Two aspects of gauge theories

higher-dimensional instantons on cones over Sasaki-Einstein spaces and Coulomb branches for 3-dimensional $\mathcal{N} = 4$ gauge theories

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Abstract

Solitons and instantons are crucial in modern field theory, which includes high energy physics and string theory, but also condensed matter physics and optics. This thesis is concerned with two appearances of solitonic objects: higher-dimensional instantons arising as supersymmetry condition in heterotic (flux-)compactifications, and monopole operators that describe the Coulomb branch of gauge theories in 2+1 dimensions with 8 supercharges.

In Part I we analyse the generalised instanton equations on conical extensions of Sasaki-Einstein manifolds. Due to a certain equivariant ansatz, the instanton equations are reduced to a set of coupled, non-linear, ordinary first order differential equations for matrix-valued functions. For the metric Calabi-Yau cone, the instanton equations are the Hermitian Yang-Mills equations and we exploit their geometric structure to gain insights in the structure of the matrix equations. The presented analysis relies strongly on methods used in the context of Nahm equations. For non-Kähler conical extensions, focusing on the string theoretically interesting 6-dimensional case, we first of all construct the relevant SU(3)-structures on the conical extensions and subsequently derive the corresponding matrix equations. For the Kähler-torsion sine-cone the problem is reduced to the Calabi-Yau cone by means of conformal invariance. For the nearly Kähler and half-flat 6-manifolds we derive new explicit solutions.

The extension of a quiver bundle construction, defined via equivariant dimensional reduction over product manifolds $M^d \times \mathrm{G/H}$ with Kähler cosets, to the case of the Sasaki-Einstein coset $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$ (and A-type orbifolds thereof) is considered in Part II. The construction of the quiver bundle via SU(3)-equivariant dimensional reduction over $M^d \times S^5$ is described in detail and turns out to be an extension of SU(3)-equivariant dimensional reduction over $M^d \times \mathbb{C}P^2$. The extension is manifest in vertex edge loops, associated to the Hopf fibration $\mathrm{U}(1) \hookrightarrow S^5 \to \mathbb{C}P^2$, and is exemplified with specific examples. As the arising quiver gauge theories are conceptually different from the *conventional* quiver gauge theories found in the physics literature, we aim to provide a first step in the comparison between these two. For that, we associate two different instanton moduli spaces on the Calabi-Yau orbifold $C(S^5/\Gamma)$: on the one hand, equivariant instantons, wherein the underlying structure of the gauge theory is fixed by the Sasakian quiver associated to S^5, and, on the other hand, translationally invariant instantons, which exhibit an appearance as moduli spaces of world volume gauge theories on D-branes located at orbifold singularities. We contrast the resulting quiver gauge theories, their relations, and construct the formal Kähler structure on the moduli space.

Recently, the Coulomb branch $\mathcal{M}_C$ of 3-dimensional $\mathcal{N} = 4$ gauge theories has experienced an algebraic description in terms of the *monopole formula* which is the Hilbert series associated to the chiral ring $\mathbb{C}[\mathcal{M}_C]$. In Part III we analyse the monopole formula and provide a geometric derivation of the set of chiral ring generators. The matter content divides the dominant Weyl chamber of the GNO-dual of the gauge group into a fan, via the weight vectors of the matter field representation. Upon intersection of the fan with the weight lattice, the monopole formula turns into a sum over affine semi-groups twisted by the Casimir invariance. This new understanding provides unprecedented insights into the combinatorics of the Hilbert series: we can explicitly sum the Hilbert series (in principle for any gauge group), we identify a sufficient set of bare monopole operators, we derive the number and degree of the corresponding dressed monopole operators, and we provide a clean approximation of the plethystic logarithm. This novel interpretation is tested against a comprehensive list of rank two examples and one rank three scenario.

key words: higher-dimensional instantons, monopole operators, gauge theories

Kurzdarstellung

Solitonen und Instantonen sind fundamentale Bestandteile moderner Feldtheorien, nicht nur in Hochenergiephysik oder Stringtheorie, sondern auch in der Theorie kondensierter Materie und Optik. In der vorliegenden Arbeit werden zwei Beispiele solcher solitonischen Objekte betrachtet: höher-dimensionale Yang–Mills-Instantonen, welche als Supersymmetriebedingungen in heterotischen (Fluss-)Kompaktifizierungen erscheinen, und Monopoloperatoren, welche für die Beschreibung des Coulomb-Zweiges von Eichtheorien in $2+1$ Dimensionen mit 8 Superladungen notwendig sind.

Der I. Teil der Arbeit ist der Analyse der verallgemeinerten Instantongleichungen auf kegelförmigen Erweiterungen von Sasaki–Einstein-Mannigfaltigkeiten gewidmet. Die Instantongleichungen können durch einen äquivarianten Ansatz auf ein System gekoppelter, nichtlinearer, gewöhnlicher Differentialgleichungen erster Ordnung für matrixwertige Funktionen reduziert werden. Für den Fall des metrischen Calabi–Yau-Kegels sind die Instantongleichungen identisch zu den hermiteschen Yang–Mills-Gleichungen, sodass wir deren geometrische Strukturen nutzen können, um die Geometrie der Matrixgleichungen zu untersuchen. Die in dieser Arbeit präsentierte Analyse basiert auf Methoden, die ihre Anwendung bereits bei den Nahm-Gleichungen gefunden haben. Im Falle von Kegelerweiterungen die nicht Kähler sind, wobei wir uns auf die stringtheoretisch interessanten 6-dimensionalen Fälle beschränken, konstruieren wir zuerst die relevanten SU(3)-Strukturen und leiten anschließend die korrespondierenden Matrixgleichungen her. Der dabei auftretende Fall Kähler-Torsion ist mittles konformer Invarianz äquivalent zum Calabi–Yau-Kegel. Für die nearly Kähler und halb-flachen 6-Mannigfaltigkeiten präsentieren wir neue explizite Lösungen.

Im Mittelpunkt des II. Teils steht die Übertragung einer Konstruktion von Köcherbündeln, welche ursprünglich durch die äquivariante dimensionale Reduktion auf Produkten $M^d \times \mathrm{G/H}$ mit einer Kähler-Faktorgruppe G/H definiert ist, auf den Fall der Sasaki-Einstein-Faktorgruppe $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$ und Orbifaltigkeiten derer. Die Konstruktion des Köcherbündels mittels SU(3)-äquivarianter dimensionaler Reduktion über $M^d \times S^5$ wird detailliert beschrieben und entpuppt sich als Erweiterung der SU(3)-äquivarianten dimensionalen Reduktion über $M^d \times \mathbb{C}P^2$. Markenzeichen der Erweiterung sind Vertexschleifen, welche zur Faserung $\mathrm{U}(1) \hookrightarrow S^5 \to \mathbb{C}P^2$ assoziiert sind, und wir verdeutlichen diesen Umstand anhand von Beispielen. Da diese so erzeugten Köchereichtheorien konzeptionell verschieden von den *gewöhnlichen* Köchereichtheorien der Hochenergiephysik sind, beabsichtigen wir einen ersten Vergleich zwischen diesen zu ziehen. Darum betrachten wir zwei verschiedene Instantonlösungsräume auf der Calabi–Yau-Orbifaltigkeit $C(S^5/\Gamma)$: zum einen den der äquivarianten Instantonen, deren zugrunde-liegende Eichtheorien strukturell durch die Sasaki-Köcher bestimmt sind, und zum anderen den der translationsinvarianten Instantonen, deren Lösungsraum eine Realisierung als Weltvolumeneichtheorien auf D-Branen findet. Für diese beiden Fälle stellen wir die resultierenden Köchereichtheorien sowie deren Relationen gegenüber und konstruieren die formale Kählerstruktur auf den Lösungsräumen.

Der Coulomb-Zweig $\mathcal{M}_C$ von 3-dimensionalen $\mathcal{N}=4$ Eichtheorien wurde kürzlich algebraisch durch die *Monopolformel* beschrieben, welche der Hilbertreihe des chiralen Rings $\mathbb{C}[\mathcal{M}_C]$ entspricht. Im III. Teil analysieren wir die Monopolformel und liefern eine geometrische Herleitung einer hinreichenden Menge von Generatoren des chiralen Rings. Der Materiegehalt der Theorie, genauer gesagt die Gewichtsvektoren der Darstellung in welcher die Materie transformiert, zerlegt die fundamentale Weylkammer des GNO-Dualen der Eichgruppe in einen Fächer. Nach Schnitt des Fächers mit dem Gewichtsgitter kann die Monopolformel als Summe über affine Halbgruppen verstanden werden, welche noch zusätzlich mit der Casimir-Invarianz nicht-trivial verflochten wird. Dieses neuartige Verständnis erlaubt weitere Einblicke in die Kombinatorik der Hilbertreihe: Man kann die Hilbertreihe explizit summieren (im Prinzip für einen beliebige

Eichtheorie), einen hinreichenden Satz von nackten Monopoloperatoren identifizieren, sowie die Anzahl und die Dimensionen der bekleideten Monopoloperatoren herleiten. Außerdem erlaubt dieses Wissen eine saubere Näherung des plethystischen Logarithmus. Die entwickelte neue Interpretation wird mittels einer umfassende Liste von Beispielen mit Eichgruppen vom Rang zwei und einer Gruppe vom Rang drei verifiziert sowie verdeutlicht.

Schlagworte: verallgemeinerte Instantonen, Monopoloperatoren, Eichtheorien

Contents overview

1 Introduction

Modern day physics rests on two pillars: general relativity and quantum field theory, both of which have been experimentally verified with remarkable precision. Relativistic quantum field theories, on the one hand, provide a valid approximation of microscopic processes at energies currently accessible in experiments. The Standard Model (SM) of Particle Physics has been completed by the discovery of its last missing ingredient — the Higgs boson [1, 2]. The SM is one of the experimentally most well-verified theories and is based on a spontaneously broken $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ gauge theory. The Standard Model covers three fundamental interactions called: electromagnetic, weak, and strong. On the other hand, general relativity (GR) successfully describes the macroscopic processes at cosmological scales and has found a beautiful verification by the recent direct experimental detection of gravitational waves [3]. GR encodes our knowledge of the fourth fundamental interaction: gravity. Nonetheless, the unification of all fundamental interactions, i.e. a consistent quantum theory of gravity, has yet to been found and is a driving force behind much of current higher energy physics research.

String theory started as the attempt to model hadronic resonances, but has been discarded shortly after due to undesired (phenomenological) implications and the formulation of quantum chromodynamics as superior description. Nevertheless, string theory turned out to be a quantum theory of gravity, having the potential to overcome the shortcomings of any ordinary quantum field theory. Pushed by two superstring revolutions, string theory developed into an active field of research and initiated an unprecedented fruitful flow of ideas between physics and mathematics.

1.1 Motivation

Let us first recollect some historic milestones of the developments of string theory and highlight their implications for gauge theories. A detailed account on the history of string theory can, for example, be found in [4]. Then, we focus on equations which are naturally associated to gauge theories and indicate their appearance in string theory.

1.1.1 A brief history of string theory

The original bosonic string theory was flawed by the necessity of 26 dimensions, the prediction of tachyons, and the absence of fermions. Ramond [5] introduced the superstring in 1971, which allowed to reduce the dimensionality to 10 and made the theory tachyon-free. The appearance of massless spin-2 particles in the vibrational modes was the nail in the coffin of any hopes that string theory could describe only strong interactions. However, these are precisely the properties of the hypothetical graviton. Although Schwarz and Scherk [6] subsequently discovered that string theory itself is a quantum theory of gravity, the field entered a decade of low attention and interest.

First superstring revolution In 1984, Green and Schwarz [7] proved that anomalies in string theories cancel; thus, rendering it consistent. More excitement arose as Gross, Harvey, Martinec, and Rohm [8–10] discovered the heterotic string in 1985. Due to the intrinsic gauge field and the large gauge groups $E_8 \times E_8$ or $\mathrm{SO}(32)$, many believed the final Grand Unified Theory (GUT) of everything was imminent.

The physics community was well-equipped with five superstring theories in 10 dimensions at their disposal — type I, IIA, IIB, heterotic $E_8 \times E_8$, and heterotic SO(32). At the same time, a plaguing question arose: which one of them provides the correct description of Nature?

Second superstring revolution Witten [11] proposed in 1995 that all of the five superstring theories were different limits of one 11-dimensional supergravity theory, called M-theory. This conjecture relies heavily on so-called dualities, which — put simply — relate different theories that describe the same physics. Shortly after, Polchinski [12] realised that higher dimensional objects, called D-branes, can and have to be included in string theory.

Such branes introduced a much richer machinery for constructing gauge theories in lower dimensions. One method, for instance, is *geometrical engineering* [13] in which branes wrap supersymmetric cycles such that the world-volume theory is a dimensionally reduced gauge theory inherited from the original D-brane. Here, the geometry of the homological cycles (together with the wrapped branes) determines the gauge group and matter content. Another method, the so-called *Hanany-Witten setups* [14], concerns world-volume theories for configurations of branes. The benefit of this approach lies in the direct translation of simple geometric quantities, such as distances and bendings, into physical quantities, like coupling constants and beta functions. Lastly, a method called *brane probes*, initiated by [15], exploits stacks of D-branes located at (isolated) orbifold singularities to deduce gauge theories that differ from simple $\mathrm{U}(N)$-theories. Depending on the choice of orbifold group Γ with respect to the R-symmetry group, i.e. $\Gamma \subset \mathrm{SU}(2)$, $\mathrm{SU}(3)$, or $\mathrm{SU}(4)$, the world-volume theory has $\mathcal{N} = 2$, 1, or 0 supersymmetries, respectively. As a remark, all of these three methods are equivalent under a sequence of dualities.

In 1997, Maldacena [16] conjectured the AdS/CFT or gauge/gravity correspondence, which would be an exact equivalence between a conformally invariant quantum field theory and a superstring theory. The archetypal example states the equivalence between $\mathcal{N}$=4 $\mathrm{SU}(N)$ super Yang-Mills in 4 dimensions and type IIB superstring theory on the 10-dimensional product $\mathrm{AdS}_5 \times S^5$. Ever since this astonishing conjecture much effort has been devoted in testing and exploiting this correspondence.

Related mathematics In the course of string theory developments, some remarkable mathematics has been unveiled. One of the most famous string theory concepts to mathematicians probably is *mirror symmetry*, which associates to almost any Calabi-Yau 3-fold M another Calabi-Yau 3-fold W such that the cohomology groups satisfy $H^{p,q}(M) = H^{3-p,q}(W)$. In particular, it implies that the Hodge-diamond of W is the flipped Hodge-diamond of M. Consequently, it is conjectured that type IIA string theory compactified on M is dual to type IIB string theory compactified on W; in other words, both are isomorphic string theories that give rise to the same physics [17, 18]. Even more is true: a Calabi-Yau manifold M allows to introduce two moduli spaces $\mathcal{M}^{1,1}$, $\mathcal{M}^{2,1}$ associated to Kähler and complex structure deformations. For a mirror pair (M, W) these moduli spaces are identified as $\mathcal{M}^{1,1}(M) = \mathcal{M}^{2,1}(W)$ and $\mathcal{M}^{1,1}(W) = \mathcal{M}^{2,1}(M)$.

Another incidence of stringy mathematics occurred in the McKay correspondence and higher dimensional generalisations thereof. Dixon, Harvey, Vafa, and Witten [19, 20] proposed a string partition function which resolved orbifolds and predicted the Euler characteristic of orbifolds. It was then realised that the DHVW proposal is an important ingredient for crepant resolutions of $\mathbb{C}^3/\Gamma$, for a finite subgroup $\Gamma \subset \mathrm{SL}(3, \mathbb{C})$, as well as for the associated McKay correspondence.

1.1.2 Gauge theoretic equations

There are several instances in which gauge theoretic equations, first discovered in physics and later formalised in mathematics, appear in string theory. Alternatively, one could just call them soliton equations, as we will encounter instantons, monopoles, and vortices as solutions. For a review of the D-brane realisations of such solitons, see [21] and references therein.

Yang-Mills instantons Yang-Mills instantons are (anti-)self-dual connections $\mathcal{A}$ on a G-principal bundle over a 4-dimensional Riemannian manifold M^4, i.e. their defining property is $\mathcal{F}_{\mathcal{A}} = \pm \star \mathcal{F}_{\mathcal{A}}$, with $\star$ the Hodge star on M^4. They were first found as finite action solutions on 4-dimensional Euclidean space compactified to the 4-sphere [22, 23]. The finiteness of the action has to be ensured by boundary conditions at $\partial\mathbb{R}^4 \cong S^3_\infty$, which then allows a classification of solutions by homotopy classes in $\pi_3(\mathrm{G})$. These give rise to the instanton number (or charge), which also equals the Pontryagin number. Moreover, the study of (anti-)self-dual connections led to a classification of 4-manifolds, as demonstrated by Donaldson [24]. By the seminal work of Atiyah, Drinfeld, Hitchin, and Manin [25] the construction of instanton solutions, for all classical groups, on $\mathbb{R}^4$ has been turned into an algebraic problem.

The so-called ADHM construction later experienced a realisation in D-brane dynamics [26, 27]. Considering Dp-branes inside D(p+4)-branes, the Dp-branes are co-dimension 4 objects and dissolve into instantons for the world-volume gauge fields of the D(p+4)-branes. From the perspective of the gauge theory on the Dp-brane, the Higgs branch of the moduli space then corresponds to the moduli space of instantons of the D(p+4) gauge group. In addition, moduli spaces of anti-self dual connections on $\mathbb{C}^2$ appear as Higgs branches [28–30] or Coulomb branches [31–33] of, for example, 3-dimensional $\mathcal{N} = 4$ gauge theories.

Generalised Yang-Mills instantons, which have been first introduced in [34], appear frequently in string theory. Most prominently, the heterotic compactifications require the gauge field to satisfy the Hermitian Yang-Mills (HYM) equations. By the seminal work of Donaldson [35, 36], and Uhlenbeck and Yau [37] solutions to the HYM equations are equivalent to (poly-)stable holomorphic bundles. Hence, the problem has been transformed from a partial differential equation into an algebo-geometric formulation.

Monopoles A (magnetic) monopole is mathematically given by a solution of the Bogomolny equations, but can also be considered as dimensional reduction of an instanton. For a G-monopole on a Riemannian 3-manifold M^3 the data of a monopole consists of a connection $\mathcal{A}$ on a G-principal bundle over M^3 and a section Φ of the adjoint bundle, called Higgs field, such that they satisfy the Bogomolny equation $\mathcal{F}_{\mathcal{A}} = \star D_{\mathcal{A}}\Phi$ with certain asymptotic conditions Φ_∞ on the Higgs field. These boundary conditions topologically classify the solutions via homotopy classes $[\Phi_\infty] \in \pi_2(\mathrm{G}/\mathrm{T})$, wherein T is a maximal torus of G. The construction is valid for all (classical) Lie groups G, and monopoles are an active research topic in mathematics, see for instance [38–41]. Remarkably, Nahm [38] provided a description of the monopole moduli space in terms of matrix equations, now known as Nahm equations.

The first finite energy solutions in a non-abelian gauge theory have been found independently by 't Hooft [42] and Polyakov [43]. The 't Hooft-Polyakov monopole is a smooth solution in contrast to the previously found singular Dirac monopole [44].

Surprisingly, monopoles and Nahm equations appear also in type IIB string theory. There, the dynamics of N D1-branes stretching between two separated D3-branes is described by the Nahm equations [45, 46].

In addition, magnetic monopoles also appear in the description of moduli spaces of supersymmetric gauge theories. More precisely, the description of the Coulomb branch of 3-dimensional $\mathcal{N} = 4$ gauge theories relies on 't Hooft monopole operators [47, 48], which lead to an insertion of a particular monopole singularity in the path integral.

Vortex equations Geometrically, vortices are absolute minima of the Yang-Mills-Higgs functional

$$S[\mathcal{A},\phi] = \int_{\mathbb{R}^2} \left(|\mathcal{F}_{\mathcal{A}}|^2 + |d_{\mathcal{A}}\phi|^2 + \frac{1}{4}\left(1-|\phi|^2\right)^2 \right) \ ,$$

for a unitary connection $\mathcal{A}$ and a smooth section ϕ of a Hermitian line bundle over $\mathbb{R}^2$. Identifying $\mathbb{R}^2$ with the complex plane, the minimum is attained if and only if $\bar{\partial}_{\mathcal{A}}\phi = 0$ and $\mathcal{F}_{\mathcal{A}} = \frac{1}{2} \star (1-|\phi|^2)$, which are the vortex equations. In analogy to instantons and monopoles, the topological charge (or vortex winding number) is now given by the equivalence class of the solution in the first homotopy group. In 1950, vortices were first introduced in [49] in the study of superconductivity, and an extensive treatment can be found in [50].

Interestingly, the setting has been generalised to vortex equations on Kähler manifolds [51,52] and on Kähler products $M^{2n} \times \mathrm{G/H}$ in the context of quiver bundles in [53–55]. Moreover, the vortex equations are a dimensional reduction of the HYM equations, which is a generalisation of the statement that the (ordinary) vortex equations on $\mathbb{R}^2$, i.e. $n = 1$, are a dimensional reduction of the (anti-)self-dual connections on $\mathbb{R}^4$. Furthermore, the vortex equations for $n = 2$ are the Seiberg-Witten monopole equations [56].

1.2 Thesis overview

This thesis is divided into three (fairly self-contained) parts, each devoted to a different aspect of gauge theories related to string theory.

Part I is centred around the generalised Yang-Mills instanton equation which arises in heterotic (flux-)compactifications to 4-dimensional theories with $\mathcal{N} = 1$ supersymmetry. Attention will be paid to three particular points: (i) We recall known conical extensions as well as explicitly construct several new conical extensions of Sasaki-Einstein manifolds, mostly 5-dimensional ones, which are of interest for flux compactifications. (ii) The instanton equation on the Calabi-Yau cone will be treated rather mathematically. We will study the moduli space of certain equivariant instantons on the Calabi-Yau cone over arbitrary Sasaki-Einstein manifolds. An equivariant ansatz for the gauge field reduces the Hermitian Yang-Mills equations to a set of matrix equations, for which we describe the geometric properties. (iii) In contrast, we establish explicit instanton solutions on 6-manifolds which are manifestly non-Kähler; thus, they might be interesting testing grounds for flux compactifications.

A different perspective is taken in Part II, in which we construct quiver gauge theories associated to the simplest 5-dimensional Sasaki-Einstein coset S^5 (and certain A-type orbifolds thereof). We first provide a detailed account of the construction of so-called quiver bundles, before we compare two different instanton moduli spaces. On the one hand, these moduli spaces describe the (non-trivial) vacua of the quiver gauge theories; while, on the other hand, they can be motivated from two different settings: (i) equivariant instantons on conical extensions, as known from the heterotic BPS-equations, and (ii) translationally invariant instantons on an orbifold, which resembles the space of classical vacua for D-branes probing orbifold singularities. Therefore, the comparison is between two moduli spaces originating from different, well-motivated phenomena.

Lastly, we explore the Coulomb branch of 3-dimensional $\mathcal{N} = 4$ gauge theories with rank two gauge groups in Part III. The techniques employed are complementary to the previous two parts, as the tools mostly borrow from algebraic geometry. We will show that the computation of the Hilbert series by means of the *monopole formula* can be greatly simplified by our novel interpretation, which also sheds light on the combinatorics of the monopole formula. The main idea relies on the insight that the conformal dimension generates a fan in the Weyl chamber of the GNO-dual group. Upon intersection with the (magnetic) weight lattice, the Hilbert basis is a unique, finite set of generators for each affine semi-group. Employing this, we provide an explicit expression of the Hilbert series for any rank two gauge group. In addition, the number and the degrees of dressed monopole operators in non-abelian theories can be understood by ratios of functions accounting for the Casimir invariance. We then supplement these findings by a comprehensive list of examples in which we demonstrate the novel interpretation.

Part I

Instantons on cones over Sasaki-Einstein manifolds

Contents

2 Introduction and motivation

Solitons and instantons are important objects in modern field theory [57–59]. For example, solitons in supergravity theories are branes of various dimensions, which describe non-perturbative states of the underlying string theories or M-theory [60–62]. Yang-Mills instantons, on the other hand, can appear as co-dimension four objects in brane systems [26, 27] or, as we will explore in this part, as supersymmetry condition in heterotic string compactifications.

Let us review compactifications of the heterotic string to theories in 4-dimensional spacetime with $\mathcal{N} = 1$ supersymmetry.

2.1 Compactifications of the heterotic string

Kaluza-Klein compactifications of the heterotic string on Calabi-Yau 3-folds have been the simplest attempt to reduce the 10 dimensions of the heterotic superstring to the four physical and to break some of the original 16 supersymmetries. The heterotic string is phenomenologically particularly interesting as it contains a gauge field even without D-branes, and its gauge group $E_8 \times E_8$ or SO(32) is large enough to accommodate the Standard Model gauge group as well as GUT-groups thereof [63].

Heterotic supergravity, as a low-energy effective field theory, preserves supersymmetry in 10 dimensions precisely if there exists at least one globally defined and nowhere-vanishing Majorana-Weyl spinor ϵ such that the supersymmetry variations of the fermionic fields (gravitino λ, dilatino ψ, and gaugino ξ) vanish, i.e. the so-called BPS equations

$$\delta_\lambda = \nabla^+ \epsilon = 0 \ , \tag{2.1a}$$

$$\delta_\psi = \gamma \left(\mathrm{d}\phi - \frac{1}{2} H \right) \epsilon = 0 \ , \tag{2.1b}$$

$$\delta_\xi = \gamma \left(\mathcal{F}_\mathcal{A} \right) \epsilon = 0 \tag{2.1c}$$

hold, wherein $\gamma(\omega) = \frac{1}{p!} \omega_{i_1 \dots i_p} \Gamma^{i_1 \dots i_p}$ is the Clifford map for a p-form ω. The bosonic field content is given by the metric g, the dilaton ϕ, the 3-form H, and the gauge field $\mathcal{A}$. Further, ∇^+ is a metric-compatible connection with torsion H.

The BPS equations (2.1) have to be supplemented by the α'-corrected Bianchi identity

$$\mathrm{d}H = \frac{\alpha'}{4} \left[\mathrm{tr} \left(R \wedge R \right) - \mathrm{tr} \left(\mathcal{F}_\mathcal{A} \wedge \mathcal{F}_\mathcal{A} \right) \right] \tag{2.2}$$

due to the Green-Schwarz anomaly cancellation mechanism [7]. Here R is the curvature of a connection ∇ on the tangent bundle[1].

The 10-dimensional space is assumed to be a product $\mathbb{M}^{p-1,1} \times M^{10-p}$, where M^{10-p} is a $(10-p)$-dimensional internal manifold and $\mathbb{M}^{p-1,1}$ is a Lorentzian manifold of signature $(p-1, 1)$. Then (2.1a) translates into the existence of an covariantly constant spinor ϵ_d on M^d, with d=$(10-p)$. Moreover, a globally defined nowhere-vanishing spinor exists only on manifolds M^d with reduced structure group (i.e. a G-structure), which in $d = 6$ amounts to an SU(3)-structure. Then a metric-compatible connection, which leaves ϵ_6 parallel and is also compatible with the

[1] Different choices for ∇, such as ∇^+, are mentioned in [64].

SU(3)-structure, always exists, but possibly has torsion. In other words, a connection with SU(3)-holonomy always exists on SU(3)-manifolds, but in general it differs from the Levi-Civita connection. As a consequence, manifolds with special holonomy or G-structure are essential in string theory compactifications. Moreover, assuming that the external space $\mathbb{M}^{p-1,1}$ is maximally symmetric plus employing the existence of a covariantly constant spinor on it, reduces the external space to a Minkowski space.

2.1.1 Calabi-Yau compactifications

Assuming maximal symmetry along $\mathbb{M}^{3,1}$ imposes the vanishing of any component of the NS-NS 3-form H and the field strength $\mathcal{F}_{\mathcal{A}}$ which are purely in $\mathbb{M}^{3,1}$ or which are mixed between external and internal space. Most importantly, if one further *assumes* that $H = 0$ in the internal space and a constant dilaton ϕ, the BPS-equations (2.1) for $\mathcal{N} = 1$ unbroken supersymmetry imply that M^6 is a Calabi-Yau manifold.

In such compactifications, it is then necessary to specify a 6-dimensional Calabi-Yau manifold as well as a Hermitian Yang-Mills (HYM) instanton on a gauge bundle over that manifold [63], because (2.1c) is precisely the HYM instanton condition on the gauge field $\mathcal{A}$. For compact Calabi-Yau 3-folds M^6 this is equivalent to specifying a (poly-)stable holomorphic vector bundle of M^6 due to [35–37].

Nevertheless, the Bianchi identity (2.2) has to be satisfied. As $\mathrm{d}H$ is exact, $\mathrm{tr}(\mathcal{F}_{\mathcal{A}} \wedge \mathcal{F}_{\mathcal{A}})$ and $\mathrm{tr}(R \wedge R)$ belong to the same cohomology class, and coincide for $\mathrm{d}H = 0$. We recall: R takes values in the Lie algebra of the holonomy group SU(3), while $\mathcal{F}_{\mathcal{A}}$ takes values in the Lie algebra of the gauge group $E_8 \times E_8$ or SO(32). Hence, a simple way to solve (2.2) (for $\mathrm{d}H = 0$) is to specify an SU(3)-subgroup of the gauge group. This is usually called *standard embedding* of the spin connection into the gauge connection $\mathcal{A}$, i.e. the gauge bundle over the Calabi-Yau 3-folds M^6 is taken to be the tangent bundle TM^6. Among the multitude of embeddings of SU(3) in, say, $E_8 \times E_8$ only some are admissible [63]; one, for instance, is SU(3) embedded in one E_8 factor with commutant E_6. The resulting unbroken gauge group[2] in 4 dimensions is $E_6 \times E_8$. This scenario offers two intriguing features: E_6 has been proposed as a grand unification group for the Standard Model. In addition, the remaining E_8 gauge factor can comprise a *hidden sector*, which is desirable for inclusion of dark matter. Nevertheless, the standard embedding is of limited usefulness from the phenomenologically point of view [62, 65], because it limits considerations to E_6 GUTs, and very few Calabi-Yau 3-folds with the right features are known.

However, the geometry of the internal space is not completely fixed by the BPS-equations, as Betti numbers do not uniquely characterise a Calabi-Yau space. The deformations of the internal space are parametrised by numbers called *moduli*, which characterise the shape and size. The moduli give rise to effective scalar fields in the low-energy effective theory. If the H-flux vanishes, there is no potential for these fields in the effective theory. As these massless scalar fields are unrestrained, they can have arbitrary magnitudes, but these moduli control the size of the internal manifold. Consequently, the entire idea of compactifications is spoiled by massless moduli, at least for vanishing H-flux. Nevertheless, it is remarkable that the requirement of $\mathcal{N} = 1$ supersymmetry determines the internal geometry (almost) entirely.

2.1.2 Flux compactifications

Allowing for a non-vanishing H-flux has two implications for the background geometry. Firstly, the 10-dimensional space M^{10} becomes a warped product in the *Einstein frame* with warping factor $\exp(2\phi)$, which again becomes a direct product in the *string frame*. Secondly, the non-vanishing 3-form still renders the internal space M^6 into a complex manifold, but generically

[2]The corresponding unbroken gauge group for SO(32) is SO(26) × U(1), which turns out to be unrealistic [63].

not Kähler any more. Moreover, the 3-form can be interpreted geometrically as torsion of the internal manifold.

String vacua with p-form fields along the extra dimensions (*flux compactifications*) have been intensively studied in recent years, we refer to [66, 67] for a review and references. In particular, heterotic flux compactifications have been known for quite some time, starting from [68–71] in the mid-1980s, and have been considered, for example, in [72–85]. The introduction of fluxes partially resolves the vacuum degeneracy problem by giving masses to problematic moduli, but they lead to non-integrable SU(3)-structures (i.e. with intrinsic torsion) on the internal compact 6-manifolds. Among these manifolds there are six-dimensional nearly Kähler and half-flat manifolds [72–76, 82–85].

On a different note, branes are sources of p-form flux fields. They can also wrap various supersymmetric cycles of special holonomy manifolds [62], and these cycles (which are calibrated submanifolds [86]) are calibrated via the p-form fluxes. Thus, fluxes play an important role in the compactification of low-energy string theories and M-theory.

The BPS-equations for the generic case of heterotic compactifications down to $\mathcal{N} = 1$ in 4 dimensions are also known as the *Strominger system*, based on [71]. Again, the curvature $\mathcal{F}_\mathcal{A}$ of a connection $\mathcal{A}$ on a gauge bundle has to satisfy the Hermitian Yang-Mills equations (2.1c), but the choice of the gauge bundle is restricted by the Bianchi identity (2.2) for $\mathrm{d}H \neq 0$.

By a theorem of Ivanov [64], a solution to the BPS equations (2.1) and the Bianchi identity (2.2) satisfies the heterotic equations of motion if and only if the connection ∇ is an SU(3)-instanton in $d = 6$. In other words, R and $\mathcal{F}_\mathcal{A}$ are treated on the same footing in a pure supergravity point of view, i.e. $\gamma(\mathcal{F}_\mathcal{A})\epsilon = \gamma(R)\epsilon = 0$. Therefore, in the spirit of [84, 85, 87, 88], we will study the instanton equation (2.1c) for non-integrable SU(3)-structures in order to provide an important ingredient for full heterotic supergravity solutions[3].

2.2 Cones and G-structures

The construction of metric cones and sine-cones over manifolds M^d with a G-structure provides a tool to generate and link different G′-structures on (d+1)-dimensional manifolds. Most prominently, Sasaki-Einstein 5-manifolds generate a Calabi-Yau structure on their metric cone and a nearly Kähler structure on their sine-cone. Recently, the study of Sasaki-Einstein manifolds [89–93] has lead to infinitely many explicit metrics on (non-compact) Calabi-Yau cones. Since there are no explicit Ricci-flat metrics known on compact Calabi-Yau manifolds, metric cones over Sasaki-Einstein spaces provide a testing ground for Calabi-Yau compactifications.

Focusing on non-Kähler backgrounds for flux compactifications, a generalisation of Sasaki-Einstein 5-manifolds is provided by hypo geometry, in particular hypo, nearly hypo and double hypo SU(2)-structures; see for instance [94]. Double hypo structures lift to nearly Kähler as well as to half-flat SU(3)-structures on the sine-cone. The described *linking* phenomenon is well-known from the cases of cylinders, cones and sine-cones over nearly Kähler 6-manifolds, which lead to different G_2-manifolds [95]. Here, we use these techniques in order to construct 6-dimensional manifolds with special SU(3)-structures that may be valuable, for example, in flux compactifications of the heterotic string.

Supergravity in d=10 dimensions allows for brane solutions which interpolate between an $\mathrm{Ad}S_{p+1} \times M^{9-p}$ near-horizon geometry and an asymptotic geometry $\mathbb{R}^{p-1,1} \times C(M^{9-p})$, where $C(M^{9-p})$ is a metric cone over M^{9-p} (see e.g. [96, 97] and references therein). These brane solutions in heterotic supergravity with Yang-Mills instantons on the metric cones $C(M^{9-p})$

[3]Choosing a different connection ∇, for example ∇^+, the BPS equations together with the Bianchi identity imply the heterotic equations of motion only up to higher α'-correction. This yields a perturbative solution in contrast to the exact solution advocated above.

have been considered in [84,85,98]. Here, we take the first step to generalise them by considering sine-cones with Kähler-torsion and nearly Kähler structures as well as cylinders with half-flat structures instead of metric cones with Kähler structures.

2.3 Generalised instanton condition

As remarked earlier, the curvature $\mathcal{F}_\mathcal{A}$ of a connection $\mathcal{A}$ on a gauge bundle has to satisfy the generalised instanton equation (2.1c). Instantons have proven to be interesting both for mathematicians and physicists. Starting from work [24] by Donaldson, anti-self-dual Yang-Mills connections provided a new topological invariant for four-manifolds. The generalisation of Yang-Mills instantons to higher dimensions ($d{>}4$) was first proposed in [34] and further studied in [35–37,99–106] (see also references therein). Some solutions for $d{>}4$ have been found, namely Spin(7)-instantons on $\mathbb{R}^8$ in [107,108] and G_2-instantons on $\mathbb{R}^7$ in [109–111]. Although recently the moduli space of contact instantons has been discussed in [112], the moduli spaces of higher-dimensional instantons are still not fully understood.

In particular, the instanton equation can be introduced on any manifold with a G-structure. On manifolds M^d with integrable G-structure, instantons have two crucial features. Firstly, they solve the Yang-Mills equation (without torsion), and, secondly, the Levi-Civita connection on TM^d already is an instanton. For generic non-integrable G-structures, the instanton equation implies the Yang-Mills equation with torsion. However, as shown in [84], on manifolds with real Killing spinors the corresponding instantons solve the Yang-Mills equation without torsion even if the G-structure has non-vanishing intrinsic torsion.

