Fractional-level WZW models

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Outline

CFTs and coset constructions

Two of the most fundamental families of CFTs are

- the Virasoro minimal models $M(p, p')$, $p, p' \in \mathbb{Z}_{\geqslant 2}$ coprime, which are unitary iff $|p' - p| = 1$, and
- the (always-unitary) Wess–Zumino–Witten models G_k, $k \in \mathbb{N}$ and G a (compact, connected, simply connected) simple Lie group.

[Goddard–Kent–Olive] famously showed that

$$M(k+2, k+3) = \frac{SU(2)_k \otimes SU(2)_1}{SU(2)_{k+1}},$$

thereby completing the classification of unitary Virasoro minimal models.

Question [Kent]: Is there a similar construction of the non-unitary models?

To get $M(p, p')$, we'd need $k = -2 + \dfrac{\min\{p, p'\}}{|p' - p|}$ ($\notin \mathbb{N}$ if $|p' - p| > 1$).

When $k \notin \mathbb{N}$, the string WZW action for G_k has problems. However, G_k still makes sense algebraically as a CFT (unless $k = -\mathsf{h}^\vee$).

[Kac–Wakimoto] singled out the admissible levels ($\ell = 1, 2, 3$ is the lacety)

$$k = -\mathsf{h}^\vee + \frac{u}{v}, \quad u, v \in \mathbb{Z}_{\geqslant 1} \text{ coprime}, \quad u \geqslant \begin{cases} \mathsf{h} & \text{if } \ell \mid v, \\ \mathsf{h}^\vee & \text{if } \ell \nmid v, \end{cases}$$

as having nice mathematical properties (character formulae).

Later, [Gorelik–Kac] extended this to the fractional levels which satisfy

$$\ell(k + \mathsf{h}^\vee) = \frac{u}{v}, \quad u \in \mathbb{Z}_{\geqslant 2} \text{ and } v \in \mathbb{Z}_{\geqslant 1} \text{ coprime},$$

these being the levels where the representation theory is not just that of the underlying affine Kac–Moody algebra.

These levels give the fractional-level WZW models G_k.

The fractional-level WZW model $SU(2)_k$

When $G = SU(2)$, fractional-level = admissible-level. Take $k = -2 + \frac{u}{v}$, with $u, v \in \mathbb{Z}_{\geqslant 2}$ coprime (this excludes $k \in \mathbb{N}$).

Then, there are $(u-1)v$ irreducible highest-weight modules, all but $u-1$ of which have infinitely many primary fields [Adamović–Milas].

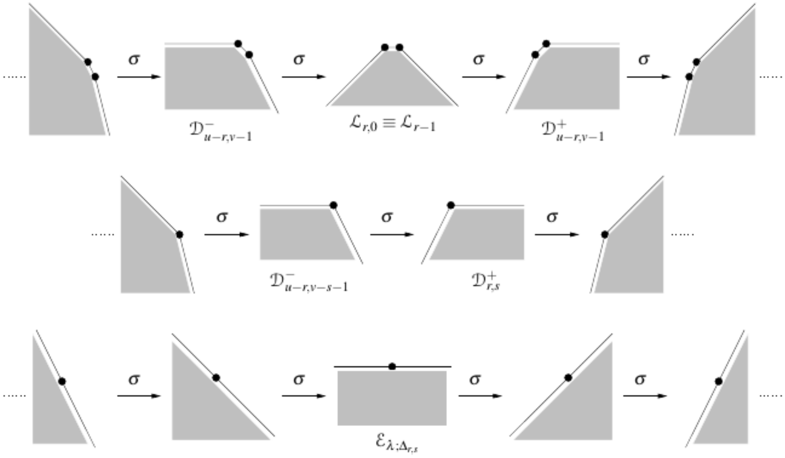
There are also $\frac{1}{2}(u-1)(v-1)$ one-parameter families of irreducibles that are not highest-weight and each has infinitely many spectral flows whose energies are unbounded-below [DR–Wood].

Worse yet, there are also infinitely many reducible but indecomposable modules on which the hamiltonian acts non-diagonalisably.

Nevertheless, this is all mathematically consistent with modular-covariant characters and non-negative fusion coefficients [Creutzig–DR].

For fractional levels, $SU(2)_k$ is a logarithmic CFT. Moreover, the coset construction of the non-unitary Virasoro minimal models works!

$SU(2)_k$-irreducibles



4D/2D correspondence

One of the prevailing themes in recent high-energy physics research is to relate $D > 2$-dimensional QFTs to 2D CFTs, *eg.* the AGT conjectures.

In 2015, [Beem *et al*] discovered a correlator-preserving map between sectors of 4D $N = 2$ gauge field theories and certain non-unitary 2D CFTs.

