Decomposition and the Gross-Taylor string

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An overview of T. Pantev, ES, arXiv:2307.08729

The purpose of this talk today is to reconcile two different perspectives on two-dimensional pure Yang-Mills theories:

1) Decomposition

(Hellerman, Henriques, Pantev, ES, Ando '06; ... ..., Nguyen, Tanizaki, Unsal '21, ...)

Two-dimensional pure Yang-Mills $= \oplus_R$ (Trivial (invertible) QFTs)

2) Gross-Taylor expansion

(Gross, Taylor '93; Cordes, Moore, Ramgoolam '94, ...)

Two-dimensional pure Yang-Mills = target-space field theory of a string field theory

Executive summary:

Decomposition appears to predict a one-form symmetry in the Gross-Taylor string theory.

Plan of the talk:

1) Review decomposition

Focusing on examples of S_n orbifolds & 2d pure YM

2) Gross-Taylor and two puzzles

Logic of Gross-Taylor:

First rewrite pure YM partition function as a sum of S_n orbifolds,
then, interpret those orbifolds as branched covers and then as SFT.

We'll see that the S_n orbifolds interlace with decomposition *perfectly*,
but two puzzles arise in the branched covers/SFT interpretation.

3) Proposed resolution

The branched cover/SFT interpretation will also be compatible
if the GT string is required to have a novel symmetry.

A short review of decomposition

In $d > 1$ spacetime dimensions,
if a local quantum field theory has a global $(d-1)$-form symmetry,
it is equivalent to a disjoint union of other local QFT's,
known in this context as `universes.'

We call this **decomposition**.

(2d: Hellerman et al '06, ...;
d>2: Tanizaki-Unsal '19, Cherman-Jacobson '20, ...)

When this happens, we say the QFT `decomposes.'
Decomposition has been explored in many examples, as I'll quickly review.
Today: understand decomposition in the Gross-Taylor expansion of 2d pure YM.

More on decomposition...

What does it mean for one local QFT to be a sum of other local QFTs?

(Helleman et al '06)

1) Existence of projection operators

The theory contains topological local operators Π_i such that

$$\Pi_i \Pi_j = \delta_{i,j} \Pi_j \qquad \sum_i \Pi_i = 1 \qquad [\Pi_i, \mathcal{O}] = 0$$

Operators Π_i simultaneously diagonalizable; state space = $\mathcal{H} = \oplus_i \mathcal{H}_i$

Correlation functions:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle \Pi_i \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle (\Pi_i \mathcal{O}_1) \cdots (\Pi_i \mathcal{O}_m) \rangle = \sum_i \langle \tilde{\mathcal{O}}_1 \cdots \tilde{\mathcal{O}}_m \rangle_i$$

2) Partition functions decompose

$$Z = \sum_{\text{states}} \exp(-\beta H) = \sum_i \sum \exp(-\beta H_i) = \sum_i Z_i$$

(on a connected spacetime)

Decomposition in 2d gauge theories

(Hellerman et al '06)

Example:

S'pose have G-gauge theory, G semisimple, with finite central $K \subset G$ acting trivially.

Statement of decomposition (in this example):

$$\mathrm{QFT}(G-\text{gauge theory}) \;=\; \coprod_{\text{char's } \hat{K}} \mathrm{QFT}\left(G/K-\text{gauge theory w/ discrete theta angles}\right)$$

Example: pure $SU(2)$ gauge theory = sum $SO(3)_+ + SO(3)_-$ pure gauge theories

where $\pm$ denote discrete theta angles (w_2)

Perturbatively, the $SU(2)$, $SO(3)_\pm$ theories are identical
— differences are all nonperturbative.

Decomposition in 2d gauge theories

(Hellerman et al '06)

Example:

S'pose have G-gauge theory, G semisimple, with finite central $K \subset G$ acting trivially.

As discussed previously, has 1-form symmetry (specifically, BK).

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Example: pure $SU(2)$ gauge theory = sum $SO(3)_+ + SO(3)_-$ pure gauge theories

where $\pm$ denote discrete theta angles (w_2)

$SU(2)$ instantons (bundles) $\subset$ $SO(3)$ instantons (bundles)

The discrete theta angles weight the non-$SU(2)$ $SO(3)$ instantons so as to cancel out of the partition function of the disjoint union.

Summing over the $SO(3)$ theories projects out some instantons, giving the $SU(2)$ theory.

Decomposition in 2d gauge theories

(Hellerman et al '06)

Example:

S'pose have G-gauge theory, G semisimple, with finite central $K \subset G$ acting trivially.

As discussed previously, has 1-form symmetry (specifically, BK).

Statement of decomposition (in this example):

$$\text{QFT}(G-\text{gauge theory}) = \coprod_{\text{char's } \hat{K}} \text{QFT}\left(G/K-\text{gauge theory w/ discrete theta angles}\right)$$

Formally, the partition function of the disjoint union can be written

$$Z = \underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \overbrace{\left(\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right] \right)}^{\text{projection operator}}$$

where we have moved the summation inside the integral.