In studying generalised instantons on conical extensions of Sasaki-Einstein manifolds M^{2n+1}, we follow the approach outlined in [113], which extends the results of [84]. As a consequence, the instanton equations have been reduced by an equivariant ansatz to a set of matrix equations. This set of equations comprises three types of equations: (i) an SU(n+1)-equivariance condition, (ii) a holomorphicity condition, and (iii) a remaining equation that strongly depends on the type of SU(n+1)-structure on the conical extension.

One of the main topics in this part is the moduli space of the reduced instanton matrix equations on the Calabi-Yau cone of an arbitrary Sasaki-Einstein manifold. Instantons on Calabi-Yau cones and their resolutions have also been studied in [114,115] and, for the particular orbifolds $\mathbb{C}^n/\mathbb{Z}_n$, in [116]. However, the settings and ansätze considered there are different: on the one hand, the authors of [114,115] considered instantons on the *tangent bundle* of a $(2n{+}2)$-dimensional Calabi-Yau cone whose structure was largely determined by the $2n$-dimensional Einstein-Kähler manifold underlying the Sasaki-Einstein manifold in between. The ansatz for the connection was adapted to the isometry of the Calabi-Yau cone, and the *seed* was the spin connection in the Einstein-Kähler space, which is an instanton. On the other hand, certain gauge backgrounds for heterotic compactifications were constructed in [116] by extending a flat connection on $\mathbb{C}P^{n-1}$ to U(1) and U(n−1)-valued instanton connections on the orbifolds. In contrast, the approach in [113], which is extended here, can conceptually take *any* instanton on the Sasaki-Einstein manifold as a starting point, and the gauge bundle does not need to be the tangent bundle anymore.

2.4 Outline

The outline of the first part of this thesis is as follows: Ch. 3 is devoted to a review of Sasaki-Einstein structures in $2n{+}1$, SU(2)-structures in 5, and SU(3)-structures in 6 dimensions. By means of cone constructions we obtain the relevant geometries in one dimension higher. We then take two perspectives: We recall the geometry of HYM-instantons and argue in Ch. 4 that the moduli space of the instanton matrix equations on the CY-cone is a finite dimensional Kähler space obtained by a Kähler quotient. This part will be detached from the physical application, as we only study the mathematical properties of these equivariant instantons. In contrast, Ch. 5 focuses on the search for explicit instanton solutions on conical 6-manifolds that are promising backgrounds for heterotic flux compactifications.

The contents of this part stem from collaborations [117, 118] with S. Bunk, T. Ivanova, O. Lechtenfeld, and A.D. Popov, and from the publication [119] of the author.

3 Geometry

In this chapter we lay the foundations for Part I and II: Sasaki-Einstein geometry is introduced as well as the relevant G-structures on various conical extensions in one dimension higher.

3.1 Sasakian geometry

Sasakian geometry can be understood as odd-dimensional analogue of Kähler geometry. In particular, an odd-dimensional manifold M^{2n+1} with a Sasakian structure is naturally sandwiched between two different types of Kähler geometry in the neighbouring dimensions $2n$ and $2n{+}2$.

Following [120], a Sasakian manifold M^{2n+1} carries a Sasakian structure comprised of the quadruplet $\mathcal{S} = (\xi, \eta, \Phi, g)$, wherein ξ is the Reeb vector field, η the dual contact 1-form, $\Phi \in \mathrm{End}(TM^{2n+1})$ a tensor, and g a Riemannian metric. The defining property for $(M^{2n+1}, \mathcal{S})$ to be Sasakian is that the metric cone $(C(M^{2n+1}), \widehat{g}) = (\mathbb{R}^+ \times M^{2n+1}, \mathrm{d}r^2 + r^2 g)$ is Kähler, i.e. the holonomy group of the Levi-Civita connection on the cone is $\mathrm{U}(n{+}1)$. The (compatible) complex structure J_c on the cone acts via $J_c(r\,\partial_r) = \xi$ and $J_c(X) = \Phi(X) - \eta(X)\, r\,\partial_r$ for any vector field X on M^{2n+1}. The corresponding Kähler 2-form is $\frac{1}{2}\mathrm{d}(r^2\eta)$.

Moreover, considering the contact subbundle $\mathcal{D} = \ker(\eta) \subset TM^{2n+1}$ one has a complex structure defined by restriction $J_t = \Phi|_{\mathcal{D}}$ and a symplectic structure $\mathrm{d}\eta$. Hence, $(\mathcal{D}, J_t, \mathrm{d}\eta)$ defines the transverse Kähler structure [120].

A *Sasaki-Einstein manifold* is Sasakian and Einstein simultaneously; thus, the defining property is that the metric cone is Calabi-Yau, i.e. the holonomy group on the cone is reduced to $\mathrm{SU}(n{+}1)$.

For the purposes of this thesis, it is convenient to understand a Sasaki-Einstein manifold M^{2n+1} in terms of an $\mathrm{SU}(n)$-structure. To this end, consider the 2-form ω defined by $\mathrm{d}\eta = -2\omega$. One can always choose a co-frame $\{e^\mu\} = (e^a, e^{2n+1})$, with $\mu = 1, 2, \ldots, 2n+1$ and $a = 1, 2, \ldots, 2n$, such that these forms are locally given by

$$\eta = e^{2n+1} \quad \text{and} \quad \omega = e^1 \wedge e^2 + e^3 \wedge e^4 + \ldots + e^{2n-1} \wedge e^{2n} \equiv \frac{1}{2}\omega_{ab} e^{ab} \tag{3.1}$$

and the metric reads

$$g = \delta_{\mu\nu} e^\mu \otimes e^\nu = \delta_{ab} e^a \otimes e^b + \eta \otimes \eta \ . \tag{3.2}$$

Moreover, there exists a canonical connection Γ^P on TM^{2n+1} which is metric-compatible, is an instanton with respect to the $\mathrm{SU}(n)$-structure, and has non-vanishing torsion[4]. The torsion components are given by [84]

$$T^{2n+1}_{a\,b} = -2\omega_{ab} \quad \text{and} \quad T^{b}_{a\,2n+1} = \frac{n+1}{n}\omega_{ab} \ . \tag{3.3}$$

Remarks Besides Sasaki-Einstein structures there exist various other Sasakian structures that are of similar interest. For example, 3-Sasakian manifolds M^{4d+3} are manifolds whose

[4] The torsion components can be related to the components of the 3-form $P = \eta \wedge \omega$; hence, the name Γ^P. However, the torsion is not completely antisymmetric itself.

metric cone is hyper-Kähler, i.e. its holonomy is a subgroup of $\mathrm{Sp}(d{+}1)$. Hyper-Kähler spaces will play a central role in Part III.

3.2 Calabi-Yau metric cone

One result that makes Sasaki-Einstein manifolds interesting for string theorists as well as mathematicians is that their metric cones are Calabi-Yau. First of all, let us recall the basic properties of a Calabi-Yau manifold M^{2n+2}: as a Calabi-Yau space is Kähler, one has the Kähler form, which is a closed $(1,1)$-form on M^{2n+2}. In addition, the Calabi-Yau condition enforces the canonical bundle to be trivial, i.e. $K_{M^{2n+2}} = \Lambda^{(n+1,0)}T^*M^{2n+2} \cong M^{2n+2} \times \mathbb{C}$. Thus, there exists a nowhere vanishing section in $K_{M^{2n+2}}$ which translates into an $(n{+}1,0)$-form on M^{2n+2}.

The metric on the metric cone $(C(M^{2n+1}), \widehat{g})$ over the Sasaki-Einstein manifold M^{2n+1} is defined as

$$\widehat{g} = \mathrm{d}r^2 + r^2 g = e^{2t}\left(\mathrm{d}t^2 + \delta_{\mu\nu} e^{\mu} \otimes e^{\nu}\right) = e^{2t}\widetilde{g} \ , \tag{3.4}$$

where the last equality employs a conformal rescaling $r = e^t$ from the metric cone with cone coordinate $r \in \mathbb{R}^+$ to the cylinder $(\mathrm{Cyl}(M^{2n+1}), \widetilde{g})$ with coordinate $t \in \mathbb{R}$. Also, we identify $\mathrm{d}t = e^{2n+2}$ and extend the index range $\hat{\mu} = 1, 2, \ldots, 2n+1, 2n+2$. The Kähler form $\widehat{\omega}$ on the cone is

$$\widehat{\omega} = r^2\omega + r\eta \wedge \mathrm{d}r = e^{2t}\left(\omega + \eta \wedge \mathrm{d}t\right) = e^{2t}\widetilde{\omega} \ , \tag{3.5}$$

which is again related to the fundamental $(1,1)$-form $\widetilde{\omega}$ on the cylinder. Next, we introduce a complexified basis on the cotangent bundle of $\mathrm{Cyl}(M^{2n+1})$ as follows

$$\theta^j = \mathrm{i}e^{2j-1} + e^{2j} \quad \text{and} \quad \bar{\theta}^j = -\mathrm{i}e^{2j-1} + e^{2j} \qquad \text{for} \quad j = 1, 2, \ldots, n+1 \ , \tag{3.6}$$

such that the metric and fundamental $(1,1)$-form read

$$\widetilde{g} = \frac{1}{2}\sum_{j=1}^{n+1}\left(\theta^j \otimes \bar{\theta}^j + \bar{\theta}^j \otimes \theta^j\right) \quad \text{and} \quad \widetilde{\omega} = -\frac{\mathrm{i}}{2}\sum_{j=1}^{n+1}\theta^j \wedge \bar{\theta}^j \ . \tag{3.7}$$

The compatible complex structure J acts via $J\theta^j = \mathrm{i}\theta^j$ and $J\bar{\theta}^j = -\mathrm{i}\bar{\theta}^j$, such that the compatibility relation is $\widetilde{\omega}(\cdot,\cdot) = \widetilde{g}(\cdot, J\cdot)$.

Let us compare the choice (3.6) with the *canonical choice* $\theta^j_{\mathrm{can}} = e^{2j-1} + \mathrm{i}e^{2j}$ and the canonical complex structure $J_{\mathrm{can}}\theta^j_{\mathrm{can}} = \mathrm{i}\theta^j_{\mathrm{can}}$. The conventions used here correspond to $J = -J_{\mathrm{can}}$ such that the $(1,0)$ and $(0,1)$-forms are interchanged, which implies that $\widetilde{\omega}(\cdot,\cdot) = \widetilde{g}(J_{\mathrm{can}}\cdot,\cdot) = -\widetilde{g}(\cdot, J_{\mathrm{can}}\cdot) = \widetilde{g}(\cdot, J\cdot)$ is consistent with the above. The reasons for this choice are that we desire a resemblance to the treatment of [40, 121, 122], while at the same time we treat $\mathrm{d}t$ as the $(2n{+}2)$-th basis 1-form instead of the 0-th.

3.3 SU(2)-structures in 5 dimensions

Sasaki-Einstein 5-manifolds are a particular case of manifolds carrying an SU(2)-structures in 5 dimensions. Let us briefly summarise their definition and introduce some notation for later convenience.

Let M^5 be 5-manifold with an SU(2)-structure, i.e. the frame bundle of M^5 can be reduced to an SU(2) principal subbundle. It has been proven in [123] that an SU(2)-structure is determined by a quadruplet $(\eta, \omega^1, \omega^2, \omega^3)$ of differential forms, wherein $\eta \in \Omega^1(M^5)$ and $\omega^\alpha \in \Omega^2(M^5)$ for

$\alpha = 1, 2, 3$. These forms satisfy

$$\omega^\alpha \wedge \omega^\beta = 2\,\delta^{\alpha\beta} Q \tag{3.8}$$

for the 4-form $Q = \frac{1}{2}\,\omega^3 \wedge \omega^3$. We note that $\eta \wedge Q \neq 0$ holds.

Moreover, it has been shown in [123] that it is always possible to choose a local orthonormal coframe $e^1, \ldots, e^5$ of forms on M^5 such that

$$\eta = -e^5\,, \qquad \omega^1 = e^{23} + e^{14}\,, \qquad \omega^2 = e^{31} + e^{24}\,, \qquad \omega^3 = e^{12} + e^{34}\,. \tag{3.9}$$

By means of the 't Hooft symbols η^α_{ab}, see for instance [57], one can express the 2-forms as

$$\omega^\alpha = \tfrac{1}{2}\,\eta^\alpha_{ab}\, e^a \wedge e^b. \tag{3.10}$$

Here again $a, b = 1, 2, 3, 4$. Among the SU(2)-structures in 5 dimensions there are several types having particularly interesting geometry. We will now recall their definitions following [94].

Sasaki-Einstein A Sasaki-Einstein 5-manifold is a manifold carrying an SU(2)-structure defined by $(\eta, \omega^1, \omega^2, \omega^3)$, where these forms are subject to

$$\mathrm{d}\,\eta = 2\,\omega^3\,, \qquad \mathrm{d}\,\omega^1 = -3\,\eta \wedge \omega^2\,, \qquad \mathrm{d}\,\omega^2 = 3\,\eta \wedge \omega^1\,. \tag{3.11}$$

Hypo An SU(2)-structure on a 5-manifold is called *hypo* if

$$\mathrm{d}\,\omega^3 = 0\,, \qquad \mathrm{d}\Big(\omega^1 \wedge \eta\Big) = 0\,, \qquad \mathrm{d}\Big(\omega^2 \wedge \eta\Big) = 0 \tag{3.12}$$

holds true. Hypo geometry, therefore, is a generalisation of Sasaki-Einstein geometry.

Nearly hypo An SU(2)-structure on a 5-manifold is called *nearly hypo* if it satisfies

$$\mathrm{d}\,\omega^1 = -3\,\eta \wedge \omega^2\,, \qquad \mathrm{d}\left(\eta \wedge \omega^3\right) = 2\,\omega^1 \wedge \omega^1\,. \tag{3.13}$$

Note that any SU(2)-structure which satisfies the first two identities of (3.11) is a nearly hypo structure.

Double hypo An SU(2)-structure on a 5-manifold is called *double hypo* if it is hypo and nearly hypo simultaneously, i.e. if it satisfies (3.12) and (3.13). Thus, the Sasaki-Einstein 5-manifolds are a subset of the double hypo manifolds.

As shown in [123], SU(2)-structures in 5 dimensions always induce a nowhere-vanishing spinor on M^5. This will be generalised Killing if and only if the SU(2)-structure is hypo, and Killing if and only if the SU(2)-structure is Sasaki-Einstein. In [84] it has been argued that in the latter case there exists a one-parameter family of metrics

$$g_{M^5} = e^{2h} \delta_{ab}\, e^a \otimes e^b + e^5 \otimes e^5\,, \tag{3.14}$$

with the real parameter h. This family is compatible with an $\mathfrak{su}(2)$-valued connection on TM^5 for which the Killing spinor is parallel. For the special value $\exp(2h) = 4/3$ the torsion of that connection is totally antisymmetric and parallel with respect to that connection, i.e. there exists a canonical $\mathfrak{su}(2)$ connection. For all values of h however, this connection is an $\mathfrak{su}(2)$ instanton on TM^5 for the respective SU(2)-structure. For $h = 0$, M^5 is a Sasaki-Einstein manifold and the torsion components of the canonical connection are as follows:

$$T^a = \tfrac{3}{4} P_{a\mu\nu} e^{\mu\nu} \qquad \text{and} \qquad T^5 = P_{5\mu\nu} e^{\mu\nu} \qquad \text{for} \quad P = \eta \wedge \omega^3 \ . \tag{3.15}$$

Remark For the treatment of SU(2)-structures in $d = 5$ we have chosen a different sign in the definition (3.9) of η compared to (3.1), implying that (3.3) (for $n = 2$) and (3.15) differ.

3.4 SU(3)-structures in 6 dimensions

As pointed out in the introduction, one of our goals is the construction of SU(3)-structures on 6-dimensional manifolds. Therefore, we introduce these structures and their characterisation via intrinsic torsion classes. In a manner similar to Sec. 3.3, an SU(3)-structure on a 6-manifold M^6 is given by a reduction of the frame bundle to an SU(3)-subbundle. An SU(3)-structure on a 6-dimensional manifold M^6 is characterised in terms of a triple (J, ω, Ω), where J is an almost complex structure, ω a $(1,1)$-form, and Ω a $(3,0)$-form with respect to J. These are subject to the algebraic relations

$$\omega \wedge \Omega = 0 \quad \text{and} \quad \Omega \wedge \bar{\Omega} = -\tfrac{4\mathrm{i}}{3}\, \omega \wedge \omega \wedge \omega \ . \tag{3.16}$$

The compatible Riemannian metric is determined by $\omega(\cdot,\cdot) = g(J(\cdot),\cdot)$, and the $(3,0)$-form can be split into its real and imaginary part, i.e. $\Omega = \Omega^+ + \mathrm{i}\,\Omega^-$. By an appropriate choice of a local frame, these forms can always be brought into the form

$$\omega = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6 \quad \text{and} \quad \Omega = (e^1 + \mathrm{i}e^2) \wedge (e^3 + \mathrm{i}e^4) \wedge (e^5 + \mathrm{i}e^6). \tag{3.17}$$

For SU(3)-structures in 6 dimensions, there exist several types of such structures with different geometric behaviour, which is mostly governed by the differentials $\mathrm{d}\omega$ and $\mathrm{d}\Omega$. SU(3)-structures in 6 dimensions have been classified in terms of their five intrinsic torsion classes [124]. These are encoded in the differentials of the defining forms in the following manner:

$$\mathrm{d}\omega = \tfrac{3}{2}\, \Im\mathrm{m}\left(\left(W_1^+ - \mathrm{i}W_1^-\right)\Omega\right) + W_3 + W_4 \wedge \omega \ , \tag{3.18a}$$

$$\mathrm{d}\Omega = \left(W_1^+ + \mathrm{i}W_1^-\right)\omega \wedge \omega + \left(W_2^+ + \mathrm{i}W_2^-\right) \wedge \omega + \Omega \wedge W_5 \ . \tag{3.18b}$$

Here $W_1^\pm$ are real functions, W_4 and W_5 are real 1-forms, $W_2^\pm$ are the real and imaginary part of a $(1,1)$-form, respectively, and W_3 is the real part of a $(2,1)$-form. Note that both W_2 and W_3 are primitive forms [66], i.e.

$$\omega \lrcorner W_2 = 0 \quad \text{and} \quad \omega \lrcorner W_3 = 0 \ , \tag{3.18c}$$

where the contraction of two forms A, B is defined via $A \lrcorner B \coloneqq \star(A \wedge \star B)$, see for instance [125]. The Nijenhuis tensor gives rise to the components W_1 and W_2 such that the almost complex structure J of any SU(3)-structure with non-vanishing W_1 or W_2 is non-integrable [124].

We now list the structures of particular relevance for this thesis.

Kähler-torsion On any almost Hermitian manifold (M, g, J) there exists a unique connection preserving this structure and having totally antisymmetric torsion [126]. This connection is called the *Kähler-torsion* (KT) *connection* or *Bismut connection* [127]. KT 6-manifolds are characterised by their torsion, which is given by

$$T = J\,\mathrm{d}\omega \ , \tag{3.19}$$

and is the real part of a $(2,1)$-form. From [126] one can see that KT manifolds are complex manifolds, i.e. they enjoy

$$W_1^{\pm} = W_2^{\pm} = 0 \ . \tag{3.20}$$

Note that in general their structure group is U(3) rather than SU(3), as they are a subclass of almost Hermitian structures. However, they may reduce to an SU(3)-structure that is contained in the U(3)-structure.

Calabi-Yau-torsion If the KT connection is traceless, its holonomy is SU(3) instead of U(3) and, in particular, the structure group is reduced to SU(3). Conversely, if one is given an SU(3)-structure (g, ω, Ω) on M^6, this is always contained in the almost Hermitian structure defined by (g, ω). The KT connection of the latter then comprises an SU(3) connection for the SU(3)-structure if and only if its U(1) part vanishes on the SU(3) subbundle. This can be written as a further condition on their torsion classes of the SU(3)-structure under consideration (see, e.g. [128]), which reads

$$2\,W_4 + W_5 = 0 \ , \tag{3.21}$$

without further restricting W_3. SU(3)-structures that are compatible with the KT connection of their almost Hermitian structure in this sense are called *Calabi-Yau-torsion* (CYT). Hence, CYT manifolds form a subset of KT manifolds, but with SU(3) structure group.

Nearly Kähler An SU(3)-structure on a 6-manifold is *nearly Kähler* if

$$W_1^{+} = W_2^{\pm} = W_3 = W_4 = W_5 = 0 \ . \tag{3.22}$$

Note that, in general, one does not need a vanishing W_1^{+}, but this can be achieved by suitable phase-transformation in Ω.

Half-flat An SU(3)-structure on a 6-manifold which satisfies

$$W_1^{+} = W_2^{+} = W_4 = W_5 = 0 \tag{3.23}$$

is called *half-flat*.

Note that generic nearly Kähler and half-flat 6-manifolds have a non-integrable almost complex structure J and that nearly Kähler manifolds are a subclass of half-flat manifolds.

3.5 Cylinders and sine-cones over 5-manifolds with SU(2)-structure

Cylinders, metric cones, and sine-cones provide a tool for constructing $(n+1)$-dimensional G-structure manifolds starting from n-dimensional H-structure manifolds with $\mathrm{H} \subset \mathrm{G}$. At first, we review the Calabi-Yau cone with canonical complex structure for completeness. Next, we focus on the Kähler-torsion sine-cone, the nearly Kähler sine-cone and the half-flat cylinder, which will provide the stage for the instanton equations considered in Ch. 5.

First, let us assume we are given a 5-dimensional manifold M^5 with an SU(2)-structure defined by (η, ω^{α}) and a Riemannian metric g_5. These tensor fields induce global tensor fields on the Cartesian product $M^5 \times I$, where I is an interval. Due to the properties (3.9) of the

SU(2)-structure on M^5, around every point of $M^5 \times I$ there is a local frame such that

$$\eta = -e^5 \ , \quad \omega^\alpha = \frac{1}{2}\,\eta^\alpha_{ab}\,e^a \wedge e^b \quad \text{and} \quad \mathrm{d}r = e^6 \ , \tag{3.24}$$

if r is the natural coordinate on the interval I. Next, we can apply transformations to these local frames; for example, we can perform a transformation like

$$e^\mu \mapsto \phi(r)\,e^\mu \quad \text{and} \quad e^6 \mapsto e^6 \ , \tag{3.25a}$$

changing the metric on $M^5{\times}I$ to the warped-product metric

$$g = \mathrm{d}r^2 + \phi(r)^2\,g_5 \qquad \text{on } M^5{\times}_\phi I \ . \tag{3.25b}$$

Still, the forms $(\phi\,\eta,\ \phi^2\,\omega^\alpha,\ \mathrm{d}r)$ have the same components as in (3.24) with respect to the transformed frames.

Afterwards, one still has the freedom of further transformations. These need to map one SU(2)-structure to another, which means that the defining forms need to have the standard components (3.9) with respect to the new frame. In addition, those transformations can be chosen to preserve the warped-product metric. In other words, these admissible transformations are given by maps from $M^5{\times}I$ to the normaliser subgroup of SU(2) in $\mathrm{GL}(6,\mathbb{R})$ (or SO(6) if one wants to preserve g), i.e.

$$L : M^5 \times I \to N_{\mathrm{GL}(6,\mathbb{R})}(\mathrm{SU}(2)) \ . \tag{3.26}$$

The crucial statement is that if we are given a set of forms (η, ω^α) on $M^5 \times I$ such that around every point in $M^5 \times I$ there is a local frame with respect to which (3.24) holds true, the forms defined by

$$\omega = \omega^3 - \eta \wedge \mathrm{d}r \ , \tag{3.27a}$$

$$\Omega^+ = \ -\,\omega^1 \wedge \mathrm{d}r + \omega^2 \wedge \eta \ , \tag{3.27b}$$

$$\Omega^- = \ -\,\omega^2 \wedge \mathrm{d}r - \omega^1 \wedge \eta \tag{3.27c}$$

take the standard components (3.17) with respect to these local frames and, therefore, define an SU(3)-structure on $M^5 \times I$. Note that ω and Ω are globally well-defined, simply because η and the ω^α are.

This provides us with a general way to construct SU(3)-structure manifolds in 6 dimensions. Namely, we push a given SU(2)-structure on M^5 forward to $M^5 \times I$ and apply transformations such that we still are given forms with components (3.24). Then we know that there exists an extension to an SU(3)-structure given by (3.27). For a generalisation of this procedure we refer to [129]. In the following subsections we apply this procedure in several cases.

3.5.1 Calabi-Yau cones

Recall the Calabi-Yau cone introduced in Sec. 3.2. We now equip it with the canonical complex structure and complete the picture for 6-dimensional SU(3)-manifolds. To prevent confusion, we deliberately change the notation and index sets for the complexified forms.

Consider a Sasaki-Einstein 5-manifold M^5 with local coframes e^μ, where $\mu = (a, 5)$ and $a = 1, 2, 3, 4$. The metric on its metric cone reads

$$g = r^2\left(\delta_{ab}\,e^a \otimes e^b + e^5 \otimes e^5\right) + \mathrm{d}r \otimes \mathrm{d}r = r^2\left(\delta_{ab}\,e^a \otimes e^b + e^5 \otimes e^5 + e^6 \otimes e^6\right) \tag{3.28}$$

with

$$e^6 = \mathrm{d}\tau = \frac{\mathrm{d}r}{r}\ . \tag{3.29}$$

The last equality in (3.28) displays the conformal equivalence to the cylinder over M^5 with the metric

$$g_{\text{cyl}} = \delta_{ab}\, e^a \otimes e^b + e^5 \otimes e^5 + e^6 \otimes e^6\ . \tag{3.30}$$

We can introduce the canonical almost complex structure J on the metric cone via

$$J\hat{\Theta}^\alpha = \mathrm{i}\hat{\Theta}^\alpha \qquad \text{for} \quad \alpha = 1,2,3 \qquad \text{with} \quad \hat{\Theta}^\alpha = \hat{e}^{2\alpha-1} + \mathrm{i}\hat{e}^{2\alpha}\ , \tag{3.31}$$

and we set $\hat{e}^{\hat{\mu}} = re^{\hat{\mu}}$ for $\hat{\mu} = 1,\dots,6$. The SU(3)-structure forms $(\hat{\omega}, \hat{\Omega})$ have the local expressions

$$\hat{\omega} = \hat{e}^1 \wedge \hat{e}^2 + \hat{e}^3 \wedge \hat{e}^4 + \hat{e}^5 \wedge \hat{e}^6 = r^2(\omega^3 + e^5 \wedge e^6)\ , \tag{3.32a}$$

$$\hat{\Omega} = \hat{\Theta}^1 \wedge \hat{\Theta}^2 \wedge \hat{\Theta}^3\ , \tag{3.32b}$$

for which a direct computation yields

$$\mathrm{d}\hat{\omega} = 0 \qquad \text{and} \qquad \mathrm{d}\hat{\Omega} = 0\ . \tag{3.33}$$

Therefore, the metric cone introduced in (3.28) is indeed Calabi-Yau as all SU(3)-torsion classes vanish.

3.5.2 Kähler-torsion sine-cones

It has been shown in [130] that the sine-cone over a d-dimensional Einstein manifold with Einstein constant $d-1$ is again an Einstein manifold with Einstein constant d. Here we will show that the sine-cone over any Sasaki-Einstein 5-manifold is not only Einstein, but additionally carries a Kähler-torsion structure[5].

Consider a Sasaki-Einstein 5-manifold M^5 and the product manifold $M^6 = M^5 \times (0, \Lambda\pi)$ with the metric

$$g = \Lambda^2 \sin^2\varphi \left(\delta_{ab}\, e^a \otimes e^b + e^5 \otimes e^5\right) + \mathrm{d}r \otimes \mathrm{d}r \tag{3.34a}$$

$$= \Lambda^2 \sin^2\varphi \left(\delta_{ab}\, e^a \otimes e^b + e^5 \otimes e^5 + e^6 \otimes e^6\right)\ , \tag{3.34b}$$

where

$$\varphi = \frac{r}{\Lambda} \quad \text{and} \quad e^6 = \mathrm{d}\tau = \frac{\mathrm{d}\varphi}{\sin\varphi}\ , \tag{3.35}$$

with the scaling parameter $\Lambda \in \mathbb{R}^+$. Equation (3.34b) shows that the metric on the sine-cone is conformally equivalent to the metric (3.30) on the cylinder over M^5.

The explicit solution of $\tau = \tau(\varphi)$ is computed to

$$\tau = \ln\left|\tan\tfrac{\varphi}{2}\right| + \text{constant}\ . \tag{3.36}$$

[5] In fact, the argument holds for the sine-cone over any $(2n+1)$-dimensional Sasaki-Einstein manifold, which we proved in [117].

Demanding that in the large volume limit $\Lambda \to \infty$ the sine-cone becomes the metric cone fixes the integration constant such that

$$\tau(\varphi) = \ln\left(2\Lambda \tan\tfrac{\varphi}{2}\right) = \ln\left(2\Lambda\sqrt{\tfrac{1-\cos\varphi}{1+\cos\varphi}}\right) \ . \tag{3.37}$$

Next, we introduce an almost complex structure J and the associated fundamental $(1,1)$-form $\tilde{\omega}$ on the sine-cone as follows ($\alpha = 1,2,3$):

$$J\tilde{\Theta}^{\alpha} = \mathrm{i}\tilde{\Theta}^{\alpha} \qquad \text{with} \quad \tilde{\Theta}^{\alpha} = \Lambda \sin\varphi \left(e^{2\alpha-1} + \mathrm{i}e^{2\alpha}\right) \ , \tag{3.38a}$$

$$J\tilde{\Theta}^{\bar{\alpha}} = -\mathrm{i}\tilde{\Theta}^{\bar{\alpha}} \qquad \text{with} \quad \tilde{\Theta}^{\bar{\alpha}} = \overline{\tilde{\Theta}^{\alpha}} \ , \tag{3.38b}$$

$$\tilde{\omega} = \Lambda^2 \sin^2\!\varphi \left(\omega^3 + e^5 \wedge e^6\right) \ , \tag{3.38c}$$

where ω^3 is defined in (3.9). It then follows that the above structure defines a Kähler-torsion structure on the sine-cone, because there exists the uniquely defined Bismut connection ∇^B, which preserves g and J, and has torsion given by

$$T^B = J\,\mathrm{d}\tilde{\omega} \ . \tag{3.39}$$

Remarks One can also introduce a globally well-defined complex $(3,0)$-form $\tilde{\Omega}$ defined as

$$\tilde{\Omega} = \tilde{\Theta}^1 \wedge \tilde{\Theta}^2 \wedge \tilde{\Theta}^3 = \Lambda^3 \sin^3\!\varphi(\omega^2 - \mathrm{i}\omega^1) \wedge \eta - \Lambda^2 \sin^2\!\varphi(\omega^1 + \mathrm{i}\omega^2) \wedge \mathrm{d}r \ . \tag{3.40}$$

Applying the exterior differential yields

$$\mathrm{d}\tilde{\omega} = 2\,\frac{\cos\varphi - 1}{\Lambda \sin\varphi}\,\tilde{\omega} \wedge \tilde{e}^6 = -\frac{2}{\Lambda}\tan\tfrac{\varphi}{2}\,\tilde{\omega} \wedge \tilde{e}^6 \ , \tag{3.41a}$$

$$\mathrm{d}\tilde{\Omega} = 3\,\frac{1-\cos\varphi}{\Lambda \sin\varphi}\,\tilde{\Omega} \wedge \tilde{e}^6 = \frac{3}{\Lambda}\tan\tfrac{\varphi}{2}\,\tilde{\Omega} \wedge \tilde{e}^6 \ , \tag{3.41b}$$

thus rendering the sine-cone over M^5 an SU(3)-structure manifold as defined in Sec. 3.4. From (3.41) we immediately see that J is integrable and

$$2W_4 + W_5 = -\frac{1}{\Lambda}\tan\tfrac{\varphi}{2}\,\tilde{e}^6 \neq 0 \quad \text{for} \quad \Lambda < \infty \ , \tag{3.42}$$

whence the Bismut connection does not preserve the SU(3)-structure unless $\Lambda = \infty$. Nevertheless, the condition $3\,W_4 + 2\,W_5 = 0$ is satisfied, which is in agreement with the conformal equivalence between the sine-cone over a Sasaki-Eintein 5-manifold and the Calabi-Yau metric cone over M^5 [124,131]. That is, the conformal equivalence of the Calabi-Yau cone and the Kähler-torsion sine-cone also maps their two SU(3)-structures into one another. We also note that $2W_4 + W_5 \to 0$ as $\Lambda \to \infty$, and the KT sine-cone becomes the Calabi-Yau metric cone. Recall from Sec. 3.4 that Kähler-torsion structures are U(3)-structures, whence one has to distinguish between the KT-structure and the SU(3)-structure.

3.5.3 Nearly Kähler sine-cones

In [94] a nearly Kähler structure on the sine-cone over a Sasaki-Einstein 5-manifold has been obtained by means of flow equations. Here, in contrast, we show that this structure can be constructed by means of a combined rotation and rescaling of the coframes of the cylinder over the Sasaki-Einstein 5-manifold. We will carry this construction out in the following three steps:

1. An SU(3)-structure on the cylinder over a Sasaki-Einstein 5-manifold M^5 can be introduced via a metric (3.30), an almost complex structure J, or the equivalent $(1,1)$-form ω, and a $(3,0)$-form Ω. These objects are

$$\omega = \omega^3 + e^5 \wedge e^6 = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6 \ , \qquad \text{with} \quad e^6 = \mathrm{d}t \ , \tag{3.43a}$$

$$J\Theta^\alpha = \mathrm{i}\Theta^\alpha \ , \qquad \text{for} \quad \Theta^\alpha = e^{2\alpha-1} + \mathrm{i}e^{2\alpha} \qquad \text{with} \quad \alpha = 1,2,3 \ , \tag{3.43b}$$

$$\Omega = \Theta^1 \wedge \Theta^2 \wedge \Theta^3 = -\omega^2 \wedge e^5 - \omega^1 \wedge e^6 + \mathrm{i}\left(\omega^1 \wedge e^5 - \omega^2 \wedge e^6\right) \ . \tag{3.43c}$$

2. Next, we consider an SO(5)-rotation of the SU(2)-structure (η, ω^α) on M^5. Let η^2 be the matrix of the 't Hooft symbols η^2_{ab}, and perform a rotation of the basis 1-forms $e^1, \dots, e^4$ as follows:

$$E = \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \end{pmatrix} \mapsto E_\varphi = \exp\left(\tfrac{\varphi}{2}\,\eta^2\right) E = \begin{pmatrix} \cos\frac{\varphi}{2} & 0 & -\sin\frac{\varphi}{2} & 0 \\ 0 & \cos\frac{\varphi}{2} & 0 & \sin\frac{\varphi}{2} \\ \sin\frac{\varphi}{2} & 0 & \cos\frac{\varphi}{2} & 0 \\ 0 & -\sin\frac{\varphi}{2} & 0 & \cos\frac{\varphi}{2} \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \end{pmatrix} . \tag{3.44}$$

In the rotated frame (e^a_φ, e^5) we define the SU(3)-structure forms to have the same components as in the unrotated frame (3.43), i.e.

$$\omega_\varphi = \omega^3_\varphi + e^5 \wedge e^6 \ , \tag{3.45a}$$

$$\Omega_\varphi = -\omega^2_\varphi \wedge e^5 - \omega^1_\varphi \wedge e^6 + \mathrm{i}\left(\omega^1_\varphi \wedge e^5 - \omega^2_\varphi \wedge e^6\right) \ , \tag{3.45b}$$

where $\omega^\alpha_\varphi = \frac{1}{2}\eta^\alpha_{\mu\nu}\, e^{\mu\nu}_\varphi$. Note that this is still an SU(3)-structure on the cylinder, because the defining forms have the standard components (3.43) with respect to the coframes e^μ_φ.

3. Last, the pullback to the sine-cone $C_s(M^5)$ along the map establishing the conformal equivalence to the cylinder yields

$$e^a_s = \Lambda\, e^a_\varphi \sin\varphi \ , \qquad e^5_s = \Lambda\, e^5 \sin\varphi \ , \qquad e^6_s = \Lambda\, e^6 \sin\varphi = \Lambda\, \mathrm{d}\varphi = \mathrm{d}r \ , \tag{3.46a}$$

$$\omega^\alpha_s = \Lambda^2\, \omega^\alpha_\varphi \sin^2\!\varphi \ , \quad \omega_s = \omega^3_s + \Lambda^2\, e^5 \wedge e^6 \sin^2\!\varphi \ , \tag{3.46b}$$

$$\Omega_s = \Lambda^3\, \Omega_\varphi \sin^3\!\varphi \tag{3.46c}$$

as an SU(3)-structure on the sine-cone. By a direct calculation we obtain

$$\mathrm{d}\omega_s = -\frac{3}{\Lambda}\,\Omega^+_s \ , \tag{3.47a}$$

$$\mathrm{d}\Omega^+_s = 0 \ , \qquad \mathrm{d}\Omega^-_s = \frac{2}{\Lambda}\,\omega_s \wedge \omega_s \ , \tag{3.47b}$$

which confirms that (3.46) induces a nearly Kähler structure on the sine-cone.

