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Arrangements of hyperplanes and Lie algebra homology

Vadim V. Schechtman [1]* **and Alexander N. Varchenko** [2]*

[1] Institute of Problems of Microelectronics Technology and High Purity Materials, Chernogolovka, Moscow region 142432, USSR
[2] Moscow Institute of Gas and Oil, Leninsky Prospekt 65, Moscow 117917, USSR

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Introduction

The paper is devoted to the study of cohomology of one-dimensional local systems over complements of hyperplanes in complex affine spaces.

It consists of two parts. The first part (Sects. 1–4) contains several results concerning arbitrary affine arrangements. It may be considered as the continuation of the theme initiated by Arnold, Brieskorn, Orlik-Solomon,... ([Br, OS, C]). In the second part (Sects. 5–7), which contains main results of the paper, we study arrangements of a special kind ("discriminantal" ones). We show that cohomology of certain local systems over them are closely connected with homology of nilpotent subalgebras of Kac-Moody type Lie bialgebras. The Gauss-Manin connection which arises in natural variations of these arrangements is given by Knizhnik-Zamolodchikov differential equations [KZ] first appeared in Conformal field theory. The main result of the paper yields (under certain

* *Current address:* Institute for Advanced Study, School of Mathematics, Princeton NJ 08540, USA

conditions) the complete set of solutions of the last equations in terms of generalized hypergeometric integrals.

0.1 Contents of Part I. Let $\mathscr{C}$ be a finite arrangement, i.e. collection of affine hyperplanes H in a complex affine space V. We put $Y(\mathscr{C})=\bigcup_{H\in\mathscr{C}} H$; $U(\mathscr{C})=V-Y(\mathscr{C})$. Denote by $\Omega^{\cdot}(U(\mathscr{C}))$ the de Rham complex of holomorphic forms over U. Let $\mathscr{A}^{\cdot}(\mathscr{C})_{\mathbb{Z}}\subset\Omega^{\cdot}(U(\mathscr{C}))$ be the DG-subalgebra over $\mathbb{Z}$ generated by forms $\iota(H)=d\log f_H$, $H\in\mathscr{C}$, where $f_H=0$ is an equation of H. By [Br] the above inclusion induces the isomorphism $\mathscr{A}^p(\mathscr{C})_{\mathbb{Z}}=H^p(U(\mathscr{C});(2\pi i)^p\mathbb{Z})$. Abusing the language, we'll denote elements $\iota(H_1)\cdot\iota(H_2)\cdot\ldots\cdot\iota(H_p)$ simply by $H_1\cdot\ldots\cdot H_p$.

According to Orlik-Solomon, [OS], relations in $\mathscr{A}^{\cdot}(\mathscr{C})_{\mathbb{Z}}$ admit a nice description in terms of the combinatorics of $\mathscr{C}$. In Sects. 1, 2 we review general algebraic facts about $\mathscr{A}^{\cdot}(\mathscr{C})_{\mathbb{Z}}$. Most of them are known.

We reprove (Theorem 1.6.5) a theorem of Bjorner stating that all groups $\mathscr{A}^p(\mathscr{C})_{\mathbb{Z}}$ are free and giving natural bases of them. We put $\mathscr{A}^{\cdot}(\mathscr{C})=\mathscr{A}^{\cdot}(\mathscr{C})_{\mathbb{Z}}\otimes\mathbb{C}$[1]. Introduce a differential d^*: $\mathscr{A}^{\cdot}(\mathscr{C})_{\mathbb{Z}}\to\mathscr{A}^{\cdot-1}(\mathscr{C})_{\mathbb{Z}}$ by formula $d^*(H_1\cdot\ldots\cdot H_p)=\sum(-1)^i H_1\ldots\hat{H}_i\ldots H_p$ if $H_1,\ldots,H_p$ are in general position, and zero otherwise. The only less standard result of § 1 is Theorem 1.7.2 that says that the complex $(\mathscr{A}^*(\mathscr{C})_{\mathbb{Z}}, d^*)$ may have only the top non-zero homology.

In Sect. 2 we introduce the *flag complex* $(\mathscr{F}^{\cdot}(\mathscr{C})_{\mathbb{Z}}, d)$. The group $\mathscr{F}^p(\mathscr{C})_{\mathbb{Z}}$ may be defined simply as the dual group $\mathscr{A}^p(\mathscr{C})^*_{\mathbb{Z}}$. Let us call an *edge* of $\mathscr{C}$ any non-empty intersection of its hyperplanes. Denote by $\mathscr{C}^i$ the set of all edges of codimension i. We show that $\mathscr{F}^p(\mathscr{C})_{\mathbb{Z}}$ may be defined as groups which have as generators all flags of $\mathscr{C}$: $L^1\supset\ldots\supset L^p$, $L^i\in\mathscr{C}^i$, subject to certain relations. d: $\mathscr{F}^p(\mathscr{C})_{\mathbb{Z}}\to\mathscr{F}^{p+1}(\mathscr{C})_{\mathbb{Z}}$ is adjoint to d^*. We put $\mathscr{F}^{\cdot}(\mathscr{C})=\mathscr{F}^{\cdot}(\mathscr{C})_{\mathbb{Z}}\otimes\mathbb{C}$.

Suppose that a map a: $\mathscr{C}\to\mathbb{C}$ is given. We call such a map a *collection of exponents* for $\mathscr{C}$. Put $\omega(a)=\sum_{H\in\mathscr{C}} a(H)H$. Define the differential $d=d(a)$: $\mathscr{A}^{\cdot}(\mathscr{C})\to\mathscr{A}^{\cdot+1}(\mathscr{C})$ to be the left multiplication by $\omega(a)$. In Sect. 3, which is the core of the first part, we introduce and study a certain map of complexes

$$S^{\cdot}=S^{\cdot}(a):\ (\mathscr{F}^{\cdot}(\mathscr{C}),d)\to(\mathscr{A}^{\cdot}(\mathscr{C}),d). \tag{0.1.1}$$

Namely, for an edge L put $S(L)=\sum a(H)H\in\mathscr{A}^1$, the sum taken over all $H\in\mathscr{C}$ containing L. The map S^p: $\mathscr{F}^p(\mathscr{C})\to\mathscr{A}^p(\mathscr{C})$ assigns to a flag $F=(L^1\supset\ldots\supset L^p)$ an element. $S^p(F)=S(L^1)\,S(L^2)\ldots S(L^p)$. Since $\mathscr{F}^p(\mathscr{C})=\mathscr{A}^p(\mathscr{C})^*$, S^p may be considered as a bilinear form on $\mathscr{F}^p(\mathscr{C})$. It is symmetric. Complexes $(\mathscr{F}^{\cdot}(\mathscr{C}), d)$, $(\mathscr{A}^{\cdot}(\mathscr{C}), d(a))$, and the form $S^{\cdot}$ are main personages of our story.

The first main result of Part I is Theorem 3.7 which calculates the determinant of $S^{\cdot}$. In particular, its zeroes are numbers $a(L):=\sum_{H\supset L} a(H)$.

In view of results of § 6, this theorem may be considered as a generalization of the Shapovalov determinant formula [Sh]

In Sect. 4 we clarify the topological meaning of objects studied in §§ 1–3. We show (Theorem 4.3) that there is a natural isomorphism $H^{\cdot}(\mathscr{F}^{\cdot}(\mathscr{C})_{\mathbb{Z}}, d)=H^{\cdot}(V, Y(\mathscr{C});\mathbb{Z})$.

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[1] This notation slightly differs from that of §§ 1–2

A collection of exponents a defines an integrable connection $\nabla(a)$ on the trivial one-dimensional bundle over $U(\mathscr{C})$ with the connection form $\sum_{H\in\mathscr{C}} a(H)d\log f_H$.

Denote by $\Omega^{\cdot}(\mathscr{L}(a))$ the corresponding holomorphic de Rham complex over $U(\mathscr{C})$. We have $H^{\cdot}(\Omega^{\cdot}(\mathscr{L}(a)))=H^{\cdot}(U(\mathscr{C});\mathscr{S}(a))$ where $\mathscr{S}(a)$ is the local system of horizontal sections of $\nabla(a)$. The inclusion $\mathscr{A}^{\cdot}(\mathscr{C})\hookrightarrow\Omega^{\cdot}(U(\mathscr{C}))$ is the map of complexes $(\mathscr{A}^{\cdot}(\mathscr{C}), d(a))\to\Omega^{\cdot}(\mathscr{L}(a))$. So we get the morphism

$$(0.1.2)\qquad \iota(a)\colon\ H^{\cdot}(\mathscr{A}^{\cdot}(\mathscr{C}), d(a))\to H^{\cdot}(U(\mathscr{C}),\mathscr{S}(a)).$$

Note that $\iota(0)$ is an isomorphism by Orlik-Solomon. The second main result of Part I is Theorem 4.6 which says that $\iota(\lambda a)$ is isomorphism for all $\lambda\in\mathbb{C}$ except for a discrete set, not containing zero. One may imagine the cohomology of $\mathscr{A}^{\cdot}(\mathscr{C})$ as the "quasiclassical" part of $H^{\cdot}(U(\mathscr{C}),\mathscr{S}(a))$. In n° 4.7 we show that (under certain assumptions) the map induced by $S^{\cdot}$ in cohomology is the "quasiclassical limit" of the canonical map from the locally finite (near $Y(\mathscr{C})$) to the ordinary cohomology of $\mathscr{S}(a)$.

Note that it would be interesting to study the Jantzen filtration on $\mathscr{A}^{\cdot}(\mathscr{C})$ induced by the form $S^{\cdot}(a)$ (see discussion in 4.8).

0.2 Contents of Part II. Denote by $\mathscr{C}_N$ the set of hyperplanes $H_{ij}\colon t_i-t_j=0$, $1\leqq i<j\leqq N$ in $\mathbb{C}^N$. Let $p_n^N\colon\mathbb{C}^{n+N}\to\mathbb{C}^n$ be the projection on the first n coordinates. For $z=(z_1,\dots,z_n)\in U(\mathscr{C}_n)$ denote by $\mathscr{C}_{n,N}(z)$ the arrangement induced by $\mathscr{C}_{n+N}$ in $(p_n^N)^{-1}(z)\cong\mathbb{C}^N$.

Let us fix the following data.

(a) A finite dimensional vector space $\mathfrak{h}$;
(b) a symmetric non-degenerate bilinear form $(\,,)$ on $\mathfrak{h}$;
(c) linearly independent functionals $\alpha_1,\dots,\alpha_r\in\mathfrak{h}^*$;
(d) arbitrary functionals $\Lambda_1,\dots,\Lambda_n\in\mathfrak{h}^*$;
(e) a non-zero complex number κ.

For every r-tuple of nonnegative integers $\lambda=(k_1,\dots,k_r)$, define a collection of exponents $a(\lambda)\colon\mathscr{C}_{n+N}\to\mathbb{C}$, where $N=\sum k_i$, as follows. To every coordinate t_i in $\mathscr{C}^{n+N}$ assign a covector $\alpha(t_i)\in\mathfrak{h}^*$: put $\alpha(t_i)=\Lambda_i$ if $1\leqq i\leqq n$; $\alpha(t_i)=-\alpha_j$ if $n+\sum_{p=1}^{j-1}k_p<i\leqq n+\sum_{p=1}^{j}k_p$.

Put $a(\lambda)(H_{ij})=(\alpha(t_i),\alpha(t_j))/\kappa$. (We induce a scalar product on $\mathfrak{h}^*$ by means of the isomorphism $\mathfrak{h}\cong\mathfrak{h}^*$ defined by $(\,,)$.) This defines also exponents on all $\mathscr{C}_{n;N}(z)$.

On the other hand, we assign to the data (a)–(c) a Lie algebra $\mathfrak{g}$ with generators f_i, e_i, $i=1,\dots,r$ and $\mathfrak{h}$ subject to Kac-Moody type relations, see 6.1. Roughly speaking, $\mathfrak{g}$ is a "Kac-Moody algebra without Serre relations". One has the decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ where $\mathfrak{n}_-$ (resp. $\mathfrak{n}_+$) denote the subalgebra generated by f_i (resp., e_i). $\mathfrak{n}_\pm$ are free. Similarly to [D, n° 3], one introduces a cobracket on $\mathfrak{g}$ (see 6.14) making it the Lie bialgebra.

A weight $\lambda\in\mathfrak{h}^*$ defines a Verma module $M(\Lambda)$ over $\mathfrak{g}$. As a $\mathfrak{n}_-$-module $M(\Lambda)$ is isomorphic to the enveloping algebra $U\mathfrak{n}_-$. Put

$M=M(\Lambda_1)\otimes\ldots\otimes M(\Lambda_n)$. The main result of § 5 is Theorem 5.13 which establishes natural isomorphisms of complexes

$$(0.2.1) \qquad C_{\bullet}(\mathfrak{n}_-, M)_\lambda \cong \mathscr{F}^{N-\bullet}(\mathscr{C}_{n;N}(z))^{\Sigma_\lambda}.$$

Here $\lambda=(k_1, \ldots, k_r)$ is as above; Σ_λ is the product of symmetric groups $\Sigma_{k_1}\times\ldots\times\Sigma_{k_r}$; it acts naturally on $\mathscr{F}^{\bullet}$, $\mathscr{A}^{\bullet}$, $(\cdot)^{\Sigma_\lambda}$ denotes invariants; $C_{\bullet}$ denotes the standard chain complex of a Lie algebra with coefficients in a module. For α $\mathfrak{g}$-module A A_λ denotes the component of weight $\Lambda-\Sigma k_i\alpha_i$: $\{x\in A \mid h\cdot x =\langle \Lambda-\Sigma k_i\alpha_i, h\rangle x$ for all $h\in\mathfrak{h}\}$, $\Lambda=\Sigma\Lambda_j$. Note that both sides of (0.2.1) and the isomorphism do not depend on data (a)–(d).

In § 6 one introduces certain symmetric bilinear form on $\mathfrak{g}$ and M, which induce the form on $C_{\bullet}(\mathfrak{n}_-, M)$. The main result is Theorem 6.6 which asserts that (0.2.1) maps this form to the form $S^{\bullet}$ on $\mathscr{F}^{\bullet}$ defined in § 3. We also define (6.15–6.17) on the dual space M^* a canonical structure of the module over the classical double $D(\mathfrak{b})=\mathfrak{b}\oplus\mathfrak{b}^*$ [D, n° 13] of the Borel subalgebra $\mathfrak{b}=\mathfrak{n}_-\oplus\mathfrak{h}$. (0.2.1) induces isomorphism of complexes

$$(0.2.2) \qquad C_{\bullet}(\mathfrak{n}_-^*, M^*)^\lambda \xrightarrow{\sim} \mathscr{A}^{N-\bullet}(\mathscr{C}_{n,N}(z))^{\Sigma_\lambda}.$$

(Theorem 6.16.2).

Here $\mathfrak{n}_-^*$ is considered as a Lie algebra by means of the above mentioned cobracket, and the structure of the $\mathfrak{n}_-^*$-module on M^* is the part of the above mentioned structure of the $D(\mathfrak{b})$-module.

We mention also Remarks 6.8.12 and 6.8.10 which allow to define explicitly the action of g on appropriate Orlik-Solomon spaces.

Note that results of §§ 5, 6 resemble the classical link between representations of semisimple Lie algebras and $\mathscr{D}$-modules over flag spaces (see for example, [BB]).

Knizhnik-Zamolodchikov equations

The canonical tensor $\Omega\in\mathfrak{b}\otimes\mathfrak{b}^*\oplus\mathfrak{b}^*\otimes\mathfrak{b}$ defines linear operators Ω on tensor products $M(\Lambda)^*\otimes M(\Lambda')^*$ through the $D(\mathfrak{b})$-module structure on $M(\Lambda^{(\prime)})^*$ (see 0.2).

Let Ω_{ij} denote the operator on $M^*=M(\Lambda_1)^*\otimes\ldots\otimes M(\Lambda_n)^*$ acting as Ω on $M(\Lambda_i)^*\otimes M(\Lambda_j)^*$ and as identity on other factors. They induce endomorphisms of $C_{\bullet}(\mathfrak{n}_-, M^*)$ which respect the weight decomposition. The main result of § 7 is Theorem 7.2.5 which asserts that the Gauss-Manin connection arising in groups $H^N(\mathscr{A}^{\bullet}_{n;N}(z))$ when a point $z\in U(\mathscr{C}_n)$ is moving, is mapped by (0.2.2) to the connection in a trivial bundle with a fiber $H_0(\mathfrak{n}_-^*, M^*)$ whose horizontal sections are given by the Knizhnik-Zamolodchikov system

$$(0.2.3) \qquad \frac{\partial}{\partial z_i}\eta(z)=\frac{1}{\kappa}\sum_{j\neq i}\frac{\Omega_{ij}\eta(z)}{z_i-z_j}, \qquad i=1, \ldots, n.$$

Theorem 4.6 mentioned in 0.1 says that for general κ integrals over covariantly constant relative cycles in $U(\mathscr{C}_{n+N})$ provide the complete space of solutions of (0.2.3).

0.3 Appearance of Lie bialgebras in the above considerations is not accidental. All this story is only the "quasiclassical" part of the whole picture. One may calculate the cohomology of the above local systems using cochains instead of differential forms. This will lead to the "quantization" of the picture. For example, $\mathfrak{g}$ will be replaced by its quantum deformation, [D]. On this way one may obtain an explicit version of Kohno's theorem on monodromy of KZ systems [K]. For details, see [SV 3].

0.4 Let us say a few words about the history of the subject. As we have already mentioned, the equations of type (0.2.3) first appeared in physics, as equations satisfied by correlation functions in certain models of Conformal field theory, [KZ]. Physicists first discovered that such correlation functions admit representation as generalized hypergeometric integrals (they call them "Feigin-Fuchs integrals") ([DF]). Our starting point was [CF] where such integral solutions of KZ equations with $\mathfrak{g}=\mathrm{sl}(2)$, and $n=4$ were written down. The case $\mathfrak{g}=\mathrm{sl}(2)$ and arbitrary n was studied in an interesting work [L]. In [SV 1, SV 2] we announced the solution for arbitrary Kac-Moody Lie algebras. Partial results in this direction were obtained also in [DJMM, M]. Note also an interesting work [Ch]. The present paper provides complete proofs of the results of [SV 1, SV 2] in a more general framework.

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Notations

$\mathbb{N}$ – the set of non-negative integers; for $n\in\mathbb{N}$, $[n]=\{1, \ldots, n\}$ ($[n]=\emptyset$ if $n=0$).
$\# I$ – the cardinality of a set I.
Σ_p – symmetric group of permutations of the set $[p]$; for $\sigma\in\Sigma_p$ $|\sigma|\in\mathbb{Z}/2\mathbb{Z}$ is the parity of σ.

For a k-vector space V, V^* denotes the dual vector space $\mathrm{Hom}_k(V, k)$. If A is an abelian group. $A^*:=\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$. □

Part I. Cohomology of local systems over complements of hyperplanes

1 Orlik-Solomon algebra

1.0 Let us fix an N-dimensional complex affine space V.

1.1 Let us call an *arrangement* a finite collection $\mathscr{C}$ of affine hyperplanes in V. An *edge* of $\mathscr{C}$ is a non-empty intersection of hyperplanes of $\mathscr{C}$; a *vertex* is a 0-dimensional edge. We'll denote by $\mathscr{C}^p$ (respectively, $\mathscr{C}_p$) the set of all edges of codimension p (resp., of dimension p). So $\mathscr{C}^1=\mathscr{C}$. We put $\mathscr{C}^0=\mathscr{C}_N=\{V\}$. We'll call $\mathscr{C}$ *central* if $\bigcap\limits_{H\in\mathscr{C}} H\neq\emptyset$.

1.2 Let us say that hyperplanes $H_1,\ldots,H_p$ are *in general position* if $\operatorname{codim}_V(H_1\cap\ldots\cap H_p)=p$. So, if $p>N$ then any p-tuple of hyperplanes is not in general position.

Let $\mathscr{C}$ be an arrangement. Define abelian groups $\mathscr{A}^p(\mathscr{C})$, $0\leqq p\leqq N$, as follows. For $p=0$ put $\mathscr{A}^p(\mathscr{C})=\mathbb{Z}$. For $p\geqq 1$ $\mathscr{A}^p(\mathscr{C})$ is generated by p-tuples $(H_1,\ldots,H_p)$, $H_i\in\mathscr{C}$, subject to the following relations:

$$(H_1,\ldots,H_p)=0 \tag{1.2.1}$$

if $H_1,\ldots,H_p$ are not in general position;

$$(H_{\sigma(1)},\ldots,H_{\sigma(p)})=(-1)^{|\sigma|}(H_1,\ldots,H_p) \tag{1.2.2}$$

for any $\sigma\in\Sigma_p$; for any $(p+1)$-tuple $H_1,\ldots,H_{p+1}$ which is not in general position and such that $H_1\cap\ldots\cap H_{p+1}\neq\emptyset$,

$$\sum_{i=1}^{p+1}(-1)^i(H_1,\ldots,\hat{H}_i,\ldots,H_{p+1})=0. \tag{1.2.3}$$

The direct sum $\mathscr{A}^{\bullet}(\mathscr{C})=\bigoplus\limits_{p=1}^{N}\mathscr{A}^p(\mathscr{C})$ is a graded skew commutative algebra with respect to the multiplication $(H_1,\ldots,H_p)\cdot(H'_1,\ldots,H'_q)=(H_1,\ldots,H_p,H'_1,\ldots,H'_q)$. We'll denote $(H_1,\ldots,H_p)$ by $H_1\cdot\ldots\cdot H_p$. $\mathscr{A}^{\bullet}(\mathscr{C})$ is called *the Orlik-Solomon algebra* of $\mathscr{C}$.

1.3 For an edge L of $\mathscr{C}$ put $\mathscr{C}(L)=\{H\in\mathscr{C}\mid H\supset L\}$. Inclusions $\mathscr{C}(L)\hookrightarrow\mathscr{C}$ induce the map

$$\bigoplus_{L\in\mathscr{C}^p}\mathscr{A}^p(\mathscr{C}(L))\to\mathscr{A}^p(\mathscr{C}). \tag{1.3.1}$$

1.3.2 Lemma. (1.3.1) *is an isomorphism.*

Proof. For $H_1,\ldots,H_p$ in general position let $L=H_1\cap\ldots\cap H_p\in\mathscr{C}^p$. Assign to $(H_1,\ldots,H_p)$ the same p-tuple considered as the element of $\mathscr{A}^p(\mathscr{C}(L))$. One easily sees that this correctly defines the map $\mathscr{A}^p(\mathscr{C})\to\oplus\mathscr{A}^p(\mathscr{C}(L))$, which is inverse to (1.3.1). $\square$

We'll denote by $\mathscr{A}^p(\mathscr{C})_L$ the image of $\mathscr{A}^p(\mathscr{C}(L))$ in $\mathscr{A}^p(\mathscr{C})$. So, we have

$$\mathscr{A}^p(\mathscr{C})=\bigoplus_{L\in\mathscr{C}^p}\mathscr{A}^p(\mathscr{C})_L. \tag{1.3.3}$$

1.4 For $H\in\mathscr{C}$ put

$$\mathscr{A}^p(\mathscr{C};H)=\oplus\mathscr{A}^p(\mathscr{C})_L\hookrightarrow\mathscr{A}^p(\mathscr{C}) \tag{1.4.1}$$

the sum is taken over all edges $L\in\mathscr{C}^p$ not contained in H. Suppose that $\mathscr{C}$ is central, with the unique vertex v. We can consider $\mathscr{C}$ as a collection of hyperplanes in the $(N-1)$-dimensional projective space $\mathbb{P}(V,v)$ of lines containing

v . Let us take H as the infinite hyperplane; we get the arrangement $\mathscr{C}-\{H\}$ in the $(N-1)$-dimensional affine space $V_H=\mathbb{P}(V,v)\setminus H$. We'll denote this arrangement $\tilde{\mathscr{C}}_H$.

We have evident identification

$$(1.4.2) \qquad \mathscr{A}^{\cdot}(\tilde{\mathscr{C}}_H)\cong\mathscr{A}^{\cdot}(\mathscr{C};H).$$

Conversely, to every arrangement $\mathscr{C}$ in V we can assign by taking its projective closure the central arrangement $\hat{\mathscr{C}}$ in the $(N+1)$-dimensional linear space V together with a distinguished ("infinite") hyperplane H_∞.

1.5 Let $\mathscr{C}$ be a central arrangement with the unique vertext v; choose $H_0\in\mathscr{C}$. Consider the map

$$(1.5.1) \qquad \mathscr{A}^{N-1}(\mathscr{C};H_0)\to\mathscr{A}^N(\mathscr{C})$$

which sends $H_1\cdot\ldots\cdot H_{N-1}$ to $H_0H_1\ldots H_{N-1}$.

1.5.2 Lemma. (1.5.1) *is an isomorphism.*

Proof. The inverse map may be defined as the composition of the map $\mathscr{A}^N(\mathscr{C})\to\mathscr{A}^{N-1}(\mathscr{C})$ which sends an N-tuple in general position $(H_1,\ldots,H_N)$, to $\sum_{i=1}^{N}(-1)^{i-1}H_1\ldots\hat{H}_i\ldots H_N$, and the projection $\mathscr{A}^{N-1}(\mathscr{C})\to\mathscr{A}^{N-1}(\mathscr{C},H_0)$ induced by the decomposition (1.3.3). □

1.6 Framings and bases. Let us call a *framing* $\mathcal{O}$ of an arrangement $\mathscr{C}$ a choice for every edge L of $\mathscr{C}$ of a hyperplane $H(L)$ containing L. For such a framing $\mathcal{O}$ define subsets $\mathcal{O}_j\subset\mathscr{C}_j$, $j=0,\ldots,N$ inductively: $\mathcal{O}_0=\mathscr{C}_0$; $\mathcal{O}_1=\{L\in\mathscr{C}_1 \mid \text{for all } v\in\mathscr{C}_0\; L\not\subset H(v)\}$, ... $\mathcal{O}_j=\{L\in\mathscr{C}_j \mid \text{for all } L'\in\mathcal{O}_0\cup\ldots\cup\mathcal{O}_{j-1},\; L\not\subset H(L')\}$, $\mathcal{O}_N=\{V\}$. Next, let $Fl_j(\mathcal{O})$ denote the set of all flags $L_0\subset L_1\subset\ldots\subset L_{j-1}$ such that $L_i\in\mathcal{O}_i$, $0\leqq i\leqq j-1$.

Define the groups $\mathscr{A}^{N-j,j}=\mathscr{A}^{N-j,j}(\mathscr{C};\mathcal{O})$, $0\leqq j\leqq N$, as follows

$$(1.6.1) \qquad \begin{aligned} &\mathscr{A}^{N,0}(\mathscr{C};\mathcal{O})=\mathscr{A}^N(\mathscr{C});\\ &\mathscr{A}^{N-j,j}(\mathscr{C};\mathcal{O})=\bigoplus_{(L_0\subset\ldots\subset L_{j-1})\in Fl_j(\mathcal{O})}\mathscr{A}^{N-j}(\mathscr{C}(L_{j-1}),H(L_{j-1})).\end{aligned}$$

Note that groups $\mathscr{A}^{N-j,j}$ depend only on the part of the framing $\mathcal{O}$, consisting of $H(L)$, $L\in C_i$, $0\leqq i\leqq j-1$.

