Higher Geometric Quantization

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Overview/Motivation

Interested in manifolds equipped with a closed, "non-degenerate" form of degree ≥ 2.

symplectic manifold (M, ω)	n**-plectic manifold** (M, ω)
ω closed, non-degen. **2-form**	ω closed, non-degen. $(n+1)$**-form**
$\omega \rightsquigarrow$ Lie algebra $(C^{\infty}, \{\cdot, \cdot\})$	$\omega \rightsquigarrow$ L_{∞}**-algebra**
classical mechanics	classical field theory

higher degree forms $\Rightarrow$ higher analogs of structures found in symplectic geometry

Overview/Motivation

How can we quantize 2-plectic manifolds?

What would be the interesting applications and examples?

1. Quantizing (sub-bundles of) $(\Lambda^2 T^* X, d\theta) \rightsquigarrow (1+1)$-QFTs.

2. Representation theory:
   - $G =$ compact simple Lie group, $\nu_k = \frac{k}{12\pi}\langle \theta_L, [\theta_L, \theta_L]\rangle$.
   - "The main open question seems to be to obtain the representation theory of LG from the canonical sheaf of groupoids on G." (Brylinski 1993)
   - Relationship with quantization of quasi-Hamiltonian G-spaces? (Meinrenken 2009)
   - Relationship with bundle gerbe approach to fusion rules? (Runkel-Suszek 2011)

Geometric Quantization via Bohr-Sommerfeld

Śniatyki (1977), Guillemin-Sternberg (1983)

Start with **integral** symplectic manifold (M, ω):

$$[\omega] \in \operatorname{im}\big(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})\big)$$

1. Choose **prequantization**:

   (M, ω, L, ∇) with $\operatorname{curv}(\nabla) = i\omega$,

2. Choose **polarization** F of M:

   A (singular) foliation F of M s.t. the regular leaves are Lagrangian.

3. Construct **Bohr-Sommerfeld variety**:

   V_{BS} = union of all Λ of F s.t. $L|_\Lambda$ admits global non-vanishing section σ_Λ with $\nabla\sigma_\Lambda = 0$.

4. Construct **quantum state space:**

$$\text{Quant} = \bigoplus_{\Lambda \subseteq V_{BS}} \mathbb{C} \cdot \sigma_\Lambda$$

Prequantization for symplectic via Deligne cohomology

Weil (1958), Kostant (1970), Brylinski (1993)

Let $D_1^\bullet := \underline{\mathrm{U}(1)}_M \xrightarrow{dlog} \Omega^1_M$.

Get exact sequence:

$$0 \to H^1(M, \mathrm{U}(1)) \to H^1(M, D_1^\bullet) \to \Omega^2_{closed}(M) \to H^2(M, \mathrm{U}(1)).$$

Principal $\mathrm{U}(1)$-bundles with connection on integral symplectic manifold (M, ω) are classified by $H^1(M, D_1^\bullet)$.

We **prequantize** (M, ω) by choosing a cocycle representing a class in $H^1(M, D_1^\bullet)$:

$$\text{good cover } \{U_i\}, \quad g_{ij} \colon U_i \cap U_j \to \mathrm{U}(1), \quad \theta_i \in \Omega^1(U_i)$$
$$\text{s.t. } (\delta g)_{ijk} = 1, \quad \sqrt{-1}(\theta_j - \theta_i) = d \log g_{ij}, \quad d\theta_i = \omega|_{U_i}$$

Pre-quantum line bundle: $L = P \times_{\mathrm{U}(1)} \mathbb{C}, \quad \nabla_i = d + \sqrt{-1}\,\theta_i$.

Prequantization for 2-plectic via Deligne cohomology

Brylinski (1993)

Let $D_2^\bullet := \underline{\mathrm{U}(1)}_M \xrightarrow{dlog} \Omega^1_M \xrightarrow{d} \Omega^2_M$.