Remarks In the limit $\Lambda \to \infty$, in which the sine-cone becomes the metric cone, this nearly Kähler structure on the sine-cone is smoothly deformed to the Calabi-Yau structure on the metric cone since

$$\lim_{\Lambda\to\infty} \mathrm{d}\omega_s = 0 \qquad \text{and} \qquad \lim_{\Lambda\to\infty} \mathrm{d}\Omega_s = 0 \ . \tag{3.48}$$

Generically, the sine-cone, as a conifold, has two singularities at $\varphi = 0$ and $\varphi = \pi$. As we see from (3.46), the SU(3)-structure cannot be extended to the tips, because all defining forms vanish at these points. Hence, the sine-cone is a nearly Kähler manifold but not extendible to the tips.

3.5.4 Half-flat cylinders

Consider a 5-dimensional manifold M^5 endowed with a Sasaki-Einstein SU(2)-structure defined by $(\eta, \omega^1, \omega^2, \omega^3)$ as in (3.11). For an arbitrary coframe e^μ compatible with the SU(2)-structure, consider the transformation

$$
\begin{aligned}
e_z^1 &= e^4 \cos\zeta + e^3 \sin\zeta \ , \quad & e_z^2 &= -e^1 \ , && (3.49a)\\
e_z^3 &= e^2 \ , & e_z^4 &= e^3 \cos\zeta - e^4 \sin\zeta \ , && (3.49b)\\
e_z^5 &= \varrho\, e^5 . &&&& (3.49c)
\end{aligned}
$$

Here $\zeta \in [0, 2\pi]$ and $\rho \in \mathbb{R}_+$ are two constant parameters. For $\varrho = 1$ this can be seen to be an SO(5)-transformation of the coframe, such that the metric on M^5 is unchanged. Nevertheless, we obtain a two-parameter family of SU(2)-structures on M^5 by defining

$$
\eta_z = \varrho\, \eta \ , \qquad \omega_z^\alpha = \tfrac{1}{2}\, \eta^\alpha_{\mu\nu}\, e_z^\mu \wedge e_z^\nu \ , \qquad g_z = \delta_{\mu\nu}\, e_z^\mu \otimes e_z^\nu \ . \tag{3.50}
$$

These objects are globally well-defined as can be seen from

$$
\begin{aligned}
\omega_z^1 &= -\omega^3 \ , && (3.51a)\\
\omega_z^2 &= \omega^1 \sin\zeta + \omega^2 \cos\zeta \ , && (3.51b)\\
\omega_z^3 &= \omega^1 \cos\zeta - \omega^2 \sin\zeta \ , && (3.51c)
\end{aligned}
$$

and, thus, yield a two-parameter family of SU(2)-structures on M^5. Note that these structures are neither hypo nor nearly hypo any more.

With these SU(2)-structures on M^5 at hand we define a two-parameter family of SU(3)-structures on the metric cylinder $(M^5 \times \mathbb{R},\ \bar{g}_z = g_z + \mathrm{d}r \otimes \mathrm{d}r)$ by

$$
\begin{aligned}
\omega_z &= \omega_z^3 - \eta_z \wedge \mathrm{d}r = \omega^1 \cos\zeta - \omega^2 \sin\zeta - \varrho\, \eta \wedge \mathrm{d}r \ , && (3.52a)\\
\Omega_z^+ &= -\omega_z^1 \wedge \mathrm{d}r + \omega_z^2 \wedge \eta_z = \varrho \left(\omega^1 \sin\zeta + \omega^2 \cos\zeta\right) \wedge \eta + \omega^3 \wedge \mathrm{d}r \ , && (3.52b)\\
\Omega_z^- &= -\omega_z^2 \wedge \mathrm{d}r - \omega_z^1 \wedge \eta_z = -\left(\omega^1 \sin\zeta + \omega^2 \cos\zeta\right) \wedge \mathrm{d}r + \varrho\, \omega^3 \wedge \eta \ , && (3.52c)
\end{aligned}
$$

which yields a two-parameter family of half-flat SU(3)-structures. The non-vanishing torsion classes can be computed to read

$$
\begin{aligned}
W_1^- &= \frac{3 + 2\varrho^2}{3\varrho} \ , \quad W_2^- = \frac{4\varrho^2 - 3}{3\varrho} \left(\omega_z^3 + 2\, \eta_z \wedge \mathrm{d}r\right) \quad \text{and}\\
W_3 &= \frac{2\varrho^2 - 3}{2\varrho} \left(\omega_z^1 \wedge \mathrm{d}r + \omega_z^2 \wedge \eta_z\right) \ .
\end{aligned} \tag{3.53}
$$

Furthermore, the conditions $\omega_z \lrcorner W_2^- = 0$ and $\omega_z \lrcorner W_3 = 0$ are satisfied for any values of the parameters ζ and ϱ.

4 Hermitian Yang-Mills instantons on Calabi-Yau cones

The generalised instanton equations on the Calabi-Yau cone above any Sasaki-Einstein manifold are the HYM equations, which exhibit a rich geometric structure. We recall the relevant concepts and apply them to a certain equivariant ansatz.

4.1 Instanton condition induced by a G-structure

(Anti-)Self-dual connections on 4-manifolds [24] were the first examples of what is nowadays known as instantons. The generalisation to higher dimensions was first proposed in [34] and has been the topic of active research ever since. Let us provide a brief summary of the relevant notions.

Suppose an n-dimensional manifold M^n is equipped with a G-structure, i.e. a reduction of the structure group of the tangent bundle TM^n to a Lie subgroup $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$. Then there exist covariantly constant sections in certain associated bundles [132]. We have already encountered several examples of G-structures in Ch. 3 and Tab. 4.1 provides archetypal examples. In particular, a Kähler structure on M^{2n} lies at the interplay of complex, Riemannian, and

structure	Lie group G	relevant section
Riemannian	$\mathrm{O}(2n)$	metric g
(almost) complex	$\mathrm{GL}(n, \mathbb{C})$	(almost) complex struture J
symplectic	$\mathrm{Sp}(2n)$	symplectic form ω

Table 4.1: *Three typical geometric structures on a $2n$-dimensional manifold M^{2n}.*

symplectic geometry, i.e. all three structures coexist and are subject to compatibility constraints. Consequently, the Lie subgroup G is obtained from the intersection of all the individual structure groups $\mathrm{GL}(n, \mathbb{C}) \cap \mathrm{O}(2n) \cap \mathrm{Sp}(2n) = \mathrm{U}(n)$.

In the cases considered, G is moreover a subgroup of $\mathrm{SO}(n)$, as we start from Riemannian manifolds with chosen orientation. Therefore, the G-structure equals a G-principal subbundle $\mathcal{Q}$ of the frame bundle $\mathrm{SO}(M^n, g)$. There exists an isomorphism $\mathrm{Ad}(\mathrm{SO}(M^n, g)) \cong \Lambda^2 T^* M^n$, which at a point in M^n reduces to the isomorphism between the Lie algebra $\mathfrak{so}(n)$ and the space of 2-forms $\Lambda^2 \mathbb{R}^n$. The restriction of the bundle isomorphism to $\mathcal{Q}$ defines a subbundle $W(\mathcal{Q}) \subset \Lambda^2 T^* M^n$, the so-called *instanton bundle*. An instanton in (M^n, g) with respect to the G-structure is a connection $\mathcal{A}$ on a principal bundle $\mathfrak{B}$ over M^n whose curvature $\mathcal{F}_{\mathcal{A}}$ satisfies

$$\mathcal{F}_{\mathcal{A}} \in \Gamma \left(W(\mathcal{Q}) \otimes \mathrm{Ad}(\mathfrak{B}) \right) \subset \Omega^2 \left(M^n, \mathrm{Ad}(\mathfrak{B}) \right) \ . \tag{4.1}$$

There exist various equivalent realisations of this mathematical definition, two of which we like to summarise here, see also [133]. Firstly, if the G-structure can be defined via a spinor ϵ then $\mathcal{A}$ is an instanton if it annihilates the spinor under the Clifford map γ, i.e.

$$\gamma(\mathcal{F}_{\mathcal{A}}) \epsilon = 0 \ . \tag{4.2}$$

Secondly, one can employ the differential forms that define the G-structure to construct a G-invariant $(n-4)$-form Q on M^n. By means of Q one defines an endomorphism of $\Lambda^2 T^* M^n$ via

$$\begin{aligned} \Lambda^2 T^* M^n &\to \Lambda^2 T^* M^n \\ \rho &\mapsto \star(Q \wedge \rho) \ , \end{aligned} \tag{4.3a}$$

where $\star$ is the Hodge star on M^n. Then $\mathcal{A}$ is an instanton if the curvatures lies in the eigenspace with eigenvalue -1, i.e.

$$\star\,(Q \wedge \mathcal{F}_{\mathcal{A}}) = -\mathcal{F}_{\mathcal{A}} \ . \tag{4.3b}$$

Summarising, the generalised instanton equations are conditions on the 2-form part of the curvature $\mathcal{F}_{\mathcal{A}}$. In the proceeding considerations, we mainly consider connections on associated vector bundles, but employ the natural principal bundle setting whenever necessary.

4.2 Hermitian Yang-Mills instantons

The instanton equations for Kähler manifolds are rather special and known under the name *Hermitian Yang-Mills* equations. For later analysis, the geometric properties of the space of connections and the HYM instanton moduli space over a Kähler manifold are recalled. This brief account is inspired by [134, 135].

Space of connections Let M^{2n} be a (closed) Kähler manifold of $\dim_{\mathbb{C}}(M) = n$ and $\mathcal{G}$ a compact matrix Lie group. Let $P(M^{2n}, \mathcal{G})$ be a $\mathcal{G}$-principal bundle over M^{2n}, $\mathcal{A}$ a connection 1-form and $\mathcal{F}_{\mathcal{A}} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ the curvature.

Let $\mathrm{Int}(P) \coloneqq P \times_{\mathcal{G}} \mathcal{G}$ be the group bundle (where $\mathcal{G}$ acts via the internal automorphism $h \mapsto ghg^{-1}$), let $\mathrm{Ad}(P) \coloneqq P \times_{\mathcal{G}} \mathfrak{g}$ be the Lie algebra bundle (where $\mathcal{G}$ acts on $\mathfrak{g}$ via the adjoint action), and $E \coloneqq P \times_{\mathcal{G}} F$ be an associated vector bundle (where the vector space F, the typical fibre, carries a $\mathcal{G}$-representation).

Denote the space of all connections on P by $\mathbb{A}(P)$, and note that all associated bundles E inherit their space of connections $\mathbb{A}(E)$ from P. On $\mathbb{A}(P)$ there is a natural action of the gauge group $\widehat{\mathcal{G}}$, i.e. the set of automorphisms on P which are trivial on the base. With

$$\widehat{\mathcal{G}} = \Gamma\left(M^{2n}, \mathrm{Int}(P)\right) \tag{4.4}$$

one has an identification with the space of global sections of the group bundle. The action is realised via

$$\mathcal{A} \to \mathcal{A}^g = \mathrm{Ad}(g^{-1})\mathcal{A} + g^{-1}\mathrm{d}g \qquad \text{for} \quad g \in \Gamma\left(M^{2n}, \mathrm{Int}(P)\right) \ . \tag{4.5}$$

The Lie algebra of the gauge group then equals

$$\widehat{\mathfrak{g}} = \Gamma\left(M^{2n}, \mathrm{Ad}(P)\right) \ , \tag{4.6}$$

and the infinitesimal gauge transformations are given by

$$\mathcal{A} \mapsto \delta\mathcal{A} = \mathrm{d}_{\mathcal{A}}\chi \coloneqq \mathrm{d}\chi + [\mathcal{A}, \chi] \qquad \text{for} \quad \chi \in \Gamma(M^{2n}, \mathrm{Ad}(P)) \ . \tag{4.7}$$

Since $\mathbb{A}(P)$ is an affine space over $\Omega^1(M^{2n}, \mathrm{Ad}(P))$, the tangent space $T_{\mathcal{A}}\mathbb{A}$ for any $\mathcal{A} \in \mathbb{A}(P)$ is canonically identified with $\Omega^1(M^{2n}, \mathrm{Ad}(P))$. Further, assuming $\mathcal{G} \hookrightarrow \mathrm{U}(N)$, for some $N \in \mathbb{N}$, implies that the trace is an Ad-invariant inner product. Hence, a metric on $\mathbb{A}(P)$ is defined via

$$\boldsymbol{g}_{|\mathcal{A}}(X_1, X_2) \coloneqq \int_{M^{2n}} \mathrm{tr}\,(X_1 \wedge \star X_2) \qquad \text{for} \quad X_1, X_2 \in T_{\mathcal{A}}\mathbb{A} \ , \tag{4.8}$$

with $\star$ the Hodge-dual on M^{2n}. Moreover, the space $\mathbb{A}(P)$ allows for a symplectic structure

$$\boldsymbol{\omega}_{|\mathcal{A}}(X_1, X_2) := \int_{M^{2n}} \operatorname{tr}(X_1 \wedge X_2) \wedge \frac{\omega^{n-1}}{(n-1)!} \qquad \text{for} \quad X_1, X_2 \in T_{\mathcal{A}}\mathbb{A} \ , \tag{4.9a}$$

with ω the Kähler form on M^{2n}. Since $\boldsymbol{\omega}$ is completely base point independent (on $\mathbb{A}$), $\boldsymbol{\omega}$ is in fact a symplectic form.

In addition, one can check that $X \wedge \frac{\omega^{n-1}}{(n-1)!} = \star J(X)$ holds for any $X \in T_{\mathcal{A}}\mathbb{A}$, where J, the (canonical) complex structure of M^{2n}, acts only the 1-form part of X. This allows to reformulate the symplectic structure as

$$\boldsymbol{\omega}_{|\mathcal{A}}(X_1, X_2) = \int_{M^{2n}} \operatorname{tr}(X_1 \wedge \star J(X_2)) \qquad \text{for} \quad X_1, X_2 \in T_{\mathcal{A}}\mathbb{A} \ . \tag{4.9b}$$

Moreover, it implies that $\boldsymbol{\omega}$ is non-degenerate as $\boldsymbol{\omega}_{|\mathcal{A}}(X_1, X_2) = \boldsymbol{g}_{|\mathcal{A}}(X_1, J(X_2))$ holds for any X_1, X_2 and any $\mathcal{A}$. Consequently, $(\mathbb{A}, \boldsymbol{g}, \boldsymbol{\omega})$ is an infinite-dimensional Riemannian, symplectic manifold, which is equipped with compatible $\widehat{\mathcal{G}}$-action.

Holomorphic structure Next, consider the restriction to connections on $E \xrightarrow{\simeq F} M$ which satisfy the so-called holomorphicity condition

$$\mathcal{F}_{\mathcal{A}}^{2,0} = 0 \quad \text{and} \quad \mathcal{F}_{\mathcal{A}}^{0,2} = 0 \ . \tag{4.10}$$

It is known that this condition is equivalent to the existence of a holomorphic structure on E, i.e. a Cauchy-Riemann operator $\bar{\partial}_E := \bar{\partial} + A^{0,1}$ that satisfies the Leibniz-rule as well as $\bar{\partial}_E \circ \bar{\partial}_E = 0$. If M^{2n} is also Calabi-Yau, then the condition (4.10) is equivalent to $\Omega \wedge \mathcal{F}_{\mathcal{A}} = 0$, where Ω is a holomorphic $(n, 0)$-form.

Define the subspace of holomorphic connections as

$$\mathbb{A}^{1,1} = \left\{ \mathcal{A} \in \mathbb{A}(E) : \mathcal{F}_{\mathcal{A}}^{0,2} = -\left(\mathcal{F}_{\mathcal{A}}^{2,0}\right)^{\dagger} = 0 \right\} \subset \mathbb{A}(E) \ . \tag{4.11}$$

This definition employs the underlying complex structure on M^{2n}, and $\dagger$ denotes complex conjugation and transposition. Moreover, one can show that $\mathbb{A}^{1,1}$ is an infinite-dimensional Kähler space, i.e. the restriction of $\boldsymbol{g}$ to $\mathbb{A}^{1,1}$ is a Hermitian metric and the symplectic form $\boldsymbol{\omega}$ is Kähler. The compatible complex structure $\boldsymbol{J}$ (with $\boldsymbol{\omega}(\cdot,\cdot) = \boldsymbol{g}(\boldsymbol{J}\cdot,\cdot)$) can be read off from (4.8) and (4.9) to be

$$\boldsymbol{J}_{|\mathcal{A}}(X) = -J(X) \qquad \text{for} \quad X \in T_{\mathcal{A}}\mathbb{A} \ , \tag{4.12}$$

i.e. it is base point independent.

Moment map The space $\mathbb{A}^{1,1}$ inherits the $\widehat{\mathcal{G}}$-action from $\mathbb{A}$, and since it has a symplectic form, i.e. the Kähler form, one can introduce a moment map

$$\begin{aligned} \mu : \mathbb{A}^{1,1} &\to \widehat{\mathfrak{g}}^* \cong \Omega^{2n}(M^{2n}, \operatorname{Ad}(P)) \\ \mathcal{A} &\mapsto \mathcal{F}_{\mathcal{A}} \wedge \frac{\omega^{n-1}}{(n-1)!} \ . \end{aligned} \tag{4.13}$$

We see that μ is $\widehat{\mathcal{G}}$-equivariant by construction. Nonetheless, for this to be a moment map of the $\widehat{\mathcal{G}}$-action, one needs to verify the defining property

$$(\phi, \mathrm{D}\mu_{|\mathcal{A}})(\psi) = \iota_{\phi^\natural} \boldsymbol{\omega}_{|\mathcal{A}}(\psi) \ , \tag{4.14}$$

where $\phi \in \Gamma(M^{2n}, \mathrm{Ad}(P))$ an element of the gauge Lie algebra, $\phi^\natural$ be the corresponding vector field on $\mathbb{A}^{1,1}$ and $\psi \in \Omega^1(M^{2n}, \mathrm{Ad}(P))$ a tangent vector at the base point $\mathcal{A}$. By D we denote the exterior derivative on $\mathbb{A}$. Moreover, the duality pairing $(\cdot,\cdot)$ of $\widehat{\mathfrak{g}}$ and its dual is defined via the integral over M^{2n} and the invariant product on $\mathfrak{g}$. We now generalise the arguments presented in [134]. Firstly, in the definition of μ only $\mathcal{F}_\mathcal{A}$ is base point dependent, and a standard computation gives

$$\mathcal{F}_{\mathcal{A}+t\,\psi} = \mathcal{F}_\mathcal{A} + t\,\mathrm{d}_\mathcal{A}\psi + \tfrac{1}{2}\,t^2\,\psi\wedge\psi \quad \text{so that} \quad \mathrm{D}\mathcal{F}_{|\mathcal{A}}(\psi) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{\mathcal{A}+t\,\psi}\right)_{|t=0} = \mathrm{d}_\mathcal{A}\psi\ . \tag{4.15a}$$

Thus, the left-hand side of (4.14) is

$$(\phi, \mathrm{D}\mu_{|\mathcal{A}})(\psi) = \int_M \mathrm{tr}\big((\mathrm{d}_\mathcal{A}\psi)\wedge\phi\big)\wedge\frac{\omega^{n-1}}{(n-1)!}\ . \tag{4.15b}$$

Secondly, the vector field $\phi^\natural$ can be read off from (4.7) to be $\phi^\natural_{|\mathcal{A}} = \mathrm{d}_\mathcal{A}\phi \in \Omega^1(M, \mathrm{Ad}(P))$. Hence, the right-hand side is

$$\iota_{\phi^\natural}\boldsymbol{\omega}_{|\mathcal{A}}(\psi) = \int_M \mathrm{tr}\big((\mathrm{d}_\mathcal{A}\phi)\wedge\psi\big)\wedge\frac{\omega^{n-1}}{(n-1)!}\ . \tag{4.15c}$$

But from $\int_M \mathrm{d}\left(\mathrm{tr}\,(\psi\wedge\phi)\wedge\frac{\omega^{n-1}}{(n-1)!}\right) = 0$ by Stokes' theorem[6] and $\mathrm{d}\omega = 0$ one has

$$\int_M \mathrm{tr}\big((\mathrm{d}_\mathcal{A}\psi)\wedge\phi\big)\wedge\frac{\omega^{n-1}}{(n-1)!} = -\int_M \mathrm{tr}\,\big(\psi\wedge(\mathrm{d}_\mathcal{A}\phi)\big)\wedge\frac{\omega^{n-1}}{(n-1)!}\ , \tag{4.15d}$$

and, therefore, the relation (4.14) holds, i.e. μ is a moment map of the $\widehat{\mathcal{G}}$-action on $\mathbb{A}^{1,1}$.

However, one can equally well use the dual map defined by

$$\begin{aligned} \mu^* : \mathbb{A}^{1,1} &\to \widehat{\mathfrak{g}} = \Omega^0(M^{2n}, \mathrm{Ad}(P)) \\ \mathcal{A} &\mapsto \omega\lrcorner\mathcal{F}_\mathcal{A}\ , \end{aligned} \tag{4.16}$$

which is equivalent to μ of (4.13) due to

$$\mathcal{F}_\mathcal{A}\wedge\omega^{n-1} = \frac{1}{n}(\omega\lrcorner\mathcal{F}_\mathcal{A})\,\omega^n\ . \tag{4.17}$$

Thus, we will no longer explicitly distinguish between μ and μ^*.

For $\Xi \in \mathrm{Centre}(\widehat{\mathfrak{g}})$, we know $\mu^{-1}(\Xi) \subset \mathbb{A}^{1,1}$ defines a sub-manifold which allows for a $\widehat{\mathcal{G}}$-action. The quotient

$$\mathbb{A}^{1,1} /\!\!/ \widehat{\mathcal{G}} \equiv \mu^{-1}(\Xi)/\widehat{\mathcal{G}} \tag{4.18}$$

is well-defined and, moreover, is a Kähler manifold, as the Kähler form and the complex structure descend from $\mathbb{A}^{1,1}$.

We recognise the zero-level set $\mu^{-1}(0)/\widehat{\mathcal{G}}$ as the *Hermitian Yang-Mills* moduli space. In other words, the HYM equations consist of the holomorphicity conditions (4.10) together with the so-called stability condition

$$\mu(\mathcal{F}_\mathcal{A}) = \mathcal{F}_\mathcal{A}\wedge\frac{\omega^{n-1}}{(n-1)!} = 0 \quad \text{or equivalently} \quad \mu^*(\mathcal{F}_\mathcal{A}) = \omega\lrcorner\mathcal{F}_\mathcal{A} = 0\ . \tag{4.19}$$

[6] For the non-compact Calabi-Yau cone of this thesis, the boundary term arising by Stokes' theorem will be cancelled be restriction to *framed* gauge transformations. See Sec. 4.3.3.

By well-known theorems [35–37], a holomorphic vector bundle admits a solution to the HYM equations if and only if the bundle is (poly-)stable in the algebraic geometry sense.

Complex group action As the $\widehat{\mathcal{G}}$-action on $\mathbb{A}^{1,1}$ preserves the Kähler structure, one can extend it to a $\widehat{\mathcal{G}}^{\mathbb{C}}$-action on $\mathbb{A}^{1,1}$. In other words, the holomorphicity condition $\mathcal{F}_{\mathcal{A}}^{0,2} = 0$ are invariant under the action of the complex gauge group

$$\widehat{\mathcal{G}}^{\mathbb{C}} = \widehat{\mathcal{G}} \otimes \mathbb{C} \, . \tag{4.20}$$

Let $\mathcal{A} \in \mathbb{A}^{1,1}$, then the orbit $\widehat{\mathcal{G}}^{\mathbb{C}}_{\mathcal{A}}$ of $\mathcal{A}$ under the $\widehat{\mathcal{G}}^{\mathbb{C}}$-action is

$$\widehat{\mathcal{G}}^{\mathbb{C}}_{\mathcal{A}} = \left\{ \mathcal{A}' \in \mathbb{A}^{1,1} \,|\, \exists q \in \widehat{\mathcal{G}}^{\mathbb{C}} : \mathcal{A}' = \mathcal{A}^q \right\} . \tag{4.21}$$

A point $\mathcal{A} \in \mathbb{A}^{1,1}$ is called *stable* if $\widehat{\mathcal{G}}^{\mathbb{C}}_{\mathcal{A}} \cap \mu^{-1}(\Xi) \neq \emptyset$, and we denote by $\mathbb{A}^{1,1}_{st}(\Xi) \subset \mathbb{A}^{1,1}$ the set of all stable points (for a given Ξ). Then, a well-known result (see for example [136]) is

$$\mathbb{A}^{1,1} /\!\!/ \widehat{\mathcal{G}} \equiv \mu^{-1}(\Xi)/\widehat{\mathcal{G}} \cong \mathbb{A}^{1,1}_{st}(\Xi)/\widehat{\mathcal{G}}^{\mathbb{C}} \, . \tag{4.22}$$

Remark A peculiarity arises for holomorphic bundles E over a compact Kähler manifold M^{2n} with non-empty boundary [137]. Due to the prescription of boundary conditions, the stability condition is automatically satisfied for a unitary connection whose curvature is of type $(1,1)$. Hence, all points in $\mathbb{A}^{1,1}$ are stable in this case.

In the following we will consider the HYM equations (4.10) and (4.19) on the non-compact Calabi-Yau cones. For these, the holomorphicity conditions still imply the existence of a holomorphic structure, while the notion of stability is not applicable anymore. Nonetheless, we will continue referring to $\omega \lrcorner \mathcal{F}_{\mathcal{A}} = 0$ as *stability-like* condition.

4.3 Equivariant instantons

The main focus of this section lies on the description of instantons on certain vector bundles E over $\mathrm{Cyl}(M^{2n+1})$, where M^{2n+1} is taken to be Sasaki-Einstein. However, instead of generic connections the set-up will be restricted to connections that arise from an instanton on the Sasaki-Einstein space M^{2n+1} by an extension $X \in \Omega^1(\mathrm{Cyl}(M^{2n+1}), \mathrm{End}(E))$. This extension has to satisfy a certain invariance condition.

The arguments presented in what follows are a generalisation of [40,121,122]: i.e. we generalise from spherically symmetric instantons on vector bundles over $C(S^3) \cong \mathbb{R}^4 \backslash \{0\}$ with an SU(2)-structure to SU(n+1)-equivariant instantons on vector bundles over $C(M^{2n+1})$ with an SU(n+1)-structure, where M^{2n+1} is an arbitrary Sasaki-Einstein manifold. Analogously to the work of Donaldson and Kronheimer, it will be necessary to consider boundary conditions for the components of the connection 1-form, i.e. for the Yang-Mills field.

4.3.1 Ansatz

Let us recall the ansatz based on [113]. Start from any Sasaki-Einstein manifold M^{2n+1}, i.e. the manifold carries an SU(n)-structure $\mathcal{Q}$ together with the canonical connection Γ^P on the tangent bundle. The metric cone is Calabi-Yau with holonomy SU(n+1), i.e. it carries an integrable SU(n+1)-structure, as mentioned earlier in Sec. 3.1. By conformal equivalence one can consider $\mathrm{Cyl}(M^{2n+1})$, which is equipped with a non-integrable SU(n+1)-structure.

Let $\mathcal{P}$ be the principal SU(n+1) bundle of $\mathrm{SO}(\mathrm{Cyl}(M^{2n+1}), \widetilde{g})$ which comprises the SU(n+1)-structure. Consider an associated complex vector bundle $E \to \mathrm{Cyl}(M^{2n+1})$ of rank p, which consequently has structure group $\mathrm{SU}(n{+}1)$. In particular, it is a Hermitian vector bundle where $\mathcal{F}^\dagger = -\mathcal{F}$ and $\mathrm{tr}(\mathcal{F}) = 0$ hold for the curvature $\mathcal{F}$ of a compatible connection. (Thereby, $\mathrm{tr}(\mathcal{F}) = 0$ equals a vanishing first Chern class, which is consistent as $\mathcal{P}$ is the associated principal $\mathrm{SU}(n{+}1)$ bundle.) For example, the (holomorphic) tangent bundle of the Calabi-Yau cone is such a bundle, but one does not have to restrict to this case.

We recall that the connection 1-forms are $\mathfrak{su}(n{+}1)$-valued 1-forms on $\mathrm{Cyl}(M^{2n+1})$ for any connection $\mathcal{A}$ on E. The ansatz for a connection is

$$\mathcal{A} = \widehat{\Gamma}^P + X \tag{4.23a}$$

where $\widehat{\Gamma}^P$ is the lifted $\mathfrak{su}(n)$-valued connection on E obtained from Γ^P, i.e. one essentially has to change the representation on the fibres. Moreover, on a patch $\mathcal{U} \subset \mathrm{Cyl}(M^{2n+1})$ with the co-frame $\{e^{\hat{\mu}}\}$ we employ the local description

$$X_{|\mathcal{U}} = X_\mu \otimes e^\mu + X_{2n+2} \otimes e^{2n+2} \ , \tag{4.23b}$$

where $X_{\hat{\mu}|x} \in \mathrm{End}(\mathbb{C}^p)$ for $x \in \mathcal{U}$. Usually X_{2n+2} is eliminated by a suitable gauge transformation, but there is no harm in not doing so.

The ansatz (4.23) is a generic connection in the sense that the $X_{\hat{\mu}}$ are base point dependent, skew-Hermitian, traceless matrices with nontrivial transformation behaviour under change of trivialisation. Hence, *any* connection $\mathcal{A}$ on E can be reached starting from $\widehat{\Gamma}^P$, simply because $\mathbb{A}$ is an affine over the space in which X lives.

Next, we investigate the matrices $X_{\hat{\mu}}$ and their transformation behaviour under a change of e. By construction, $X_{\hat{\mu}} e^{\hat{\mu}}$ is the local representation of an Ad-equivariant 1-form X on the gauge principal bundle, which here coincides with the $\mathrm{SU}(n{+}1)$-subbundle $\mathcal{P}$. Note that, $\mathcal{P}$ contains a principal $\mathrm{SU}(n)$-subbundle $\mathcal{Q}$; the latter is the pullback of the $\mathrm{SU}(n)$-structure on M^{2n+1} and is the appropriate bundle for the connection Γ^P. Now let e and e' be two local sections of $\mathcal{Q} \subset \mathcal{P}$ over some $\mathcal{U} \subset \mathrm{Cyl}(M^{2n+1})$ related by an $\mathrm{SU}(n)$-transformation $L : \mathcal{U} \to \mathrm{SU}(n)$. The components $X'_{\hat{\mu}}$ and $X_{\hat{\mu}}$ of X with respect to e' and e are related via

$$X'_\mu = Ad(L^{-1}) \circ X_\nu \, \rho(L)^\nu_\mu \quad \text{and} \quad X'_{2n+2} = Ad(L^{-1}) \circ X_{2n+2} \ . \tag{4.24}$$

Here ρ is the dual of the representation of $\mathrm{SU}(n)$ on $\mathbb{R}^{2n+1}$ which is the typical fibre of TM^{2n+1}. It coincides with the representation $\mathrm{Ad}_{\mathrm{SU}(n+1)} : \mathrm{SU}(n) \to \mathrm{End}(\mathfrak{m})$, where $\mathfrak{su}(n{+}1) = \mathfrak{su}(n) \oplus \mathfrak{m}$.

Since $\mathrm{SU}(n)$ is a closed subgroup of $\mathrm{SU}(n{+}1)$, one can choose an $\mathrm{SU}(n)$-invariant decomposition

$$\mathfrak{su}(n{+}1) = \mathfrak{su}(n) \oplus \mathfrak{m} \qquad \text{with} \qquad \begin{aligned} \mathfrak{su}(n{+}1) &= \mathrm{span}\left\{ I_A \,|\, A = 1, \ldots, (n{+}1)^2{-}1 \right\} \ , \\ \mathfrak{su}(n) &= \mathrm{span}\left\{ I_\alpha \,|\, \alpha = 2n{+}2, \ldots, (n{+}1)^2{-}1 \right\} \ , \\ \mathfrak{m} &= \mathrm{span}\left\{ I_\mu \,|\, \mu = 1, \ldots, 2n{+}1 \right\} \ , \end{aligned} \tag{4.25}$$

and denote by $\widehat{I}_A$ the generators in a representation on the fibres $E_x \cong \mathbb{C}^p$. By the $\mathrm{Ad}_{\mathrm{SU}(n)}$-invariant splitting, one has the following commutation relations:

$$\left[\widehat{I}_\alpha, \widehat{I}_\beta\right] = f_{\alpha\beta}{}^\gamma \widehat{I}_\gamma \ , \quad \left[\widehat{I}_\alpha, \widehat{I}_\mu\right] = f_{\alpha\mu}{}^\nu \widehat{I}_\nu \ , \quad \left[\widehat{I}_\mu, \widehat{I}_\nu\right] = f_{\mu\nu}{}^\alpha \widehat{I}_\alpha + f_{\mu\nu}{}^\sigma \widehat{I}_\sigma \ , \tag{4.26}$$

for $\alpha, \beta, \gamma = 2n{+}2, \ldots, (n{+}1)^2{-}1$ and $\mu, \nu, \sigma = 1, \ldots, 2n{+}1$. A suitable choice of these structure constants can be found in [84, 113].

Generically, only X is well-defined globally, rather than the component maps $X_{\hat{\mu}}$. The latter strongly depend on the choice of the local frame e and, therefore, we have no control over their behaviour in general. In other words, one could try to find local solutions, but it is a priori not clear that they glue together to a global solution. That would be different, if the components $X_{\hat\mu}$ were independent of the trivialisation of the involved bundles, that is, if the $X_{\hat\mu}$ were invariant under the aforementioned transformations (4.24) that change the local frames. Furthermore, since $\mathrm{SU}(n)$ is connected, this is equivalent to the infinitesimal version of the invariance, i.e.

$$\left[\widehat{I}_\alpha, X_\mu\right] = \rho_*(I_\alpha)_\mu^\nu\, X_\nu = f_{\alpha\mu}{}^\nu X_\nu \quad \text{and} \quad \left[\widehat{I}_\alpha, X_{2n+2}\right] = 0 \ , \tag{4.27}$$

for $\mu,\nu = 1,\dots,2n{+}1$ and $\alpha = 2n{+}2,\dots,(n{+}1)^2{-}1$. Note that this simplification implies that the $X_{\hat\mu}$ are independent of the choice of frame adapted to the $\mathrm{SU}(n)$-structure $\mathcal{Q}$. As additional simplification, we choose them to vary with the cone or cylinder direction only. Condition (4.27) appeared, for example, in [138, 139] on coset spaces, where equivariant connections have been constructed. We will in the following refer to (4.27) as the *equivariance condition*, for the same reasons. The representation-theoretic content of (4.27) is that the matrix-valued functions $X_{\hat\mu}$ have to transform in a representation of $\mathfrak{su}(n)$.

Computing the curvature $\mathcal{F}_\mathcal{A}$ for the ansatz (4.23) together with an X satisfying the *equivariance condition* (4.27) then yields

$$\begin{aligned}\mathcal{F}_\mathcal{A} = \mathcal{F}_{\widehat\Gamma^P} &+ \frac{1}{2}\left([X_a,X_b] + T_{a\,b}^{2n+1}X_{2n+1}\right) e^a\wedge e^b + \left([X_a,X_{2n+1}] + T_{a\,2n+1}^{b}X_b\right) e^a\wedge e^{2n+1} \\ &+ \left([X_a,X_{2n+2}] - \tfrac{\mathrm{d}}{\mathrm{d}t}X_a\right) e^a\wedge e^{2n+2} \\ &+ \left([X_{2n+1},X_{2n+2}] - \tfrac{\mathrm{d}}{\mathrm{d}t}X_{2n+1}\right) e^{2n+1}\wedge e^{2n+2} \ ,\end{aligned} \tag{4.28}$$

with $\mathcal{F}_{\widehat\Gamma^P}$ is the curvature of $\widehat\Gamma^P$, and $a,b = 1,\dots,2n$. The torsion components of $\widehat\Gamma^P$ are denoted by $T_{\mu\nu}^{\rho}$. The HYM instanton equations (4.10) and (4.19) reduce for the ansatz to a set of matrix equations for the $X_{\hat\mu}$, which we spell out in a moment. Let us note that these equations have already been derived in [113], for the choice $X_{2n+2} = 0$. Moreover, $\mathcal{F}_{\widehat\Gamma^P}$ already satisfies the HYM equations, as the connection $\widehat\Gamma^P$ is the lift of an $\mathrm{SU}(n)$-instanton, and the corresponding $\mathrm{SU}(n)$-principal bundle is a subbundle in the $\mathrm{SU}(n{+}1)$-principal bundle associated to E.

