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CATEGORIES AND COHOMOLOGY THEORIES

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§0. INTRODUCTION

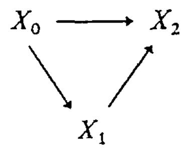
IN THIS paper I shall describe a method of associating a spectrum, and hence a cohomology theory, to a category with a composition-law of a suitable kind. The work is one possible formulation of Quillen's ideas about algebraic K-theory, and I am very grateful to him for explaining them to me.

Two applications of the method are included. The first is to prove the theorem of Barratt, Priddy and Quillen [3], [13] relating the space $\Omega^\infty S^\infty$ to the classifying-spaces of the symmetric groups. This asserts that the cohomology theory arising from the category of finite sets (under disjoint union) is stable cohomotopy.

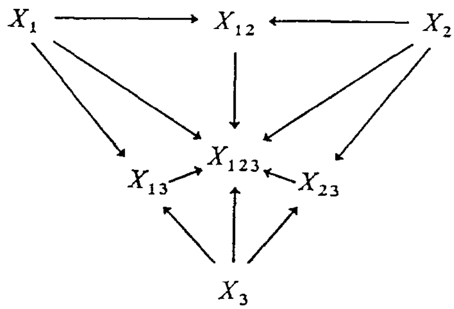
The second application is to prove the theorems of Boardman and Vogt [5] asserting that various classifying-spaces are infinite-loop-spaces. One advantage of the present treatment is that it includes the case of BU with the H-space structure arising from the tensor product.

One can outline the method as follows. A topological abelian group A has a classifying-space BA. If one uses a suitable model BA is itself a topological abelian group with a classifying-space B^2A, and so on. The sequence A, BA, B^2A, ... is a spectrum, and defines a cohomology theory h^*. The theories so arising are "classical": in fact $h^q(X) = \bigoplus_{n \geq 0} H^{q+n}(X; \pi_n A)$. In this paper I shall introduce a generalization of the notion of topological abelian group which leads to generalized cohomology theories. Roughly speaking, instead of giving a composition-law on a space A one gives for each n a space A_n and a homotopy-equivalence $p_n\colon A_n \to \underbrace{A \times \cdots \times A}_{n}$ and a "n-fold composition-law" $m_n\colon A_n \to A$. The maps p_n and m_n are required to satisfy certain conditions corresponding to associativity and commutativity of the composition-law. Such structures I have called "Γ-spaces". A Γ-space for which all the maps p_n are isomorphisms is simply a topological abelian monoid. A Γ-space has a classifying-space which is again a Γ-space, so it defines a spectrum and hence a cohomology theory.

If $\mathscr{C}$ is a category one can associate to it a space $|\mathscr{C}|$, its "nerve", described in [16]. This is a CW-complex which has a 0-cell for each object of $\mathscr{C}$, a 1-cell for each morphism, a 2-cell for each commutative diagram



in $\mathscr{C}$, and so on. Now suppose, for example, that sums exist in $\mathscr{C}$. Then $|\mathscr{C}|$ does not quite acquire a composition-law, but it is precisely a Γ-space. For example, let $\mathscr{C}_3$ be the category of diagrams in $\mathscr{C}$ of the form



in which each straight line (such as $X_1 \to X_{123} \leftarrow X_{23}$) is an expression of the middle object as a sum of the ends. Then there is an equivalence of categories $\mathscr{C}_3 \to \mathscr{C} \times \mathscr{C} \times \mathscr{C}$, taking the above diagram to (X_1, X_2, X_3), which induces a homotopy-equivalence p_3: $|\mathscr{C}_3| \to |\mathscr{C}| \times |\mathscr{C}| \times |\mathscr{C}|$; and there is a functor $\mathscr{C}_3 \to \mathscr{C}$, taking the diagram to X_{123}, which induces a "composition-law" m_3: $|\mathscr{C}_3| \to |\mathscr{C}|$. Thus $\mathscr{C}$ determines a Γ-space, and hence a cohomology theory.

The plan of the paper is as follows. §1 introduces Γ-spaces and shows how they lead to spectra. §2 shows how Γ-spaces arise from categories, and gives the main examples. §3 makes precise the relationship between Γ-spaces and spectra: this is used to prove the Barratt–Priddy–Quillen theorem (3.5), but it is otherwise technical and uninteresting. §4 concerns the relation between a topological monoid and the loops on its classifying-space. Together with §2 it proves the delooping theorems of Boardman and Vogt, but it is important mainly because it allows one to restate the Barratt–Priddy–Quillen theorem as a "K-theoretical" description of stable cohomotopy which is useful in applications (cf. [17]). §5 shows how ring-like categories lead to ring-spectra, a question I hope to pursue elsewhere.

Throughout the paper I have used "space" to mean "compactly generated space" [19], and products of spaces are always formed in that category.

§1. Γ-SPACES

Suppose A is an abelian group. If θ is a map which to each integer $i \in \{1, 2, \ldots, m\}$ associates a subset $\theta(i)$ of $\{1, 2, \ldots, n\}$ let us define θ^*: $A^n \to A^m$ by $\theta^*(a_1, \ldots, a_n) = (b_1, \ldots, b_m)$, where $b_i = \sum_{j \in \theta(i)} a_j$. (If $\theta(i)$ is empty this means that $b_i = 0$.) Obviously the composition-law of A is described (very wastefully) by giving all such maps θ^*. That motivates the following pair of definitions, in which the set of subsets of a set T is denoted by $\mathcal{P}(T)$, and the set $\{1, 2, \ldots, n\}$ is denoted by $\mathbf{n}$.

Definition 1.1. Γ is the category whose objects are all finite sets, and whose morphisms from S to T are the maps $\theta: S \to \mathcal{P}(T)$ such that $\theta(\alpha)$ and $\theta(\beta)$ are disjoint when $\alpha \neq \beta$. The composite of $\theta: S \to \mathcal{P}(T)$ and $\phi: T \to \mathcal{P}(U)$ is $\psi: S \to \mathcal{P}(U)$, where $\psi(\alpha) = \bigcup_{\beta \in \theta(\alpha)} \phi(\beta)$.

Definition 1.2. A Γ*-space* is a contravariant functor A from Γ to topological spaces such that

(i) $A(\mathbf{0})$ is contractible, and

(ii) for any n the map $p_n: A(\mathbf{n}) \to \underbrace{A(\mathbf{1}) \times \cdots \times A(\mathbf{1})}_{n}$ induced by the maps $i_k: \mathbf{1} \to \mathbf{n}$ in Γ, where $i_k(1) = \{k\} \subset \mathbf{n}$, is a homotopy-equivalence.

One should think of a Γ-space A as a kind of structure with $A(\mathbf{1})$ as "underlying space".

Before defining the classifying-space of a Γ-space I must make two remarks. The first is that a Γ-space is a simplicial space with additional structure. Recall [16] that a simplicial space is a contravariant functor $A: \Delta \to$ (spaces), where Δ is the category whose objects are the finite *ordered* sets $[m] = \{0, 1, \ldots, m\}$, and whose morphisms are all non-decreasing maps. There is a covariant functor $\Delta \to \Gamma$ which takes $[m]$ to $\mathbf{m}$ and $f: [m] \to [n]$ to $\theta(i) = \{j \in \mathbf{n}: f(i-1) < j \leq f(i)\}$. Using this functor one can regard Γ-spaces as simplicial spaces.

Secondly, recall ([12], [16]) that a simplicial space A has a *realization* $|A|$ as a topological space. (The realization I shall use here is not quite the usual one: it is discussed in Appendix A.) If A is a Γ-space its realization will mean the realization of the simplicial space it defines. (A more intrinsic definition will be given in (3.2).)

Definition 1.3. If A is a Γ-space, its classifying-space is the Γ-space BA such that, for any finite set S, $BA(S)$ is the realization of the Γ-space $T \mapsto A(S \times T)$.

To validate this definition one must check that $(S, T) \mapsto S \times T$ is a functor from $\Gamma \times \Gamma$ to Γ, and also that BA satisfies the conditions of Definition 1.2. The latter follows from the equivalence $A(\mathbf{n} \times \mathbf{m}) \to A(\mathbf{m})^n$, using (A.2) (ii) and (iii) of the Appendix.

If A is a Γ-space the spaces $A(\mathbf{1})$, $BA(\mathbf{1})$, $B^2A(\mathbf{1})$, ... form a spectrum, denoted by $\mathbf{B}A$. For the realization of A contains a subspace, its "1-skeleton", naturally homotopy-equivalent to the suspension of $A(\mathbf{1})$, giving up to homotopy a map $SA(\mathbf{1}) \to |A| = BA(\mathbf{1})$.

To say when $A(\mathbf{1})$ is the loop-space of $BA(\mathbf{1})$ one needs to observe that the Γ-structure of A (in fact even its simplicial structure) defines a composition-law on $A(\mathbf{1})$ up to homotopy. This law is the composite $A(\mathbf{1}) \times A(\mathbf{1}) \xrightarrow{p_2^{-1}} A(\mathbf{2}) \xrightarrow{m_2} A(\mathbf{1})$, where p_2^{-1} is an arbitrary homotopy-inverse to the equivalence p_2 mentioned in the definition of a Γ-space, and m_2 is induced by the morphism $m_2: \mathbf{1} \to \mathbf{2}$ in Γ taking 1 to $\{1, 2\}$. The composition-law makes $A(\mathbf{1})$ into an H-space.