By 1.5.2, we have isomorphisms

$$(1.6.2) \qquad \varphi_j\colon\ \mathscr{A}^{N-j,j}\xrightarrow{\ \sim\ }\mathscr{A}^{N-j+1,j-1};$$

where for $x=\sum x_{L_0\subset\ldots\subset L_{j-1}}\in\mathscr{A}^{N-j,j}$

$$x_{(L_0\subset\ldots\subset L_{j-1})}\in\mathscr{A}^{N-j}(\mathscr{C}(L_{j-1}),H(L_{j-1})),\qquad (L_0\subset\ldots\subset L_{j-1})\in Fl_j(\mathcal{O}),$$
$$\varphi_j(x)_{(L_0\subset\ldots\subset L_{j-2})}=\sum_{\substack{L_{j-1}\supset L_{j-2}\\ L_{j-1}\in\mathcal{O}_{j-1}}}x_{(L_0\subset\ldots\subset L_{j-1})}\cdot H(L_{j-1}).$$

By definition,

$$\mathscr{A}^{0,N}(\mathscr{C},\mathcal{O})=\bigoplus_{F\in Fl_N(\mathcal{O})}\mathbb{Z}. \tag{1.6.3}$$

Define the set

$$B(\mathcal{O})=\{H(L_0)\cdot H(L_1)\cdot\ldots\cdot H(L_{N-1})\}\subset\mathscr{A}^N(\mathscr{C}) \tag{1.6.4}$$

indexed by all complete flags $L_0\subset\ldots\subset L_{N-1}\in Fl_N(\mathcal{O})$.
From the above follows the

1.6.5 Theorem. [Bj] *The group $\mathscr{A}^N(\mathscr{C})$ is free over $\mathbb{Z}$, admitting $B(\mathcal{O})$ as a base.*

1.6.6 Example. Suppose that the set $\mathscr{C}$ is linearly ordered. Define $H(L)$ to be the minimal element of $\mathscr{C}(L)$. This gives the framing of $\mathscr{C}$. The corresponding base of $\mathscr{A}^N(\mathscr{C})$ was considered in [GZ, Theorem 1].

1.6.7 Corollary. *$H_1\cdot\ldots\cdot H_p\neq 0$ in $\mathscr{A}^p(\mathscr{C})$ iff $H_1,\ldots,H_p$ are in general position.*

Proof. The "only if" part is evident. For the converse, we can suppose that $p=N$. If $H_1,\ldots,H_N$ are in general position then we can choose a framing of $\mathscr{C}$ such that $H_i=H(H_1\cap\ldots\cap H_i)$, $i=1,\ldots,N$. So, by 1.6.5 $H_1\ldots H_N$ is a member of a base. □

1.6.8 Corollary. *$\mathscr{A}^p(\mathscr{C})=0$ iff $\mathscr{C}^p=\emptyset$.*

Note that if $\mathscr{C}^p=\emptyset$ then $\mathscr{C}^q=\emptyset$ for all $q>p$.

1.6.9 By additivity 1.3.2 all groups $\mathscr{A}^p(\mathscr{C})$ are free. Define the Euler characteristics

$$\chi(\mathscr{C})=\sum_{i=0}^{N}(-1)^i rk\,\mathscr{A}^{N-i}(\mathscr{C}).$$

1.7 Let $\mathscr{C}$ be an arrangement. Define maps $d^*=d^{*p}$: $\mathscr{A}^p(\mathscr{C})\to\mathscr{A}^{p-1}(\mathscr{C})$ by the rule

$$d^*(H_1\cdot\ldots\cdot H_p)=\sum_{i=1}^{p}(-1)^{i-1}H_1\cdot\ldots\cdot\hat{H}_i\cdot\ldots\cdot H_p \tag{1.7.1}$$

if $H_1,\ldots,H_p$ are in general position and zero otherwise (cf. 1.6.7).
Evidently, $d^*\circ d^*=0$.
Let $n=\sup\{p\,|\,\mathscr{A}^p(\mathscr{C})\neq 0\}=\sup\{p\,|\,\mathscr{C}^p\neq\emptyset\}$ (cf. 1.6.8).

1.7.2 Theorem. *$H^i(\mathscr{A}^\bullet(\mathscr{C}),d^*)=0$ for $i\neq n$.*

1.7.3 Remark. If $\mathscr{C}$ is central then $(\mathscr{A}^\bullet(\mathscr{C}),d^*)$ is acyclic, [OS]. In fact, the multiplication by any $H\in\mathscr{C}$ gives a contracting homotopy.

1.7.4 Corollary. *The group $H^n(\mathscr{A}^\bullet(\mathscr{C}),d^*)=\ker d^{*n}$ is free. Its rank is equal to the Euler characteristics $(-1)^{n-N}\chi(\mathscr{C})$ (cf. 1.6.9).*

1.8 Proof of 1.7.2 If $n<N$, consider the set $\mathscr{C}^n$. It is a set of parallel subspaces. Let us consider a subspace $W\subset V$ of dimension n transversal to these subspaces.

The arrangement $\mathscr{C} \cap W$ in W induced by $\mathscr{C}$ contains vertices, and we have (in the evident sense) $\mathscr{C} \cong (\mathscr{C} \cap W) \times (V/W)$; $\mathscr{A}^{\cdot}(\mathscr{C}) = \mathscr{A}^{\cdot}(\mathscr{C} \cap W)$.

Thus, we may, and will, suppose that $n = N$.

For any $H \in \mathscr{C}$ introduce operators of degree zero $\Delta_H : \mathscr{A}^{\cdot}(\mathscr{C}) \to \mathscr{A}^{\cdot}(\mathscr{C})$,

$$\Delta_H(x) = H \cdot d^* x + d^*(H x). \tag{1.8.1}$$

1.8.2 Lemma. *Let* $x = H_1 \dots H_p \in \mathscr{A}^p(\mathscr{C})$, $x \neq 0$, $H_0 \in \mathscr{C}$. *Then*

$$x - \Delta_{H_0} x = \begin{cases} 0 & \text{if } H_0 x \neq 0 \\ \sum_{i=0}^{p} (-1)^i H_0 \cdot \ldots \cdot \hat{H}_i \cdot \ldots \cdot H_p & \text{if } H_0 x = 0 \end{cases}$$

In the last case $\operatorname{Supp}(x - \Delta_{H_0} x) = \operatorname{Supp} H_0 \cdot \ldots \cdot \hat{H}_i \cdot \ldots \cdot H_p$ *for all i.*

Note that $H_0 x = 0$ means that H_0 is parallel to (or contains) the edge $H_1 \cap \ldots \cap H_p$ (by 1.6.7).

1.8.3 Now consider the projective closure of $\mathscr{C}$ (cf. 1.4). Namely let $\mathbb{P} = V \cup \bar{H}_\infty$ be a projective closure of V – the union of V and the infinite hyperplane $\bar{H}_\infty \cong \mathbb{P}^{N-1}(\mathbb{C})$. For $H \subset V$ denote by $\bar{H} \subset \mathbb{P}$ its closure.

For a p-tuple $H_I = (H_1, \ldots, H_p)$, $H_i \subset V$, put

$$\operatorname{Supp}(H_I) = \bar{H}_1 \cap \bar{H}_2 \cap \ldots \cap \bar{H}_p \cap \bar{H}_\infty. \tag{1.8.4}$$

For a formal linear combination $x = \sum a_I H_I$, let $\operatorname{Supp} x$ be the smallest projective subspace of H containing all $\operatorname{Supp} H_I$ with $a_I \neq 0$.

For $x \in \mathscr{A}^p(\mathscr{C})$, and $\bar{L} \subset \bar{H}_\infty$, let us say that $\operatorname{Supp} x \subset \bar{L}$ if one can write x as a linear combination of monomials $x = \sum a_I H_I$, such that $\operatorname{Supp}(\sum a_I H_i) \subset \bar{L}$.

From 1.8.2 follows that if $\operatorname{Supp} x \subset \bar{L}$ then

$$\operatorname{Supp}(x - \Delta_H x) \subset \bar{L} \cap \bar{H}. \tag{1.8.5}$$

Now suppose that $x \in \mathscr{A}^p(\mathscr{C})$, $d^* x = 0$, $p < n$. Suppose that $x \neq 0$. Let us choose some representative $x = \sum a_I H_I$. We'll prove that x is a d^* – boundary by induction on $\dim \operatorname{Supp}(\sum a_I H_I)$. There exists $H \in \mathscr{C}$ such that $\operatorname{Supp}(\sum a_I H_I) \not\subset \bar{H} \cap \bar{H}_\infty$. We have $\Delta_H x = d^* H x$, so $x = d^* H x + y$ where $d^* y = 0$ and, by (1.8.5) $\operatorname{Supp} y \subset \bar{H}_\infty \cap \operatorname{Supp}(\sum a_I H_I)$. By induction hypothesis then $y = d^* z$ for some z. This completes the proof of 1.7.2. □

1.9 Let $\mathscr{C}$ be an arrangement. Denote by $\mathscr{E}_p(\mathscr{C})$ an abelian group whose generators are all p-tuples $(H_1, \ldots, H_p)$ of distinct hyperplanes of $\mathscr{C}$ such that $H_1 \cap \ldots \cap H_p \neq \emptyset$; subject to the skew-symmetry relations

$$(H_{\sigma(1)}, \ldots, H_{\sigma(p)}) = (-1)^{|\sigma|} (H_1, \ldots, H_p). \tag{1.9.1}$$

Put $\mathscr{E}_0(\mathscr{C}) = \mathbb{Z}$. $\mathscr{E}_\cdot(\mathscr{C}) := \bigoplus_p \mathscr{E}_p(\mathscr{C})$ is a graded skew commutative algebra with respect to the multiplication $(H_1, \ldots, H_p) \cdot (H'_1, \ldots, H'_q) = (H_1, \ldots, H_p, H'_1, \ldots, H'_q)$. We'll denote $(H_1, \ldots, H_p)$ by $H_1 \ldots H_p$.

For an edge $L\in\mathscr{C}^q$ let $\mathscr{E}_p(\mathscr{C})_L$ denote the subgroup of $\mathscr{E}_p(\mathscr{C})$ generated by $H_1\dots H_p$ such that $H_1\cap\dots\cap H_p=L$. Put

(1.9.2) $$\mathscr{E}^q_p(\mathscr{C})=\bigoplus_{L\in\mathscr{C}^q}\mathscr{E}_p(\mathscr{C})_L.$$

So, we have

(1.9.3) $$\mathscr{E}_p(\mathscr{C})=\bigoplus_{0\leqq q\leqq p}\mathscr{E}^q_p(\mathscr{C}).$$

Introduce the differential ∂: $\mathscr{E}_p(\mathscr{C})\to\mathscr{E}_{p-1}(\mathscr{C})$ by the formula

$$\partial(H_1\dots H_p)=\sum(-1)^{i-1}H_1\cdot\ldots\cdot\hat{H}_i\cdot\ldots\cdot H_p.$$

Clearly, $\partial^2=0$.

Define maps $\psi=\psi_p: \mathscr{E}_p(\mathscr{C})\to\mathscr{A}^p(\mathscr{C})$ by the rule

(1.9.4) $$\psi_p(H_1\dots H_p)=\begin{cases}H_1\dots H_p\in\mathscr{A}^p(\mathscr{C}) & \text{if } \operatorname{codim} H_1\cap\dots\cap H_p=p\\ 0 & \text{otherwise.}\end{cases}$$

We get a map of complexes

(1.9.5) $$\psi\colon\ (\mathscr{E}_\bullet(\mathscr{C}),\partial)\to(\mathscr{A}^\bullet(\mathscr{C}),d^*).$$

The following result will be needed in § 4.

1.9.6 Theorem. *ψ is quasiisomorphism.*

Proof. First let us fix p and let us consider the complex

(1.9.7) $$\dots\xrightarrow{\partial'}\mathscr{E}^p_{p+2}\xrightarrow{\partial'}\mathscr{E}^p_{p+1}\xrightarrow{\partial'}\mathscr{E}^p_p\xrightarrow{\Psi}\mathscr{A}^p\longrightarrow 0$$

with ∂' induced by ∂.

1.9.8 Lemma. (1.9.7) *is a resolution of $\mathscr{A}^p$.*

Proof. By 1.3.2 we are reduced to the case of central $\mathscr{C}$ which is proven in [GZ, Appendix, p. 33].

Now let

(1.9.9) $$\partial''\colon\ \mathscr{E}^q_p\to\mathscr{E}^{q-1}_{p-1}$$

be also induced by ∂. Clearly $\mathscr{E}^q_p$, ∂', ∂'' is a bicomplex whose associated simple complex is $(\mathscr{E}_\bullet,\partial)$. So, 1.9.6 follows from 1.9.8 by a standard spectral sequence argument. □

2 *Flag complex*

We save the assumptions of § 1.

2.1 Let $\mathscr{C}$ be an arrangement. For $0\leqq p\leqq N$ denote by $Fl^p(\mathscr{C})$ the set of all flags $(L^0\supset L^1\supset\dots\supset L^p)$, $L^i\in\mathscr{C}^i$, $0\leqq i\leqq p$. Denote by □ the unique element of

$Fl^0(\mathscr{C})$. Denote by $\tilde{\mathscr{F}}^p(\mathscr{C})$ the free abelian group on $Fl^p(\mathscr{C})$, and by $\mathscr{F}^p(\mathscr{C})$ the quotient of $\tilde{\mathscr{F}}^p(\mathscr{C})$ by the following relations.

For every i, $0<i<p$, and a flag with a gap $\hat{F}=(L^0\supset L^1\supset\ldots\supset L^{i-1}\supset L^{i+1}\supset\ldots\supset L^p)$, $L^j\in\mathscr{C}^j$,

$$\sum_{F\supset\hat{F}} F=0 \tag{2.1.1}$$

in $\mathscr{F}^p(\mathscr{C})$, where the summing is extended over all $F=(\tilde{L}^0\supset\tilde{L}^1\supset\ldots\supset\tilde{L}^p)\in Fl^p(\mathscr{C})$ such that $\tilde{L}^j=L^j$ for all $j\neq i$.

For $L\in\mathscr{C}^p$ let $\mathscr{F}^p(\mathscr{C})_L\subset\mathscr{F}^p(\mathscr{C})$ denote the subgroup generated by $(L^0\supset\ldots\supset L^p)$ with $L^p=L$. Clearly $\mathscr{F}^p(\mathscr{C})_L=\mathscr{F}^p(\mathscr{C}(L))$, and

$$\mathscr{F}^p(\mathscr{C})=\bigoplus_{L\in\mathscr{C}^p}\mathscr{F}^p(\mathscr{C})_L \tag{2.1.2}$$

(cf. 1.3.2).

2.2 Define differentials $d\colon\mathscr{F}^p(\mathscr{C})\to\mathscr{F}^{p+1}(\mathscr{C})$ by

$$d(L^0\supset\ldots\supset L^p)=\sum_{\substack{L^{p+1}\in\mathscr{C}^{p+1}\\ L^{p+1}\subset L}}(L^0\supset\ldots\supset L^p\supset L^{p+1}). \tag{2.2.1}$$

From the relation (2.1.1) follows that $d^2=0$.

2.3 Define maps $\varphi=\varphi^p\colon\mathscr{A}^p(\mathscr{C})\to\mathscr{F}^p(\mathscr{C})^*$ as follows. For $(H_1,\ldots,H_p)$ in general position, $H_i\in\mathscr{C}$, put

$$F(H_1,\ldots,H_p)=(H_1\supset H_{12}\supset H_{123}\supset\ldots\supset H_{12\ldots p})\in Fl^p(\mathscr{C}) \tag{2.3.1}$$

where $H_{1\ldots i}:=H_1\cap\ldots\cap H_i$.

For a flag $F\in Fl^p(\mathscr{C})$ define a functional $\delta_F\in\tilde{\mathscr{F}}^p(\mathscr{C})^*$ as

$$\delta_F(F')=\begin{cases}1 & \text{if } F'=F\\ 0 & \text{otherwise.}\end{cases}$$

For $(H_1,\ldots,H_p)$ in general position, put

$$\varphi(H_1\ldots H_p)=\sum_{\sigma\in\Sigma_p}(-1)^{|\sigma|}\delta_{F(H_{\sigma(1)},\ldots,H_{\sigma(p)})}. \tag{2.3.2}$$

This defines the map $\mathscr{A}^p\to\tilde{\mathscr{F}}^{p*}$. Clearly, its image lies in the subgroup $\mathscr{F}^{p*}\subset\tilde{\mathscr{F}}^{p*}$, so we get maps

$$\varphi^p\colon\mathscr{A}^p(\mathscr{C})\to\mathscr{F}^p(\mathscr{C})^*. \tag{2.3.3}$$

2.3.4 Lemma. *φ^p define maps of complexes*

$$\varphi^\cdot\colon(\mathscr{A}^\cdot(\mathscr{C}),d^*)\to(\mathscr{F}^\cdot(\mathscr{C})^*,d^*)$$

where d^* in the left hand side is (1.7.1), *and d^* in the r.h.s. is adjoint to* (2.2.1).

2.4 Theorem. (a) *All groups $\mathscr{F}^p(\mathscr{C})$ are free.*
(b) *All maps φ^p are isomorphisms.*

The Theorem will be proved a bit later, after some preliminaries.

2.5 For $H \in \mathscr{C}$ put

(2.5.1)
$$\mathscr{F}^p(\mathscr{C}; H) = \bigoplus_{L \not\subset H} \mathscr{F}^p(\mathscr{C})_L \subset \mathscr{F}^p(\mathscr{C})$$

(cf. 1.4.1).

Suppose that $\mathscr{C}$ is central, with a unique vertex v. In notations of n° 1.4, we have evident isomorphisms

(2.5.2)
$$\mathscr{F}^\cdot(\mathscr{C}; H) \cong \mathscr{F}^\cdot(\tilde{\mathscr{C}}_H).$$

Define the map

(2.5.3)
$$\mathscr{F}^{N-1}(\mathscr{C}; H) \to \mathscr{F}^H(\mathscr{C})$$

which sends $(L^0 \supset \ldots \supset L^{N-1})$ to $(L^0 \supset \ldots \supset L^{N-1} \supset v)$. Using relations (2.1.1) one easily sees that (2.5.3) is epimorphic. (We'll prove a bit later that it is isomorphism).

2.6 Proof of 2.4 By (2.1.2) and 1.3.2 one may suppose that $\mathscr{C}$ is central with a unique vertex, and $p = N$. Let us prove 2.4 by induction on N. Consider the square

(2.6.1)
$$\begin{array}{ccc} \mathscr{F}^{N-1}(\mathscr{C}; H) & \xrightarrow{(2.5.3)} & \mathscr{F}^N(\mathscr{C}) \\ {\scriptstyle \varphi(\mathscr{C}, H)^*} \big\downarrow \wr & & \big\downarrow {\scriptstyle \varphi^*} \\ \mathscr{A}^{N-1}(\mathscr{C}; H)^* & \xrightarrow[(1.5.1)^{-1}]{\sim} & \mathscr{A}^N(\mathscr{C})^* \end{array}$$

Here vertical maps are adjoint to (2.3.3). One verifies that (2.6.1) commutes.

By (2.5.2), (1.4.2) and induction hypothesis, $\varphi(\mathscr{C}, H)^*$ is iso; by (2.5) the upper horizontal map is epi. It follows that φ^* and (2.5.3) are isomorphisms, and we are done by 1.6.5. □

2.7 Remark. Let $\mathscr{C}$ be a central arrangement with the unique vertex. Let K be the simplicial set associated with the ordered set of edges of $\mathscr{C}$, as in [OS, § 4].

From definitions follows the isomorphism $\mathscr{F}^N(\mathscr{C})^* \cong \tilde{H}_{N-2}(K)$.

In particular, 2.6.b follows from [OS, 4.3].

2.8 Corollary. *Let $n = \max\{p \mid \mathscr{F}^p(\mathscr{C}) \neq 0\}$. Then $H^p(\mathscr{F}^\cdot(\mathscr{C}), d) = 0$ for $p \neq n$.*

Proof. Follows from 1.7.2 and 2.6. □

2.9 In the course of the proof of 2.6 we have proven the

2.9.1 Corollary. *Let $\mathscr{C}$ be a central arrangement with the unique vertex v, $H \in \mathscr{C}$. Then the map* (2.5.3) *is an isomorphism.*

Let $\mathscr{C}$ be arbitrary, $\mathcal{O}$ a framing of $\mathscr{C}$. Denote by $Fl^p(\mathcal{O})$ the set of all flags $(L^0 \supset \ldots \supset L^p)$ with $L^i \in \mathcal{O}_{N-i}$ (see 1.6).

2.9.2 Theorem. *The set $Fl^p(\mathcal{O})$ forms the base of the free abelian group $\mathscr{F}^p(\mathscr{C})$.*

Proof. This is deduced from 2.8.1 in the same way as 1.6.5 from 1.5.2. □

3 Contravariant form

We save the assumptions of §§ 1, 2.

3.0 From now on up to the end of the paper we'll change slightly notations: $\mathscr{A}^\bullet(\mathscr{C})$, $\mathscr{F}^\bullet(\mathscr{C})$ will be denoted by $\mathscr{A}^\bullet(\mathscr{C})_{\mathbb{Z}}$, $\mathscr{F}^\bullet(\mathscr{C})_{\mathbb{Z}}$, and $\mathscr{A}^\bullet(\mathscr{C})$, $\mathscr{F}^\bullet(\mathscr{C})$ will denote the complexifications $\mathscr{A}^\bullet(\mathscr{C})_{\mathbb{C}} = \mathscr{A}^\bullet(\mathscr{C})_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, $\mathscr{F}^\bullet(\mathscr{C})_{\mathbb{C}} = \mathscr{F}^\bullet(\mathscr{C})_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$.

3.1 Let $\mathscr{C}$ be an arrangement. Suppose that a map a: $\mathscr{C} \to \mathbb{C}$ is given. We'll call such a map a *collection of exponents* for $\mathscr{C}$. For an edge L put

$$a(L) = \sum_{\substack{H \supset L, \\ H \in \mathscr{C}}} a(H). \tag{3.1.1}$$

Put

$$\omega = \omega(a) = \sum_{H \in \mathscr{C}} a(H) H \in \mathscr{A}^1(\mathscr{C}). \tag{3.1.2}$$

Define differentials $d = d^p = d^p(a)$: $\mathscr{A}^p(\mathscr{C}) \to \mathscr{A}^{p+1}(\mathscr{C})$ by the rule

$$d(x) = \omega(a) \cdot x. \tag{3.1.3}$$

It is clear that $d^2 = 0$.

3.2 Define maps

$$S = S^p = S^p(a)\colon\ \mathscr{F}^p(\mathscr{C}) \to \mathscr{A}^p(\mathscr{C}) \tag{3.2.1}$$

as follows. For $L \in \mathscr{C}^i$ put

$$S(L) = \sum_{\substack{H \supset L, \\ H \in \mathscr{C}}} a(H) H \in \mathscr{A}^1(\mathscr{C}); \tag{3.2.2}$$

for a flag $F = (L^0 \supset L^1 \supset \ldots \supset L^p) \in Fl^p(\mathscr{C})$ put

$$S(F) = S(L^1) \cdot S(L^2) \cdot \ldots \cdot S(L^p). \tag{3.2.3}$$

In other words,

$$S(F) = \sum a(H_1)\, a(H_2) \ldots a(H_p)\, H_1 \cdot \ldots \cdot H_p, \tag{3.2.4}$$

the sum is taken over all p-tuples $(H_1, \ldots, H_p)$ such that $H_i \supset L^i$, $H_i \in \mathscr{C}$, $1 \leqq i \leqq p$.

One easily verifies that (3.2.4) correctly defines maps (3.2.1).

3.2.5 Lemma. *$S^\bullet$ defines the map of complexes $S^\bullet = S^\bullet(a)$: $(\mathscr{F}^\bullet(\mathscr{C}), d) \to (\mathscr{A}^\bullet(\mathscr{C}), d(a))$.*

3.3 If we identify $\mathscr{A}^\bullet(\mathscr{C})$ with $\mathscr{F}^\bullet(\mathscr{C})^*$ by means of isomorphisms (2.3.3) then S^p may be regarded as bilinear forms on $\mathscr{F}^p$. Let us calculate their value on a pair of flags (F, F').

Let us call a p-tuple $\bar{H}=(H_1, \ldots, H_p)$, $H_i \in \mathscr{C}$, *adjacent* to a flag F if there exists $\sigma \in \Sigma_p$ such that $F=F(H_{\sigma(1)}, \ldots, H_{\sigma(p)})$ (see (2.3.1)). Note that such a permutation is unique; denote it $\sigma_{\bar{H}}(F)$.

Then for $F, F' \in Fl^p(\mathscr{C})$

$$S^p(F, F') = \frac{1}{p!} \sum (-1)^{\sigma_{\bar{H}}(F)\sigma_{\bar{H}}(F')} a(H_1) \ldots a(H_p) \tag{3.3.1}$$

the summing is taken over all $\bar{H}=(H_1, \ldots, H_p)$ which are adjacent to F and F'.

3.3.2 Corollary. *Forms $S^\bullet$ are symmetric.*

3.4 For a flag $F=(L^0 \supset L^1 \supset \ldots \supset L^p) \in Fl^p(\mathscr{C})$ put

$$a(F) = \prod_{i=1}^{p} a(L^i). \tag{3.4.1}$$

Suppose that $a(L) \neq 0$ for all edges L of codimension p. For a p-tuple $\bar{H}=(H_1, \ldots, H_p)$, $H_i \in \mathscr{C}$, in general position put

$$R^p(H) = \sum_{\sigma \in \Sigma_p} (-1)^{|\sigma|} a(F(\sigma\bar{H}))^{-1} F(\sigma\bar{H}) \in \mathscr{F}^p(\mathscr{C}) \tag{3.4.2}$$

where $\sigma\bar{H}=(H_{\sigma(1)}, \ldots, H_{\sigma(p)})$.