Matrix equations: real basis The resulting instanton matrix equations in the real basis $\{e^{\hat\mu}\}$ are the holomorphicity conditions

$$[X_{2j-1},X_{2k-1}] - [X_{2j},X_{2k}] = 0 \ , \tag{4.29a}$$
$$[X_{2j-1},X_{2k}] + [X_{2j},X_{2k-1}] = 0 \ , \tag{4.29b}$$
$$[X_{2j-1},X_{2n+2}] + [X_{2j},X_{2n+1}] = \tfrac{\mathrm{d}}{\mathrm{d}t}X_{2j-1} + \tfrac{n+1}{n}X_{2j-1} \ , \tag{4.29c}$$
$$[X_{2j},X_{2n+2}] - [X_{2j-1},X_{2n+1}] = \tfrac{\mathrm{d}}{\mathrm{d}t}X_{2j} + \tfrac{n+1}{n}X_{2j} \ , \tag{4.29d}$$

for $j,k = 1,\dots,n$ and the stability-like condition

$$\tfrac{\mathrm{d}}{\mathrm{d}t}X_{2n+1} + 2nX_{2n+1} = \sum_{k=1}^{n+1}[X_{2k-1},X_{2k}] \ . \tag{4.29e}$$

Solving (4.29) subject to (4.27) is usually a sophisticated task, as one has to find a suitable ansatz that satisfies (4.27) and that at the same time reduces to non-trivial, solvable equations for (4.29). For instance, in the *temporal gauge* $X_{2n+2} = 0$, a possible choice of ansatz [113] is

$$X_a(\tau) = \psi(\tau)\widehat{I}_a \quad \text{and} \quad X_{2n+1}(\tau) = \chi(\tau)\widehat{I}_{2n+1} \ . \tag{4.30}$$

This ansatz satisfies the equivariance condition and reduces the above matrix equations to two coupled ordinary differential equations of first order,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\psi = \frac{n+1}{n}\psi(\chi-1) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\chi = 2n(\psi^2-\chi) \tag{4.31}$$

for the functions $\psi(\tau)$ and $\chi(\tau)$. Coincidentally, these equations have already been studied in [84]; although there they have been derived from a different, less generic starting point.

Matrix equations: complex basis For the intents and purposes here, it is more convenient to switch to the complex basis $\{\theta^j, \bar{\theta}^j\}$ defined in (3.6) and introduce

$$Y_j \coloneqq \frac{1}{2}\left(X_{2j} - \mathrm{i}X_{2j-1}\right) \quad \text{and} \quad Y_{\bar{j}} \coloneqq \frac{1}{2}\left(X_{2j} + \mathrm{i}X_{2j-1}\right) \qquad \text{for} \quad j = 1, 2, \ldots, n+1 \ . \tag{4.32}$$

Hence, $Y_{\bar{j}} = -(Y_j)^\dagger$ since $X_{\hat{\mu}}(t) \in \mathfrak{su}(n+1)$ for all $t \in \mathbb{R}$. For the $Y_j : \mathbb{R} \to \mathrm{End}(\mathbb{C}^p)$ one finds the holomorphicity conditions

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Y_j + \tfrac{n+1}{n}Y_j = 2\left[Y_j, Y_{n+1}\right] \quad \text{and} \quad \left[Y_j, Y_k\right] = 0 \qquad \text{for} \quad j,k = 1, \ldots, n \ , \tag{4.33a}$$

and the adjoint equations thereof. The stability-like condition reads

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(Y_{n+1} + Y_{n+1}^\dagger\right) + 2n\left(Y_{n+1} + Y_{n+1}^\dagger\right) + 2\sum_{j=1}^{n+1}\left[Y_j \, , Y_j^\dagger\right] = 0 \ . \tag{4.33b}$$

The equivariance conditions for the complex matrices are

$$\left[\widehat{I}_\alpha, Y_j\right] = -\mathrm{i}f_{\alpha 2j-1}^{\ \ 2j}Y_j \quad \text{and} \quad \left[\widehat{I}_\alpha, Y_{n+1}\right] = 0 \ , \tag{4.34}$$

for $j, = 1, \ldots, n$. For these calculations we have used a choice of structure constants $f_{\alpha\mu}^{\ \ \nu} = 0$ if μ or $\nu = 2n+1$ and $f_{\alpha a}^{\ \ b} \propto \omega_{ab}$, see for instance [84, 113]. Later, in Ch. 5 we actually need to solve the equivariance condition and we provide such a choice in (5.14) and (5.15). Here, we *assume* the equivariance condition to be solved, cf. the last comment in Sec. 4.3.5.

Change of trivialisation The remaining non-trivial effects of a change of trivialisation of the bundle E over $\mathrm{Cyl}(M^{2n+1})$ are given by the set of functions $\{g(t) |\, g : \mathbb{R} \to \mathrm{SU}(p)\}$ that act as

$$X_\mu \mapsto \mathrm{Ad}(g)X_\mu \qquad \text{for} \quad \mu = 1, \ldots, 2n+1 \quad \text{and} \quad X_{2n+2} \mapsto \mathrm{Ad}(g)X_{2n+2} - \left(\tfrac{\mathrm{d}}{\mathrm{d}t}g\right)g^{-1} \ , \tag{4.35}$$

which follows from[7] $\mathcal{A} \mapsto \mathcal{A}^g = \mathrm{Ad}(g)\mathcal{A} - (\mathrm{d}g)g^{-1}$ and $g = g(t)$. Due to their adjoint transformation behaviour, the X_μ are sometimes called *Higgs fields*, for example in quiver gauge theories. The inhomogeneous transformation of X_{2n+2} is crucial to be able to *gauge away* this connection component. Furthermore, these gauge transformations (and their complexification) will be used to study the solutions of the matrix equations.

Yang-Mills with torsion The instanton equations (on the cone *and* the cylinder) are equivalently given by

$$\star\,\mathcal{F}_\mathcal{A} = -\frac{\omega^{n-1}}{(n-1)!} \wedge \mathcal{F}_\mathcal{A} \ , \tag{4.36}$$

[7] We have simply replaced g in (5.11) by g^{-1}.

where ω is the corresponding $(1,1)$-form ($\mathrm{d}\omega = 0$ on the cone, but $\mathrm{d}\omega \neq 0$ on the cylinder). An immediate consequence is that the instanton equation for the integrable $\mathrm{SU}(n{+}1)$-structure implies the Yang-Mills equations, while this is not true for the $\mathrm{SU}(n{+}1)$-structure with torsion. In detail

$$\text{cone:} \qquad (4.36) \quad \Rightarrow \quad \mathrm{d}_{\mathcal{A}} \star \mathcal{F}_{\mathcal{A}} = 0 \qquad \text{Yang-Mills}\ , \tag{4.37a}$$

$$\text{cylinder:} \ (4.36) \quad \Rightarrow \quad \mathrm{d}_{\mathcal{A}} \star \mathcal{F}_{\mathcal{A}} + \frac{\omega^{n-2}}{(n-2)!} \wedge \mathrm{d}\omega \wedge \mathcal{F}_{\mathcal{A}} = 0 \quad \text{Yang-Mills with torsion}\ . \tag{4.37b}$$

These *torsionful Yang-Mills* equations (4.37b), which arise in the context of non-integrable G-structures (with intrinsic torsion), have been studied in the literature before [125, 128, 140–143]. In particular, the torsion term does not automatically vanish on instantons because $\mathrm{d}\omega$ contains $(2,1)$ and $(1,2)$-forms. This is, for instance, in contrast to the nearly Kähler case discussed in [144], in which nearly Kähler instantons were found to satisfy the ordinary Yang-Mills equations.

It is known that the appropriate functional for the torsionful Yang-Mills equations consists of the ordinary Yang-Mills functional plus an additional Chern-Simons term

$$S_{\mathrm{YM+T}}(\mathcal{A}) = \int_{\mathrm{Cyl}(M^{2n+1})} \mathrm{tr}\left(\mathcal{F}_{\mathcal{A}} \wedge \star \mathcal{F}_{\mathcal{A}}\right) + \frac{\omega^{n-1}}{(n-1)!} \wedge \mathrm{tr}\left(\mathcal{F}_{\mathcal{A}} \wedge \mathcal{F}_{\mathcal{A}}\right)\ , \tag{4.38}$$

which is a gauge-invariant functional. The properties of $S_{\mathrm{YM+T}}$ are the following: firstly and most importantly, instanton connections satisfying (4.36) have $S_{\mathrm{YM+T}}(\mathcal{A}) = 0$, i.e. the *action is finite.* Secondly, the stationary points of (4.38) are the vanishing locus of the torsionful Yang-Mills equations (up to boundary terms). For this, we use $\mathcal{F}_{\mathcal{A}+z\Psi} = \mathcal{F}_{\mathcal{A}} + z\mathrm{d}_{\mathcal{A}}\Psi + \frac{1}{2}z^2 \Psi \wedge \Psi$ for any $\Psi \in T_{\mathcal{A}}\mathbb{A}(E)$ and compute the variation

$$\begin{aligned}
\delta S_{\mathrm{YM+T}}(\mathcal{A}) &\coloneqq \frac{\mathrm{d}}{\mathrm{d}z} S_{\mathrm{YM}}(\mathcal{A} + z\Psi)\Big|_{z=0} \\
&= \int_{\mathrm{Cyl}(M^{2n+1})} 2\ \mathrm{tr}\left(\mathrm{d}_{\mathcal{A}}\Psi \wedge \star \mathcal{F}_{\mathcal{A}}\right) + 2\frac{\omega^{n-1}}{(n-1)!} \wedge \mathrm{tr}\left(\mathcal{F}_{\mathcal{A}} \wedge \mathrm{d}_{\mathcal{A}}\Psi\right) \\
&= 2\int_{\mathrm{Cyl}(M^{2n+1})} \mathrm{tr}\left[\Psi \wedge \left(\mathrm{d}_{\mathcal{A}} \star \mathcal{F}_{\mathcal{A}} + \frac{\omega^{n-2}}{(n-2)!} \wedge \mathrm{d}\omega \wedge \mathcal{F}_{\mathcal{A}}\right)\right] \\
&\qquad + 2\int_{\mathrm{Cyl}(M^{2n+1})} \mathrm{d}\ \mathrm{tr}\left[\Psi \wedge \left(\star \mathcal{F}_{\mathcal{A}} + \frac{\omega^{n-1}}{(n-1)!} \wedge \mathcal{F}_{\mathcal{A}}\right)\right]\ .
\end{aligned} \tag{4.39}$$

The boundary term cannot simply vanish as the cylinder is never a closed manifold. Hence, if one assumes M^{2n+1} to be closed, the vanishing of the boundary term requires certain assumptions on the fall-off rate of $\mathcal{F}_{\mathcal{A}}$ for $t \to \pm\infty$. Moreover, it is interesting to observe that the boundary term in (4.39) vanishes for instanton configurations.

4.3.2 Rewriting the instanton equations

Real equations Returning to the reduced instanton equations for the X-matrices (4.29), the linear terms can be eliminated via a change of coordinates:

$$X_{2j-1} \eqqcolon e^{-\frac{n+1}{n}t}\, \mathcal{X}_{2j-1}\ , \qquad X_{2j} \eqqcolon e^{-\frac{n+1}{n}t}\, \mathcal{X}_{2j} \qquad \text{for} \quad j = 1, \dots, n\ , \tag{4.40a}$$

$$X_{2n+1} \eqqcolon e^{-2nt}\, \mathcal{X}_{2n+1}\ , \qquad X_{2n+2} \eqqcolon e^{-2nt}\, \mathcal{X}_{2n+2}\ , \tag{4.40b}$$

$$s = -\frac{1}{2n} e^{-2nt} \in \mathbb{R}^-\ , \qquad \lambda_n(s) \coloneqq \left(\frac{-1}{2ns}\right)^{2-\frac{n+1}{n^2}}\ . \tag{4.40c}$$

Note that the exponent $2 - \frac{n+1}{n^2}$ vanishes for $n = 1$ and is strictly positive for any $n > 1$. The matrix equations (4.29) now read as follows:

$$[\mathcal{X}_{2j-1}, \mathcal{X}_{2k-1}] - [\mathcal{X}_{2j}, \mathcal{X}_{2k}] = 0 \ , \qquad [\mathcal{X}_{2j-1}, \mathcal{X}_{2k}] + [\mathcal{X}_{2j}, \mathcal{X}_{2k-1}] = 0 \ , \tag{4.41a}$$

$$[\mathcal{X}_{2j-1}, \mathcal{X}_{2n+2}] + [\mathcal{X}_{2j}, \mathcal{X}_{2n+1}] = \tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{X}_{2j-1} \ , \quad [\mathcal{X}_{2j}, \mathcal{X}_{2n+2}] - [\mathcal{X}_{2j-1}, \mathcal{X}_{2n+1}] = \tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{X}_{2j} \ , \tag{4.41b}$$

for $j, k = 1, \dots, n$ and

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{X}_{2n+1} = \lambda_n(s) \sum_{k=1}^{n} [\mathcal{X}_{2k-1}, \mathcal{X}_{2k}] + [\mathcal{X}_{2n+1}, \mathcal{X}_{2n+2}] \ . \tag{4.41c}$$

Complex equations Completely analogously, the change of coordinates for the complex equations is performed via

$$Y_j =: e^{-\frac{n+1}{n}t}\mathcal{Y}_j \qquad \text{for} \quad j = 1, \dots, n \quad \text{and} \quad Y_{n+1} =: e^{-2nt}\mathcal{Z} \ . \tag{4.42}$$

We will refer to this set of matrices simply by $(\mathcal{Y}, \mathcal{Z})$. In summary, the instanton equations are now comprised by the *complex equations*

$$[\mathcal{Y}_j, \mathcal{Y}_k] = 0 \quad \text{and} \quad \tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{Y}_j = 2\,[\mathcal{Y}_j, \mathcal{Z}] \qquad \text{for} \quad j, k = 1, \dots, n \ , \tag{4.43a}$$

and the *real equation*

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\left(\mathcal{Z} + \mathcal{Z}^\dagger\right) + 2\left[\mathcal{Z}, \mathcal{Z}^\dagger\right] + 2\lambda_n(s)\sum_{j=1}^{n}\left[\mathcal{Y}_j, \mathcal{Y}_j^\dagger\right] = 0 \ . \tag{4.43b}$$

These equations are reminiscent of those encountered in the considerations of the instantons on $\mathbb{R}^4\backslash\{0\}$ of [40,121,122], and, in fact, they reduce to the same system for $n = 1$, but in general on a Calabi-Yau 2-fold $\mathbb{C}^2/\Gamma$. To see this, we recall [120] that all 3-dimensional Sasaki-Einstein spaces are given by S^3/Γ, where Γ is a finite subgroup of SU(2) (and commutes with $\mathrm{U}(1) \subset \mathrm{SU}(2)$) which acts freely and isometrically from the left on $S^3 \cong \mathrm{SU}(2)$.

Remarks The equivariance conditions for the rescaled matrices $\{\mathcal{X}_{\hat{\mu}}\}$ or $(\{\mathcal{Y}_j\}, \mathcal{Z})$ are exactly the same as (4.27) or (4.34), respectively.

Moreover, the rescaling has another salient feature: the matrices $\{\mathcal{X}_{\hat{\mu}}\}$ or $(\{\mathcal{Y}_j\}, \mathcal{Z})$ (as well as their derivatives) are bounded (see for instance [121]); in contrast, the original connection components will develop a pole at the origin $r = 0$. This will become apparent once the boundary conditions are specified. For further details, see App. A.1.

In addition, we observe that the exponents on the rescaling (4.40) reflect the torsion components (3.3). The choice of a flat starting point $\Gamma = 0$ would lead to Nahm-type equations straight away, but solutions to the resulting matrix equations would not interpolate between any (non-trivial) lifted instantons from M^{2n+1} and instantons on the Calabi-Yau space $C(M^{2n+1})$, cf. [113, 128].

Real gauge group The full set of instanton equations (4.43) is invariant under the action of the gauge group

$$\widehat{\mathcal{G}} \coloneqq \{g(s) | g : \mathbb{R}^- \to \mathrm{U}(p)\} \ , \tag{4.44}$$

wherein the action is defined via

$$\mathcal{Y}_j \mapsto \mathcal{Y}_j^g \coloneqq \mathrm{Ad}(g)\mathcal{Y}_j \qquad \text{for} \quad j = 1, \dots, n \ , \tag{4.45a}$$

$$\mathcal{Z} \mapsto \mathcal{Z}^g := \mathrm{Ad}(g)\mathcal{Z} - \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}s}g\right) g^{-1} . \tag{4.45b}$$

Note that only the real equation (4.43b) requires $g^{-1} = g^\dagger$ for it to be gauge invariant. Moreover, one can always find a gauge transformation $g \in \widehat{\mathcal{G}}$ such that $\mathcal{Z}^g = (\mathcal{Z}^g)^\dagger$ (Hermitian) or, equivalently, $X^g_{2n+2} = 0$.

In summary, these properties follow from (4.35) as the X-matrices are extensions to a connection. However, the gauge group (4.44) still contains $\{g(s)|g(s) = \phi(s)\mathbb{1}_{p\times p} \text{ with } \phi : \mathbb{R}^- \to \mathrm{U}(1)\}$ as non-trivial centre, such that (4.35) corresponds to the quotient of $\widehat{\mathcal{G}}$ by its centre.

Complex gauge group Moreover, the complex equations (4.43a) allow for an action of the complexified gauge group

$$\widehat{\mathcal{G}}^{\mathbb{C}} \equiv \{g(s)|g : \mathbb{R}^- \to \mathrm{GL}(p,\mathbb{C})\} \ , \tag{4.46}$$

given by

$$\mathcal{Y}_k \mapsto \mathrm{Ad}(g)\mathcal{Y}_k \ , \qquad \mathcal{Y}_{\bar{k}} \mapsto \mathrm{Ad}((g^{-1})^\dagger)\mathcal{Y}_{\bar{k}} \ , \qquad \text{for} \quad k = 1, \dots, n \ , \tag{4.47a}$$

$$\mathcal{Z} \mapsto \mathrm{Ad}(g)\mathcal{Z} - \tfrac{1}{2}\left(\tfrac{\mathrm{d}}{\mathrm{d}s}g\right) g^{-1} \ , \qquad \bar{\mathcal{Z}} \mapsto \mathrm{Ad}((g^{-1})^\dagger)\bar{\mathcal{Z}} + \tfrac{1}{2}(g^{-1})^\dagger \left(\tfrac{\mathrm{d}}{\mathrm{d}s}g^\dagger\right) \ . \tag{4.47b}$$

The extension to $\widehat{\mathcal{G}}^{\mathbb{C}}$-invariance for the holomorphicity conditions exemplifies the generic situation discussed in Sec. 4.2.

Boundary conditions We observe that a trivial solution of (4.41) is

$$\mathcal{X}_{2n+2}(s) = 0 \quad \text{and} \quad \mathcal{X}_\mu(s) = T_\mu \qquad \text{with} \quad [T_\mu, T_\nu] = 0 \qquad \text{for} \quad \mu,\nu = 1, \dots, 2n+1 \ , \tag{4.48}$$

where the (constant) T_μ are elements in the Cartan subalgebra of $\mathfrak{su}(p)$; i.e. the (real) $(p-1)$-dimensional space spanned by the diagonal, traceless matrices with purely imaginary values. Later, we will be *modelling* the general solution based on this particular solution.

To begin with, we observe from the rescaling (4.40) of the $X_{\hat{\mu}}$ that these matrices become singular as $r \to 0$ ($t \to -\infty$ or $s \to -\infty$). Following [121, 145], it is appropriate to choose the boundary conditions for X_μ to be[8]

$$s \to 0 : \quad X_\mu(s) \to 0 \qquad \text{for} \quad \mu = 1, \dots, 2n+1 \quad \text{and} \tag{4.49a}$$

$$s \to -\infty : \quad \exists\, g_0 \in \mathrm{U}(p) \ \text{ such that } \ \mathcal{X}_\mu(s) \to \mathrm{Ad}(g_0)T_\mu \qquad \text{for} \quad \mu = 1, \dots, 2n+1 \ . \tag{4.49b}$$

One can show [121] that this implies the existence of the limit of $\mathcal{X}_\mu$ for $s \to 0$. Hence, the solutions extend to the interval $(-\infty, 0]$, see also App. A.1. Thus, we are led to consider (4.41) for matrices $\mathcal{X}_\mu(s)$ over $(-\infty, 0]$ with one remaining boundary condition:

$$\exists\, g_0 \in \mathrm{U}(p) \text{ such that } \forall \mu = 1, \dots, 2n+1 : \lim_{s\to-\infty} \mathcal{X}_\mu(s) = \mathrm{Ad}(g_0)\, T_\mu \ . \tag{4.50}$$

We have to give a little thought on the choice of boundary values T_μ. If the T_μ lie in a Cartan subalgebra of $\mathfrak{su}(p)$, then the generic model solution to (4.29) subject to (4.50) is given by

$$X_a(t) = e^{-\frac{n+1}{n}t} T_a + S_a, \qquad X_{2n+1}(t) = e^{-2nt} T_{2n+1} + S_{2n+1} \quad \text{and} \quad X_{2n+2}(t) = 0 \, , \tag{4.51}$$

[8] One does not need to worry about X_{2n+2}, as it can always be gauged away.

with elements $S_\mu \in \mathfrak{su}(p)$ that commute with each T_μ and satisfy the algebra[9]

$$[S_{2j-1}, S_{2k-1}] - [S_{2j}, S_{2k}] = 0 \ , \qquad [S_{2j-1}, S_{2k}] + [S_{2j}, S_{2k-1}] = 0 \ , \tag{4.52a}$$

$$[S_{2j}, S_{2n+1}] = \tfrac{n+1}{n} S_{2j-1} \ , \qquad [S_{2j-1}, S_{2n+1}] = -\tfrac{n+1}{n} S_{2j} \ , \tag{4.52b}$$

$$\sum_{k=1}^{n} [S_{2k-1}, S_{2k}] = 2n S_{2n+1} \ , \tag{4.52c}$$

for $k, j = 1, \dots, n$. The rescaled matrices take the following form:

$$\mathcal{X}_a(s) = T_a + \frac{S_a}{(-2ns)^{\frac{n+1}{2n^2}}} , \qquad \mathcal{X}_{2n+1}(s) = T_{2n+1} + \frac{S_{2n+1}}{-2ns} \quad \text{and} \quad \mathcal{X}_{2n+2} = 0 \ . \tag{4.53}$$

For simplification, we can *require* the T_μ to be a *regular tuple*, i.e. the intersection of the centralisers of the T_μ consists only of the Cartan subalgebra of $\mathfrak{su}(p)$. Then all of the S_μ have to vanish such that the T_μ alone provide the only model for the behaviour of the $\mathcal{X}_\mu$ near $s \to -\infty$.

Moreover, since one has first order differential equations, it suffices to impose this one boundary condition, here at $s = -\infty$. Thus, the values of $\mathcal{Y}_k$ at $s = 0$ are completely determined by the solution. Following [121], we observe that (4.43a) implies that $\mathcal{Y}_k(s)$ lies entirely in a single adjoint orbit $\mathcal{O}(k)$ of the complex group $\mathrm{GL}(p, \mathbb{C})$, for each $k = 1, \dots, n$. By further *imposing* that $\mathcal{T}_k = \frac{1}{2}(T_{2k} - \mathrm{i}T_{2k-1})$ is a *regular pair* in the complexified Cartan subalgebra, i.e. the centraliser of $\mathcal{T}_k$ in $\mathfrak{gl}(p, \mathbb{C})$ is only the Cartan subalgebra itself, for each $k = 1, \dots, n$, one obtains that $\mathcal{Y}_k(s{=}0) \in \mathcal{O}(k)$. Thus, the values at $s = 0$ are in a conjugacy class of $\mathcal{T}_k$ because the orbits are closed. Moreover, only the conjugacy class has a gauge-invariant meaning.

Nonetheless, the boundary conditions (4.50) clearly show that the original connection (4.23) develops the following poles at the origin $r = 0$ of the Calabi-Yau cone:

$$\lim_{r\to 0} r^{\frac{n+1}{n}} X_a = \mathrm{Ad}(g_0)\, T_a \quad \text{for} \quad a = 1, \dots, 2n \quad \text{and} \quad \lim_{r \to 0} r^{2n} X_{2n+1} = \mathrm{Ad}(g_0)\, T_{2n+1} \ . \tag{4.54}$$

Note that the case $n = 1$ is reminiscent to the *instantons with poles* considered in [121].

4.3.3 Geometric structure

Consider the space of $\mathfrak{su}(n{+}1)$-valued connections $\mathbb{A}(E)$ in which any element can be parametrised as in (4.23). Due to the ansatz of Sec. 4.3.1, we restrict ourselves to the subspace $\mathbb{A}_{\mathrm{equi}}(E) \subset \mathbb{A}(E)$ of connections which satisfy (4.27). Specialising the considerations of Sec. 4.2, we will now establish certain (formal) geometric structures.

Kähler structure The first step is to establish a Kähler structure on $\mathbb{A}_{\mathrm{equi}}(E)$. Since $\mathbb{A}_{\mathrm{equi}}(E)$ is a subset of the space of all connection $\mathbb{A}(E)$, one can simply obtain the geometric structures by restriction. A tangent vector

$$\boldsymbol{y} = \sum_{j=1}^{n+1} \left(\boldsymbol{y}_j \theta^j + \boldsymbol{y}_{\bar{j}} \bar{\theta}^j \right) \tag{4.55}$$

at a point $\mathcal{A} \in \mathbb{A}_{\mathrm{equi}}(E)$ is defined by the linearisation of (4.33) for paths $\boldsymbol{y}_j : \mathbb{R} \to \mathfrak{su}(p)$. Their gauge transformations are

$$\boldsymbol{y}_j \to \boldsymbol{y}_j^g \coloneqq \mathrm{Ad}(g) \boldsymbol{y}_j \qquad \text{for} \quad j = 1, \dots, n+1 \ . \tag{4.56}$$

[9]For $n = 1$, this coincides with the usual $\mathfrak{su}(2)$ relations $[S_1, S_2] = 2S_3$, $[S_2, S_3] = 2S_1$, and $[S_1, S_3] = -2S_2$.

Taking the generic expressions for the metric (4.8) and the symplectic structure (4.9), we can specialise to the case at hand by transition to the cylinder and neglecting the volume integral of M^{2n+1}. Thus, for a metric on $\mathbb{A}_{\text{equi}}$ we obtain

$$\boldsymbol{g}_{|\mathcal{A}}\left(\boldsymbol{y}^{(1)},\boldsymbol{y}^{(2)}\right) \equiv 2\int_{\mathbb{R}} \mathrm{d}t\ e^{2nt}\ \mathrm{tr}\left\{\sum_{j=1}^{n+1}\left(\boldsymbol{y}_j^{(1)\dagger}\boldsymbol{y}_j^{(2)}+\boldsymbol{y}_j^{(1)}\boldsymbol{y}_j^{(2)\dagger}\right)\right\}\ . \tag{4.57}$$

Similarly, the symplectic form reads as

$$\boldsymbol{\omega}_{|\mathcal{A}}\left(\boldsymbol{y}^{(1)},\boldsymbol{y}^{(2)}\right) \equiv -2\mathrm{i}\int_{\mathbb{R}} \mathrm{d}t\ e^{2nt}\ \mathrm{tr}\left\{\sum_{j=1}^{n+1}\left(\boldsymbol{y}_j^{(1)\dagger}\boldsymbol{y}_j^{(2)}-\boldsymbol{y}_j^{(1)}\boldsymbol{y}_j^{(2)\dagger}\right)\right\}\ . \tag{4.58}$$

Moreover, a complex structure $\boldsymbol{J}$ on $\mathbb{A}(E)_{\text{equiv}}$ has been given in (4.12). Keeping in mind that (3.6) implies $J=-J_{\text{can}}$, we obtain

$$\boldsymbol{J}_{|\mathcal{A}}(\boldsymbol{y}) = J(\boldsymbol{y}) = \mathrm{i}\sum_{j=1}^{n+1}\left(\boldsymbol{y}_j\theta^j - \boldsymbol{y}_{\bar{j}}\bar{\theta}^{j}\right) \tag{4.59}$$

As before, the symplectic form $\boldsymbol{\omega}$ and the metric $\boldsymbol{g}$ are compatible, i.e $\boldsymbol{g}(\boldsymbol{J}\cdot,\cdot)=\boldsymbol{\omega}(\cdot,\cdot)$. We note that both structures are gauge-invariant by construction.

Moment map The subspace of holomorphic connections $\mathbb{A}^{1,1}_{\text{equi}}(E)\subset\mathbb{A}_{\text{equi}}(E)$ is defined by the condition (4.43a). This condition only restricts the allowed endmorphism-valued 1-forms, because $\widehat{\Gamma}^P$ is already a $(1,1)$-type connection, since it is an HYM-instanton. Again, the metric $\boldsymbol{g}$ and Kähler form $\boldsymbol{\omega}$ descend to $\mathbb{A}^{1,1}_{\text{equi}}(E)$ from the corresponding objects on $\mathbb{A}_{\text{equi}}(E)$. Moreover, on the Kähler space $\mathbb{A}^{1,1}_{\text{equi}}(E)$, one defines a moment map

$$\begin{aligned}\mu:\quad &\mathbb{A}^{1,1}_{\text{equi}}(E)\to\widehat{\mathfrak{g}}_0=\mathrm{Lie}(\widehat{\mathcal{G}}_0)\\ &(\mathcal{Y},\mathcal{Z})\mapsto \mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathcal{Z}+\mathcal{Z}^\dagger\right)+2\left[\mathcal{Z},\mathcal{Z}^\dagger\right]+2\ \lambda_n(s)\sum_{k=1}^{n}\left[\mathcal{Y}_k,\mathcal{Y}_k^\dagger\right]\right)\ ,\end{aligned} \tag{4.60}$$

where $\widehat{\mathcal{G}}_0$ is the corresponding *framed* gauge group. That is

$$\widehat{\mathcal{G}}_0 := \left\{g(s)|g:\mathbb{R}^-\to\mathrm{U}(p)\ ,\ \lim_{s\to0}g(s)=\lim_{s\to-\infty}g(s)=1\right\}\ . \tag{4.61}$$

It is an important observation that on the non-compact Calabi-Yau cone (and the conformally equivalent cylinder) one has to compensate the appearing boundary terms in Stokes' theorem by the transition to the framed gauge transformations. The details of the proof that (4.60) satisfies conditions (4.14) are given in the App. A.2. Here, we just note that the map (4.60) maps the matrices $(\mathcal{Y},\mathcal{Z})$ into the correct space: the factor of i renders the expression anti-Hermitian, while the boundary conditions (4.49) together with the gauge choice $\mathcal{Z}=-\mathcal{Z}^\dagger$ yield the vanishing of $\mu\left(\mathcal{Y},\mathcal{Z}\right)$ at $s\to0$ and $s\to-\infty$.

The part of instanton moduli space that is connected with the lift $\widehat{\Gamma}^P$ (in the sense of our ansatz (4.23)) is then readily obtained by the Kähler quotient

$$\mathcal{M}_{\Gamma^P}=\mu^{-1}(0)/\widehat{\mathcal{G}}_0\ . \tag{4.62}$$

Stable points Alternatively, one can describe this part of the moduli space via the stable points

$$\mathbb{A}^{1,1}_{st}(E) \equiv \left\{ \widehat{\Gamma}^P + X \in \mathbb{A}^{1,1}(E) \middle| \, (\widehat{\mathcal{G}}^{\mathbb{C}}_0)_{(\mathcal{Y},\mathcal{Z})} \cap \mu^{-1}(0) \neq \emptyset \right\} , \tag{4.63}$$

where the tuple $(\mathcal{Y}, \mathcal{Z})$ is obtained from X via complex linear combinations and rescaling as before. The moduli space arises then by taking the $\widehat{\mathcal{G}}^{\mathbb{C}}_0$-quotient

$$\mathbb{A}^{1,1}_{st}(E)/\widehat{\mathcal{G}}^{\mathbb{C}}_0 \cong \mathcal{M}_{\Gamma^P} . \tag{4.64}$$

We argue in the next couple of paragraphs that it suffices to solve the complex equations (4.43a), because the solution to the real equation (4.43b) follows from a framed complex gauge transformation. More precisely: for every point in $\mathbb{A}^{1,1}_{\text{equi}}(E)$ there exists a unique point in the complex gauge orbit such that the real equation is satisfied. In other words, every point in $\mathbb{A}^{1,1}_{\text{equi}}(E)$ is stable.

4.3.4 Solutions to matrix equations

Solutions to complex equation In the spirit of [40], one can also understand the complex equations as being locally trivial. That is, one can take (4.47) and demand the gauge transformed $\mathcal{Z}$ to be zero

$$\mathcal{Z}^g = \mathrm{Ad}(g)\mathcal{Z} - \tfrac{1}{2}\left(\tfrac{\mathrm{d}}{\mathrm{d}s}g\right) g^{-1} \stackrel{!}{=} 0 \qquad \Rightarrow \qquad \mathcal{Z} = \frac{1}{2} g^{-1} \frac{\mathrm{d}}{\mathrm{d}s} g . \tag{4.65}$$

From the holomorphicity equations (4.43a) one obtains

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{Y}^g_k = 0 \quad \text{and} \quad \mathcal{Y}^g_k = \mathrm{Ad}(g_0)\, \mathcal{T}_k \quad \text{with} \quad [\mathcal{T}_j, \mathcal{T}_k] = 0 , \tag{4.66}$$

for $j,k = 1, \ldots, n$ and g_0 is a *constant* gauge transformation[10]. Consequently, the general local solution of the complex equations (4.43a) is

$$\mathcal{Y}_k = \mathrm{Ad}(g^{-1})\, \mathcal{T}_k \quad \text{with} \quad [\mathcal{T}_j, \mathcal{T}_k] = 0 \quad \text{and} \quad \mathcal{Z} = \frac{1}{2} g^{-1} \frac{\mathrm{d}}{\mathrm{d}s} g , \tag{4.67}$$

for *any* $g \in \widehat{\mathcal{G}}^{\mathbb{C}}$. A solution to the commutator constraint is obtained by choosing $\mathcal{T}_k$ for $k = 1, \ldots, n$ as elements of a Cartan subalgebra of the Lie algebra $\mathfrak{gl}(p, \mathbb{C})$, which are all diagonal (complex) $p{\times}p$ matrices.

Solutions to real equation In any case, one can in principle solve the complex equations; now, the real equation (4.43b) needs to be solved as well. Following the ideas of [40], the considerations are split in two steps: (i) a variational description and (ii) a differential inequality. We provide the details of (i) in this paragraph, while we postpone the details of (ii) to the App. A.4. Let us recall that the complete set of instanton equations is gauge-invariant under $\widehat{\mathcal{G}}$. Thus, define for each $g \in \widehat{\mathcal{G}}^{\mathbb{C}}$ the map

$$h = h(g) = g^\dagger g : \mathbb{R}^- \to \mathrm{GL}(p, \mathbb{C})/\mathrm{U}(p) . \tag{4.68}$$

[10] This g_0 can also be gauged away to 1.

The quotient $\mathrm{GL}(p,\mathbb{C})/\mathrm{U}(p)$ can be identified with the set of positive, self-adjoint $p \times p$ matrices. Then, fix a tuple $(\mathcal{Y}, \mathcal{Z})$ and define the functional $\mathcal{L}_\epsilon[g]$ for g

$$\mathcal{L}_\epsilon[g] = \tfrac{1}{2} \int_{-\frac{1}{\epsilon}}^{-\epsilon} \mathrm{d}s \ \mathrm{tr} \left(\left| \mathcal{Z}^g + (\mathcal{Z}^\dagger)^g \right|^2 + 2\lambda_n(s) \sum_{k=1}^{n} |\mathcal{Y}_k^g|^2 \right) \qquad \text{for} \quad 0 < \epsilon < 1 \ , \tag{4.69}$$

where $(\mathcal{Y}^g, \mathcal{Z}^g)$ denotes the gauge-transformed tuple. For the variation of (4.69) it suffices to consider variations with $\delta g = \delta g^\dagger$ around $g = 1$, but of course $\delta g \neq 0$. Then the gauge transformations (4.47) imply

$$\delta \mathcal{Z} = [\delta g, \mathcal{Z}] - \tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s} \delta g \quad \text{and} \quad \delta \mathcal{Y}_k = [\delta g, \mathcal{Y}_k] \qquad \text{for} \quad k = 1, \ldots, n \ . \tag{4.70}$$

The variation then leads to

$$\delta_g \mathcal{L}_\epsilon = -\mathrm{i} \int_{-\frac{1}{\epsilon}}^{-\epsilon} \mathrm{d}s \ \mathrm{tr} \left\{ \mu(\mathcal{Y}, \mathcal{Z}) \ \delta g \right\} \ , \tag{4.71}$$

i.e. critical points of (4.69) are precisely the zero-level set of the moment map. Next, we take the solution (4.67) and insert it as a starting point for $\mathcal{L}_\epsilon$. Thus, one obtains a functional for h

$$\begin{aligned} \mathcal{L}_\epsilon[h] &= \tfrac{1}{2} \int_{-\frac{1}{\epsilon}}^{-\epsilon} \mathrm{d}s \ \left\{ \tfrac{1}{4} \mathrm{tr} \left(h^{-1} \frac{\mathrm{d}h}{\mathrm{d}s} \right)^2 + 2\lambda_n(s) \sum_{k=1}^{n} \mathrm{tr} \left(h \mathcal{T}_k h^{-1} \mathcal{T}_k^\dagger \right) \right\} \\ &= \tfrac{1}{2} \int_{-\frac{1}{\epsilon}}^{-\epsilon} \mathrm{d}s \ \left\{ \tfrac{1}{4} \mathrm{tr} \left(h^{-1} \frac{\mathrm{d}h}{\mathrm{d}s} \right)^2 + V \right\} \ . \end{aligned} \tag{4.72}$$

Following [40], the potential $V(h) = 2\lambda_n(s) \sum_{k=1}^{n} \mathrm{tr} \left(h \mathcal{T}_k h^{-1} \mathcal{T}_k^\dagger \right)$ is positive[11], implying that for *any* boundary values $h_-, h_+ \in \mathrm{GL}(p,\mathbb{C})/\mathrm{U}(p)$ there exists a continuous path[12]

$$h : \left[-\tfrac{1}{\epsilon}, -\epsilon \right] \to \mathrm{GL}(p,\mathbb{C})/\mathrm{U}(p) \qquad \text{with} \quad h(-\tfrac{1}{\epsilon}) = h_- \quad \text{and} \quad h(-\epsilon) = h_+ \ , \tag{4.73}$$

which is smooth in $I_\epsilon = \left(-\frac{1}{\epsilon}, -\epsilon \right)$ and minimises the functional. Hence, for any choice of gauge transformation g such that $g^\dagger g = h$ one has that

$$\left(\{\mathcal{T}_k\}_{k=1,\ldots,n} \, , 0 \right)^g = \left(\{\mathrm{Ad}(g) \mathcal{T}_k\}_{k=1,\ldots,n} \, , -\tfrac{1}{2} (\tfrac{\mathrm{d}}{\mathrm{d}s} g) g^{-1} \right) \tag{4.74}$$

satisfies the real equation in I_ϵ for any $0 < \epsilon < 1$. From now on, we restrict the attention to $h_+ = h_- = 1$, i.e. h is framed.

