The Geometry of Scattering Amplitudes

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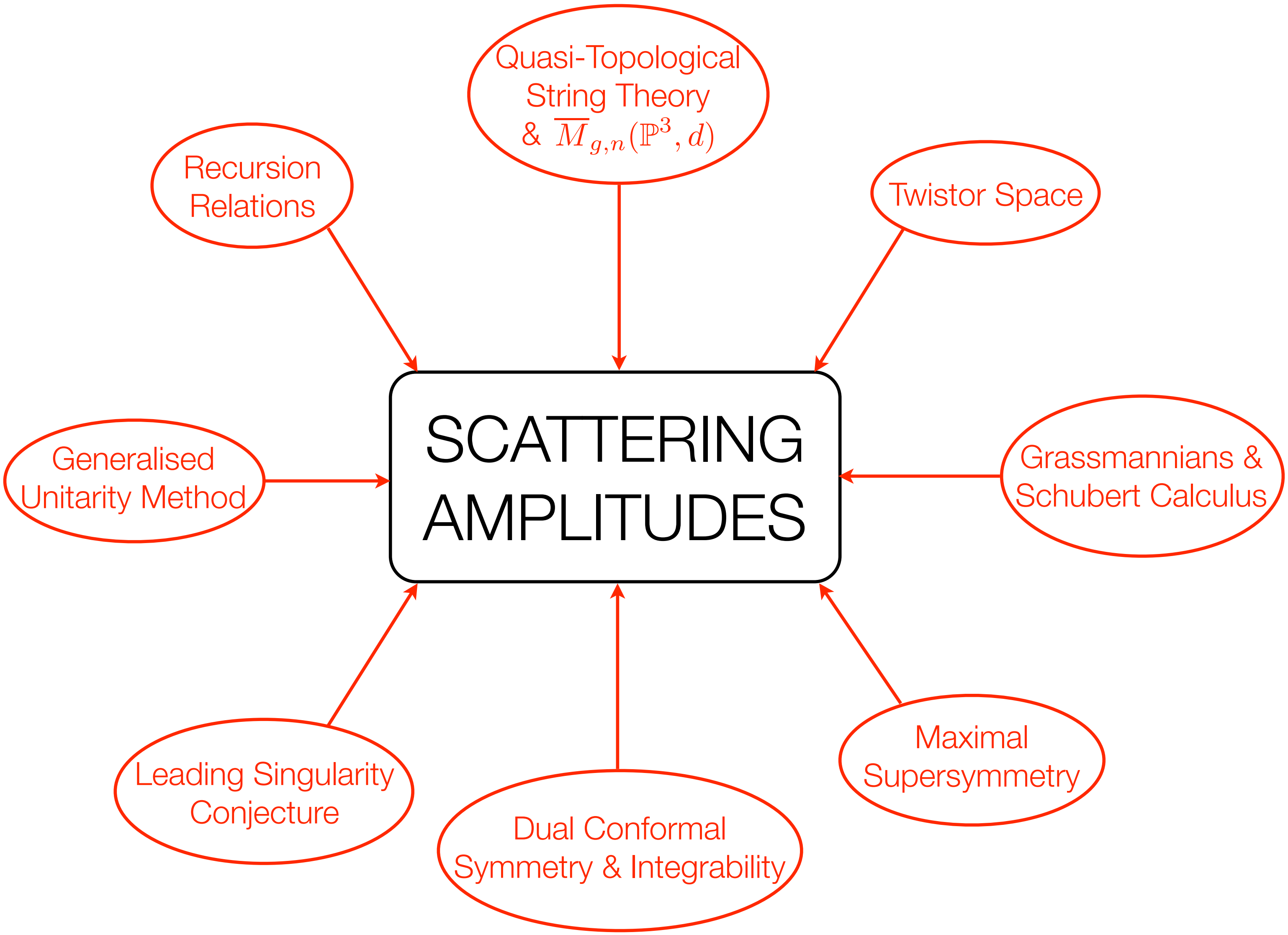
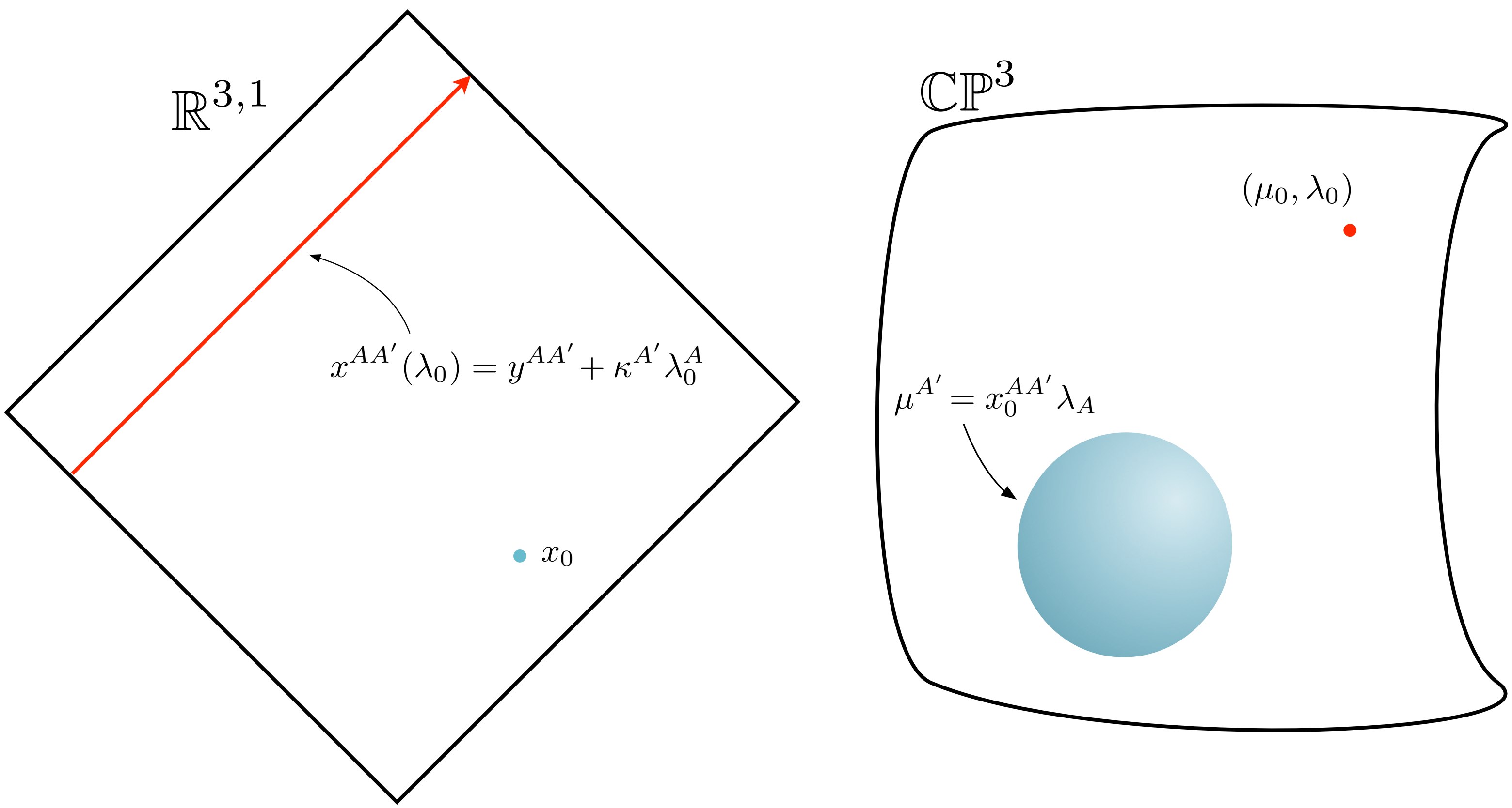
based on work in progress with L. Mason



- The topology of the diagram is certainly not accurate!

Twistor space

Twistor space is a copy of $\mathbb{CP}^3$ with homogeneous coordinates $W_\alpha = (\lambda_A, \mu^{A'})$



- Points in space-time are Riemann spheres in twistor space
- Points in twistor space are null rays in space-time (really β-planes in complexified space-time)

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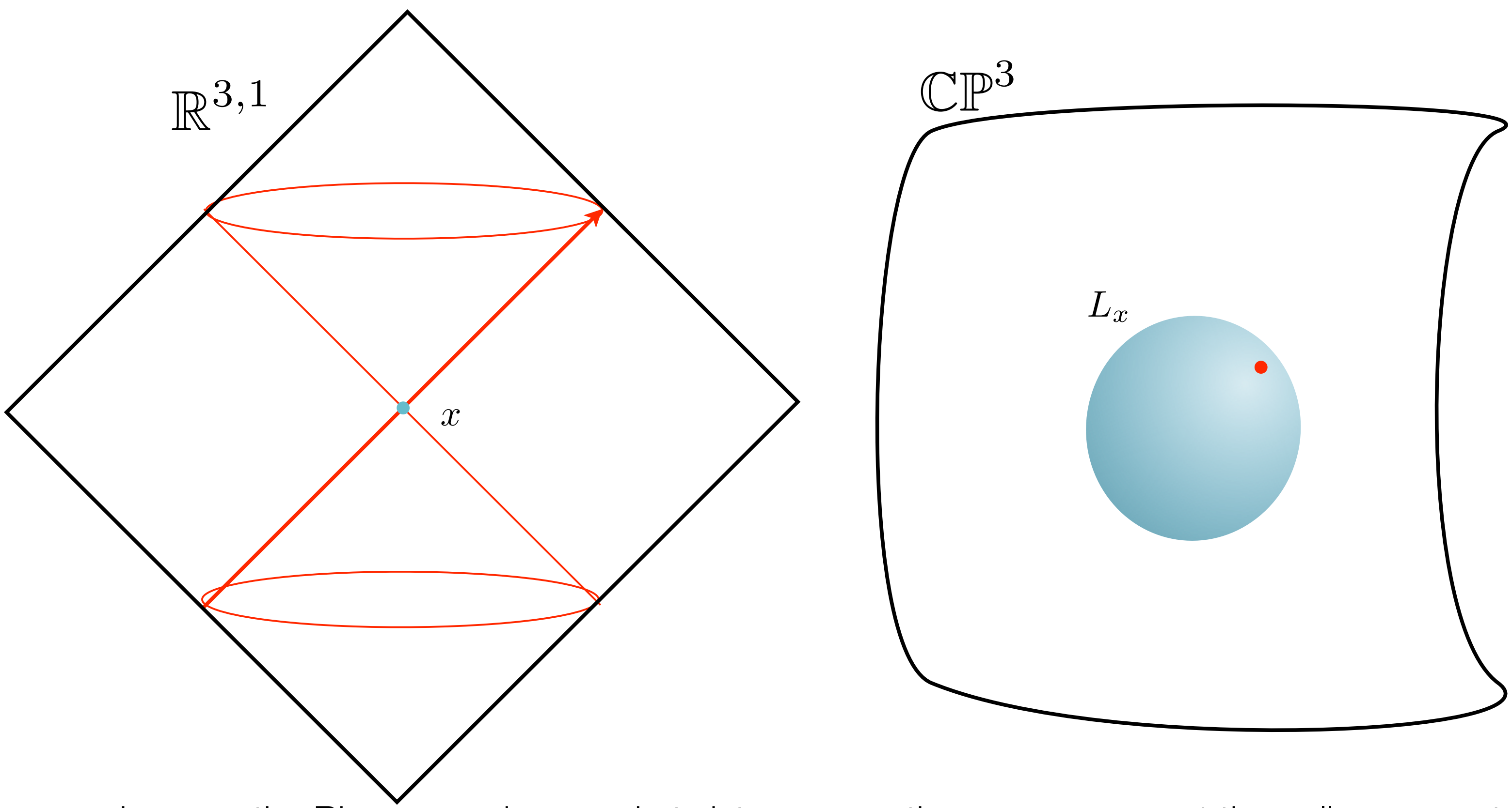