Even the 4D geometric invariants (Schur indices and Higgs branches) may be expressed in terms of 2D data (characters and associated varieties).

Of the many 2D CFTs identified as corresponding to a 4D theory, most are fractional-level WZW models (or are closely related to them).

In particular, the Deligne exceptional series arises in this way.

G	$SU(2)$	$SU(3)$	G_2	$SO(8)$	F_4	E_6	E_7	E_8
k	$-\frac{4}{3}$	$-\frac{3}{2}$	$-\frac{5}{3}$	-2	$-\frac{5}{2}$	-3	-4	-6

There is therefore some good fun to be had in comparing results from 4D and 2D calculations. However, we need to understand the representation theory of the fractional-level WZW models, which is not at all easy...

For $SU(2)_k$, we are in good shape; for higher-rank G, we know very little.

An exception is that we know the irreducible highest-weight modules for all admissible levels [Arakawa]. Unfortunately, these are vastly outnumbered.

For the purposes of CFT, we want to understand representations that

- are weight: the Cartan subalgebra of $\mathfrak{g}$ acts diagonalisably;
- and have finite multiplicities: so characters are defined.

It turns out [Futorny–Tsylke] that every irreducible of this type is the spectral flow of a relaxed highest-weight module (which we define next).

It would of course be nice to understand some indecomposables too...

Relaxed highest-weight modules

Recall that a highest-weight vector of an affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is:

1. an eigenvector of the Cartan subalgebra $\mathfrak{h} \oplus \mathbb{C}K$ that is
2. annihilated by all the $J_n \in \widehat{\mathfrak{g}}$ with $n > 0$ and
3. by all the $J_0 \in \widehat{\mathfrak{g}}$ where J is a positive root vector of $\mathfrak{g}$.

A highest-weight module is then just a module generated by a single highest-weight vector.

To define relaxed highest-weight vectors and modules, we merely relax (*ie.* omit!) condition 3. above [Feigin–Semikhatov–Tipunin, DR–Wood].

A relaxed highest-weight $\widehat{\mathfrak{g}}$-module is just like a highest-weight $\widehat{\mathfrak{g}}$-module except that its primary fields need not form an highest-weight $\mathfrak{g}$-module.

In particular, its conformal dimensions are bounded below.

Of course, the primary fields of an irreducible relaxed highest-weight $\widehat{\mathfrak{g}}$-module still form an irreducible weight $\mathfrak{g}$-module.

They also still determine the irreducible completely — this is our key!

We therefore need to:

1. understand the irreducible weight $\mathfrak{g}$-modules with finite multiplicities
2. and then classify which of them give primary fields in G_k.

Surprisingly, 1. was only completed in 2000 [Fernando, Mathieu].

Our contribution to 2. is to provide an algorithm which explicitly completes the classification, given that one has already classified the highest-weight $\mathfrak{g}$-modules corresponding to primaries in G_k.

This highest-weight classification is already known for all admissible levels [Joseph, Arakawa], so this is (in principle) a powerful result.

The Fernando–Mathieu classification

The irreducible weight $\mathfrak{g}$-modules come in n-parameter families.

These correspond to certain parabolic subalgebras of $\mathfrak{g}$, namely those whose Levi factor $\mathfrak{l}$ has simple ideals only of types A and C. n is the product of the ranks of these simple ideals.

eg., $\mathfrak{g} = \mathfrak{sl}_3$ has a parabolic with $\mathfrak{l} = \mathfrak{gl}_2$ and simple ideal $\mathfrak{sl}_2$ of type A.

Such simple ideals have irreducible dense modules, so we

1. tensor the modules together (with a 1-dim module for $\mathfrak{z}(\mathfrak{l})$),
2. parabolically induce this $\mathfrak{l}$-module to a $\mathfrak{g}$-module,
3. and take the (unique) irreducible quotient.

[Fernando, Mathieu] say that all irreducible weight $\mathfrak{g}$-modules with finite multiplicities arise in this manner.

The irreducible highest-weight modules arise by taking the parabolic to be a Borel subalgebra (so $\mathfrak{l}$ is the Cartan subalgebra).

Algorithm (Kawasetsu-DR arXiv:1906.02935).

Assume that we know the irreducible highest-weight modules for G_k.