The uniqueness of the solution h on each interval I_ϵ and the existence of the limit h_∞ for $\epsilon \to 0$ follows from the aforementioned differential inequality similar to [40] and the discussion of [121, Lem. 3.17]. The details are presented in App. A.4. The relevant (framed) gauge transformation[13] is then simply given by $g = \sqrt{h_\infty}$.

However, we need to emphasise two crucial points. Firstly, the construction of a solution for the limit $\epsilon \to 0$ relies manifestly on the use of the boundary conditions (4.50), and the fact that these give rise to a (constant) solution of *both* the complex equations *and* the real equation.

[11] Note that $\lambda_n(s)$ is strictly positive and smooth on $\left(-\frac{1}{\epsilon}, -\epsilon \right)$ for any $0 < \epsilon < 1$.

[12] See for instance the note under [40, Cor. 2.13]: One knows that $\mathrm{GL}(p,\mathbb{C})/\mathrm{U}(p)$ satisfies all necessary conditions for the existence of a unique stationary path between any two points.

[13] We use the *unique* principal root of the positive Hermitian matrix h, which is a continuous operation. Consequently, the framing of h implies the framing of g.

Secondly, the corresponding complex gauge transformation $g = g(h_\infty)$ is only determined up to unitary gauge transformations, i.e. it is not unique. This ambiguity in the choice of g can be removed, when we recall that a $\widehat{\mathcal{G}}$ gauge transformation suffices to eliminate X_{2n+2}. Hence, one can demand that the gauge-transformed system $(\mathcal{Y}^g, \mathcal{Z}^g)$ of a solution $(\mathcal{Y}, \mathcal{Z})$ satisfies $\mathcal{Z}^g = (\mathcal{Z}^g)^\dagger$. This fixes $g = g(h)$ uniquely, see also App. A.4.4 for further details.

Result In summary, it is sufficient to search for solutions $(\mathcal{Y}', \mathcal{Z}')$ of the complex equations (4.43a) on the interval $(-\infty, 0]$ such that the boundary conditions (4.50) are satisfied. Then one has the existence of a unique complex gauge transformation g such that

(i) $(\mathcal{Y}, \mathcal{Z}) = (\mathcal{Y}', \mathcal{Z}')^g$ satisfies (4.43b) ,

(ii) $\mathcal{Z}$ is Hermitian (i.e. $\mathcal{X}_{2n+2} = 0$) and

(iii) g is bounded and framed.

In other words, it suffices to solve the complex equations subject to some boundary conditions and the real equation will be satisfied automatically.

Moreover, the above indicates that any point in $\mathbb{A}^{1,1}_{\text{equi}}$ is stable, which we recall to be exactly the condition that every complex gauge orbit intersects $\mu^{-1}(0)$. We believe that this circumstance holds because we restricted ourselves to the space of equivariant connections. The benefit is then, that one, in principle, only has to show the solvability of the holomorphicity conditions in order to solve the instanton (matrix) equations. Nevertheless, one still has to find an ansatz that satisfies the equivariance conditions (4.27).

4.3.5 Further comments

Before concluding this chapter, we can further exploit the results collected so far as well as to illustrate another perspective on the reduced instanton equations.

Relation to coadjoint orbits Let us denote by $\mathcal{M}_n(E)$ the moduli space of solutions to the complex and real equations satisfying the boundary conditions (4.50) (with suitable regularity) as well as the equivariance condition. From the considerations above, we can establish the following map

$$\begin{aligned} \mathcal{M}_n(E) &\to \mathcal{O}_{\text{diag}}(\mathcal{Y}_1, \dots, \mathcal{Y}_n) \\ (\mathcal{Y}, \mathcal{Z}) &\mapsto (\mathcal{Y}_1(0), \dots, \mathcal{Y}_n(0)) \end{aligned} \tag{4.75}$$

where $\mathcal{O}_{\text{diag}}(\mathcal{Y}_1, \dots, \mathcal{Y}_n)$ is defined as follows: The n objects $\mathcal{Y}_k$ can be understood as element of $\mathfrak{gl}(p, \mathbb{C}) \otimes \mathbb{C}^n$, because the gauge group $\mathrm{GL}(p, \mathbb{C})$ does not act separately on each $\mathcal{Y}_k$, but it acts the *same* on every $\mathcal{Y}_k$. In other words, consider $(\mathrm{GL}(p, \mathbb{C}))^{\times n}$ which has a natural $\mathrm{GL}(p, \mathbb{C})^{\times n}$ action. Here, we select the diagonal embedding $\mathrm{GL}(p, \mathbb{C}) \hookrightarrow \mathrm{GL}(p, \mathbb{C})^{\times n}$, which gives rise to the relevant action (4.47a). Then we clearly see

$$\begin{aligned} \mathcal{O}_{\text{diag}}(\mathcal{Y}_1, \dots, \mathcal{Y}_n) &:= \{(\mathrm{Ad}(g)\mathcal{Y}_1(0), \dots \mathrm{Ad}(g)\mathcal{Y}_n(0)) \mid g \in \mathrm{GL}(p, \mathbb{C})\} \\ &\subset \prod_{j=1}^{n} \{\mathrm{Ad}(g_j)\mathcal{Y}_j(0) \mid g_j \in \mathrm{GL}(p, \mathbb{C})\} = \mathcal{O}_{\mathcal{T}_1} \times \cdots \times \mathcal{O}_{\mathcal{T}_n} \end{aligned} \tag{4.76}$$

where $\mathcal{O}_{\mathcal{T}_k}$ denotes the adjoint orbit of $\mathcal{T}_k$ in $\mathfrak{gl}(p, \mathbb{C})$. Analogous to [121], the map (4.75) is injective due to the uniqueness of the corresponding solution of the real and complex equations. In contrast, the surjectivity is less clear. By the construction of the local solution (4.67), one finds that any element of $\mathcal{O}_{\mathrm{diag}}(\mathcal{Y}_1,\dots,\mathcal{Y}_n)$ gives rise to a solution of the complex and real equation, but it is unclear if this solution satisfies the required asymptotic. The Nahm equations of [121] can be cast as gradient flow, which together with boundedness of the matrix-valued functions gives the existence of the limit $s\to-\infty$ and that the limit must be a critical point of the gradient flow function. This is not the case for our Nahm-type equations and, hence, we do not know whether the map (4.75) is surjective.

Moreover, one knows that the orbit of an element $\mathcal{T}_k$ of the Cartan subalgebra is of the form $\mathrm{GL}(p,\mathbb{C})/\mathrm{Stab}(\mathcal{T}_k)$ and $\mathrm{Stab}(\mathcal{T}_k)$ is the maximal torus of $\mathrm{GL}(p,\mathbb{C})$ because each $\mathcal{T}_k$ is assumed to be a regular element. The product of the regular semi-simple coadjoint orbits is a complex symplectic manifold. Each orbit is equipped with the so-called Kirillov-Kostant-Souriau symplectic form [146] and the product thereof gives the symplectic structure on the total space. As a manifold the orbit $\mathcal{O}_{\mathrm{diag}}(\mathcal{Y}_1,\dots,\mathcal{Y}_n)$ is just $\mathrm{GL}(p,\mathbb{C})/\mathrm{Stab}\left(\mathcal{Y}_1(0),\dots,\mathcal{Y}_n(0)\right)$, wherein

$$\mathrm{Stab}\left(\mathcal{Y}_1(0),\dots,\mathcal{Y}_n(0)\right)=\bigcap_{j=1}^{n}\mathrm{Stab}\left(\mathcal{Y}_j(0)\right)=\bigcap_{j=1}^{n}\mathrm{Stab}(\mathcal{T}_j) \tag{4.77}$$

and the intersection of the stabilisers of the $\mathcal{T}_j$ is the complexified maximal torus, by the regularity assumption. Hence, the complex dimension[14] is

$$\dim_{\mathbb{C}}\left(\mathcal{O}_{\mathrm{diag}}(\mathcal{Y}_1,\dots,\mathcal{Y}_n)\right)=\dim_{\mathbb{R}}(\mathrm{U}(p))-\mathrm{rk}(\mathrm{U}(p))=p(p-1)\ , \tag{4.78}$$

which always is a multiple of 2. The diagonal orbit is also Kähler, as it is a complex sub-manifold of a (hyper-)Kähler product. Analogous to [121], the map (4.75) is holomorphic such that it describes an embedding of the framed moduli space $\mathcal{M}_n(E)$ into the diagonal orbit, which is a finite dimensional Kähler manifold.

Relation to quiver representations The instanton matrix equations can be seen to define quiver representations, depending on the chosen $\mathrm{SU}(n{+}1)$-representation on the typical fibre $\mathbb{C}^p$. Then, by the employed ansatz, we decompose this representation with respect to $\mathrm{SU}(n)$ into

$$\mathbb{C}^p\Big|_{\mathrm{SU}(n)}=\bigoplus_{w\in K}\mathbb{C}^{n_w}\ , \tag{4.79}$$

where $\mathbb{C}^{n_w}$ carries an n_w-dimensional irreducible $\mathrm{SU}(n)$-representation. More explicitly, w should be understood as pair of labels: let ϕ label the irreducible $\mathrm{SU}(n)$-representations and recall that the centraliser of $\mathrm{SU}(n)$ inside $\mathrm{SU}(n{+}1)$ is a $\mathrm{U}(1)$. Then each representation space $\mathbb{C}^{n_w}$ carries also a $\mathrm{U}(1)$-representation characterised by a *charge* q. Therefore, the decomposition is labelled by pairs $w=(\phi,q)$.

As a consequence, the equivariance condition (4.27) dictates the decomposition of the X_μ-matrices into homomorphisms

$$X_\mu=\bigoplus_{w,w'\in K}(X_\mu)_{w,w'}\qquad\text{with}\quad (X_\mu)_{w,w'}\in\mathrm{Hom}\left(\mathbb{C}^{n_w},\mathbb{C}^{n_{w'}}\right)\ . \tag{4.80}$$

The quiver representation then arises as follows: the set Q_0 of vertices is the set $\{\mathbb{C}^{n_w}|w\in K\}$ of vector spaces and the set Q_1 of arrows is given by the non-vanishing homomorphisms $\{(X_\mu)_{w,w'}|w,w'\in K\ ,\ \mu=1,\dots,2n+1\}$.

The reduced instanton equations (or matrix equations) then lead to relations on the quiver representation. Examples for the arising quiver diagrams as well as their relations for the case $n=1$ and $M^3=S^3$ can be found in [147], for $n=2$ and $M^5=S^5$ in Part II of this thesis,

[14]In fact, as each $\mathcal{T}_j$ is a regular pair, each regular semi-simple $\mathcal{O}_{\mathcal{T}_k}$ has the same dimension as the diagonal orbit.

and $M^5 = T^{1,1}$ in [148]. To study the representations of a quiver one would rather use the constructions of [147–149], instead of the ansatz employed here. This is because once the bundle E and the action of $\mathrm{SU}(n+1)$ on the fibres is chosen, there is no freedom to change the quiver representation any more.

5 Instantons on non-Kähler conical 6-manifolds

Having constructed several 6-dimensional conical manifolds in Sec. 3.5, we now turn our attention to instanton equations on such spaces. Here, we restrict ourselves to the tangent bundle of the generically non-complex spaces.

5.1 Definition and reduction of instanton equations on conical 6-manifolds

As these spaces are non-Kähler, the generalised instanton equations do not coincide with the HYM equations and we need to revise the definition of an instanton with respect to the non-integrable SU(3)-structures.

5.1.1 Instanton condition

Firstly, we need to specify the instanton condition for non-integrable SU(3)-structures. Let M^6 be a 6-manifold with a connection $\mathcal{A}$ on the tangent bundle. Then the curvature 2-form $\mathcal{F}_{\mathcal{A}}$ satisfies the Bianchi identity $D_{\mathcal{A}}\mathcal{F}_{\mathcal{A}} = 0$, where $D_{\mathcal{A}}$ is the covariant differential associated to $\mathcal{A}$. As before, we can perform the type-decomposition of a form with respect to any almost complex structure J, yielding

$$\mathcal{F}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}}^{2,0} + \mathcal{F}_{\mathcal{A}}^{1,1} + \mathcal{F}_{\mathcal{A}}^{0,2} \ . \tag{5.1}$$

For a given SU(3)-structure (ω, Ω) on a 6-manifold and a curvature 2-form $\mathcal{F}_{\mathcal{A}}$, the instanton equation can be defined in two steps: first, the *pseudo-holomorphicity condition* reads

$$\Omega \wedge \mathcal{F}_{\mathcal{A}} = 0 \quad \Leftrightarrow \quad \mathcal{F}_{\mathcal{A}}^{0,2} = 0 \ , \tag{5.2a}$$

and, second, applying the covariant differential to (5.2a), and using the Bianchi identity as well as (5.2a) yields

$$\mathrm{d}\Omega \wedge \mathcal{F}_{\mathcal{A}} = \left[\left(W_1^+ + \mathrm{i}W_1^-\right)\omega \wedge \omega + \left(W_2^+ + \mathrm{i}W_2^-\right) \wedge \omega\right] \wedge \mathcal{F}_{\mathcal{A}} = 0 \ . \tag{5.2b}$$

The last equation, although a mere consequence of (5.2a), depends strongly on the type of SU(3)-manifold under consideration. For example, on nearly Kähler manifolds one has

$$\mathrm{d}\Omega \propto \omega \wedge \omega \quad \xRightarrow{(5.2b)} \quad \omega \wedge \omega \wedge \mathcal{F}_{\mathcal{A}} = 0 \quad \Leftrightarrow \quad \omega \lrcorner \mathcal{F}_{\mathcal{A}} = 0 \ , \tag{5.3}$$

whereas on half-flat SU(3)-manifolds this is not true as $\mathrm{d}\Omega \neq \kappa\, \omega \wedge \omega$. For Calabi-Yau, on the other hand, (5.2b) is trivial as $\mathrm{d}\Omega = 0$, and the condition $\omega \lrcorner \mathcal{F}_{\mathcal{A}} = 0$ is added as an additional stability-like condition.

Lastly, for Kähler-torsion spaces we define an instanton via the condition

$$\star\,\mathcal{F}_{\mathcal{A}} = -(\star Q) \wedge \mathcal{F}_{\mathcal{A}} \qquad \text{for} \quad Q = \frac{1}{2}\omega \wedge \omega \ , \tag{5.4}$$

wherein ω is the fundamental $(1,1)$-form of the Hermitian space. If, on the other hand, the KT-structure really is a CYT-structure, one can equivalently employ (5.2).

5.1.2 Ansatz

We follow the ansatz outlined previously in Sec. 4.3.1, but we are forced to adjust the setting. Let M^6 be the SU(3)-structure manifold of interest, constructed as a conical extension of a Sasaki-Einstein 5-manifold M^5. Note that we do *not* assume M^6 to be conformally equivalent to $\mathrm{Cyl}(M^5)$. Suppose we are given an SU(2)-instanton Γ on the tangent bundle TM^6 with curvature R_Γ. As before, we then generalise this instanton Γ by extending it to a connection $\mathcal{A}$ with curvature $\mathcal{F}_{\mathcal{A}}$ by the ansatz[15]

$$\mathcal{A} = \Gamma + X_\mu e^\mu \qquad \text{with} \quad \Gamma = \Gamma^i \widehat{I}_i \ , \tag{5.5}$$

where $\mu = 1, \ldots, 5$ and $i = 6, 7, 8$. In addition, $\widehat{I}_i$ is a representation of the $\mathfrak{su}(2)$-generators I_i on the fibres $\mathbb{R}^6$ of the bundle, and Γ^i are the components of an $\mathfrak{su}(2)$-connection on the tangent bundle of M^6. Furthermore, the X_μ are now $\mathrm{End}(\mathbb{R}^6)$-valued functions on some interval, specified by the conical construction of M^6.

The computation of $\mathcal{F}_{\mathcal{A}}$ with the ansatz for $\mathcal{A}$ yields

$$\begin{aligned} \mathcal{F}_{\mathcal{A}} = R_\Gamma + \mathrm{d}X_\mu \wedge e^\mu + T^\mu_{6\nu} X_\mu e^6 \wedge e^\nu + \tfrac{1}{2}\left([X_\mu, X_\nu] + T^\varsigma_{\mu\nu} X_\varsigma \right) e^\mu \wedge e^\nu \\ + \Gamma^i \left([\widehat{I}_i, X_\mu] - f^\nu_{i\mu} X_\nu \right) \wedge e^\mu \ . \end{aligned} \tag{5.6}$$

Herein, T denotes the torsion of the connection Γ, which is *not* assumed to be the canonical connection Γ^P. By the same arguments as before, the consistency condition (4.27) reduces (5.6) to

$$\mathcal{F}_{\mathcal{A}} = R_\Gamma + (\dot{X}_\mu + T^\nu_{6\mu} X_\nu)\, e^6 \wedge e^\mu + \tfrac{1}{2}\left([X_\mu, X_\nu] + T^\varsigma_{\mu\nu} X_\varsigma \right) e^\mu \wedge e^\nu \ , \tag{5.7}$$

as we choose the X_μ to vary on the cone direction only. Hence, the dot denotes the derivative in the cone direction. In any case, the instanton condition is the requirement that the 2-form part of $\mathcal{F}_{\mathcal{A}}$ takes values in a certain subbundle of $\Lambda^2 T^* M^6$, which we called the *instanton bundle* in Sec. 4.1. Following the ideas of [113], we anticipate that 2-forms of the general form $e^6 \wedge e^\sigma + \frac{1}{2} N^\sigma_{\mu\nu} e^\mu \wedge e^\nu$, with N to be determined from the geometry under consideration, are local sections of this instanton bundle, we add a zero to the above expression and obtain

$$\begin{aligned} \mathcal{F} = R_\Gamma + (\dot{X}_\mu + T^\nu_{6\mu} X_\nu)\left(e^6 \wedge e^\mu + \tfrac{1}{2} N^\mu_{\sigma\rho}\, e^\sigma \wedge e^\rho\right) \\ + \tfrac{1}{2}\left([X_\mu, X_\nu] + T^\varsigma_{\mu\nu} X_\varsigma - N^\sigma_{\mu\nu} (\dot{X}_\sigma + T^\rho_{6\sigma} X_\rho) \right) e^\mu \wedge e^\nu \ . \end{aligned} \tag{5.8}$$

From Sec. 4.3.1 we know R_Γ to be an instanton; while the second term in (5.8) is already an instantons by construction. Thus, we are left to require that the last term satisfies the instanton equation; this leads us to

$$[X_\mu, X_\nu] + T^\varsigma_{\mu\nu} X_\varsigma = N^\sigma_{\mu\nu} (\dot{X}_\sigma + T^\rho_{6\sigma} X_\rho) + \mathcal{N}_{\mu\nu} \ , \tag{5.9}$$

[15] As usual, one component can be set to zero. Mimicking temporal gauge, we set the component in the cone direction to zero.

where $\mathcal{N}$ has to be an instanton on M^6 that compensates for the $\mathfrak{su}(2)$-component of the left-hand-side commutator. Hence, $\mathcal{N}$ can only be a linear combination of the three instantons [84] ${f_{\mu\nu}}^i e^\mu \wedge e^\nu$ for $i = 6, 7, 8$, which depends on the cone coordinate. That is,

$$[X_\mu, X_\nu] + T^{\varsigma}_{\mu\nu} X_\varsigma = N^{\sigma}_{\mu\nu} \left(\dot{X}_\sigma + T^{\rho}_{6\sigma} X_\rho\right) + {f_{\mu\nu}}^i \mathcal{N}_i \; . \tag{5.10}$$

In summary, we are searching for $\mathfrak{m}$-valued matrices X_μ that solve equations (4.27) and (5.10), as these will give rise to new instantons on the considered manifolds.

5.1.3 Remarks on the instanton equation

Before proceeding with the particular cases of the nearly Kähler sine-cone and the half-flat cylinder, one needs to clarify an important point regarding the transformations of coframes mentioned in Sec. 3.5.

The SU(2)-structure on the Sasaki-Einstein 5-manifold is understood as an SU(2)-principal bundle $\mathcal{Q}$, a subbundle of the frame bundle $F(TM^5)$. The warped product $M^5 \times_\phi I$ of (3.25) is equipped with an SU(3)-structure via (3.27) and the corresponding principal bundle is denoted by $\mathcal{P} \subset F(T(M^5 \times_\phi I))$. We refer to Fig. 5.1 for an illustration. However, $\mathcal{P}$ is not the SU(3)-structure one is interested in, i.e. in our cases it is neither nearly Kähler nor half-flat. The constructions of Sec. 3.5.3 and 3.5.4 rely on transformations of the coframes on M^5: they generate a different SU(2)-structure $\mathcal{Q}'$ that can be extended to the desired SU(3)-structure $\mathcal{P}'$ on the warped product. An important observation is the following: for a G-structure $\mathcal{Q}$ the bundle $\mathcal{Q}'$ defined via $\mathcal{Q}' = R_L \mathcal{Q}$ is a G-structure if and only if L is a map from the base to the normaliser $N_{\mathrm{GL}(6,\mathbb{R})}(\mathrm{G})$, cf. (3.26) or the treatment in [129].

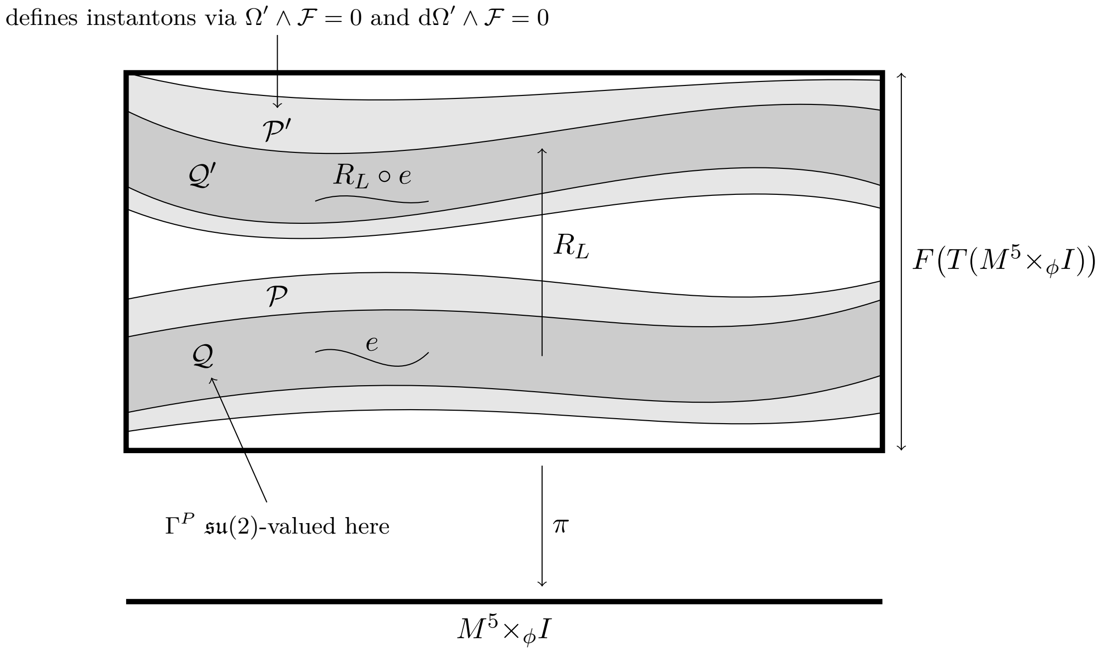


Figure 5.1: *A schematic depiction of the different principal bundles involved in the definition of the instanton condition: $\mathcal{Q}$ and $\mathcal{P}$ are the* SU(2)*- and* SU(3)*-bundles, respectively, which originate from the Sasaki-Einstein structure on M^5. The transformation L defines the principal bundles $\mathcal{Q}'$ and $\mathcal{P}'$, which again are* SU(2) *and* SU(3)*-bundles, respectively. All bundles under consideration are understood as principal subbundles of the frame bundle $F(T(M^5 \times_\phi I))$.*

The crux of the instanton equation is the following: the defining forms (ω', Ω') stem from $\mathcal{P}'$, whereas the canonical connection Γ^P belongs to $\mathcal{Q}$ and is trivially lifted to an instanton on $\mathcal{P}$. Let us denote by $e \in \Gamma(\mathcal{U}, \mathcal{Q})$ an adapted frame for $\mathcal{Q}$. Then by construction $e' =: (R_L \circ e) \in \Gamma(\mathcal{U}, \mathcal{Q}')$ is an adapted frame for $\mathcal{Q}'$. By standard results, the connection 1-forms of $\mathcal{A}$ transform under a change of section as

$$e'^* \mathcal{A} = \mathrm{Ad}(L^{-1}) \circ e^* \mathcal{A} + L^{-1} \mathrm{d}L \; . \tag{5.11}$$

The employed extension $\mathcal{A} = \Gamma^P + X$ relies on the splitting (4.25) such that X corresponds to $\mathfrak{m}$-valued 1-forms. However, this only holds in the frame e, due to the following: Starting with Γ^P on $\mathcal{Q}$, one has a purely $\mathfrak{su}(2)$-valued connection. Applying any transformation L to $\mathcal{Q}$, Γ^P is generically not an $\mathfrak{su}(2)$-valued connection on $\mathcal{Q}'$. This is due to the fact that $L^{-1}\mathrm{d}L$, in general, takes values in the Lie-algebra of $N_{\mathrm{GL}(6,\mathbb{R})}(\mathrm{SU}(2))$ instead of $\mathfrak{su}(2)$. Therefore, one cannot simply take $e'^*\Gamma^P$ as a starting point for some ansatz like $e'^*\mathcal{A} = e'^*\Gamma^P + X'_\mu e'^\mu$.

For the cases under consideration, L depends (at most) on the cone direction r. Hence, one has that $\mathrm{Ad}(L^{-1}) \circ e^*\mathcal{A}$ is $\mathfrak{su}(2)$-valued and $L^{-1}\mathrm{d}L \propto \mathrm{d}r$, but generically not $\mathfrak{su}(2)$-valued. The immediate consequences are the following:

- On the nearly Kähler sine-cone, for instance, one has to perform all calculations in the frame e, because for the derivation of Sec. 5.1 we employed a section of the bundle on which Γ^P is an $\mathfrak{su}(2)$-valued connection. We will, however, compute $e'^*\Gamma^P$ explicitly in Sec. 5.3.2 and demonstrate that it yields an $\mathfrak{su}(3)$-valued instanton on the sine-cone.

- In contrast, the transformation for the half-flat cylinder (3.49) is, although a 2-parameter family, base point independent. Therefore, one is allowed to consider the frames e as well as e' for this instanton equation, as $e^*\Gamma^P$ and $e'^*\Gamma^P$ are $\mathfrak{su}(2)$-valued connection 1-forms. However, this raises the question whether the two extensions $X_\mu e^\mu$ and $X'_\mu e'^\mu$ are in any sense comparable. The coframe transformations are only required to be $N_{\mathrm{GL}(6,\mathbb{R})}(\mathrm{SU}(2))$-valued, which implies that the $\mathfrak{m}$-piece will, in general, not be mapped into $\mathfrak{m}$ or even $\mathfrak{su}(3)$. Hence, one cannot simply compare both extensions, but it is admissible to consider both cases.

In summary, these remarks were not relevant for the cases studied for example in [84,113] or our earlier results of Sec. 4.3.1, because the construction of the G-structures on the warped product $M^5 \times_\phi I$ followed immediately from the chosen frame on M^5. In other words, no (base point dependent) transformation of coframes was necessary. Even on the Kähler-torsion sine-cone, the relevant rescaling (3.38) does not affect the computations due to the conformal equivalence to the cylinder. However, for the nearly Kähler sine-cone and the half-flat cylinder the situation is more involved and a careful analysis is mandatory.

5.2 Instantons on Kähler-torsion sine-cone

Consider the Λ-scaled sine-cone $C_\Lambda(M^5)$ with a KT-structure as constructed in Sec. 3.5.2. The instanton equations for the Kähler-torsion space can be written as

$$\star \mathcal{F}_\mathcal{A} = -(\star Q) \wedge \mathcal{F}_\mathcal{A} \qquad \text{with} \quad Q = \frac{1}{2} \omega \wedge \omega \; . \tag{5.12}$$

In the coframe $\{e^\mu\}$ of the cylinder, the space of admissible 2-forms is spanned by

$$e^5 \wedge e^6 - \frac{1}{4} \eta^3_{ab} \, e^a \wedge e^b \quad \text{and} \quad e^a \wedge e^6 - \eta^{3\,a}{}_b \, e^a \wedge e^5 \; , \tag{5.13}$$

which one finds either by direct computation or by employing the projectors from $\mathfrak{so}(6)$ to $\mathfrak{su}(3)$ of [110]. A 6-dimensional representation of $\mathfrak{m}$ can be chosen as in [84, 113],

$$(\widehat{I}_5)_a^b = \tfrac{1}{2}\,\eta^3_{ab}\ , \qquad -(\widehat{I}_5)_5^6 = (\widehat{I}_5)_6^5 = 1\ , \tag{5.14a}$$

$$-(\widehat{I}_a)_b^6 = (\widehat{I}_a)_6^b = \delta_a^b\ , \qquad (\widehat{I}_a)_b^5 = -(\widehat{I}_a)_5^b = \eta^3_{ab}\ , \tag{5.14b}$$

from which one obtains the structure constants

$$f_{5a}{}^{b} = \tfrac{3}{2}\,\eta^{3}{}_{a}{}^{b} \qquad \text{and} \qquad f_{ab}{}^{5} = 2\,\eta^3_{ab}\ . \tag{5.15}$$

The torsion components of the canonical $\mathfrak{su}(2)$-connection Γ^P in the unrotated frame e^μ read

$$T^5_{ab} = -2\,\eta^3_{ab} = -f_{ab}{}^{5}\ , \tag{5.16a}$$

$$T^a_{b5} = -\tfrac{3}{2}\,(\eta^3)^a{}_b = -f_{b5}{}^{a}\ . \tag{5.16b}$$

Inserting the chosen representation and employing the ansatz (5.5) into (5.12) one derives the non-vanishing components $N^\rho_{\mu\nu}$ of the parametrisation as

$$N_{b\,5}{}^{a} = \frac{2}{3} f_{b5}{}^{a} \quad \text{and} \quad N^5_{ab} = \frac{1}{4} f_{ab}{}^{5}\ . \tag{5.17}$$

Therefore, the reduced instanton equations are given by

$$\left[\widehat{I}_i, X_\mu\right] = f_{i\mu}{}^{\nu} X_\nu\ , \tag{5.18a}$$

$$[X_a, X_b] = 2\eta^3_{ab}\left(X_5 + \frac{1}{4}\dot{X}_5\right) + f_{ab}{}^{i}\mathcal{N}_i\ , \tag{5.18b}$$

$$[X_5, X_a] = \frac{3}{2}\eta^{3}{}_{a}{}^{b}\left(X_b + \frac{2}{3}\dot{X}_b\right)\ , \tag{5.18c}$$

where the dot denotes the derivative in the cylinder direction τ. We observe that the equations (5.18) are identical to those considered on the Calabi-Yau cone in [113], see also (4.29). This is not surprising, as the instanton equations are conformally invariant and the CY cone is conformally equivalent to the KT sine-cone. Hence, once a solution to (5.18) is known, the corresponding solutions on the cone and sine-cone differ as the functional dependence of the (sine-)cone coordinate of τ enters. A possible solution of (5.18) has already been discussed in Sec. 4.3.1.

5.3 Instantons on nearly Kähler sine-cones

5.3.1 Matrix equations — part I

The set-up for the nearly Kähler sine-cone has been described in Sec. 3.5.3. In particular, we are investigating extensions of the connection Γ^P on the sine-cone in this subsection. M^6 being a nearly Kähler manifold, the instanton equation with respect to the coframe e^μ is equivalent to

$$\omega\wedge\omega\wedge\mathcal{F}_{\mathcal{A}} = 0 \qquad \Leftrightarrow \qquad \omega^{\hat{\mu}\hat{\nu}}(\mathcal{F}_{\mathcal{A}})_{\hat{\mu}\hat{\nu}} = 0\ , \tag{5.19a}$$

$$\Omega\wedge\mathcal{F}_{\mathcal{A}} = 0 \qquad \Leftrightarrow \qquad \Omega^{\hat{\varsigma}\hat{\mu}\hat{\nu}}(\mathcal{F}_{\mathcal{A}})_{\hat{\mu}\hat{\nu}} = 0 \qquad \text{for} \quad \hat{\varsigma} = 1,\ldots,6\ . \tag{5.19b}$$

The seven equations (5.19) restrict the space of admissible 2-forms, and the instanton bundle, which is locally isomorphic[16] to the subspace $\mathfrak{m}$, is spanned by

$$\begin{aligned} & e^5 \wedge e^6 - \tfrac{\Lambda \sin\varphi}{4} \left(\sin\varphi\, \eta^1_{ab} + \cos\varphi\, \eta^3_{ab} \right) e^a \wedge e^b \quad \text{and} \\ & e^a \wedge e^6 - \Lambda \sin\varphi \left(\sin\varphi\, \eta^{1\,a}{}_b + \cos\varphi\, \eta^{3\,a}{}_b \right) e^b \wedge e^5 \ . \end{aligned} \tag{5.20}$$

This can be seen either by direct computation or by the explicit form of the projectors from $\mathfrak{so}(6)$ to $\mathfrak{su}(3)$ of [110]. Here we have used the Riemannian metric to pull up one of the indices of η^3, and from here on we use $e^6 = \mathrm{d}r$.

With the chosen representation (5.14) and (5.15) for $\mathfrak{m}$ and by inserting the ansatz

$$\mathcal{A} = \Gamma^P + X_\mu\, e^\mu \tag{5.21}$$

into (5.19), one obtains the non-vanishing components $N^\rho_{\mu\nu}$ of the parametrization (5.10) as follows:

$$N^5_{ab} = \tfrac{\Lambda \sin\varphi}{2} \left(\sin\varphi\, \eta^1_{ab} + \cos\varphi\, \eta^3_{ab} \right) \quad \text{and} \quad N^a_{b5} = \Lambda \sin\varphi \left(\sin\varphi\, \eta^{1\,a}{}_b + \cos\varphi\, \eta^{3\,a}{}_b \right) \ . \tag{5.22}$$

Finally, the matrix equations for X_μ read

$$[\widehat{I}_i, X_\mu] = f_{i\mu}{}^{\nu}\, X_\nu \ , \tag{5.23a}$$

$$[X_a, X_b] = \frac{\Lambda \sin\varphi}{2} \left(\sin\varphi\, \eta^1_{ab} + \cos\varphi\, \eta^3_{ab} \right) \dot{X}_5 + 2\, \eta^3_{ab} X_5 + f_{ab}{}^{i} \mathcal{N}_i \ , \tag{5.23b}$$

$$[X_5, X_a] = \Lambda \sin\varphi \left(\sin\varphi\, \eta^1{}_a{}^b + \cos\varphi\, \eta^3{}_a{}^b \right) \dot{X}_b + \frac{3}{2}\, \eta^3{}_a{}^b X_b \ , \tag{5.23c}$$

where the first line is just the equivariance condition (4.27). The dot-notation means $\dot{Y} \equiv \frac{\mathrm{d}}{\mathrm{d}r} Y$. An obvious solution to (5.23) is $X_\mu \equiv 0$, which yields the instanton solution $\mathcal{A} = \Gamma^P$ that is the lift of the instanton Γ^P from M^5 to the sine-cone $C_s(M^5)$.