PROPOSITION 1.4. *If A is a Γ-space and $A(\mathbf{1})$ is k-connected, then $BA(\mathbf{1})$ is $(k+1)$-connected. Furthermore $A(\mathbf{1})$ is the loop-space of $BA(\mathbf{1})$ if and only if the H-space $A(\mathbf{1})$ has a homotopy inverse.*

Note. The H-spaces $B^kA(\mathbf{1})$ for $k \geq 1$ automatically have homotopy inverses. For an H-space X has a homotopy inverse if the set of connected components $\pi_0(X)$ is a group and X has a numerable covering by sets which are contractible in X. (This follows from [7, (6.3,4)].) The realization of a simplicial space has such a numerable covering if the space of 0-simplexes is contractible. This was pointed out to me by D. Puppe.

(1.4) is really a statement about simplicial spaces, so I shall state it in its proper generality:

PROPOSITION 1.5. *Let* $[n] \mapsto A_n$ *be a simplicial space such that*

(i) A_o *is contractible, and*

(ii) $p_n = \prod_{k=1}^{n} i_k^*\colon A_n \to A_1 \times \cdots \times A_1$ *is a homotopy-equivalence, where* $i_k\colon [1] \to [n]$ *is defined by* $i_k(0) = k - 1$, $i_k(1) = k$.

Then (a) *if* A_1 *is k-connected* $|A|$ *is* $(k + 1)$*-connected, and* (b) *the map* $A_1 \to \Omega|A|$ *adjoint to* $SA_1 \to |A|$ *is a homotopy-equivalence if and only if* A_1 *has a homotopy inverse.*

Proof. (a) The realization $|A|$ has a natural filtration

$$A_o = |A|_{(o)} \subset |A|_{(1)} \subset \cdots \subset |A|$$

such that $|A|_{(p)}/|A|_{(p-1)} \simeq S^p(\underbrace{A_1 \wedge \cdots \wedge A_1}_{p})$ (cf. Remark 1 at the end of Appendix A). But SA_1 is $(k + 1)$-connected, $S^2(A_1 \wedge A_1)$ is $(2k + 1)$-connected and so on.

(b) I shall exhibit a commutative diagram

$$\begin{array}{ccc} A_1 & \to & |PA| \\ \downarrow & & \downarrow \\ A_o & \to & |A|, \end{array}$$

where $|PA|$ is contractible, and show it is homotopy-cartesian if A_1 has a homotopy inverse. ("Homotopy-cartesian" means that the induced map from A_1 to the homotopy-theoretic fibre-product of A_o and $|PA|$ over $|A|$ is a homotopy-equivalence.)

PA is the usual "simplicial path-space" of A, i.e. it is the composite $A \circ P$, where $P\colon \Delta \to \Delta$ takes $[n]$ to $[n + 1]$ and $\theta\colon [m] \to [n]$ to $\phi\colon [m + 1] \to [n + 1]$, where $\phi(0) = 0$ and $\phi(i) = \theta(i - 1) + 1$ if $i > 0$. The map $PA \to A$ comes from the transformation $\mathrm{id} \to P$ induced by the face-operator $\partial_o\colon [n] \to [n + 1]$. The contractibility of $|PA|$ comes from the standard simplicial null-homotopy $PA \times J \to PA$, where J is the simplicial unit interval. That the diagram is homotopy-cartesian follows from

PROPOSITION 1.6. *Let* $f\colon A' \to A$ *be a map of simplicial spaces such that for each* $\theta\colon [m] \to [n]$ *in* Δ *the diagram*

$$\begin{array}{ccc} A'_n & \xrightarrow{\theta^*} & A'_m \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_m} \\ A_n & \xrightarrow{\theta^*} & A_m \end{array}$$

is homotopy-cartesian. Then the diagrams

$$\begin{array}{ccc} \Delta^n \times A'_n & \to & |A'| \\ \downarrow & & \downarrow \\ \Delta^n \times A_n & \to & |A| \end{array}$$

are homotopy-cartesian for each n.

The application of (1.6) to prove (1.5) is as follows. By assumption $(PA)_n \simeq A_1^{n+1}$, and the diagram in the hypothesis of (1.6), in the case $\theta\colon [0] \to [n]$ with $\theta(0) = n$, is equivalent to

$$\begin{array}{ccc} A_1^{n+1} & \xrightarrow{m_{n+1}} & A_1 \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow \\ A_1^n & \longrightarrow & \text{point,} \end{array}$$

where pr is projection on to the last n factors, and m_{n+1} is the composition-law. This diagram is homotopy-cartesian if and only if the composition-law has a homotopy-inverse. If so, the cartesianness for other morphisms θ follows trivially.

Proof of 1.6. In view of the discussion in Appendix A it suffices to give the proof using the realization-functor $A \mapsto \|A\|$. Let $\|A\|_{(n)}$ be the part of $\|A\|$ which is the image of $\coprod_{k \leq n} \Delta^k \times A_k$. It is homeomorphic to the double mapping-cylinder of $(\|A\|_{(n-1)} \leftarrow \dot{\Delta}^n \times A_n \to \Delta^n \times A_n)$. One proves by induction on m that

$$\begin{array}{ccc} \Delta^m \times A'_m & \to & \|A'\|_{(m)} \\ \downarrow & & \downarrow \\ \Delta^m \times A_m & \to & \|A\|_{(m)} \end{array}$$

is homotopy-cartesian. If this is true when $m = n - 1$ then it follows from the hypothesis of (1.6) that the left-hand square of

$$\begin{array}{ccccc} \|A'\|_{(n-1)} & \leftarrow & \dot{\Delta}^n \times A'_n & \to & \Delta^n \times A'_n \\ \downarrow & & \downarrow & & \downarrow \\ \|A\|_{(n-1)} & \leftarrow & \dot{\Delta}^n \times A_n & \to & \Delta^n \times A_n \end{array}$$

is homotopy-cartesian (the other square being trivially so). Then the inductive step follows from

LEMMA 1.7. *Let* $\begin{array}{ccccc} Y_1 & \leftarrow & Y_o & \to & Y_2 \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \leftarrow & X_o & \to & X_2 \end{array}$

be a commutative diagram of spaces in which the squares are homotopy-cartesian. Let $Y \to X$ *be the induced map of double mapping-cylinders. Then*

$$\begin{array}{ccc} Y_i & \to & Y \\ \downarrow & & \downarrow \\ X_i & \to & X \end{array}$$

is homotopy-cartesian for $i = 0, 1, 2$.

This lemma is well-known when all the spaces are CW-complexes. A proof in the general case is given in [14].

To see now that
$$\begin{array}{ccc} \Delta^n \times A'_n & \to & \|A'\| \\ \downarrow & & \downarrow \\ \Delta^n \times A_n & \to & \|A\| \end{array}$$
is homotopy-cartesian one needs, because $\|A\|$ is equivalent to the infinite telescope of $(\|A\|_{(0)} \to \|A\|_{(1)} \to \|A\|_{(2)} \to \cdots)$,

LEMMA 1.8. *Let*
$$\begin{array}{ccccccc} Y_0 & \to & Y_1 & \to & Y_2 & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ X_0 & \to & X_1 & \to & X_2 & \to & \cdots \end{array}$$
be a commutative diagram such that each square is homotopy-cartesian. Let $Y \to X$ be the induced map or telescopes. Then
$$\begin{array}{ccc} Y_n & \to & Y \\ \downarrow & & \downarrow \\ X_n & \to & X \end{array}$$
is homotopy-cartesian for each n.

(1.8) follows at once from (1.7) because the telescope of $(X_0 \to X_1 \to X_2 \to \cdots)$ is the double mapping-cylinder of $\left(\coprod_{i \text{ even}} X_i \leftarrow \coprod_{\text{all } i} X_i \to \prod_{i \text{ odd}} X_i\right)$, where the left-hand map takes X_{2i} to X_{2i} by the identity, and X_{2i-1} to X_{2i} by the given map; and the right-hand map is analogous.

§2. CATEGORIES WITH COMPOSITION LAWS

I refer to [16] for a discussion of the "space" or "nerve" $|\mathscr{C}|$ of a category $\mathscr{C}$. The main facts to recall are that $|\mathscr{C} \times \mathscr{C}'| \simeq |\mathscr{C}| \times |\mathscr{C}'|$, and that an equivalence of categories $\mathscr{C} \to \mathscr{C}'$ induces a homotopy-equivalence $|\mathscr{C}| \to |\mathscr{C}'|$. "Category" will always mean *topological category* in the sense of [16], i.e. the set of objects and the set of morphisms have topologies for which the structural maps are continuous.

The reason for introducing Γ-spaces is that they arise naturally from categories, as we shall now see.