- As W varies over the Riemann sphere L_x in twistor space, the rays sweep out the null cone centered on x in space-time

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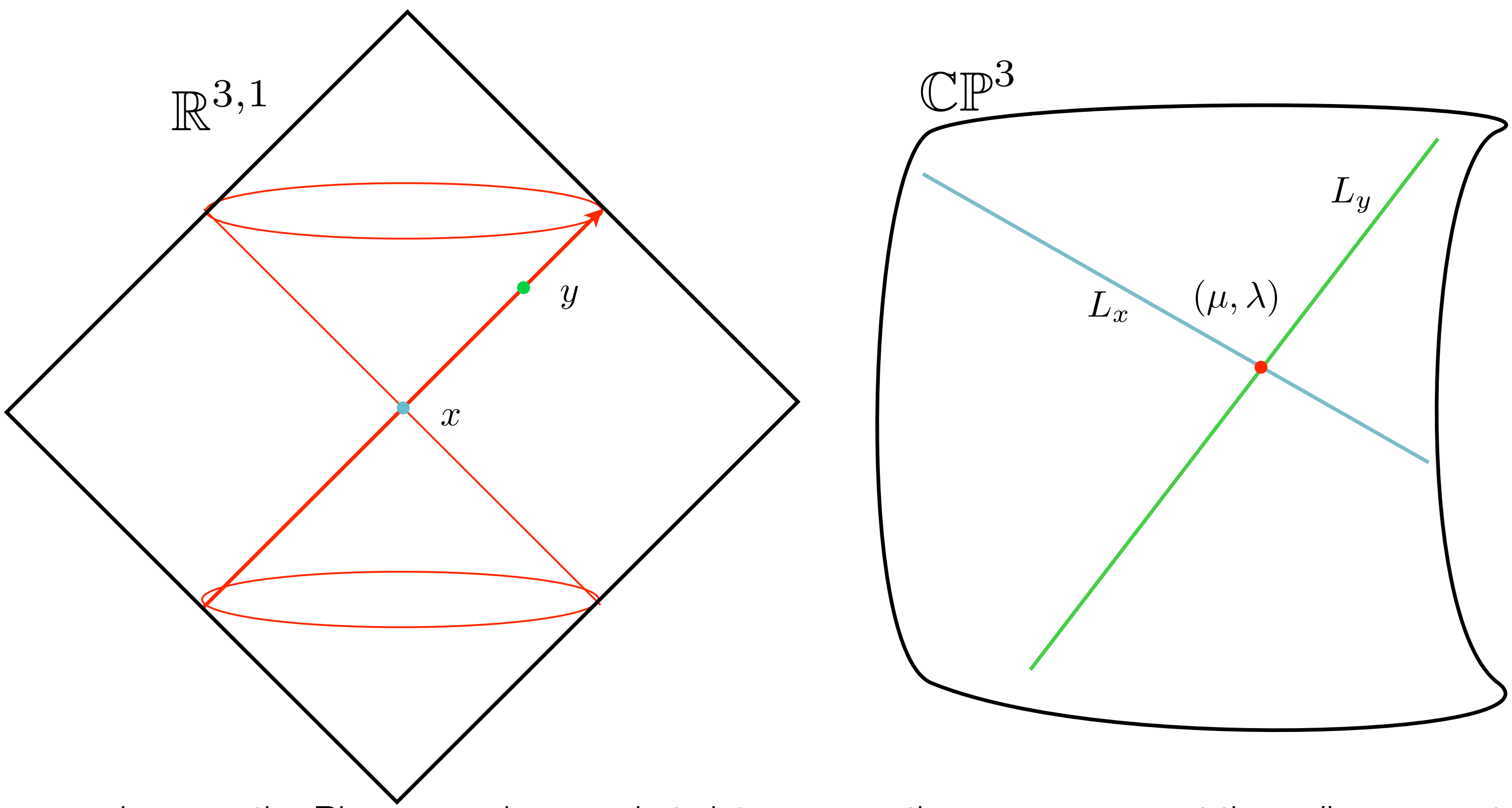

- As W varies over the Riemann sphere L_x in twistor space, the rays sweep out the null cone centered on x in space-time
- If two twistor lines intersect, their corresponding space-time points are null-separated

$$\mu^{A'} = x^{AA'}\lambda_A \quad \text{and} \quad \mu^{A'} = y^{AA'}\lambda_A \quad \Leftrightarrow \quad (x-y)^{AA'}\lambda_A = 0$$

The twistor & momentum representations

Twistor space provides a convenient way - the Penrose transform - to describe the general solution of massless linear field equations such as $\Box\phi = 0$.

Momentum space

$$\phi(x) = \int \mathrm{d}^4 p \ \mathrm{e}^{\mathrm{i}p\cdot x} \ \delta(p^2)\Phi(\lambda, \tilde{\lambda})$$

$\Phi(\lambda, \tilde{\lambda})$ an arbitrary function

$\Box\phi = 0$ ensured by restriction to null cone

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Twistor space

$$\phi(x) = \oint \langle \lambda \, \mathrm{d}\lambda \rangle \, f(W)|_{L_x}$$

$f(W)$ is (locally) a holomorphic function of weight -2

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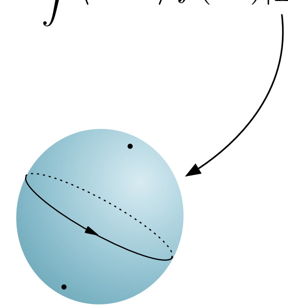

$$L_x = \{(\lambda_A, \mu^{A'}) \in \mathbb{CP}^3 : \mu^{A'} = x^{AA'} \lambda_A\}$$

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Twistor space

$$\phi(x) = \oint \langle \lambda \, \mathrm{d}\lambda \rangle \, f(W)|_{L_x}$$

$f(W)$ arbitrary

$\Box\phi = 0$ ensured by holomorphy of $f(W)$

$$\frac{\partial f(W)}{\partial x^{BB'}} = \lambda_B \frac{\partial f}{\partial \mu^{B'}}$$

and therefore

$$\frac{\partial^2 f}{\partial x^{BB'} \partial x_{BB'}} = \underbrace{\lambda^B \lambda_B}_{=0} \frac{\partial^2 f}{\partial \mu^{B'} \partial \mu_{B'}}$$

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- Both representations have easy generalisations to other helicities

$$\Phi(\lambda, \tilde{\lambda}) \longrightarrow \lambda_A \cdots \lambda_D \Phi(\lambda, \tilde{\lambda}) \text{ or } \tilde{\lambda}_{A'} \cdots \tilde{\lambda}_{D'} \Phi(\lambda, \tilde{\lambda})$$

$$f_{-2}(W) \longrightarrow \lambda_A \cdots \lambda_D f_{2h-2}(W) \text{ or } \frac{\partial}{\partial \mu^{A'}} \cdots \frac{\partial}{\partial \mu^{D'}} f_{2h-2}(W)$$

"half Fourier transform"

The twistor & momentum representations

Twistor space provides a convenient way - the Penrose transform - to describe the general solution of massless linear field equations such as $\Box\phi = 0$.

Momentum space	Twistor space
$\phi(x) = \int \mathrm{d}^4 p \; \mathrm{e}^{\mathrm{i}p\cdot x}\, \delta(p^2)\Phi(\lambda,\tilde{\lambda})$	$\phi(x) = \oint \langle\lambda\, \mathrm{d}\lambda\rangle\, f(W)\|_{L_x}$
$\Phi(\lambda,\tilde{\lambda})$ an arbitrary function	$f(W)$ arbitrary
$\Box\phi = 0$ ensured by restriction to null cone	$\Box\phi = 0$ ensured by holomorphy of $f(W)$

- Both representations have easy generalisations to other helicities
- Twistor space makes conformal properties manifest - *cf* $K^{AA'} = \mu^{A'}\dfrac{\partial}{\partial\lambda_A}$ *vs* $\dfrac{\partial^2}{\partial\tilde{\lambda}_{A'}\partial\lambda_A}$
- Off-shell, either drop restriction to momentum null cone, or drop holomorphy requirement

$\Phi'(p)$ *vs* $f(W,\overline{W})$ $\Rightarrow$ Twistor theory more complicated off-shell

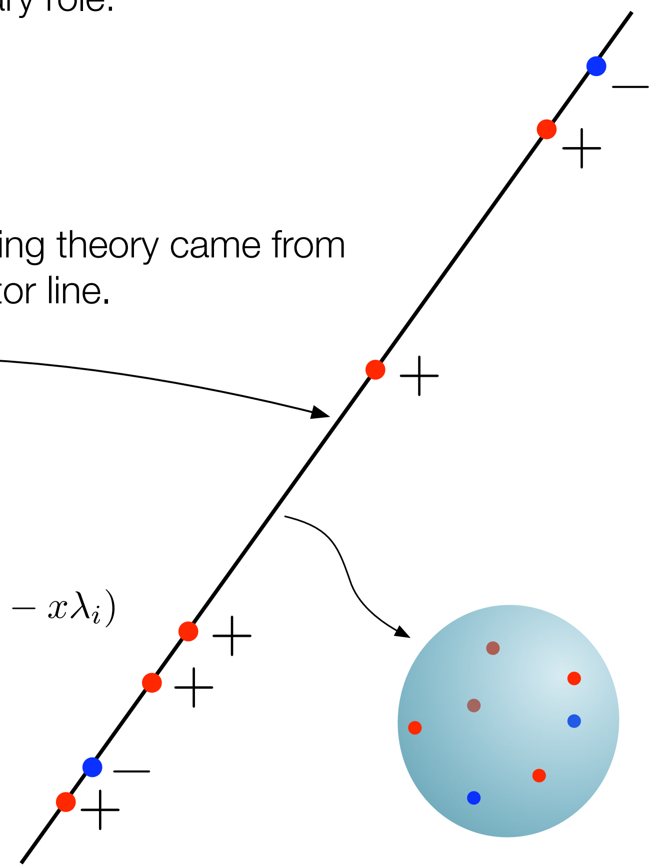
Twistor theory makes intimate use of null separation, so (with hindsight!) it's not surprising that it's better suited to on-shell methods for calculating amplitudes than to a traditional approach based on Feynman diagrams.

