma_{a,b}$.
- The face opposite vertex $(0, a)$ is contained in $\Lambda_1^2 \times \Delta^n$ if $a = 0$, or is a face of $\tau_{a-1,b}$ if $a > 0$.
- The face opposite vertex $(1, a) = (1, b)$ is not contained in $\Delta^2 \times \partial\Delta^n$, nor in $\Lambda_1^2 \times \Delta^n$, nor in any $\sigma_{i,j}$ (because of the vertices $(0, a)$ and $(2, b)$), nor in any $\tau_{i,b}$ with $i < b$ (because of the vertex $(0, a)$), nor in any $\tau_{i,j}$ with $i \leq j < b$ (because of the vertex $(0, a)$).

$\square$

59.4. **A pushout-product version of Joyal lifting.** We now give a proof of (29.12): we will prove the case of $(i, j) = (0, 0)$, i.e., given $p\colon C \to D$ an inner fibration of quasicategories, $n \geq 1$, and

$$
\begin{array}{ccc}
\Delta^1 \times \{0\} \overset{f}{\rightarrowtail} (\{0\} \times \Delta^n) \cup_{\{0\}\times\partial\Delta^n} (\Delta^1 \times \partial\Delta^n) \longrightarrow C \\
\downarrow \qquad\qquad\qquad\qquad\qquad \nearrow \qquad\qquad \downarrow p \\
\Delta^1 \times \Delta^n \longrightarrow D
\end{array}
$$

such that f represents an isomorphism in C, we will construct a lift. (Note that if $n = 0$ such a lift does not generally exist.)

For each $0 \leq a \leq n$ let τ_a be the $(n + 1)$-simplex of $\Delta^1 \times \Delta^n$ defined by

$$\tau_a = \langle (0, 0), \dots, (0, a),\ (1, a), \dots, (1, n) \rangle.$$

Let $K = (\{0\} \times \Delta^n) \cup (\Delta^1 \times \partial\Delta^n)$. We obtain an ascending filtration of $\Delta^1 \times \Delta^n$ by starting with K and attaching simplices in the following order:

$$\tau_n, \tau_{n-1}, \dots, \tau_1, \tau_0.$$

The τs range through all non-degenerate $(n + 1)$-simplices of $\Delta^1 \times \Delta^n$, so $K \cup \bigcup \tau_a = \Delta^1 \times \Delta^n$.

The claim is that each attachment is along a specified horn inclusion. More precisely, for $a \in [n]$ the simplex τ_a is attached to $K \cup \bigcup_{k>a} \tau_k$ along the horn at the vertex $(0, a)$ in τ_a, i.e., via a $\Lambda_a^{n+1} \subset \Delta^n$ horn inclusion. Note that if when $a > 0$ this is an inner horn, while when $a = 0$ this is the inclusion $\Lambda_0^{n+1} \subset \Delta^n$ and the edge $\Delta^1 \times \{0\}$ is the leading edge of τ_0.

Given the claim, we thus construct the desired lift by inductively choosing a lift defined on each τ_a relative to the given lift on its Λ_a^{n+1}-horn. When $a > 0$ such a lift exists because p is an inner fibration, while when $a = 0$ a lift exists by Joyal lifting (28.13).

The proof of the claim amounts to the following list of elementary observations. For τ_a:

- Every codimension-one face is contained in $\Delta^1 \times \partial\Delta^n$ except: the face opposite vertex $(0, a)$, and the face opposite vertex $(1, a)$.
- The face opposite vertex $(1, a)$ is contained in $\{0\} \times \Delta^n$ if $a = n$, or is a face of τ_{a+1} if $a < n$.
- The face opposite vertex $(0, a)$ is not contained in $\Delta^1 \times \partial\Delta^n$, nor in $\{0\} \times \Delta^n$, nor in any τ_i with $i > a$ (beacuse of the vertex $(1, a)$).

The proof of the other case $(i, j) = (1, n)$ proceeds analogously; in that case simplices are attached to K in the order: $\tau_0, \tau_1, \ldots, \tau_n$.

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