This is an interference effect between universes: **multiverse interference**

Decomposition in 2d gauge theories

(Hellerman et al '06)

projection operator

$$Z = \sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right] = \int [DA] \exp(-S) \left(\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right] \right)$$

Disjoint union

Decomposition in 2d gauge theories

(Hellerman et al '06)

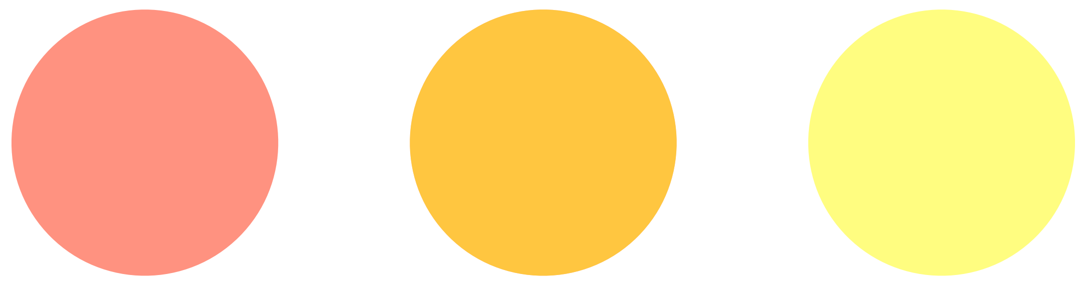
One effect is a projection on nonperturbative sectors:

$$\underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \overbrace{\left(\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right] \right)}^{\text{projection operator}}$$

Disjoint union of several QFTs / universes = \`One' QFT with a restriction on nonperturbative sectors = \`multiverse interference'

Schematically,
two theories combine to form a distinct third:

universe ($SO(3)_+$)



universe ($SO(3)_-$)

multiverse interference effect ($SU(2)$)

Since 2005, decomposition has been checked in many examples in many ways. Examples:

- GLSM's: mirrors, quantum cohomology rings (Coulomb branch) (T Pantev, ES '05; Gu et al '18-'20)
- Orbifolds: partition f'ns, massless spectra, elliptic genera (T Pantev, ES '05; Robbins et al '21)
- Open strings, K theory (Hellerman et al hep-th/0606034)
- Susy gauge theories w/ localization (ES 1404.3986)
- Nonsusy pure Yang-Mills ala Migdal (ES '14; Nguyen, Tanizaki, Unsal '21)
- Adjoint QCD_2 (Komargodski et al '20)
- Numerical checks (lattice gauge thy) (Honda et al '21)
- Versions in d-dim'l theories w/ (d-1)-form symmetries (Tanizaki, Unsal, '19; Cherman, Jacobson '20)

This list is incomplete; apologies to those not listed.

Applications include:

- Sigma models with target stacks & gerbes (T Pantev, ES '05)
- Predictions for Gromov-Witten theory (checked by H-H Tseng, Y Jiang, E Andreini, etc starting '08)
- Nonperturbative constructions of geometries in GLSMs (Caldararu et al 0709.3855, Hori '11, ..., Romo et al '21)
- Elliptic genera (Eager et al '20)
- Anomalies in orbifolds (Robbins et al '21)

Today: decomposition in the Gross-Taylor string....

Two examples of decomposition will play an important role in this talk:

- 2d pure Yang-Mills (decomposing to invertibles)
- 2d Dijkgraaf-Witten theory

The role of the first is clear:
we're trying to reconcile decomposition of 2d pure Yang-Mills
with its description ala Gross-Taylor.

Now, part of the Gross-Taylor story is a rewriting of the 2d pure YM partition function as a sum of 2d Dijkgraaf-Witten theories, so its decomposition will also play a role.

We'll discuss each in turn.

Example: 2d pure Yang-Mills (decomposing to invertibles)

Recall from (Migdal '75, Drouffe '78, Lang et al '81, Menotti et al '81, Rusakov '90) that 2d pure Yang-Mills has been solved exactly.

The partition function $Z(\Sigma)$ on a closed Riemann surface Σ of genus p and area A is

$$Z(\Sigma) = \sum_R \left(\dim R\right)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2} C_2(R)\right)$$

where

R is an irrep of the gauge group

$C_2(R)$ is the quadratic Casimir of R

How does it decompose? ....

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Decomposes into theories associated with irreps R:

$$Z(\Sigma) = \sum_R Z_R \qquad Z_R = \left(\dim R\right)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2} C_2(R)\right)$$

(It can also decompose along center symmetries,
but the decomposition along irreps will be the focus of the rest of this talk.)

How to interpret those constituent theories?...

Example: 2d pure Yang-Mills (decomposing to invertibles)

2d pure YM is a disjoint sum of trivial (`invertible') field theories, associated to the irreps R:

(Nguyen, Tanizaki, Unsal '21)

$$Z(\Sigma) = \sum_R Z_R \qquad Z_R = \left(\dim R\right)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2} C_2(R)\right)$$

The constituent invertible field theories are ~ classical theories, with 1d Fock space (only vacuum), indexed by counterterms:

$$S = \int_\Sigma \sqrt{-g}\,\left(aR + b\right) \qquad Z = \exp\left(a\chi(\Sigma) + b \cdot \text{Area}\right)$$

so the universe associated to irrep R (partition function Z_R)

has $\quad a(R) = \ln \dim R, \qquad b(R) = -\frac{g_{YM}^2}{2} C_2(R)$

when interpret as invertible field theory.

Next: Dijkgraaf-Witten…

Example: 2d Dijkgraaf-Witten theory

This is a fancy name for an orbifold of a point: [point/G] for G finite

In cases w/o discrete torsion, operators are twist fields associated to conjugacy classes.