Definition 2.1. A Γ-category is a contravariant functor $\mathscr{C}$ from Γ to categories such that

(i) $\mathscr{C}(\mathbf{0})$ is equivalent to the category with one object and one morphism;

(ii) for each n the functor $p_n: \mathscr{C}(\mathbf{n}) \to \mathscr{C}(\mathbf{1}) \times \underset{\leftarrow n \rightarrow}{\cdots} \times \mathscr{C}(\mathbf{1})$ induced by the morphisms $i_k: \mathbf{1} \to \mathbf{n}$ in Γ (cf. (1.2)) is an equivalence of categories.

COROLLARY 2.2. *If $\mathscr{C}$ is a Γ-category, $|\mathscr{C}|$ is a Γ-space.*

Here $|\mathscr{C}|$ means the functor $S \mapsto |\mathscr{C}(S)|$.

Γ-categories arise in the following way. Let $\mathscr{C}$ be a category in which sums exist (though no choices of them are given). If S is a finite set, let $\mathscr{P}(S)$ denote the category of subsets of S and their inclusions—this should not cause confusion with the earlier use of $\mathscr{P}(S)$. Let

$\mathscr{C}(S)$ denote the category whose objects are the functors from $\mathscr{C}(S)$ to which take disjoint unions to sums, and whose morphisms are *isomorphisms* of functors. That means, for example, that an object of $\mathscr{C}(\mathbf{2})$ is a diagram $A_1 \to A_{12} \leftarrow A_2$ in $\mathscr{C}$ which has the universal property for expressing A_{12} as a sum of A_1 and A_2. The morphisms of Γ were so defined that the morphisms from S to T in Γ correspond precisely to functors from $\mathscr{P}(S)$ to $\mathscr{P}(T)$ which preserve disjoint unions. Thus $S \mapsto \mathscr{C}(S)$ is a contravariant functor from Γ to categories. It satisfies the conditions of Definition 2.1 because, for example, the forgetful functor $\mathscr{C}(\mathbf{2}) \to \mathscr{C} \times \mathscr{C}$, which takes $(A_1 \to A_{12} \leftarrow A_2)$ to (A_1, A_2) is an equivalence of categories.

Similarly, if products exist in $\mathscr{C}$, one has a Γ-category $S \mapsto \mathscr{C}^{\Pi}(S)$ associated to $\mathscr{C}$ with its product as composition: one defines $\mathscr{C}^{\Pi}(S)$ as the category of contravariant functors $\mathscr{P}(S) \to \mathscr{C}$ which take disjoint unions to products. For a third example, if $\mathscr{C}$ is the category of modules over a commutative ring one can define a Γ-category $\mathscr{C}^{\otimes}$ associated to $\mathscr{C}$ and its tensor-product: an object of $\mathscr{C}^{\otimes}(\mathbf{2})$ is a quadruple $(M_1, M_2, M_{12}; \alpha_{12})$, where α_{12}: $M_1 \times M_2 \to M_{12}$ is a bilinear map satisfying the universal property for a tensor-product, and so on.

The "most fundamental" Γ-space is that arising from the category $\mathscr{S}$ of finite sets under disjoint union. Let us choose a model for $\mathscr{S}$ in which there is one object $\mathbf{n}$ for each natural number. Then $|\phi(\mathbf{1})|$ is $\coprod_{n \geq 0} B\Sigma_n$, where Σ_n is the nth symmetric group. I shall call this Γ-space $B\Sigma$. We shall not need to know anything about $|\mathscr{S}(\mathbf{k})|$ for $k > 1$, but in fact

$$|\mathscr{S}(\mathbf{2})| = \coprod_{m,\, n \geq 0} (E\Sigma_m \times E\Sigma_n \times E\Sigma_{m+n})/(\Sigma_m \times \Sigma_n),$$

and in general

$$|\mathscr{S}(\mathbf{k})| = \coprod_{m_1, \ldots, m_k} \Big(\prod_{\sigma \subset \mathbf{k}} E\Sigma_{m_\sigma}\Big)/(\Sigma_{m_1} \times \cdots \times \Sigma_{m_k}),$$

where $m_\sigma = \sum_{\alpha \in \sigma} m_\alpha$.

The following generalization of $B\Sigma$ will occur later. Let F be a contravariant functor from the category of finite sets and inclusions to the category of topological spaces. Let $\mathscr{S}_F$ be the topological category whose objects are pairs (S, x), where S is a finite set and $x \in F(S)$, and whose morphisms from (S, x) to (T, y) are injections θ: $S \to T$ such that $\theta^* y = x$. Then one can form $\mathscr{S}_F(\mathbf{k})$ by analogy with $\mathscr{S}(\mathbf{k})$, and $\mathbf{k} \mapsto |\mathscr{S}(\mathbf{k})|$ is a Γ-space providing that for each S and T the map $F(S \amalg T) \to F(S) \times F(T)$ is a homotopy-equivalence. The most important case is when $F(\mathbf{n}) = X^n$ for some fixed space X: then I shall call the resulting Γ-space $B\Sigma_X$.

$$B\Sigma_X(\mathbf{1}) \quad \text{is} \quad \coprod_{n \geq 0} (E\Sigma_n \times X^n)/\Sigma_n.$$

The category of finite-dimensional real vector-spaces under $\oplus$ leads, as indicated above, to a Γ-space A such that

$$A(\mathbf{1}) = \coprod_{n \geq 0} BG_n, \quad \text{where} \quad G_n = Gl_n(R).$$

In this case

$$A(\mathbf{2}) = \coprod_{n \geq 0} (EG_m \times EG_n \times EG_{m+n})/(G_m \times G_n),$$

and so on. The explicitness of this formula makes clear that one can construct such a Γ-space whenever one is given naturally for each finite set S a topological group $G(S)$ containing the symmetric group $\Sigma(S)$ of S, and a family of associative natural transformations $G(S) \times G(T) \to G(S \amalg T)$. In fact $G(S)$ need be only a topological monoid, providing $\pi_o G(S)$ is a group. The minimal example of this situation is when $G(S)$ is a wreath-product $\Sigma(S) \int G$ for some fixed topological group G. Then the Γ-space is the same as $B\Sigma_{BG}$ introduced above. More important examples are:

(a) $\coprod_{n \geq 0} BPL_n$, where PL_n is the realization of the simplicial group whose k-simplexes are the fibre-preserving PL isomorphisms $\Delta^k \times \mathbb{R}^n \to \Delta^k \times \mathbb{R}^n$ over Δ^k;

(b) $\coprod_{n \geq 0} BTop_n$, defined the same way, but replacing PL isomorphisms by homeomorphisms;

(c) $\coprod_{n \geq 0} BF_n$, where F_n is the monoid of proper homotopy-equivalences $\mathbb{R}^n \to \mathbb{R}^n$.

The relation of the last three spaces to $\mathbb{Z} \times BPL$, $\mathbb{Z} \times BTop$, and $\mathbb{Z} \times BF$ will be discussed in §4.

The most complicated example I shall mention arises when one wants to describe the tensor-product composition-law on $1 \times BO \subset \mathbb{Z} \times BO$. Let $\mathscr{C}$ be the category of chain-complexes of finite length of finite-dimensional real vector-spaces, with chain-homotopy-equivalences as morphisms. More precisely, if $n = \{n_i\}_{i \in \mathbb{Z}}$ is a sequence of positive integers almost all zero let K_n be the space of chain-complexes E such that $E^i = \mathbb{R}^{n_i}$. (K_n is a real algebraic variety, so has the homotopy-type of a CW-complex.) Then $\mathrm{ob}(\mathscr{C}) = \coprod_n K_n$. The space $\mathrm{mor}(\mathscr{C})$ is a similar real algebraic variety. In the Appendix to [17] it is proved that isomorphism-classes of bundles on a compact space X for the topological groupoid $\mathscr{C}$ correspond precisely to elements of $KO(X)$, which implies that $|\mathscr{C}|$ represents the functor KO, i.e. that $|\mathscr{C}| \simeq \mathbb{Z} \times BO$. If $\mathscr{C}_1$ is the full subcategory of $\mathscr{C}$ spanned by the objects in K_n with $\Sigma(-1)^i n_i = 1$ then $|\mathscr{C}_1| \simeq 1 \times BO$. The tensor-product of complexes makes $\mathscr{C}_1$ into a Γ-category.

§3. THE SYMMETRIC GROUPS AND STABLE HOMOTOPY

We have seen that a Γ-space gives rise to a spectrum. Conversely we shall see now that a spectrum gives rise to a Γ-space; this section is devoted to working out a precise correspondence between Γ-spaces and spectra. We shall begin by making definite conventions about spectra.

A spectrum will mean a sequence of spaces with base-points $\mathbf{X} = \{X_o, X_1, \ldots\}$ together with closed embeddings $X_k \to \Omega X_{k+1}$ for each $k \geq 0$. Such a spectrum gives rise to a loop-spectrum $\omega\mathbf{X}$, defined by $(\omega\mathbf{X})_k = \bigcup_{n \geq 0} \Omega^n X_{n+k}$. A *morphism* from a spectrum $\mathbf{X}$ to a spectrum $\mathbf{Y}$ will mean a *strict* morphism from $\mathbf{X}$ to $\omega\mathbf{Y}$, i.e. a sequence of maps f_k: $X_k \to (\omega\mathbf{Y})_k$ such that the diagrams

$$\begin{array}{ccc} X_k & \xrightarrow{f_k} & (\omega\mathbf{Y})_k \\ \downarrow & & \downarrow \\ \Omega X_{k+1} & \xrightarrow{\Omega f_{k+1}} & \Omega(\omega\mathbf{Y})_{k+1} \end{array}$$

commute.