Get exact sequence:

$$0 \to H^2(M, \mathrm{U}(1)) \to H^2(M, D_2^\bullet) \to \Omega^3_{closed}(M) \to H^3(M, \mathrm{U}(1)).$$

$H^2(M, D_1^\bullet)$ classifies $\mathrm{U}(1)$-**gerbes with 2-connection**.

We **prequantize** an integral 2-plectic manifold (M, ω) by choosing a cocycle representing a class in $H^2(M, D_2^\bullet)$:

$$\text{good cover } \{U_i\}, \quad h_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{U}(1),$$
$$A_{ij} \in \Omega^1(U_i \cap U_j), \quad B_i \in \Omega^2(U_i)$$
$$\text{s.t. } (\delta h)_{ijkl} = 1, \quad \sqrt{-1}(A_{jk} - A_{ik} + A_{ij}) = d \log h_{ijk},$$
$$B_j - B_i = dA_{ij}, \quad dB_i = \omega|_{U_i}$$

Pre-quantum “2-line bundle” ?

$\mathrm{U}(1)$-gerbes and 2-bundles

Bartels (2004), Baez-Schreiber(2007), Wockel (2011)

A $\mathrm{U}(1)$-gerbe $\mathcal{G}$ on M is a stack locally isomorphic to $\mathcal{B}\mathrm{U}(1)$, the stack of principal $\mathrm{U}(1)$-bundles over M. They are classified by $H^2(M, \underline{\mathrm{U}(1)})$.

We can think of $\mathcal{G}$ as the "sheaf of sections" of a **principal 2-bundle** on M.

A **smooth 2-space** is a small category C s.t. C_0, C_1, and $\mathrm{C}_1 {}_s\times_t \mathrm{C}_1$ are smooth manifolds, and all structure maps are smooth.

Morphisms between smooth 2-spaces are **smooth functors**, and 2-morphisms are **smooth natural transformations**.

A **strict Lie 2-group** is a 2-space equipped with a smooth strict monoidal structure s.t. all objects and morphisms are invertible.

Main example: $\mathbf{BU(1)} = \quad \mathrm{U}(1) \rightrightarrows *$.

Principal 2-bundles

Baez-Schreiber(2007), Wockel (2011)

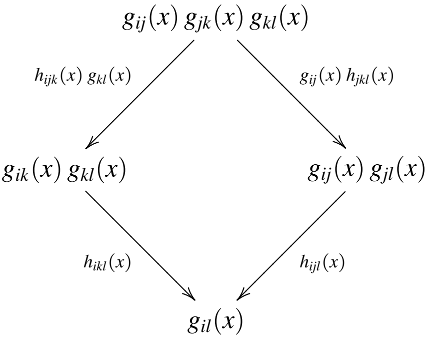
Let G be a Lie 2-group. A **principal** G **2-bundle** over a manifold M is a smooth G 2-space P equipped with smooth functor $\pi\colon$ P $\to M$ s.t. there exists a (good) open cover $\{U_i\}$ of M and *equivalences* of G 2-spaces $\tau_i\colon \mathrm{P}_{U_i} \to U_i \times \mathrm{G}$ with $\pi|_{U_i} = \mathrm{pr}_1 \circ \tau_i$.

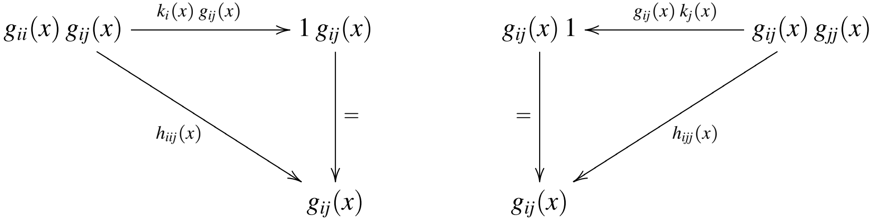
A G**-valued 2-cocycle** on M consists of the following data:

- a good cover $\{U_i\}_{i\in I}$ of M,
- smooth functors $g_{ij}\colon U_i \cap U_j \to \mathrm{G}$ for all $i,j \in I$,
- smooth natural isomorphisms $h_{ijk}\colon g_{ij} \cdot g_{jk} \Rightarrow g_{ik}$ for all $i,j,k \in I$,
- smooth natural isomorphisms $k_i\colon g_{ii} \Rightarrow 1_G$ for all $i \in I$,

such that for all $x \in U_{ijkl}$:

Principal 2-bundles





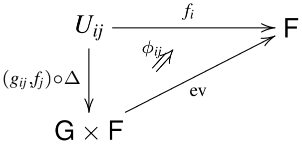
Sections of 2-bundles

If $\mathsf{G} = \mathrm{BU}(1)$, then the functors g_{ij} are trivial, and h_{ijk} becomes a $\underline{\mathrm{U}(1)}$-valued Čech 2-cocycle.

Theorem (Bartels, Wockel): Principal $\mathrm{BU}(1)$ 2-bundles are classified by $H^2(M, \underline{\mathrm{U}(1)})$.

If the Lie 2-group G acts via automorphisms on a 2-space F, then a G-valued 2-cocycle $(U_i, g_{ij}, h_{ijk}, k_i)$ can be used to build an **associated 2-bundle** E $\to M$ whose typical fiber is F (Bartels 2004).

A **global section** of E is a collection of functors $f_i\colon U_i \to$ F, and natural isomorphisms ϕ_{ij}



such that:

Sections of 2-bundles

$$
\begin{array}{ccc}
g_{ij}(x)f_j(x) & \xrightarrow{\phi_{ij}} & f_i(x) \\
{\scriptstyle g_{ij}\phi_{jk}^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \phi_{ik}^{-1}} \\
g_{ij}(x)g_{jk}(x)f_k(x) & \xrightarrow[h_{ijk}]{} & g_{ik}(x)f_k(x)
\end{array}
\qquad\qquad
\begin{array}{ccc}
g_{ii}(x)f_i(x) & \xrightarrow{\phi_{ii}} & f_i(x) \\
& {\scriptstyle k_i}\searrow & \big\downarrow{\scriptstyle =} \\
& & f_i(x)
\end{array}
$$

A **morphism** between global sections $\{f_i\colon U_i \to \mathsf{F}\} \to \{f_i'\colon U_i \to \mathsf{F}\}$ consists of natural transformations $\alpha_i\colon f_i \Rightarrow f_i'$ which "intertwine" ϕ_{ij} and ϕ_{ij}'. Hence **global sections of** E **form a category.**

To summarize the story so far:

- We prequantize an integral 2-plectic manifold (M, ω) by equipping it with a Deligne 2-cocycle, giving us $\mathrm{BU}(1)$ 2-cocycle (plus a 2-connection).
- We know how to construct the category of global sections of an associated 2-bundle.
- Hence, we just need to find a 2-space which"categorifies" $(\mathbb{C}, \langle\cdot,\cdot\rangle)$.

Hilb categorifies $(\mathbb{C}, \langle\cdot,\cdot\rangle)$

Baez, HDA2: 2-Hilbert Spaces (1997)

Let Hilb denote the category whose objects are fin. dim. Hilbert spaces, and whose morphisms are linear maps.

$$\text{Hilb} \quad \simeq \quad \coprod_{n,m} \mathrm{Mat}_{\mathbb{C}}(n \times m) \rightrightarrows \mathbb{N}$$

$\mathbb{C}$	Hilb
$+, -, \times$	$\oplus, \mathrm{coker}, \otimes$
generated by 1	generated by $\mathbb{C}$
$\langle\cdot,\cdot\rangle : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$	$\mathrm{Hom}(\cdot,\cdot) :$ Hilb $\times$ Hilb $\to$ Hilb
$\mathrm{U}(1) \circlearrowleft \mathbb{C}$	$\mathrm{BU}(1) \circlearrowleft$ Hilb

Smooth action of $\mathrm{BU}(1)$:

$$\mathrm{BU}(1) \times \text{Hilb} \longrightarrow \text{Hilb}$$

$$\left(* \overset{g}{\curvearrowright} * \,, \quad n \xrightarrow{(A_{ij})} m \right) \mapsto \; n \xrightarrow{g\cdot(A_{ij})} m$$

Global sections of a “2-line bundle”

Given a Čech 2-cocycle $h_{ijk}\colon U_i \cap U_j \cap U_k \to U(1)$ on a good cover of M, we obtain a $\mathrm{BU}(1)$-valued 2-cocycle $(U_i, g_{ij}, h_{ijk}, k_i)$.