Consider the ansatz

$$X_a = \psi(r) \left(\exp(\xi\, \eta^3) \right)_a^{\ b} \widehat{I}_b \ , \qquad \text{for} \quad \xi \in [0, 2\pi) \qquad \text{and} \qquad X_5 = \chi(r) \widehat{I}_5 \ , \tag{5.24}$$

which respects equivariance due to $[\eta^\alpha, \bar{\eta}^\beta] = 0$. Here, ξ is a parameter, and $\psi(r)$, $\chi(r)$ are two functions depending only on the cone direction r. Inserting (5.24) into (5.23) yields

$$\mathcal{N}_i = \psi^2(r)\, \widehat{I}_i \qquad \text{for} \quad i = 6, 7, 8 \tag{5.25}$$

as well as the following differential equations:

$$\tfrac{\Lambda}{2}\, \dot{\chi}(r)\, \sin(2\varphi) = 4 \left(\psi^2(r) - \chi(r) \right) \qquad \text{and} \qquad \tfrac{\Lambda}{2} \dot{\psi}(r) \sin(2\varphi) = \tfrac{3}{2} \psi(r) \left(\chi(r) - 1 \right) \ . \tag{5.26a}$$

These are subject to the constraints

$$\tfrac{\Lambda}{2}\, \dot{\psi}(r)\, \sin^2\varphi = \tfrac{\Lambda}{2}\, \dot{\chi}(r)\, \sin^2\varphi = 0 \ . \tag{5.26b}$$

As a matter of fact, these equations (5.26) hold for any value of $\xi \in [0, 2\pi)$. The solutions to (5.26) are readily obtained to be the following:

- <u>$(\psi, \chi) = (0, 0)$:</u> This is, of course, the trivial solution of (5.23), but is still required for

[16] One employs the identification $\mathfrak{so}(6) \simeq \Lambda^2(\mathbb{R}^6)$ to obtain 2-forms from antisymmetric 6×6-matrices.

consistency as it confirms that Γ^P satisfies the instanton condition on M^6.

- $(\psi, \chi) = (1, 1)$: Here we obtain an extension of the original instanton Γ^P. Despite being an instanton, this newly obtain instanton is a mere lift of an instanton in M^5 as it does not have any dependence on the cone direction.
- $(\psi, \chi) = (-1, 1)$: Again, we obtain an extension which is a lift of an M^5-instanton. Note that the existence of this solutions follows from $\xi \mapsto \xi + \pi$, because $\left(\exp(\pi\ \eta^3)\right)_a{}^b = -\delta_a{}^b$.

Hence, we have a one-parameter family of $\mathfrak{su}(3)$-valued instantons given by

$$\mathcal{A} = \Gamma^P + \Big(\exp(\xi\ \eta^3)\Big)_a{}^b\, \widehat{I}_b \otimes e^a + \widehat{I}_5 \otimes e^5\ . \tag{5.27}$$

To summarise, the ansatz solving the matrix equations (5.23) generates isolated instanton solutions which can all be interpreted as lifts of connections living on M^5. The non-trivial solutions are $\mathfrak{su}(3)$-valued connections; whereas the trivial solution is a purely $\mathfrak{su}(2)$-valued connection.

Remarks Firstly, the family of solutions (5.27) can be seen to be contained in a gauge orbit if we recall that $(\eta^3)_\mu^\nu \propto f_{5\mu}{}^\nu = \mathrm{ad}(I_5)_\mu^\nu$ and whence $\exp(\xi\ \eta^3) \propto \mathrm{Ad}(\exp(I_5))$. Nevertheless, this gauge symmetry clarifies the origin of the ψ-reflection symmetry of the solutions.

Secondly, in the same manner as in the previous Sec. 5.2 we can equivalently provide the matrix equations on the conformally equivalent cylinder with coordinate τ as follows:

$$[\widehat{I}_i, X_\mu] = f_{i\mu}{}^\nu X_\nu\ , \tag{5.28a}$$

$$[X_a, X_b] = \frac{1}{2}\Big(\sin\varphi\, \eta^1_{ab} + \cos\varphi\, \eta^3_{ab}\Big)\frac{\mathrm{d}}{\mathrm{d}\tau}X_5 + 2\,\eta^3_{ab}X_5 + f_{ab}{}^i\mathcal{N}_i\ , \tag{5.28b}$$

$$[X_5, X_a] = \Big(\sin\varphi\, \eta^1{}_a{}^b + \cos\varphi\, \eta^3{}_a{}^b\Big)\frac{\mathrm{d}}{\mathrm{d}\tau}X_b + \frac{3}{2}\,\eta^3{}_a{}^b X_b\ . \tag{5.28c}$$

Further, the limit $\Lambda \to \infty$ (with $\varphi{=}\frac{r}{\Lambda} \to 0$ and keeping r fixed) transforms the sine-cone into the Calabi-Yau cone, as mentioned in Sec. 3.5.3. In this limit, the matrix equations (5.28) take the following form:

$$[X_a, X_b] = f_{ab}{}^5\left(X_5 + \frac{1}{4}\,\dot{X}_5\right) + f_{ab}{}^i\mathcal{N}_i \quad \text{and} \quad [X_5, X_a] = f_{5a}{}^b\left(X_b + \frac{2}{3}\,\dot{X}_b\right)\ , \tag{5.29}$$

which are exactly the same equations as on the Kähler-torsion sine-cone of our earlier results (5.18). Applying the τ-dependent version of the ansatz (5.24) yields

$$\dot{\chi}(\tau) = 4\Big(\psi^2(\tau) - \chi(\tau)\Big) \qquad \text{and} \qquad \dot{\psi}(\tau) = \tfrac{3}{2}\psi(\tau)\left(\chi(\tau) - 1\right)\ . \tag{5.30}$$

Obviously, all constant solutions found above are still instantons on the CY-cone, but the reduced equations do not automatically enforce constant χ and ψ. Finally, note that (5.30) is, of course, equivalent to (5.26) in the limit $\Lambda \to \infty$ as the constraint on the derivatives vanishes.

Thirdly, the sine-cone is a conifold with two conical singularities, here at $\varphi = 0$ and $\varphi = \pi$. One observes that the coefficient functions, i.e. $\cos\varphi$ and $\sin\varphi$, of (5.23) as well as our solutions are well-behaved at the singular points. However, recall the remark from Sec. 3.5.3 that the defining sections of the SU(3)-structure become trivial at these singular points; hence, the instanton condition is not well-defined there. Yet, in principal one could extend the gauge field to these points.

5.3.2 Nearly Kähler canonical connection

In this section we construct the canonical $\mathfrak{su}(3)$-connection of the nearly Kähler sine-cone. It turns out that we obtain an instanton for the SU(3)-structure that is not the lift of an instanton on M^5; furthermore, this instanton is of the form (5.21) presented above. On the 5-manifold M^5 the Maurer-Cartan equations read

$$\mathrm{d}e^a = -(\Gamma^P)^a_b \wedge e^b + \tfrac{1}{2}\, T^a_{\mu\nu}\, e^\mu \wedge e^\nu \ , \tag{5.31a}$$

$$\mathrm{d}e^5 = -(\Gamma^P)^5_5 \wedge e^5 + \tfrac{1}{2}\, T^5_{\mu\nu}\, e^\mu \wedge e^\nu \ , \tag{5.31b}$$

where the torsion components are given by (cf. [84, 113])

$$T^a_{b5} = -\tfrac{3}{2}\, \eta^{3a}{}_b \qquad \text{and} \qquad T^5_{ab} = -2\, \eta^3_{ab} \ . \tag{5.32}$$

In particular, the last identity implies $(\Gamma^P)^5_5 = 0$ due to the Sasaki-Einstein relation $\mathrm{d}e^5 = -2\,\omega^3$.

Next, we are interested in the Maurer-Cartan equations for the frame e^μ_s resulting from the rotation (3.44) and rescaling (3.46) of the SU(2)-structure. With respect to coframes e adapted to $\mathcal{Q}$, the canonical $\mathfrak{su}(2)$-connection Γ^P has components

$$(\Gamma^P)^\nu_\mu = (\Gamma^P)^i f_{i\mu}{}^\nu \qquad \text{with} \quad (f_{ia}{}^b) \propto \bar{\eta}^{\alpha(i)} \ , \tag{5.33}$$

where $\alpha(i) = i-5$ and $\bar{\eta}^\alpha$ are the anti-self-dual 't Hooft tensors. Noting that $[\eta^\alpha, \bar{\eta}^\beta] = 0$ for all α, β, we see that the components of the canonical $\mathfrak{su}(2)$-connection are unaffected by the homogeneous part of the transformation (5.11) with

$$L(r) = \Lambda \sin(\varphi) \begin{pmatrix} \exp(\frac{\varphi}{2}\, \eta^2)_{4\times 4} & 0_{4\times 2} \\ 0_{2\times 4} & \mathbb{1}_{2\times 2} \end{pmatrix} \in N_{\mathrm{GL}(6,\mathbb{R})}(\mathrm{SU}(2)) \ , \tag{5.34}$$

which realises the rotation (3.44) and the rescaling (3.46). In detail, the transformation reads $(\Gamma^P)^a_b = L^a_c\, (\Gamma^P)^c_d\, (L^{-1})^d_b$. A straightforward computation yields

$$\begin{aligned} \mathrm{d}e^a_s = -(\Gamma^P)^a_b \wedge e^b_s &- \frac{\cot\varphi}{\Lambda}\left(e^a_s \wedge e^6_s + \eta^{3a}{}_b\, e^b_s \wedge e^5_s\right) - \frac{\cot\varphi}{2\Lambda} \eta^{3a}{}_b\, e^b_s \wedge e^5_s \\ &- \frac{1}{2\Lambda}\left(\eta^{2a}{}_b\, e^b_s \wedge e^6_s - \eta^{1a}{}_b\, e^b_s \wedge e^5_s\right) + \frac{1}{\Lambda}\eta^{1a}{}_b\, e^b_s \wedge e^5_s \ , \end{aligned} \tag{5.35a}$$

$$\mathrm{d}e^5_s = -\frac{\cot\varphi}{\Lambda}\left(e^5_s \wedge e^6_s + \eta^3_{ab}\, e^a_s \wedge e^b_s\right) + \frac{1}{\Lambda}\eta^1_{ab}\, e^a_s \wedge e^b_s \ , \tag{5.35b}$$

$$\mathrm{d}e^6_s = 0 \ . \tag{5.35c}$$

It is important to realise that, although the components $(\Gamma^P)^a_b$ used in (5.35) coincide with the components of the lift of the canonical connection on the Sasaki-Einstein 5-manifold to the cylinder; the transformed coframe e^μ_s is used since we are on the nearly Kähler sine-cone. Thus, with respect to that frame $(\Gamma^P)^a_b$ no longer comprises the canonical $\mathfrak{su}(2)$-connection. However, it forms a different $\mathfrak{su}(2)$-valued connection $\Gamma_{\mathfrak{su}(2)}$. This is because the inhomogeneous term in (5.11), which results from the change of basis, has been split off.

Introducing an almost complex structure J via demanding

$$\Theta^1_s = e^1_s + \mathrm{i}e^2_s \ , \quad \Theta^2_s = e^3_s + \mathrm{i}e^4_s \quad \text{and} \quad \Theta^3_s = \mathrm{i}(e^5_s + \mathrm{i}e^6_s) \tag{5.36}$$

to be $(1,0)$-forms yields

$$\mathrm{d}\begin{pmatrix}\Theta_s^1\\ \Theta_s^2\\ \Theta_s^3\end{pmatrix} = -\underbrace{\begin{pmatrix}\hat{\Gamma}_{\mathfrak{su}(2)}{}^1_1 + \frac{\mathrm{i}\cot\varphi}{2\Lambda}e_s^5 & \hat{\Gamma}_{\mathfrak{su}(2)}{}^1_2 & -\frac{\cot\varphi}{\Lambda}\Theta_s^1 - \frac{1}{2\Lambda}\Theta_s^{\bar{2}}\\ \hat{\Gamma}_{\mathfrak{su}(2)}{}^2_1 & \hat{\Gamma}_{\mathfrak{su}(2)}{}^2_2 + \frac{\mathrm{i}\cot\varphi}{2\Lambda}e_s^5 & -\frac{\cot\varphi}{\Lambda}\Theta_s^2 + \frac{1}{2\Lambda}\Theta_s^{\bar{1}}\\ \frac{\cot\varphi}{\Lambda}\Theta_s^{\bar{1}} + \frac{1}{2\Lambda}\Theta_s^2 & \frac{\cot\varphi}{\Lambda}\Theta_s^{\bar{2}} - \frac{1}{2\Lambda}\Theta_s^1 & -\frac{\mathrm{i}\cot\varphi}{\Lambda}e_s^5\end{pmatrix}}_{\text{canonical } \mathfrak{su}(3)\text{-connection } \hat{\Gamma}_{\mathfrak{su}(3)} \text{ on sine-cone}} \wedge \begin{pmatrix}\Theta_s^1\\ \Theta_s^2\\ \Theta_s^3\end{pmatrix} \underbrace{-\frac{1}{\Lambda}\begin{pmatrix}\Theta_s^{\bar{2}\bar{3}}\\ \Theta_s^{\bar{3}\bar{1}}\\ \Theta_s^{\bar{1}\bar{2}}\end{pmatrix}}_{\text{NK-torsion } \hat{T}} . \tag{5.37}$$

Here we used the shorthand notation $\Theta^{\bar{\alpha}\bar{\beta}} \equiv \Theta^{\bar{\alpha}} \wedge \Theta^{\bar{\beta}}$.

The connection 1-forms $\hat{\Gamma}_{\mathfrak{su}(2)}{}^\beta_\alpha$ with $\alpha, \beta = 1,2$ are defined via the components $(\Gamma^P)_a^{\ b}$ by employing (5.31) and (5.35) as well as the change to the complex basis (5.36). We use the hat to indicate that we are considering the connection forms with respect to the complex basis Θ_s rather than the real basis e_s. Thus, the corresponding Maurer-Cartan equations read

$$\mathrm{d}\Theta_s^\alpha = -\hat{\Gamma}_{\mathfrak{su}(3)}{}^\alpha_\beta \wedge \Theta_s^\beta + \hat{T}^\alpha \qquad \text{and} \qquad \mathrm{d}\Theta_s^{\bar{\alpha}} = -\hat{\Gamma}_{\mathfrak{su}(3)}{}^{\bar{\alpha}}_{\bar{\beta}} \wedge \Theta_s^{\bar{\beta}} + \hat{T}^{\bar{\alpha}} . \tag{5.38}$$

Note that $\Gamma_{\mathfrak{su}(3)} = \mathrm{diag}(\hat{\Gamma}_{\mathfrak{su}(3)}, \hat{\Gamma}^*_{\mathfrak{su}(3)})$ is indeed a connection on TM^6, which can be seen from (5.38) and the fact that $\hat{T}$ transforms as a tensor. Furthermore, $\Gamma_{\mathfrak{su}(3)}$ is an instanton because it satisfies the conditions of [84, Prop. 3.1].

The above result (5.37) can be brought into a more suggestive form by rewriting it as

$$\begin{aligned}
\hat{\Gamma}_{\mathfrak{su}(3)} = \hat{\Gamma}_{\mathfrak{su}(2)} &+ \frac{1}{2\Lambda}\begin{pmatrix}0 & 0 & -2\cot\varphi\\ 0 & 0 & 1\\ 2\cot\varphi & -1 & 0\end{pmatrix} e_s^1 + \frac{\mathrm{i}}{2\Lambda}\begin{pmatrix}0 & 0 & -2\cot\varphi\\ 0 & 0 & -1\\ -2\cot\varphi & -1 & 0\end{pmatrix} e_s^2\\
&+ \frac{1}{2\Lambda}\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & -2\cot\varphi\\ 1 & 2\cot\varphi & 0\end{pmatrix} e_s^3 + \frac{\mathrm{i}}{2\Lambda}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & -2\cot\varphi\\ 1 & -2\cot\varphi & 0\end{pmatrix} e_s^4 \qquad (5.39\text{a})\\
&+ \frac{\mathrm{i}}{2\Lambda}\begin{pmatrix}\cot\varphi & 0 & 0\\ 0 & \cot\varphi & 0\\ 0 & 0 & -2\cot\varphi\end{pmatrix} e_s^5\\
= \hat{\Gamma}_{\mathfrak{su}(2)} &+ B_\mu \otimes e_s^\mu , \qquad (5.39\text{b})
\end{aligned}$$

which reflects exactly the X_μ-ansatz from (5.21). One can check that the matrices B_μ satisfy the equivariance condition (4.27). Thus, as $\Gamma_{\mathfrak{su}(3)}$ is a connection on TM^6, one can infer by the same arguments as in Sec. 5.1 that $\Gamma_{\mathfrak{su}(2)}$ is a well-defined connection on TM^6. An alternative way to see that is to check that the inhomogeneous part, which has been split off in the transformation law (5.11) for the components of Γ^P, glues to globally well-defined 1-forms with values in the adjoint bundle of $\mathcal{P}$. This, however, holds due to the fact that the transformation L given in (5.34) commutes with the SU(2) subgroup of GL(6, ℝ), i.e. takes values in centraliser $C_{\mathrm{GL}(6,\mathbb{R})}(\mathrm{SU}(2))$.

Note that in the limit $\Lambda \to \infty$ (i.e. $\varphi = \frac{r}{\Lambda} \to 0$) the torsion on $C(M^5)$ vanishes, and $\hat{\Gamma}_{\mathfrak{su}(3)}$ coincides with the connection corresponding to the $\chi = \psi = 1$ case of [84], which has been stated to be the Levi-Civita connection of the cone. Furthermore, this is consistent with the observation that as $\hat{\Gamma}_{\mathfrak{su}(3)}$ preserves the metric and as in the above limit its torsion vanishes, $\hat{\Gamma}_{\mathfrak{su}(3)}$ has to converge to the Levi-Civita connection of the CY-cone.

5.3.3 Matrix equations — part II

As pointed out above, there are two different $\mathfrak{su}(2)$-valued connections on the nearly Kähler sine-cone. On the one hand, there is the lift of the canonical connection Γ^P of the Sasaki-Einstein

5-manifold; on the other hand, there is $\Gamma_{\mathfrak{su}(2)}$. A comparison shows that the respective curvature 2-forms coincide, i.e.

$$R_{\Gamma^P} = R_{\Gamma_{\mathfrak{su}(2)}} \ . \tag{5.40}$$

This stems from the fact that the generators of the two transformations (3.44) and (3.46), which lead from the cylinder to the sine-cone, commute with $\mathfrak{su}(2)$. In other words, the inhomogeneous part of (5.11) yields an abelian flat part proportional to e^6_s. As a consequence, $\Gamma_{\mathfrak{su}(2)}$ is another $\mathfrak{su}(2)$-valued instanton on the sine-cone, since Γ^P is an instanton itself[17]. Therefore, we can use $\Gamma_{\mathfrak{su}(2)}$ in the procedure described in Sec. 5.1: One extends $\Gamma_{\mathfrak{su}(2)}$ by some suitable 1-form $X_\mu \, e^\mu_s$ and investigates the conditions on X_μ such that the new connection is an instanton on the sine-cone.

However, we have to adjust the equations (5.23) due to the different torsion of $\Gamma_{\mathfrak{su}(2)}$. Denoting by T the torsion of Γ^P, the torsion of $\Gamma_{\mathfrak{su}(2)}$ reads

$$T^{\hat{\mu}}_{\mathfrak{su}(2)} = T^{\hat{\mu}} + \frac{1}{\Lambda}\Big(\delta^{\hat{\mu}}_{\hat{\nu}} \, \cot\varphi + \tfrac{1}{2}\, \eta^{2\hat{\mu}}{}_{\hat{\nu}}\Big)\, e^6_s \wedge e^{\hat{\nu}}_s \ , \tag{5.41}$$

where we defined $\eta^2_{\hat{\mu}\hat{\nu}} = \eta^2_{ab}$ for $\hat{\mu},\hat{\nu} = a,b \in \{1,\dots,4\}$ and $\eta^2_{\hat{\mu}\hat{\nu}} = 0$ whenever $\hat{\mu} \geq 5$ or $\hat{\nu} \geq 5$. The components of N are the same as in Sec. 5.3.1 and, by inserting everything into (5.8), we obtain the matrix equations

$$[\widehat{I}_\alpha, X_\mu] = f_{\alpha\mu}{}^{\nu}\, X_\nu \ , \tag{5.42a}$$

$$[X_a, X_b] = \frac{1}{2}\, \eta^3_{ab} \dot{X}_5 + \frac{1}{2\Lambda}\left(5 \cot\varphi\, \eta^3_{ab} - 4\,\eta^1_{ab}\right)\, X_5 + f_{ab}{}^{\alpha}\, \mathcal{N}_\alpha \ , \tag{5.42b}$$

$$[X_5, X_a] = \eta^3{}_a{}^b \dot{X}_b + \frac{1}{2\Lambda}\, \left(5\, \cot\varphi\, \eta^3{}_a{}^b - 3\,\eta^1{}_a{}^b - \eta^3{}_a{}^c\, \eta^2{}_c{}^b\right)\, X_b \ , \tag{5.42c}$$

with the notation $\dot{Y} = \frac{\mathrm{d}}{\mathrm{d}r} Y$. Next, we use the matrices in (5.39) to construct the extension of $\Gamma_{\mathfrak{su}(2)}$. Recall that we had defined auxiliary matrices B_μ that solve the equivariance condition (4.27) by writing (5.39) in the form

$$\hat{\Gamma}_{\mathfrak{su}(3)} = \hat{\Gamma}_{\mathfrak{su}(2)} + B_\mu e^\mu_s \ , \tag{5.43}$$

and that the B_μ explicitly depend on $\varphi = \frac{r}{\Lambda}$. Hence, we may set

$$X_a \coloneqq \psi(r)\, B_a \qquad \text{and} \qquad X_5 \coloneqq \chi(r)\, B_5 \tag{5.44}$$

as in the usual procedure[18]. The equivariance condition enforces the same coefficient function $\psi(r)$ for all four B_a. Inserting this X_μ-ansatz in the matrix equations (5.42), one can first of all read off

$$\mathcal{N}_i = \psi(r)^2 \frac{1 + 4\cot^2\!\varphi}{4\Lambda^2} I_i \ , \qquad \text{for} \quad i = 6,7,8 \ , \tag{5.45}$$

which is compatible with the assumptions on $\mathcal{N}$ used in Sec. 5.1. Using this explicit form, we obtain the algebraic equation

$$\psi(r)^2 - \chi(r) = 0 \ . \tag{5.46}$$

This then reduces the remaining equations to

$$\dot{\chi}(r) = \dot{\psi}(r) = 0 \qquad \text{and} \qquad \psi(r)\big(\chi(r) - 1\big) = 0 \ . \tag{5.47}$$

[17] We recall from Sec. 5.1.3: Γ^P is a connection on the SU(2)-bundle $\mathcal{Q}$, whereas $\Gamma_{\mathfrak{su}(2)}$ is a connection on the SU(2)-bundle $\mathcal{Q}'$.

[18] Note that in (5.39) we have $B_\mu \in \mathrm{End}(\mathbb{C}^3)$. Here we used the identification $\mathbb{C} \simeq \mathbb{R}^2$ to obtain $B_\mu \in \mathrm{End}(\mathbb{R}^6)$, which is necessary for the ansatz (5.5).

Let us now comment on the three solutions to this system:

- $(\psi,\chi)=(0,0)$: To start with, there is the obvious trivial solution of (5.42). This is required for consistency, since $\Gamma_{\mathfrak{su}(2)}$ is an instanton.
- $(\psi,\chi)=(1,1)$: This second solution is very important because it reproduces $\Gamma_{\mathfrak{su}(3)}$ from Sec. 5.3.2. We already knew from [84, Prop. 3.1] that this particular connection is an instanton on the nearly Kähler sine-cone, but here we confirmed it directly, using techniques completely different than those employed in [84]. In addition, this provides us with another way of constructing the canonical connection of the nearly Kähler sine-cone than the one we followed in Sec. 5.3.2, namely as the extension of an $\mathfrak{su}(2)$-valued instanton.
- $(\psi,\chi)=(-1,1)$: Thirdly, there is again the solution which results from the invariance of (5.42) under the simultaneous sign-flip $X_a\mapsto -X_a$ for $a=1,2,3,4$. Nevertheless, this solution is an additional instanton.

In summary, the solutions we obtained here are isolated $\mathfrak{su}(3)$- and $\mathfrak{su}(2)$-valued connections on M^6 that cannot be traced back to lifts of connections on M^5. In contrast to e.g. [95], there are no instanton solutions that interpolate between these isolated instantons. (At least not in the subset of the moduli space that we can explore with these techniques.)

Remarks Firstly, the CY-limit $\Lambda\to\infty$ of (5.42) is given by

$$[X_a,X_b]=f_{ab}{}^5\left(X_5+\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\tau}X_5\right)+f_{ab}{}^i\mathcal{N}_i \quad\text{and}\quad [X_5,X_a]=f_{5a}{}^b\left(X_b+\frac{2}{3}\frac{\mathrm{d}}{\mathrm{d}\tau}X_b\right) \ , \tag{5.48}$$

wherein one requires the rescaling $X_\mu\mapsto\frac{1}{r}X_\mu$, which can be seen from $X_\mu e^\mu_s\to X_\mu\, re^\mu$ for $\Lambda\to\infty$. Further, recall that in the limit $\Lambda\to\infty$ we have $\mathrm{d}\tau=\frac{1}{r}\mathrm{d}r$. The above matrix equations coincide with the ones obtained in the Kähler-torsion case (5.18) of Sec. 5.2 as well as with the limit (5.29) of Sec. 5.3.1. The two reductions of Sec. 5.3.1 and 5.3.3 used the different $\mathfrak{su}(2)$-instantons Γ^P and $\Gamma_{\mathfrak{su}(2)}$ as starting point; however, in the above limit the difference

$$\Gamma^P-\Gamma_{\mathfrak{su}(2)}\xrightarrow{\Lambda\to\infty}\mathbb{1}\otimes\frac{\mathrm{d}r}{r}\in\Omega^1(M^6,\mathrm{End}(\mathbb{R}^6)) \tag{5.49}$$

becomes an abelian flat part, which contributes to the instanton equation via the altered torsion.

Secondly, note the explicit impact of the conical singularities at $\varphi=0$ or $\varphi=\pi$ in the matrix equations (5.42) as well as the B_μ-matrices of (5.39). However, we do not have to consider these singularities, as the instanton equation is not well-defined at the tips.

5.3.4 Transfer of solutions

The previous subsections considered the nearly Kähler sine-cone from two perspectives: in Sec. 5.3.1 we extended the instanton Γ^P, which is a connection on $\mathcal{Q}$, whereas Sec. 5.3.3 was concerned with $\Gamma_{\mathfrak{su}(2)}$, which is an $\mathfrak{su}(2)$-valued connection on $\mathcal{Q}'$, as a starting point for our ansatz (5.5). The local representations of these are related via a transformation L as considered in (5.34). Due to the properties of L we arrive at the following statement (c.f. Sec. 5.1.3):

$$e'^*\Gamma_{\mathfrak{su}(2)}=e'^*\Gamma^P-L^{-1}\mathrm{d}L=e^*\Gamma^P \ , \tag{5.50}$$

implying that $\Gamma_{\mathfrak{su}(2)}$ and Γ^P have the same components with respect to their adapted coframes e' and e. Observe that the inhomogeneous part that is split off in the connection 1-form enters in the torsion (5.41) of $\Gamma_{\mathfrak{su}(2)}$, thus altering the matrix equations. However, from (5.8) one can check that the local expressions of the respective field strengths of the extension of both Γ^P and $\Gamma_{\mathfrak{su}(2)}$ by $X_\mu \otimes e'^\mu = X_\mu L^\mu_\nu \otimes e^\nu$ coincide. Consequently, every instanton extension X_μ of $\Gamma_{\mathfrak{su}(2)}$ gives rise to an instanton extension $X_\nu L^\nu_\mu$ of Γ^P and vice versa. In other words, we have the relation

$$X_\mu \text{ solves } (5.42) \quad \xLeftrightarrow{1:1} \quad X_\nu L^\nu_\mu \text{ solves } (5.23) \ . \tag{5.51}$$

As a remark, the above is true whenever L takes values in the centraliser $C_{\mathrm{GL}(6,\mathbb{R})}(\mathrm{SU}(2))$, as then $L^{-1}\mathrm{d}L$ gives rise to a well-defined equivariant 1-form. However, one should not naively expect that the solutions obtained in Sec. 5.3.1 and 5.3.3 are related via (5.51), as this does not necessarily transform the employed ansätze into one another.

The benefit from observation (5.51) is that we can generate further instanton solutions from our previous ones. On the one hand, we can apply the above to (5.27) and obtain the ansatz

$$X_a = \frac{\psi(r)}{\Lambda \sin(\frac{r}{\Lambda})} \left(\exp\left(\frac{r}{2\Lambda}\eta^2\right) \exp(\xi\eta^3) \right)_a^{\ b} \widehat{I}_b \quad \text{and} \quad X_5 = \frac{\chi(r)}{\Lambda \sin(\frac{r}{\Lambda})} \widehat{I}_5 \ , \tag{5.52}$$

which inserted into (5.42) has precisely the solutions $(\psi, \chi) = (0,0), (\pm 1, 1)$, just as one would expect from the above arguments. This is another non-constant instanton extension for $\Gamma_{\mathfrak{su}(2)}$.

On the other hand, the same can be done for (5.44) in the other direction. There one derives the ansatz

$$X_a = \psi(r)\, \Lambda \sin(\tfrac{r}{\Lambda}) \exp\left(-\frac{r}{2\Lambda}\eta^2\right)_a^{\ b} B_b(r) \quad \text{and} \quad X_5 = \chi(r)\, \Lambda \sin(\tfrac{r}{\Lambda}) B_5(r) \ . \tag{5.53}$$

Rewritten in a linear combination of the $\widehat{I}_\mu$, the ansatz (5.53) is given as

$$\begin{aligned}
X_1 &= \psi(r) \left(\cos^3\left(\tfrac{r}{2\Lambda}\right) \widehat{I}_1 - \sin^3\left(\tfrac{r}{2\Lambda}\right) \widehat{I}_3 \right) \ , \\
X_2 &= \psi(r) \left(\cos^3\left(\tfrac{r}{2\Lambda}\right) \widehat{I}_2 + \sin^3\left(\tfrac{r}{2\Lambda}\right) \widehat{I}_4 \right) \ , \\
X_3 &= \psi(r) \left(\cos^3\left(\tfrac{r}{2\Lambda}\right) \widehat{I}_3 + \sin^3\left(\tfrac{r}{2\Lambda}\right) \widehat{I}_1 \right) \ , \\
X_4 &= \psi(r) \left(\cos^3\left(\tfrac{r}{2\Lambda}\right) \widehat{I}_4 - \sin^3\left(\tfrac{r}{2\Lambda}\right) \widehat{I}_2 \right) \ , \\
X_5 &= \chi(r) \, \cos\left(\tfrac{r}{\Lambda}\right) \widehat{I}_5 \ .
\end{aligned} \tag{5.54}$$

One can check that this, again, produces the solutions $(\psi, \chi) = (0,0), (\pm 1, 1)$. The two non-trivial instanton solutions correspond to non-constant extensions of Γ^P.

5.4 Instantons on half-flat cylinders

Let us now return to the half-flat 6-manifolds constructed in Sec. 3.5.4 and apply the ansatz developed above to the instanton equation on these spaces. The instanton equation on spaces with non-vanishing W_2 was introduced in (5.2). In a local coframe adapted to the SU(3)-structure, imposing the pseudo-holomorphicity condition

$$\Omega_z \wedge \mathcal{F}_{\mathcal{A}} = 0 \tag{5.55}$$

yields the same set of six equations as we have obtained in the nearly Kähler case. But the additional equation implied by the pseudo-holomorphicity condition reads

$$\mathrm{d}\Omega_z \wedge \mathcal{F}_{\mathcal{A}} = 0 \qquad \Leftrightarrow \qquad (\mathcal{F}_{\mathcal{A}})_{12} + (\mathcal{F}_{\mathcal{A}})_{34} + \tfrac{4}{3}\varrho^2\, (\mathcal{F}_{\mathcal{A}})_{56} = 0 \tag{5.56}$$

in the rotated frame e_z. Note that for $\varrho = \pm\frac{\sqrt{3}}{2}$ this coincides with the nearly Kähler instanton equation of Sec. 5.3.1, although the SU(3)-structure is not nearly Kähler (see for example the torsion classes (3.53)).

It is important to recall that the lift of the canonical connection of the Sasaki-Einstein M^5 provides an instanton on the cylinder that one can extend by some X in our ansatz to $\mathfrak{su}(3)$-valued connections, being defined either on $\mathcal{P}$ or $\mathcal{P}'$. We will do so in two set-ups: firstly, we formulate the matrix equations in the frame e^μ, and, secondly, the analogous computation is performed in the adapted frame e_z^μ for the half-flat SU(3)-structure.

5.4.1 Matrix equations — part I

In the unrotated frame e^μ the instanton bundle is locally spanned by

$$e^5\wedge e^6 - \tfrac{\varrho}{3}\left(\cos\zeta\,\eta^1_{ab} - \sin\zeta\,\eta^2_{ab}\right)e^a\wedge e^b \quad \text{and} \quad e^a\wedge e^6 - \varrho\left(\cos\zeta\,\eta^{1a}{}_b - \sin\zeta\,\eta^{2a}{}_b\right)e^b\wedge e^5\ , \tag{5.57}$$

from which we can extract the components of $(N^\rho_{\mu\nu})$ to be

$$N^5_{ab} = \frac{2\varrho}{3}\left(\cos\zeta\,\eta^1_{ab} - \sin\zeta\,\eta^2_{ab}\right) \quad \text{and} \quad N^a_{b5} = \varrho\left(\cos\zeta\,\eta^{1a}{}_b - \sin\zeta\,\eta^{2a}{}_b\right)\ . \tag{5.58}$$

As the torsion components are unchanged, we can directly give the matrix equations

$$[\widehat{I}_\alpha, X_\mu] = f_{\alpha\mu}{}^\nu\,X_\nu\ , \tag{5.59a}$$

$$[X_a, X_b] = \frac{2\varrho}{3}\left(\cos\zeta\,\eta^1_{ab} - \sin\zeta\,\eta^2_{ab}\right)\dot{X}_5 + 2\,\eta^3_{ab}\,X_5 + f_{ab}{}^\alpha\,\mathcal{N}_\alpha\ , \tag{5.59b}$$

$$[X_5, X_a] = \varrho\left(\cos\zeta\,\eta^1{}_a{}^b - \sin\zeta\,\eta^2{}_a{}^b\right)\dot{X}_b + \frac{3}{2}\,\eta^3{}_a{}^b\,X_b\ . \tag{5.59c}$$

The ansatz

$$X_a = \psi(r)\left(\exp(\xi\ \eta^3)\right)_a^{\ b}\hat{I}_b \qquad \text{for} \quad \xi\in[0,2\pi) \quad \text{and} \quad X_5 = \chi(r)\hat{I}_5 \tag{5.60}$$

satisfies, again, the equivariance condition of (5.59). We then obtain

$$\mathcal{N}_i = \psi^2(r)\ \hat{I}_i\ , \qquad \text{for} \quad i = 6,7,8 \tag{5.61}$$

as well as the set of equations

$$\dot{\psi}(r) = \dot{\chi}(r) = 0\ , \qquad \psi^2(r) = \chi(r)\ , \qquad \text{and} \qquad \psi(r)\left(\chi(r)-1\right) = 0\ , \tag{5.62}$$

for the two functions ψ and χ, and the equations hold for all values of ξ.

- $(\psi,\chi) = (0,0)$: The trivial solution appears again for consistency.
- $(\psi,\chi) = (\pm 1,1)$: These two extensions of the lift of Γ^P are newly obtained instantons; however, they correspond to lifts of M^5-instantons because they are independent of the cylinder direction. Recall that $(\psi,\chi) = (-1,1)$ can be generated from $(\psi,\chi) = (+1,1)$ by the shift $\xi \mapsto \xi + \pi$.

Identically to the nearly Kähler case, one obtains the one-parameter family (5.27) as a solution. As a matter of fact, these instanton solutions are identical to the ones obtained in Sec. 5.3.1. The explanation is as follows: firstly, note that nearly Kähler 6-manifolds are a subset of half-flat 6-manifolds; thus, any nearly Kähler instanton solution must necessarily appear in the half-flat scenario. Secondly, the matrix equations (5.23) and (5.59) differ only in their derivative parts, i.e. in the coefficients of $\dot{X}_\mu$, which implies that both sets have coinciding constant solutions.