One easily verifies that this formula correctly defines the map

$$R^p = R^p(a) \colon \mathscr{A}^p(\mathscr{C}) \to \mathscr{F}^p(\mathscr{C}). \tag{3.4.3}$$

3.4.4 Lemma. *R^p is inverse to S^p.*

Proof. Since $\mathscr{A}^p$ and $\mathscr{F}^p$ are spaces of equal dimensions, it suffices to prove that $S^p R^p = \mathrm{id}$. Let $H_1, \ldots, H_p$ be in general position, $L = H_1 \cap \ldots \cap H_p$. Note that

$$\left(\sum_{i=1}^{p} (-1)^{p-i} H_1 \ldots \hat{H}_i \ldots H_p\right) \cdot \sum_{H \supset L} a(H)\, H = a(L)\, H_1 \ldots H_p \tag{3.4.4.1}$$

as follows from (1.2.3). Now let us prove that $S^p R^p(H_1 \ldots H_p) = H_1 \ldots H_p$ by induction on p. We have

$$S^p R^p(H_1 \ldots H_p) = \sum_{i=1}^{p} (-1)^{p-i} \frac{1}{a(L)} S^{p-1} R^{p-1}(H_1 \ldots \hat{H}_i \ldots H_p)$$

$$\cdot \sum_{H \supset L} a(H)\, H \overset{\text{(induction)}}{=} \left(\sum_{i=1}^{p} (-1)^{p-i} \frac{1}{a(L)} H_1 \ldots \hat{H}_i \ldots H_p\right)$$

$$\cdot \sum_{H \supset L} a(H)\, H \overset{(3.4.4.1)}{=} H_1 \ldots H_p.$$

3.4.5 Corollary. *If* $a(L)\neq 0$ *for all edges* L *of codimension* $\leqq p$ *then* $S^p(a)$ *is iso.* (cf. Theorem 3.7).

3.5 Remark. Let us consider groups

$$\bar{\mathscr{F}}^p(\mathscr{C})=\mathscr{F}^p(\mathscr{C})/\ker S^p(a)\cong S^p(\mathscr{F}^p(\mathscr{C}))\subset\mathscr{A}^p(\mathscr{C}).$$

S^p induces non-degenerate symmetric forms S^p on $\bar{\mathscr{F}}^p$; differentials d, d^* on $A^\cdot$ induce differentials on $\bar{\mathscr{F}}^\cdot=\oplus\bar{\mathscr{F}}^p$ which are adjoint with respect to $S^\cdot$. If all $a(H)$, $H\in\mathscr{C}$, are real then S^p are real and we have usual "Hodge" decomposition $H^p(\mathscr{F}^\cdot(\mathscr{C}))\cong\{x\in\bar{\mathscr{F}}^p(\mathscr{C})\,|\,dx=0 \text{ and } d^*x=0\}$, etc. cf. 6.13, [Kos].

Determinant formula. All $S^p(a)$ are linear operators between vector spaces of equal dimension, depending polynomially on $\{a(H)\}_{H\in\mathscr{C}}$. So, det $S^p(a)$ is a polynomial in $\{a(H)\}$ well defined up to a non-zero multiplicative constant. We'll write down the formula for it.

3.6 Let $\mathscr{C}$ be a central arrangement in $\mathbb{C}^N$ with a unique vertex. Choose some $H\in\mathscr{C}$, and put

$$e(\mathscr{C})=\chi(\tilde{\mathscr{C}}_H) \tag{3.6.1}$$

where $\tilde{\mathscr{C}}_H$ is as in 1.4 (cf. 1.7.4). From 4.1.5 follows that this number doesn't depend on the choice of H.

Now let $\mathscr{C}$ be arbitrary. For an edge L of $\mathscr{C}$ put

$$e(L)=e(\mathscr{C}(L)). \tag{3.6.2}$$

(We consider $\mathscr{C}(L)$ as a central arrangement in the normal space to L in V).

Let $\mathscr{C}_L$ be an arrangement in L induced by $\mathscr{C}$, i.e. $\mathscr{C}_L=\{H\cap L\}_{H\in\mathscr{C},H\not\supset L}$. Put

$$d^p(L)=\dim\mathscr{A}^p(\mathscr{C}_L). \tag{3.6.3}$$

3.7 Theorem. *Let* $\mathscr{C}$ *be an arrangement in* $V\cong\mathbb{C}^N$ *with exponents* $a\colon\mathscr{C}\to\mathbb{C}$. *Then for all* $p\geqq 0$

$$\det S^{N-p}(a)=\mathrm{const}\cdot\prod_{i\geqq 0}\prod_{L\in\mathscr{C}_i}a(L)^{e(L)d^{i-p}(L)} \tag{3.7.1}$$

(*We put* $e(V)=0$; *so* det $S^0=1$.)

Choose a framing of $\mathscr{C}$. *This gives bases in* $\mathscr{A}^p(\mathscr{C})$, $\mathscr{F}^p(\mathscr{C})$, *see* 1.6, 2.9.2. *With respect to these bases,* det $S^{N-p}(a)$ *is equal to the r.h.s. of* (3.7.1) *with* Const $=1$.

3.7.2 Example. Consider an arrangement $\mathscr{C}$ on the plane consisting of n lines going through a point, with exponents $a_1,\dots,a_n$. Then det $S^2=\mathrm{Const}\cdot a_1\dots a_n\left(\sum_{i=1}^n a_i\right)^{n-2}$.

3.8 Corollary. *Put*

$$\det S^{\cdot}(a) = \prod_{p=0}^{N} (\det S^{p}(a))^{(-1)^{N-p}}. \tag{3.8.1}$$

Then

$$\det S^{\cdot}(a) = \mathrm{Const} \cdot \prod_{L} a(L)^{e(L)\chi(\mathscr{C}_L)} \tag{3.8.2}$$

where the product is taken over all edges of $\mathscr{C}$.

This formula should be compared with [V, Theorem 1.1]. The proof of 3.7 will occupy the rest of the section. It will consist of three steps.

3.9 Step I. *S and Laplace operators*

Let $\mathscr{C}$ be a central arrangement with a unique vertex v in $V = \mathbb{C}^N$.

Matrix elements of $S^N(a)$ are homogeneous degree N polynomials in $a(H)$. In this n⁰ we'll show that $S^N(a)$ may be decomposed into a product of N operators whose matrix elements are linear in $a(H)$. (In fact, each framing of $\mathscr{C}$ induces such a decomposition.) These last operators have remarkable form (Const – Laplace operator).

Choose $H_0 \in \mathscr{C}$

3.9.1 Lemma. *The rectangle*

$$\begin{tikzcd}
\mathscr{F}^{N-1}(\tilde{\mathscr{C}}_{H_0}) \arrow[r, "S^{N-1}(\tilde{\mathscr{C}}_{H_0})"] \arrow[d, "\wr"] & \mathscr{A}^{N-1}(\tilde{\mathscr{C}}_{H_0}) \arrow[d, "\wr"] \\
\mathscr{F}^{N-1}(\mathscr{C}; H_0) \arrow[d, "\wr"] & \mathscr{A}^{N-1}(\mathscr{C}; H_0) \arrow[d, "(-1)^{N-1}_{d}"] \\
\mathscr{F}^{N}(\mathscr{C}) \arrow[r, "S^{N}(\mathscr{C})"] & \mathscr{A}^{N}(\mathscr{C})
\end{tikzcd}$$

(2.5.2)

(2.5.3)

commutes. Here the right vertical arrow is the restriction of $(-1)^{N-1}d$: $\mathscr{A}^{N-1}(\mathscr{C}) \to \mathscr{A}^{N}(\mathscr{C})$ *to* $\mathscr{A}^{N-1}(\mathscr{C}; H_0) \subset \mathscr{A}^{N-1}(\mathscr{C})$.

Proof. Follows immediately from (3.2.3). □

Now consider the (non-central!) arrangement $\tilde{\mathscr{C}} = \tilde{\mathscr{C}}_{H_0}$ in the $(N-1)$-dimensional space. Let $\tilde{d}$, $\tilde{d}^*$ be differentials in $\mathscr{A}^{\cdot}(\tilde{C})$; $\tilde{\Delta} = \tilde{d}\tilde{d}^* + \tilde{d}^*\tilde{d}$. In particular, $\tilde{\Delta}^{N-1} = \tilde{d}^{N-2}\tilde{d}^{*N-1}$.

3.9.2 Lemma. *The diagram*

$$\begin{tikzcd}
\mathscr{A}^{N-1}(\mathscr{C}; H_0) \arrow[r, "d^{N-1}"] \arrow[d, "\wr"] & \mathscr{A}^{N}(\mathscr{C}) \arrow[d, "\wr"] \\
\mathscr{A}^{N-1}(\tilde{\mathscr{C}}) \arrow[r, "a(v) - \tilde{\Delta}^{N-1}"] & \mathscr{A}^{N-1}(\tilde{\mathscr{C}})
\end{tikzcd}$$

(1.4.2) (1.5.1)

commutes.

Proof. For $H_1 \ldots H_{N-1} \in \mathscr{A}^{N-1}(\mathscr{C}; H_0)$ $d(H_1 \ldots H_{N-1}) = \sum_{H \in \mathscr{C}} a(H) H H_1 \ldots H_{N-1}$. On the other hand, by (1.2.3)

$$HH_1 \ldots H_{N-1} = H_0 H_1 \ldots H_{N-1} - H_0 H \sum_{i=1}^{N-1} (-1)^{i-1} H_1 \ldots \hat{H}_i \ldots H_{N-1},$$

substituting this, we easily deduce the desired assertion. □

3.10 Step II. *Spectrum of Laplace operators*

Let $\hat{\mathscr{C}}$ be a central arrangement with exponents a in $V \cong \mathbb{C}^N$ with a unique vertex v. Fix some $H_0 \in \hat{\mathscr{C}}$; put $\mathscr{C} = \tilde{\mathscr{C}}_{H_0}$ (see 1.4); let d, d^* be differentials in $\mathscr{A}^{\cdot}(\mathscr{C})$, $\Delta = dd^* + d^* d$. We'll identify $\mathscr{C}$ with $\hat{\mathscr{C}} \setminus \{H_0\}$.

For every p put $\tilde{\mathscr{A}}^p = d^*(\mathscr{A}^{p+1}(\mathscr{C})) = \ker(d^*\colon \mathscr{A}^p(\mathscr{C}) \to \mathscr{A}^{p-1}(\mathscr{C})) \subset \mathscr{A}^p(\mathscr{C})$. For any edge L of $\mathscr{C}$ contained in H_0, define the subspace $\mathscr{A}^p(L) \subset \tilde{\mathscr{A}}^p$:

$$\mathscr{A}^p(L) = \{x \in \tilde{\mathscr{A}}^p \mid \mathrm{Supp}\, x \subset L\} \tag{3.10.1}$$

(see 1.8).

Suppose that $L = H_1 \cap \ldots \cap H_r \cap H_0$, $H_i \in \mathscr{C}$. Put

$$\mathscr{A}^p(L)' = (E - d^* H_1)(E - d^* H_2) \ldots (E - d^* H_r)\, \tilde{\mathscr{A}}^p, \tag{3.10.2}$$

where E denotes the identity operator.

3.10.3 Lemma. $\mathscr{A}^p(L) = \mathscr{A}^p(L)'$. *In particular,* $\mathscr{A}^p(L)'$ *depend only on* L.

Proof. Clearly, $\mathscr{A}^p(L)' \subset \tilde{\mathscr{A}}^p$. By (1.8.2) $\mathscr{A}^p(L)' \subset \mathscr{A}^p(L)$. On the other hand, if $x \in \mathscr{A}^p$, $\mathrm{Supp}\, x \subset H \cap H_0$, then $Hx = 0$, whence $(E - d^* H)x = x$, so $\mathscr{A}^p(L) \subset \mathscr{A}^p(L)'$. □

Clearly, if $L' \subset L$, then $\mathscr{A}^p(L') \subset \mathscr{A}^p(L)$. Put

$$\mathscr{G}r^p(L) = \mathscr{A}^p(L) / \sum_{L' \subsetneq L} \mathscr{A}^p(L') \tag{3.10.4}$$

By 1.8.2 and (3.10.1) subspaces $\mathscr{A}^p(L)$ are stable under Δ.

3.10.5 Lemma. *Operator* $a(v) \cdot E - \Delta$ *induces on* $\mathscr{G}r^p(L)$ *the multiplication by* $a(L)$.

Proof. $\Delta|_{\tilde{\mathscr{A}}^p} = \sum_{H \in \mathscr{C}} a(H)\, d^* H|_{\tilde{\mathscr{A}}^p}$. If $H \supset L$ then $H|_{\tilde{\mathscr{A}}^p} = 0$; if $H \not\supset L$ then $(E - d^* H)\, \tilde{\mathscr{A}}^p(L) \subset \tilde{\mathscr{A}}^p(L \cap H)$, so $d^* H$ acts as identity on $\mathscr{G}r^p(L)$, hence Δ acts as $\sum_{H \not\supset L} a(H)$. □

Let $L \in \hat{\mathscr{C}}^{q+1}$, $L \subset H_0$. Consider $\hat{\mathscr{C}}(L)$ as a central arrangement in the normal space to L in V; let $\tilde{\mathscr{C}} = \widetilde{\hat{\mathscr{C}}(L)}_{H_0}$ be the corresponding q-dimensional affine arrangement (1.4). Put

$$\mathscr{E}(L) = \ker(d^*\colon \mathscr{A}^q(\tilde{\mathscr{C}}) \to \mathscr{A}^{q+1}(\tilde{\mathscr{C}})). \tag{3.10.6}$$

By 1.7.4

$$e(L)=\dim \mathscr{E}(L). \tag{3.10.7}$$

Let $\hat{\mathscr{C}}_L$ be the arrangement induced by $\hat{\mathscr{C}}$ on L.

Define maps (for $i \geqq 1$)

$$\mu^i(L)\colon\ \mathscr{E}(L) \otimes \mathscr{A}^i(\tilde{\mathscr{C}}_L) \to \mathscr{G}r^{q+i-1}(L) \tag{3.10.8}$$

as follows. For $x \in \mathscr{E}(L) \subset \mathscr{A}^q(\hat{\mathscr{C}}(L))$, $y=(H_1 \cap L)\cdot(H_2 \cap L)\cdot \ldots \cdot (H_i \cap L) \in \mathscr{A}^i(\tilde{\mathscr{C}}_L)$ $x \cdot d^*(H_1 \ldots H_i)$ lies in $\mathscr{A}^{q+i-1}(L)$. By definition, $\mu^i(L)(x \otimes y)$ is the image of this element in $\mathscr{G}r^{q+i-1}(L)$.

3.10.9 Lemma. *Maps $\mu^i(L)$ are epimorphic.*

Proof. Put $p=q+i-1$. Consider $z=H_1 \ldots H_{p+1} \in \mathscr{A}^{p+1}(\mathscr{C})$, $z \neq 0$, $H \in \mathscr{C}$. Put $z_i = H_1 \ldots \hat{H}_i \ldots H_{p+1}$.

Set $I=I(H)=\{i \in [p+1] \mid H z_i \neq 0\}$, $J=J(H)=[p+1] \setminus I(H)$. For every $I' \subset [p+1]$ $H \cap H_0 \supset (\bigcap_{i \in I'} H_i) \cap H_0$ iff $I' \supset I$. If $I=\{i_1, \ldots, i_r\}$, $i_1 < i_2 < \ldots < i_r$; $J=\{j_1, \ldots, j_s\}$, $j_1 < \ldots < j_s$, put $z_J = H_{j_1} \ldots H_{j_s}$; $z_I = H_{i_1} \ldots H_{i_r}$ (so $z = \pm z_I z_J$). Put $\tilde{d}^*(H z_I) := z_I + \sum_{k=1}^{r} (-1)^k H H_{i_1} \ldots \hat{H}_{i_k} \ldots H_{i_r}$ (note that $H z_I = 0$ in $\mathscr{A}^{\cdot}(\mathscr{C})$!). One can easily show that

$$(E - d^* H)\, d^* z = \pm \tilde{d}^*(H z_I)\, d^*(z_J). \tag{3.10.9.1}$$

Analogously if we have t hyperplanes $H^{(1)}, \ldots, H^{(t)} \in \mathscr{C}$ such that $I(H^{(1)}) \subset I(H^{(2)}) \subset \ldots \subset I(H^{(t)})$, put $K^i = I(H^{(i)}) \setminus I(H^{(i-1)})$, $K^1 = I(H^{(1)})$. Then

$$\begin{aligned}(E - d^* H^{(t)})(E - d^* H^{(t-1)}) \ldots (E - d^* H^{(1)})\, d^* z \\ = \pm \tilde{d}^*(H^{(1)} z_{K^1})\, \tilde{d}^*(H^{(2)} z_{K^2}) \ldots \tilde{d}^*(H^{(t)} z_{K^t})\, d^* z_{J(H^{(t)})}.\end{aligned} \tag{3.10.9.2}$$

The lemma follows from this and from Definition 3.10.2. □

3.11 Step III. End of the proof of Theorem 3.7 We'll prove (3.7.1) by induction on N. Maps S^p respect decompositions 1.3.2 and (2.1.2); hence we may, and will, suppose that $\mathscr{C}$ is central with a unique vertex v, and $p=0$.

Choose $H_0 \in \mathscr{C}$, and consider $\tilde{\mathscr{C}} = \mathscr{C}_{H_0}$. By induction det $S^{N-1}(\tilde{\mathscr{C}})$ is equal to

$$\text{Const} \prod_{i \geqq 0} \prod_{L \in \tilde{\mathscr{C}}_i} a(L)^{e(L)\, d^i(L)}.$$

So by 3.9.1 and 3.9.2 we have to show that

$$\det(a(v) \cdot E - \tilde{\Delta}^{N-1}) \sim \prod_{i \geqq 0} \prod_{\substack{L \in \mathscr{C}_i \\ L \subset H_0}} a(L)^{e(L)\, d^i(L)}. \tag{3.11.1}$$

Clearly, $a(v)\cdot E-\tilde{\Delta}^{N-2}$ acts as multiplication by $a(v)^{e(v)}$ on the subgroup $\tilde{A}^{N-1}=\ker(d^*\colon \mathscr{A}^{N-1}(\tilde{\mathscr{C}})\to\mathscr{A}^{N-2}(\tilde{\mathscr{C}}))\subset\mathscr{A}^{N-1}(\tilde{\mathscr{C}})$. So, it remains to show that

(3.11.2) $$\det(a(v)\cdot E-\tilde{\Delta}^{N-2})|_{\operatorname{Im} d^*}\sim\prod_{i\geqq 1}\prod_{\substack{L\in\mathscr{C}_j\\ L\subset H_0}} a(L)^{e(L)\,d^i(L)}.$$

By 3.10.5 and 3.10.9 the left hand side of (3.11.2) divides the right hand side. On the other hand, because of independence of $\det S^N$ on the choice of H_0, and by induction on N, the r.h.s. divides the l.h.s., so they are proportional. This completes the proof of Theorem 3.7. □

3.12 Corollary. *Maps* $\mu^i(L)$ (3.10.8) *are isomorphisms.*

4 Topology

Let $\mathscr{C}$ be an arrangement in V, $\dim V=N$. Put $Y(\mathscr{C})=\bigcup_{H\in\mathscr{C}} H\subset V$; $U(\mathscr{C})=V\setminus Y(\mathscr{C})$.

4.1 Orlik-Solomon theorem. Denote by $\Omega^{\cdot}(U(\mathscr{C}))$ the de Rham complex of holomorphic forms over $U(\mathscr{C})$. One has canonical isomorphism

(4.1.1) $$H^{\cdot}(\Omega^{\cdot}(U(\mathscr{C})))\cong H^{\cdot}(U(\mathscr{C});\mathbb{C}).$$

For any $H\in\mathscr{C}$ choose an affine equation $\ell_H=0$ of H; assign to H a l-form

(4.1.2) $$\iota(H)=d\log\ell_H\in\Omega^1(U(\mathscr{C})).$$

$\iota(H)$ doesn't depend on the choice of ℓ_H. This assignment defines an inclusion of graded $\mathscr{C}$-algebras

(4.1.3) $$\iota\colon\ \mathscr{A}^{\cdot}(\mathscr{C})\to\Omega^{\cdot}(U(\mathscr{C}))$$

which is compatible with differentials if we imply the zero differential on $\mathscr{A}^{\cdot}(\mathscr{C})$ [Br, OS].

4.1.4 Theorem (Brieskorn-Orlik-Solomon). ι *is a quasi-isomorphism, i.e. it induces the isomorphism*

$$\mathscr{A}^{\cdot}(\mathscr{C})=H^{\cdot}(U(\mathscr{C});\mathbb{C})$$

which maps $\mathscr{A}^p(\mathscr{C})_{\mathbb{Z}}$ *isomorphically to* $H^p(U(\mathscr{C});(2\pi i)^p\mathbb{Z})$.

Proof. Follows from [Br, OS] and 1.3.2. □

4.1.5 Corollary.

$$\chi(\mathscr{C})=\sum_{i=0}^{N}(-1)^{N-i}\,rk\,H^i(U(\mathscr{C});\mathbb{Z}).$$

4.2 For a topological space X denote by $C_{\cdot}(X)$ (resp., $C^{\cdot}(X)$) the complex of integer-valued singular chains (resp., cochains) of X. Put $C_{\cdot}(V,Y(C)):=C_{\cdot}(V)/C_{\cdot}(Y(\mathscr{C}))$, $C^{\cdot}(V,Y(\mathscr{C})=(C_{\cdot}(V,Y(\mathscr{C})))^*$.

The aim of this and the next n^0 is to construct (well defined up to a homotopy) quasiisomorphisms

$$C^{\cdot}(V, Y(\mathscr{C})) \to (\mathscr{F}^{\cdot}(\mathscr{C})_{\mathbb{Z}}, d).$$

For an edge L of $\mathscr{C}$ put

$$L^0 = L \setminus \Big(\bigcup_{\substack{H \in \mathscr{C} \\ H \not\supset L}} H \cap L \Big); \qquad V^0 := U(\mathscr{C}). \tag{4.2.1}$$

Let us call a *marking* x of $\mathscr{C}$ a choice of a point $x(L) \in L^0$ for every edge L (including V). Fix such a marking x. To every flag $F = (L^1 \supset \ldots \supset L^p)$, $L^i \in \mathscr{C}^{j_i}$; let us assign the singular p-simplex $\Delta(F) = \Delta_x(F)$: $\Delta^p \to V$, where $\Delta^p = \{(t_0, \ldots, t_p) \in \mathbb{R}^{p+1} \mid t_i \geqq 0; \sum t_i = 1\}$ is the standard oriented p-simplex. Namely, if $x_i = (0, \ldots 0, 1, 0 \ldots 0)$ with 1 on i-th place, put

$$\Delta(F)(x_i) = x(L^i), \qquad \Delta(F)(0) = x(V), \tag{4.2.2}$$

and extend this map to Δ^p by $\mathbb{R}$-affinity.

Now define maps

$$u^p = u^p_x \colon \; C^p(V, Y(\mathscr{C})) \to \mathscr{F}^p(\mathscr{C})_{\mathbb{Z}} \tag{4.2.3}$$

by the rule

$$u^p(\varphi) = \sum_{F \in Fl^p(\mathscr{C})} \varphi(\Delta(F)) \tag{4.2.4}$$

for $\varphi \in C^p(V, Y(\mathscr{C}))$. One verifies directly that u^p define the map of complexes

$$u^{\cdot} = u^{\cdot}_x \colon \; C^{\cdot}(V, Y(\mathscr{C})) \to (\mathscr{F}^{\cdot}(\mathscr{C})_{\mathbb{Z}}, d). \tag{4.2.5}$$

4.2.6 Lemma. *Any pair of markings x, x' defines a canonical homotopy*

$$h(x, x') \colon \; C^{\cdot}(V, Y(\mathscr{C})) \to \mathscr{F}^{\cdot}(\mathscr{C})_{\mathbb{Z}}[-1] \tag{4.2.6.1}$$

between u_x and $u_{x'}$. In particular, the map induced by (4.2.5) *in homology is independent of the choice of the marking.*

For a triple of markings x, x', x'' $h(x, x') + h(x', x'')$ is homotopic to $h(x, x'')$. (Recall that for a graded group $A = \oplus A^p$ $A[n]$ denotes the shifted graded group $(A[n])^p = A^{p+n}$.)

Proof. To construct $h(x, x')$, it is sufficient to suppose that $x(L) = x'(L)$ for all edges L except for a one. Suppose for example that $x(L) = x'(L)$ for all L except for V. Then we can assign to every p-flag $F = (L^1 \ldots L^p)$ a $(p+1)$-simplex $\hat{\Delta}(F)$ – an affine span of $(x'(V), x(V), x(L^1), \ldots, x(L^p))$. This defines the map (4.2.6.1). Details are left to the reader. $\square$

4.3 Theorem. *Maps* (4.2.5) *are quasiisomorphisms.*

Proof. We'll prove that the adjoint maps

$$u^{\cdot *} \colon \; (\mathscr{A}^{\cdot}(\mathscr{C})_{\mathbb{Z}}, d^*) \to C_{\cdot}(V, Y(\mathscr{C})) \tag{4.3.1}$$

are quasiisomorphisms.

Let us choose some linear order on $\mathscr{C}$. Consider the bicomplex $C_{\bullet\bullet}$ with

$$C_{pq} = \bigoplus_{i_1 < \ldots < i_q} C_p(H_{i_1} \cap \ldots \cap H_{i_q});$$
$$C_{p0} = C_p(V).$$

Differentials $C_{pq} \to C_{p-1,q}$ are boundary operators is $C_\bullet$, and $C_{pq} \to C_{p,q-1}$ are Čech differentials. Denote by Tot $C_{\bullet\bullet}$ the corresponding simple complex.

We have evident map Tot $C_{\bullet\bullet} \to C_\bullet(V, Y(\mathscr{C}))$ which is quasiisomorphism by Mayer-Vietoris. The spectral sequence associated with $C_{\bullet\bullet}$ degenerates at E^1 and E^1 is isomorphic to the complex $(\mathscr{E}_\bullet(\mathscr{C}), \partial)$, studied in 1.9. So, by 1.9.6 we have canonical isomorphisms $H^\bullet(\mathscr{A}^\bullet(\mathscr{C})_{\mathbb{Z}}, d^*) = H_\bullet(V, Y(\mathscr{C}))$.

Any marking of $\mathscr{C}$ induces the map $\mathscr{A}^\bullet(\mathscr{C})_{\mathbb{Z}} \to \operatorname{Tot} C_{\bullet\bullet}$ whose composition with Tot $C_{\bullet\bullet} \to C_\bullet(V, Y(\mathscr{C}))$ is equal to the map (4.3.1). One easily sees that the induced map in homology is inverse to the one constructed by using the spectral sequence. □

4.4 By 4.1.4 and 2.4 we have natural isomorphisms

$$\mathscr{F}^p(\mathscr{C})_{\mathbb{Z}} \cong H_p(U(\mathscr{C}), \mathbb{Z}). \tag{4.4.1}$$

Let us construct these isomorphisms explicitly. Namely, to each flag $F = (L^1 \supset \ldots \supset L^p)$ we associate a map (defined up to a homotopy)

$$c(F)\colon (S^1)^p \to U(\mathscr{C}) \tag{4.4.2}$$

from a p-dimensional torus to $U(\mathscr{C})$. The image under $c(F)$ of a canonical generator of $H_p((S^1)^p, \mathbb{Z})$ in $H_p(U(\mathscr{C}), \mathbb{Z})$ will give the map (4.4.1).