- For each non-empty subset of the simple roots, check if the corresponding parabolic subalgebra $\mathfrak{p}$ has Levi factor $\mathfrak{l}$ with simple ideals $\mathfrak{s}_i$ of types A or C.
- If so, project the highest weight λ of each irreducible highest-weight G_k-module along each simple ideal $\mathfrak{s}_i$ of $\mathfrak{l}$. Check if *all* projections correspond to "bounded" $\mathfrak{s}_i$-modules (conditions given by [Mathieu]).
- If so, take the family of irreducible dense $\mathfrak{s}_i$-modules containing the bounded $\mathfrak{s}_i$-module, for all i, and tensor them together, along with a (uniquely determined) $\mathfrak{z}(\mathfrak{l})$-module.
- Induce the result to $\mathfrak{g}$ and take simple quotients. We thereby obtain families of irreducible $\mathfrak{g}$-modules that coincide with the primaries of families of irreducible G_k-modules.
- Up to twisting by the Weyl group W, these families constitute a complete set of irreducible relaxed highest-weight modules for G_k (that are not already highest-weight).

Example: Relaxed $SO(8)_{-2}$-modules.

The DES model $SO(8)_{-2}$ has five irreducible highest-weight modules L_i, $i = 0, \dots, 4$. The vacuum module L_0 is invariant under W-twists while L_2 has 24 twists and the others have 8 each.

The algorithm now gives

simple root subset	$\mathfrak{l}$	# parabolic families
$\{1\}, \{3\}, \{4\}$	$\mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 3}$	24 each
$\{2\}$	$\mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 3}$	72
$\{1,2\}, \{2,3\}, \{2,4\}$	$\mathfrak{sl}_3 \oplus \mathfrak{gl}_1^{\oplus 2}$	32 each
$\{1,3\}, \{1,4\}, \{3,4\}$	$\mathfrak{sl}_2^{\oplus 2} \oplus \mathfrak{gl}_1^{\oplus 2}$	no families
$\{1,3,4\}$	$\mathfrak{sl}_2^{\oplus 3} \oplus \mathfrak{gl}_1$	no families
$\{1,2,3\}, \{1,2,4\}, \{2,3,4\}$	$\mathfrak{sl}_4 \oplus \mathfrak{gl}_1$	8 each

and each of these families gives either a 1-, 2- or 3-parameter family of irreducible relaxed highest-weight modules for $SO(8)_{-2}$.

The reducible but indecomposable case

We also have some existence results for reducible but indecomposable relaxed highest-weight modules for G_k that are relevant for constructing staggered modules / projective covers in logarithmic CFT.

Every family of irreducible (non-highest-weight) relaxed highest-weight modules for G_k constructed by our algorithm may be "completed" to include some reducible but indecomposable modules.

Theorem (Kawasetsu-DR arXiv:1906.02935).

If one of these reducible but indecomposable modules has the property that some $J_0 \in \widehat{\mathfrak{g}}$, J a root vector of $\mathfrak{g}$, acts bijectively on the space of primaries of the module, then it is also a G_k-module.

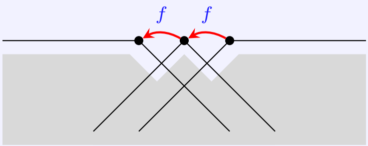
This means that the above algorithm also constructs many examples of reducible but indecomposable relaxed highest-weight modules for G_k.

Example: Reducible but indecomposable $SO(8)_{-2}$-modules.

Recall the irreducible highest-weight modules $\mathsf{L}_0, \dots, \mathsf{L}_4$, of $SO(8)_{-2}$.

For the simple root subset $\{1\}$, $\mathfrak{l} \cong \mathfrak{sl}_2 \oplus \mathfrak{gl}_1^{\oplus 3}$ and the completion of the corresponding family of irreducible $\mathfrak{l}$-modules has reducible but indecomposable modules formed by gluing three irreducibles together.

The corresponding family of irreducible $\mathfrak{so}_8$-modules then does too:



The maximal proper submodule now corresponds to a reducible but indecomposable module formed by gluing two highest-weight modules:

$$0 \longrightarrow \mathsf{L}_1 \longrightarrow \mathsf{M} \longrightarrow \mathsf{L}_2 \longrightarrow 0.$$

M is an $SO(8)_{-2}$-module (our theorem), solving a conjecture of Arakawa.

Outlook

- These results allow us to explore relaxed module theory in general. In particular, all admissible levels are now accessible (in principle).
- Character calculations indicate that relaxed $L_k(\mathfrak{g})$-modules are inextricably linked to highest-weight (and maybe relaxed) modules of the W-algebras obtained by quantum hamiltonian reduction.
- We therefore need to simultaneously investigate W-algebra representation theory [*cf.* talk by Fehily], possibly for all nilpotents!
- Reducible but indecomposable relaxed highest-weight modules (and their spectral flows) will be used to construct projective covers.
- An ultimate goal would be to compute the fusion rules of all fractional-level WZW models and W-algebras.
- The future of these CFTs is looking good...

"*Only those who attempt the absurd will achieve the impossible.*"

— M C Escher

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