There are two other places where on-shell methods play a primary role:

- Modern recursion relations / generalised unitarity methods
- String theory

The first hint of a relation between twistors and some form of string theory came from Nair, who noticed that MHV amplitudes are supported on a twistor line.

$$\mathcal{A}^{(0)}_{\mathrm{MHV}}(W_1,\ldots,W_n) = \int \frac{\mathrm{d}^{4|8}x}{\langle 12\rangle\cdots\langle n1\rangle}\prod_{i=1}^{n}\bar{\delta}^{2|4}(\mu_i - x\lambda_i)$$



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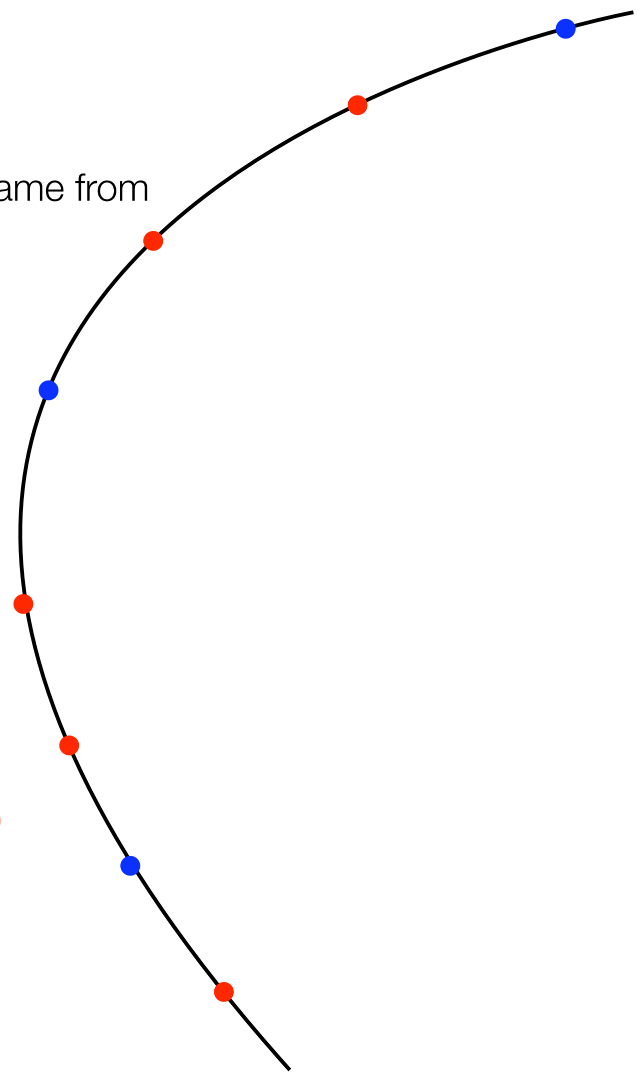
Witten used Nair's observation as the basis of his twistor-string theory, in which N^{k-2}MHV amplitudes are supported on holomorphic twistor curves of degree

$$d = k - 1 + g$$

and genus

$$h \leq g$$

at g-loops.



$$A(W,\chi) = g^+(W) + \chi_a \Gamma^a(W) + \ldots + \frac{\epsilon^{abcd}\chi_a\chi_b\chi_c\chi_d}{4!} g^-(W)$$

$$\chi(\sigma) \in \mathbb{C}^4 \times H^0(\Sigma, \mathcal{L}) \qquad \text{where} \qquad \mathcal{L} = W^*\mathcal{O}(1)$$

$$k = h^0(\Sigma, \mathcal{L}) = \deg(\mathcal{L}) + 1 - g \qquad \text{generically}$$

Twistor-string theory

The twistor-string can be interpreted as a twisted (0,2) model[Mason, DS]. The path integral is similar to that of the heterotic string and gives

$$\int \mathrm{d}\mu \, \frac{\det'(\bar\partial_{W^*E})}{\det'(\bar\partial_{W^*(N_{C|\mathbb{PT}^*})})} \exp\left(-\frac{A(C)}{2\pi} + \mathrm{i}\int_C B\right)$$

Worldsheet map $W : \Sigma \to \mathbb{PT}^*$ and worldsheet (0,2) gravity

$$\frac{\det'(\bar\partial_{T_\Sigma})\det'(\partial_{W^*(\overline{T}_{\mathbb{PT}^*})})}{\det'(\Delta_{W^*(T_{\mathbb{PT}^*})})} = \frac{\det'(\bar\partial_{T_\Sigma})}{\det'(\bar\partial_{W^*(T_{\mathbb{PT}^*})})} = \frac{1}{\det'(\bar\partial_{W^*(N_{C|\mathbb{PT}^*})})}$$

since $0 \to \mathcal{O} \to \mathbb{C}^{4|4} \times \mathcal{O}(1) \to T_{\mathbb{PT}^*} \to 0$

and $0 \to T_\Sigma \to W^*T_{\mathbb{PT}^*} \to W^*N_{C|\mathbb{PT}^*} \to 0$

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Left-movers
($E \to \mathbb{PT}^*$ a holomorphic v.b.)

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Area of curve

$$\int \mathrm{d}\mu \, \frac{\det{}'(\bar{\partial}_{W^*E})}{\det{}'(\bar{\partial}_{W^*(N_{C|\mathbb{PT}^*})})} \exp\left(-\frac{A(C)}{2\pi} + \mathrm{i}\int_C B\right)$$

NS B-field

Twistor-string theory

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Integral over space $\overline{M}_{g,0}(\mathbb{PT}^*, d)$ of zero-modes, of (virtual) dimension $4d$.

c.f. 2875 isolated lines on $Q_5 \subset \mathbb{P}^4$

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As usual, vertex operators correspond to infinitesimal deformations of background structure. These are

$E \to \mathbb{PT}^* \quad \Leftrightarrow \quad H^1(\mathbb{PT}^*, \mathrm{End}\, E) \quad \Leftrightarrow \quad \mathcal{N}=4$ SYM multiplet

by Penrose-Ward transform

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$\mathbb{C}\text{−str} \quad \Leftrightarrow \quad H^1(\mathbb{PT}^*, T_{\mathbb{PT}^*}) \quad \Leftrightarrow \quad \mathcal{N}=4$ sd conformal sugra multiplet

Flux $H = \mathrm{d}B \quad \Leftrightarrow \quad H^1(\mathbb{PT}^*, \Omega^2_{\mathrm{cl}}) \quad \Leftrightarrow \quad \mathcal{N}=4$ asd conformal sugra multiplet (!)

Twistor-string theory contains *conformal* supergravity[Berkovits, Witten] and is therefore (probably) non-unitary.

At tree-level one can “extract” the pure SYM piece by hand[Witten; Roiban, Spradlin, Volovich; Dolan, Goddard; Vergu]

$$\int \mathrm{d}\mu \, \ln\det'(\bar\partial_{W^*E})$$

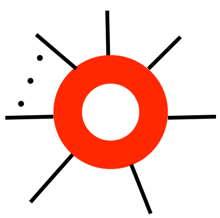
... but at loop level the situation looks bleak.

guarantees only single-trace contributions

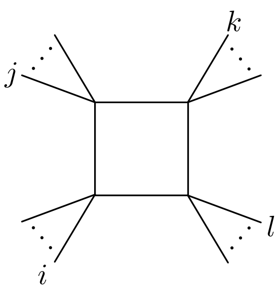
Generalised unitarity & leading singularities

Although twistor-string theory itself is badly behaved, it led to a resurgence of interest in computing scattering amplitudes using unitarity-based methods[Bern, Dixon, Kosower; many others!].

not present in $\mathcal{N} = 4$ SYM

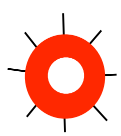


$= \sum c_{ijkl} \times$



$+ \cdots$

$c_{ijkl} = \int_\Gamma \mathrm{d}^4 p$



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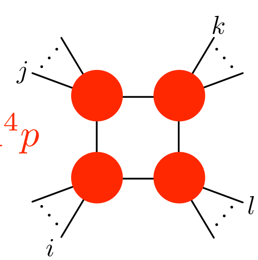
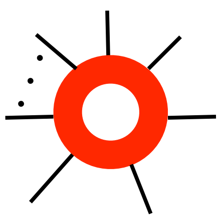


are 1-loop “leading singularities”

where $\Gamma = \{ p \in \mathbb{C}^4 \; : \; |p^2| = |(p+K_1)^2| = |(p+K_2)^2| = |(p-K_4)^2| = \varepsilon \}$

Generalised unitarity & leading singularities

Although twistor-string theory itself is badly behaved, it led to a resurgence of interest in computing scattering amplitudes using unitarity-based methods[Bern, Dixon, Kosower; many others!].