To define the Γ-space associated to a spectrum observe that if P is a space with a base-point p_o there is a covariant functor from Γ to spaces which takes the finite set S to P^S and the map $\theta: S \to \mathscr{P}(T)$ to $\theta_*: P^S \to P^T$, where $\{\theta_*(x)\}_\beta = x_\alpha$ if $\beta \in \theta(\alpha)$, and $\{\theta_*(x)\}_\beta = p_o$ if $\beta \notin \theta(\alpha)$ for all $\alpha \notin S$. Similarly if $\mathbf{P}$ is a spectrum one can define a covariant functor $S \mapsto \mathbf{P}^S$ from Γ to spectra such that $(\mathbf{P}^S)_k = (P_k)^S$.

Definition 3.1. The Γ-space $\mathbf{AX}$ associated to a spectrum $\mathbf{X}$ is $\mathbf{n} \mapsto (\mathbf{AX})(\mathbf{n}) = \mathrm{Mor}(\underbrace{\mathbf{S} \times \cdots \times \mathbf{S}}_{n}; \mathbf{X})$, where $\mathbf{S}$ is the sphere-spectrum.

To check that $\mathbf{AX}$ is a Γ-space, i.e. that $(\mathbf{AX})(\mathbf{n}) \to \{(\mathbf{AX})(\mathbf{1})\}^n$ is a homotopy-equivalence, observe that

$$(\mathbf{AX})(\mathbf{n}) = \mathrm{Mor}(\mathbf{S} \times \cdots \times \mathbf{S}; \mathbf{X}) \simeq \mathrm{Mor}(\mathbf{S} \vee \cdots \vee \mathbf{S}; \mathbf{X}) = \{\mathrm{Mor}(\mathbf{S}: \mathbf{X})\}^n.$$

The pair of functors (Γ-spaces) ⇄ (spectra) that we have defined are adjoint. To see this one defines the spectrum associated to a Γ-space in a new way.

If A is a Γ-space and P is a space with base-point one can define a Γ-space $P \otimes A$ such that, for a finite set S, $(P \otimes A)(S)$ is the quotient of the disjoint union $\coprod_{n \geq 0} P^n \times A'(\mathbf{n} \times S)$ by the equivalence-relation which identifies $(p, \theta^* a) \in P^m \times A'(\mathbf{m} \times S)$ with $(\theta_* p, a) \in P^n \times A'(\mathbf{n} \times S)$ for all $\theta: \mathbf{m} \to \mathbf{n}$ in Γ. (A' is the thickening τA of A described in Appendix A.)

PROPOSITION 3.2. *For $n \geq 0$, $B^n A \simeq S^n \otimes A$, where S^n is the n-sphere.*

I shall postpone the proof. In virtue of the proposition there are maps $(S^n)^m \times A(\mathbf{m}) \to B^n A(\mathbf{1})$ for each n and m, and thus a map of spectra $(\underbrace{\mathbf{S} \times \cdots \times \mathbf{S}}_{n}) \times A(\mathbf{m}) \to \mathbf{B}A$. This leads to

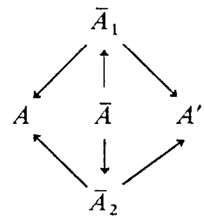
PROPOSITION 3.3. *For every Γ-space A there is a natural map $A \to \mathbf{AB}A$ of Γ-spaces, and for any spectrum $\mathbf{X}$ a natural morphism $\mathbf{BAX} \to \mathbf{X}$, making the functors $\mathbf{A}$ and $\mathbf{B}$ adjoint.*

Proof. To define $A \to \mathbf{AB}A$ one must define for each m a map $A(\mathbf{m}) \to \mathrm{Mor}(\mathbf{S}^m; \mathbf{B}A)$. This we have just found.

To define $\mathbf{BAX} \to \mathbf{X}$ one must give for each n and m compatible maps $(S^n)^m \times \mathrm{Mor}(S^m; \mathbf{X}) \to (\omega\mathbf{X})_n$. These are provided by evaluation.

The adjointness of the functors is equivalent to the fact that the composites $\mathbf{B} \to \mathbf{BAB} \to \mathbf{B}$ and $\mathbf{A} \to \mathbf{ABA} \to \mathbf{A}$ are the identity, which is easy to check.

To make it supple enough for our purposes we need to add more morphisms to the category of Γ-spaces. Let us call a map $A \to A'$ of Γ-spaces an *equivalence* if for each S in Γ the map $A(S) \to A'(S)$ is a Hurewicz fibration with contractible fibres. We shall formally adjoin (cf. [8]) inverses of all equivalences to the category. To be precise, we define a *weak* *morphism* of Γ-spaces from A to A' as a diagram $(A \leftarrow \bar{A} \rightarrow A')$ in which $\bar{A} \rightarrow A$ is an equivalence. Two such diagrams $(A \leftarrow \bar{A}_1 \rightarrow A')$ and $(A \leftarrow \bar{A}_2 \rightarrow A')$ are identified if there are equivalences $\bar{A} \rightarrow \bar{A}_1$, $\bar{A} \rightarrow \bar{A}_2$ such that



commutes. The composite of $(A \leftarrow \bar{A} \rightarrow A')$ and $(A' \leftarrow \bar{A}' \rightarrow A'')$ is $(A \leftarrow B \rightarrow A'')$, where $B = \bar{A} \times_{A'} \bar{A}'$.

Let $\mathscr{A}$ be the category of Γ-spaces and homotopy-classes of weak morphisms. The significance of this category is that a map $A \rightarrow A'$ of Γ-spaces such that $A(\mathbf{1}) \rightarrow A'(\mathbf{1})$ is a homotopy-equivalence is invertible in $\mathscr{A}$. (In fact its inverse is $(A' \leftarrow A \times_{A'} PA' \rightarrow A)$, where PA' is the Γ-space of unbased paths in A'.) We shall relate $\mathscr{A}$ to the category $\mathscr{S}\!\not\!h$ of spectra and homotopy-classes of morphisms.

If $A \rightarrow A'$ is an equivalence of Γ-spaces then $\mathbf{B}A \rightarrow \mathbf{B}A'$ is a homotopy-equivalence, so $\mathbf{B}$ induces a functor $\mathbf{B}\colon \mathscr{A} \rightarrow \mathscr{S}\!\not\!h$. Similarly $\mathbf{A}$ induces $\mathbf{A}\colon \mathscr{S}\!\not\!h \rightarrow \mathscr{A}$, and the functors $\mathscr{A} \rightleftarrows \mathscr{S}\!\not\!h$ are still adjoint. The outcome of this discussion is the following reformulation of (1.4).

PROPOSITION 3.4. (a) *There are adjoint functors* $\mathbf{B}\colon \mathscr{A} \rightarrow \mathscr{S}\!\not\!h$ *and* $\mathbf{A}\colon \mathscr{S}\!\not\!h \rightarrow \mathscr{A}$. *For any Γ-space A the spectrum $\mathbf{B}A$ is connective*, i.e. $\pi_q(B^q A) = 0$ *for $p < q$; and for any spectrum $\mathbf{X}$ the H-space $\mathbf{AX}(\mathbf{1})$ has a homotopy inverse.*

(b) *If A is a Γ-space the adjunction $A \rightarrow \mathbf{AB}A$ is an isomorphism in $\mathscr{A}$ if and only if $A(\mathbf{1})$ has a homotopy inverse.*

(c) *If $\mathbf{X}$ is a spectrum the adjunction $\mathbf{BAX} \rightarrow \mathbf{X}$ is an isomorphism in $\mathscr{S}\!\not\!h$ if and only if $\mathbf{X}$ is connective.*

Now we can prove very simply the theorem of Barratt–Priddy–Quillen.

PROPOSITION 3.5. *If $B\Sigma$ is the Γ-space described in §2 arising from the category of finite sets, then the associated spectrum $\mathbf{B}(B\Sigma)$ is equivalent to the sphere-spectrum $\mathbf{S}$.*

Proof. It is enough to give an isomorphism of functors of $\mathbf{X}$.

$$\mathrm{Hom}_{\mathscr{S}\!\not h}(\mathbf{B}(B\Sigma); \mathbf{X}) \cong \mathrm{Hom}_{\mathscr{S}\!\not h}(\mathrm{S}; \mathbf{X}) = \pi_o(X).$$

By adjointness it suffices to give $\mathrm{Hom}_{\mathscr{A}}(B\Sigma; \mathbf{AX}) \cong \pi_o(\mathbf{X})$; and so, because $\pi_o(\mathbf{X}) = \pi_o(\mathbf{AX}(\mathbf{1}))$, to give an isomorphism $\mathrm{Hom}_{\mathscr{A}}(B\Sigma; A) = \pi_o(A(\mathbf{1}))$ functorial for Γ-spaces A.