We define $c(F)$ by induction on p. For $p=1$ it is a small circle around L^1. For $p>1$ $c(F)$ is a product of a circle around L^1 by $c(F')$ where $F' = (L^2 \supset \ldots \supset L^p)$ considered as a flag in L^1.

Alternatively, one may define $c(F)$ as follows. Suppose that $p=N$. Choose some $H \in \mathscr{C}$ such that $H \cap L^{N-1} = L^N$. Such a choice defines a diffeomorphism of $U(\mathscr{C}) \cap$(small neighbourhood of L^N) with $\mathbb{C}^* \times U(\tilde{\mathscr{C}})$ where $\tilde{\mathscr{C}} = \mathscr{C}_H$ (see 1.4). We put $c(F)$ to be the product of a circle around zero in $\mathbb{C}^*$ by $c(L^1 \supset \ldots \supset L^{N-1})$: $(S^1)^{p-1} \to U(\tilde{\mathscr{C}})$ defined by induction.

From the second description one can show that

$$\int_{c(F)} \imath(H_1 \ldots H_p) = (-1)^{|\sigma|} \tag{4.4.3}$$

if $F = F(H_{\sigma(1)}, \ldots, H_{\sigma(p)})$ for some $\sigma \in \Sigma_p$, and zero otherwise. It follows that the assignment $F \mapsto c(F)$ induces the isomorphism (4.4.1). Note that we have obtained the geometric interpretation of isomorphisms 2.3.3.

4.5 Now suppose that a collection of exponents $a\colon \mathscr{C} \to \mathbb{C}$ is given. Denote by $\mathscr{L}(a)$ a line bundle over $U(\mathscr{C})$ with an integrable connection which is trivial as a bundle and whose connection

$$d(a)\colon \mathscr{O} \to \Omega^1 \tag{4.5.1}$$

is equal to $d + \imath(\omega) = d + \Sigma_H\, a(H)\, d \log \ell_H$, d being the de Rham differential. Denote by $\Omega^\bullet(\mathscr{L}(a))$ the complex of $U(\mathscr{C})$-sections of the holomorphic de Rham complex of $\mathscr{L}(a)$. Thus, $\Omega^{\cdot}(\mathscr{L}(a))=\Omega^{\cdot}(U(\mathscr{C}))$ as a graded vector space, with the differential $d(a)$,

$$d(a)(x)=dx+\imath(\omega)\wedge x. \tag{4.5.2}$$

We'll denote by $\mathscr{S}(a)$ the sheaf (over $U(\mathscr{C})$) of horizontal sections of $\mathscr{L}(a)$. Thus, we have the de Rham isomorphism

$$H^{\cdot}(\Omega^{\cdot}(\mathscr{L}(a)))=H^{\cdot}(U(\mathscr{C});\mathscr{S}(a)). \tag{4.5.3}$$

The inclusion (4.1.3) induces the map of complexes

$$\imath(a)\colon\ (\mathscr{A}^{\cdot}(\mathscr{C}),d(a))\to\Omega^{\cdot}(\mathscr{L}(a)). \tag{4.5.4}$$

When $a\equiv 0$, we return to the situation of n° 4.1. Simple examples for $\dim V=1$ show that (4.5.4) is not always a quasiisomorphism.

Let us consider the family $\imath(\lambda a)$ depending on $\lambda\in\mathbb{C}$ (by definition, $(\lambda a)(H)=\lambda a(H)$). Here is the main result of this section.

4.6 Theorem. There exists an open set $W\subset\mathbb{C}$ containing 0, *which is a complement of the set of zeroes of some holomorphic function and such that $\imath(\lambda a)$ is a quasiisomorphism for all $\lambda\in W$.*

Proof. Let us choose a finite subcomplex $X\subset U(\mathscr{C})$ such that this inclusion is a homotopy equivalence. Let us consider the cochain complex of X with coefficients in $\mathscr{S}(\lambda a)$: $C^{\cdot}=C^{\cdot}(X;\ \mathscr{S}(\lambda a)|_X)$. Let us choose over each cell of X branches of $\log\ell_H$ for all $H\in\mathscr{C}$. This gives branches of $\ell^{\lambda a}:=\prod_{H\in\mathscr{C}}\exp(\lambda a(H)\log\ell_H)$, hence a base in each finite dimensional space C^p, enumerated by p-cells of X. Differentials $d(\lambda)\colon C^p\to C^{p+1}$ are matrices with entries of the form $\pm\exp(\lambda b)$, $b\in\mathbb{C}$.

By the de Rham theorem, the integration map

$$\textstyle\int\colon\ \Omega^{\cdot}(\mathscr{L}(\lambda a))\to C^{\cdot} \tag{4.6.1}$$

is a quasiisomorphism. Denote by

$$I(\lambda)\colon\ (\mathscr{A}^{\cdot}(\mathscr{C}),\lambda d(a))\to(C^{\cdot},d(\lambda)) \tag{4.6.2}$$

the composition of $\imath(\lambda a)$ and (4.6.1).

$I(\lambda)$ is a map of complexes of finite dimensional vector spaces. $I(\lambda)$ may be represented as a convergent in some neighbourhood of 0 power series

$$I(\lambda)=\sum_{i=0}^{\infty}I_i\cdot\lambda^i. \tag{4.6.3}$$

The same is true for $d(\lambda)$:

$$d(\lambda)=\sum_{i=0}^{\infty}d_i\cdot\lambda^i. \tag{4.6.4}$$

4.6.5 Lemma. *Let $(A^{\bullet}, d(\lambda))$ be a complex of finite dimensional complex vector spaces with the differential depending on a complex parameter λ holomorphic in a neighbourhood of zero. Suppose that $(A^{\bullet}, d(0))$ is acyclic.*

Then $(A^{\bullet}, d(\lambda))$ is acyclic for sufficiently small λ.

Proof. Let $d(\lambda)=\sum_{i\geqq 0} d_i\lambda^i$ be the Taylor series of $d(\lambda)$ at 0. We have $d(\lambda)^2=0$ which means that

$$(4.6.5.1)\qquad \sum_{p=0}^{i} d_p d_{i-p}=0$$

for all $i\geqq 0$.

Let us consider the space of cocycles $Z^p(\lambda)=\mathrm{Ker}(d^p(\lambda)\colon A^p\to A^{p+1})$. One can see that $Z^p(\lambda)$ for sufficiently small λ is generated by $x(\lambda)=\sum_{i\geqq M} x_i\lambda^i$, $M\in\mathbb{Z}$, such that $d(\lambda)x(\lambda)=0$. Let $x(\lambda)$ be such a cocycle; we have to show that it is a coboundary for small λ. Multiplying it by λ^M we may suppose that $M=0$, i.e. $x(\lambda)=\sum_{i\geqq 0} x_i\lambda^i$. We have

$$(4.6.5.2)\qquad \sum_{p=0}^{i} d_p x_{i-p}=0$$

for all $i\geqq 0$.

Let us look for $y(\lambda)=\sum_{i\geqq 0} y_i\lambda^i$ such that

$$(4.6.5.3)\qquad d(\lambda)\,y(\lambda)=x(\lambda).$$

Step I. There exists a formal solution of (4.6.5.3). In fact, (4.6.5.3) is equivalent to the infinite system

$$(4.6.5.3)_i\qquad \sum_{p=0}^{i} d_p y_{i-p}=x_i,$$

$i\geqq 0$. Let us solve it by induction on i. We have $d_0x_0=0$, so $x_0=d_0y_0$ for some y_0 since $(A^{\bullet}, d_0)$ is acyclic. Suppose we have solved (4.6.5.3) for $i<k$. We have to find y_k such that

$$(4.6.5.4)\qquad x_k=d_k y_0+d_{k-1}y_1+\ldots+d_0 y_k.$$

It suffices to show that

$$d_0\left(x_k-\sum_{h=1}^{k} d_h y_{k-n}\right)=0.$$

We have by induction

$$d_0 x_k = -\sum_{i=1}^{k} d_i x_{k-i} = -\sum_{i=1}^{k} \sum_{p=0}^{k-i} d_i d_p y_{k-i-p}$$

$$= -\sum_{n=1}^{k} \left(\sum_{p=0}^{k-1} d_{n-p} d_p \right) y_{k-n} = \sum_{n=1}^{k} d_0 d_n y_{k-n},$$

and we are done.

Step II. We can choose a solution $y(\lambda)$ to (4.6.5.3) convergent in a neighbourhood of zero.

Indeed, since $x(\lambda)$, $d(\lambda)$ are convergent for small λ, there exist constants A, $M>1$ such that $|x_i|$, $|d_i|<AM^i$ for all i. On every step of solving (4.6.5.4) we solve an equation of the form $x=d_0 y$. We may suppose that, given x such that $d_0 x=0$, we choose y such that $|y|\leq M|x|$. Then one shows by induction that $|y_i|<A\, 2^i M^{2i+1}$. The Lemma is proved. □

4.6.6 Corollary. *Let $f(\lambda)$: $(A^{\bullet}, d(\lambda))\to(A'^{\bullet}, d'(\lambda))$ be a map of complexes of finite dimensional complex vector spaces with $f(\lambda)$, $d(\lambda)$, $d'(\lambda)$ depending holomorphically on λ for small λ. Suppose that $f(0)$ is a quasiisomorphism.*

Then $f(\lambda)$ is a quasiisomorphism for small λ.

Proof. Apply 4.6.5 to the cone of $f(\lambda)$. □

Now, applying the above corollary to $I(\lambda)$ (4.6.2) we get the assertion of Theorem 4.6. □

4.6.7 Remark. The idea of the above proof arose from the discussion with Misha Kapranov who explained to us the work [N]. We are very gratefull to him.

4.6.8 Remark. Theorem 4.6 proves a weak form of a conjecture made by Aomoto [A].

4.7 Remarks on relative cohomology. Let us denote by $C_{\bullet\ell f}(U(\mathscr{C});\ \mathscr{S}(a))$ the complex of complex singular chains whose symplices may have a boundary on $Y(\mathscr{C})$. Let $C^{\bullet}_{\ell f}$ be the corresponding cochain complex. By definition

$$H^{\bullet}(C^{\bullet}_{\ell f}(U(\mathscr{C});\mathscr{S}(a)))=H^{\bullet}(V, j_!\,\mathscr{S}(a)) \tag{4.7.1}$$

where j: $U(\mathscr{C})\hookrightarrow V$.

Fix a marking x of $\mathscr{C}$ as in 4.2, and branches of $\log \ell_H$, $h\in\mathscr{C}$ in a neighbourhood of $x(V)$. Then the rule (4.2.4) defines the map of complexes

$$u^{\bullet}(a)\colon\ C^{\bullet}_{\ell f}(U(\mathscr{C});\mathscr{S}(a))\to(\mathscr{F}^{\bullet}(\mathscr{C}), d). \tag{4.7.2}$$

It seems very probable that it is always a quasiisomorphism (cf. 4.3).

Consider a symplex $\Delta(F)$ for some $F\in Fl^p(\mathscr{C})$. Put as usually $\ell^{\lambda a}=\prod_H \exp(\lambda a(H)\log \ell_H)$.

4.7.3 Lemma. *Suppose that* $a(F) \neq 0$ *(cf.* (3.4.1)*). For any* $H_1, \ldots, H_p \in \mathscr{C}$

$$\int_{\Delta(F)} \iota(H_1 \ldots H_p)\, \ell^{\lambda a} = \frac{A}{\lambda^p} + O(\lambda^{-p+1}) \qquad (\lambda \to 0)$$

where

$$A = \frac{(-1)^{|\sigma|}}{a(F)}$$

if $F = F(H_{\sigma(1)}, \ldots, H_{\sigma(p)})$ *for some* $\sigma \in \Sigma_p$, *and* $A = 0$ *otherwise (cf.* 3.4.2*).*

Proof. The case $p=1$ is elementary. The general case is reduced to the p-th power of this one, using the monoidal transformation associated with F. □

Suppose that $\mathscr{C}$ has a vertex, $\dim V = N$. Consider the canonical map

$$\tilde{S}(\lambda a)\colon\ H^N(V, j_! \mathscr{S}(\lambda a)) \to H^N(V, j_* \mathscr{S}(\lambda a)).$$

Suppose that (4.7.2) is a quasiisomorphism, and $a(L) \neq 0$ for every edge L of $\mathscr{C}$.

If we identify $H^N(V, j_! \mathscr{S}(\lambda a))$ with $H^N(\mathscr{F}^\cdot(\mathscr{C}))$ and $H^N(V, j_* \mathscr{S}(\lambda a))$ with $H^N(\mathscr{A}^\cdot(\mathscr{C}), d(\lambda a))$ for small λ by 4.6, then from 4.7.3 and 3.4.4 follows that

$$\tilde{S}(\lambda a) = \lambda^N S^N(a) + O(\lambda^{N-1}) \qquad (\lambda \to 0) \tag{4.7.4}$$

(Note that $S^N(\lambda a) = \lambda^N S^N(a)$.)

Presumably, the same holds without the assumption that $a(L) \neq 0$ for all L.

4.8 Remarks on Jantzen filtration. Using the form $S(a)$, one can define the Jantzen filtration on groups $\mathscr{A}^\cdot(\mathscr{C})$, cf. [J]. For arrangements studied in Part II it gives in particular the usual Jantzen filtration on Verma modules. Consider an arrangement $\hat{\mathscr{C}}$ defined after (1.4.2). Put $a(H_\infty) = -\sum_{H \in \mathscr{C}} a(H)$. We have $\mathscr{A}^\cdot(\hat{\mathscr{C}}) \cong \mathscr{A}^\cdot(\mathscr{C}) \oplus \mathscr{A}^\cdot(\mathscr{C})$ [1]. Suppose that conditions of 4.6 fulfilled, so that we can identify $H^\cdot(\mathscr{A}^\cdot(\mathscr{C}), d(a))$ with $H^\cdot(U(\mathscr{C}), \mathscr{S}(a))$. It seems then plausible that the filtration on $H^\cdot(\mathscr{A}^\cdot(\mathscr{C})$ [1]) induced by the Jantzen filtration on $H^\cdot(\mathscr{A}^\cdot(\hat{\mathscr{C}}))$ coincides with the weight filtration on $H^\cdot(U(\mathscr{C});\ \mathscr{S}(a))$ when the last one is defined. cf. [BB, § 5].

In particular, it would be very interesting to compare $H^\cdot(\mathscr{F}^\cdot(\mathscr{C}))$ (cf. 3.5) with the Goresky-MacPherson cohomology $H^\cdot(U(\mathscr{C}); j_{!*} \mathscr{S}(a))$.

Part II. Discriminantal arrangements and Lie algebra homology

5 *Free Lie algebras*

5.1 Discriminantal arrangements. Let $\mathbb{C}^N$ be an affine complex space with coordinates $t_1, \ldots, t_N$. We'll denote by $\mathscr{C}_N$ the arrangement in $\mathbb{C}^N$ consisting of hyperplanes

$$H_{ij}\colon\ t_i - t_j = 0; \qquad 1 \leqq i < j \leqq N. \tag{5.1.1}$$

So, $\mathscr{C}_1 = \phi$.

Thus, $U(\mathscr{C}_N)$ is the space of N-tuples of distinct points in $\mathbb{C}^1$. Let $z = (z_1, \ldots, z_n) \in U(\mathscr{C}_n)$. We'll denote by $\mathscr{C}_{n;N}(z)$ an arrangement in $\mathbb{C}^N$ consisting of hyperplanes

$$(5.1.2) \qquad H_i^j\colon\ t_i - z_j = 0, \quad 1 \leqq i \leqq N; \quad 1 \leqq j \leqq n$$
$$H_{ij}; \quad 1 \leqq i < j \leqq N.$$

Thus,

$$(5.1.3) \qquad U(\mathscr{C}_{n;N}(z)) = p^{-1}(z),$$

where $p\colon U(\mathscr{C}_{n+N}) \to U(\mathscr{C}_n)$ is the projection on the first n coordinates. We put $\mathscr{C}_{0,N} := \mathscr{C}_N$.

5.2 The symmetric group Σ_N acts on $\mathbb{C}^N$ by the permutation of coordinates. This induces the action of Σ_N on sets of edges of $\mathscr{C}_{n,N}(z)$, $n \geqq 0$. Introduce the action of Σ_N on groups $\mathscr{A}^{\bullet}(\mathscr{C}_{n,N}(z))$, $\mathscr{F}^{\bullet}(\mathscr{C}_{n,N}(z))$ by the rules

$$(5.2.1) \qquad \sigma(H_1 \cdot \ldots \cdot H_p) = (-1)^{|\sigma|} \sigma H_1 \cdot \ldots \cdot \sigma H_p$$

$$(5.2.2) \qquad \sigma(L^1 \supset \ldots \supset L^p) = (-1)^{|\sigma|} (\sigma L^1 \supset \ldots \supset \sigma L^p).$$

Isomorphisms $\varphi^{\bullet}$ (2.3.3) respect this action.

5.3 Fix an integer $r \geqq 1$, and denote by $\mathfrak{n}$ the free Lie algebra on generators $f_1, \ldots, f_r$, and by $U\mathfrak{n}$ its envelopping algebra. For an N-tuple $I = (i_1, \ldots, i_N) \in [r]^N$ put

$$(5.3.1) \qquad f_I = f_{I_N} f_{I_{N-1}} \cdots f_{i_1} \in U\mathfrak{n}$$

$$(5.3.2) \qquad [f_I] = [f_{i_N}, [f_{i_{N-1}}, [\ldots [f_{i_2}, f_{i_1}] \ldots] \in \mathfrak{n}.$$

More generally, for n tuples $I_1, \ldots, I_n$; $I_j \in [r]^{N_j}$, put

$$(5.3.3) \qquad f_{I_1, \ldots, I_n} = f_{I_1} \otimes \ldots \otimes f_{I_n} \in (U\mathfrak{n})^{\otimes n}.$$

Let $\lambda = (k_1, \ldots, k_r) \in \mathbb{N}^r$. Put $|\lambda| := \sum_{j=1}^{r} k_j = N$. Denote by $\mathscr{P}(\lambda)$ the set of all N-tuples $I = (i_1, \ldots, i_N) \in [r]^N$ such that $\operatorname{card}\{p \mid i_p = j\} = k_j$ for all j, $1 \leqq j \leqq r$. Set $(U\mathfrak{n})_\lambda$ (resp., $\mathfrak{n}_\lambda$) to be the subspace of $U\mathfrak{n}$ (resp., of $\mathfrak{n}$) generated by all f_I (resp., $[\mathrm{f}_I]$), $I \in \mathscr{P}(\lambda)$. Clearly, $\mathfrak{n}_\lambda = \mathfrak{n} \cap (U\mathfrak{n})_\lambda$.

More generally, let $\Lambda^p\mathfrak{n}$ denote the p-th exterior power. Denote by $[\Lambda^p\mathfrak{n} \otimes (U\mathfrak{n})^{\otimes n}]_\lambda$ the subspace of $\Lambda^p\mathfrak{n} \otimes (U\mathfrak{n})^{\otimes n}$ generated by all

$$(5.3.4) \qquad [f_{J_1}] \wedge [f_{J_2}] \wedge \ldots \wedge [f_{J_p}] \otimes f_{I_1, \ldots, I_n}$$

such that $J_1 \cup \ldots \cup J_p \cup I_1 \cup \ldots \cup I_n \in \mathscr{P}(\lambda)$.

We have weight decompositions

$$(5.3.5) \qquad \Lambda^p\mathfrak{n} \otimes (\mathrm{U}\mathfrak{n})^{\otimes n} = \bigoplus_{\lambda \in \mathbb{N}^r} [\Lambda^p\mathfrak{n} \otimes (U\mathfrak{n})^{\otimes n}]_\lambda.$$

5.4 For a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module M we'll denote by $C_\bullet(\mathfrak{g}, M)$ the standard chain complex of $\mathfrak{g}$ with coefficients in M. So,

$$C_p(\mathfrak{g}, M) = \Lambda^p \mathfrak{g} \otimes M; \tag{5.4.1}$$

$$d(g_p \wedge \ldots \wedge g_1 \otimes m) = \sum_{i=1}^{p} (-1)^{i-1} g_p \wedge \ldots \wedge \hat{g}_i \wedge \ldots \wedge g_1 \otimes g_i m + \sum_{1 \leq i < j \leq p} (-1)^{i+j} g_p \wedge \ldots \wedge \hat{g}_j \wedge \ldots \wedge \hat{g}_i \wedge \ldots \wedge g_1 \wedge [g_j, g_i] \otimes m. \tag{5.4.2}$$

In notations of 5.3, we have the weight decompositions

$$C_\bullet(\mathfrak{n}, (U\mathfrak{n})^{\otimes n}) \cong \bigoplus_{\lambda \in \mathbb{N}^r} C_\bullet(\mathfrak{n}, (U\mathfrak{n})^{\otimes n})_\lambda \tag{5.4.3}$$

5.5 Let $\lambda = (k_1, \ldots, k_r) \in \mathbb{N}^r$, $N = |\lambda|$. Put

$$\Sigma_\lambda = \Sigma_{k_1} \times \ldots \times \Sigma_{k_r}. \tag{5.5.1}$$

We consider Σ_λ as a subgroup of $\Sigma_{|\lambda|}$, with Σ_{k_j} acting by permutations on $\left\{\sum_{p=1}^{j-1} k_p + 1, \sum_{p=1}^{j-1} k_p + 2, \ldots, \sum_{p=1}^{j} k_p\right\} \subset [N]$. So, Σ_λ acts on $\mathscr{F}^\bullet(\mathscr{C}_{|\lambda|})$, $\mathscr{A}^\bullet(\mathscr{C}_{|\lambda|})$, etc. Fix $z = (z_1, \ldots, z_n) \in U(\mathscr{C}_n)$ and put $\mathscr{C}_{n;N} = \mathscr{C}_{n;N}(z)$.

The aim of this § is to construct canonical isomorphisms

$$\Psi = \Psi_{(\lambda, p, n)}\colon C_p(\mathfrak{n}, (U\mathfrak{n})^{\otimes n}) \xrightarrow{\sim} \mathscr{F}^{N-p}(\mathscr{C}_{n;N}) \tag{5.5.2}$$

defined for all p if $n \geq 1$, and for $p \geq 1$ if $n = 0$. These isomorphisms will be compatible with differentials in $C_\bullet$ and $\mathscr{F}^\bullet$.

We'll construct (5.5.2) after some preliminaries on the geometry of the arrangement $\mathscr{C}_{n;N}$.

5.6 Edges and flags of $\mathscr{C}_{n;N}$. For a non-empty subset $J = \{j_1, \ldots, j_p\} \subset [N]$ put

$$L_J = H_{j_1 j_2} \cap H_{j_2 j_3} \cap \ldots \cap H_{j_{p-1} j_p} \in \mathscr{C}^{p-1}_{n;N}. \tag{5.6.1}$$

So, if $p = 1$, then $L_J = \mathbb{C}^N$. For $i \in [n]$ put

$$L^i_J = H^i_{j_1} \cap H^i_{j_2} \cap \ldots \cap H^i_{j_p} \in \mathscr{C}^p_{n;N}. \tag{5.6.2}$$

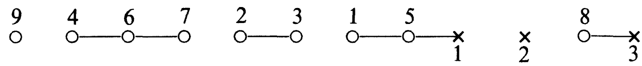
Put $L^i_\emptyset = \mathbb{C}^N$.

Given non-intersecting subsets $J_1, \ldots, J_p; I_1, \ldots, I_n \subset [N]$, put

$$L_{J_1, \ldots, J_p; I_1, \ldots, I_n} = \left(\bigcap_{k=1}^{p} L_{J_k}\right) \cap \left(\bigcap_{i=1}^{n} L^i_{I_i}\right). \tag{5.6.3}$$

It is clear that edges of the shape (5.6.3) exhaust all edges of $\mathscr{C}_{n;N}$. It is convenient to picture an edge (5.6.3) as a graph of the following sort.

It has two kinds of vertices: N circles and n crosses which are enumerated by numbers from 1 to N and from 1 to n respectively. For each J_k (resp. I_k) we successively connect by edges all circles o_j, $j \in J_k$ (resp., all circles o_i, $i \in I_k$ and a cross x_k) in the order induced from [N] (we put x_k to the end of the chain). So we get a picture like this:

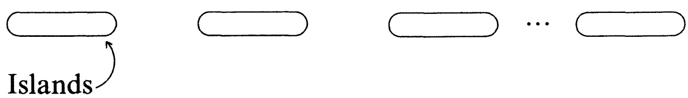


We'll call connected components of a graph *islands*. Islands that do not contain a cross are called *swimming*; they correspond to sets J_k; those that contain a cross are called *fixed*; the corresponding to sets I_k. All the picture is called *an archipelago.*

Edges of $\mathscr{C}_{n;N}$ are in one-to-one correspondence with archipelagos. Given an edge L, to define an edge $L' \subset L$ of codimension 1 is the same as to connect two swimming islands of L or to fasten a swimming islands to a fixed one. (It is not allowed to connect two fixed islands.)

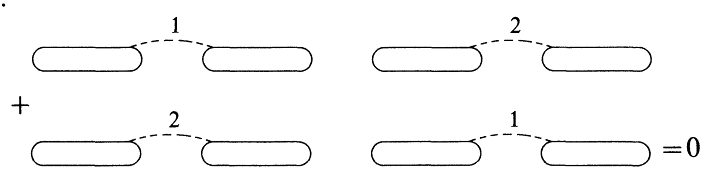
A flag is a growing archipelago.

5.6.4 Relations in flag groups $\mathscr{F}^{\bullet}(\mathscr{C}_{n;N})$. Suppose we have a non-complete flag $F = (L^1 \supset \ldots \supset L^i \supset L^{i+2} \supset \ldots \supset L^p)$. Let us draw the archipelago of L^i:



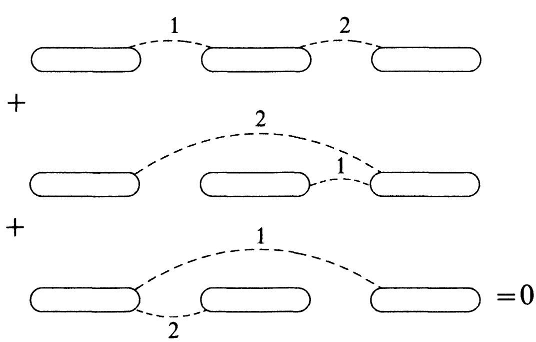
(we don't distinguish fixed and swimming islands). To pass from L^i to L^{i+2} we have to make twice a connection of two islands. This gives two types of relations in $\mathscr{F}^{\bullet}$:

(5.6.4) (a)



and

(5.6.4) (b)



where we labelled connecting edges to represent the order of connections.

5.7 Multiplication of flags. Given two subsets $J \subset [N]$, $I \subset [n]$, denote by $\mathscr{C}_{I;J} \subset \mathscr{C}_{n;N}$ the subset consisting of all hyperplanes $H_{j_1 j_2}$ with $j_1, j_2 \in J$, and H^i_j with $j \in J$, $i \in I$.

