Explicit String bundles

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If we look back at the historical development of bundles, the notion of a principal H-bundle for H a Lie group arose via considerations of homogeneous spaces G/H, and the defining bundle $G \to G/H$. Here H is a closed subgroup of G, and this will be a general assupmption for the talk. For instance, we can conside Stiefel manifolds, Grassmann manifolds, projective spaces, Minkowski space, spheres, ... I am to leverage this in order to address the

Challenge: Write down a (nontrivial) 2-bundle. Equivalently, write down a Čech cocycle with values in an interesting crossed module $(K \xrightarrow{t} H, H \times K \xrightarrow{a} K)$.

Recall (Breen 1994) that the cocycle equations are

$$h_{ij}^{\alpha} h_{jk}^{\beta} = t(k_{ijk}^{\alpha\beta\gamma}) h_{ik}^{\gamma}$$
$$t(h_{ij}^{a}, k_{jkl}^{\beta\lambda\varepsilon}) k_{ijl}^{\alpha\varepsilon\delta} = k_{ijk}^{\alpha\beta\gamma} k_{ikl}^{\gamma\lambda\delta}.$$

where the h_{ij}^{α} are H-valued functions, the $k_{ijk}^{\alpha\beta\gamma}$ are K-valued functions and the two sorts of indices label open sets of the base space. We shall return to this momentarily. Note that at this point we haven't even started to consider connections, which are necessary for gauge theory (and in fact we won't even go so far today).

Note: I am *not* going to use good open covers (that is, those such that non-empty finite intersections are contractible), since in many geometric situations there are naturally arising open covers that are not good. Instead, I will be using *truncated globular hypercovers* (these are open covers with particular properties), and I will define these in a moment. For now, suffice it to say, this is why there are two different sorts of indices on the cocycle.

Christian Saemann asked (Feb 2013): I want a 2-bundle on (conformally compactified) $\mathbb{R}^{5,1}$ (recall that this is $S^5 \times S^1$). So let's try lifting the frame bundle of $S^5 \times S^1$ to a *String* bundle. Note that the S^1 factor contributes nothing (its frame bundle is trivial) so just work over S^5. Note that the frame bundle of S^5 is most definitely not trivial.

The frame bundle $FS^5 \to S^5$ is classified by a map $S^5 \supset S^4 \to SO(5)$, called the *clutching* or *transition function.* Since S^5 is 4-connected, the first Stiefel-Whitney class w_1 necessarily vanishes, as does the characteristic class $p_1/2 \in H^4(S^5, \mathbb{Z})$ that is the obstruction to lifting to a String bundle. Thus we can be assured that the lift we are after does exist. From the vanishing of w_1 we know the transition function lifts to a function $S^4 \to Spin(5)$, and so defines a class in $\pi_4(Spin(5))$, which is the group $\mathbb{Z}/2\mathbb{Z}$ (Mimura-Toda 1964). Since FS^5

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is not trivial, the transition function needs to represent the non-trivial homotopy class. We want to write down an explicit function in coordinates, rather than use some abstract representative.

To approach this, we first use the exceptional isomorphism $Spin(5) \simeq Sp(2)$, where $Sp(2)$ is the group of 2×2 unitary quaternionic matrices. The non-trivial class in $\pi_4(Sp(2))$ is represented by a map $S^4 \to S^3 \simeq Sp(1) \hookrightarrow Sp(2)$ and here $Sp(1)$ is the group of unit quaternions. The map between spheres is (up to homotopy) the suspension of the Hopf map $S^3 \to S^2$, which is not a priori a smooth map, and the inclusion is $q \mapsto \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$. Note that this implies that FS^5 lifts to an $Sp(1)$-bundle, and this is what we shall assue without further comment.

The first task is then to write down a smooth, non-null-homotopic smooth map $S^4 \to Sp(1)$. We shall use quaternionic coordinates on $S^4 = \mathbb{HP}^1$, that is, homogeneous coordinates $[p;q]$ where at least one of p, q is non-zero.

Proposition: The smooth function

$$T[p;q] = \frac{2p\bar{q}i\bar{p}q - |p|^4 + |q|^4}{|p|^4 + |q|^4} \quad (\in Sp(1))$$

represents the non-trivial class of $\pi_4(Sp(1))$, and hence is the transition function for FS^5.

Now we want to shift perspective a little bit, and note that the function T gives rise to a smooth *functor* from the Čech groupoid $U \times_{S^5} U \rightrightarrows U$ over S^5 coming from the open cover by two discs[2] $U := D_+ \coprod D_- \to S^5$. For future notational convenience, write $U^{[2]} = U \times_{S^5} U$.