Correlation functions: On a Riemann surface Σ of genus p,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \frac{1}{|G|} \sum_{s_1, t_1, \cdots, s_p, t_p \in G} \delta\left(\mathcal{O}_1 \cdots \mathcal{O}_n \prod_{i=1}^{p} s_i t_i s_i^{-1} t_i^{-1} \right)$$

where $$\delta(g) = \begin{cases} 1 & g = 1 \\ 0 & g \neq 1 \end{cases}$$

For example, the partition function is

$$Z = \frac{1}{|G|} \sum_{s_1, t_1, \cdots, s_p, t_p \in G} \delta\left(\prod_{i=1}^{p} s_i t_i s_i^{-1} t_i^{-1} \right)$$

How does it decompose? ....

Example: 2d Dijkgraaf-Witten theory

This theory also decomposes into a disjoint sum of trivial (`invertible') field theories, associated to the irreps r.

Projection operators P_r exist: $$P_r = \frac{\dim r}{|G|} \sum_{g \in G} \chi_r\left(g^{-1}\right) g$$

This can also be written as a sum over conjugacy classes, but this form is simpler.

These are projection operators in the sense that $P_r P_s = \delta_{r,s} P_r \,, \quad \sum_r P_r = 1$

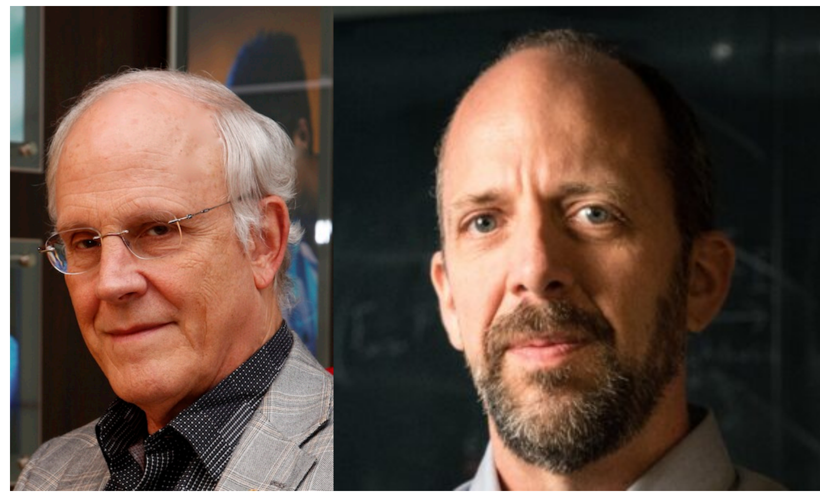
Correlation functions in the universe associated to irrep r are

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_r = \langle \mathcal{O}_1 \cdots \mathcal{O}_n P_r \rangle = \frac{1}{|G|} \sum_{s_1, t_1, \cdots, s_p, t_p \in G} \delta\left(\mathcal{O}_1 \cdots \mathcal{O}_n \left(\prod_{i=1}^{p} s_i t_i s_i^{-1} t_i^{-1} \right) P_r \right)$$

Note $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \sum_r \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_r$

Next: Gross-Taylor...

Next, we turn to the Gross-Taylor expansion of 2d pure $SU(N)$ Yang-Mills.



They argued that at large N, this is a target-space SFT of some other 2d string theory, via a series expansion of the partition functions.

Let's review. On a closed Riemann surface Σ_T of genus p and area A,

$$Z\left(\Sigma_T\right) \;=\; \sum_R \left(\dim R\right)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R)\right)$$

Strictly speaking, to get the right large N asymptotics, we need to write irreps R in terms of coupled representations. For sake of time, and b/c it doesn't significantly affect our result, I'll gloss over that step.

Basic strategy: rewrite the sum over $SU(N)$ irrep data,
as a sum over S_n's and S_n irrep data,
where n is the num' boxes in Young tableau for irrep R,
and then interpret in terms of branched covers of Σ_T

$$Z(\Sigma_T) = \sum_R (\dim R)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R)\right)$$

Let's rewrite in terms of irreps & characters of the finite symmetric group S_n

Expand the terms using Schur-Weyl duality:

$SU(N)$ data (fixed irrep R) →
$$(\dim R(Y))^m = \left(\frac{N^n \dim r(Y)}{|S_n|}\right)^m \frac{\chi_{r(Y)}\left((\Omega_n)^m\right)}{\dim r(Y)}$$
← S_n data

where

$Y =$ Young tableau associated with $SU(N)$ irrep R

$n =$ num' boxes in Young tableau Y

$r(Y) = S_n$ irrep associated to Y (and hence $R = R(Y)$)

$$\Omega_n = \sum_{\sigma \in S_n} N^{K_\sigma - n} \sigma$$

$K_\sigma =$ num' cycles in the cycle decomposition of $\sigma \in S_n$

$$Z\left(\Sigma_T\right) = \sum_R \left(\dim R\right)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R)\right)$$

$$\left(\dim R(Y)\right)^m = \left(\frac{N^n \dim r(Y)}{|S_n|}\right)^m \frac{\chi_{r(Y)}\left((\Omega_n)^m\right)}{\dim r(Y)}$$

Use the identity $\sum_{s,t\in G} \chi_r\left(sts^{-1}t^{-1}\right) = \left(\frac{|G|}{\dim r}\right)^2 \dim r$ to show

$$\left(\dim R(Y)\right)^m = N^{nm} \left(\frac{\dim r(Y)}{|S_n|}\right)^{m+2p} \sum_{s_1,t_1,\cdots,s_p,t_p\in S_n} \frac{\chi_r\left((\Omega_n)^m \prod_{i=1}^p s_i t_i s_i^{-1} t_i^{-1}\right)}{\dim r(Y)}$$

$$= N^{nm} \left(\frac{\dim r(Y)}{|S_n|}\right)^{m+2p-1} \sum_{s_1,t_1,\cdots,s_p,t_p\in S_n} \frac{\delta\left((\Omega_n)^m \left(\prod_{i=1}^p s_i t_i s_i^{-1} t_i^{-1}\right) P_{r(Y)}\right)}{\dim r(Y)}$$