5.4.2 Matrix equations — part II

In contrast to the previous subsection, here the focus is on the formulation of the instanton equations in the adapted coframe e^μ_z for the SU(3)-structure on the cylinder. As with respect to these, the SU(3)-structure forms have their standard components, one only has to compute the components of its torsion with respect to the transformed basis.

The space $\mathfrak{m}$ is now spanned by the 2-forms

$$e^5_z \wedge e^6_z - \tfrac{1}{3}\varrho^2\, \eta^3_{ab}\, e^a_z \wedge e^b_z \qquad \text{and} \qquad e^a_z \wedge e^6_z - {\eta^{3a}}_b\, e^b_z \wedge e^5_z \,, \tag{5.63}$$

which follows from direct evaluation of (5.55) and (5.56). In the coframe e_z the torsion components of the lifted canonical connection of the Sasaki-Einstein manifold are

$$\tilde{T}^5_{ab} = 2\varrho\, \eta^1_{ab} \qquad \text{and} \qquad \tilde{T}^a_{b5} = \tfrac{3}{2\varrho}\, {\eta^{1a}}_b \,. \tag{5.64}$$

In addition, we need the tensor N that appeared in (5.10). Since the instanton equations take a slightly different form here, its components now read

$$N^a_{\mu\nu} = \tfrac{2}{3}\, {f_{\mu\nu}}^a \qquad \text{and} \qquad N^5_{\mu\nu} = \tfrac{1}{3}\varrho^2\, {f_{\mu\nu}}^5 \,, \tag{5.65}$$

wherein we have used the same $\mathfrak{su}(3)$ structure constants as in (5.15). With these alterations (5.10) can be written as

$$\begin{aligned} [\widehat{I}_\alpha, X_\mu] &= {f_{\alpha\mu}}^\nu\, X_\nu \,, &\text{(5.66a)}\\ [X_a, X_b] &= -2\varrho\, \eta^1_{ab}\, X_5 + \tfrac{2}{3}\varrho^2\, {f_{ab}}^5\, \dot{X}_5 + \mathcal{N}_\alpha\, {f_{ab}}^\alpha \,, &\text{(5.66b)}\\ [X_a, X_5] &= \tfrac{3}{2\varrho}\, {\eta^1}_a{}^b\, X_b + \tfrac{2}{3}\, {f_{a5}}^b\, \dot{X}_b \,. &\text{(5.66c)} \end{aligned}$$

One can employ the following ansatz:

$$X_a = \psi(r)\left(\exp(\xi\, \eta^1)\exp(\theta\, \eta^2)\right)_a^{\;b} \widehat{I}_b \,, \qquad \text{for} \quad \theta,\xi \in [0,2\pi) \quad \text{and} \quad X_5 = \chi(r)\widehat{I}_5 \,, \tag{5.67}$$

which, again, satisfies the equivariance condition. The insertion of (5.67) into (5.66) yields for the $\mathfrak{su}(2)$-part

$$\mathcal{N}_i = \psi^2 I_i \,, \tag{5.68}$$

as the projection of $[X_a, X_b]$ onto $\mathfrak{su}(2)$ in $\mathfrak{su}(3)$ is independent of θ and ξ. Further, for the functions ψ and χ one derives the set of equations

$$\begin{aligned} \dot{\chi} &= \frac{3}{\varrho^2}\, \psi^2 \left(\cos^2\theta - \sin^2\theta\right) \,, &\text{(5.69a)}\\ \chi &= \frac{2}{\varrho}\, \psi^2\, \cos\theta\, \sin\theta \,, &\text{(5.69b)}\\ \dot{\psi}\, \cos\theta &= \frac{3}{2}\, \psi \left(\frac{1}{\varrho}\, \sin\theta + \chi\, \cos\theta\right) \,, &\text{(5.69c)}\\ \dot{\psi}\, \sin\theta &= -\frac{3}{2}\, \psi \left(\frac{1}{\varrho}\, \cos\theta + \chi\, \sin\theta\right) \,. &\text{(5.69d)} \end{aligned}$$

Note that the equations are independent of ξ. These equations are mutually compatible only for $\theta{=}\frac{\pi}{4}$ or $\theta{=}\frac{3\pi}{4}$. For these values of θ the first two equations yield $\dot{\psi} = \dot{\chi} = 0$ and the last two equations coincide. The system (5.69) admits, besides the trivial solution $(\psi, \chi) = (0, 0)$, only the following solutions:

$$\theta = \frac{\pi}{4} \quad : \qquad \psi = \pm 1 \ , \quad \chi = +\frac{1}{\varrho} \ , \tag{5.70a}$$

$$\theta = \frac{3\pi}{4} \quad : \qquad \psi = \pm 1 \ , \quad \chi = -\frac{1}{\varrho} \ . \tag{5.70b}$$

Hence, we again have a whole family of solutions given by

$$A = \Gamma + \Big(\exp(\xi\ \eta^1) \exp(\theta\ \eta^2) \Big)_a^{\ b} \ \widehat{I}_b \otimes e^a_z \pm \frac{1}{\varrho} \ \widehat{I}_5 \otimes e^5_z \ , \qquad \text{for} \quad \theta \in \{ \tfrac{\pi}{4}, \tfrac{3\pi}{4} \} \ , \ \xi \in [0, 2\pi) \ . \tag{5.71}$$

As the corresponding instantons on the cylinder over M^5 do neither depend on the cone coordinate nor contain $\mathrm{d}r$, they are actually lifts of instantons on M^5, which live on the pull-back bundle of the SU(3)-bundle on the slices of the cylinder.

6 Conclusions and outlook

Summarising this first part of the thesis, we follow the separated presentation of the Calabi-Yau cone of Ch. 4 and non-Kähler conical extensions of Ch. 5.

Hermitian Yang-Mills on Calabi-Yau cone It is known that the instanton moduli space over a Kähler manifold is a Kähler space. Therefore, we argued that the moduli space of certain invariant connections inherits this property.

The ansatz presented in Sec. 4.3 relied on two steps: firstly, we chose an instanton Γ^P as a starting point and, secondly, we imposed an equivariance condition and simplified to dependence on the cone-direction only. Therefore, we restrict ourselves to a subset of all accessible connections. Hence, by this construction one can only reach a particular part of the full instanton moduli space by the solutions of the instanton matrix equations.

The arguments subsequently presented in Sec. 4.3 show that the reduced instanton matrix equations can be treated similarly to the Nahm equations with regular boundary conditions. As a consequence, one gains local solvability of the holomorphicity conditions together with the fact that any solution can be uniquely gauge-transformed into a solution of the stability-like condition. Moreover, the structure of the (framed) moduli space shares, at least locally, all features of a Kähler space due to the Kähler quotient construction. In addition, we showed that the framed moduli space is mapped into a finite dimensional orbit with a Kähler structure.

Some open questions remain and we hope to address them in future research. For example, the properties of the map (4.75) have to be studied in more detail; in particular, if there is any hope to restore surjectivity without the gradient flow present for the Nahm equations. Similar to [121, 122], one could explore the implications of non-regular boundary conditions. Moreover, the solutions to the equivariance condition (4.27) have to be investigated in more detail.

In addition, it is of interest to extend the ansatz presented here from cones to their smooth resolutions as in [113, 115]. Moreover, focusing on Sasaki-Einstein coset spaces and the solutions to the equivariance condition it seems natural to consider quiver gauge theories which can be associated to Calabi-Yau cones along the lines of [147]. The latter will be addressed in Part II for the cone over the 5-sphere (and orbifolds thereof).

Instantons on non-Kähler conical 6-manifolds We investigated the geometry of cylinders, cones and sine-cones over 5-dimensional SU(2)-manifolds in Sec. 3.5. On the resulting 6-dimensional conical manifolds we formulated generalised instanton equations and reduced them to matrix equations via the ansatz (5.5). In particular, we focused on Kähler-torsion structures as well as nearly Kähler and half-flat SU(3)-manifolds.

Firstly, we demonstrated in Sec. 3.5.2 that the sine-cone over a Sasaki-Einstein 5-manifold is not only an Einstein space, but, moreover, a Kähler-torsion space. The instanton equations where reduced to matrix equations (5.18), which turned out to be identical to the equations on Calabi-Yau cone due to conformal equivalence. Although we discussed the HYM-equations thoroughly in Ch. 4, the geometric interpretation is *not* the same for the KT-case. The reason behind this is the non-Kähler nature of a generic KT-structure, such that the notion of a moment map is not appropriate.

Secondly, we constructed in Sec. 3.5.3 a nearly Kähler 6-manifold as a sine-cone over an arbitrary Sasaki-Einstein 5-manifold by means of a rotation of the SU(2)-structures on the slices. Employing the ansatz (5.21), the instanton equation was reduced to the set (5.23) of matrix equations, for which we found a family of non-trivial, but constant solutions. All of these correspond to lifts of M^5-instantons to $C_s(M^5)$. In addition, in Sec. 5.3.2 we obtained an instanton solution on the manifold $C_s(M^5)$ by the construction of its $\mathfrak{su}(3)$-valued canonical connection. We decomposed this connection $\Gamma_{\mathfrak{su}(3)}$ into another $\mathfrak{su}(2)$-valued instanton $\Gamma_{\mathfrak{su}(2)}$ plus an additional part resembling the ansatz used before. Using this decomposition and, again, carrying the reduction of the instanton equation out, we obtained a set of four equations for two functions which parametrise the ansatz. Its three solutions, for which the scalar functions take certain constant values, correspond to three instantons on the nearly Kähler sine-cone that cannot be constructed as lifts of instanton connections on M^5. As a by-product, we explicitly confirmed the nearly Kähler canonical connection to be an instanton. In addition, observing a correspondence between the solutions, we transferred the solutions of the two cases to new r-dependent instanton extensions of Γ^P as well as $\Gamma_{\mathfrak{su}(2)}$. Remarkably, the extension found for Γ^P does not seem to correspond to a lift of an instanton from M^5.

Furthermore, we introduced a two-parameter family of half-flat structures on the cylinder over a generic Sasaki-Einstein 5-manifold in Sec. 3.5.4. Again employing the ansatz (5.5) on these cylindrical half-flat 6-manifolds, we were able to deduce the matrix equations (5.66) on the two local frames $e^{\hat{\mu}}$ and $e_z^{\hat{\mu}}$. Moreover, we provided families of constant, but non-trivial solutions. In that case, the instantons obtained this way do correspond to lifts of instantons on M^5.

It would be interesting to extend the methods presented here, i.e. the reduction of the instanton equation to matrix equations and the construction of higher-dimensional G-structure manifolds from lower-dimensional ones, to other scenarios that appear in string theory. For example, in M-theory desirable (internal) manifolds are 7-dimensional and are endowed with a G_2-structure. Therefore, the study of certain SU(3)-structures seems to be promising, as one could hope to obtain interesting G_2-geometries as well as explicit instanton solutions via the procedures employed here.

Returning to the heterotic supergravity point of view, we expect that our solutions to the instanton equations can be lifted to full solutions of the heterotic equations of motions via the BPS equations (2.1) and the Bianchi identity (2.2). The gaugino equation (2.1c) is already solved by the instanton solutions above. The remaining equations should be solvable in a manner similar to [84, 85, 87, 88], which may look as follows:

1. The dilatino equation (2.1b) may be solved by a suitable ansatz such as choosing the dilaton $\phi = \phi(\tau)$ and the 3-form $H \propto \frac{\mathrm{d}\phi}{\mathrm{d}\tau} P$ where P is the canonical 3-form on the Sasaki-Einstein 5-manifold.

2. The gravitino equation (2.1a) requires a spin connection with SU(3)-holonomy and torsion H. Therefore, one can take an ansatz similar to (5.5) from which we know it to be an SU(3)-instanton. The remaining task is then to check the correct torsion for this connection. One choice might be the canonical connection $\Gamma_{\mathfrak{su}(3)}$ on the nearly Kähler sine-cone, whose torsion is by definition skew-symmetric, and we know $\Gamma_{\mathfrak{su}(3)}$ is an instanton.

3. The theorem of Ivanov requires a connection ∇ on TM^6 which is an instanton. Here, the instantons constructed in the first part of this thesis provide a valuable choice, i.e. by an extension of the canonical connection. Then the connection ∇, together with the gauge connection $\mathcal{A}$, needs to satisfy the Bianchi identity (2.2).

Finally, one has to solve the differential equations that appear for the degrees of freedom in the different ansätze for H, ∇^+, and ∇. We hope to report on this process and embed our solutions into heterotic supergravity in the future.

A Appendix: Details

In this appendix, we provide the proofs of the statements made in Sec. 4.3.2–4.3.4. Although the steps are similar to those performed in [40, 121, 122], we believe that these are necessary because the reduced instanton matrix equations are generalisations of the Nahm equations.

A.1 Boundedness of rescaled matrices

Recall the boundary conditions (4.50) for the original matrices

$$t \to +\infty : \qquad X_\mu \to 0 \ , \tag{A.1a}$$

$$t \to -\infty : \qquad e^{\frac{n+1}{n}t} X_a \to \mathrm{Ad}(g_0)\, T_a \quad \text{and} \quad e^{2nt} X_{2n+1} \to \mathrm{Ad}(g_0)\, T_{2n+1} \ . \tag{A.1b}$$

Evaluating the asymptotic behaviour for $t \to +\infty$ of (4.29), one finds the leading behaviour of (the real and imaginary part) of each matrix element to be

$$\frac{\mathrm{d}}{\mathrm{d}t}(X_a)_{AB} + \frac{n+1}{n}(X_a)_{AB} \simeq 0 \qquad \to \qquad (X_a)_{AB} \sim e^{-\frac{n+1}{n}t} \text{ as } t \to \infty \ , \tag{A.2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(X_{2n+1})_{AB} + 2n(X_{2n+1})_{AB} \simeq 0 \qquad \to \qquad (X_{2n+1})_{AB} \sim e^{-2nt} \text{ as } t \to \infty \ , \tag{A.2b}$$

because the commutator terms vanish faster than linear order. These results imply the following:

(i) The rescaled matrices $\mathcal{X}_\mu$ of (4.40) are bounded for $s \to 0$.

(ii) The commutators $e^{\frac{n+1}{n}t}[X_a, X_{2n+1}]$ are integrable over $(0, \infty)$.

(iii) The derivatives $\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{n+1}{n}t} X_a\right)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{2nt} X_{2n+1}\right)$ are integrable, which follows by the use of the equations (4.29).

In conclusion, the $\mathcal{X}_{\hat\mu}$ as well as their derivatives are bounded.

A.2 Well-defined moment map

We need to prove (4.14) for μ defined in (4.60). Recall that $\mu(\mathcal{A}) := \mathcal{F}_\mathcal{A} \wedge \frac{\widehat{\omega}^{n-1}}{(n-1)!}$ and that we identified μ^* with μ. Moreover, it is crucial to use the closed Kähler 2-form from the cone, i.e. $\widehat{\omega} = e^{2t}\widetilde{\omega}$ on the cylinder. We will work with the original connection components Y_k defined in (4.32).

For the left-hand-side we proceed as follows: Let $\phi \in \widehat{\mathfrak{g}}_0$ and $\Psi = \Psi_k \theta^k - \Psi_k^\dagger \bar\theta^k$ be a tangent vector at $\mathcal{A}$. The duality pairing of Lie- and dual Lie-algebra is realised by the integration over the cylinder and the subsequent invariant product on $\mathfrak{u}(p)$.

$$(\phi, \mathrm{D}\mu_{|\mathcal{A}})\Psi = \int_{\mathrm{Cyl}(M^{2n+1})} \mathrm{tr}\left\{\phi \frac{\mathrm{d}}{\mathrm{d}z}\mathcal{F}_{\mathcal{A}+z\Psi}\big|_{z=0}\right\} \wedge \frac{\widehat{\omega}^n}{n!} \tag{A.3a}$$

$$= \int_{\mathbb{R}} \mathrm{d}t\ e^{2nt}\ \mathrm{tr}\Bigg\{\phi \cdot \mathrm{i}\Bigg[\frac{\mathrm{d}}{\mathrm{d}t}(\Psi_{n+1} + \Psi_{n+1}^{\dagger}) + 2n(\Psi_{n+1} + \Psi_{n+1}^{\dagger}) + 2\sum_{k=1}^{n+1}\Big(\Big[\Psi_k, Y_k^{\dagger}\Big] + \Big[Y_k, \Psi_k^{\dagger}\Big]\Big)\Bigg]\Bigg\} \cdot \int_{M^{2n+1}} \mathrm{vol}\ . \tag{A.3b}$$

Hence, for the dual moment map one can neglect the volume integral over M^{2n+1} and the dual pairing is defined via the integral over t.

To compute the right-hand-side of (4.14) we need to take a step back and derive the symplectic form on $\mathbb{A}$ from (4.9) as follows:

$$\boldsymbol{\omega}_{|\mathcal{A}}(\Psi, \Xi) = -\int_{\mathrm{Cyl}(M^{2n+1})} \mathrm{tr}\,(\Psi \wedge \Xi) \wedge \frac{\widehat{\omega}^n}{n!} \tag{A.4a}$$

$$= -2\mathrm{i} \int_{\mathbb{R}} \mathrm{d}t\ e^{2nt}\ \mathrm{tr} \sum_{k=1}^{n+1} \Big\{\Psi_k^{\dagger}\Xi_k - \Psi_k \Xi_k^{\dagger}\Big\} \cdot \int_{M^{2n+1}} \mathrm{vol}\ . \tag{A.4b}$$

Again, we can drop the volume of the Sasaki-Einstein space. Next, we need the infinitesimal gauge transformation generated by an (framed) Lie-algebra element ϕ. From (4.45) we obtain

$$\phi^{\#} = \frac{\mathrm{d}}{\mathrm{d}z} Y_j^{g=\exp(z\phi)}\Big|_{z=0} = \begin{cases} [\phi, Y_j] & , j = 1, \ldots, n \\ [\phi, Y_{n+1}] - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\phi & , j = n+1\ , \end{cases} \tag{A.5}$$

which then leads us to

$$\iota_{\phi^{\#}} \boldsymbol{\omega}_{|\mathcal{A}}(\Psi) = -2\mathrm{i} \int_{\mathbb{R}} \mathrm{d}t\ e^{2nt}\ \mathrm{tr}\Bigg\{ \sum_{k=1}^{n} \Big\{[\phi, Y_k]^{\dagger}\, \Psi_k - [\phi, Y_k]\, \Psi_k^{\dagger}\Big\} + \Big([\phi, Y_{n+1}] - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\phi\Big)^{\dagger} \Psi_{n+1} - \Big([\phi, Y_{n+1}] - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\phi\Big)\, \Psi_{n+1}^{\dagger}\Bigg\} \tag{A.6a}$$

$$= \int_{\mathbb{R}} \mathrm{d}t\ e^{2nt}\ \mathrm{tr}\Bigg\{\phi \cdot \mathrm{i}\Bigg[\frac{\mathrm{d}}{\mathrm{d}t}(\Psi_{n+1} + \Psi_{n+1}^{\dagger}) + 2n(\Psi_{n+1} + \Psi_{n+1}^{\dagger}) + 2\sum_{k=1}^{n+1}\Big(\Big[\Psi_k, Y_k^{\dagger}\Big] + \Big[Y_k, \Psi_k^{\dagger}\Big]\Big)\Bigg]\Bigg\} - \mathrm{i}\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t}\Big\{e^{2nt}\mathrm{tr}\phi(\Psi_{n+1} + \Psi_{n+1}^{\dagger})\Big\}\ . \tag{A.6b}$$

A close inspection of the boundary term reveals that

$$\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t}\Big\{e^{2nt}\ \mathrm{tr}\Big(\phi(\Psi_{n+1} + \Psi_{n+1}^{\dagger})\Big)\Big\} = e^{2nt}\ \mathrm{tr}\Big(\phi(\Psi_{n+1} + \Psi_{n+1}^{\dagger})\Big)\Big|_{t\to-\infty}^{t\to+\infty} \tag{A.7}$$

vanishes provided $\lim_{t\to\pm\infty}\phi(t) = 0$, i.e. the map defined in (4.60) is a moment map for the action of the *framed gauge group* $\widehat{\mathcal{G}}_0 = \{g(t) | g : \mathbb{R} \to \mathrm{U}(p), \text{s.t. } \lim_{t\to\pm\infty} g(t) = 1\}$.

A.3 Notation

We need to introduce some notation, which is relevant for the proofs later.

∂,$\bar{\partial}$-operators Following [40], we define the following ∂, $\bar{\partial}$-operators on $\mathbb{C}^p$-valued functions f on $\mathbb{R}^-$:

$$\mathrm{d}_{\mathcal{Z}} f = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s} f + \mathcal{Z} f \ , \qquad \bar{\mathrm{d}}_{\mathcal{Z}} f = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s} f - \mathcal{Z}^\dagger f \ , \tag{A.8a}$$

$$\mathrm{d}_j f = \mathcal{Y}_j f \ , \qquad \bar{\mathrm{d}}_j f = -\mathcal{Y}_j^\dagger f \ , \tag{A.8b}$$

and on matrix-valued functions γ on $\mathbb{R}^-$

$$\mathrm{d}_{\mathcal{Z}} \gamma = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s} \gamma + [\mathcal{Z}, \gamma] \ , \qquad \bar{\mathrm{d}}_{\mathcal{Z}} \gamma = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s} \gamma - \left[\mathcal{Z}^\dagger, \gamma\right] \ , \tag{A.8c}$$

$$\mathrm{d}_j \gamma = [\mathcal{Y}_j, \gamma] \ , \qquad \bar{\mathrm{d}}_j \gamma = -\left[\mathcal{Y}_j^\dagger, \gamma\right] \ . \tag{A.8d}$$

These operators will give rise to the $\bar{\partial}$-operators associated to the connection $\mathcal{A}$. For that we take the covariant derivative $\mathrm{d}_{\mathcal{A}} = \mathrm{d} + \widehat{\Gamma}^P + Y_j \theta^j + Y_{\bar{j}} \bar{\theta}^j$ and define $\bar{\partial}_{\mathcal{A}} = \bar{\partial} + (\widehat{\Gamma}^P)^{(0,1)} + Y_{\bar{j}} \bar{\theta}^j$. Hence, the above definitions are understood as components of $\bar{\partial}_{\mathcal{A}}$. However, our notation and conventions differ slightly from [40] in the sense that we work with the equivalent $\partial_{\mathcal{A}}$-operator. In detail, the cone direction s in [40] is considered as 0-th coordinate such that the *canonical* complex structure is defined via the choice of $(1,0)$-forms $\mathrm{d}s + \mathrm{i}e^1$ and $e^2 + \mathrm{i}e^3$ ($\{e^p,\ p = 1,2,3\}$ a co-frame on $\mathbb{R}^3$). In contrast, we designated the cone coordinate as e^{2n+2} and choose the $(1,0)$-forms as in (3.6) in order to avoid unnecessary factors of i. With respect to the canonical choice $e^{2j-1} + \mathrm{i}e^{2j}$ our complex structure is simply $J = -J_{\mathrm{can}}$, implying that we interchanged $(1,0)$ and $(0,1)$-forms. Consequently, we consider the $\partial_{\mathcal{A}}$-operator.

Gauge transformations For the ∂-operators the action of the complex automorphisms is defined via

$$\mathrm{d}_j^g \coloneqq g \circ \mathrm{d}_j \circ g^{-1} \quad \text{and} \quad \mathrm{d}_{\mathcal{Z}}^g \coloneqq g \circ \mathrm{d}_{\mathcal{Z}} \circ g^{-1} \ . \tag{A.9}$$

From these definitions, we obtain

$$g^{-1} \mathrm{d}_{\mathcal{Z}}^g g = \mathrm{d}_{\mathcal{Z}} \ , \qquad g^{-1} \bar{\mathrm{d}}_{\mathcal{Z}}^g g = \bar{\mathrm{d}}_{\mathcal{Z}} + h^{-1} \bar{\mathrm{d}}_{\mathcal{Z}} h \ , \tag{A.10a}$$

$$g^{-1} \mathrm{d}_j^g g = \mathrm{d}_j \ , \qquad g^{-1} \bar{\mathrm{d}}_j^g g = \bar{\mathrm{d}}_j + h^{-1} \bar{\mathrm{d}}_j h \ . \tag{A.10b}$$

for $h \coloneqq g^\dagger g$.

Complex equations For the complex equations it holds

$$[\mathrm{d}_j, \mathrm{d}_k] = 0 \qquad \Leftrightarrow \qquad [\mathcal{Y}_j, \mathcal{Y}_k] = 0 \ , \tag{A.11a}$$

$$[\mathrm{d}_{\mathcal{Z}}, \mathrm{d}_j] = 0 \qquad \Leftrightarrow \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s} \mathcal{Y}_j = [\mathcal{Y}_j, \mathcal{Z}] \ , \tag{A.11b}$$

where the right-hand-side is understood as acting on $\mathbb{C}^p$- or matrix-valued functions. For the integrability of $\partial_{\mathcal{A}}$, i.e. $\partial_{\mathcal{A}}^2 = 0$, we need besides (A.11) also $\partial_{\widehat{\Gamma}^P}^2 = 0$ and (4.34) to hold. Fortunately, $\widehat{\Gamma}^P$ is an HYM-instantons and, thus, defines an integrable ∂-operator. Moreover, by construction we restricted to matrix-valued functions $\mathcal{Y}_j$ and $\mathcal{Z}$ that satisfy the equivariance. In summary, the complex equations are the integrability conditions for $\partial_{\mathcal{A}}$.

Real equation Recall the definition (4.60) of the moment map $\mu(\mathcal{Y},\mathcal{Z})$. The expression is identical to the action of the operator[19]

$$\Upsilon(\mathcal{Y},\mathcal{Z}) := 2\left(\left[\bar{\mathrm{d}}_{\mathcal{Z}},\mathrm{d}_{\mathcal{Z}}\right]+\lambda_n(s)\sum_{j=1}^{n}\left[\bar{\mathrm{d}}_j,\mathrm{d}_j\right]\right) \tag{A.12}$$

in the usual sense. This operator behaves under complex gauge transformations as follows:

$$g^{-1}\left(\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\right)g = \Upsilon(\mathcal{Y},\mathcal{Z}) - 2\left(\mathrm{d}_{\mathcal{Z}}(h^{-1}\bar{\mathrm{d}}_{\mathcal{Z}}h)+\lambda_n(s)\sum_{j=1}^{n}\mathrm{d}_j(h^{-1}\bar{\mathrm{d}}_j h)\right) . \tag{A.13}$$

A.4 Adaptation of proofs

A.4.1 Differential inequality

Let $\{\kappa_i\}_{i=1,\dots,p}$ be the *positive* eigenvalues (still functions of s) of h on I_ϵ. Define

$$\Phi(h) := \ln\left(\max_{i=1,\dots,p}\kappa_i\right) , \tag{A.14}$$

which is well-defined. The claim is that the inequalities

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(h) \geq -2\left(\|\Upsilon(\mathcal{Y},\mathcal{Z})\|+\|\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\|\right) , \tag{A.15a}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(h^{-1}) \geq -2\left(\|\Upsilon(\mathcal{Y},\mathcal{Z})\|+\|\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\|\right) \tag{A.15b}$$

hold in a weak sense.

Proof: Following [40], it is sufficient to consider the case where all eigenvalues of h are distinct for each s. Further, by a unitary gauge transformation one finds in each $\mathrm{GL}(p,\mathbb{C})/\mathrm{U}(p)$-equivalence class an element g (which corresponds to a given h) such that

$$g = \mathrm{diag}(e^{t_1},\dots,e^{t_p}) \qquad \text{with} \quad t_1(s)>t_2(s)>\dots>t_p(s) \quad \forall s\in I_\epsilon . \tag{A.16}$$

Hence, one obtains $h=\mathrm{diag}(e^{2t_1},\dots,e^{2t_p})$ and $h^{-1}=\mathrm{diag}(e^{-2t_1},\dots,e^{-2t_p})$ such that $\Phi(h)=2t_1$ and $\Phi(h^{-1}) = -2t_p$. Next, we compute

$$\bar{\mathrm{d}}_{\mathcal{Z}}h = \mathrm{diag}(e^{2t_j}\tfrac{\mathrm{d}}{\mathrm{d}s}t_j) - \left[\mathcal{Z}^\dagger,h\right] , \tag{A.17a}$$
$$h^{-1}\bar{\mathrm{d}}_{\mathcal{Z}}h = \mathrm{diag}(\tfrac{\mathrm{d}}{\mathrm{d}s}t_j) + \mathcal{Z}^\dagger - h^{-1}\mathcal{Z}^\dagger h , \tag{A.17b}$$
$$\mathrm{d}_{\mathcal{Z}}(h^{-1}\bar{\mathrm{d}}_{\mathcal{Z}}h) = \mathrm{diag}\left(\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}t_j\right) + \left[\mathcal{Z},\mathrm{diag}(\frac{\mathrm{d}}{\mathrm{d}s}t_j)\right] + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathcal{Z}^\dagger - h^{-1}\mathcal{Z}^\dagger h\right) + \left[\mathcal{Z},\mathcal{Z}^\dagger - h^{-1}\mathcal{Z}^\dagger h\right] . \tag{A.17c}$$

Now, we consider the diagonal elements

$$\left(\mathrm{d}_{\mathcal{Z}}(h^{-1}\bar{\mathrm{d}}_{\mathcal{Z}}h)\right)_{(a,a)} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}t_a + \sum_{b\neq a}|\mathcal{Z}_{ab}|^2\left\{\left(1-e^{2(t_a-t_b)}\right)-\left(1-e^{-2(t_a-t_b)}\right)\right\} , \tag{A.18}$$

where we used

$$\left(\left[\mathcal{Z},\mathrm{diag}(\frac{\mathrm{d}}{\mathrm{d}s}t_j)\right]\right)_{(a,a)} = 0 \quad \text{and} \quad \left(\mathcal{Z}^\dagger - h^{-1}\mathcal{Z}^\dagger h\right)_{(a,a)} = 0 . \tag{A.19}$$

Similarly, one derives

$$\left(\mathrm{d}_j(h^{-1}\bar{\mathrm{d}}_j h)\right)_{(a,a)} = \sum_{b\neq a}|(\mathcal{Y}_j)_{ab}|^2\left\{\left(1-e^{2(t_a-t_b)}\right)-\left(1-e^{-2(t_a-t_b)}\right)\right\} . \tag{A.20}$$

[19] This object is analogous to $\widehat{F}$ of [40, Eq. (1.10)].

Then, one proceeds by computing

$$\begin{aligned}(\Upsilon(\mathcal{Y},\mathcal{Z})-\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g))_{(a,a)} &= \left(\Upsilon(\mathcal{Y},\mathcal{Z})-g^{-1}\left(\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\right)g\right)_{(a,a)} \\ &= 2\left(\mathrm{d}_{\mathcal{Z}}(h^{-1}\bar{\mathrm{d}}_{\mathcal{Z}}h)+\lambda_n(s)\sum_{j=1}^{n}\mathrm{d}_j(h^{-1}\bar{\mathrm{d}}_jh)\right)_{(a,a)} \\ &= \frac{\mathrm{d}^2}{\mathrm{d}s^2}t_a+2\sum_{b\neq a}\left(|\mathcal{Z}_{ab}|^2+\lambda_n(s)\sum_{j=1}^{n}|(\mathcal{Y}_j)_{ab}|^2\right)\left\{\left(1-e^{2(t_a-t_b)}\right)-\left(1-e^{-2(t_a-t_b)}\right)\right\}.\end{aligned} \tag{A.21}$$

To get the estimate for $\Phi(h)=2t_1$ we take $a=1$ and use that $\left\{\left(1-e^{2(t_1-t_b)}\right)-\left(1-e^{-2(t_1-t_b)}\right)\right\}<0$ as $t_1>t_b$ for all $b>1$. Then

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}s^2}t_1 &\geq -\left(\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)-\Upsilon(\mathcal{Y},\mathcal{Z})\right)_{(1,1)} \geq -\left(|\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)_{(1,1)}|+|\Upsilon(\mathcal{Y},\mathcal{Z})_{(1,1)}|\right) \\ &\geq -\left(\|\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\|+\|\Upsilon(\mathcal{Y},\mathcal{Z})\|\right) \\ &\Rightarrow \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(h)\geq -2\left(\|\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\|+\|\Upsilon(\mathcal{Y},\mathcal{Z})\|\right).\end{aligned} \tag{A.22}$$

Similarly, the estimate for $\Phi(h^{-1})$ is obtained by taking $a=p$ and $\left\{\left(1-e^{2(t_p-t_b)}\right)-\left(1-e^{-2(t_p-t_b)}\right)\right\}>0$ for all $b<p$. Hence, we obtain

$$(\Upsilon(\mathcal{Y},\mathcal{Z})-\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g))_{(p,p)}\geq\frac{\mathrm{d}^2}{\mathrm{d}s^2}t_p \quad\Rightarrow\quad \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(h^{-1})\geq -2\left(\|\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\|+\|\Upsilon(\mathcal{Y},\mathcal{Z})\|\right). \tag{A.23}$$

Thus, the claim (A.15) holds. □

A.4.2 Uniqueness

Suppose that $(\mathcal{Y},\mathcal{Z})$ is a solution to the complex equations on I_ϵ. Let us assume that we have two complex gauge transformations g_1 and g_2 such that

(i) $\mu(\mathcal{Y}^{g_1},\mathcal{Z}^{g_1})=0$ and $\mu(\mathcal{Y}^{g_2},\mathcal{Z}^{g_2})=0$ in I_ϵ

(ii) $h_1=g_1^\dagger g_1$ and $h_2=g_2^\dagger g_2$ satisfying $h_1|_{\partial I_\epsilon}=h_2|_{\partial I_\epsilon}$.

Then $h_1=h_2$ in I_ϵ.

Proof: We can suppose $g_2=1$ such that $h_2=1$ in I_ϵ and ∂I_ϵ. Hence, $g\equiv g_1$ and $h|_{\partial I_\epsilon}=1$. Since $\Upsilon(\mathcal{Y},\mathcal{Z})=0$ and $\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)=0$, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(h)=2\frac{\mathrm{d}^2}{\mathrm{d}s^2}t_1\geq 0 \text{ in } I_\epsilon\ ,\ t_1|_{\partial I_\epsilon}=0 \quad\text{and}\quad \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(h^{-1})=-2\frac{\mathrm{d}^2}{\mathrm{d}s^2}t_p\geq 0 \text{ in } I_\epsilon\ ,\ t_p|_{\partial I_\epsilon}=0\,. \tag{A.24}$$

By (weak) convexity, it follows $t_1\leq 0$ in I_ϵ and $t_p\geq 0$ in I_ϵ, but we now arrive at $0\geq t_1>t_2>\ldots>t_p\geq 0$. Hence, $t_j=0$ in I_ϵ and $h=1$ in I_ϵ (modulo unitary transformations). □

A.4.3 Boundedness

Next, we need to show the boundedness of $\mu(\mathcal{Y},\mathcal{Z})$. The only critical term is $\lambda_n(s)$, which diverges for $s\to 0$. However, it is straightforward to derive the pole structure of the gauge transformed operator Υ to be

$$\begin{aligned}g^{-1}\left(\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\right)g\Big|_{\text{pole}} &= \Upsilon(\mathcal{Y},\mathcal{Z})\Big|_{\text{pole}}-2\lambda_n\sum_{j=1}^{n}\mathrm{d}_j\left(h^{-1}\bar{\mathrm{d}}_jh\right) \\ &= 2\lambda_n\sum_{j=1}^{n}\left[\mathcal{Y}_j,h^{-1}\mathcal{Y}_j^\dagger h\right]_{s\to 0}.\end{aligned} \tag{A.25}$$

But recall that we will consider framed gauge transformation, i.e. $h = 1$ at the boundaries, and $\mathcal{Y}(s=0)$ are elements of a Cartan subalgebra. Hence, the potential pole vanishes for any gauge transformation once the correct boundary conditions (4.50) are imposed. Thus, $\mu(\mathcal{Y},\mathcal{Z})$ is bounded.

A.4.4 Limit $\epsilon \to 0$

Finally, we need to show that the limit $\epsilon \to 0$ exists, for which we follow [121,122]. Let $(\mathcal{Y},\mathcal{Z})$ be *any* solution of the complex equation, then for each $\epsilon > 0$ there exists a unique *complex* gauge transformation g_ϵ such that $(\mathcal{Y}^{g_\epsilon},\mathcal{Z}^{g_\epsilon})$ satisfies the real equation in I_ϵ. Associate $h_\epsilon = g_\epsilon^\dagger g_\epsilon$.

We start by constructing a solution $(\widehat{\mathcal{Y}},\widehat{\mathcal{Z}})$ of the complex equations with the properties

$$(\widehat{\mathcal{Y}}_j,\widehat{\mathcal{Z}})(s) = \begin{cases} (\tau_j, 0) & \text{for} \quad s = -\epsilon \,, \\ (\mathcal{T}_j, \mathcal{T}_{n+1}) & \text{for} \quad -\frac{1}{\epsilon} < s < -1 \,, \end{cases} \tag{A.26}$$

where $(\mathcal{T}_j,\mathcal{T}_{n+1})$ correspond to the complex linear combinations of the T_μ of the boundary condition (4.50), i.e. they lie in a Cartan subalgebra of $\mathfrak{su}(n+1)$. The τ_j are arbitrary points in the complex orbits $\mathcal{O}(\mathcal{T}_j)$, because we know that the boundary values at $s \to 0$ are in gauge orbits of the $\mathcal{T}_j$.