Given subsets J, $J' \subset [N]$; I, $I' \subset [n]$ such that $J \cap J' = \emptyset$; $I \cap I' = \emptyset$, define maps

$$(5.7.1) \qquad \circ\colon Fl^p(\mathscr{C}_{I;J}) \times Fl^q(\mathscr{C}_{I',J'}) \to Fl^{p+q}(\mathscr{C}_{I \cup I'; J \cup J'})$$

as follows. For $F = F(H_1, \ldots, H_p) \in Fl^p(\mathscr{C}_{I;J})$, $F' = F(H'_1, \ldots, H'_q) \in Fl^q(\mathscr{C}_{I';J'})$,put $F \circ F' = F(H_1, \ldots, H_p, H'_1, \ldots, H'_q)$. It is clear that this correctly defines (5.7.1).

5.7.2 Lemma. (i) *The map* (5.7.1) *correctly defines the map*

$$(5.7.2.1) \qquad \mathscr{F}^p(\mathscr{C}_{I;J}) \otimes \mathscr{F}^q(\mathscr{C}_{I';J'}) \to \mathscr{F}^{p+q}(\mathscr{C}_{I \cup I'; J \cup J'}).$$

(ii) *For all* $x \in \mathscr{F}^p(\mathscr{C}_{I;J})$, $y \in \mathscr{F}^q(\mathscr{C}_{I';J'})$

$$(5.7.2.2) \qquad x \circ y = (-1)^{pq} y \circ x.$$

Proof. Follows easily from Remarks 5.6.4. □

5.8 Suppose that $\lambda = (1, 1, \ldots, 1)$. So, $N = r$, and $\Sigma_\lambda = \{e\}$. In this n° we'll define isomorphisms (5.5.2) for this case.

5.8.1 Let us define *commutators* $g \in \mathfrak{n}$ *of length* $\ell = \ell(g)$, $1 \leqq \ell \leqq N$, by induction on ℓ as follows. Commutators of length 1 are all f_i, $1 \leqq i \leqq N$. A commutator of length $\ell > 1$ is an expression of the form $g = [g_1, g_2]$, where g_i is a commutator of length ℓ_i, $\ell = \ell_1 + \ell_2$, and the sets of f_i's which are contained in g_1 and g_2, do not intersect. We'll denote the set of f_i's contained in g, by $|g|$. So, $\ell(g) = \# |g|$.

5.8.2 Now let us assign to every commutator g a number $b(g) \in \mathbb{Z}/2\mathbb{Z}$ ("the bracket sign" of g) as follows. Set $b(f_i) = 0$; $b([g_1, g_2]) = b(g_1) + b(g_2) + \ell(g_1) \bmod 2$.

5.8.3 Example. For $J \subset [N]$, $b([f_J]) = (\# J - 1) \bmod 2$.

5.8.4 Let g be a commutator. Let us assign to g the flag $Fl(g) \in Fl^{\ell(g)-1}(\mathscr{C}_{\emptyset;|g|})$ as follows. Put $Fl(f_i) = \square$ (the flag of length 0). If $g = [g_2, g_1]$, and $Fl(g_i)$ we know, consider the flag $Fl(g_1) \circ Fl(g_2)$ of length $\ell(g) - 2$. $Fl(g)$ is, by definition, $Fl(g_1) \circ Fl(g_2)$, completed by the edge $L_{|g|}$.

5.8.5 Example. For $J = \{j_1, \ldots, j_l\}$, $Fl([f_J]) = F(H_{j_1 j_2}, H_{j_2 j_3}, \ldots, H_{j_{\ell-1} j_\ell})$.

Finally, put

$$(5.8.6) \qquad F(g) = (-1)^{b(g)} Fl(g) \in F^{\ell(g)-1}(\mathscr{C}_{\emptyset;|g|}).$$

5.8.7 For $I = \{i_1, \ldots, i_\ell\} \subset [N]$, $1 \leqq i \leqq n$, set

$$F^i(f_I) = F(H^i_{i_1}, H^i_{i_2}, \ldots, H^i_{i_\ell}) \in \mathscr{F}^\ell(\mathscr{C}_{\{i\};I}).$$

Now let $z \in C_p(\mathfrak{n}, (U, \mathfrak{n})^{\otimes n})_\lambda$,

$$(5.8.8) \qquad z = g_p \wedge g_{p-1} \wedge \ldots \wedge g_1 \otimes f_{I_n} \otimes f_{I_{n-1}} \otimes \ldots \otimes f_{I_1}$$

where all g_i are commutators, $\ell(g_i)=\ell_i$, $|g_i|=J_i$. Let $\{f_{i_1}, \ldots, f_{i_N}\}$ be the list of f_i's in z read from right to left. Define $\sigma(z)\in\Sigma_N$ by $\sigma(z)(j)=i_j$. Put

$$\Psi_p(z)=(-1)^{|\sigma(z)|+\sum_{i=1}^{p} i(\ell_i-1)} F^1(f_{I_1})\circ F^2(f_{I_2})\circ \ldots \tag{5.8.9}$$
$$\ldots \circ F^n(f_{I_n})\circ F(g_1)\circ \ldots \circ F(g_p).$$

5.8.10 Example. $(n=1)$. $\Psi_N(f_N\wedge f_{N-1}\wedge \ldots \wedge f_{k+1}\otimes f_k f_{k-1}\ldots f_1)=F(H_1^1, \ldots, H_k^1)$.

5.8.11 Lemma. *The rule* (5.8.9) *correctly defines maps*

$$\Psi_p\colon\ C_p(\mathfrak{n}, (U\mathfrak{n})^{\otimes n})_\lambda \to \mathcal{F}^{N-p}(\mathcal{C}_{n;N}).$$

Proof. We have to verify that Ψ_p respects the Jacobi identity and skew commutativity in g_i. This follows from relations 5.6.4(b) and 5.6.4(a) respectively. $\square$

The next lemma will explain rather combersome choice of the sign in (5.8.9). Let us introduce operators

$$\partial_{j_1 j_2}, \partial_j^i\colon\ C_p(\mathfrak{n}, (U\mathfrak{n})^{\otimes n})\to C_{p-1}(\mathfrak{n}, (U\mathfrak{n})^{\otimes n}),$$
$$1\leqq j_1<j_2\leqq p;\quad 1\leqq j\leqq p;\quad 1\leqq i\leqq n:\qquad \text{for } z=g_p\wedge \ldots \wedge g_1\otimes x_n\otimes \ldots \otimes x_1.$$

$$\partial_{j_1 j_2}(z)=g_p\wedge \ldots \wedge \hat{g}_{j_2}\wedge \ldots \wedge \hat{g}_{j_1}\wedge \ldots g_1\wedge [g_{j_2}, g_{j_1}]\otimes x_n\otimes \ldots \otimes x_1, \tag{5.8.12}$$

$$\partial_j^i(z)=g_p\wedge \ldots \wedge \hat{g}_j\wedge \ldots \wedge g_1\otimes x_n \ldots \otimes g_j x_i\otimes \ldots \otimes x_1. \tag{5.8.13}$$

So,

$$d(z)=\sum_{j_1<j_2} (-1)^{j_1+j_2}\partial_{j_1 j_2}(z)+\sum_{i,j}(-1)^{j-1}\partial_j^i(z) \tag{5.8.14}$$

(cf. (5.4.2)).

Let z be as in (5.8.8). By definition, $\Psi(z)=(-1)^{s(z)}Fl(z)$, $Fl(z)\in Fl^{N-p}$. The last edge of $Fl(z)$ is $L_{J_1, \ldots, J_p; I_1, \ldots, I_n}=L(z)$. Let $\partial_{j_1 j_2}L(z)$ (resp., $\partial_j^i L(z)$) denote the codimension 1 subedge of $L(z)$ obtained by the connection of islands J_{j_1} and J_{j_2} (resp., J_j and I_i) (see 5.6). Let $\partial_{j_1 j_2} Fl(z)$ ($\partial_j^i Fl(z)$) denote the flag obtained by adding to $Fl(z)$ the edge $\partial_{j_1 j_2}L(z)$ ($\partial_j^i L(z)$); put $\partial_{j_1 j_2}\Psi(z)=(-1)^{s(z)}\partial_{j_1 j_2}Fl(z)$, $\partial_j^i\Psi(z)=(-1)^{s(z)}\partial_j^i Fl(z)$.

5.8.15 Lemma. (i) $\Psi(f_N\wedge f_{N-1}\wedge \ldots \wedge f_1\otimes 1\otimes \ldots \otimes 1)=\square$.

(ii) *For z of the form* (5.8.8) *and all* j_1, j_2, j, i

$$\Psi(\partial_{j_1 j_2}(z))=(-1)^{j_1+j_2}\partial_{j_1 j_2}\Psi(z);$$
$$\Psi(\partial_j^i(z))=(-1)^{j-1}\partial_j^i\Psi(z).$$

Proof. Direct check. $\square$

5.8.16 Corollary. *$\Psi_\bullet=(\Psi_p)$ defines the map of complexes* $\Psi_\bullet\colon C_\bullet(\mathfrak{n}, (U\mathfrak{n})^{\otimes n})_\lambda \to \mathcal{F}^{N-\bullet}(\mathcal{C}_{n;N})$.

Conversely, properties 5.8.15(i) and (ii) determine the map Ψ uniquely, and allow to define the inverse map.

Namely, let us assign to every flag $F \in F\ell^{N-p}(\mathscr{C}_{n;N})$ an element $\Psi'(F) \in C_p(\mathfrak{n}, (U\mathfrak{n})^{\otimes n})_\lambda$ of the form

$$\Psi'(F) = (-1)^{s(F)} g_p \wedge g_{p-1} \wedge \dots \wedge g_1 \otimes x_n \otimes \dots \otimes x_1 \tag{5.8.17}$$

where all g_j are commutators, and x_i are products of commutators. This element will have the property: if $J_i = |g_i|$, and I_i = set of f_j's contained in x_i, then the last edge of F is $L_{J_1, \dots, J_p; I_1, \dots, I_n}$.

We'll construct $\Psi'(F)$ by induction on $N-p$. Put

$$\Psi'(\square) = f_N \wedge f_{N-1} \wedge \dots \wedge f_1 \otimes 1 \otimes 1 \otimes \dots \otimes 1. \tag{5.8.18}$$

Now let $F = (L^0 \supset \dots \supset L^{N-p})$, $F' = (L^0 \supset \dots \supset L^{N-p-1})$; $L^{N-p-1} = L_{J'_1, \dots, J'_{p+1}; I'_1, \dots, I'_n}$. If L^{N-p} is obtained from L^{N-p-1} by connecting two islands J'_{j_1} and J'_{j_2} $(j_1 < j_2)$ (resp., J'_j, I_i), then put

$$\Psi'(F) = (-1)^{j_1 + j_2} \partial_{j_1 j_2}(\Psi'(F')) \tag{5.8.19}$$

(resp.,

$$\Psi'(F) = (-1)^{j-1} \partial^i_j(\Psi'(F'))). \tag{5.8.20}$$

One verifies, using (5.6.4), that these rules correctly define maps

$$\Psi' \colon \mathscr{F}^{N-p}(\mathscr{C}_{n;N}) \to C_p(\mathfrak{n}, (U\mathfrak{n})^{\otimes n})_\lambda. \tag{5.8.21}$$

By 5.8.15 they are inverse to Ψ. So we get

5.9 Theorem. *Suppose that* $\lambda = (1, 1, \dots, 1)$. *The maps* $\underset{\bullet}{\Psi}$ (5.8.16) *define isomorphisms of complexes*

$$\underset{\bullet}{\Psi} \colon C_\bullet(\mathfrak{n}, (U\mathfrak{n})^{\otimes n})_\lambda \xrightarrow{\sim} \mathscr{F}^{N-\bullet}(\mathscr{C}_{n;N}).$$

5.10 Remark. In the previous considerations we can readily put $N = \infty$, $\mathfrak{n}$ to be the free Lie algebra on $f_1, f_2, \dots$; $\lambda = (1, 1, \dots)$. We get the isomorphism $\underset{\bullet}{\Psi} \colon C_{\infty - \bullet}(\mathfrak{n}, (U\mathfrak{n})^{\otimes n})_\lambda \xrightarrow{\sim} \mathscr{F}^\bullet(\mathscr{C}_{n;\infty})$.

5.11 Now let $\lambda = (k_1, \dots, k_r) \in N^r$ be arbitrary, $N = |\lambda|$. Introduce a free Lie algebra $\tilde{\mathfrak{n}}$ with generators $\tilde{f}_1, \dots, \tilde{f}_N$. The inclusion $\Sigma_\lambda \hookrightarrow \Sigma_N$ defined in 5.5 induces the action of Σ_λ on $\tilde{\mathfrak{n}}$ by permutation of $\tilde{f}_i$'s, whence the action of Σ_λ on $C_\bullet(\tilde{\mathfrak{n}}, (U\tilde{\mathfrak{n}})^{\otimes n})$.

Define the map of Lie algebras $\mathfrak{n} \to \tilde{\mathfrak{n}}$ by putting

$$f_i \mapsto \sum_{j=1}^{k_i} \tilde{f}_{k(i)+j}, \tag{5.11.1}$$

where $k(i)=\sum_{p=1}^{i-1} k_p$. It induces the map of complexes $C_{\cdot}(\mathfrak{n},(U\mathfrak{n})^{\otimes n})\to C_{\cdot}(\tilde{\mathfrak{n}},(U\tilde{\mathfrak{n}})^{\otimes n})^{\Sigma_\lambda}$.
Let

$$(5.11.2) \qquad s\colon\ C_{\cdot}(\mathfrak{n},(U\mathfrak{n})^{\otimes n})_\lambda\to C_{\cdot}(\tilde{\mathfrak{n}},(U\tilde{\mathfrak{n}})^{\otimes n})_{\tilde{\lambda}}^{\Sigma_\lambda}$$

where $\tilde{\lambda}=\underbrace{(1, 1, \dots, 1)}_{N}$, be the composition of the previous map with the projection onto the $\tilde{\lambda}$-component.

On the other hand, define the map of Lie algebras $\tilde{\mathfrak{n}}\to\mathfrak{n}$ by putting

$$(5.11.3) \qquad \tilde{f}_j\to f_i \quad \text{if } k(i)<j\leqq k(i+1).$$

It induces the map $C_{\cdot}(\tilde{\mathfrak{n}}, (U\tilde{\mathfrak{n}})^{\otimes n})_{\tilde{\lambda}}\to C_{\cdot}(\mathfrak{n}, (U\mathfrak{n})^{\otimes n})_\lambda$. Let

$$(5.11.4) \qquad \pi\colon\ C_{\cdot}(\tilde{\mathfrak{n}},(U\tilde{\mathfrak{n}})^{\otimes n})_{\tilde{\lambda}}^{\Sigma_\lambda}\to C_{\cdot}(\mathfrak{n},(U\mathfrak{n})^{\otimes n})_\lambda$$

be its restriction to Σ_λ-invariants.

The compositions $s\pi$ and πs are equal to the multiplication by $\#\,\Sigma_\lambda$. Hence, we get

5.12 Proposition. *The map* s (5.11.2) *is an isomorphism.*

We may apply the results of 5.8, 5.9 to $\tilde{\mathfrak{n}}$. Note that the isomorphism 5.9 is compatible with the action of Σ_λ. So, we get

5.13 Theorem. *The composition of* s (5.11.2) *and* Ψ (5.9) *applied to* $\tilde{\mathfrak{n}}$, $\tilde{\lambda}$, *gives the isomorphism of complexes*

$$\Psi=\Psi_{(\lambda,n)}\colon\ C_{\cdot}(\mathfrak{n},(U\mathfrak{n})^{\otimes n})_\lambda\xrightarrow{\sim}\mathscr{F}^{N-\cdot}(\mathscr{C}_{n;N})^{\Sigma_\lambda}.$$

5.14 Examples. 1. Let $n=1, r=1$. $\Psi(f_1^m)=\sum_{\sigma\in\Sigma_m}(-1)^{|\sigma|}F(H^1_{\sigma(1)},\dots,H^1_{\sigma(m)})$.
2. $n=1, r=2$. $\Psi(f_1f_2f_1)=F(H^1_2, H^1_3, H^1_1)-F(H^1_1, H^1_3, H^1_2)$.

6 Contravariant form, II

6.1 Fix a finite dimensional complex vector space $\mathfrak{h}$ together with a non-degenerate symmetric bilinear form (,) on it. Denote by b: $\mathfrak{h}^*\to\mathfrak{h}$ the isomorphism induced by (,). By means of b we transfer the form (,) to $\mathfrak{h}^*$; we'll denote it also by (,).

Fix a finite set of linear independent covectors $\{\alpha_1, \dots, \alpha_r\}\subset\mathfrak{h}^*$. We'll call α_i *simple roots.* Put $h_i=b(\alpha_i)$; $b_{ij}=(\alpha_i, \alpha_j)=\langle h_i, \alpha_j\rangle$; $B=(b_{ij})_{i,j=1}^r$.

Denote by $\mathfrak{g}=\mathfrak{g}(B)$ the Lie algebra with generators e_i, f_i, $i=1, \ldots, r$, and $h\in\mathfrak{h}$; subject to the relations

$$\begin{gathered}[e_i, f_j]=\delta_{ij} h_i;\\ [h, e_i]=\langle h, \alpha_i\rangle e_i; \quad [h, f_i]=-\langle h, \alpha_i\rangle f_i\\ [h, h']=0\end{gathered} \tag{6.1.1}$$

for all $i, j=1, \ldots, r$, $h, h'\in\mathfrak{h}$.

We'll denote by $\mathfrak{n}=\mathfrak{n}_-$ (resp., by $\mathfrak{n}_+$) the Lie subalgebra of $\mathfrak{g}$ generated by f_i (resp., e_i), $i=1, \ldots, r$. These subalgebras are free [K, Theorem 9.2(b)]. We'll identify $\mathfrak{n}$ with the Lie algebra studied in § 5. In particular, the decomposition $\mathfrak{n}=\bigoplus_{\lambda\in\mathbb{N}^r}\mathfrak{n}_\lambda$ is defined as in loc. cit. We have the natural decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$. Put $\mathfrak{b}=\mathfrak{n}\oplus\mathfrak{h}$; it is a Lie subalgebra of $\mathfrak{g}$.

6.2 *Bilinear forms*

6.2.1 Lemma-definition. *There is a unique bilinear form $K(,)$ on $\mathfrak{g}$ such that*
(i) *K coincides with $(,)$ on $\mathfrak{h}$; K is zero on $\mathfrak{n}_-$ and $\mathfrak{n}_+$; $\mathfrak{h}$ and $\mathfrak{n}_-\oplus\mathfrak{n}_+$ are orthogonal;*
(ii) *$K(f_i, e_j)=K(e_j, f_i)=\delta_{ij}$, $i, j=1, \ldots, r$;*
(iii) *K is $\mathfrak{g}$-invariant, i.e.*

$$K([x, y], z)=K(x, [y, z])$$

for all $x, y, z\in\mathfrak{g}$.

This form is symmetric.

Proof. The same as in [K, § 2.2]. □

Denote by τ: $\mathfrak{g}\to\mathfrak{g}$ the Lie algebra automorphism which is (-1) on $\mathfrak{h}$, and maps f_i to $-e_i$, e_i to $-f_i$ ("the Chevalley involution"). We have

$$K(\tau x, \tau y)=K(x, y) \tag{6.2.2}$$

for all $x, y\in\mathfrak{g}$.

Define the bilinear form S on $\mathfrak{g}$ by the rule

$$S(x, y)=K(\tau x, y). \tag{6.2.3}$$

From (6.2.2) follows that S is symmetric. We have

$$S([x, y], z)=S(x, [\tau y, z]). \tag{6.2.4}$$

This follows from the invariance of K and (6.2.2).

The subspaces $\mathfrak{n}_-$, $\mathfrak{h}$, $\mathfrak{n}_+$ are pairwise orthogonal with respect to S. The restriction of S to $\mathfrak{n}$ is uniquely determined by the properties

$$S(f_i, f_j)=-\delta_{ij} \tag{6.2.5}$$

$$S([f_i, x], y)=S(x, [e_i, y]). \tag{6.2.6}$$

The subspaces $\mathfrak{n}_\lambda\subset\mathfrak{n}$, $\lambda\in\mathbb{N}^r$ are pairwise orthogonal with respect to S.

6.3 Relation with Kac-Moody algebras. Suppose that $b_{ii} \neq 0$ for all i. Put $a_{ij} = \frac{2b_{ij}}{b_{ii}}$; $A=(a_{ij})_{i,j=1}^{r}$. Our algebra $\mathfrak{g}(B)$ is isomorphic to the algebra $\tilde{\mathfrak{g}}(A)$ defined in [K, § 1.2]; the isomorphism maps generators $\tilde{e}_i \in \tilde{\mathfrak{g}}(A)$ to e_i; $\tilde{f}_i$ to $\frac{2}{b_{ii}} \cdot f_i$, and is identity on $\mathfrak{h}$. This isomorphism maps the form K to the form $(\cdot|\cdot)$ defined in [K, Ch. 2].

We have $\operatorname{Ker} K = \operatorname{Ker} S \subset \mathfrak{g}$. The quotient $\overline{\mathfrak{g}(B)} = \mathfrak{g}(B)/\operatorname{Ker} S$ is the Kac-Moody Lie algebra associated with A. Suppose that A is a generalized Cartan matrix, that is, $a_{ij} \in \mathbb{Z}$, $a_{ij} \leqq 0$ for $i \neq j$. Then the Gabber-Kac theorem [K, Theorem 9.11] asserts that $\operatorname{Ker} S$ is generated by Serre elements $(ad f_i)^{n_{ij}}(f_j)$ and $(ad e_i)^{n_{ij}}(e_j)$, $n_{ij} = -a_{ij}+1$, $i \neq j$.

6.4 Verma modules. Let $\Lambda \in \mathfrak{h}^*$. Denote by $M(\Lambda)$ ("a Verma module") the $\mathfrak{g}$-module generated by the unique vector v subject to relations $\mathfrak{n}_+ v = 0$; $hv = \langle h, \Lambda \rangle v$, $h \in \mathfrak{h}$. $M(\Lambda)$ is isomorphic to $U\mathfrak{n}$ as the $\mathfrak{n}$-module. We'll use the notations of § 5 for $M(\Lambda)_\lambda$. If $\lambda = (k_1, \ldots, k_r)$ then $M(\Lambda)_\lambda = \left\{ x \in M(\Lambda) \,|\, hx = \left\langle h, \Lambda - \sum_{i=1}^{r} k_i \alpha_i \right\rangle \cdot x \text{ for all } h \in \mathfrak{h} \right\}$.

The contragradient module $M(\Lambda)^*$ is by definition the dual space to $M(\Lambda)$ with the $\mathfrak{g}$-module structure defined by the rule $\langle x, g\varphi \rangle = \langle -\tau(g)x, \varphi \rangle$, $x \in M(\Lambda)$, $\varphi \in M(\Lambda)^*$, $g \in \mathfrak{g}$.

6.4.1 Lemma-definition. *There is a unique bilinear form S on $M(\Lambda)$ such that*

$$S(v, v) = 1; \qquad S(f_i x, y) = S(x, e_i y);$$

$S(e_i x, y) = S(x, f_i y)$ for all $x, y \in M(\Lambda)$; $i \in [r]$. S is symmetric. Subspaces $M(\Lambda)_\lambda$ are pairwise orthogonal with respect to S.

Proof is left to the reader. □

The form S induces the map of g-modules S: $M(\Lambda) \to M(\Lambda)^*$. $\ker S$ is the maximal proper submodule of $M(\Lambda)$, [K, 9.2]. We put $L(\Lambda) := M(\Lambda)/\ker S$. It is the irreducible g-module with the highest weight Λ.

More generally, define the symmetric form S on spaces $\Lambda^p \mathfrak{n} \otimes M(\Lambda_1) \otimes \ldots \otimes M(\Lambda_n)$, $\Lambda_i \in \mathfrak{h}^*$, by the rule

$$(6.4.2) \qquad S(g_1 \wedge \ldots \wedge g_p \otimes x_1 \otimes \ldots \otimes x_n, g'_1 \wedge \ldots \wedge g'_p \otimes x'_1 \otimes \ldots \otimes x'_n) = \det(S(g_i, g'_j)) \cdot \prod_{i=1}^{n} S(x_i, x'_i).$$

So we get the map of graded spaces

$$(6.4.3) \quad S\colon C_\bullet(\mathfrak{n}, M(\Lambda_1) \otimes \ldots \otimes M(\Lambda_n)) \to (C_\bullet(\mathfrak{n}, M(\Lambda_1) \otimes \ldots \otimes M(\Lambda_n)))^*.$$

Below we'll introduce the differential on the r.h.s. of (6.4.3) so that S will be map of complexes.

We'll call the form S the *contravariant* or the *Shapovalov form*.

6.5 Fix a non-zero complex parameter κ. For $\lambda=(k_1, \dots, k_r)\in\mathbb{N}^r$, $|\lambda|=N$, and $\Lambda_1, \dots, \Lambda_n\in\mathfrak{h}^*$, define the set of exponents $a=a(B, \Lambda_1, \dots, \Lambda_n)$: $\mathscr{C}_{n;N}\to\mathbb{C}$:

$$\begin{aligned} a(H_{ij})&=(-\alpha_{\pi(i)}, -\alpha_{\pi(j)})/\kappa \\ a(H_i^m)&=(-\alpha_{\pi(i)}, \Lambda_m)/\kappa. \end{aligned} \tag{6.5.1}$$

Here π: $[N]\to[r]$ is defined by the rule $\pi(j)=i$ if $\sum_{p=1}^{i-1} k_p<j\leqq \sum_{p=1}^{i} k_p$.

Note that a is Σ_λ-invariant, i.e. $a(\sigma H)=a(H)$ for all $H\in\mathscr{C}_{n;N}$, $\sigma\in\Sigma_\lambda$. It follows that S: $\mathscr{F}^\cdot(\mathscr{C}_{n;N})\to\mathscr{A}^\cdot(\mathscr{C}_{n;N})$ is Σ_λ-equivariant.

Put for brevity $M=M(\Lambda_1)\otimes\dots\otimes M(\Lambda_n)$.

Let us denote by

$$\eta=\eta_{(\lambda,p,n)}\colon C_p(\mathfrak{n}, M)^*_\lambda\to\mathscr{A}^{N-p}(\mathscr{C}_{n;N})^{\Sigma_\lambda} \tag{6.5.2}$$

the isomorphism, which is the composition of the inverse to the conjugate to $\Psi_{(\lambda,p,n)}$ (5.5.2), and φ^{-1} (2.3.3).