A global section of the assoc. 2-line bundle $\mathsf{L} \to M$:

$$\{f_i\colon U_i \to \text{Hilb}\} \rightsquigarrow \text{vector bundles } \{E_i = U_i \times \mathbb{C}^n \to U_i\}.$$

$$\begin{array}{ccc} U_{ij} & \xrightarrow{\quad f_i \quad} & \text{Hilb} \\ {\scriptstyle (g_{ij},f_j)\circ\Delta}\downarrow & \overset{\phi_{ij}}{\Nearrow} & \nearrow{\scriptstyle \mathrm{ev}} \\ \mathrm{BU}(1) \times \text{Hilb} & & \end{array} \quad \rightsquigarrow \quad \phi_{ij}\colon E_j|_{U_{ij}} \xrightarrow{\sim} E_i|_{U_{ij}}$$

$$\begin{array}{ccc} g_{ij}(x) f_j(x) & \xrightarrow{\quad \phi_{ij} \quad} & f_i(x) \\ {\scriptstyle g_{ij}\phi_{jk}^{-1}}\downarrow & & \downarrow{\scriptstyle \phi_{ik}^{-1}} \\ g_{ij}(x) g_{jk}(x) f_k(x) & \xrightarrow[\quad h_{ijk} \quad]{} & g_{ik}(x) f_k(x) \end{array} \quad \rightsquigarrow \quad \phi_{ik}^{-1} \circ \phi_{ij} \circ \phi_{jk} = \operatorname{diag}(\underbrace{h_{ijk}, \ldots, h_{ijk}}_{n})$$

Twisted vector bundles

Prop: The category of global sections of L $\to M$ is equivalent to the category of (h_{ijk})-twisted complex vector bundles on M.

Now add connection data:

Let $\xi = \big(\{U_i\}, h_{ijk}, A_{ij}, B_i\big)$ be a Deligne 2-cocycle on M. A $\boldsymbol{\xi}$**- twisted vector bundle with connection** over M consists of the following data:

- $\mathbb{C}$-vector bundles with connection: $(E_i \to U_i, \nabla_i)$,
- isomorphisms: $\phi_{ij} \colon E_j|_{U_{ij}} \xrightarrow{\sim} E_i|_{U_{ij}}$ s.t. $\phi_{ij}\nabla_j - \nabla_i\phi_{ij} = \sqrt{-1} \cdot A_{ij} \otimes \phi_{ij}$
- $\phi_{ik}^{-1} \circ \phi_{ij} \circ \phi_{jk} = \mathrm{diag}(\underbrace{h_{ijk}, \ldots, h_{ijk}}_{n})$ on U_{ijk}.

A **morphism** $(E_i, \nabla_i, \phi_{ij}) \xrightarrow{\{f_i\}} (E'_i, \nabla'_i, \phi'_{ij})$ consists of maps:

$$f_i \colon (E_i, \nabla_i) \to (E'_i, \nabla'_i) \text{ s.t. } f_i \circ \phi_{ij} = \phi'_{ij} \circ f_j.$$

Brylinski (1998), B-C-M-M-S (2002), Karoubi (2010).

Prequantum 2-line stack

Prop: If (M, ω) is an integral 2-plectic manifold equipped with a Deligne 2-cocycle ξ with $\mathrm{curv}([\xi]) = \omega$, then there exists a stack $\mathcal{L}^\xi$ over M whose category of global sections $\mathcal{L}^\xi(M)$ is equivalent to the category of ξ-twisted vector bundles on M.