Recall that $B\Sigma(\mathbf{1}) = \coprod_{n \geq 0} B\Sigma_n$; and let $\epsilon = B\Sigma_1$, a single point of $B\Sigma(\mathbf{1})$. Under any weak morphism $B\Sigma \rightarrow A$ the point ϵ is associated with a definite connected component of $A(\mathbf{1})$, so one has a transformation $\mathrm{Hom}_{\mathscr{A}}(B\Sigma; A) \rightarrow \pi_o(A(\mathbf{1}))$.

To construct its inverse, suppose a is a point of $A(\mathbf{1})$. Let F_n be the homotopy-theoretical fibre of p_n; $A(\mathbf{n}) \to A(\mathbf{1})^n$ at the point $(a, a, \ldots, a)$. This is a contractible space, and $\mathbf{n} \mapsto F_n$ is a contravariant functor on the category of finite sets and inclusions. As in §2, construct the category $\mathscr{S}_F$ of pairs $(\mathbf{n}, x \in F_n)$, and the associated Γ-space $B\Sigma_F$. An object of $\mathscr{S}_F(\mathbf{k})$ can be described as a pair (θ, x), where $\theta\colon \mathbf{k} \to \mathbf{n}$ is a morphism in Γ such that $\theta(\mathbf{k}) = \mathbf{n}$, and $x \in F_n$. By the composition $F_n \longrightarrow A(\mathbf{n}) \xrightarrow{\theta^*} A(\mathbf{k})$ each such object determines a point of $A(\mathbf{k})$. If two objects of $\mathscr{S}_F(\mathbf{k})$ are related by a morphism they have the same image in $A(\mathbf{k})$, so there is a map $|\mathscr{S}_F(\mathbf{k})| \to A(\mathbf{k})$, i.e. a morphism $B\Sigma_F \to A$. But the forgetful map $B\Sigma_F \to B\Sigma$ is an isomorphism in $\mathscr{A}$, so we have associated to $a \in A(\mathbf{1})$ a morphism $B\Sigma \to A$ in $\mathscr{A}$. To show that it depends only on the component of $A(\mathbf{1})$ in which a lies, let a' be another point in the same component, and let α be a path from a to a'. Form $B\Sigma_{F'}$ like $B\Sigma_F$, but using a' instead of a; and form $B\Sigma_{F''}$ using the homotopy-theoretical fibre-product of $([0, 1]^n \xrightarrow{\alpha^n} A(\mathbf{1})^n \longleftarrow A(\mathbf{n}))$ instead of F_n. Then there is a commutative diagram

$$\begin{array}{ccccc} & & B\Sigma_F & & \\ & \overset{\cong}{\swarrow} & \uparrow & \searrow & \\ B\Sigma & \underset{\cong}{\longleftarrow} & B\Sigma_{F''} & \longrightarrow & A \\ & \underset{\cong}{\nwarrow} & \downarrow & \nearrow & \\ & & B\Sigma_{F'} & & \end{array}$$

in which the indicated maps are isomorphisms in $\mathscr{A}$. That completes the proof.

The same argument proves more generally.

PROPOSITION 3.6. *If X is a space, and $B\Sigma_X$ is the Γ-space described in §2 with $B\Sigma_X(\mathbf{1}) = \coprod_{n \geq 0} (E\Sigma_n \times X^n)/\Sigma_n$, then $\mathbf{B}(B\Sigma_X)$ is equivalent to $\mathbf{S}X$, the suspension-spectrum of X.*

Note. $(\mathbf{S}X)_n$ is defined as $S^n(X_+)$, the n-fold reduced suspension of the union of X and a disjoint base-point.

Proof. One modifies the foregoing argument to prove that $\mathrm{Hom}_{\mathscr{A}}(B\Sigma_X; A) = [X; A(\mathbf{1})]$ for any Γ-space A. To do so, given $f\colon X \to A(\mathbf{1})$, define F_n as the homotopy-theoretical fibre-product of $(X^n \xrightarrow{f^n} A(\mathbf{1})^n \xleftarrow{p_n} A(\mathbf{n}))$, and proceed as before.

I shall end this section with the proof of (3.2). For the sake of clarity I shall ignore the question of thickening; i.e. in the following discussion the realization of a simplicial space has its classical meaning, and $(P \otimes A)(S)$ means a certain quotient of $\coprod_n P^n \times A(\mathbf{n} \times S)$. Then we shall see that $S^n \otimes A$ is homeomorphic to B^nA. The correct arguments proceed as in Appendix A, and lead to $S^n \otimes A \simeq B^nA$.

When $n = 1$ one has to prove that $\left(\coprod_{n \geq 0} \Delta^n \times A(\mathbf{n} \times S)\right)/\Delta = \left(\coprod_{n \geq 0} (S^1)^n \times A(\mathbf{n} \times S)\right)/\Gamma$ in what I hope is obvious notation. A point of Δ^n can be represented by $(t_1, \ldots, t_n)$, where $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1$. Regarding S^1 as $[0, 1]$ with its ends identified one can think of $(t_1, \ldots, t_n)$ as a point of $(S^1)^n$; thus one has a map $\Delta^n \to (S^1)^n$ inducing a map between the foregoing spaces. It is surjective because Γ contains the permutations of $\mathbf{n}$, so that any point of $(S^1)^n \times A(\mathbf{n} \times S)$ is equivalent to one with the coordinates $(t_1, \ldots, t_n)$ in ascending order. Injectivity is also easy to check, and the inverse map is obviously continuous on the images of the sets $\Delta^n \times A(\mathbf{n} \times S)$, with respect to which $S^1 \otimes A$ has the weak topology.

The case $n > 1$ follows now by

LEMMA 3.7. *If X and Y are spaces with base-points, and A is a Γ-space, then* $X \otimes (Y \otimes A) = (X \wedge Y) \otimes A$.

Proof. It is enough to show that $\left(\coprod_{m,n \geq 0} X^m \times Y^n \times A(\mathbf{m} \times \mathbf{n})\right)/\Gamma = \left(\coprod_{q \geq 0} (X \wedge Y)^q \times A(\mathbf{q})\right)/\Gamma$. Define a map from left to right by taking $(x_1, \ldots, x_m; y_1, \ldots, y_n; a)$ to $(\{x_i \wedge y_j; a) \in (X \wedge Y)^{mn} \times A(\mathbf{m} \times \mathbf{n})$. Define a map from right to left by taking $(\{x_i \wedge y_i\}; a)$ to $(\{x_i\}; \{y_i\}; \theta^* a) \in X^q \times Y^q \times A(\mathbf{q} \times \mathbf{q})$, where $\theta: \mathbf{q} \times \mathbf{q} \to \mathbf{q}$ is $S \mapsto S \cap$ (diagonal). I shall omit the verification that the two maps are well-defined and inverse to each other.

§4. MAKING π_o INTO A GROUP: THE GROTHENDIECK CONSTRUCTION IN HOMOTOPY THEORY†

We have associated to a Γ-space A a spectrum $\mathbf{B}A = \{B_o, B_1, \ldots\}$. It has the property that $B_k \to \Omega B_{k+1}$ is a homotopy-equivalence when $k \geq 1$, because then B_k is connected. But $B_o \to \Omega B_1$ is a homotopy-equivalence only if the H-space B_o has a homotopy inverse, which is usually not the case. Then one needs to identify the infinite-loop-space ΩB_1. The reason this is important is that in practice one begins with a category $\mathscr{C}$ with a composition-law, and ends with a cohomology theory $k_{\mathscr{C}}^*$ represented by the spectrum $\{B_k\}$, where $B_o = |\mathscr{C}|$. One would like to know the relationship between the monoid-valued functor $X \mapsto [X; |\mathscr{C}|]$ and the group-valued functor $X \mapsto k_{\mathscr{C}}^o(X) = [X; \Omega B_1]$. Of course there is a natural transformation $[X; |\mathscr{C}|] \to k_{\mathscr{C}}^o(X)$. In practice $[X; |\mathscr{C}|]$ is the monoid of isomorphism-classes of $\mathscr{C}$-bundles on X, in some appropriate sense. The following proposition seems to cover all the interesting cases.

PROPOSITION 4.1. *Suppose that* $|\mathscr{C}|$ *is of the homotopy-type of a CW-complex, and that* $\pi_o(|\mathscr{C}|)$ *contains a cofinal free abelian monoid. Then the transformation* $[\ \ ; |\mathscr{C}|] \to k^o$ *is universal among transformations* $\Theta: [\ \ ; |\mathscr{C}|] \to F$, *where F is a representable abelian-group-valued homotopy-functor on compact spaces, and* Θ *is a transformation of monoid-valued functors.*

The proof will be given at the end of this section.

Returning to Γ-spaces, I shall associate naturally to a Γ-space A another Γ-space A' with a map $A \to A'$ with the following properties:

(a) $\pi_o(A')$ is the abelian group associated to the monoid $\pi = \pi_o(A)$; and

(b) $\mathbf{B}A \to \mathbf{B}A'$ is a weak equivalence of spectra.