Consider the square

$$\begin{tikzcd} C_\cdot(\mathfrak{n}, M)_\lambda \arrow[r, "S"] \arrow[d, "\psi_\cdot"', "\wr"] & C_\cdot(\mathfrak{n}, M)^*_\lambda \arrow[d, "\eta_\cdot", "\wr"'] \\ \mathscr{F}^{N-\cdot}(\mathscr{C}_{n;N})^{\Sigma_\lambda} \arrow[r, "S"] & \mathscr{A}^{N-\cdot}(\mathscr{C}_{n;N})^{\Sigma_\lambda} \end{tikzcd} \tag{6.5.3}$$

where the upper S is (6.4.5), and the lower S is (3.2.1).

The main result of this section is

6.6 Theorem. *One has for all p*

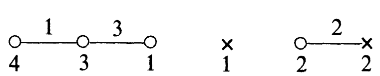
$$\eta_p S=(-1)^N\kappa^N S\psi_p.$$

This Theorem will be proven below in several steps.

6.7 First, let us introduce certain diagram notation for elements of groups $\mathscr{A}^p(\mathscr{C}_{n;N})$, similar to that of § 5.

Namely, we'll draw a monomial $H_1\cdot\ldots\cdot H_p\in\mathscr{A}^p(\mathscr{C}_{n;N})$ as the following non-oriented graph. It has $n+N$ vertices: N circles enumerated from 1 to N and n crosses enumerated from 1 to n. It has p edges enumerated from 1 to p: the i-th edge connects circles α and β if $H_i=H_{\alpha\beta}$, and the α-th circle with the β-th cross if $H_i=H_\alpha^\beta$. So, if we forget the numbers of edges, we obtain the picture of an edge $H_1\cap\dots\cap H_p\in\mathscr{C}^p_{n;N}$, cf. 5.6.

Example. $N=4$; $n=2$.



$=H_{34}H_2^2H_{13}$

6.8 Suppose that $\lambda=(1, 1, \dots, 1)$, $N=|\lambda|=r$, and $n=1$. The aim of this n° is to prove 6.6 for this case and for $p=0$. Put $\Lambda=\Lambda_1$, $M=M(\Lambda)$. The space M_λ has as a base the set of monomials $f_\sigma=f_{\sigma(N)}f_{\sigma(N-1)}\dots f_{\sigma(1)}\cdot v$, $\sigma\in\Sigma_N$.

Denote by $\delta_\sigma \in M_\lambda^*$ the functional equal to 1 on $f_{\sigma(1)} f_{\sigma(2)} \dots f_{\sigma(N)} v$ and 0 on other monomials. (NB! Pay attention to the inverse order!).

6.8.1 Lemma. *η: $M_\lambda^* \to \mathscr{A}^N(\mathscr{C}_{1;N})$ maps δ_σ to*

$$(-1)^{|\sigma|} \underset{\sigma(1)}{\circ} \overset{1}{\text{———}} \underset{\sigma(2)}{\circ} \overset{2}{\text{———}} \underset{\sigma(3)}{\circ} \quad \cdots \quad \underset{\sigma(N-1)}{\circ} \overset{N-1}{\text{———}} \underset{\sigma(N)}{\circ} \overset{N}{\text{———}} \underset{1}{\times}$$

Proof. Since η is Σ_N-equivariant, it suffices to prove it for $\sigma = 1$. This case is easily deduced from the definitions. □

For any subset $I \subset [N]$, put $\lambda_I = \{k_1, \dots, k_N\}$, where $k_i = 1$ if $i \in I$, and 0 otherwise. Set $M_I := M_{\lambda_I}$. For example, $M_{[N]} = M_\lambda$.

Let $\mathscr{C}_I \subset \mathscr{C}_{1;N}$ be the subarrangement, consisting of all hyperplanes H_{ij} and H_i^1 with $i, j \in I$. Put $\mathscr{F}_I = \mathscr{F}^p(\mathscr{C}_I)$, $\mathscr{A}_I = \mathscr{A}^p(\mathscr{C}_I)$, where $p = \# I$. We have $\mathscr{F}_I = \mathscr{F}^p(\mathscr{C}_{1;N})$; $\mathscr{A}_I = \mathscr{A}^p(\mathscr{C}_{1;N})$. Define maps

$$\psi = \psi_I\colon\ M_I \to \mathscr{F}_I \tag{6.8.2}$$

by the following rule. Let $I = \{i_1, \dots, i_p\}$, $i_1 < \dots < i_p$.

Let $J = [N] \setminus I$; $J = \{j_1, \dots, j_{N-p}\}, j_1 < j_2 < \dots < j_{N-p}$. Denote by σ_I the permutation $(i_1, \dots, i_p, j_1, j_2, \dots, j_{N-p}) \in \Sigma_N$. For $\sigma \in \Sigma_p$ put $f_\sigma = f_{i_{\sigma(p)}} \cdot f_{i_{\sigma(p-1)}} \cdot \dots f_{i_{\sigma(1)}} \cdot v \in M_I$. Set

$$\Psi_I(f_\sigma) = (-1)^{|\sigma| + |\sigma_I|} F(H^1_{i_{\sigma(1)}}, H^1_{i_{\sigma(2)}}, \dots, H^1_{i_{\sigma(p)}}). \tag{6.8.3}$$

In other words,

$$\Psi_I(f_\sigma) = \Psi_\lambda(f_{j_{N-p}} \wedge f_{j_{N-p-1}} \wedge \dots \wedge f_{j_1} \otimes f_\sigma) \tag{6.8.4}$$

(cf. 5.8.10). From 5.9 follows that Ψ_I are isomorphisms.

Let

$$\eta = \eta_I\colon\ M_I^* \to \mathscr{A}_I \tag{6.8.5}$$

be the composition $M_I^* \xrightarrow{\Psi_I^{*-1}} \mathscr{F}_I^* \xrightarrow{\sim} \mathscr{A}_I$. Denote by $\delta_\sigma \in M_I^*$ the δ-functional equal to 1 on $f_{i_{\sigma(1)}} f_{i_{\sigma(2)}} \dots f_{i_{\sigma(N)}} \cdot v$ and to 0 on other monomials.

From 6.8.1 follows that

$$\eta_I(\delta_0) = (-1)^{|\sigma| + |\sigma_I|} \underset{i_{\sigma(1)}}{\circ} \overset{1}{\text{——}} \underset{i_{\sigma(2)}}{\circ} \overset{2}{\text{——}} \circ \quad \cdots \quad \underset{i_{\sigma(p-1)}}{\circ} \overset{p-1}{\text{——}} \underset{i_{\sigma(p)}}{\circ} \overset{p}{\text{——}} \times \tag{6.8.6}$$

Let $j \in J$. Put $I' = I \cup \{j\}$. Define the operator

$$\omega_j\colon\ \mathscr{A}_I \to \mathscr{A}_{I'} \tag{6.8.7}$$

by the formula

$$\omega_j(x) = x \cdot [\sum_{i \in I} a(H_{ij}) H_{ij} + a(H_j^1) H_j^1]. \tag{6.8.8}$$

Let us consider the square

$$(6.8.9)\qquad \begin{array}{ccc} M_I^* & \xrightarrow{e_j^*} & M_{I'}^* \\ {\scriptstyle \eta_I}\downarrow\wr & & \downarrow{\scriptstyle \eta_{I'}} \\ \mathscr{A}_I & \xrightarrow{\omega_j} & \mathscr{A}_{I'} \end{array}$$

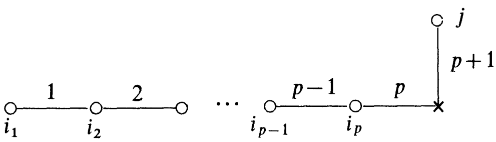
where e_j^* is the conjugate to the multiplication by e_j.

6.8.10 Lemma. *If $j=j_q$ then*

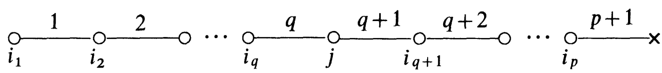
$$\eta_{I'}\, e_j^* = (-1)^{q+1}\kappa\,\omega_j\,\eta_I.$$

Proof. This follows easily from the relation in $A_\bullet$:

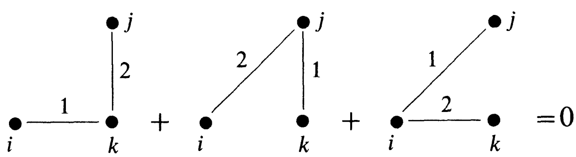
6.8.11 Lemma.



$$=\sum_{q=0}^{p}(-1)^{p-q}$$



Proof of 6.8.11 Induction by p, using the Orlik-Solomon relation



($\bullet = \circ$ or $\times$). $\square$

Now let us consider the Shapovalov form S: $M_\lambda \to M_\lambda^*$. For $\sigma\in\Sigma_N$ $S(f_\sigma)$ is the image of the canonical generator of $M_\emptyset^*$ under the composition

$$M_\emptyset^* \xrightarrow{e_{\sigma(1)}^*} M_{\{\sigma(1)\}}^* \xrightarrow{e_{\sigma(2)}^*} M_{\{\sigma(1),\sigma(2)\}}^* \longrightarrow \dots \xrightarrow{e_{\sigma(N)}^*} M_\lambda^*.$$

From this remark and from 6.8.10 follows 6.6 for $\lambda=(1, 1, \dots, 1)$ and $p=0$.

6.8.12 Remark. In assumptions of 6.8.11 identify spaces $\mathscr{A}_I$, $\mathscr{A}_{I'}$ with the corresponding spaces of differential forms. Define operators $\mathrm{res}_{H_{ij}}$ (resp., $\mathrm{res}_{H_j^1}$): $\mathscr{A}_{I'} \to \mathscr{A}_I$ as follows. For a form $\omega=\omega(t_i)_{i\in I}$ take its residue along the hyperplane $t_j=t_i$ (resp., $t_j=z_1$) and then put $t_j=t_i$ (resp., $t_j=z_1$).

One verifies easily that after identification (6.8.5) the operator $\mathrm{res}_{H_j^1}+\sum_{i\in I}\mathrm{res}_{H_{ij}}$ corresponds (up to a sign) to f_j^* (cf. 6.8.10).

6.9 Proof of 6.6 for $\lambda=(1, 1, \dots, 1)$, $n=0$ and $p=1$. We'll reduce to the case $n=1$, $p=0$, using the following trick.

We have the isomorphism of graded algebras

$$\mathscr{A}^{\cdot}(\mathscr{C}_{0;N}) \xrightarrow{\sim} \mathscr{A}^{\cdot}(\mathscr{C}_{1;N-1}) \tag{6.9.1}$$

which maps H_{ij} to $H_{i-1,j-1}$ if $i, j > 1$, and H_{1i} to H^1_{i-1}. It induces the isomorphism of flag groups. On the other hand, consider the algebra $\tilde{\mathfrak{g}} = \mathfrak{g}(\tilde{B})$ corresponding to the $(r-1) \times (r-1)$ matrix $B = (b_{ij}) := b_{i+1,j+1})_{i,j=1}^{r-1}$. Let $\tilde{M} = M(\tilde{\Lambda})$ be the $\tilde{\mathfrak{g}}$-module with the highest weight $\tilde{\Lambda} = -\alpha_1$. Put $\tilde{\lambda} = \underbrace{(1, 1, \ldots, 1)}_{r-1 \text{ times}}$. We have the commutative square of isomorphisms

$$\begin{array}{ccc} \mathfrak{n}_\lambda & \xrightarrow[\sim]{\Psi} & \mathscr{F}^{N-1}(\mathscr{C}_{0;N}) \\ \chi \downarrow \wr & & \wr \downarrow (6.9.1)^* \\ \tilde{M}_{\tilde{\lambda}} & \xrightarrow[\sim]{\Psi} & \mathscr{F}^{N-1}(\mathscr{C}_{1;N}) \end{array} \tag{6.9.2}$$

where χ maps $[f_{i_N}, (f_{i_{N-1}}, \ldots [f_{i_2}, f_1] \ldots]$ (all $i_j \geq 2$) to $\tilde{f}_{i_N - 1} \cdot \tilde{f}_{i_{N-1}-1} \cdot \ldots \cdot \tilde{f}_{i_2 - 1} \cdot \tilde{v}$.

One easily verifies by induction on N that the isomorphism χ maps the form S on $\mathfrak{n}_\lambda$ to (-1) times the form S on $\tilde{M}_{\tilde{\lambda}}$. The isomorphism (6.9.1) is compatible with the form S. Hence, the case $n=0$, $p=1$ follows from the case $n=1$, $p=0$ proven in 6.8.

6.10 Proof of 6.6 for $\lambda = (1, \ldots, 1)$, $n=1$ *and arbitrary* p. For a subset $I \subset [N]$ put $\mathfrak{n}_I = \mathfrak{n}_{\lambda_I}$ where λ_I is defined as in 6.8. We have the natural S-orthogonal decomposition

$$(\Lambda^p \mathfrak{n} \otimes M)_\lambda = \oplus (\mathfrak{n}_{J_1} \otimes \ldots \otimes \mathfrak{n}_{J_p} \otimes M_I)$$

the sum is taken over all decompositions of $[N]$ in the disjoint union $J_1 \amalg \ldots \amalg J_p \amalg I$.

Analogously, we have the S-orthogonal decomposition

$$\mathscr{F}^{N-p}(\mathscr{C}_{1;N}) = \oplus \mathscr{F}^{N-p}(\mathscr{C}_{1;N})_{L_{J_1, \ldots, J_p; I}}$$

(see 5.6.3). By definition, ψ respects these decompositions, and 6.6 follows from the particular cases proven in 6.8, 6.9.

The case of $\lambda = (1, \ldots, 1)$, arbitrary p and arbitrary n is proven similarly and we leave it to the reader.

Thus, we have proven 6.6 for $\lambda = (1, \ldots, 1)$.

6.11 Proof of 6.6 for an arbitrary λ. Let $\lambda = (k_1, \ldots, k_r)$, $N = |\lambda|$. Fix some map $\pi: [N] \to [r]$ with $\# \pi^{-1}(j) = k_j$ for all j. Introduce the new $N \times N$ matrix $\tilde{B} = (\tilde{b}_{ij})$, where $\tilde{b}_{ij} = b_{\pi(i)\pi(j)}$. Let $\tilde{\mathfrak{g}} = \mathfrak{g}(\tilde{B})$ be the corresponding Lie algebra with generators $\tilde{e}_i, \tilde{f}_i, \tilde{h} \in \tilde{\mathfrak{h}}$.

Let $\Lambda \in \mathfrak{h}^*$ be a weight. Choose a weight $\tilde{\Lambda} \in \tilde{\mathfrak{h}}^*$ such that $\langle \tilde{h}_i, \tilde{\Lambda} \rangle = \langle h_{\pi(i)}, \Lambda \rangle$ for all $i \in [N]$. Put $M = M(\Lambda)$; $\tilde{M} = M(\tilde{\Lambda})$. Put $\tilde{\lambda} = \underbrace{(1, 1, \ldots, 1)}_{N \text{ times}}$. For any subset $I \subset [N]$ define M_I as in 6.8. Set $\lambda(I) = (k'_1, \ldots, k'_r)$ where $k'_j = \# (I \cap \pi^{-1}(j))$.

Define maps

$$\pi_I\colon\ \tilde{M}_I \to M_{\lambda(I)} \tag{6.11.1}$$

be the formula $\pi_I(f_{i_1}\dots f_{i_p}) = f_{\pi(i_1)}\dots f_{\pi(i_p)}$. On the other hand, define averaging maps

$$s_I\colon\ M_{\lambda(I)} \to \tilde{M}_I \tag{6.11.2}$$

by the formula

$$s_I(f_{j_1}\dots f_{j_p}) = \mathrm{pr}(s(f_{j_1})\, s(f_{j_2})\dots s(f_{j_p})) \tag{6.11.3}$$

where $s(f_j) = \sum_{i\in\pi^{-1}(j)} \tilde{f}_1$; pr: $\tilde{M} \mapsto \tilde{M}_I$ is the projection to the homogeneous component (cf. 5.11.2).

In other words,

$$s_I(f_{j_1}\dots f_{j_p}) = \sum \tilde{f}_{i_1}\dots \tilde{f}_{i_p} \tag{6.11.4}$$

the sum being taken over all p-tuples of pairwise distinct integers $(i_1, \dots, i_p)$ such that $\pi(i_k) = j_k$ for all k.

Maps s_I induce isomorphisms

$$M_{\lambda(I)} \xrightarrow{\sim} \tilde{M}_I^{\Sigma_{\lambda(I)}} \tag{6.11.5}$$

where $\Sigma_{\lambda(I)}$ acts on M_I in the evident way (cf. 5.11).

6.11.6 Lemma ("The projection formula"). *Let $I \subset [N]$, $i \in I$; put $I' = I - \{i\}$. The square*

$$\begin{tikzcd} M_{\lambda(I)} \arrow[r, "s_I"] \arrow[d, "e_{\pi(i)}"'] & \tilde{M}_I \arrow[d, "\tilde{e}_i"] \\ M_{\lambda(I')} \arrow[r, "s_{I'}"] & \tilde{M}_{I'} \end{tikzcd}$$

commutes.

Proof. Direct verification. □

6.11.7 Lemma. *Squares*

$$\begin{tikzcd} M_\lambda \arrow[r, "s_N"] \arrow[d, "S_M"'] & \tilde{M}_\lambda^{\Sigma_\lambda} \arrow[d, "S_M"] \\ M_\lambda^* \arrow[r, "s_N^{*-1}"] & \tilde{M}_\lambda^{*\Sigma_\lambda} \end{tikzcd}$$

commute.

Proof. Follows from the previous lemma by induction of N (cf. the end of 6.8). □

This Lemma proves 6.6 for $p=0$, $n=1$ and any λ. The case of general p, n follows from the commutativity of squares

$$\begin{array}{ccc} C_\bullet(\mathfrak{n}, M)_\lambda & \xrightarrow{s} & C_\bullet(\tilde{\mathfrak{n}}, \tilde{M}) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ C_\bullet(\mathfrak{n}, M)^*_\lambda & \xrightarrow{s} & C_\bullet(\tilde{\mathfrak{n}}, \tilde{M}) \end{array} \tag{6.11.8}$$

which in turn follows from 6.11.7 (cf. 6.9 and 6.10).

The proof of 6.6 is complete. □

6.12 Remark. In view of 6.6, the determinant formula 3.7 applied to arrangements $\mathscr{C}_{n;N}$ appears to be a variant of the Shapovalov determinant formula [Sh].

6.13 In conditions of 6.6, put $\bar{\mathfrak{g}}=\mathfrak{g}/\ker S$; it is the Kac-Moody Lie algebra associated with the matrix A defined in 6.3. Put $\bar{\mathfrak{n}}=\mathfrak{n}/\ker S\subset\bar{\mathfrak{g}}$; $L(\Lambda_i)=M(\Lambda_i)/\ker S$ – these are irreducible $\bar{\mathfrak{g}}$-modules; $L=L_1\otimes\ldots\otimes L_n$. As in § 3, denote by $\bar{\mathscr{F}}^\bullet$ the image $S(\mathscr{F}^\bullet)\subset\mathscr{A}^\bullet$.

Corollary. *The maps $\Psi_\bullet$ induce isomorphisms of complexes*

$$C_\bullet(\bar{\mathfrak{n}}, L)_\lambda \xrightarrow{\sim} \bar{F}^{N-\bullet}(\mathscr{C}_{n;N})^{\Sigma_\lambda}$$

for any $\lambda\in N^r$.

In the rest of this section we'll introduce a structure of a Lie bialgebra on $\mathfrak{b}[D]$ and a structure of a $\mathfrak{b}$-comodule on $M=M(\Lambda_1)\otimes\ldots\otimes M(\Lambda_n)$. Complexes $\mathscr{A}^{N-\bullet}(\mathscr{C}_{n;N})$ are identified with homogeneous parts of the standard complex $C_\bullet(\mathfrak{n}^*, M^*)$.

6.14 Cobracket. **6.14.1 Lemma-definition.** *There exists a unique map ν: $\mathfrak{b}\to\mathfrak{b}\wedge\mathfrak{b}$ such that*
(i) $\nu(h)=0$; $\nu(f_i)=\frac{1}{2}f_i\wedge h_i$ *for all* $h\in\mathfrak{h}$; $i=1, \ldots, r$;
(ii) $\nu([x, y])=x\nu(y)-y\nu(x)$ *for all* $x, y\in\mathfrak{b}$, *where the action of* $\mathfrak{b}$ *on* $\mathfrak{b}\wedge\mathfrak{b}$ *is the adjoint one*: $a(b\wedge c)=[a, b]\wedge c+a\wedge[b, c]$ (*i.e.* ν *is a* 1-*cocycle of* $\mathfrak{b}$ *with values in* $\mathfrak{b}\wedge\mathfrak{b}$).

Proof. Left to the reader (cf. [D, Example 3.2]). □

Let us define the map μ: $N^r\to\mathfrak{h}$ by the formula $\mu(k_1, \ldots, k_r)=\sum_{i=1}^{r} k_i h_i$.

Note that $\nu([h, x])=h\cdot\nu(x)$ for $h\in\mathfrak{h}$, $x\in\mathfrak{b}$. Hence the cobracket ν preserves the weight decomposition (with respect to the $\mathfrak{h}$-action).

6.14.2 Lemma. *For every* $x\in\mathfrak{n}_\lambda$
(i) $\nu(x)=-\frac{1}{2}\mu(\lambda)\wedge x+\nu(x)_-$ *where* $\nu(x)_-\in\mathfrak{n}\wedge\mathfrak{n}\subset\mathfrak{b}\wedge\mathfrak{b}$.
(ii) *For every* i

$$\nu([f_i, x])_- =f_i\cdot\nu(x)_- -(h_i, \mu(\lambda))f_i\wedge x=f_i\cdot\nu(x)_- +f_i\wedge [h_i, x].$$

Proof. For $x=f_i$ (i) is 6.14.1 (i). Suppose that we know (i) for x. We have $\nu([f_i, x])=f_i\cdot\nu(x)-\frac{1}{2}x\cdot(f_i\wedge h_i)=\frac{1}{2}[f_i, x]\wedge(\mu(\lambda)+h_i)-(h_i, \mu(\lambda))f_i\wedge x+f_i\cdot\nu(x)_-$.

This implies (i) for $[f_i, x]$ and (ii). $\square$

6.14.3 Lemma. *For every* $x\in\mathfrak{n}$

$$S(\nu(x), a\wedge b)=\begin{cases}\frac{1}{2}S(x,[a,b]) & \text{if } a\in\mathfrak{n}, \quad b\in\mathfrak{h}\\ S(x,[a,b]) & \text{if } a, b\in\mathfrak{n}.\end{cases}$$

S is defined on $\mathfrak{b}\wedge\mathfrak{b}$ by the rule

$$S(a\wedge b, c\wedge d)=\det\begin{pmatrix}S(a,c) & S(a,d)\\ S(b,c) & S(b,d)\end{pmatrix} \quad \text{(cf. (6.4.2))}.$$

Proof. The first equality is the direct consequence of 6.14.2(i). Let us prove the second one. For $x=f_i$ the both sides are zero. Note that the form S on $\mathfrak{b}\wedge\mathfrak{b}$ has the property

$$S(x\cdot y, z)=S(y, -\tau(x)\cdot z) \tag{6.14.4}$$

if $\tau(x)\cdot z\in\mathfrak{b}$. Suppose that we know 6.14.3 for $x\in\mathfrak{n}_\lambda$. We have

$$\begin{aligned}S(\nu([f_i, x]), a\wedge b)&=S(f_i\cdot\nu(x)-x\cdot\nu(f_i), a\wedge b)\\ &=S(\nu(x), e_i(a\wedge b))-S(\tfrac{1}{2}f_i\wedge[x, h_i], a\wedge b) \quad \text{if } a, b\in\mathfrak{n}.\end{aligned}$$

Put $\lambda_i^0=(0, \ldots, 0, 1, 0, \ldots, 0)$, *so* $\mu(\lambda_i^0)=h_i$.

1st case. $a\in\mathfrak{n}_{\lambda_i^0}$, $b\in\mathfrak{n}_\lambda$. If $\lambda=\lambda_i^0$ then both sides of 6.14.3 for f_i, x are zero. If $\lambda\neq\lambda_i^0$, we have

$$\begin{aligned}S(\nu([f_i, x]), a\wedge b)&=S(\nu(x), [e_i, a]\wedge b+a\wedge[e_i, b])-\tfrac{1}{2}(h_i, \mu(\lambda))\,S(f_i, a)\,S(x, b)\\ &=\tfrac{1}{2}S(x, [[e_i, a], b])+S(x, [a, [e_i, b]])+\tfrac{1}{2}S(x, [e_i, [a, b]])\\ &=S(x, [e_i, [a, b]])=S([f_i, x], [a, b]).\end{aligned}$$

2nd case. $a\in\mathfrak{n}_{\lambda'}$, $b\in\mathfrak{n}_{\lambda''}$, $\lambda', \lambda''\neq\lambda_i^0$. Then

$$\begin{aligned}S(\nu([f_i, x]), a\wedge b)&=S(\nu(x), e_i\cdot(a\wedge b))\\ &=S(x, [e_i, [a, b]])\\ &=S([f_i, x], [a, b]).\end{aligned}$$

So, the second equality 6.14.3 is proven for $[f_i, x]$. $\square$

6.15 Comultiplication. Suppose that n weights $\Lambda_1, \ldots, \Lambda_n\in\mathfrak{h}^*$ are given. Put $M_i=M(\Lambda_i)$ with the generator v_i; $M=M_1\otimes\ldots\otimes M_n$, $v=v_1\otimes\ldots\otimes v_n$, $\Lambda=\sum_{i=1}^{n}\Lambda_i$.

Let $\mathfrak{b}$ act on $\mathfrak{b}\otimes M$ by the rule $a\cdot(b\otimes m)=[a, b]\otimes m+a\otimes b\cdot m$.

For $1\leqq i\leqq n$, c, $b\in\mathfrak{g}$, $m=x_1\otimes\ldots\otimes x_n\in M$, put $b^{(i)}m=x_1\otimes\ldots\otimes x_{i-1}\otimes bx_i\otimes x_{i+1}\otimes\ldots\otimes x_n$; $b^{(i)}(c\otimes m)=[b, c]\otimes m+c\otimes b^{(i)}m$.

6.15.1 Lemma-definition. *There exists a unique* $\mathbb{C}$*-linear map* $\nu_M\colon M\to\mathfrak{b}\otimes M$ *such that*

(i) $\nu_M(h\cdot x)=h\cdot\nu_M(x)$ *for any* $h\in\mathfrak{h}$, $x\in M$;
(ii) $\nu_M(x)=\frac{1}{2}(b(\Lambda)-\mu(\lambda))\otimes x+\nu_M(x))$, $\nu_M(x))\in\mathfrak{n}\otimes M$, *for any* $x\in M_\lambda$ *where* b: $\mathfrak{h}^*\xrightarrow{\sim}\mathfrak{h}$ *is defined in* 6.1;
(iii) $\nu_M(f_i^{(j)}\, x)=f_i^{(j)}\cdot\nu_M(x)_- -f_i\otimes h_i^{(j)}\, x$ *for* $1\leqq i\leqq r$, $1\leqq j\leqq n$.