One more step....

$$Z\left(\Sigma_T\right) = \sum_R \left(\dim R\right)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R)\right)$$

So far:

$$\left(\dim R(Y)\right)^m = N^{nm} \left(\frac{\dim r(Y)}{|S_n|}\right)^{m+2p-1} \sum_{s_1,t_1,\cdots,s_p,t_p \in S_n} \frac{\delta\left((\Omega_n)^m \left(\prod_{i=1}^p s_i t_i s_i^{-1} t_i^{-1}\right) P_{r(Y)}\right)}{\dim r(Y)}$$

Use the identity

$$\frac{C_2(R(Y))}{N} = n + \frac{2}{N} \frac{\chi_{r(Y)}(T_2)}{\dim r(Y)} - \frac{n^2}{N^2}$$

to write

$$\left(\dim R(Y)\right)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R)\right)$$

$$= N^{n(2-2p)} \left(\frac{\dim r(Y)}{|S_n|}\right) \sum_{s_1,t_1,\cdots,s_p,t_p \in S_n} \frac{\delta\left((\Omega_n)^{2-2p} \left(\prod_{i=1}^p s_i t_i s_i^{-1} t_i^{-1}\right) P_{r(Y)}\right)}{\dim r(Y)} \exp\left(-g_{YM}^2 \frac{A}{2} n\right)$$

$$+ \text{ subleading}$$

Finally, we have the Gross-Taylor series expansion.

The partition function of two-dimensional pure $SU(N)$ Yang-Mills

$$Z\left(\Sigma_T\right) = \sum_R \left(\dim R\right)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R)\right)$$

has now been rewritten in terms of S_n's and S_n irrep data:

$$\left(\dim R(Y)\right)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R)\right)$$

→ $SU(N)$ data (fixed irrep R)

S_n data ↙

$$= N^{n(2-2p)} \left(\frac{\dim r(Y)}{|S_n|}\right) \sum_{s_1,t_1,\cdots,s_p,t_p \in S_n} \frac{\delta\left((\Omega_n)^{2-2p}\left(\prod_{i=1}^p s_i t_i s_i^{-1} t_i^{-1}\right) P_{r(Y)}\right)}{\dim r(Y)} \exp\left(-g_{YM}^2 \frac{A}{2} n\right)$$

$+$ subleading

Strictly speaking, we need to break up each irrep R into coupled reps; however, the analysis is nearly identical, and the expression above emerges as one of two chiral components.

Next: interpretation....

Let's interpret:

$$\left(\dim R(Y)\right)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R)\right)$$

← Partition function of a single universe in the decomposition of 2d pure YM.

$$= N^{n(2-2p)} \left(\frac{\dim r(Y)}{|S_n|}\right) \sum_{s_1,t_1,\cdots,s_p,t_p \in S_n} \frac{\delta\left((\Omega_n)^{2-2p}\left(\prod_{i=1}^p s_i t_i s_i^{-1} t_i^{-1}\right) P_{r(Y)}\right)}{\dim r(Y)} \exp\left(-g_{YM}^2 \frac{A}{2} n\right)$$

$$+ \text{ subleading}$$

The RHS (above) is a sum of 2d Dijkgraaf-Witten correlation functions for group S_n.
In fact, note that the correlation functions have projectors $P_{r(Y)}$
— these are correlation functions in the universe associated to $r(Y)$!

Takeaway: the partition function of a single universe in the decomposition of 2d pure YM, is a sum of correlation functions in a single universe of 2d Dijkgraaf-Witten for S_n .

Perfect match!   Next: Gross-Taylor and 2d strings....

So far: written partition function of a single universe of 2d pure $SU(N)$ Yang-Mills as a sum of correlation functions in a single universe of 2d Dijkgraaf-Witten for S_n

Decomposition meshes perfectly!

Next: interpret in terms of branched covers of the Riemann surface Σ_T

Interpretation of S_n Dijkgraaf-Witten in terms of branched n-covers

(Gross, Taylor '93)

For simplicity, let's take the Riemann surface $\Sigma_T = S^2$

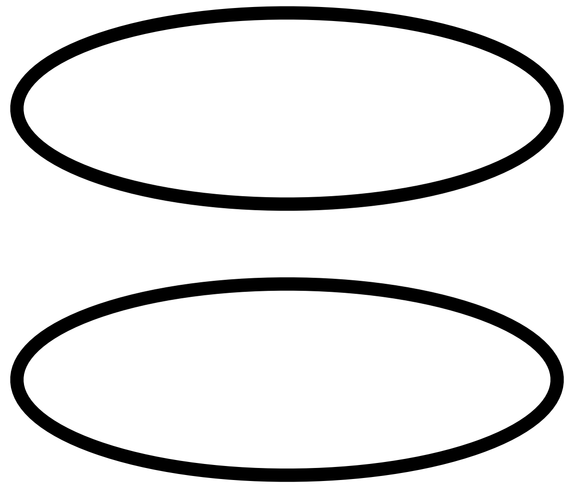
If there are no insertions, then, identify the cover with a disjoint union $\coprod_n S^2$