The construction of A' from A is simple but not very practical, so it is important that one can construct more practically another space T_A such that $A(\mathbf{1}) \to A'(\mathbf{1})$ factorizes through T_A and $T_A \to A'(\mathbf{1})$ induces an isomorphism of homology. In other words, $A'(\mathbf{1})$ is

† This section is greatly indebted to discussions with Quillen. Cf. also [2], [4], [15].

obtained from T_A by abelianizing the fundamental group in Quillen's sense. In the basic example where $A(\mathbf{1})$ is $\coprod_{n\geq 0} B\Sigma_n$, one has $T_A \simeq \mathbb{Z} \times B\Sigma_\infty$. Similarly, for any ring R, the category of finitely generated projective R-modules leads to a Γ-space A for which $T_A \simeq K_o(R) \times BGL_\infty(R)$.

To construct A', recall from §1 the functor $P\colon \Delta \to \Delta$ taking $[k]$ to $[k+1]$. Define $PA\colon \Delta \times \Gamma \to$ (spaces) by $PA[[k], \mathbf{m}) = A(P([k]) \times \mathbf{m})$, where $P([k])$ is regarded as an object of Γ by means of the standard functor $\Delta \to \Gamma$. Then $A'(\mathbf{m})$ is defined as the realization of the simplicial space $[k] \mapsto A'_k(\mathbf{m}) = PA([k], \mathbf{m}) \overset{h}{\times}_{A(\mathbf{k}\times\mathbf{m})} PA([k], \mathbf{m})$, where $\overset{h}{\times}$ denotes the homotopy-theoretical fibre-product.

One obtains a transformation $A \to A'$ as the composite

$$A(\mathbf{m}) = A(\mathbf{m}) \times_{A(\mathbf{0})} A(\mathbf{0}) \subset A(\mathbf{m}) \overset{h}{\times}_{A(\mathbf{0})} A(\mathbf{0}) \to A(\mathbf{m}) \overset{h}{\times}_{A(\mathbf{0})} A(\mathbf{m}) = A'_o(\mathbf{m}) \subset A'(\mathbf{m}),$$

where the third step is induced by the canonical map $A(\mathbf{0}) \to A(\mathbf{m})$.

In case the construction of A' seems mysterious I should explain that it is the transliteration into the context of Γ-spaces of the following method for trying to construct homotopy-theoretically the enveloping group of a topological monoid.

Let M be a topological monoid. Let C_M be the topological category whose space of objects is $M \times M$, and whose space of morphisms from (m_1, m_2) to (m'_1, m'_2) is

$$\{m \in M\colon m_1 m = m'_1,\ m_2 m = m'_2\}.$$

(Thus the complete space of morphisms of C_M is $M \times M \times M$.) M is embedded in $|C_M|$ by identifying $m \in M$ with the object $(m, 1)$; and one can show that $|C_M|$ is homotopy-equivalent to ΩBM providing M is sufficiently homotopy-commutative. By definition $|C_M|$ is the realization of a simplicial space whose k-simplexes are $M^{k+2} = M^{k+1} \times_{M^k} M^{k+1}$; and, because realization commutes with fibre-products, $|C_M| = EM \times_{BM} EM$.

I shall prove that $\mathbf{B}A \to \mathbf{B}A'$ is a weak homotopy-equivalence by calculating the homology of $A'(\mathbf{1})$ with coefficients in a field F. If H is the Pontrjagin ring $H_*(A(\mathbf{1}))$ then the simplicial module obtained by taking the homology term by term of the simplicial space whose realization is $A'(\mathbf{1})$ is precisely the bar-construction for calculating $\mathrm{Tor}_*^H(H \otimes H, F)$, where $H \otimes H$ is regarded as an H-module by the diagonal map of the Hopf algebra H.

LEMMA 4.2. (i) $(H \otimes H) \otimes_H F = H[\pi^{-1}]$, *where* $\pi = \pi_o(A(\mathbf{1})) \subset H$.

(ii) $\mathrm{Tor}_i^H(H \otimes H, F) = 0$ if $i > 0$.

In view of the lemma (which will be proved below) the standard spectral sequence [16] converging to $H' = H_*(A'(\mathbf{1}))$ is trivial, and $H' = H[\pi^{-1}]$. Now compare the standard spectral sequences converging to $H_*(BA(\mathbf{1}))$ and $H_*(BA'(\mathbf{1}))$. Their E^1-terms are the bar-constructions for calculating $\mathrm{Tor}_*{}^H(F, F)$ and $\mathrm{Tor}_*{}^{H'}(F, F)$. But because Tor commutes with localization, and $F[\pi^{-1}] = F$, these are the same, and so $H_*(BA(\mathbf{1})) \xrightarrow{\cong} H_*(BA'(\mathbf{1}))$. As both spaces are connected H-spaces it follows that $BA(\mathbf{1}) \to BA'(\mathbf{1})$ is a weak homotopy-equivalence; and similarly $B^k A(\mathbf{1}) \to B^k A'(\mathbf{1})$ for $k > 1$.

Proof of 4.2. (i) The Hopf algebra $H[\pi^{-1}]$ has an involution ("inversion") c extending the obvious involution in dimension 0. Define $H \otimes H \to H[\pi^{-1}]$ by $x \otimes y \mapsto x.c(y)$. This induces a surjection $(H \otimes H) \otimes_H F \to H[\pi^{-1}]$. A map in the other direction is defined by $x.p^{-1} \mapsto x \otimes p$; one checks by induction on dimension that it is the inverse.

(ii) Because Tor commutes with localization and π acts trivially on F it suffices to prove this when H is replaced by $H[\pi^{-1}]$. But because $H[\pi^{-1}]$ has an involution $H[\pi^{-1}] \otimes H[\pi^{-1}]$ is as $H[\pi^{-1}]$-module via the diagonal action isomorphic to $H[\pi^{-1}] \otimes H[\pi^{-1}]$ with the left-hand $H[\pi^{-1}]$-action. But the latter is obviously flat.

I shall construct T_A for simplicity only in the special case (which includes all the examples I know) where $A(\mathbf{1})$ is a topological monoid M and one can find discrete submonoid μ of M whose image in $\pi = \pi_0 M$ is cofinal. Then $A'(\mathbf{1})$ is up to homotopy the space of the category C_M described above, and it contains the space of the category $C_{M,\mu}$ whose space of objects is $M \times \mu$ and whose morphisms from (m_1, m_2) to (m'_1, m'_2) are the set $\{m \in \mu: m_1 m = m'_1, m_2 m = m'_2\}$. Define $T_{M,\mu} = |C_{M,\mu}|$. The spectral sequence for $H_*(T_{M,\mu})$ begins with the bar-construction for $\mathrm{Tor}_*^{F[\mu]}(H \otimes F[\mu], F)$, where $H = H_*(M)$. By a lemma like (4.2) (but simpler) one has $(H \otimes F[\mu]) \otimes_{F[\mu]} F \cong H[\mu^{-1}]$, and $\mathrm{Tor}_i = 0$ for $i > 0$. Comparing with the spectral sequence for $H_*(|C_M|)$, and observing that $H[\mu^{-1}] = H[\pi^{-1}]$, one finds that $H_*(T_{M,\mu}) \xrightarrow{\cong} H_*(A'(\mathbf{1}))$.

The most important case of the construction of $T_{M,\mu}$ is when $\mu = \mathbb{N}$, the natural numbers. Then if e is the generator of μ the space $T_{M,\mu}$ is (up to homotopy) just the telescope of the sequence

$$M \xrightarrow{\times e} M \xrightarrow{\times e} M \xrightarrow{\times e} \cdots.$$

I conclude this section with the proof of (4.1). Again I shall assume for simplicity that $|\mathscr{C}|$ is a topological monoid M. By a direct limit argument one is reduced at once to the case where $\pi = \pi_0 M$ is finitely generated, and then one can suppose that the cofinal monoid is $\mathbb{N}$, and can lift it back to a submonoid μ of M, generated by $e \in M$.

Let Θ: [; $M] \to F$ be a transformation of the kind envisaged, where F is represented by an H-space B. If X is a compact space then $[X; T_{M,\mu}]$ is the direct limit of the sequence of sets

$$[X; M] \to [X; M] \to [X; M] \to \cdots,$$

where each map is addition of the constant map $E \in [X; M]$ with value $e \in M$. But $\Theta(E)$ has an additive inverse in $F(X)$, so Θ induces a transformation $[X; T_{M,\mu}] \to F(X)$. By [1] (1.5) this is induced by a map $\Theta: T_{M,\mu} \to B$, unique up to weak homotopy. By obstruction-theory, because $H_*(T_{M,\mu}) \xrightarrow{\cong} H_*(|C_M|)$, this extends uniquely to $\Theta: |C_M| \to B$. But $|C_M|$ represents $k^o_{\mathscr{C}}$, so one has shown that transformations [; $|\mathscr{C}|] \to F$ extend uniquely to $k^o_{\mathscr{C}} \to F$, proving (4.1).

§5. RING SPECTRA

It often happens that a category has two composition-laws analogous to those of a ring. For example, in the category of finite sets one has disjoint union and also the cartesian product, and in the category of finitely generated modules over a commutative ring one has the direct sum and also the tensor-product. In these cases the spectrum associated to the category is a ring spectrum, as the following discussion shows.