Since we now have an explicit Čech cocycle (this is precisely what the above functor is) for FS^5, we can talk about lifting this to a Čech cocycle for the 2-group $String_{Sp(1)}$. But what *is* this? There are many models for String 2-groups, and we shall take the crossed module $(\widehat{\Omega Sp(1)} \to PSp(1))$, where $PSp(1)$ is the group of smooth paths $[0,1] \to Sp(1)$ based at $1 \in Sp(1)$, and $\widehat{\Omega Sp(1)}$ is the universal central extension of the subgroup $\Omega Sp(1) \subset PSp(1)$ of loops (Baez-Crans-Schreiber-Stevenson 2007). Note that the abstract details of what I'm considering doesn't rely on this choice of model. Notice that $(\widehat{\Omega Sp(1)} \to PSp(1))$ comes with a map to the crossed module $(1 \to Sp(1))$, and that the former gives rise to a groupoid (which I shall call $String(3)$, as $Sp(1) \simeq Spin(3)$), namely the action groupoid for $\widehat{\Omega Sp(1)}$ acting on $PSp(1)$ via the given homomorphism, and a 2-groupoid $\mathbf{B}String(3)$ with a single object (using the 2-group structure). More generally, we can repeat these constructions with any compact, simple, simply connected Lie group G to get a 2-group $String_G$. Also, given an inclusion of Lie groups[3] $H \to G$ gives an inclusion of

[2] One should take these as open discs, and so the intersection would be $S^4 \times (-\varepsilon, \varepsilon)$; we extend T to this slightly larger subspace by taking it constant in the direction of the interval.

[3] that induces an isomorphism $H^3(G,\mathbb{Z}) \to H^3(H,\mathbb{Z})$; the examples listed below all satisfy this, as can be calculated via the long-exact sequence in homotopy.

Lie 2-groups $String_H \to String_G$.

In the Čech groupoid $U^{[2]} \rightrightarrows U$ we don't have $U^{[2]}$ a disjoint union of contractible opens, so we take an open cover $V \to U^{[2]}$ where V *is* such a disjoint union (or, at least, acyclic enough). Since the non-trivial part of $U^{[2]}$ is $D_+ \cap D_- \sim \mathbb{HP}^1$, we will take V to be the two $\mathbb{H}$ charts $\mathbb{H}_+$ and $\mathbb{H}_-$ given by non-vanishing of each of the two homogenous coordinates. Then if we take the fibred product $V^{[2]} = V \times_{U^{[2]}} V$ we get a Lie 2-groupoid $V^{[2]} \rightrightarrows V \rightrightarrows U$, which I call a *truncated globular hypercover*.[4] The nontrivial component of $V^{[2]}$ (it contains boring bits like D_+) is the intersection $\mathbb{H}_+ \cap \mathbb{H}_- = \mathbb{H}^\times$. Notice that if we wanted to use a good open cover then U would necessarily have had more open sets, and so more overlaps. In some sense we have made a trade-off in the number of open sets and the slight increase in complexity of the description. Also, we can finally see where the two sorts of indices in the cocycle equation above come from: the indices $i, j, \ldots$ label open sets appearing in U, and the indices $\alpha, \beta, \ldots$ label the open sets appearing in V.

So, finally, a Čech cocycle on S^5 with values in $String(3)$ is 'just' a 2-functor

$$(V^{[2]} \rightrightarrows V \rightrightarrows U) \to \mathbf{B}String(3).$$

If we break this down, it is determined by components

$$\begin{aligned} &V \to PSp(1) \\ &V^{[2]} \to \Omega\widehat{PSp}(1) \end{aligned}$$

and since we have so few open sets in the globular hypercover, functoriality follows automatically. In our particular case, we want the first map to lift the given $V \to U^{[2]} \to Sp(1)$.

Recall that V is (essentially) $\mathbb{H}_+ \coprod \mathbb{H}_-$, we define the lift in two parts:

$$\begin{aligned} T_+(q) &= \left(s \mapsto \frac{|q|^4 - s^2 + 2\bar{q}iq}{|q|^4 + s^2} \right) \\ T_-(p) &= \left(s \mapsto \frac{|p|^4 s^2 - 1 + 2\bar{p}ip}{|p|^4 s^2 + 1} \cdot \left(\frac{s-i}{s+i} \right)^2 \right) \end{aligned}$$

To define the remaining component of the 2-functor, we first take the difference of these two maps to get a function $\mathbb{H}^\times \to \Omega Sp(1)$

$$T_\Omega(q) = \left(s \mapsto \frac{(s+Q)(sQ-1)}{(s-Q)(sQ+1)} \cdot \left(\frac{s-i}{s+i} \right)^2 \right), \quad \text{where } Q = \bar{q}iq.$$

Now we need to lift this map through the projection $\widehat{\Omega Sp(1)} \to \Omega Sp(1)$ (this is not a priori possible, but one calculates the possible obstructions and they vanish). To do this, we need a workable description of what $\widehat{\Omega Sp(1)}$ is. There are

[4] This may look familiar if you've seen bundle 2-gerbes before.

multiple papers constructing this e.g. Mickelsson, Murray, Murray-Stevenson. We shall use the description of it as the quotient group

$$\frac{P\Omega Sp(1) \rtimes U(1)}{\widetilde{\Omega^2 Sp(1)}}$$

The precise embedding of the simply-connected covering group $\widetilde{\Omega^2 Sp(1)}$ is not important, just that we can represent elements as equivalence classes of pairs consisting of paths in $\Omega Sp(1)$ and elements of $U(1)$.