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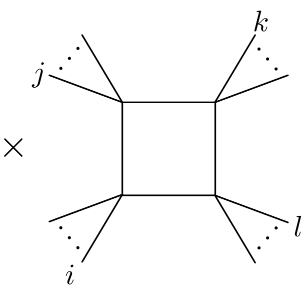
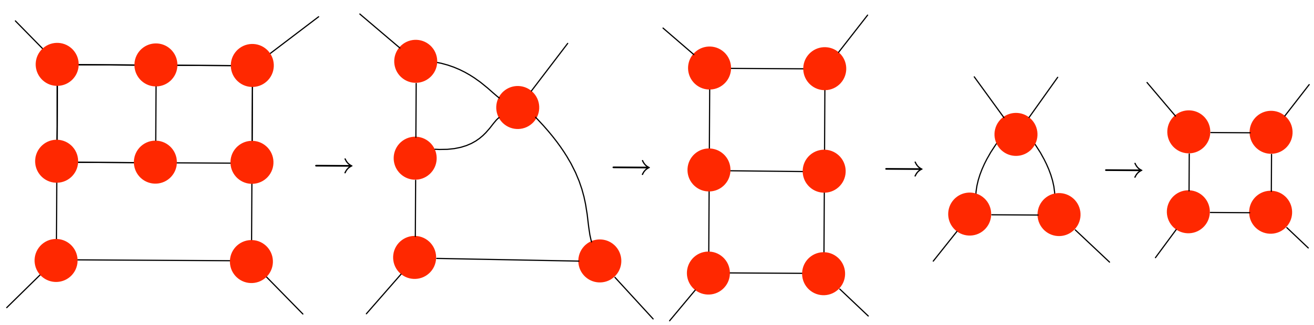


$$+ \cdots$$

The *leading singularity method*[Buchbinder, Cachazo, DS, Spradlin, Volovich, Wen] conjectures that all coefficients of similar higher-loop expansions can be fixed in the same way.



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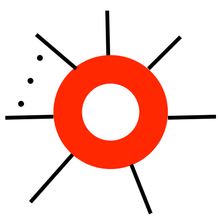

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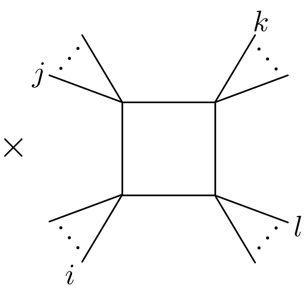

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YM amplitudes have universal IR divergence structure ($s_{ij}^{-\epsilon}/\epsilon^2$ at 1-loop in dim reg). The individual boxes have different IR properties, so their coefficients have to satisfy many constraints. One such constraint recovers the tree amplitude - realising this led to the BCF(W) recursion relations.

All tree amplitudes in $\mathcal{N} = 4$ SYM

By combining BCFW recursion with dual superconformal invariance, last year Drummond & Henn were able to obtain all n-point tree amplitudes in maximal SYM (and hence in pure YM).

Their solution is

$$\mathcal{A}^{(0)}_{\mathrm{MHV}} = \frac{\delta^{4|8}(\sum |i\rangle [i|)}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

$$\mathcal{A}^{(0)}_{\mathrm{NMHV}} = \mathcal{A}^{(0)}_{\mathrm{NMHV}} \times \sum_{2 \leq a,b < n} R_{n;ab}$$

$$R_{n;ab} := \frac{\langle a\, a{-}1 \rangle \langle b\, b{-}1 \rangle \delta^{0|4} \left(\langle n | x_{na} x_{ab} | \theta_{bn} \rangle + \langle n | x_{nb} x_{ba} | \theta_{an} \rangle \right)}{x^2_{ab} \langle n | x_{nb} x_{ba} | a \rangle \langle n | x_{nb} x_{ba} | a{-}1 \rangle \langle n | x_{na} x_{ab} | b \rangle \langle n | x_{na} x_{ab} | b{-}1 \rangle}$$

where $x_{ij} := p_i + p_{i+1} + \cdots + p_{j-1}$

and $\theta_{ij} := \lambda_i \eta_i + \cdots + \lambda_{j-1} \eta_{j-1}$

$R_{n;ab}$ is invariant under both dual superconformal and (on the support of $\mathcal{A}^{(0)}_{\mathrm{MHV}}$) usual superconformal transformations

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more complicated version of $R_{n;ab}$

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$$\mathcal{A}^{(0)}_{\text{N}^2\text{MHV}} = \mathcal{A}^{(0)}_{\text{MHV}} \times \sum_{2\leq a_1,b_1<n} R_{n;a_1b_1} \left(\sum_{a_1<a_2,b_2\leq b_1} R^{0;a_1b_1}_{n;b_1a_1;a_2b_2} + \sum_{b_1\leq a_2,b_2<n} R^{a_1b_1;0}_{n;a_2b_2} \right)$$

$$\mathcal{A}^{(0)}_{\text{N}^3\text{MHV}} = \mathcal{A}^{(0)}_{\text{MHV}} \times \sum_{2\leq a_1,b_1<n} R_{n;a_1b_1}$$

$$\times \left\{ \begin{array}{l} \displaystyle\sum_{a_1<a_2,b_2\leq b_1} R^{0;a_1b_1}_{n;b_1a_1;a_2b_2} \left(\sum_{a_1<a_3,b_3\leq b_2} R^{0;b_1a_1,a_2b_2}_{n;b_1a_1;b_2a_2;a_3b_3} + \sum_{b_2\leq a_3,b_3\leq b_1} R^{b_1a_1,a_2b_2;a_1b_1}_{n;b_1a_1;a_3b_3} + \sum_{b_1\leq a_3,b_3<n} R_{n;a_3b_3} \right) \\ \displaystyle\qquad + \sum_{b_1\leq a_2,b_2<n} R^{a_1b_1;0}_{n;a_2b_2} \left(\sum_{a_2<a_3,b_3\leq b_2} R^{0;a_2b_2}_{n;b_2a_2;a_3b_3} + \sum_{b_2\leq a_3,b_3<n} R^{a_2b_2;0}_{n;a_3b_3} \right) \end{array} \right\}$$

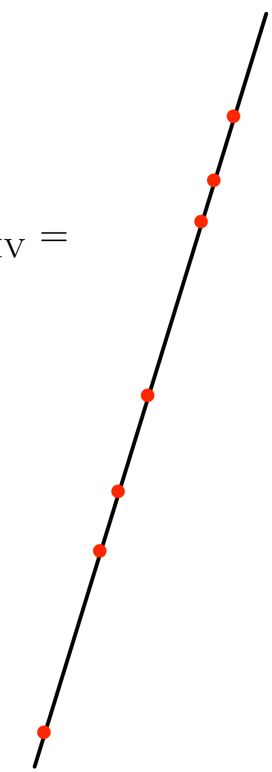
etc.

All tree amplitudes in twistor space

It's interesting to look at the twistor space support of this expression for the tree amplitudes. This can be done either by translating the BCFW recursion procedure into twistor space[Mason & DS] or by translating the Drummond & Henn solution directly[Korchemsky & Sokatchev].

Of course,

$$\mathcal{A}^{(0)}_{\mathrm{MHV}} =$$



and you might expect that the N^{k-2}MHV terms are each supported on curves of degree

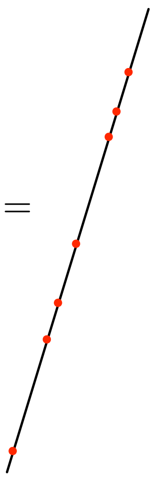
$$d = k - 1$$

using Witten's formula at genus zero.

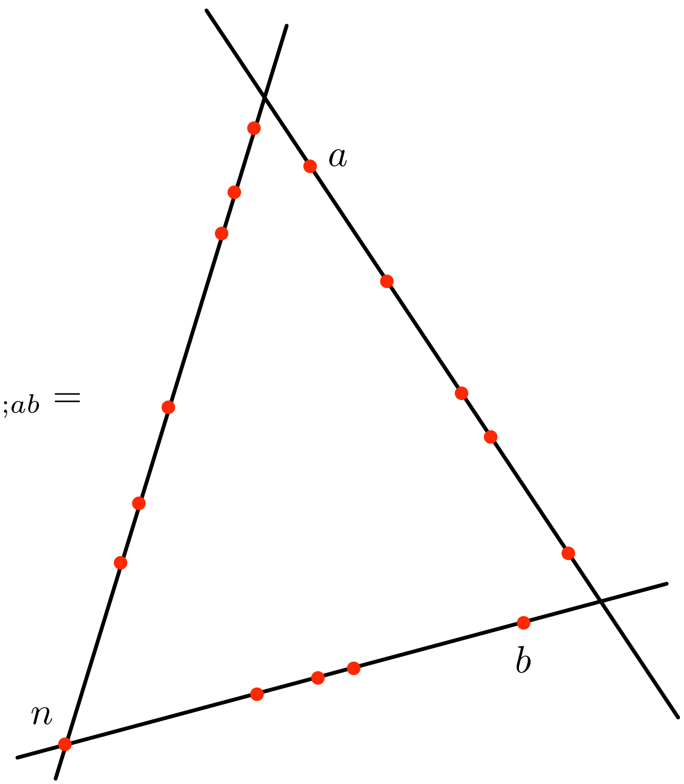
The true story is even more interesting...

All tree amplitudes in twistor space

$$\mathcal{A}^{(0)}_{\mathrm{MHV}} =$$



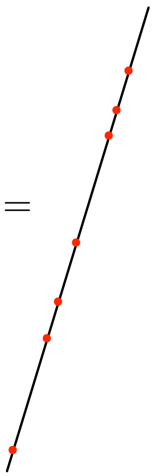
$$\mathcal{A}^{(0)}_{\mathrm{MHV}} R_{n;ab} =$$



At NMHV we find a (reducible) degree 3 curve of genus 1, in agreement with the prediction $d = k - 1 + g$.
In fact, it's well-known that this term also arises as a 3-mass box coefficient and so "knows" about 1-loop.