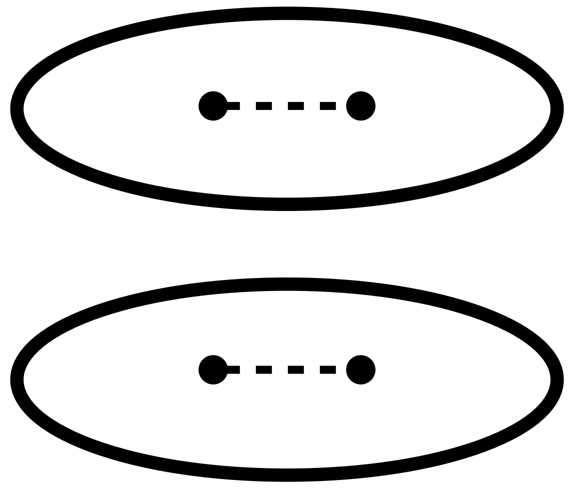
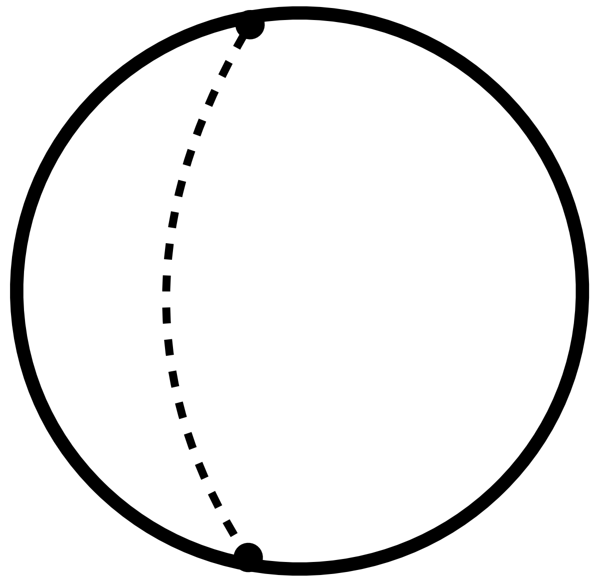
An insertion of $g \in S_n$ corresponds to a branch point of monodromy g,
that ties the n sheets of the cover together.

Let's see some examples....

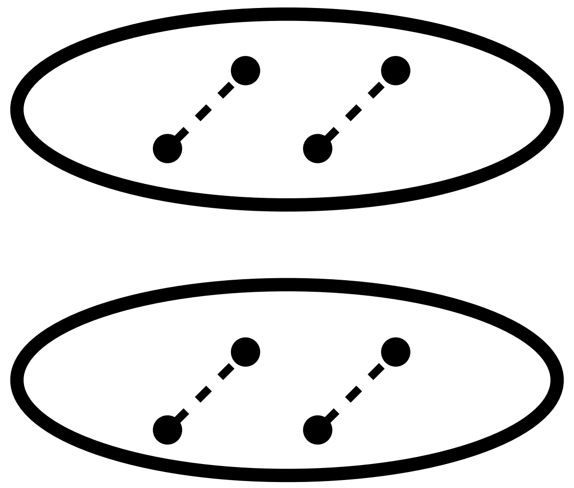
Interpretation of S_n Dijkgraaf-Witten in terms of branched n-covers

Examples: $\Sigma_T = S^2,\ n = 2$: double covers of S^2

$\langle 1 \rangle =$  $= \coprod_2 S^2$

$\langle g^2 \rangle =$  $=$  $= S^2$

S^2 as branched double cover of S^2; branch pts at poles, and wraps.

$\langle g^4 \rangle =$  $=$  $= T^2$

Let's apply to the (original) Gross-Taylor expansion:

$$\sum_R \left(\dim R(Y)\right)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R)\right)$$

$$= \sum_{n=0}^{\infty} \sum_r N^{n(2-2p)} \left(\frac{\dim r(Y)}{|S_n|}\right) \sum_{s_1,t_1,\cdots,s_p,t_p \in S_n} \frac{\delta\left((\Omega_n)^{2-2p}\left(\prod_{i=1}^p s_i t_i s_i^{-1} t_i^{-1}\right)\right)}{\dim r(Y)} \exp\left(-g_{YM}^2 \frac{A}{2} n\right)$$

$$+ \text{ subleading}$$

This is the expansion of the full YM theory — includes sum over all representations (so the projectors $P_{r(Y)}$ sum out — we'll return to them when we look at individual universes).

$$\Omega_n = \sum_{\sigma \in S_n} N^{K_\sigma - n} \sigma$$

Powers of N:

$$n(2-2p) + \sum_j \left(K_{\sigma_j} - n\right) = n\chi\left(\Sigma_T\right) + \sum_j \left(K_{\sigma_j} - n\right)$$

$$= \chi\left(\Sigma_W\right) \quad \text{(Riemann-Hurwitz theorem)}$$

where Σ_W is a branched n-fold cover of Σ_T

Let's apply to the (original) Gross-Taylor expansion:

$$\sum_R \left(\dim R(Y)\right)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R)\right)$$
$$= \sum_{n=0}^{\infty} \sum_{s_i, t_i \in S_n} \sum_{L=0}^{\infty} \sum_{v_1,\cdots,v_L \in S_n} N^{\chi(\Sigma_W)} (\#)\, \delta\left(v_1 \cdots v_L \left(\prod_{i=1}^{p} [s_i, t_i]\right)\right) \exp\left(-\frac{A}{\alpha'_{GT}} n\right)$$
$$+ \text{ subleading}$$

where

$\Sigma_W =$ branched n-fold cover of Σ_T, branched over L points

$$\alpha'_{GT} = \frac{2}{g_{YM}^2}$$

$\# =$ misc' numerical factors, which match Euler char' of space of maps

This is the form expected if 2d pure YM is the SFT of a sigma model $\Sigma_W \to \Sigma_T$, at large N

Now let's turn to the decomposition.