Definition 5.1. A *multiplication* on a Γ-space A is a contravariant functor $\hat{A}: \Gamma \times \Gamma \to$ (spaces) together with natural transformations

$$i_1: \hat{A}(S, T) \to A(S), \qquad i_2: \hat{A}(S, T) \to A(T),$$
$$m: \hat{A}(S, T) \to A(S \times T),$$

functorial for S and T in Γ, such that for each S and T the map $i_1 \times i_2: \hat{A}(S, T) \to A(S) \times A(T)$ is a homotopy-equivalence.

A multiplication on A defines a pairing of spectra $\mathbf{B}A \wedge \mathbf{B}A \to \mathbf{B}A$. More generally, it defines for any pointed spaces X and Y a pairing $(X \otimes A) \wedge (Y \otimes A) \to (X \wedge Y) \otimes A$, where I have written $X \otimes A$ for $(X \otimes A)(\mathbf{1})$ and so on. To see this, let $(X, Y) \otimes A$ denote the quotient of $\coprod_{n, m \geq 0} (X^n \times Y^m \times \hat{A}(\mathbf{n}, \mathbf{m}))$ by the equivalence-relation generated by $((x, y), (\theta, \phi)^*a) \sim ((\theta_* x, \phi_* y), a)$. In view of (5.1) there is a homotopy-equivalence $(X, Y) \otimes \hat{A} \to (X \otimes A) \times (Y \otimes A)$ induced by $i_1 \times i_2$. On the other hand m induces $(X, Y) \otimes \hat{A} \to X \otimes (Y \otimes A) \cong (X \wedge Y) \otimes A$. Inverting the homotopy-equivalence gives one $(X \otimes A) \times (Y \otimes A) \to (X, Y) \otimes A$, which is trivial on $(X \otimes A) \vee (Y \otimes A)$, and so defines the desired pairing.

In the examples mentioned above one can define a multiplication on the relevant Γ-spaces. It arises from the existence of "multilinear" morphisms in the categories. For example, let $\mathscr{C}$ be the category of finitely generated projective modules over a fixed commutative ring. Define $\mathscr{C}(S, T)$, for finite sets S and T, as the category whose objects are quadruples $(X, Y, Z; \mu)$, where $X: \mathscr{P}(S) \to \mathscr{C}$ and $Y: \mathscr{P}(T) \to \mathscr{C}$ and $Z: \mathscr{P}(S \times T) \to \mathscr{C}$ are functors which take disjoint unions to direct sums, and μ is a collection of natural bilinear maps $\mu_{\sigma\tau}: X(\sigma) \times Y(\tau) \to Z(\sigma \times \tau)$ expressing $Z(\sigma \times \tau)$ as the tensor-product of $X(\sigma)$ and $Y(\tau)$. Then $(S, T) \mapsto |\mathscr{C}(S, T)|$ is a multiplication on $S \mapsto |\mathscr{C}(S)|$.

To obtain ring-spectra in which the multiplication is strongly homotopy-associative and homotopy-commutative one must begin with sequences $A_1, A_2, \ldots,$ where A_k is a contravariant functor from $\underbrace{\Gamma \times \cdots \times \Gamma}_{k}$ to spaces, A_1 being a Γ-space, A_2 a multiplication on A_1, and so on, together with appropriate transformations between them, one for each morphism in Γ. From this data one obtains ring-spectra possessing a *spectrum* of units. I shall return to this question elsewhere.

APPENDIX A. THE REALIZATION OF SIMPLICIAL SPACES†

If $A = \{A_n\}$ is a simplicial space one usually defines its realization [16] as $|A| = \left(\coprod_{n \geq 0} \Delta^n \times A_n\right)/\sim$, where the equivalence relation $\sim$ is generated by $(\xi, \theta^* a) \sim (\theta_* \xi, a)$ for all $\xi \in \Delta^m$, $A \in A_n$, and $\theta: [m] \to [n]$ in Δ. The functor $A \mapsto |A|$ commutes with products, which is very convenient, but has two disadvantages:

† This appendix is indebted to discussions with D. B. A. Epstein, D. Puppe, V. Puppe, and R. Vogt. I am very grateful for their help.

(i) it can take one out of the category $\mathscr{W}$ of spaces of the homotopy-type of a *CW*-complex, and

(ii) a map $A \to A'$ of simplicial spaces such that $A_n \to A'_n$ is a homotopy-equivalence for each n does not necessarily induce a homotopy-equivalence $|A| \to |A'|$.

To remedy the disadvantages one can use a modified realization $A \mapsto \|A\|$. Any θ: $[m] \to [n]$ in Δ can be factorized canonically $[m] \xrightarrow{\phi} [r] \xrightarrow{\psi} [n]$ with ϕ surjective and ψ injective—such ψ and ϕ are called "face" and "degeneracy" maps respectively. One can construct $|A|$ in two stages, by first attaching the spaces $\Delta^n \times A_n$ to each other by using the relations $(\xi, \theta^* a) \sim (\theta_* \xi, a)$ for all injective θ, and then collapsing the "degenerate parts" by using the surjective θ. The space obtained after the first stage I shall call $\|A\|$. If $\|A\|_n$ is the part of it coming from $\Delta^k \times A_k$ for $k \leq n$ then $\|A\|_n$ is obtained from $\|A\|_{n-1}$ by attaching $\Delta^n \times A_n$ by a map defined on the subspace $\dot{\Delta}^n \times A_n \subset \Delta^n \times A_n$, where $\dot{\Delta}^n$ is the boundary of Δ^n. And $\|A\|$ is the homotopy-direct-limit of the $\|A\|_n$.

PROPOSITION A.1. *The functor $A \mapsto \|A\|$ has the properties*

(i) *if each A_n is in $\mathscr{W}$ then so is $\|A\|$,*

(ii) *if $A \to A'$ is a simplicial map such that $A_n \xrightarrow{\simeq} A'_n$ for each n then $\|A\| \xrightarrow{\simeq} \|A'\|$,*

(iii) $\|A \times A'\| \xrightarrow{\simeq} \|A\| \times \|A'\|$ *for any A, A',*

(iv) *if A is good (as defined below) then $\|A\| \xrightarrow{\simeq} |A|$.*

Of these (i) and (ii) are obvious, and (iii) is a theorem of D. B. A. Epstein which will be proved below, as will (iv).

There are many disadvantages of the functor $A \mapsto \|A\|$: for example the filtration of $\|A\|$ by the $\|A\|_n$ is not what one expects.

Another approach is to thicken the spaces A_n slightly before constructing the realization. There are n surjective maps $[n] \to [n-1]$, to which correspond n copies $A_{n,i}$ of A_{n-1} embedded as retracts in A_n. If σ is a subset of $\{1, \ldots, n\}$ let $A_{n,\sigma} = \bigcap_{i \in \sigma} A_{n,i}$. (Thus $A_{n,\phi} = A_n$.) Define $\tau_n A$ as the generalized mapping-cylinder of the lattice $\{A_{n,\sigma}\}$, i.e. as the union of the subspaces $[0, 1]^\sigma \times A_{n,\sigma}$ of $[0, 1]^n \times A_n$, with the limit topology. Then $\tau_n A$ is homotopy-equivalent to A_n, and it is easy to see that $[n] \mapsto \tau_n A$ is a simplicial space τA with a map $\tau A \to A$. There is a natural inclusion $\|A\| \subset |\tau A|$: in fact $\|A\|$ is the realization of the simplicial space $[n] \mapsto \coprod_{\sigma \subset \mathbf{n}} A_{n,\sigma}$.

PROPOSITION A.2. *The functor $A \mapsto |\tau A|$ has the four properties listed above for $A \mapsto \|A\|$, and furthermore*

(v) $\|A\| \xrightarrow{\simeq} |\tau A|$ *for any A, and*

(vi) *τA is good for any A.*

Before defining "good" and proving (A.1) and (A.2) I shall mention another realization-functor. Let simp(A) be the category of simplexes of A: an object is a pair $([n], a)$ with $a \in A_n$, and a morphism $([n], a) \to ([m], b)$ is a morphism θ: $[n] \to [m]$ in Δ such that $\theta^* b = a$.

One topologizes the objects as $\coprod_n A_n$, and the morphisms as $\coprod_\theta A_m$, so that simp(A) is a topological category. (In some sense $|\text{simp}(A)|$ is a barycentric subdivision of $|A|$.)

PROPOSITION A.3. $|\text{simp}(A)| \simeq |\tau A| \simeq \|A\|$ *for any* A.

There seems no doubt that the appropriate homotopy-type to call the realization of A is that of the three spaces of (A.3). *Goodness* is a condition ensuring that the naive realization $|A|$ has the desired homotopy-type.† To be precise

Definition A.4. A simplicial space A is *good* if for each n and i the inclusion $A_{n,i} \to A_n$ is a closed cofibration.

Remarks.

(i) Each of the three functors $A \mapsto \|A\|$, $A \mapsto |\tau A|$, $A \mapsto |\text{simp}(A)|$ first replaces A by a good simplicial space then forms the naive realization of that.

(ii) The product of two good simplicial spaces is good.

(iii) The simplicial space arising from a topological monoid is good if and only if the monoid is locally contractible at 1. ("Locally" refers to a "halo" [7].)