All tree amplitudes in twistor space

$\mathcal{A}^{(0)}_{\text{MHV}} =$



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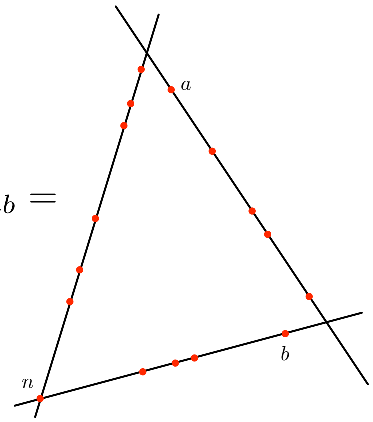
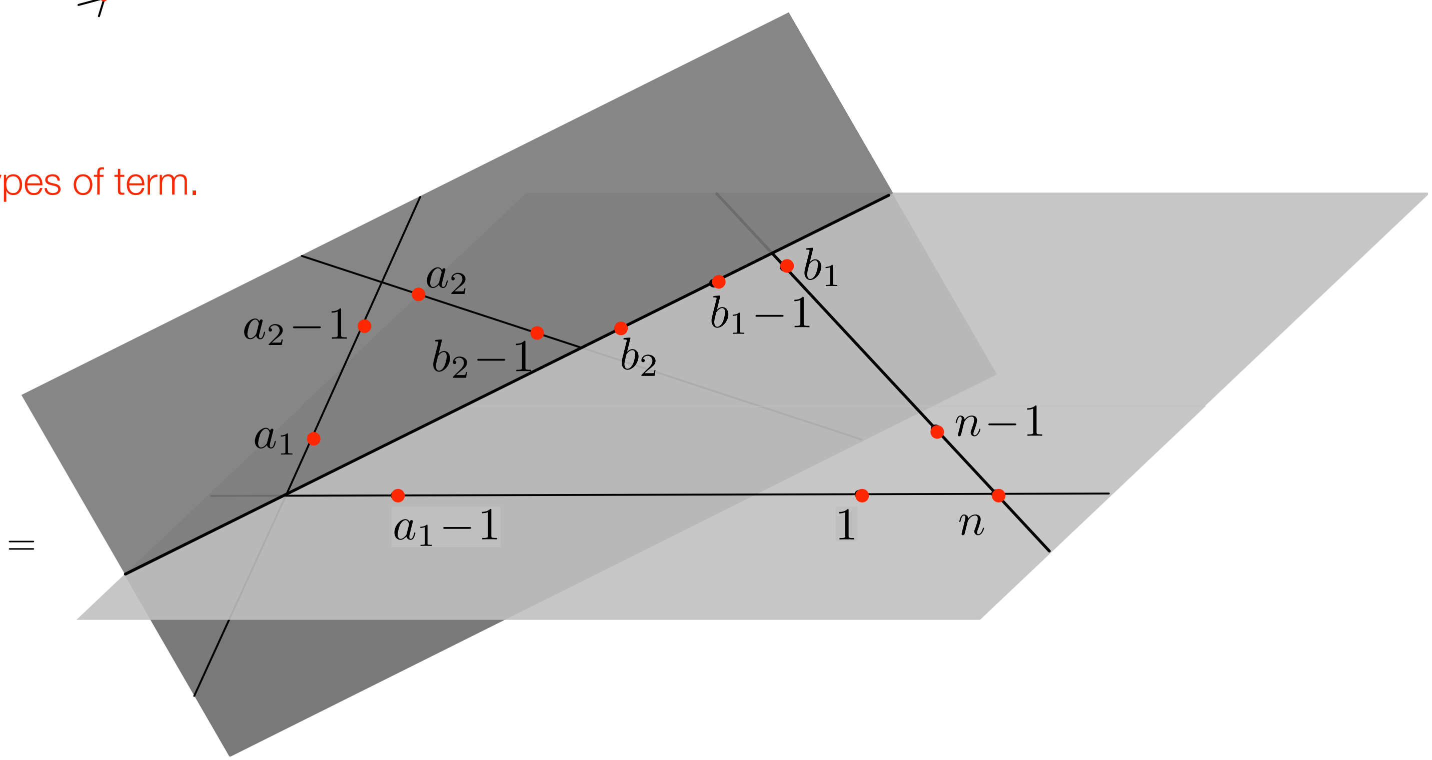


At N²MHV there are two types of term.

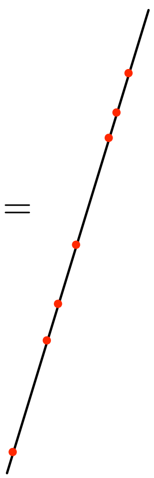
The first is

$\mathcal{A}^{(0)}_{\text{MHV}} R_{n;a_1 b_1} R^{0;a_1 b_1}_{n;b_1 a_1;a_2 b_2} =$

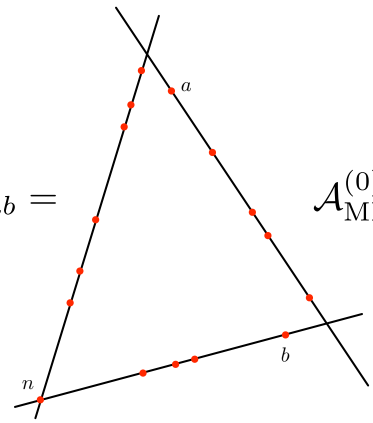


All tree amplitudes in twistor space

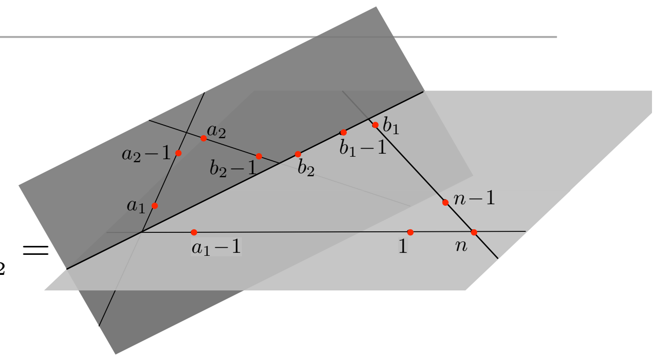
$\mathcal{A}^{(0)}_{\rm MHV} =$



$\mathcal{A}^{(0)}_{\rm MHV} R_{n;ab} =$



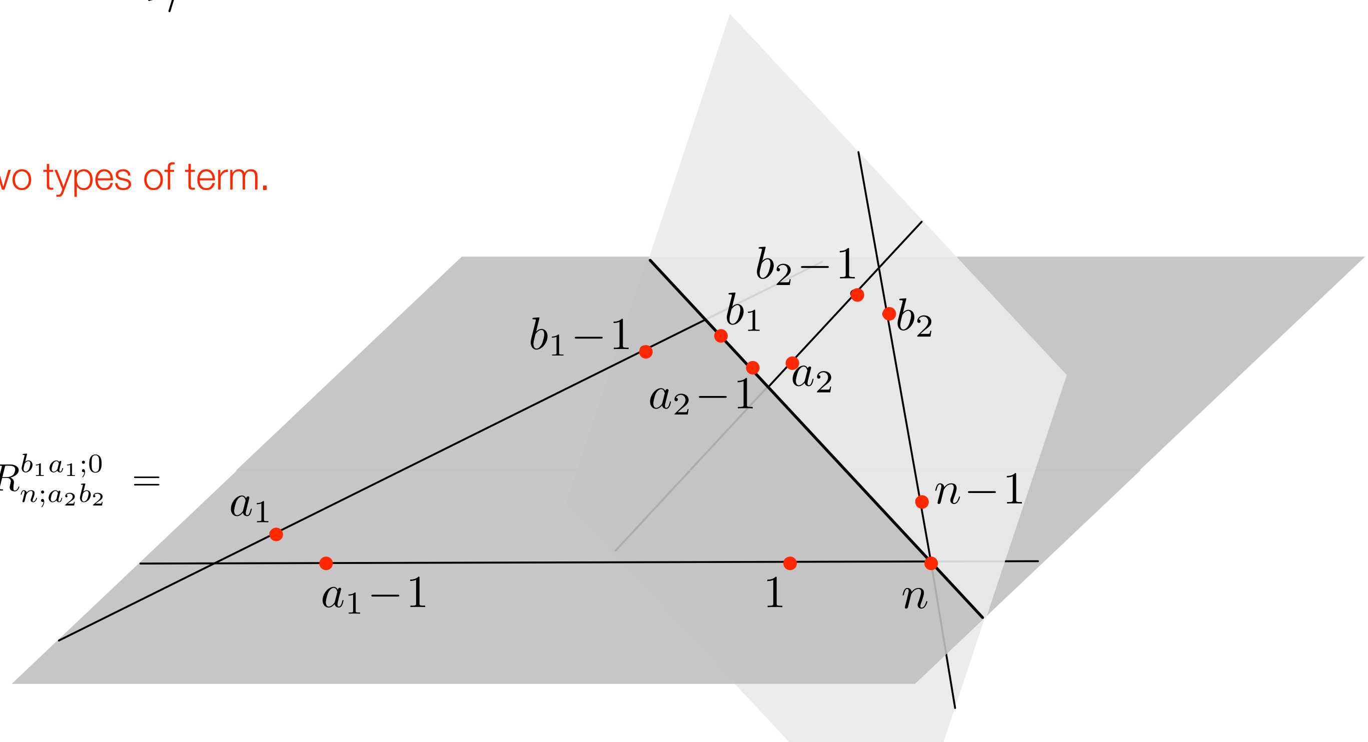
$\mathcal{A}^{(0)}_{\rm MHV} R_{n;a_1b_1} R^{0;a_1b_1}_{n;b_1a_1;a_2b_2} =$



At N[2]MHV there are two types of term.

while the second is

$\mathcal{A}^{(0)}_{\rm MHV} R_{n;a_1b_1} R^{b_1a_1;0}_{n;a_2b_2} =$



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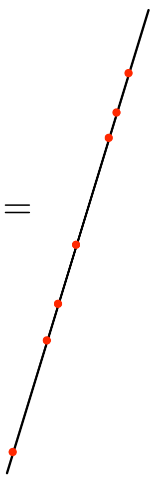

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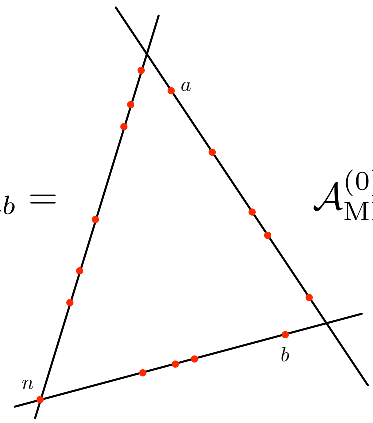

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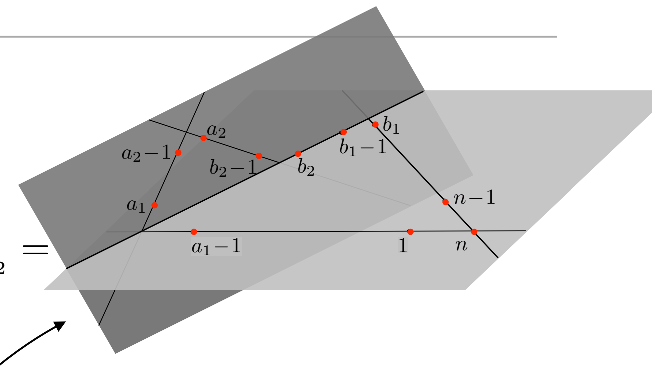

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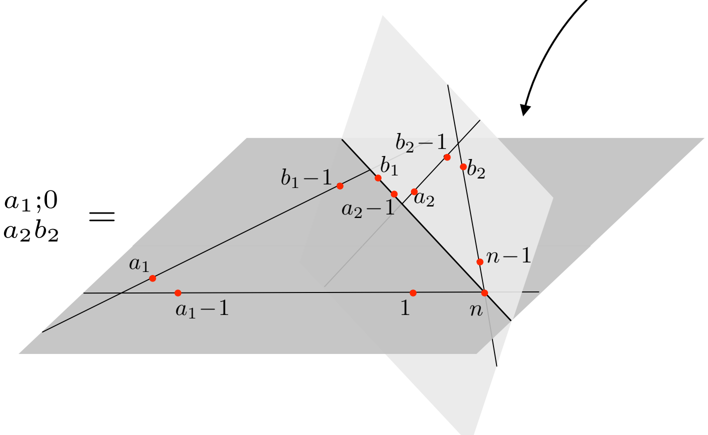

These N^2MHV terms localise on (reducible) degree 5, genus 2 curves, again in agreement with the twistor-string prediction

$$d = k - 1 + g$$

for two loops.