The partition function of a single universe of 2d pure YM is

$$\left(\dim R(Y)\right)^{2-2p} \exp\left(-g_{YM}^2 \frac{A}{2N} C_2(R)\right)$$
$$= \sum_{s_i,t_i \in S_n} \sum_{L=0}^{\infty} \sum_{v_1,\cdots,v_L \in S_n} N^{\chi(\widetilde{\Sigma}_W)} (\#)\, \delta\left(v_1 \cdots v_L \left(\prod_{i=1}^{p} [s_i, t_i]\right) \underline{P_{r(Y)}}\right) \exp\left(-\frac{A}{\alpha'_{GT}} n\right)$$
$$+ \text{ subleading}$$

- Restrict to single $SU(N)$ irrep $R(Y)$
- which fixes $n =$ num' boxes in Young diagram Y for irrep $R(Y)$ = covering map deg'
- plus added factor of projector $P_{r(Y)}$ in the delta function

This means:

1) Sigma model is restricted to maps of a single degree (n)

2) Presence of projector $P_{r(Y)}$ implies add'l contributions not present previously

So, we have puzzles to explain in the expansion of a single YM universe:

1) Sigma model is restricted to maps of a single degree (n)

2) Presence of projector $P_{r(Y)}$ implies add'l contributions not present previously

In broad brushstrokes, both phenomena are typical in decomposition:

- Restrictions on instantons / nonperturbative sectors
- Individual universes can receive contributions which cancel out in sums over universes

as we saw previously in the $SU(2) = SO(3)_+ \coprod SO(3)_-$ example.

However, the details here are more extreme:

- Restrictions are usually to a subset of instantons, not to a single instanton degree
- Here the extra contributions would expand possible worldsheets beyond smooth Riemann surfaces

Let's examine in detail....

1) Sigma model is restricted to maps of a single degree (n)

In a 2d NLSM, this is a restriction to (worldsheet) instantons of a single degree.

This is more extreme than we ordinarily see in decomposition.

Furthermore,
labelling field configurations by instanton number
is typically just an artifact of a semiclassical expansion,
and ordinarily does not have an intrinsic meaning in QFT.

Proposal:
the Gross-Taylor string has a symmetry for which map degree is a conserved quantity.

But map degree is a 2-form ($\phi^* \omega$),
so such a symmetry would be either a 1-form or (-1)-form symmetry.

1) Sigma model is restricted to maps of a single degree (n)

Proposal:
the Gross-Taylor string has a symmetry for which map degree is a conserved quantity.

But map degree is a 2-form ($\phi^*\omega$),
so such a symmetry would be either a 1-form or (-1)-form symmetry.

To make this more concrete,
next I'll walk through a related example, where precisely this happens:
2d pure Maxwell theory.

1) Sigma model is restricted to maps of a single degree (n)

2d pure Maxwell theory:

Pure Maxwell theory in any dimension has a global $BU(1)$ (1-form) symmetry:

$$A \mapsto A + \Lambda$$

and Noether current $J^e = *F$, associated to operator $U_\alpha(p) = \exp(i\alpha * F(p))$

In 2d, it also has a magnetic (-1)-form symmetry,

with current $J^m = F$, associated to operator $U_\beta(\Sigma) = \exp\left(i\beta \int_\Sigma F\right)$

So, the symmetries are of the same form as proposed for Gross-Taylor,
making it a useful prototype....

1) Sigma model is restricted to maps of a single degree (n)

2d pure Maxwell theory:

$$Z(\Sigma) = \int [DA] \exp(-S) \qquad \text{for} \qquad S = \frac{1}{g_{YM}^2} \int_\Sigma F^{\mu\nu} F_{\mu\nu} + i\theta \int_\Sigma F$$

$$\propto \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2}{g_{YM}^2 A} + i\theta n \right) \qquad \text{where} \quad n \sim c_1 \sim \int F$$

After Poisson resummation,

$$Z(\Sigma) = \sum_{m=-\infty}^{\infty} \exp\left(-\frac{g_{YM}^2 A}{4} (\theta + 2\pi m)^2 \right)$$

This is the form of the exact expression for pure YM.

(Paniak, Szabo '02; Gross, Matytsin, '94; Minahan, Polychronakos, '93; Caselle et al '93; Fine '90)

Decomposes into universes indexed by m (irreps of $U(1)$), *Poisson dual* to $n \sim c_1$.

1) Sigma model is restricted to maps of a single degree (n)

2d pure Maxwell theory:

$$Z(\Sigma) = \sum_{m=-\infty}^{\infty} \exp\left(-\frac{g_{YM}^2 A}{4}(\theta + 2\pi m)^2\right) \qquad \propto \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2}{g_{YM}^2 A} + i\theta n\right)$$

Decomposes into universes indexed by m (irreps of $U(1)$), *Poisson dual* to $n \sim c_1$.