The **pre-quantum category** $\mathcal{L}^\xi(M)$ has the structure of a Hilb**-module** in the sense of Yetter's "Categorical linear algebra":

$$\mathrm{Hilb} \times \mathcal{L}(M) \to \mathcal{L}(M)$$

$$\big(n, (E_i, \nabla_i, \phi_{ij})\big) \mapsto \big(\underbrace{E_i \oplus \cdots \oplus E_i}_{n}, \underbrace{\nabla_i \oplus \cdots \oplus \nabla_i}_{n}, \underbrace{\phi_{ij} \oplus \cdots \oplus \phi_{ij}}_{n}\big).$$

Remark: If $\mathcal{L}^\xi(M)$ admits a non-trivial section, then characteristic class of $\xi = (h_{ijk}, A_{ij}, B_i)$ must be torsion.

Bohr-Sommerfeld relative cohomology

(M, ω) is **symplectic**

Recall: A **polarization** is a (singular) foliation F of M s.t. the regular leaves are Lagrangian.

Given an embedding of a leaf $\Lambda \xrightarrow{f} M$, we can consider the **relative Deligne cohomology** $H^1(\Lambda, M; D_1^\bullet)$.

A cocycle (ζ, ξ) representing a class in $H^1(\Lambda, M; D_1^\bullet)$ corresponds to a line bundle with connection $(L \to M, \nabla)$ equipped with a global non-vanishing section σ_Λ of the **pullback bundle** $f^*(L, \nabla) \to \Lambda$.

The **curvature** of $[(\zeta, \xi)]$ is a relative closed 2-form (η_Λ, ω) i.e. $d\eta_\Lambda = f^*\omega$, $d\omega = 0$, and

$$\mathrm{curv}(L) = i\omega, \quad \nabla\sigma_\Lambda = i\eta_\Lambda \otimes \sigma_\Lambda$$

The curvature is **integral**:

$$\int_\gamma \eta_\Lambda - \int_\Sigma \omega \in \mathbb{Z}$$

for all chains $\gamma\colon \Delta^1 \to \Lambda$, $\Sigma\colon \Delta^2 \to M$, with $\partial\gamma = 0, f(\gamma) = \partial\Sigma$.

Bohr-Sommerfeld via relative cohomology

(M, ω) is **symplectic**

Recall: V_{BS} = union of all Λ of F s.t. $L|_\Lambda$ admits global non-vanishing section σ_Λ with $\nabla \sigma_\Lambda = 0$.

Prop: If $\Lambda \subseteq V_{BS}$, then $(\sigma_\Lambda, L, \nabla)$ represents a class in $H^1(\Lambda, M; D_1^\bullet)$ with curvature $(0, \omega)$.

Remark: If M is simply-connected, then $\Lambda \subseteq V_{BS}$ iff for all 1-cycles $\gamma \colon \Delta^1 \to \Lambda$

$$\int_\Sigma \omega \in \mathbb{Z} \quad \text{with } f(\gamma) = \partial \Sigma.$$

A possible generalization?

A **generalized polarization** of (M, ω) is a (singular) foliation F of M s.t. each leaf Λ is equipped with a 1-form η_Λ satisfying $d\eta_\Lambda = \omega|_\Lambda$.

Twisted Hermitian line bundles

Let $\xi = \big(\{U_i\}, h_{ijk}, A_{ij}, B_i\big)$ be a Deligne 2-cocycle on an arbitrary manifold M.

Interested in particular sections of $\mathcal{L}^\xi$:

A ξ-**twisted Hermitian line bundle** with connection over M is a collection of Hermitian line bundles with connection $(L_i, \nabla_i) \to U_i$, with isomorphisms $\phi_{ij} \colon L_j \xrightarrow{\sim} L_i$ s.t. $\phi_{ij}\nabla_j - \nabla_i\phi_{ij} = \sqrt{-1} \cdot A_{ij} \otimes \phi_{ij}$, and $\phi_{ik}^{-1} \circ \phi_{ij} \circ \phi_{jk} = h_{ijk}$.