One calculates the final answer to be as follows. For any $q \in \mathbb{H}^\times$, let q_t be any path (in $\mathbb{H}^\times$) $1 \rightsquigarrow q$, and the lift to the central extension is

$$T_{\widehat{\Omega}}(q) = [T_\Omega(q_t), 1].$$

This is independent of the choice of path and is smooth. This function, together with $T_\pm$, defines the Čech cocycle we are interested in. We know that this cocycle is not a coboundary, since geometrically realising everything we get a map $S^5 \to BString(3)$ that picks out the nontrivial class in $\pi_5(BString(3)) \simeq \pi_5(BSpin(3)) \simeq \pi_4(Spin(3)) \simeq \pi_4(Sp(1)) = \mathbb{Z}/2\mathbb{Z}$. One can also check (easily, as there are so few open sets involved in the open covers), that these functions satisfy the cocycle equations displayed at the beginning of the notes.

Now this is just one example, and a pretty exceptional example at that, as the dimensions involved are right on the boundary of where the obstructions vanish, not to mention the use of quaternions. One can take a more global approach that leads to many more examples as follows. The total space of the frame bundle FS^5, as an $Sp(1)$-bundle, is nothing other than the homogenous bundle $SU(3) \to SU(3)/Sp(1) = S^5$, using the embedding $Sp(1) \simeq SU(2) \to SU(3)$ as a block matrix. One can calculate that $String_{SU(3)}/String(3) \simeq SU(3)/Sp(1)$, so that the underlying groupoid of $String_{SU(3)}$ is the 'total space' of the $String(3)$ bundle. Another way to view this is to consider the transitive $String_{SU(3)}$ action on S^5 via the projection to $SU(3)$; then $String(3)$ is the stabiliser of the basepoint.

This picture generalises to any $String_G$ acting on G/H for $H < G$, and at this point we can use any model of $String_G$, including non-strict models, and even 2-groups in differentiable stacks, which have underlying Lie groupoids. There are a number of interesting exceptional examples which should be amenable to the same treatment as above, for instance:

- $String_{G_2} \to G_2/SU(3) = S^6$
- $String_{Spin(7)} \to Spin(7)/G_2 = S^7$
- $String_{Sp(2)} \to Sp(2)/Sp(1) = S^7$
- $String_{F_4} \to F_4/Spin(9) = \mathbb{OP}^2$

The first three of these have explicit transition functions written down by Püttmann (arXiv:1101.5147). $\mathbb{OP}^2$ admits a cover by three $\mathbb{R}^{16}$ charts, and is 7-connected.

Exercise: write down transition functions for the $Spin(9)$ bundle on $\mathbb{OP}^2$, and lift them to $String(9) = String_{Spin(9)}$-valued transition functions using a globular hypercover.

The astute reader will have realised that this method only gives a single example on each homogeneous space with that particular structure group, which in the case of S^5 is ok as there is only one nontrivial $String(3)$ bundle. But, for instance, $String_{SU(3)}$ bundles on S^6 are classified by an integer (and in fact the example above is a generator). However, using the Eckmann-Hilton argument, one can show that over a sphere S^{k+1}, given a G-bundle with transition function $t\colon S^k \to G$ representing a generator of $g \in \pi_k(G)$, we can obtain the transition functions for the bundles corresponding to elements g^n by taking the pointwise power $t^n\colon S^k \to G$ for any $n \in \mathbb{Z}$. The same will be true for the lifted 2-bundles, where we take pointwise powers of the 2-group-valued functor $(V^{[2]} \rightrightarrows V) \to String_H$. Thus, for spheres at least, we can in principle give Čech cocycle descriptions for all String bundles.

As a final note, the abstract picture in the penultimate paragraph is not restricted to smooth geometry: one can equally well take holomorphic 2-groups, assuming one has them. However, in current work with Raymond Vozzo we have found that the basic gerbe on a simple, simply-connected complex reductive Lie group, which is holomorphic (Brylinski 1994, 2000), is also multiplicative, so defines a weak 2-group in complex analytic stacks. This means we can define holomorphic String bundles on complex homogeneous spaces, which can be plugged into the higher twistor correspondence of Saemann-Wolf.