Partition function of a single universe is $\exp\left(-\frac{g_{YM}^2 A}{4}(\theta + 2\pi m)^2\right)$

Analogue of the Witten effect:

Shifting $\theta \mapsto \theta + 2\pi$ is equivalent to changing the universe: $m \mapsto m + 1$

1) Sigma model is restricted to maps of a single degree (n)

2d pure Maxwell theory:

$$Z(\Sigma) \; = \; \sum_{m=-\infty}^{\infty} \exp\left(-\frac{g_{YM}^2 A}{4} (\theta + 2\pi m)^2 \right) \qquad \propto \; \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2}{g_{YM}^2 A} + i\theta n \right)$$

This is a prototype for the Gross-Taylor proposal:
there's a decomposition, into universes indexed by m,
which is Poisson dual to the bundle degree.

In Gross-Taylor, we propose there exists a symmetry which allows us to pick out sectors of single map degree (single worldsheet instanton number),
which is analogous.

1) Sigma model is restricted to maps of a single degree (n)

So far, we've proposed that the Gross-Taylor string admits an extra symmetry.

Can that be seen directly?

There are (at least) 2 proposals in the literature for the Gross-Taylor string:

1) Cordes-Moore-Ramgoolam: GT string = modification of A model TFT

Standard kinetic terms; localizes on holomorphic maps $\{\bar{\partial} x = 0\}$

2) Horava: GT string = twisted NLSM with nonstandard kinetic terms

Localizes on harmonic maps $\{\partial \bar{\partial} x = 0\}$

The desired symmetry is not immediately visible in either;
might be realized nonlinearly, or, maybe there exists a third version.

Review: puzzles to explain in the expansion of a single YM universe:

1) Sigma model is restricted to maps of a single degree (n)

We've argued this implies the GT string has a new symmetry.

2) Presence of projector $P_{r(Y)}$ implies add'l contributions not present previously

We'll study this problem next.

2) Presence of projector $P_{r(Y)}$ implies add'l contributions not present previously

Example: $\Sigma_T = S^2 \; (p=0), \;\; n=2$

$$Z = \frac{N^{2n}}{n!}\delta\left(\left(\Omega_n\right)^2 P_r\right) \quad = \frac{N^{2n}}{n!}\delta\left((1)P_r + 2\left(\frac{1}{N}\right)vP_r + \left(\frac{1}{N}\right)^2 v^2 P_r\right)$$

$$= \frac{N^4}{2!}\delta\left(P_r\right) + 2\frac{N^3}{2!}\delta\left(vP_r\right) + \frac{N^2}{2!}\delta\left(v^2 P_r\right)$$

$$= \frac{N^4}{4} \pm \frac{N^3}{2} + \frac{N^2}{4}$$

$\Sigma_W = S^2 \coprod S^2$, $\chi(\Sigma_W) = 4$ — ???? — $\Sigma_W = S^2$, $\chi(\Sigma_W) = 2$

The N^3 term is new — not present in original GT — present here only b/c of P_r.

How to interpret? $N^\chi = N^3$ so $\chi = 3$, but no closed string worldsheet has χ odd

2) Presence of projector $P_{r(Y)}$ implies add'l contributions not present previously

How to interpret? No closed string worldsheet has χ odd

Some options:

- Expand out the projector P_r

In the previous example, we'd get a term prop' to $N^3\delta(vv)$.
From the delta, should be S^2, but wrong Euler characteristic.

- Open string?

Subleading corrections were interpreted in the old literature as nonpert' corrections;
open string worldsheets could have odd χ

But these terms aren't all subleading, so expect them to be perturbative,
hence not from open worldsheets.

2) Presence of projector $P_{r(Y)}$ implies add'l contributions not present previously

How to interpret? No closed string worldsheet has χ odd

Another possible option: stacky worldsheets

Returning to previous example ($\Sigma_T = S^2, n = 2$):

$$Z = \frac{N^4}{2!}\delta\left(P_r\right) + 2\frac{N^3}{2!}\delta\left(vP_r\right) + \frac{N^2}{2!}\delta\left(v^2 P_r\right)$$

$$= \frac{N^4}{4} \pm \frac{N^3}{2} + \frac{N^2}{4}$$

Interpret as 2 copies of S^2 with a single $\mathbb{Z}_2$ orbifold point ($\mathbb{P}^1_{[1,2]}$)

$$\chi\left(\mathbb{P}^1_{[1,2]}\right) = 3/2 \qquad \chi\left(\mathbb{P}^1_{[1,2]} \coprod \mathbb{P}^1_{[1,2]}\right) = (2)(3/2) = 3$$

matches power of N!

2) Presence of projector $P_{r(Y)}$ implies add'l contributions not present previously

How to interpret? No closed string worldsheet has χ odd

Another possible option: stacky worldsheets

For $\Sigma_T = S^2$, there is a systematic construction of stacky Σ_W's
(here, Riemann surfaces w/ orbifold points)
that gives matching powers of N.

Idea: Given $\delta(v_1 \cdots v_L)$, write each $v_i \in S_n$ as a product of cycles.
On jth copy of S^2, if j appears in a cycle of length k, insert $\mathbb{Z}_k$

Example: S'pose $n = 6$ and $v = (12)(345)(6)$
Then, insert $\mathbb{Z}_2$ on 2 copies, $\mathbb{Z}_3$ on 3 copies, smooth pt on last copy.