Proof. Left to the reader. □

6.15.2 Lemma. *For any* $x,\ y\in M$, $a\in\mathfrak{b}$

$$S(\nu_M(x), a\otimes y)=\begin{cases}\frac{1}{2}S(x, a\,y) & \text{if } a\in\mathfrak{h},\\ S(x, a\,y) & \text{if } a\in\mathfrak{n}.\end{cases}$$

Here S *is defined on* $\mathfrak{b}\otimes M$ *by the rule* $S(a\otimes x, b\otimes y)=S(a, b)\, S(x, y)$ (*cf.* (6.4.2)).

Proof. The first equality follows from 6.15.1(ii). Suppose that $a\in\mathfrak{n}$, $x\in M_{\lambda_1}\otimes\ldots\otimes M_{\lambda_n}$, and we have proven 6.15.2 for a, x and any y. Let us prove it for $f_i^{(j)}x$. We have

$$\begin{aligned}S(\nu_M(f_i^{(j)}\cdot x), a\otimes y)&=S(\nu_M(f_i^{(j)}\cdot x)_-, a\otimes y)\\ &=S(f_i^{(j)}\cdot\nu_M(x)_-, a\otimes y)-(h_i, b(\Lambda_j)-\mu(\lambda_j))\, S(f_i\otimes x, a\otimes y).\end{aligned}$$

1st case. $a\in\mathfrak{n}_{\lambda_i^0}$. We may suppose that $a=f_i$. We have

$$\begin{aligned}S(f_i^{(j)}\,\nu_M(x)_-, f_i\otimes y)&=S(\nu_M(x)_-, e_i^{(j)}\cdot(f_i\otimes y))\\ &=S(\nu_M(x)_-, f_i\otimes e_i^{(j)}\, y)=S(x, f_i\, e_i^{(j)}\, y)\\ &=S(x, [-h_i^{(j)}+e_i^{(j)}\, f_i^{(j)}]\, y).\end{aligned}$$

We may have two non-trivial possibilities

1st subcase. $y\in M_{\lambda_1}\otimes\ldots\otimes M_{\lambda_n}$. Then

$$\begin{aligned}S(f_i^{(j)}\,\nu_M(x)_-, f_i\otimes y)&=-(h_i, b(\Lambda_j)-\mu(\lambda_j))\, S(x, y)\\ &\quad+S(x, e_i^{(j)}\, f_i\, y),\\ S(f_i\otimes x, a\otimes y)&=-S(x, y).\end{aligned}$$

Hence,

$$S(\nu_M(f_i^{(j)}\, x), f_i\otimes y)=S(x, e_i^{(j)}\, f_i\, y)=S(f_i^{(j)}\, x, f_i\, y).$$

2nd subcase. $y\in M_{\lambda'_1}\otimes\ldots\otimes M_{\lambda'_n}$ where $\lambda'_j=\lambda_j+\lambda_i^0$; $\lambda'_k=\lambda_k-\lambda_i^0$ for some $k\neq j$, and $\lambda'_p=\lambda_p$ for all $p\neq k, j$. Then $S(f_i\otimes x, f_i\otimes y)=0$,

$$\begin{aligned}S(x, f_i\, e_i^{(j)}\, y)&=S(x, f_i^{(k)}\, e_i^{(j)}\, y)=S(x, e_i^{(j)}\, f_i^{(k)}\, y)\\ &=S(f_i^{(j)}\, x, f_i^{(k)}\, y)=S(f_i^{(j)}\, x, f_i\, y),\end{aligned}$$

hence $S(\nu_M(f_i^{(j)}\, x), f_i\otimes y)=S(f_i^{(j)}\, x, f_i\, y)$.

2nd case. $a\in\mathfrak{n}_{\lambda'}$. $\lambda'\neq\lambda_i^0$. This case is treated similarly, and we leave to the reader.
Lemma is proven. □

6.15.3 Remark. Let us give a pair of more explicit formulas for ν_M. First, for any $x \in M$

$$\nu_M(x)_- = -\sum_{i=1}^{r} f_i \otimes e_i x + \nu_M(x)_{--}, \tag{6.15.3.1}$$

where $\nu_M(x)_{--} \in \mathfrak{n}_{--} \otimes M$, $\mathfrak{n}_{--} = \bigoplus_{|\lambda|>1} \mathfrak{n}_\lambda$.

For a N-tuple of pairwise distinct integers $(i(1), \ldots, i(N))$, $1 \leqq i(j) \leqq r$; and $(a(1), \ldots, a(N)) \in [n]^N$

$$\begin{aligned}(6.15.3.2)\qquad & \nu_M(f_{i(N)}^{(a(N))} \cdot f_{i(N-1)}^{(a(N-1))} \ldots f_{i(1)}^{(a(1))} v) \\ &= -\sum_{p=1}^{N} \sum_{1 \leqq j_1 < \ldots < j_p \leqq N} [f_{i(j_p)}^{(a(j_p))}, [f_{i(j_{p-1})}^{(a(j_{p-1}))}, \ldots, \\ &\cdot [f_{i(j_2)}^{(a(j_2))}, f_{i(j_1)}^{(a(j_1))}] \ldots] \otimes e_{i(j_1)} f_{i(N)}^{(a(N))} \ldots \widehat{f_{i(j_p)}^{(a(j_p))}} \\ &\ldots f_{i(j_2)}^{(a(j_2))} \ldots f_{i(1)}^{(a(1))} v.\end{aligned}$$

Both formulas are proved by induction on $|\lambda|$.

6.16 From 3.4.5 follows that for general matrices B (i.e. for matrices B belonging to a Zarisky dense subset of the set of all $r \times r$-matrices) the map S: $\mathfrak{b} \to \mathfrak{b}^*$ is an isomorphism. Analogously, for general B and $\Lambda_1, \ldots, \Lambda_n$ S: $M \to M^*$ are isomorphisms. In this case properties 6.14.3 (resp., 6.15.2) uniquely determine ν: $\mathfrak{b} \to \mathfrak{b} \wedge \mathfrak{b}$ (resp., ν_M: $\mathfrak{b} \to \mathfrak{b} \otimes M$).

It follows easily that for general B, $\Lambda_i \nu$ is a Lie cobracket (i.e. satisfies the dual of the Jacobi identity), hence $\mathfrak{b}$ is a Lie bialgebra, and M is a $\mathfrak{b}$-comodule. Hence, the same is true for any B, Λ_i. So, we get

6.16.1 Proposition. *The map* ν: $\mathfrak{b} \to \mathfrak{b} \wedge \mathfrak{b}$ 6.14.1 *defines a structure of a Lie bialgebra on* $\mathfrak{b}$. *The map* ν_M: $M \to \mathfrak{b} \otimes M$ *defines a structure of a* $\mathfrak{b}$*-comodule on* M.

Alternatively, one can say that ν, ν_M define the structures of a Lie algebra on $\mathfrak{b}^*$ and of a $\mathfrak{b}^*$-module on M^*.

Consider $\mathfrak{n}^*$ as a subspace of $\mathfrak{b}^*$ by means of the projection $\mathfrak{b} \to \mathfrak{n}$. $\mathfrak{n}^*$ is a Lie subalgebra of $\mathfrak{b}^*$.

Warning. $\mathfrak{n}$ with its bracket and a cobracket induced by the above bracket is not a Lie bialgebra!

Consider the standard complex $C_\bullet(\mathfrak{n}^*, M^*) \cong C_\bullet(\mathfrak{n}, M)^*$. From 6.6 follows

6.16.2 Theorem. *The map* $\eta_\bullet$ *induces isomorphisms of complexes*

$$C_\bullet(\mathfrak{n}^*, M^*)_\lambda \xrightarrow{\sim} (\mathscr{A}^{N-\bullet}(\mathscr{C}_{n;N})^{\Sigma_\lambda}, d)$$

for all λ.

6.17 Let $D(\mathfrak{b})$ be the double of $\mathfrak{b}$. Recall [D, n° 13] that this is a Lie algebra equal to $\mathfrak{b} \oplus \mathfrak{b}^*$ as a space, with the bracket on $\mathfrak{b}$ and $\mathfrak{b}^*$ defined by the Lie algebra structure on $\mathfrak{b}$ and $\mathfrak{b}^*$, and for $b \in \mathfrak{b}$, $\ell \in \mathfrak{b}^*$ $[\ell, b] = \tilde{\ell} + \tilde{b}$, $\tilde{\ell} \in \mathfrak{b}^*$, $\tilde{b} \in \mathfrak{b}$ are defined by the rules $\tilde{\ell}(c) = \ell([b, c])$, $c \in \mathfrak{b}$; $\ell'(\tilde{b}) = [\ell', \ell]$ (b), $\ell' \in \mathfrak{b}^*$.

Consider the space M^* as in 6.16 and define the action of $\mathfrak{b}\oplus\mathfrak{b}^*$ on it. Namely, for $b\in\mathfrak{b}$, $t\in M^*$, put $bt(x)=t(-bx)$; the action of $\mathfrak{b}^*$ is defined as in 6.16.

6.17.1 Proposition. *The above rule defines on M^* the structure of a $D(\mathfrak{b})$-module.*

Proof. We have to verify that $[\ell, b]\ t=\ell bt-b\ell t$ for all $\ell\in\mathfrak{b}^*$, $b\in\mathfrak{b}$, $t\in M^*$. As in 6.16 we have to prove this for general B, Λ_i. Suppose that $\ell(\cdot)=S(\cdot, c)$, $c\in\mathfrak{b}$, $t(\cdot)=S(\cdot, n)$, $n\in M$. Then $[\ell, b]=\tilde{\ell}+\tilde{b}$ where $\tilde{b}=[\tau c, b]$ if $[\tau c, b]\in\mathfrak{b}$ and 0 otherwise, $\tilde{\ell}(\cdot)=S(\cdot, [c, \tau b])$ if $[c, \tau b]\in\mathfrak{b}$ and 0 otherwise. Hence, $[b, \ell]\ t(\cdot)=S(\cdot, [\tau b, c]\ n)$. On the other hand, $(b\ell-\ell b)\ t(\cdot)=S(\cdot, (\tau bc-c\tau b)\ n)=S(\cdot, [\tau b, c]\ n)$. $\square$

7 *Knizhnik-Zamolodchikov equations*

In this section we calculate the Gauss-Manin connection in the top cohomology of complexes studied in the previous section.

7.1 Let us save the assumptions and notations from 6.1. Fix $\lambda=(k_1, \dots, k_r)\in\mathbb{N}^r$; put $N=|\lambda|=\sum_{i=1}^{r} k_i$; fix n weights $\Lambda_1, \dots, \Lambda_n\in\mathfrak{h}^*$, $n\geqq 2$. Fix an epimorphism $\pi\colon [N]\to[r]$ with $\#\ \pi^{-1}(i)=k_i$ for all i. Put $\alpha(j)=\alpha_{\pi(j)}$. Consider the arrangement $\mathscr{C}_{n+N}$ in $\mathbb{C}^{n+N}$ with coordinates $(z_1, \dots, z_n;\ t_1, \dots, t_N)$. Define the collection of exponents

$$a=a(\Lambda_1, \dots, \Lambda_n;\kappa)\colon\ \mathscr{C}_{n+N}\to\mathbb{C}$$

where $\kappa\in\mathbb{C}$, $\kappa\neq 0$ is a complex parameter, by the rule

$$\begin{aligned} a(z_i-z_j=0)&=(\Lambda_i, \Lambda_j)/\kappa;\\ a(t_i-z_j=0)&=-(\alpha(i), \Lambda_j)/\kappa;\\ a(t_i-t_j=0)&=(\alpha(i), \alpha(j))/\kappa \end{aligned} \tag{7.1.1}$$

(cf. (6.5.1)).

Denote by $\mathscr{L}_\lambda=\mathscr{L}_\lambda(\Lambda_1, \dots, \Lambda_n;\ \kappa)$ the corresponding line bundle with an integrable connection over $U(\mathscr{C}_{n+N})$, cf. (4.5), and $\mathscr{S}_\lambda$ the local system of its horizontal sections.

Let $p\colon U(\mathscr{C}_{n+N})\to U(\mathscr{C}_n)$ be the projection on the first n coordinates. Abusing the language, we'll denote by $z=(z_1, \dots, z_n)$ a generic point of $U(\mathscr{C}_n)$. Thus, $p_{(z)}^{-1}=U(\mathscr{C}_{n;N}(z))$ (cf. 5.1). We put $\mathscr{L}_\lambda(z)=\mathscr{L}_\lambda|_{p^{-1}(z)}$.

Denote by $\Omega^\cdot(\mathscr{L}_\lambda)$ the direct image p_* of the relative de Rham complex of $\mathscr{L}_\lambda$. It is a complex of vector bundles over $U(\mathscr{C}_n)$, whose fiber at z is $\Omega^\cdot(\mathscr{L}_\lambda)_z=\Omega^\cdot(\mathscr{L}_\lambda(z))$ (cf. 4.5). Cohomology bundles

$$\mathscr{H}^i(\Omega^\cdot(\mathscr{L}_\lambda))=R^i p_*(\mathscr{L}_\lambda)$$

are supplied with the Gauss-Manin integrable connection ∇_{GM}.

Denote by $\Omega^\cdot_{\mathrm{OS}}(\mathscr{L}_\lambda(z))$ (the complex of Orlik-Solomon forms) the image of (4.5.4); put $H^\cdot_{\mathrm{OS}}(\mathscr{L}_\lambda(z))=\mathrm{Im}(H^\cdot(\Omega^\cdot_{\mathrm{OS}}(\mathscr{L}_\lambda(z)))\to H^\cdot(\Omega^\cdot(\mathscr{L}_\lambda(z))))$. This defines a subcomplex of bundles $\Omega^\cdot_{\mathrm{OS}}(\mathscr{L}_\lambda)\hookrightarrow\Omega^\cdot(\mathscr{L}_\lambda)$. Put $R^\cdot_{\mathrm{OS}}\,p_*(\mathscr{L}_\lambda)=\mathrm{Im}(H^\cdot(\Omega^\cdot_{\mathrm{OS}}(\mathscr{L}_\lambda))\to H^\cdot(\Omega^\cdot(\mathscr{L}_\lambda)))$.

It is convenient to represent local sections of $\Omega^j(\mathscr{L}_\lambda)$ in the form $\varphi(t, z)$ $\ell_\lambda(t, z)\, dt_{i_1} \wedge \ldots \wedge dt_{i_j}$ where

$$\ell_\lambda(t, z) = \prod_{m > m'} (z_m - z_{m'})^{(\Lambda_m, \Lambda_{m'})/\kappa} \cdot \prod_{i, m} (t_i - z_m)^{(-\alpha(i), \Lambda_m)/\kappa} \cdot \prod_{i > i'} (t_i - t_{i'})^{(\alpha(i), \alpha(i'))/\kappa}; \tag{7.1.2}$$

$\varphi(t, z)$ is a holomorphic function.

Let $\Sigma_\lambda = \Sigma_{k_1} \times \ldots \times \Sigma_{k_r}$ act on $[N]$ as follows: the i'th factor Σ_{k_i} acts on $\pi^{-1}(i)$ by permutations, where we identify $\pi^{-1}(i)$ with $[k_i]$ using the order $\pi^{-1}(i)$ induced from [N]. This induces the fiberwise action an of Σ_λ on $\Omega^{\cdot}(\mathscr{L}_\lambda)$ by the rule

$$\sigma[\varphi(t, z)\, \ell_\lambda\, dt_{i_1} \wedge \ldots \wedge dt_{i_j}] = \sigma\varphi(t, z)\, \ell_\lambda\, dt_{i_1} \wedge \ldots \wedge dt_{i_j}$$

where $\sigma\varphi$ is obtained from φ by permutation of t_i' s. This action respects $\Omega^{\cdot}_{\mathrm{OS}}$. Evidently, the Gauss-Manin connection is Σ_λ-equivariant.

7.2 Put $M_i = M(\Lambda_i)$; $M = M_1 \otimes \ldots \otimes M_n$.

For a vector space V denote by $\Omega(V) \in V \otimes V^*$ the tautological element. For $\lambda \in \mathbb{N}^r$ put $\Omega_\lambda^- := \Omega(\mathfrak{n}_\lambda) \in \mathfrak{n}_\lambda \otimes \mathfrak{n}_\lambda^*$; $\Omega_\lambda^+ := \Omega(\mathfrak{n}_\lambda^*) \in \mathfrak{n}_\lambda^* \otimes \mathfrak{n}_\lambda$; $\Omega^0 = \frac{1}{2}(\Omega(\mathfrak{h}) + \Omega(\mathfrak{h}^*)) \in \mathfrak{h} \otimes \mathfrak{h}^* \oplus \mathfrak{h}^* \otimes \mathfrak{h}$. For any Λ, $\Lambda' \in h$ these elements act on $M(\Lambda)^* \otimes M(\Lambda')^*$ by means of $\mathfrak{b} \oplus \mathfrak{b}^*$-action introduced in 6.17. Note that on $(M(\Lambda)^* \otimes M(\Lambda')^*)_\lambda$ only $\Omega_{\lambda'}^{\pm}$ with $\lambda' \leq \lambda$ act non-trivially. (We say that $\lambda' = (k'_1, \ldots, k'_r) \leqq \lambda = (k_1, \ldots, k_r)$ iff $k'_i \leqq k_i$ for all i). So we may form an infinite sum

$$\Omega = \sum_\lambda \Omega_\lambda^- + \Omega^0 + \sum_\lambda \Omega_\lambda^+ \tag{7.2.1}$$

which acts on $M(\Lambda)^* \otimes M(\Lambda')^*$.

Let us denote by Ω_{ij} the operator on $M^* = M_1^* \otimes \ldots \otimes M_n^*$ acting as Ω on $M_i^* \otimes M_j^*$ and as the identity on other factors. These operators respect the weight decomposition. By the same letter we'll denote the adjoint operators acting on M.

Let Ω_{ij} act on spaces $\Lambda^p \mathfrak{n}^* \otimes M^*$ as the identity on $\Lambda^p \mathfrak{n}^*$ and as above on M^*.

7.2.2 Lemma. *The above action is respects the differential in $C_\bullet(\mathfrak{n}^*, M^*)$.*

Proof. This follows from the invariance of Ω. □

Clearly, this action respects the weight decomposition.

Denote by $\mathscr{M}$, $\mathscr{M}_\lambda$, $C_\bullet(\mathfrak{n}^*, \mathscr{M}^*)$, etc. trivial bundles over $U(\mathscr{C}_n)$ with fibers M, M_λ, $C_\bullet(\mathfrak{n}^*, M^*)$, etc. respectively.

7.2.3 Definition. *The Knizhnik-Zamolodchikov connection* ∇_{KZ} on $C_\bullet(\mathfrak{n}^*, \mathcal{M}^*)$ is defined by

$$\nabla_{KZ}\left(\frac{\partial}{\partial z_i}\right)=\frac{\partial}{\partial z_i}-\frac{1}{\kappa}\sum_{j\neq i}\frac{\Omega_{ij}}{z_i-z_j}.$$

From the invariance of Ω follows that ∇_{KZ} is integrable, cf. [Ko].

Note that ∇_{KZ} respects the differential in $C_\bullet$, so it induces the integrable connection on cohomology bundles, which will be denoted by the same letter.

Theorem 6.16.2 defines canonical isomorphisms of complexes of vector bundles

(7.2.4) $$\eta_\bullet\colon\ C_\bullet(\mathfrak{n}^*, \mathcal{M}^*)_\lambda \xrightarrow{\sim} \Omega^{N-\bullet}_{\mathrm{OS}}(\mathcal{L}_\lambda)^{\Sigma_\lambda}$$

Put $\mathscr{V}^*=\mathscr{H}_0(\mathfrak{n}^*, \mathcal{M}^*)$; $\mathscr{V}^*_\lambda=H_0(C_\bullet(\mathfrak{n}^*, \mathcal{M}^*)_\lambda)$. Here is the main result of this Section.

7.2.5 Theorem. *The subsheaf* $R^N_{\mathrm{OS}}\, p_*(\mathcal{L}_\lambda)\subset R^N p_*(\mathcal{L}_\lambda)$ *is stable with respect to* ∇_{GM}. *The isomorphism*

$$\mathscr{H}_0(\eta_\bullet)\colon\ \mathscr{V}^*_\lambda \xrightarrow{\sim} R^N_{\mathrm{OS}}\, p_*(\mathcal{L}_\lambda)^{\Sigma_\lambda}$$

maps the Knizhnik-Zamolodchikov connection to the Gauss-Manin connection.

(It is natural to suggest that the same is true for all $\mathscr{H}_i(\eta_\bullet)$.) Let us formulate a more exact assertion.

7.2.5′ Theorem. *There exist canonical maps of vector bundles*

$$\eta_i'\colon\ \mathcal{M}^*_\lambda\to\Omega^{N-1}(\mathcal{L}_\lambda)^{\Sigma_\lambda}$$

$i=1, \ldots, n$ such that for every $x\in\mathcal{M}^*_\lambda$

$$\frac{\partial\eta(x)}{\partial z_i}=\frac{1}{\kappa}\sum_{j\neq i}\frac{\eta(\Omega_{ij}x)}{z_i-z_j}+d\eta_i'(x)$$

where d is the differential in $\Omega^\bullet(\mathcal{L}_\lambda)$.

Clearly, 7.2.5 follows from 7.2.5′. 7.2.5′ will be proven in n°s 7.3–7.10.

7.2.6 Corollary. *There exists a constant A such that for* $|\kappa|>A$ *the inclusion* $R^N_{\mathrm{OS}}\, p_*(\mathcal{L}_\lambda)\subset R^N p_*(\mathcal{L}_\lambda)$ *becomes an equality.*

Proof. Fix $z^0\in U(\mathscr{C}_n)$. By Theorem 4.6 there exists A such that

(*) $$H^N_{\mathrm{OS}}(p^{-1}(z), \mathcal{L}_\lambda(z))=H^N(p^{-1}(z), \mathcal{L}_\lambda(z))$$

for $z=z^0$ and $|\kappa|>A$. Since $R^N_{\mathrm{OS}}\, p_*$ is stable with respect to ∇_{GM} it follows that it is a vector bundle; hence the (*) holds for every $z\in U(\mathscr{C}_n)$. □

Let us denote by $\mathscr{S}_\lambda$ the local system of horizontal sections of $\mathcal{L}_\lambda$; put $\mathscr{S}_\lambda(z)=\mathscr{S}_\lambda|_{p^{-1}(z)}$, $z\in U(\mathscr{C}_n)$. Homology groups $H_N(p^{-1}(z), \mathscr{S}_\lambda(z)^*)$ form a vector bundle H_N over $U(\mathscr{C}_n)$ with an integrable connection. In fact, $\mathscr{H}_N$ is dual to $R^N p_*(\mathcal{L}_\lambda)$.

For $\Delta(z) \in H_N(p^{-1}(z), \mathscr{S}_\lambda(z)^*)$ put

(7.2.7)
$$\chi_\Delta(z) = \int_{\Delta(z)} \eta \in V_\lambda.$$

7.2.8 Corollary. *If $\Delta = (\Delta(z))$ is a local horizontal section of $\mathscr{H}_N$, then a V_λ-valued function $\chi_\Delta(z)$ satisfies the* KZ *differential equations*

(7.2.8.1)
$$\frac{\partial \chi_\Delta(z)}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq 1} \frac{\Omega_{ij} \cdot \chi_\Delta(z)}{z_i - z_j},$$

$i = 1, \ldots, n.$

There exists $A > 0$ such that for all κ, $|\kappa| > A$, functions χ_Δ for all Δ form a complete space of solutions of 7.2.8.1. □

Now let us consider the Kac-Moody algebra $\bar{\mathfrak{g}} = \mathfrak{g}(B)/\ker S$ (cf. 6.3); put $\overline{M(\Lambda)} = M(\Lambda)/\ker S$, $\bar{M}_i = \overline{M(\Lambda_i)}$ these are $\bar{\mathfrak{g}}$-modules (Verma modules); let $M(\Lambda)^*$ be contragradient modules. The Killing form K induces the non-degenerate pairing between root spaces $\bar{\mathfrak{g}}_\alpha$ and $\bar{\mathfrak{g}}_{-\alpha}$ (cf. [K, Theorem 2.2 (d)].

Let $\Omega_\alpha \in \bar{\mathfrak{g}}_\alpha \otimes \bar{\mathfrak{g}}_{-\alpha}$ be the corresponding element. Let us form a (possibly infinite) sum

(7.2.9)
$$\bar{\Omega} = \sum_\alpha \bar{\Omega}_\alpha$$

as in (7.2.1). Put $\bar{M} = \bar{M}_1 \otimes \ldots \otimes \bar{M}_n$; define operators $\bar{\Omega}_{ij}$ on $\bar{M}$ and $\bar{M}^*$ as after 7.2.1. Put

(7.2.10)
$$\bar{V}_\lambda = \bigcap_{i=1}^{r} \ker(e_i : \bar{M} \to \bar{M}).$$

The projection $M_\lambda \to \bar{M}_\lambda$ induces the map $V_\lambda \to \bar{V}_\lambda$.

Let $\bar{\chi}_\Delta(z)$ be the image of $\chi_\Delta(z)$ (7.2.7) in $\bar{V}_\lambda$.

7.2.11 Corollary. *In conditions of* (7.2.8) *$\bar{\chi}_\Delta(z)$ satisfies the system of differential equations*

(7.2.11.1)
$$\frac{\partial}{\partial z_i} \bar{\chi}_\Delta(z) = \frac{1}{\kappa} \sum_{j \neq i} \frac{\bar{\Omega}_{ij}}{z_i - z_j} \bar{\chi}_\Delta(z).$$

Proof. Follows from 7.2.8 and an easy assertion from linear algebra:

7.2.11.2 Lemma. *For $\lambda = (k_1, \ldots, k_r)$, $\alpha = -\sum_{i=1}^{r} k_i \alpha_i$ the action of Ω_λ^+ on $\overline{M(\Lambda)} \otimes \overline{M(\Lambda')}$ coincides with the action of $\bar{\Omega}_\alpha$.* □

7.2.12 Remark. Just Eqs. (7.2.11.1) were discovered by Knizhnik and Zamolodchikov in [KZ].

7.3 Let us review the definition of $\eta = \eta_0$ from (7.2.4).

Let $\mathscr{P}(\lambda; n)$ denote the set of all pairs (γ, ε), where γ: $[N]\to[r]$ is a map such that $\#\gamma^{-1}(i)=k_i$ for all i; ε: $[N]\to[n]$ is any non-decreasing map. To every $(\gamma, \varepsilon)\in\mathscr{P}(\lambda; n)$ we associate an element

$$f(\gamma,\varepsilon)=f_{I_1}\otimes\ldots\otimes f_{I_n}\in(M_1\otimes\ldots\otimes M_n)_\lambda \tag{7.3.1}$$

where $I_m=(\gamma(a), \gamma(a+1), \ldots, \gamma(b))$ if $\varepsilon^{-1}(m)=[a, b]:=\{i\,|\,a\leqq i\leqq b\}$, $a\leqq b$; $f_{I_m}=f_{\gamma(a)}f_{\gamma(a+1)}\cdots f_{\gamma(b)}\, v_m\in M_M$; $f_{I_m}=v_m$ (the vaccum vector of M_m) if $\varepsilon^{-1}(m)=\emptyset$. The set $\{f(\gamma, \varepsilon)\}$, $(\gamma, \varepsilon)\in\mathscr{P}(\lambda; n)$ forms a base of M. Denote by $\{\delta(\gamma, \varepsilon)=\delta_{I_1}\otimes\ldots\otimes\delta_{I_n}\}$ the dual base of M^*.