The general story holds: each term of an N^{k-2}MHV tree amplitude localises on a degree

$$d = 2k - 1$$

curve of genus $g = k$.

Tree amplitudes & multi-loop leading singularities

Why should these contributions to the tree amplitudes know anything about multi-loops?
Because they're really leading singularities!

For example, consider the N^{2}MHV term

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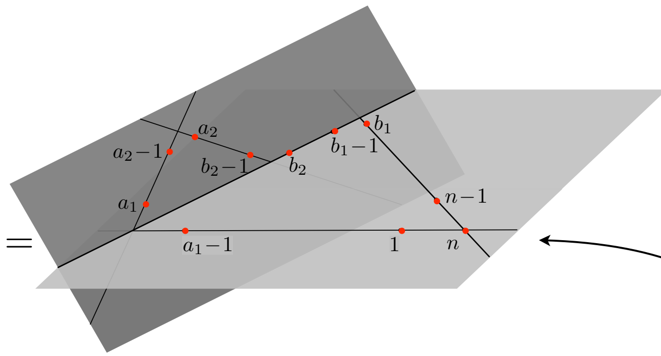
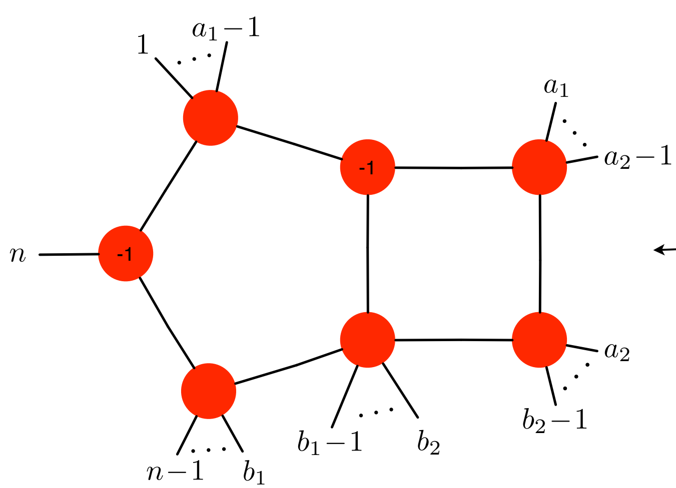


Intersecting lines in twistor space imply null separation in (possibly complex) space-time

The twistor support tells us which channel to consider in momentum space:



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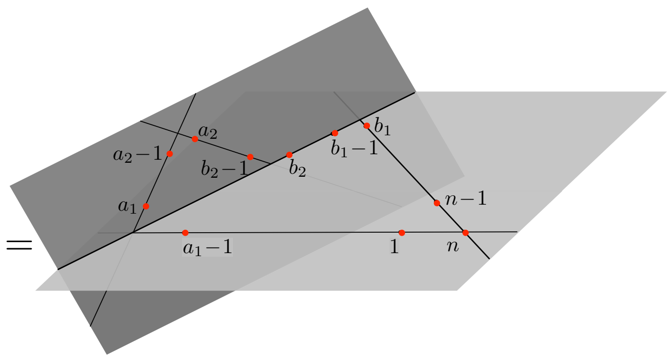

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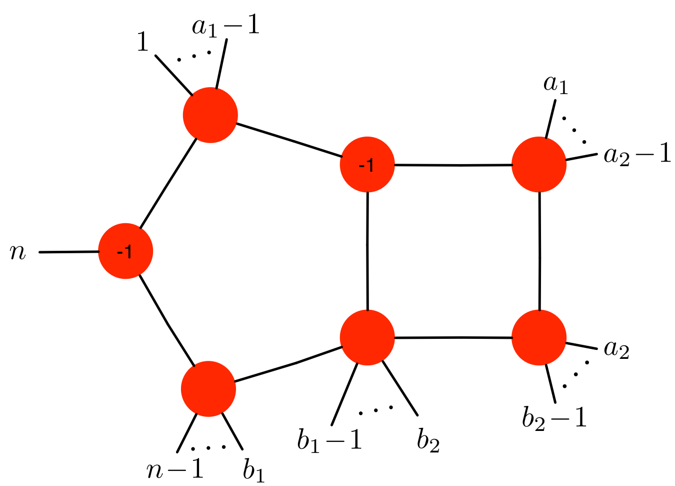
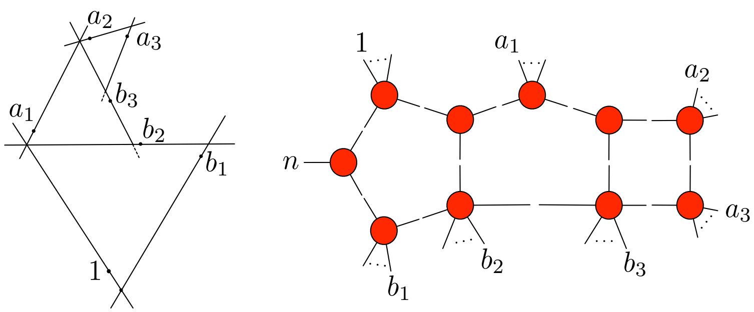
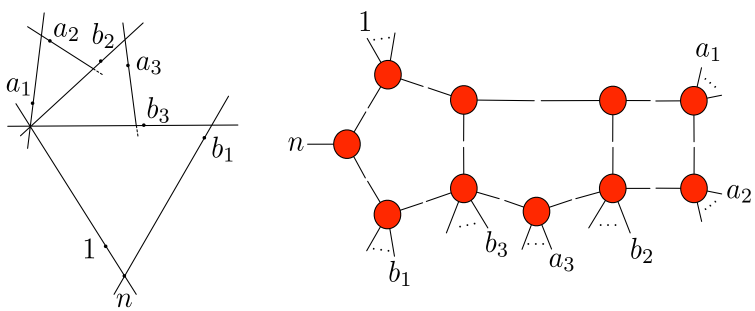


It's easy to check directly what this leading singularity actually is, and one indeed recovers $\mathcal{A}^{(0)}_{\mathrm{MHV}} R_{n;a_1 b_1} R^{0;a_1 b_1}_{n;b_1 a_1;a_2 b_2}$.

Here is the twistor support of each term contributing to the n-particle N^{3}MHV tree. Once again, each one has its own identity as a leading singularity of the 3-loop N^{3}MHV amplitude in the displayed channel in momentum space.



$$= \mathcal{A}^{(0)}_{\rm MHV} \times R_{n;a_1 b_1} R^{0;a_1 b_1}_{n;b_1 a_1;a_2 b_2} R^{0;b_1 a_1 a_2 b_2}_{n;b_1 a_1 b_2 a_2 a_3 b_3}$$



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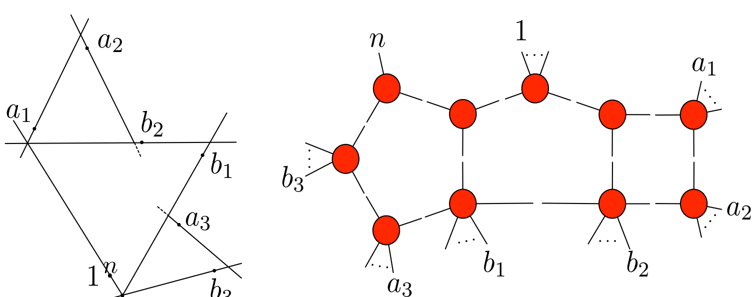
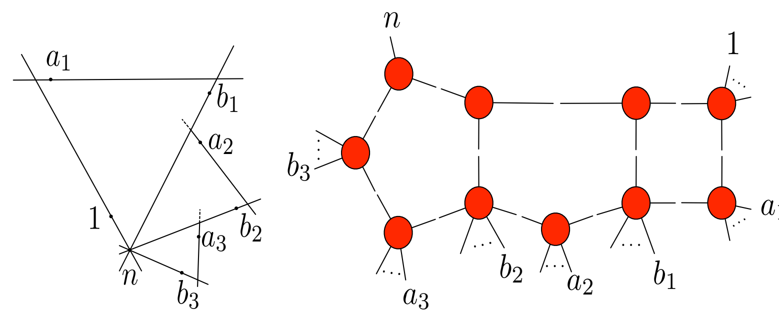


$$= \mathcal{A}^{(0)}_{\rm MHV} \times R_{n;a_1 b_1} R^{0;a_1 b_1}_{n;b_1 a_1;a_2 b_2} R^{a_1 b_1;0}_{n;a_3 b_3}$$



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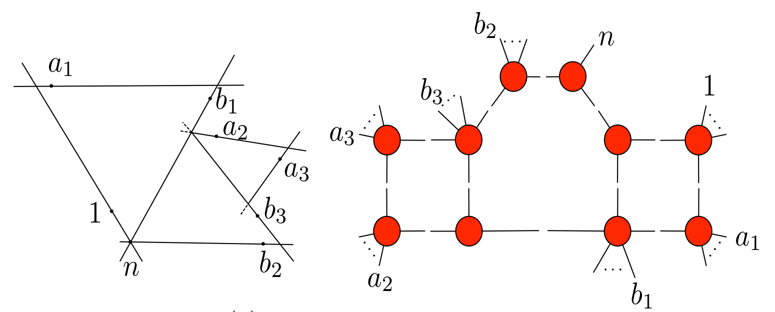