Can show $$\chi = n(2-2p) + \sum_j (K_{v_j} - n)$$ which matches power of N

2) Presence of projector $P_{r(Y)}$ implies add'l contributions not present previously

How to interpret? No closed string worldsheet has χ odd

Another possible option: stacky worldsheets

Issues:

- Construction only understood for S^2, not higher genus
- Construction not unique — orb' points can be redistributed across sheets of cover
- Have not tried to compare Hurwitz moduli spaces in general cases

In the same spirit, at least on $\Sigma_T = S^2$,
one can reinterpret the terms as contributions from `stacky' copies of Σ_T,
meaning, copies with orbifold points.

This is in the spirit of the decomposition:
instead of a sigma model summing over maps $\Sigma_W \to \Sigma_T$,
this would reflect a decomposition, to trivial field theories
(corresponding to copies of Σ_T).

Summary: reconciling decomposition & GT string pictures of 2d pure YM

1) Reviewed decomposition

Focusing on examples of S_n orbifolds & 2d pure YM

2) Gross-Taylor and the puzzles

Logic of Gross-Taylor:

First rewrote pure YM partition function as a sum of S_n orbifolds,
then, interpreted those orbifolds as branched covers and then as SFT.

We saw that the S_n orbifolds interlace with decomposition perfectly,
but two puzzles arise in the branched covers/SFT interpretation.

3) Proposed resolution

The branched cover/SFT interpretation will also be compatible
if the GT string is required to have a novel symmetry.

Thank you for your time!

Before going on, let's quickly check these claims for pure $SU(2)$ Yang-Mills in 2d.

The partition function Z, on a Riemann surface of genus g, is

(Migdal, Rusakov)

$$Z(SU(2)) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$

Sum over all SU(2) reps

$$Z(SO(3)_+) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$

Sum over all SO(3) reps

(Tachikawa '13)

$$Z(SO(3)_-) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$

Sum over all SU(2) reps that are not SO(3) reps

Result: $Z(SU(2)) = Z(SO(3)_+) + Z(SO(3)_-)$ as expected.

(Later we'll review a more extreme decomposition of 2d pure YM, which we'll compare to GT.)

1) Sigma model is restricted to maps of a single degree (n)

In a 2d NLSM, this is a restriction to (worldsheet) instantons of a single degree.

In decomposition, one often sees restrictions on instanton degrees.

For example, in the $SU(2) = SO(3)_+ \coprod SO(3)_-$ example,
$SU(2)$ instantons are a subset of $SO(3)$ instantons.

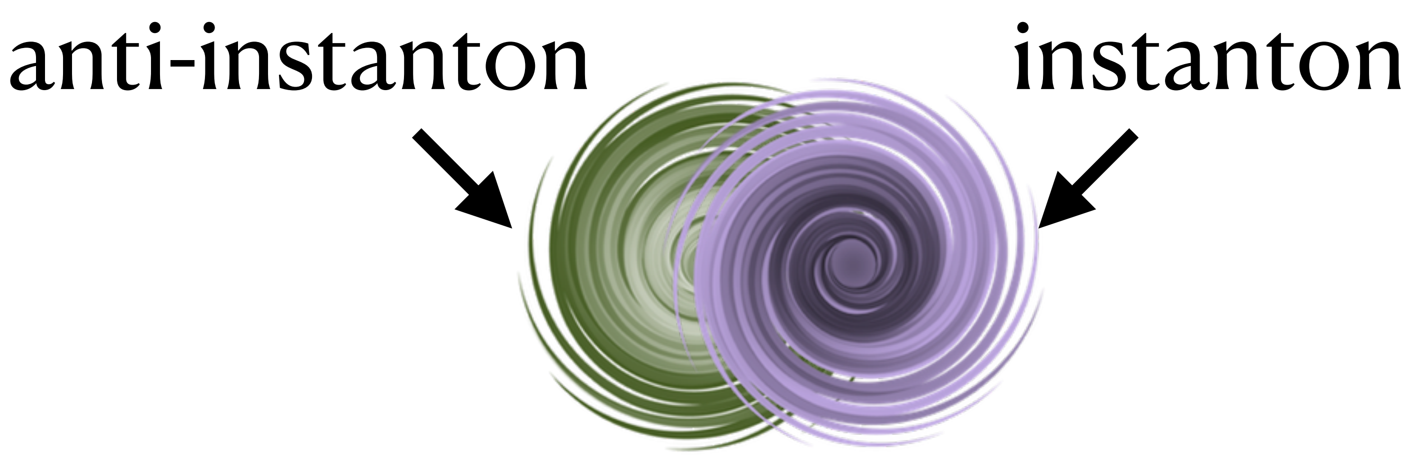
However, in that case, and most other examples,
one restricts to a subset of instantons,
not to instantons of a single degree.

Let's take a moment to review some underlying physics....

Suppose we try to require that the total instanton number always vanish in our QFT.

Start with a field configuration with no net instantons.

Now, move them far away from one another:



Nonzero instanton number here!

Total instanton number : 0

Nonzero instanton number here!

If physics is local ("cluster decomposition"),
then in those widely-separated regions, the theories have instantons.
So, even if we start with no net instantons,
cluster decomposition implies we get instantons!

Cluster decomposition:

For this reason, Steven Weinberg taught us:

All local quantum field theories must sum over all instantons, so as to preserve cluster decomposition.

Loophole: Disjoint unions of QFTs also violate cluster decomposition (ex: multiple dimension zero operators), but in principle are straightforward to deal with.

So, if a theory with a restriction on instantons is also a disjoint union, of theories which are well-behaved, then all is OK.