Denote by $\mathscr{R}=\mathscr{R}_{n;N}$ the subring of the field of rational functions $\mathbb{C}(z_1, \ldots, z_n; t_1, \ldots, t_N)$ generated by all fractions $(t_i-t_j)^{-1}$, $(t_i-z_j)^{-1}$, $(z_i-z_j)^{-1}$. To every $(\gamma, \varepsilon)\in\mathscr{P}(\lambda; n)$ we assign $\varphi(\gamma, \varepsilon)\in\mathscr{R}$ as follows.

Choose a bijection ρ: $[N]\to[N]$ such that $\gamma(a)=\gamma(b)$ implies $\pi\rho(a)=\pi\rho(b)$. So, ρ maps $\gamma^{-1}(i)$ isomorphically onto $\pi^{-1}(i)$. Put

$$\varphi_m=\varphi_m(t,z)=\left(\prod_{i=a}^{b-1}(t_{\rho(i)}-t_{\rho(i+1)})^{-1}\right)(t_{\rho(b)}-z_m)^{-1} \tag{7.3.2}$$

if $\varepsilon^{-1}(m)=[a, b]$, $a\leqq b$; if $\varepsilon^{-1}(m)=\emptyset$, put $\varphi_m=1$.
Set

$$\varphi(\gamma,\varepsilon)=\sum_\sigma \sigma(\varphi_1\ldots\varphi_n) \tag{7.3.3}$$

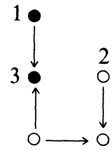
the sum being taken over all $\sigma\in\Sigma_\lambda$, where Σ_λ acts on $\mathscr{R}$ by permutations of t_i's as described above.

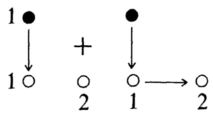
By definition.

$$\eta(\delta(\gamma,\varepsilon))=\frac{1}{(2\pi i)^N}\,\varphi(\gamma,\varepsilon)\cdot\ell_\lambda\cdot dt_1\wedge\ldots\wedge dt_N. \tag{7.3.4}$$

The map just constructed coincides with (6.5.2) for $p=0$, and "standard" π defined as after (6.5.1). This follows from 6.8.1. The case of arbitrary π is reduced to this one after renumbering of t_i's.

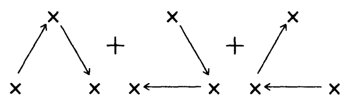
7.4 Now we begin the proof of 7.2.5'. First let us introduce a diagram notation (similar to that of § 6) for some elements of the ring $\mathscr{R}=\mathscr{R}_{n;N}$ (see 7.3). Consider an element x of $\mathscr{R}$ which is a product of pairwise distinct factors $(t_i-t_j)^{-1}$, $(t_i-z_j)^{-1}$, or $(z_i-z_j)^{-1}$. We'll depict as a directed graph: it has $n+N$ coloured and numbered vertices: n white ones which are in $1-1$ correspondence with the set $z_1, \ldots, z_n$, and N black ones which are in $1-1$ correspondence with $t_1, \ldots, t_N$. The arrows are in $1-1$ correspondence with inverse differences belonging to x: the i-th black vertex is connected with the j-th black vertex $\underset{i}{\bullet}\to\underset{j}{\bullet}$ iff there is a factor $(t_i-t_j)^{-1}$ in x, the same with $\underset{i}{\bullet}\to\underset{j}{\circ}$, $\underset{i}{\circ}\to\underset{j}{\circ}$. Sometimes we'll omit the isolated vertices which are not a boundary of any arrow. If we do not want to indicate the colour of a vertice, we'll depict it as a cross x.

Example.  $=(t_1-t_3)^{-1}(z_1-t_3)^{-1}(z_1-z_2)^{-1}(z_2-t_2)^{-1}$

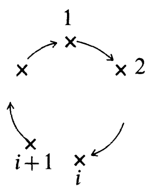
 $=(t_1-z_1)^{-1}+(t_1-z_1)^{-1}(z_1-z_2)^{-1}.$

Abusing the language, sometimes we'll denote by the same picture an element $\varphi(t, z)\in\mathscr{R}$ and differential form $\varphi(t, z)\,\ell_\lambda(t, z)\cdot dt_1\wedge\ldots\wedge dt_N$.

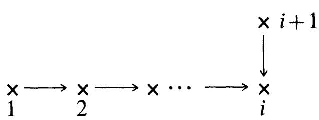
Here is the key relation in $\mathscr{R}$:

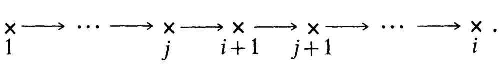
7.4.1 Lemma.  $=0.$

It implies

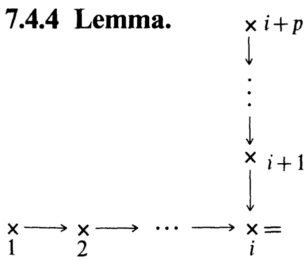
7.4.2 Circle lemma. $\sum_{i=1}^{p}$  $=0$

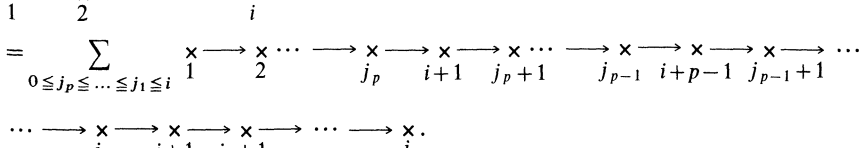
(*In i-th picture the arrow connecting i-th and* $(i+1)$*-th vertices is omitted.*)

7.4.3 Lemma. 

$=\sum_{i=1}^{p}$ 

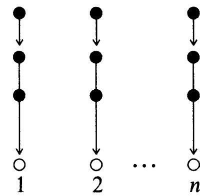
7.4.4 Lemma.



$=\sum_{0\leqq j_p\leqq\ldots\leqq j_1\leqq i}$ 

Proof. Follows from 7.4.3. □

7.5 Suppose that $\lambda=(1, 1, \ldots, 1)$, so $N=r$, $\Sigma_\lambda=\{e\}$; suppose that π: $[N]\to[r]$ is identity. In notation of 7.3, let us consider a form $\eta=\eta(\gamma, \varepsilon)$ corresponding to some $(\gamma, \varepsilon)\in\mathscr{P}(\lambda; n)$. The diagram corresponding to η has the following shape:

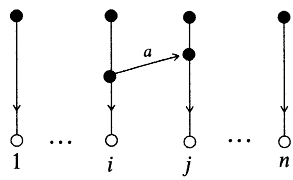
(7.5.1) $\eta=$ 

Let us call connected components of it *branches* (they are similar to islands of § 5). The m-th branch has the form

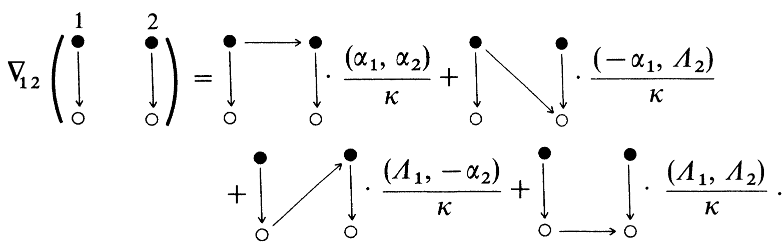
$$\underset{\gamma(a)}{\bullet}\longrightarrow\underset{\gamma(a+1)}{\bullet}\longrightarrow\cdots\longrightarrow\underset{\gamma(b)}{\circ}\longrightarrow\underset{m}{\circ}$$

if $\varepsilon^{-1}(m)=[a, b]$. Sometimes we'll draw it without marking internal black vertices: $\underset{\gamma(a)}{\bullet}\longrightarrow\underset{m}{\circ}$.

For an arrow $\underset{s}{\times}\xrightarrow{a}\underset{t}{\times}$ let us call its *weight* the number $w(a)=(w(s), w(t))/\kappa$, where $w(\underset{i}{\bullet})=-\alpha_i$, $w(\underset{i}{\bullet})=\Lambda_i$. For $i, j\in[n]$, $i\neq j$ let us introduce differential form $\nabla_{ij}\eta$ as follows. Its diagram is the sum of all diagrams obtained from (7.5.1) by adding one arrow connecting a point in i-th branch with a point in j-th branch, multiplied by the weight of an arrow:

(7.5.2) $\nabla_{ij}\,\eta=\sum_a$  $\cdot\, w(a)$

7.5.3 Example.



So we get linear operators

(7.5.4) $$\nabla_{ij}\colon\ \Omega^N_{\mathrm{OS}}(\mathscr{L}_\Lambda)\to\Omega^N_{\mathrm{OS}}(\mathscr{L}_\Lambda).$$

For $\eta=\varphi\cdot\ell_\lambda\, dt_1\wedge\ldots\wedge dt_N\in\Omega^N(\mathscr{L}_\lambda)$ put

$$\frac{\partial}{\partial t_i}\lrcorner\,\eta=(-1)^{i-1}\varphi\cdot\ell_\lambda\, dt_1\wedge\ldots\wedge\widehat{dt_i}\wedge\ldots\wedge dt_N.$$

Here is the first key point.

7.5.5 Lemma.

$$\frac{\partial}{\partial z_i}\eta(\gamma,\varepsilon)=\sum_{j\neq i} \nabla_{ij}\,\eta(\gamma,\varepsilon)-d((\textstyle\sum \partial/\partial t_p)\lrcorner\,\eta(\gamma,\varepsilon))$$

where the sum in the second summand is taken over all p lying in the i-th branch. i.e. $p\in\{\gamma(a),\gamma(a+1),\ldots,\gamma(b)\}$, $\varepsilon^{-1}(i)=[a,b]$.

Proof. By differentiation. □

7.5.6 Now suppose that λ is arbitrary. Then forms $\eta(\gamma,\varepsilon)$ are sums of forms of the type (7.5.1).

Operators ∇_{ij} defined by (7.5.2) are Σ_λ-equivariant, and one has the same formula as in 7.5.5, with the evident modification of the exact form summand.

Theorem 7.2.5′ will follow from

7.5.7 Lemma. *For every* $x\in M_\lambda^*$

$$\nabla_{ij}\,\eta(x)=\frac{1}{\kappa}\,\frac{\eta(\Omega_{ij}x)}{z_i-z_j}.$$

The proof will occupy next n°s.

We may, and shall suppose that $n=2$. Put $\nabla=\nabla_{12}$.

7.6 First we'll calculate the action of $\Omega=\Omega_{12}$. We'll use notation: $f_{i_1\ldots i_N}=f_{i_1}\cdots f_{i_N}$; $[f_{i_1\ldots i_N}]=[f_{i_1}[f_{i_2}[\cdots[f_{i_{N-1}},f_{i_N}]\ldots]$, the same with e_i's.

7.6.1 Lemma. *Let* $f=f_{i_1\ldots i_N}$, $e=e_{j_1\ldots j_q}\in U\mathfrak{g}$, $x=f\cdot v\in M(\Lambda)$. *Then* $ex=0$ *if* $q>N$; *and*

$$ex=\sum_{1\leq p_1<\ldots<p_q\leq N} f_{i_1}\cdots\hat{f}_{i_{p_1}}\cdots\hat{f}_{i_{p_q}}$$
$$\cdot[\ldots[e,f_{i_{p_1}}],f_{i_{p_2}}],]\ \ldots,f_{i_{p_q}}]\cdot f_{i_{p_q+1}}\cdots f_{i_N}v$$

if $q\leq N$.

Proof. We have $ex=efv=[e,f]v$. By the Leibniz rule $[a,bc]=[a,b]c+b[a,c]$, we have

$$[e,f]v=\sum_{p=1}^{N} f_{i_1}\cdots f_{i_{p-1}}[e,f_{i_p}]f_{i_{p+1}}\cdots f_{i_N}v.$$

This proves 7.6.1 for $q=1$. For $q>1$ use induction on q: $[e,f_{i_p}]$ is the sum of products of $q-1$ e_j's, and we may use the induction hypothesis. □

7.6.2 Lemma. *We have an identity in* $U\mathfrak{g}$:

$$[\ldots[e_{i_1\ldots i_N}],f_{j_1}],f_{j_2}],\ldots],f_{j_N}]=K([e_{i_1\ldots i_N}],[f_{j_1\ldots j_N}])\,h_{j_N}.$$

Proof. $K([e_{i_1\ldots i_N}],\ [f_{j_1\ldots j_N}])=K(\tilde{e},\ f_{j_N})$ where $\tilde{e}=[\ldots[[e_{i_1\ldots i_N}],\ f_{j_2}],\ f_{j_1}],\ldots],\ f_{j_{N-1}}]=\sum a_i e_i$. Hence, $K(\tilde{e},f_{j_N})=a_{j_N}$. On the other hand the left hand side of 7.6.2 is $[\tilde{e},f_{j_N}]=a_{j_N}h_{j_N}$. □

In notations of 7.2, put $\bar{\Omega}_q = \sum_{|\alpha|=q} \bar{\Omega}_\alpha$, the sum is taken over all roots α of length q, i.e. equal to a sum of q simple roots. Here is the key

7.6.3 Lemma. *We have an equality in* $\bar{M}_1 \otimes \bar{M}_2$:

$$\bar{\Omega}_q(f_{i_1 \dots i_N} v_1 \otimes y)$$

$$= \sum_{1 \leqq j_1 < \dots < j_q \leqq N} f_{i_2 \dots \hat{i}_{j_1} \dots \hat{i}_{j_q} \dots i_N} v_1 \otimes [f_{i_1 \dots j_q}]\, y\left(\alpha_{i_{j_q}}, \Lambda_1 - \sum_{j=j_q+1}^{N} \alpha_{i_j}\right),$$

$v_1 \in \bar{M}_1$ *the generator*, $y \in \bar{M}_2$.

Proof. Fix a root α, $|\alpha| = q$. Choose a base $\{e_\alpha^{(i)}\}$ in $\bar{\mathfrak{g}}_\alpha$ consisting of simple commutators $[e_{j_1 \dots j_q}]$. Let $\{f_\alpha^{(i)}\}$ be the dual base in $\bar{\mathfrak{g}}_{-\alpha}$. We have

$$\bar{\Omega}_\alpha \cdot (f_{i_1 \dots i_N} v_1 \otimes y) = \Big(\sum_i e_\alpha^{(i)} \otimes f_\alpha^{(i)}\Big) \cdot (f_{i_1 \dots i_N} v_1 \otimes y)$$

$$\overset{7.6.1}{=} \sum_{1 \leqq j_1 < \dots < j_q \leqq N} \sum_i f_{i_1 \dots \hat{i}_{j_1} \dots \hat{i}_{j_q} \dots i_N} [\dots [e_\alpha^{(i)}, f_{i_{j_1}}],$$

$$, \dots, f_{i_{j_q}}] f_{i_{j_q+1} \dots i_N} v_1 \otimes f_\alpha^{(i)} y = (\text{we put } [f_J] = [f_{j_1 \dots j_q}])$$

$$\overset{7.6.2}{=} \sum_{1 \leqq j_1 < \dots < j_q \leqq N} \sum_i f_{i_1 \dots \hat{i}_{j_1} \dots \hat{i}_{j_q} \dots i_N} h_{i_{j_q}} f_{i_{j_q+1} \dots i_N}$$

$$\otimes K(e_\alpha^{(i)}, [f_J]) f_\alpha^{(i)} y.$$

But $\sum_i K(e^{(i)}, [f_J]) f_\alpha^{(i)} = [f_J]$ if $[f_J] \in \bar{\mathfrak{g}}_{-\alpha}$, and zero otherwise. The lemma follows. $\square$

7.7 Suppose that K is non-degenerate on $\mathfrak{g}$. For a positive root $\alpha = \sum k_i' \alpha_i$ consider two operators: $\bar{\Omega}_\alpha \in \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ acting $M_1 \otimes M_2$, and $\bar{\Omega}_\alpha^* = \Omega_{\lambda'}^+ \in (\mathfrak{g}_{-\alpha})^* \otimes \mathfrak{g}_{-\alpha}$, $\lambda' = (k_1', \dots, k_r')$ acting on $M_1^* \otimes M_2^*$ as in 7.2.

7.7.1 Lemma. $\bar{\Omega}_\alpha^*$ *is adjoint to* $\bar{\Omega}_\alpha$.

Proof. For $f \in \mathfrak{g}_{-\alpha}$, $\varphi \in M_i^*$, $x \in M_i$, $\langle f \cdot \varphi, x \rangle = \langle \varphi, -f \cdot x \rangle$. On the other hand, if $\delta = S(\cdot, f) \in \mathfrak{g}_{-\alpha}^*$, $\langle \delta \cdot \varphi, x \rangle = \langle \varphi, -\tau f \cdot x \rangle$. This follows from the commutativity of

$$\begin{array}{ccccc}
\mathfrak{n}^* & \otimes & M_i^* & \longrightarrow & M_i^* \\
 & \uparrow{\scriptstyle S \otimes \mathrm{id}\ \wr} & & & \uparrow{\scriptstyle \otimes} \\
\mathfrak{n} & \otimes & M_i^* & \text{——} & M_i^* \\
 & \uparrow{\scriptstyle \mathrm{id} \otimes S} & & & \uparrow{\scriptstyle S} \\
\mathfrak{n} & \otimes & M_i & \text{——} & M_i
\end{array}$$

If $\bar{\Omega}_\alpha = \sum e_\alpha^{(i)} \otimes f_\alpha^{(i)}$, then $\bar{\Omega}_\alpha^* = \sum \delta_\alpha^{(i)} \otimes f_\alpha^{(i)}$ where $\delta_\alpha^{(i)} = S(\cdot, \tau e^{(i)})$. The lemma follows. $\square$

7.8 *Insertions.* Given two sequences of distinct symbols $(a_1, \dots, a_k)$; $(b_1, \dots, b_{N-k})$, and $1 \leqq m \leqq k$, we'll denote by $I(a_1, \dots, a_k; b_1, \dots, b_{N-k}; m)$ the set of all insertions of the second sequence into the first one, i.e. the set of all sequences $(c_1, \dots, c_N)$ such that

(i) $\{c_1, \dots, c_N\} = \{a_1, \dots, a_k\} \cup \{b_1, \dots, b_{N-k}\}$;
(ii) $c_{p_j} = a_j$ for some $1 \leqq p_1 < p_2 < \dots < p_k = N$; $c_{q_j} = b_j$ for some $1 \leqq q_1 < q_2 < \dots < q_{N-k} < N$;
(iii) $q_{N-k} + 1 = p_m$.

We have $\# I(a_1, \dots, a_k;\ b, \dots, b_{N-k};\ m) = \binom{m-1+N-k}{m-2}$. Put $I(a_1, \dots, a_k; b_1, \dots, b_{N-k}) = \bigcup_{m=1}^{k} I(a_1, \dots, a_k; b_1, \dots, b_{N-k}; m)$.

Let us identify elements of Σ_N with sequences $(i_1, \dots, i_N) \in [N]^N$ by the rule $\sigma \in \Sigma_N \leftrightarrow (\sigma(1), \dots, \sigma(N))$. For $1 \leqq k \leqq N$ denote by $T_k(N) \subset \Sigma_N$ the subset consisting of sequences $(i_1, \dots, i_N)$ such that $i_1 < i_2 < \dots < i_{k-1} < i_k = N > i_{k+1} > \dots > i_N$. Put $T(N) = \bigcup_{k=1}^{N} T_k(N)$. We have $\# T_k(N) = \binom{N-1}{k-1}$; $\# T(N) = 2^{N-1}$.

We have the identity in $U\mathfrak{n}$:

$$(7.8.1) \qquad [f_{1\dots N}] = \sum_{k=1}^{N} (-1)^{N-k} \sum_{(i_1, \dots, i_N) \in T_k(N)} f_{i_1 \dots i_N}.$$

Set $S_k(N) = I(1, 2, \dots, k; N, N-1, \dots, k+1)$

7.8.2 *Example.* $N=3$. T_1: $\{(3\ 2\ 1)\}$; T_2: $\{(1\ 3\ 2),\ (2\ 3\ 1)\}$; T_3: $\{(1\ 2\ 3)\}$. S_1: $\{(3\ 2\ 1)\}$; S_2: $\{(1\ 3\ 2), (3\ 1\ 2)\}$; S_3: $\{1\ 2\ 3)\}$.

7.8.3 Lemma. *$S_k(N) = T_k(N)^{-1}$, i.e. $\sigma \in S_k(N) \Leftrightarrow \sigma^{-1} \in T_k(N)$.*

Proof. Left to the reader. □

Note that if $(i_1, \dots, i_N) \in S_k(N)$ then $i_N = k$.

7.9 Let us return to the assumptions of 7.7. The base of $(M_1 \otimes M_2)_\lambda$ constitute all monomials $\{f_{i_1 \dots i_p} v_1 \otimes f_{i_{p+1} \dots i_N} v_2\}$. Let $\{\delta_{i_1 \dots i_p} \otimes \delta_{i_{p+1} \dots i_N}\}$ be the dual base of $(M_1^+ \otimes M_2^*)_\lambda$.

Set $\Omega_q^+ = \sum_{|\mu|=q} \Omega_\mu^+$.

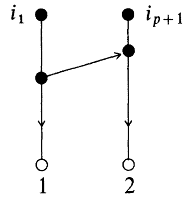
7.9.1 Lemma. $\Omega_q^+ \cdot (\delta_{i_1 \dots i_p} \otimes \delta_{i_{p+1} \dots i_N}) = \sum_{m=1}^{p+1} \sum_{k=1}^{q} \sum_{\sigma \in T_k(q)} \sum \delta_{j_1 \dots j_{p+q}} \otimes \delta_{i_{p+q+1} \dots i_N}$ $(-1)^{q-k} \cdot \left(\alpha_{i_{p+k}}, \Lambda_1 - \sum_{j=m}^{p} \alpha_{i_j}\right)$ *where the internal summation is taken over all* $(j_1, \dots, j_{p+q}) \in I(i_1, \dots, i_p; i_{p+\sigma^{-1}(1)}, \dots, i_{p+\sigma^{-1}(q)}; m)$.

Proof. This follows from 7.6.3 and 7.7.1. □

7.10 End of the proof of 7.5.7

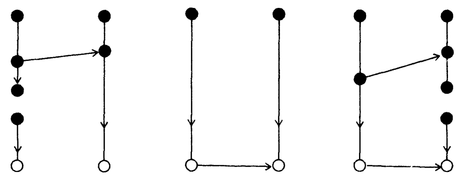
First suppose that $\lambda=(1, 1, \ldots, 1)$. Let us consider the form $\nabla\eta(x)$, $x=\delta_{i_1\ldots i_p}\otimes\delta_{i_{p+1}\ldots i_N}$. Diagrams (7.5.2) included in it have the shape

(7.10.1)



Using the Circle Lemma 7.4.2, we may present (7.10.1) as a sum (with ± 1 coefficients) of diagrams of 3 types:

(7.10.2)

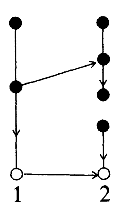


(a) (b) (c)

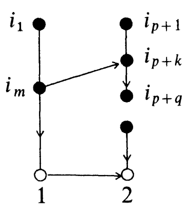
A diagram of type (b) is equal to $\frac{\eta(x)}{z_1-z_2}$. One checks directly that the sum of all diagrams of type (b) (with their coefficients) is equal to $\frac{1}{\kappa}\frac{\eta(\Omega^0 x)}{z_1-z_2}$.

Let us consider a diagram of type (c)

(7.10.3)



$$=\frac{1}{z_1-z_2}$$



By using 7.4.4 twice we transform the connected component to the standard shape $\underset{1}{\bullet\!\longrightarrow\!\circ}$

Suppose that K is non-degenerate on $\mathfrak{g}$. From 7.9.1 follows that the sum of diagrams (7.10.3) is equal to $\frac{1}{\kappa}\frac{\eta(\Omega^+ x)}{z_1-z_2}$ (the summation in the r.h.s. of 7.9.1 corresponds to the twofold application to 7.4.4 to (7.10.3)). Analogously, the sum of diagrams of type (7.10.2) (a) is equal to $\frac{1}{\kappa}\frac{\eta(\Omega^- x)}{z_1-z_2}$. This proves 7.5.7 when K is non-degenerate and $\lambda=(1, 1, \ldots, 1)$. The case of an arbitrary λ follows by symmetrization.

Both sides of 7.5.7 are polynomial functions on matrix elements of $B=(b_{ij})$, and by (3.4.5) they are equal for a Zarisky dense subset of matrices B, hence they are equal for all B. This completes the proof of 7.5.7 and of Theorem 7.2.5'. □

7.11 Remark. One can express Theorem 7.2.5′ in a more simple way. The map η defines the map

$$\tilde{\eta}:\ M_\lambda^* \to \Omega^N(U(\mathscr{C}_{n+N})).$$

The proof of Theorem 7.2.5′ shows that the following assertion holds

Theorem 7.2.5″ *For all* $x \in M_\lambda^*$

$$d\tilde{\eta}(x) = \frac{1}{\kappa} \sum_{i<j} \tilde{\eta}(\Omega_{ij}\, x) \frac{d(z_i - z_j)}{z_i - z_j}.$$

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Note added in proof

The Aomoto conjecture mentioned in 4.6.8, has been proved recently in Esnault, H., Schechtman, V., Viehweg, E.: Cohomology of local systems on the complement of hyperplanes. IAS (Preprint 1991). It implies a more precise form of 7.2.8. In Varchenko, A.: Bilinear form of real configuration of hyperplanes. IAS (Preprint 1991) the bilinear form B is defined for any arrangement $\mathscr{C}$ in a real affine space and any collection of exponents $b: \mathscr{C} \to \mathbb{C}$. The form B is a "quantum" analog of the contravariant form S, see [SV3]. In a distinguished basis any element of the matrix of the form B is a product of the exponents of some hyperplanes of the arrangement. It is proven that the determinant of B is a product of the binomials of the shape $(1-(\Pi b)^2)$, where Πb is a product of the exponents of some hyperplanes. The contravariant form S is an appropriate limit of the form B when all exponents tend to 1.