Proofs. Of the statements to be proved (A.1) (i) and (ii) and (A.2) (vi) are easy, and the rest of (A.1) and (A.2) follows from them and (A.1) (iv). Thus (A.1) (ii) implies that $\|\tau A\| \xrightarrow{\simeq} \|A\|$; but τA is good, so $\|\tau A\| \simeq |\tau A|$ giving (A.2) (v), and hence (A.2) (i), (ii), and (iv). And $\tau A \times \tau A'$ is good, so, by (A.2)(ii) and (iv), because $\tau_n(A \times A') \xrightarrow{\simeq} \tau_n A \times \tau_n A'$, $|\tau(A \times A')| \simeq |\tau A \times \tau A'| \simeq |\tau A| \times |\tau A'|$, giving (A.2) (iii), and hence (A.1) (iii), So one is reduced to proving (A.1) (iv). The proof of this will prove simultaneously that $|\text{simp}(A)| \xrightarrow{\simeq} |A|$ when A is good, and hence that $|\text{simp}(A) \simeq |\tau A|$ for all A, proving (A.3).

If A is a simplicial space and F is a *covariant* functor from Δ to spaces, written $[n] \mapsto F_n$, I shall write $F(A)$ for $\left(\coprod_{n \geq 0} F_n \times A_n\right)/\sim$, where $\sim$ is the equivalence relation $(\xi, \theta^* a) \sim (\theta_* \xi, a)$ for θ in Δ.

LEMMA A.5. *If* F, F': $\Delta \to$ *(spaces) are two covariant functors, and* T: $F \to F'$ *is a transformation, then* T: $F(A) \to F'(A)$ *for any good simplicial space* A *providing*

(i) $F_n \xrightarrow{\simeq} F'_n$ *for each* n, *and*

(ii) $F(B_n) \to F_n$ *and* $F'(B_n) \to F'_n$ *are cofibrations for each* n, *where* B_n *is the boundary of the* n-*simplex, regarded as a simplicial set in the usual way.*

Applying the lemma when F is $[n] \mapsto \|[n]\|$ and F' is $[n] \mapsto |[n]| = \Delta^n$ gives one $\|A\| \xrightarrow{\simeq} |A|$ when A is good. When F is $[n] \mapsto |\text{simp}([n])|$ and F' is $[n] \mapsto \Delta^n$ one gets $|\text{simp}(A)| \to |A|$. In each case the cofibration-condition is satisfied because any monomorphism of simplicial *sets* becomes a cofibration on realization.

† Similar conditions have been studied by May [11].

Proof of A.5. $F(\dot{A})$ is $\varinjlim F_n(A)$, where $F_n(A)$ is the push-out (amalgamated sum) of the diagram

$$F_{n-1}(A) \leftarrow (F_n \times A_n^d) \cup (F(B_n) \times A_n) \xrightarrow{i} F_n \times A_n,$$

where $A_n^d = \cup A_{n,i}$ is the degenerate part of A_n. The hypotheses imply that i is a cofibration [10]. Similarly for F'. One concludes inductively that $F_n(A) \to F_n'(A)$ for all n; and hence that $F(A) \to F'(A)$ because $F_{n-1}(A) \to F_n(A)$ and $F_{n-1}'(A) \to F_n'(A)$ are cofibrations.

Remarks.

(1). In the body of this paper "realization" means the functor $A \mapsto |\tau A|$; but I have nevertheless written it just $|A|$. Occasionally this might lead to confusion: for example in the proof of (1.5) when I write $|A|_{(p)}/|A|_{(p-1)} \simeq S^p(A_1 \wedge \cdots \wedge A_1)$ one must remember that for this to be true for badly behaved spaces A must first be replaced by its thickening τA.

(2) If A is the simplicial space associated to a topological monoid M, i.e. if $A_n = M^n$, then τA is associated to the "whiskered monoid" $M' = M \cup_{\{1\}} [0, 1]$ (cf. [11]).

(3) If A is a Γ-space then τA (formed from the simplicial structure of A only) is naturally a Γ-space. In fact the degeneracy operations in the categories Δ and Γ can be identified with each other.

APPENDIX B. RELATIONSHIP WITH THE APPROACH OF BOARDMAN–VOGT AND MAY

I shall begin with some general remarks. Let us call a topological category whose space of objects is discrete a *category of operators*. If K is such a category I shall call a continuous contravariant functor A from K to spaces a *K-diagram*. ("Continuous" means that $\mathrm{Mor}_K(S; T) \times A(T) \to A(S)$ is continuous for all objects S, T of K.) If one has a continuous functor $\pi: K \to M$ between categories of operators one can regard an M-diagram B as a K-diagram π^*B. And if A is a K-diagram one can form an M-diagram $\pi_* A$ by defining $(\pi_* A)(S)$ as the realization of the simplicial space whose k-simplexes are

$$\coprod_{T_0, \ldots, T_k \in \mathrm{ob}(K)} \mathrm{Mor}_M(S; \pi T_0) \times \mathrm{Mor}_K(T_0; T_1) \times \cdots \times \mathrm{Mor}_K(T_{k-1}; T_k) \times A(T_k).$$

There are natural transformations $\pi_* \pi^* B \to B$ and $A \to \pi^* \pi_* A$.

A map of K-diagrams $A \to A'$ will be called an *equivalence* if $A(S) \to A'(S)$ is a homotopy-equivalence for each object S of K. A functor $\pi: K \to M$ between categories of operators will be called an *equivalence* if $\mathrm{ob}\, K \xrightarrow{\simeq} \mathrm{ob}\, M$ and $\mathrm{mor}\, K \xrightarrow{\simeq} \mathrm{mor}\, M$.

PROPOSITION B.1. *If $\pi: K \to M$ is an equivalence then $A \to \pi^*\pi_* A$ and $\pi_* \pi^* B \to B$ are equivalences for any K-diagram A and M-diagram B.*

PROPOSITION B.2. *For any category of operators K there is an equivalence $\pi: \hat{K} \to K$ with the following property: if A is a K-diagram, and there is given a homotopy equivalence h_s: $A(S) \to A'(S)$ for each object S of K, then there is a $\hat{K}$-diagram A'' with an equivalence $\pi^*A \to A''$ such that $A''(S) = A'(\pi S)$ for each S.*

These two propositions can be used to relate Γ-spaces to spaces with an "operad-action" in the sense of May [11]. An operad furnishes an example of a category of operators K whose objects are the natural numbers, with an equivalence $\pi: K \to \Gamma$. An action of the operad K on a space X gives a K-diagram A such that $A(\mathbf{n}) = X^n$. It follows from (B.1) that the associated Γ-diagram $\pi_* A$ is a Γ-space. On the other hand if A is a Γ-space and one applies (B.2) with $K = \Gamma$ to the homotopy-equivalences p_n: $A(\mathbf{n}) \to A(\mathbf{1})^n$ one obtains a Γ-diagram in which all the spaces are products of copies of $A(\mathbf{1})$. This has the essential properties of one of May's operad-actions.

The proof of (B.1) is very simple. If in the expression for $(\pi^*\pi_* A)(S) = (\pi_* A)(\pi S)$ one replaces $\mathrm{Mor}_M(\pi S; \pi T_o)$ by $\mathrm{Mor}_K(S; T_o)$ one does not change its homotopy-type. But then one has precisely the space of the category S/A of pairs (θ, a) with $\theta: S \to T$ in K and $a \in A(T)$. This space collapses to $A(S)$. An analogous argument applies to $\pi_* \pi^* B$.

The proof of (B.2) is of more interest. I call the category $\hat{K}$ the "explosion" of K; it occurs in various connections: in fact a $\hat{K}$-diagram is precisely what should be called a "homotopy-commutative K-diagram." (The construction has been studied by R. Leitch [9].) By definition $\mathrm{Mor}_{\hat{K}}(S; T)$ is the space of the category of paths in K from S to T. Such a path means a pair (n, θ) where n is a natural number and θ: $[n] \to K$ is a functor such that $\theta(0) = S$ and $\theta(n) = T$. A morphism $(n, \theta) \to (m, \phi)$ is a ψ: $[n] \to [m]$ in Δ such that $\phi\psi = \theta$. Composition of morphisms in $\hat{K}$ is induced by concatenation of paths.

To prove (B.2) one begins by observing that the space $\mathrm{Mor}_{\hat{K}}(S; T)$ is a union made up of one $(n - 1)$-dimensional cube for each path of n steps from S to T. Thus a point of it can be represented.

$$(S = S_o \xrightarrow{\theta_1} S_2 \xrightarrow{\theta_2} \cdots \xrightarrow{\theta_n} S_n = T; t, \ldots, t_{n-1}) \text{ with } 0 \le t_i \le 1.$$

One associates to this point the map

$$h_{S_o} A(\theta_o) \ell_{S_1}(t_1) A(\theta_1) \ldots \ell_{S_{n-1}}(t_{n-1}) A(\theta_n) k_{S_n}: A'(S_n) \to A'(S_o),$$

where k_S is a homotopy-inverse to h_S, and $\ell_S(t)$, for $0 \le t \le 1$, is a homotopy from $h_S k_S$ to the identity.

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