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A Grassmannian interlude

The Grassmannian conjecture[Arkani-Hamed, Cachazo, Cheung, Kaplan] states that *all leading singularities of planar N^{k-2}MHV amplitudes (at arbitrary loop order) can be obtained as residues of the contour integral*

$$\oint \frac{\mathrm{D}^{k(n-k)}C}{(1,2,\ldots,k)(2,3,\ldots,k+1)\cdots(n,1,\ldots,k-1)}\left[\int\prod_{r=1}^{k}\mathrm{d}^{4|4}Y_r\prod_{i=1}^{n}\delta^{4|4}(W_i - C_{ri}Y_r)\right]$$

around a contour localising on some codimension $(k-2)(n-k-2)$ cycle in $G(k,n)$.

$$\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{k1} & C_{k2} & \cdots & C_{kn} \end{pmatrix}$$

is a $k \times n$ matrix and defines a k-plane $C \subset \mathbb{C}^n$

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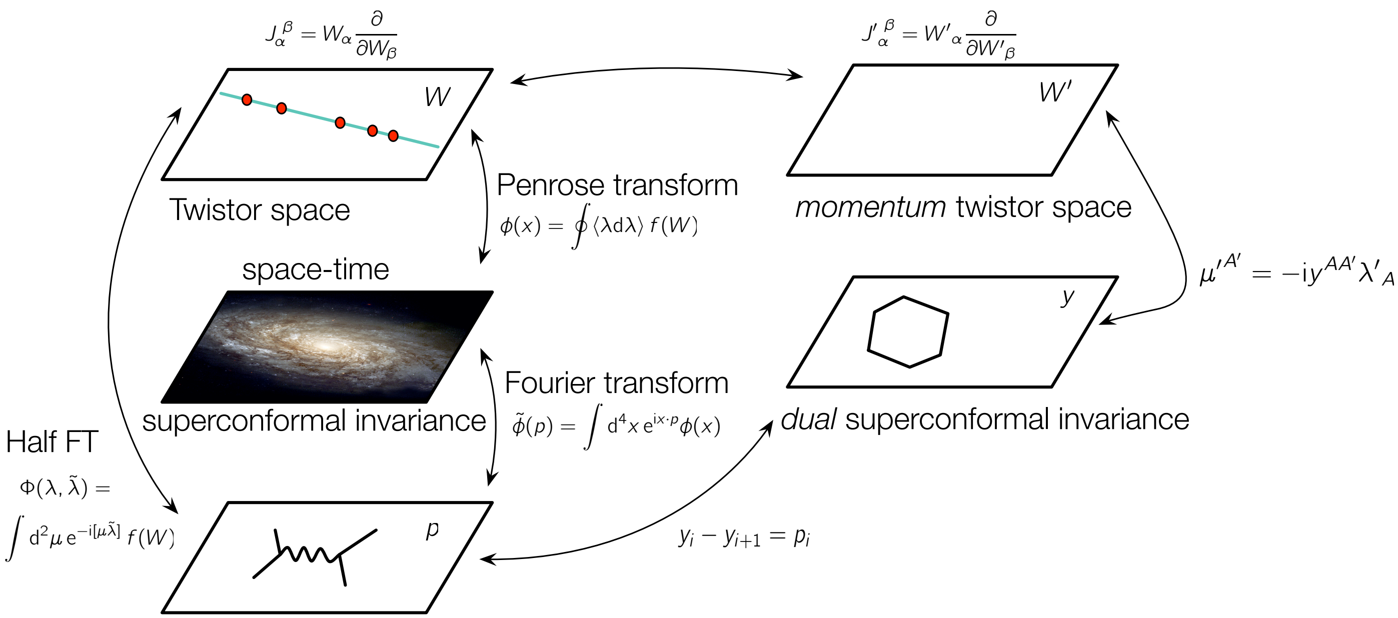
To see how it works, it's helpful to look at an analogous formula in *momentum* twistor space[Hodges; Mason, DS] where *dual* superconformal invariance is manifest.



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In particular, for NMHV we have $G(k,n) \to G(1,n) = \mathbb{P}^{n-1}$, so we should integrate

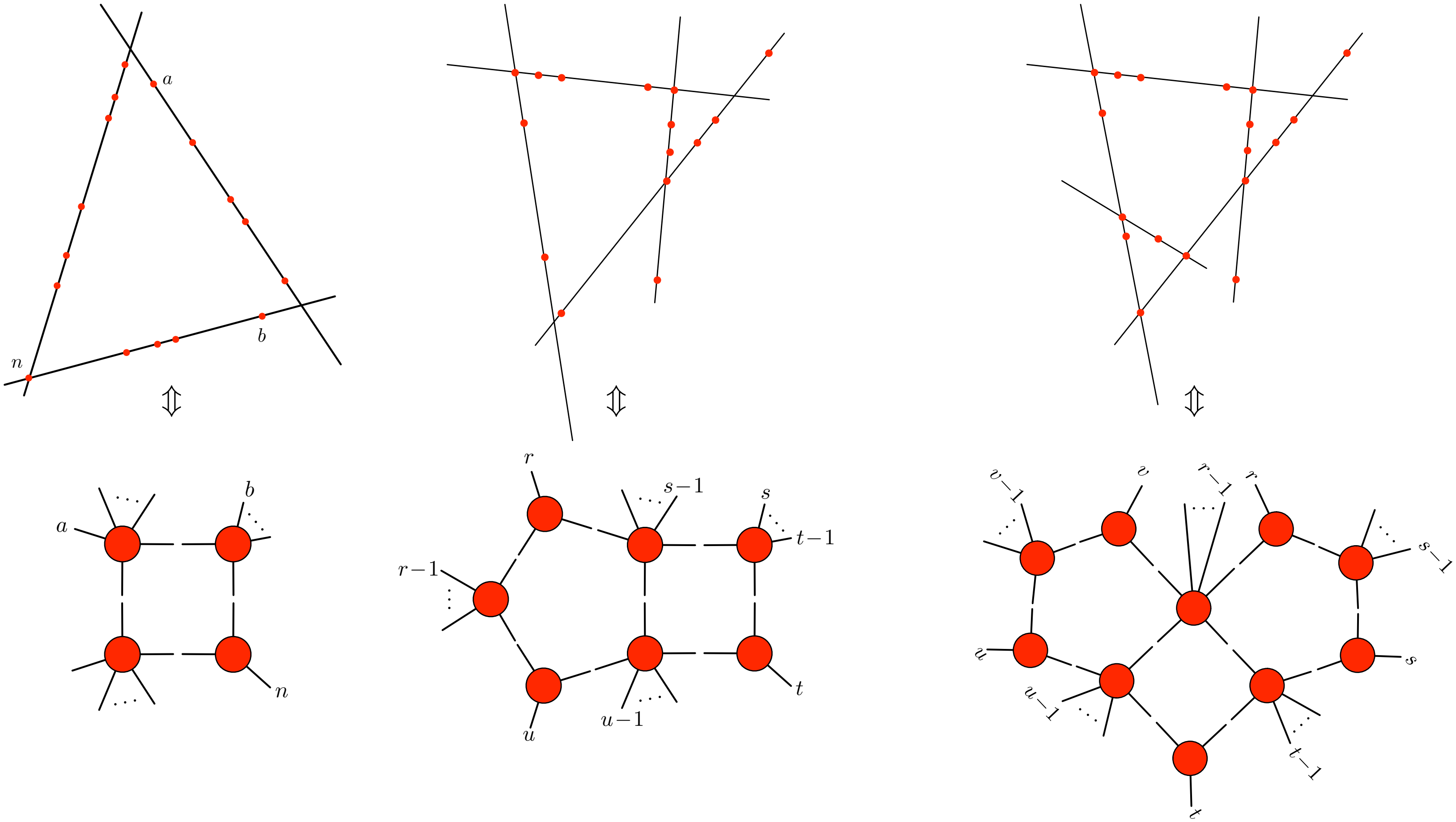
$$\oint_{\Gamma\subset\mathbb{P}^{n-1}} \frac{\mathrm{D}^{n-1}C}{C^1C^2\cdots C^n}\, \delta^{4|4}\left(\sum_{i=1}^{n} C^iW^i\right)$$ around an $(n-5)$-dimensional contour.

Each factor of the contour just sets one of the homogeneous coordinates to zero, so localises on a smaller projective space.

$$\int_{\mathbb{P}^4} \frac{\mathrm{D}^4C}{C^aC^bC^cC^dC^e}\, \delta^{4|4}(C^aW^a + \cdots + C^eW^e) \;=\; \frac{\delta^{0|4}(\chi^a\epsilon(b,c,d,e) + \text{cyclic})}{\epsilon(a,b,c,d)\epsilon(b,c,d,e)\epsilon(c,d,e,a)\epsilon(d,e,a,b)\epsilon(e,a,b,c)}$$

The Grassmannian provides a rich source (all?) of leading singularities.

For example, at NMHV every possible contour choice leads to one of the terms



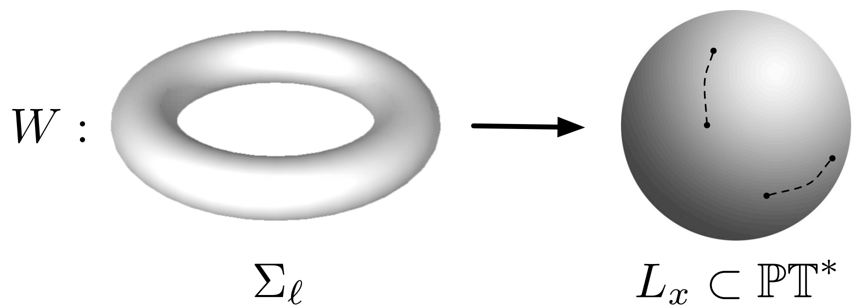
so all NMHV leading singularities are determined by the NMHV leading singularities at 3 loops (or 2 loops if $n < 10$, or 1 loop if $n < 7$).

Based on looking at the twistor support of “generic” residues in the Grassmannian, we think that all leading singularities of N^{p}MHV amplitudes are determined in terms of their leading singularities up to $3p$ loops (for $n \gg p$).