The 2-form $\sqrt{-1}Q = \mathrm{curv}(\nabla_i) + \sqrt{-1}B_i$ is globally well-defined on M and is called the **twisted curvature** of $\sigma = (L_i, \nabla_i, \phi_{ij})$.

Note: Twisted Hermitian line bundles are the sections of $\mathcal{L}^\xi$ which **trivialize the corresponding gerbe** $\mathcal{G}$.

Relative Deligne cohomology for 2-plectic

Shahbazi (2005)

Let (M, ω) be 2-plectic.

Given a map $N \xrightarrow{f} M$, we can consider the **relative Deligne cohomology** $H^2(N, M; D_2^\bullet)$.

A cocycle (ζ, ξ) representing a class in $H^2(N, M; D_2^\bullet)$ corresponds to a gerbe $\mathcal{G} \to M$ with 2-connection equipped with a twisted Hermitian line bundle σ_N in the category $\mathcal{L}^{f^*\xi}(N)$.

The **curvature** of $[(\zeta, \xi)]$ is a relative closed 3-form (Q_N, ω) i.e. $dQ_N = f^*\omega$, $d\omega = 0$, and

$$\mathrm{curv}(\mathcal{G}) = i\omega, \quad \mathrm{Twcurv}(\sigma_N) = iQ_N.$$

The curvature is **integral**:

$$\int_\Gamma Q_N - \int_\Sigma \omega \in \mathbb{Z}$$

for all chains $\Gamma \colon \Delta^2 \to N$, $\Sigma \colon \Delta^3 \to M$, with $\partial\Gamma = 0, f(\Gamma) = \partial\Sigma$.

Bohr-Sommerfeld for 2-plectic

Let (M, ω) be a 2-plectic manifold. A **polarization** of (M, ω) is a (singular) foliation F of M s.t. each leaf $\iota\colon \Lambda \hookrightarrow M$ is equipped with a 2-form Q_Λ satisfying $dQ_\Lambda = \iota^*\omega$.

Let (M, ω, ξ, F) be a 2-plectic manifold equipped with Deligne 2-cocycle ξ (whose curvature is ω) and a polarization F.

The **Bohr-Sommerfeld variety** V_{BS} is the union of all leaves Λ of F s.t. the stack $\mathcal{L}^{\iota^*\xi}$ admits a global twisted Hermitian line bundle σ_Λ with twisted curvature iQ_Λ.

Remark: If $H_2(M, \mathbb{Z}) = 0$, then $\Lambda \subset V_{BS}$ iff (Q_Λ, ω) is integral.

The quantum state category Quant

Def: Let C be a category equipped with a Hilb-module structure $(n, x) \mapsto nx$. Let S be a collection of objects of M. The **sub-module of** C **generated by** S is the full subcategory of C consisting of all objects that are isomorphic to a finite product of objects of the form $a_1x_1, a_2x_2, \ldots, a_nx_n$, where each a_i is an object of Hilb and each x_i is in S.

For a quantized 2-plectic manifold, the Hilb-module **Quant** is the submodule of $\mathcal{L}(V_{BS})$ generated by the twisted Hermitian line bundles $\{\sigma_\Lambda\}_{\Lambda \subset V_{BS}}$ i.e.

$$\text{Quant} = \bigoplus_{\Lambda \subseteq V_{BS}} \text{Hilb} \cdot \sigma_\Lambda$$

Example: $\mathrm{SU}(2)$

- $M = \mathrm{SU}(2)$, $\omega_k = \frac{k}{12\pi}\langle\theta_L, [\theta_L, \theta_L]\rangle$
- Prequantize (M, ω_k) with canonical gerbe $\mathcal{G}_k$
- Polarization given by conjugacy classes C_λ equipped with 2-form Q_λ, and $\lambda \cdot w \in A^k_{\mathrm{Weyl}} = [0, k] \cdot w$
- Q_λ is in V_{BS} if and only if $\lambda \cdot w$ is an integral weight.
- Isomorphism classes of Quant are correspond to f.d. representations of $\mathfrak{su}(2)$ which decompose into irreps. V_λ with $\lambda = 0, 1, 2, \ldots, (k-1)$.