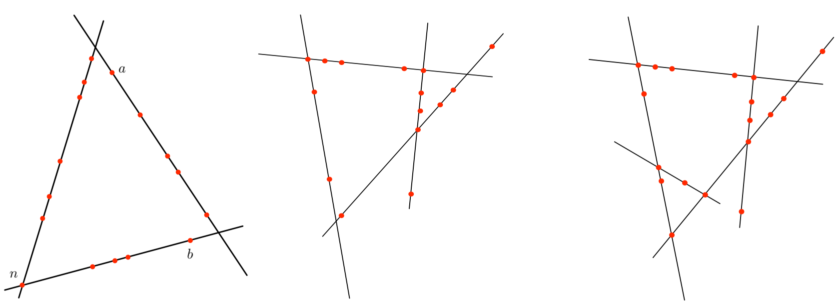
Higher loops and multiple covers

How can it be that higher-loop leading singularities are determined in terms of lower-loop ones when, for fixed N^pMHV, the degree of their twistor support $d = p + 1 + g$ depends on g?

Consider the MHV case. We expect 1-loop amplitudes to be associated with degree 2 maps from a genus 1 worldsheet. *There are no degree 2, genus 1 holomorphic curves in twistor space*, so (even away from the boundary of the moduli space) the image of this map must be a double cover of a line.



Likewise, the leading singularities of higher-loop amplitudes map onto the same twistor line configurations



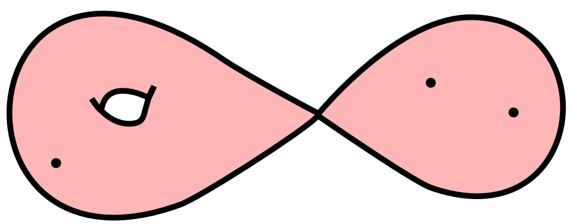
but the line components can each be multiply covered.

Leading singularities as stable maps

The intersecting line configurations we've seen are naturally interpreted as boundary components of the moduli space of stable maps.

The boundary components of this moduli space are specified by the *dual graph* of the source curve (worldsheet), together with a specification of the degree of the map on each irreducible component.

For example, represent



by

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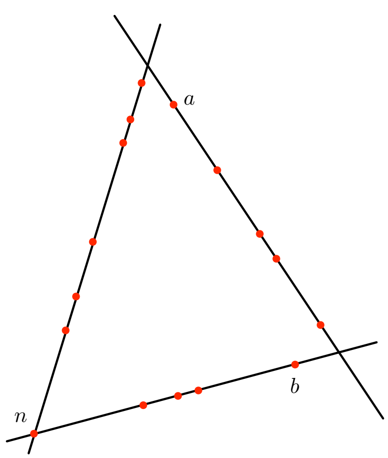
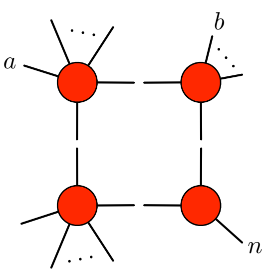
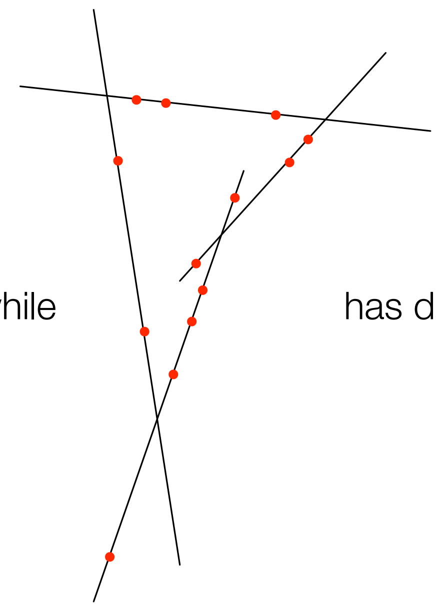
It's very revealing to draw these dual graphs (labelled by degrees) for maps whose image is a line configuration in twistor space corresponding to some leading singularity



has dual graph



while



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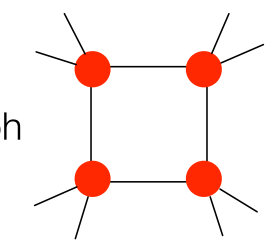

The momentum space leading singularity channels can equivalently be thought of as the dual graphs of the twistor-string worldsheet, illustrating the way in which the curve has become singular.

Twistor-strings revisited

Despite the failings of the original models, the fact that we're seeing exactly the algebraic curves expected by twistor-string theory - *even at loop level* - clearly means something's right.

But what?

$k \equiv h^0(\Sigma_g, \mathcal{O}(d))$

$$\int \mathrm{d}\mu \prod_{r=1}^{k} \mathrm{d}^{4|4} W_r \prod_{i=1}^{n} \mathrm{d}\sigma_i \; K(\sigma_{i+1}, \sigma_i) \; \mathrm{tr}\left(\mathrm{ev}_1^* A_1(W) \wedge \ldots \wedge \mathrm{ev}_n^* A_n(W)\right)$$

(free fermion) propagator at genus g

top meromorphic form on $\mathcal{L}_d \to \overline{M}_{g,n}$

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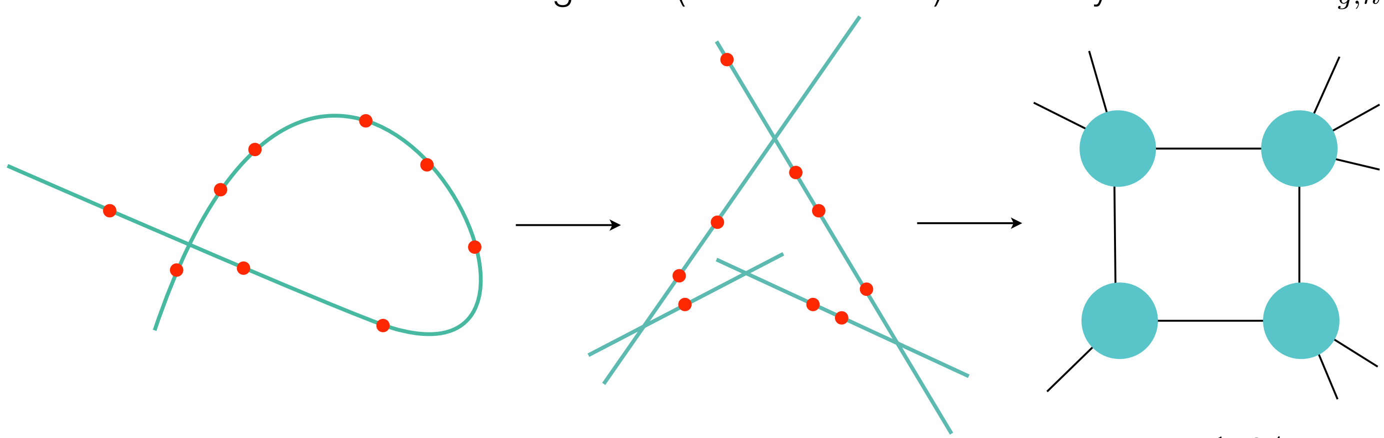
The path integral is to be treated as a contour integral. To extract leading singularities, we want to be able to choose a contour that localises the integral on (intersections of) boundary divisors in $\overline{M}_{g,n}(\mathbb{PT}^*, d)$.

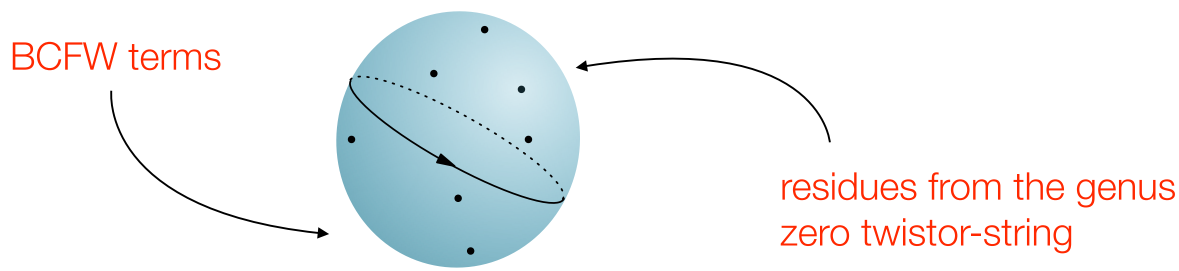


As in momentum space, this will be possible provided our contour contains an $(S^1)^{\otimes 4g}$, each factor of which encircles a boundary divisor, *and provided the integrand has a simple pole on these boundaries*.

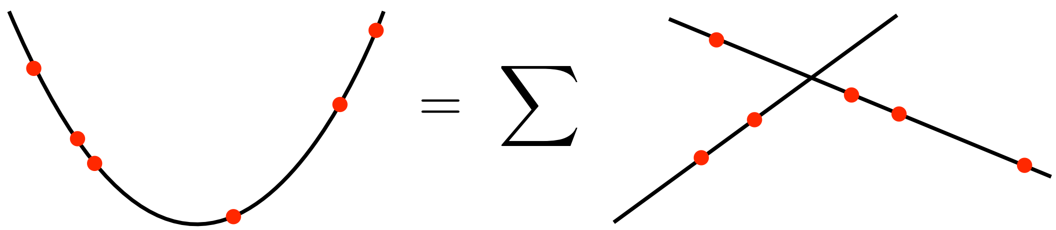
Conjecture: twistor-string theory actually gets all-loop leading singularities right.

Conclusions

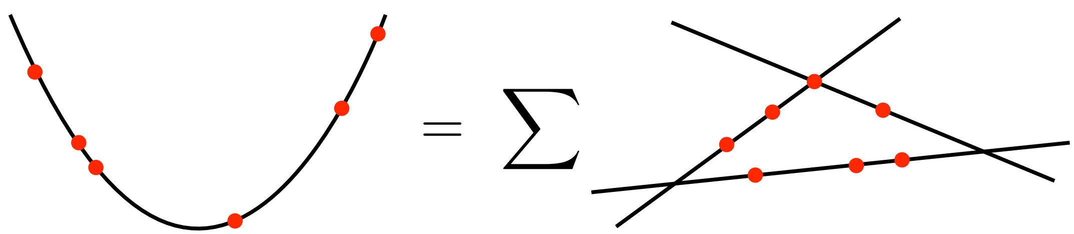
There's recently been much interest[Spradlin, Volovich; Dolan, Goddard] in studying the relation of twistor-string theory to the Grassmannian contour integral.



We propose that, unlike the conjectured equivalence[Gukov, Motl, Neitzke] of genus zero twistor-string theory to MHV diagrams



the equivalence to the Drummond & Henn form of the tree amplitudes is more naturally thought of as a story about degenerations of higher genus worldsheets.

The relations  are really IR equations.