The existence of such a solution follows from the local triviality of the complex equations. Note that this solution is *constant* in $(-\frac{1}{\epsilon},-1)$ and $\mu(\widehat{\mathcal{Y}},\widehat{\mathcal{Z}}) = 0$ for $-\frac{1}{\epsilon} < s < -1$.

The claim then is: Starting from $(\widehat{\mathcal{Y}},\widehat{\mathcal{Z}})$ as above, for each $\epsilon > 0$ there exists a unique gauge transformation g_ϵ such that

(i) $(\widehat{\mathcal{Y}}^{g_\epsilon},\widehat{\mathcal{Z}}^{g_\epsilon})$ satisfies the real equation everywhere in I_ϵ,

(ii) $(\widehat{\mathcal{Y}}^{g_\epsilon},\widehat{\mathcal{Z}}^{g_\epsilon})$ has the correct boundary conditions (4.50),

(iii) $g = 1$ at the boundaries and $\widehat{\mathcal{Z}}^{g_\epsilon}$ is Hermitian,

(iv) $\Phi(h_\epsilon)$, $\Phi(h_\epsilon^{-1})$ are uniformly bounded.

Thus, by the existence of a uniform bound, one has the existence of a C^∞ limit $h_\infty \coloneqq \lim_{\epsilon\to 0} h_\epsilon$ such that $g_\infty \coloneqq \sqrt{h_\infty}$ has all desired properties on the negative half-line.

Proof: The existence and the uniqueness of such a g_ϵ follows from the above. Using the differential inequalities (A.15) and the boundedness of μ we arrive at

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(h_\epsilon) \geq \begin{cases} -2\|\Upsilon(\widehat{\mathcal{Y}},\widehat{\mathcal{Z}})\| \geq -2C \,, & \text{for} \quad -1 < s < -\epsilon \,, \\ 0 \,, & \text{for} \quad -\frac{1}{\epsilon} < s < -1 \,. \end{cases} \tag{A.27}$$

Moreover, since $h_\epsilon = 1$ on ∂I_ϵ, the eigenvalues have to vanish, which implies $\Phi(h_\epsilon) = 0 = \Phi(h_\epsilon^{-1})$ at ∂I_ϵ. Consider the bounded, continuous, non-negative function

$$\begin{aligned} f_\epsilon(s) &= \begin{cases} -C(s+1)(s+\epsilon) & \text{for} \quad -1 < s < -\epsilon \\ 0 & \text{for} \quad -\frac{1}{\epsilon} < s < -1 \end{cases} \\ \text{with} \quad \frac{\mathrm{d}^2}{\mathrm{d}s^2} f_\epsilon &= \begin{cases} -2C & \text{for} \quad -1 < s < -\epsilon \\ 0 & \text{for} \quad -\frac{1}{\epsilon} < s < -1 \end{cases} \end{aligned} \tag{A.28}$$

in a weak sense. But then, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\Phi(h_\epsilon) - f_\epsilon\right) \geq 0 \text{ in } I_\epsilon \quad \text{and} \quad \Phi(h_\epsilon) - f_\epsilon = 0 \text{ at } \partial I_\epsilon \,. \tag{A.29}$$

By convexity, $\Phi(h_\epsilon) - f_\epsilon \leq 0$ in I_ϵ, which then implies

$$\Phi(h_\epsilon) = 2t_1 \leq \begin{cases} -C(s+1)(s+\epsilon) \leq -Cs(s+1) \,, & \text{for} \quad -1 < s < -\epsilon \,, \\ 0 \,, & \text{for} \quad -\frac{1}{\epsilon} < s < -1 \,. \end{cases} \tag{A.30}$$

Applying the very same reasoning to $\Phi(h_\epsilon^{-1})$, we obtain $\Phi(h^{-1}) - f_\epsilon \leq 0$ in I_ϵ, and thus

$$-\Phi(h_\epsilon^{-1}) = 2t_p \geq \begin{cases} Cs(s+1)\,, & \text{for} \quad -1 < s < -\epsilon\,, \\ 0\,, & \text{for} \quad -\frac{1}{\epsilon} < s < -1\,. \end{cases} \tag{A.31}$$

In conclusion, the eigenvalues of h_ϵ are uniformly bounded as follows:

$$\frac{1}{2}f \geq t_1 > \ldots > t_p \geq -\frac{1}{2}f \qquad \text{for} \quad f(s) = \begin{cases} -Cs(s+1)\,, & \text{for} \quad -1 < s < -\epsilon\,, \\ 0 & \text{for} \quad -\frac{1}{\epsilon} < s < -1 \end{cases} \tag{A.32}$$

independently of ϵ.

Lastly, we need to show that the limit $\epsilon \to 0$ yields a smooth function on compact subsets of $(-\infty, 0]$. Recall from [40, 121, 122] that the real equation is a dimensional reduction of an elliptic equation. For any compactly contained open set $\mathcal{U} \subset (-\infty, 0]$, we find an $N \in \mathbb{N}$ such that $\mathcal{U} \subset (-N, -\frac{1}{N})$. By previous arguments h_ϵ is smooth in I_ϵ such that for all $\epsilon < \frac{1}{N}$ the function h_ϵ is smooth on $(-N, -\frac{1}{N})$ as well as uniformly bounded. By ellipticity or following the steps in [40, Lem. 2.20], the n-th derivative of h_ϵ is bounded by the lower derivatives (and possibly Υ). Thus, the sequence h_ϵ is not only uniformly bounded, but also all of its derivatives are uniformly bounded on $(-N, -\frac{1}{N})$ for all $\epsilon < \frac{1}{N}$. Hence, the limit function h_∞ is smooth on all compactly contained subsets of $(-\infty, 0]$, which coincides with the C^∞ topology. □

Part II

Sasakian quiver gauge theories and instantons on cones over 5-dimensional lens spaces

Contents

7 Introduction and motivation

Quiver gauge theories arise in two distinct scenarios. On the one hand, the world-volume theories of a set of D-branes located at an orbifold singularity lead to gauge theories which differ from the U(N) gauge theory, living on the world-volume of a stack on N D-branes, due to the action of the orbifold group. First explored in [15], the full information of such (supersymmetric) gauge theories is encoded in two objects: (i) the superpotential, and (ii) the quiver diagram. The quiver diagram is a directed graph with a particular interpretation. Each node represents a gauge group, determined by the number of branes stacked on top of each other, while the arrows correspond to matter fields transforming in the bifundamental representation of the two gauge nodes to which they are attached.

On the other hand, the procedure of equivariant dimensional reduction over Kähler product manifolds of the form $M^d \times \mathrm{G/H}$ (with $\mathrm{H} \subset \mathrm{G}$ both compact), as introduced in [54], leads to *quiver bundles* together with (non-abelian) vortices and Yang-Mills-Higgs theories, wherein all information is again encoded in a quiver diagram. However, the arising quiver diagram has to be interpreted differently. In particular, the form of the diagram is entirely determined by the representation theory of G and H, such that nodes correspond to the irreducible representations of H and arrows indicate morphisms between two such H-representations. The procedure of [54] then introduces a representation of such graphs in the category of holomorphic vector bundles, such that nodes correspond to holomorphic vector bundles equipped with connections, and arrows are understood as bundle morphisms, also called Higgs fields. To be more precise, G is a connected, simply connected, semi-simple complex Lie group and P is a parabolic subgroup, such that G/P is a flag manifold, i.e. a Kähler space. Moreover, M^d is a compact Kähler manifold. In this particular set-up, one can introduce *vortex equations* as generalisations of the Hermitian Yang-Mills equations and a notion of *stability* allows for a corresponding version of the *Hitchin-Kobayashi correspondence* [55].

Such equivariant dimensional reductions are particularly interesting for compactifications in string theory, as the *dynamical degrees of freedom*, i.e. the gauge fields and the Higgs fields, vary only over M^d, while the dependence on the *internal space* G/H is compensated by gauge transformations due to the equivariant construction. Quiver gauge theories based on this construction have been studied for the internal coset $\mathbb{C}P^1$ in [53, 150–153], or $\mathbb{C}P^1 \times \mathbb{C}P^1$ in [154] and for Kähler cosets SU(3)/H in [155, 156].

However, there are a couple of questions that need to be addressed. For instance, is there a relation between the two appearances of quiver gauge theories? Or can the equivariant dimensional reduction be generalised to non-Kähler cosets?

7.1 Finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ versus $\mathrm{SU}(2)$-equivariance

An attempt to answer these questions was first provided in [147], in which the Kähler-coset G/H has been replaced by the 3-dimensional Sasaki-Einstein orbifold S^3/Γ, wherein Γ is a ADE subgroup of $\mathrm{SL}(2,\mathbb{C})$. The motivation behind can be summarised in two points: firstly, by an extension of the construction of [54, 55] one can associate a *new* class of quiver gauge theories to the Sasaki-Einstein orbifolds S^3/Γ via SU(2)-equivariant reduction. Since the quiver structure is solely determined by the representation theory of G and H, the resulting quiver theories for a Sasaki-Einstein coset G/H have been dubbed *Sasakian quiver gauge theories.*

Secondly, it was hoped that the *bridging property* of Sasaki-Einstein spaces[20] might clarify the relation between the two different types of quiver gauge theories introduced before. In detail, to any Kähler-Einstein manifold one can associate a U(1)-bundle whose total space is a Sasaki-Einstein manifold, with the metric cone being Calabi-Yau. Here, the *bridging* is schematically given by

$$\mathbb{C}P^1 \cong \frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \xrightarrow[\text{bundle}]{\mathrm{U}(1)} S^3/\Gamma \xrightarrow[\text{cone}]{\text{metric}} C(S^3/\Gamma) \cong \mathbb{C}^2/\Gamma \ . \tag{7.1}$$

For each Kähler space, there are distinct quiver gauge theories of physical significance. Starting with SU(2)-equivariant dimensional reduction on $\mathbb{C}P^1$ (and orbifolds thereof), the resulting quiver diagrams are of Dynkin type A_{k+1}, as studied in [150]. In contrast, considering the ADE-orbifolds $\mathbb{C}^2/\Gamma$, there is a deep connection between various mathematical aspects and string theory, which we briefly summarise now.

7.1.1 Du Val-Kleinian singularities

One aspect concerns the singularity resolution of Calabi-Yau orbifolds $\mathbb{C}^n/\Gamma$, where $\Gamma \subset \mathrm{SL}(n,\mathbb{C})$ is a finite subgroup. A resolution (X,π) of $\mathbb{C}^n/\Gamma$ is a non-singular complex manifold X of dimension n together with a map $\pi : X \to \mathbb{C}^n/\Gamma$ that induces a biholomorphism between open dense sets. As it is desirable to obtain a resolution that preserves the Calabi-Yau condition, i.e. triviality of the canoncial bundle, the resolution has to be *crepant*, which means that the canoncial bundles are isomorphic $K_X \cong \pi^*(K_{\mathbb{C}^n/\Gamma})$. The amount of information known about such crepant resolutions thereby depends drastically on n.

For $n = 2$, the quotient singularities $\mathbb{C}^2/\Gamma$ for finite subgroups $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ have been classified by Klein [157] in 1884. The resulting five families are in correspondence with the Dynkin diagrams of type A, D, and E. Hence, they are named ADE-singularities, but sometimes also Kleinian or Du Val singularities. The resolutions thereof have been studied by Du Val [158–160]. As a result, for $n = 2$ a crepant resolution exists and is unique. The topology of the resolution is thereby entirely determined by the finite group Γ.

Remarkably, for the crepant resolution $\pi : X \to \mathbb{C}^2/\Gamma$, the exceptional divisor $\pi^{-1}(0)$ is the dual of the Dynkin diagram[21]. In other words, the $\mathbb{C}P^1$ blow-ups intersect in the fashion of Dynkin diagrams for simply laced Lie algebras of type ADE.

7.1.2 McKay correspondence

Another reason for naming the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ as ADE is given by the McKay correspondence, which is the bijection between the set of irreducible representations of Γ and the set of vertices of an extended Dynkin diagram of type ADE. To begin with, we recall the generic McKay quiver $Q(\Gamma, V_R)$, where V_R is a representation of the finite group Γ. If $(R_0, R_1, \dots, R_r)$ denotes the set of irreducible representations of Γ with R_0 the trivial representation, then one associates the $(r{+}1)\times(r{+}1)$ adjacency matrix $A = (a_{ij})$ with $i, j = 0, 1, \dots, r$ via the tensor product decomposition $R_i \otimes V_R = \oplus_{i=0}^{r} a_{ij} R_j$. The nodes of the McKay quiver are labelled by the irreducible representations, while there are a_{ij} arrows from the i-th node to the j-th node. In general, the matrix A is non-symmetric and the quiver graph has loops as long as the trivial representation appears in the decomposition of V_R. McKay [161] then observed that for the self-dual 2-dimensional representation $V_R = \mathbb{C}^2$, induced by the embedding $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$, the adjacency matrix satisfies $A = 2 \cdot \mathbb{1} - \widetilde{C}$, wherein $\widetilde{C}$ is the Cartan matrix of the extended Dynkin

[20] Recall from the cone constructions of Ch. 3 that Sasaki-Einstein spaces are naturally sandwiched between various Kähler and non-Kähler spaces in one dimension lower and higher.

[21] For a variety M with isolated singularity s and resolution X, the exceptional divisor is the pre-image $\pi^{-1}(s) \subset X$ of the singularity under $\pi : X \to M$.

diagram associated to Γ. In this case, the McKay quiver $Q(\Gamma,\mathbb{C}^2)$ is the Dynkin diagram of the affine ADE Lie algebra itself.

7.1.3 Hyper-Kähler quotient

It was proven by Kronheimer [162, 163] that the resolutions of $\mathbb{C}^2/\Gamma$ are smooth hyper-Kähler manifolds of real dimension four, by means of the ADHM construction understood as hyper-Kähler quotient. In other words, the resolutions equal the moduli space $\mathcal{M}_\xi$ of translationally-invariant HYM-instantons on the vector bundle $\mathbb{C}^2 \times V_\Gamma \to \mathbb{C}^2/\Gamma$, with V_Γ the regular representation of Γ.

7.1.4 D-brane probes

Pioniered by the work in [15] for (abelian) A-type singularities and completed by [164] for the full ADE family, D-branes in type IIB at ALE-singularities probe the geometry. By saying that, one has typically two aspects in mind. Firstly, the quiver diagram of the $\mathcal{N}=2$, $d=6$ world-volume gauge theory on D-branes at the singularity of the ADE-orbifold $\mathbb{C}^2/\Gamma$ is the ADE McKay-quiver [164], i.e. the affine Dynkin diagram. Secondly, the space of classical vacua of that quiver gauge theory, the vanishing locus of the D- and F-terms, is described by the ADHM equations. Thus, the space of vacua is a hyper-Kähler space given by a hyper-Kähler quotient. Moreover, the space of vacua is not any hyper-Kähler space — it is precisely $\mathcal{M}_\xi$. Therefore, the classical vacua of D-branes placed at an ADE-singularity, with Fayet-Iliopoulos parameter ξ, describe the resolution $\mathcal{M}_\xi$ of the underlying ADE-orbifold.

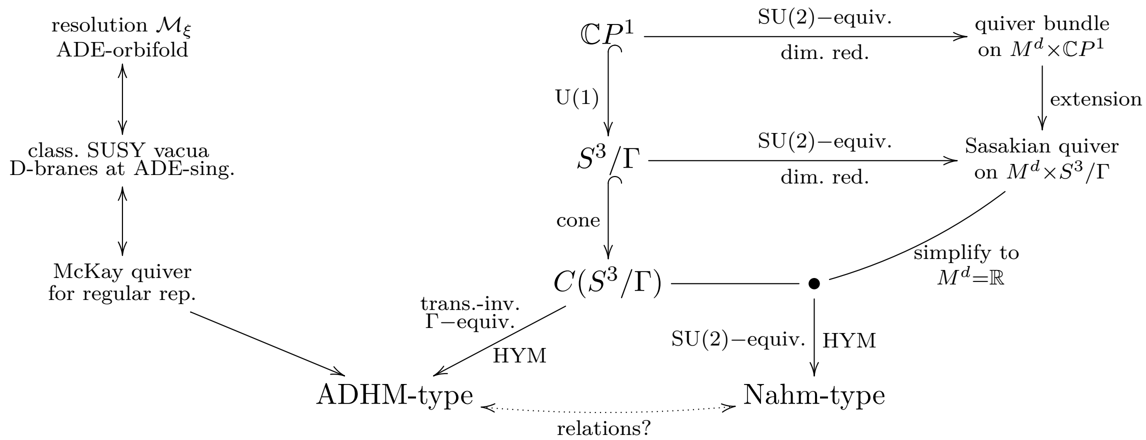


__Figure 7.1:__ The set-up of [147]: The aim is to compare two different HYM-instanton moduli spaces over $C(S^3/\Gamma)\cong\mathbb{C}^2/\Gamma$.

7.1.5 Comparison

In order to contrast the McKay quivers with the quiver bundles obtained for $\mathbb{C}P^1$, the Sasakian quiver gauge theory on S^3/Γ is taken as intermediate step, see Fig. 7.1. The corresponding quiver graphs are extensions of those for $\mathbb{C}P^1$ by vertex edge loops due to the horizontal components in the U(1)-fibration $S^3\to\mathbb{C}P^1$. Next, considering SU(2)-equivariant HYM-instantons on the cone $C(S^3/\Gamma)$ allows to study quiver gauge theories whose structure is determined by S^3/Γ due to the equivariance, but one introduces an additional dependence on the cone coordinate. The resulting instanton equations are of Nahm-type, in contrast to the ADHM-type equations for *constant* matrices from above.

As shown in [147] the comparison of the moduli spaces reveals surprising similarities although both stem from fundamentally different set-ups. For particular choices of boundary conditions, both moduli spaces are Kleinian singularities. Hence, both spaces have the same singularity structure. Nevertheless, the moduli space of SU(2)-equivariant instantons is always of A-type despite the orbifold group Γ ranging through all ADE possibilities.

7.2 Finite subgroups of $\mathrm{SL}(3,\mathbb{C})$ versus $\mathrm{SU}(3)$-equivariance

As a next step, one can continue the comparison of these different instanton moduli spaces in one (complex) dimension higher, which will be the content of this second part. By the very same reasoning, we now consider the following Kähler and Sasaki-Einstein spaces

$$\mathbb{C}P^2 \cong \frac{\mathrm{SU}(3)}{S(\mathrm{U}(1)\times\mathrm{U}(2))} \xrightarrow[\text{bundle}]{\mathrm{U}(1)} S^5/\mathbb{Z}_{q+1} \xrightarrow[\text{cone}]{\text{metric}} C(S^5/\mathbb{Z}_{q+1}) \cong \mathbb{C}^3/\mathbb{Z}_{q+1}\ . \tag{7.2}$$

The considerations of the SU(3)-equivariant dimensional reduction and the quiver bundle over $\mathbb{C}P^2$ have been performed in [155]. We now aim for two tasks: (i) construction of *new* Sasakian quiver gauge theories associated with the coset S^5=SU(3)/SU(2), and (ii) comparison of the translationally invariant and $\mathbb{Z}_{q+1}$-equivariant HYM-instantons on the Calabi-Yau 3-orbifold $\mathbb{C}^3/\mathbb{Z}_{q+1}$ with the SU(3)-equivariant HYM-instantons on the cone $C(S^5/\mathbb{Z}_{q+1})$. The overall picture is sketched in Fig. 7.2.

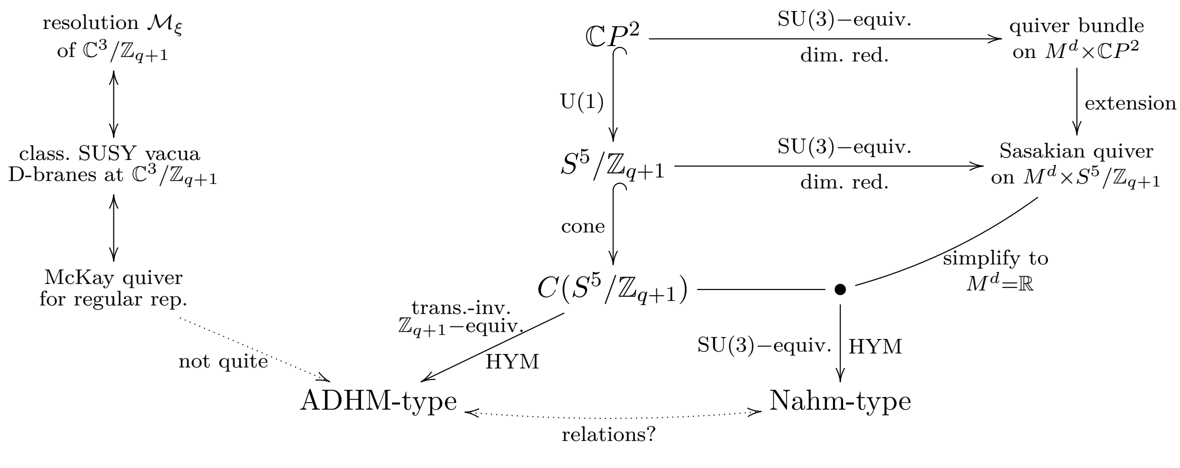


Figure 7.2: *The set-up for this part of the thesis: The aim is to compare two different HYM-instanton moduli spaces over* $C(S^5/\mathbb{Z}_{q+1}) \cong \mathbb{C}^3/\mathbb{Z}_{q+1}$.

Again, the two appearing instanton moduli spaces have different physical realisations. Equivariant instantons appear frequently in heterotic compactifications on the warped products over manifolds with G-structure, see Part I. The moduli space $\mathcal{M}_\xi$ of translationally-invariant instantons appears in special circumstances as vacua of world volume theories of D-branes at $\mathbb{C}^3/\mathbb{Z}_{q+1}$. Moreover, the same moduli space describes partial resolutions $\mathcal{M}_\xi \to \mathbb{C}^3/\mathbb{Z}_{q+1}$ of the CY-orbifold. Let us briefly recall the details.

7.2.1 Orbifold resolutions

The finite subgroups of $\mathrm{SL}(3,\mathbb{C})$ have been classified by Blichfeldt [165] in 1917, and were found to constitute ten families. For the $n = 3$ case, a crepant resolution for $\mathbb{C}^n/\Gamma$ exists, but is not unique. It has been shown that all crepant resolutions of $\mathbb{C}^3/\Gamma$ have Euler and Betti numbers given by the *stringy* (or DHVW) Euler and Betti numbers conjectured in [19, 20]. As proven in [166], the (partial) resolution $\mathcal{M}_\xi$ of $\mathbb{C}^3/\Gamma$ equals the moduli space of translationally-invariant and Γ-equivariant HYM-instantons on the trivial bundle $\mathbb{C}^3 \times V_R \to \mathbb{C}^3$, where V_R is again the regular representation of Γ. On a different account [167], the same moduli space $\mathcal{M}_\xi$ is identified with a representation moduli of the McKay quiver.

7.2.2 D-brane probes

In type II string theory [168,169], it was found that the Higgs branch of the world-volume theory on D-branes at singularities of $\mathbb{C}^3/\Gamma$ equals the resolved space $\mathcal{M}_\xi$, where ξ are the physical FI-parameters. Thus, the vacuum moduli space emerges as Kähler quotient with moment maps determined by the FI-parameters. As before, the D-branes probe the geometry, which is now Kähler rather than hyper-Kähler.

However, as we will see later, the comparison of the two constructed quiver gauge theories on $C(S^5/\Gamma)$ to the McKay quiver and the resolutions of the orbifold singularity is not quite appropriate; nevertheless, it is a intriguing phenomenon and severs us as motivation.

7.3 Outline

The remainder of this second part is organised as follows: In Ch. 8 we give a detailed account of the geometry of the orbifold $S^5/\mathbb{Z}_{q+1}$ employing its realisation as both a coset space and as a Sasaki-Einstein manifold. In Ch. 9 we provide a detailed description of the quiver gauge theory induced via SU(3)-equivariant dimensional reduction over $S^5/\mathbb{Z}_{q+1}$, including explicit constructions of the quiver bundles and their connections as well as the form of the action functional. We then describe the Higgs branch vacuum states of quiver gauge theories on the cone $C(S^5/\mathbb{Z}_{q+1})$ as SU(3)-equivariant solutions to the Hermitian Yang-Mills equations in Ch. 10 and as translationally-invariant solutions in Ch. 11. In Ch. 12 we compare the two quiver gauge theories in some detail, including a contrasting of their quiver bundles and explicit constructions of their instanton moduli spaces as Kähler quotients. Four appendices at the end of this Part II contain technical details and results which are employed in the main text.

The contents of this part stem from a collaboration [149] with O. Lechtenfeld, A.D. Popov, and R.J. Szabo.

8 Sasaki-Einstein 5-sphere and orbifolds thereof

In this chapter we introduce the basic geometrical constructions that we shall need throughout this part of the thesis. Sasaki-Einstein manifolds have been introduced in Sec. 3.1 and we continue to employ the definitions provided there.

Given a Riemannian manifold M^d and a finite group Γ acting isometrically on M^d, one can, loosely speaking, define the Riemannian space of Γ-orbits M^d/Γ, which is called an *orbifold* or sometimes *V-manifold*, see for instance [120]. The notion of fibre bundle can be adapted to the category of orbifolds, and we follow [120] in calling them *V-bundles*. Any quasi-regular Sasaki-Einstein manifold M^{2n+1} is a principal U(1) V-bundle over its transverse space M^{2n}.

8.1 Sphere S^5

The 5-dimensional sphere S^5 has two realisations: Firstly, as the coset space $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$ and, secondly, as a principal U(1)-bundle over the complex projective plane $\mathbb{C}P^2$. As such, we have the chain of principal bundles

$$\mathrm{SU}(3) \xrightarrow{\mathrm{SU}(2)} S^5 \xrightarrow{\mathrm{U}(1)} \mathbb{C}P^2 \ . \tag{8.1}$$

Our description of S^5 will be based on the principal U(1)-bundle over $\mathbb{C}P^2$, and we will construct a flat connection on the principal SU(2)-bundle over S^5 by employing this feature.

8.1.1 Connections on $\mathbb{C}P^2$

Let us consider a local section U over a patch $\mathcal{U}_0$ of $\mathbb{C}P^2$ for the principal bundle $\mathrm{SU}(3) \to \mathbb{C}P^2$. For this, let $\mathrm{G} = \mathrm{SU}(3)$ and $\mathrm{H} = S(\mathrm{U}(2){\times}\mathrm{U}(1)) \subset \mathrm{G}$, and consider the principal bundle associated to the coset G/H given by

$$\mathrm{G} = \mathrm{SU}(3) \xrightarrow{\mathrm{H}=S(\mathrm{U}(2)\times\mathrm{U}(1))} \mathrm{G}/\mathrm{H} \cong \mathbb{C}P^2 \ . \tag{8.2}$$

By the definition of the complex projective plane

$$\mathbb{C}P^2 = \mathbb{C}^3 / \sim\, = \Big\{ [z^1 : z^2 : z^3] \in \mathbb{C}^3 \mid [z^1 : z^2 : z^3] \sim [\lambda\, z^1 : \lambda\, z^2 : \lambda\, z^3] \ , \ \lambda \in \mathbb{C}^* \Big\} \ , \tag{8.3}$$

one introduces on the patch $\mathcal{U}_0 = \{[z^1 : z^2 : z^3] \in \mathbb{C}P^2 \mid z^3 \neq 0\}$ the coordinates

$$Y := \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \sim \begin{pmatrix} z^1/z^3 \\ z^2/z^3 \end{pmatrix} \ . \tag{8.4}$$

Define a local section on $\mathcal{U}_0$ of the principal bundle (8.2) via [155]

$$\begin{aligned} U : \mathcal{U}_0 &\longrightarrow \mathrm{SU}(3) \\ Y &\longmapsto U(Y) \coloneqq \frac{1}{\gamma} \begin{pmatrix} \bar{\Lambda} & \bar{Y} \\ -\bar{Y}^\dagger & 1 \end{pmatrix} , \end{aligned} \tag{8.5}$$

with the definitions

$$\bar{\Lambda} \coloneqq \gamma \, \mathbb{1}_2 - \frac{1}{\gamma+1} \bar{Y} \bar{Y}^\dagger \quad \text{and} \quad \gamma \coloneqq \sqrt{1 + Y^\dagger Y} \, . \tag{8.6}$$

From these two definitions, one observes the properties

$$\bar{\Lambda}^\dagger = \bar{\Lambda} \, , \quad \bar{\Lambda}^2 = \gamma^2 \, \mathbb{1}_2 - \bar{Y} \bar{Y}^\dagger \, , \quad \bar{\Lambda} \bar{Y} = \bar{Y} \quad \text{and} \quad \bar{Y}^\dagger \bar{\Lambda} = \bar{Y}^\dagger \, . \tag{8.7}$$

It is immediate from (8.7) that U as defined in (8.5) is SU(3)-valued. One can define a flat connection A_0 on the bundle (8.2) via

$$A_0 = U^\dagger \, \mathrm{d} U \equiv \begin{pmatrix} B & \bar{\beta} \\ -\beta^\top & -a \end{pmatrix} , \tag{8.8}$$

with the definitions

$$B \coloneqq \frac{1}{\gamma^2} \left(\bar{\Lambda} \, \mathrm{d}\bar{\Lambda} + \bar{Y} \, \mathrm{d}\bar{Y}^\dagger - \frac{1}{2} \mathbb{1}_2 \, \mathrm{d}\big(Y^\dagger Y\big) \right) , \tag{8.9a}$$

$$\bar{\beta} \coloneqq \frac{1}{\gamma^2} \bar{\Lambda} \, \mathrm{d}\bar{Y} \quad \text{and} \quad \beta^\top \coloneqq \frac{1}{\gamma^2} \mathrm{d}\bar{Y}^\dagger \bar{\Lambda} \, , \tag{8.9b}$$

$$a \coloneqq -\frac{1}{2\gamma^2} \left(\bar{Y}^\dagger \, \mathrm{d}\bar{Y} - \mathrm{d}\bar{Y}^\dagger \, \bar{Y} \right) = -\bar{a} \, . \tag{8.9c}$$

The fact that $U_{|(Y,\bar{Y})} \in \mathrm{SU}(3)$ directly implies the vanishing of the curvature 2-form F_0, which is equivalent to the set of relations

$$\mathrm{d}B + B \wedge B = \bar{\beta} \wedge \beta^\top \quad \text{and} \quad \mathrm{d}a = -\beta^\top \wedge \bar{\beta} = \beta^\dagger \wedge \beta \, , \tag{8.10a}$$

$$\mathrm{d}\bar{\beta} + B \wedge \bar{\beta} = \bar{\beta} \wedge a \quad \text{and} \quad \mathrm{d}\beta^\top + \beta^\top \wedge B = a \wedge \beta^\top \, . \tag{8.10b}$$

As elaborated in [155, 156], B can be regarded as a $\mathfrak{u}(2)$-valued connection 1-form and a as a $\mathfrak{u}(1)$-valued connection. Consequently, one can introduce an $\mathfrak{su}(2)$-valued connection $B_{(1)}$ by removing the trace of B. An explicit parametrisation yields

$$B_{(1)} \coloneqq B - \frac{1}{2} \operatorname{tr}(B) \, \mathbb{1}_2 \equiv \begin{pmatrix} B_{11} & \bar{B}_{12} \\ -B_{12} & -B_{11} \end{pmatrix} \qquad \text{with} \quad \operatorname{tr}(B) = a \, , \; B_{11} = -\bar{B}_{11} \, . \tag{8.11}$$

The geometry of $\mathbb{C}P^2$ including the properties of the SU(3)-equivariant 1-forms β^i, the instanton connection $B_{(1)}$ and the monopole connection a are described in App. B.1.

8.1.2 Connections on S^5

Consider now the principal SU(2)-bundle

$$\mathrm{G} = \mathrm{SU}(3) \xrightarrow{\mathrm{K}=\mathrm{SU}(2)} \mathrm{G}/\mathrm{K} = S^5 \, , \tag{8.12}$$

where $\mathrm{K} \subset \mathrm{G}$. Then the section U from (8.5) can be modified as

$$\begin{aligned} \hat{U} : \mathcal{U}_0 \times [0, 2\pi) &\longrightarrow \mathrm{SU}(3) \\ (Y, \varphi) &\longmapsto \hat{U}(Y, \varphi) \coloneqq U(Y) \begin{pmatrix} \mathrm{e}^{\,\mathrm{i}\,\varphi}\, \mathbb{1}_2 & 0 \\ 0 & \mathrm{e}^{\,-2\,\mathrm{i}\,\varphi} \end{pmatrix} \equiv U(Y)\, Z(\varphi) \ , \end{aligned} \tag{8.13}$$

which is a local section of the bundle (8.12) on the patch $\mathcal{U}_0 \times [0, 2\pi)$ with coordinates $\{y^1, y^2, \varphi\}$. Note that $Z^{-1} = Z^\dagger = \bar{Z}$ and $\det(Z) = 1$, and furthermore $Z(\varphi)\, Z(\psi) = Z(\psi)\, Z(\varphi) = Z(\psi+\varphi)$, which implies that Z realises the embedding $\mathrm{U}(1) \hookrightarrow \mathrm{SU}(3)$. As a consequence, we know that $\hat{U}_{|(Y,\bar{Y},\varphi)} \in \mathrm{SU}(3)$ holds. The modified (flat) connection $\hat{A}$ on the bundle (8.12) and the corresponding curvature $\hat{F}$ are given as

$$\hat{A} \coloneqq \hat{U}^\dagger\, \mathrm{d}\hat{U} = \mathrm{Ad}(Z^{-1}) A_0 + Z^\dagger\, \mathrm{d}Z = \begin{pmatrix} B + \mathrm{i}\, \mathbb{1}_2\, \mathrm{d}\varphi & \bar{\beta}\, \mathrm{e}^{\,-3\,\mathrm{i}\,\varphi} \\ -\beta^\top\, \mathrm{e}^{\,3\,\mathrm{i}\,\varphi} & -(a + 2\,\mathrm{i}\, \mathrm{d}\varphi) \end{pmatrix} \ , \tag{8.14a}$$

$$\begin{aligned} \hat{F} &\coloneqq \mathrm{d}\hat{A} + \hat{A} \wedge \hat{A} = \mathrm{Ad}(Z^{-1}) F_0 \\ &= \begin{pmatrix} \mathrm{d}B + B \wedge B - \bar{\beta} \wedge \beta^\top & \left(\mathrm{d}\bar{\beta} + B \wedge \bar{\beta} - \bar{\beta} \wedge a\right)\, \mathrm{e}^{\,-3\,\mathrm{i}\,\varphi} \\ -\left(\mathrm{d}\beta^\top + \beta^\top \wedge B - a \wedge \beta^\top\right)\, \mathrm{e}^{\,3\,\mathrm{i}\,\varphi} & -\mathrm{d}a - \beta^\top \wedge \bar{\beta} \end{pmatrix} = 0 \ . \end{aligned} \tag{8.14b}$$

Again the flatness of $\hat{A}$ yields the same set of identities (8.10), because $\hat{F}$ differs from F only by the adjoint action of Z^{-1}.

8.1.3 Contact geometry of S^5

By construction, the base space of (8.12) is a 5-sphere. Now, the aim is to choose a basis of the cotangent bundle T^*S^5 over the patch $\mathcal{U}_0 \times [0, 2\pi)$ such that one recovers the Sasaki-Einstein structure on S^5. For this, we start with the identifications

$$\beta^1_\varphi \coloneqq \beta^1\, \mathrm{e}^{\,3\,\mathrm{i}\,\varphi} \equiv e^1 + \mathrm{i}\, e^2 \ , \quad \beta^2_\varphi \coloneqq \beta^2\, \mathrm{e}^{\,3\,\mathrm{i}\,\varphi} \equiv e^3 + \mathrm{i}\, e^4 \quad \text{and} \quad \kappa\, e^5 \coloneqq \tfrac{1}{2}\, a + \mathrm{i}\, \mathrm{d}\varphi \ , \tag{8.15}$$

where $\kappa \in \mathbb{C}$ is a constant to be determined. The 1-forms β^i originate from the complex cotangent space $T^*_{(Y,\bar{Y})}\mathbb{C}P^2$ at a point $(Y, \bar{Y}) \in \mathcal{U}_0 \subset \mathbb{C}P^2$. Next, we define the forms

$$\omega_1 \coloneqq e^{14} + e^{23} \ , \quad \omega_2 \coloneqq e^{31} + e^{24} \ , \quad \omega_3 \coloneqq e^{12} + e^{34} \quad \text{and} \quad \eta \coloneqq e^5 \ . \tag{8.16}$$

In the basis (8.15), one obtains

$$\begin{aligned} \omega_1 = \tfrac{1}{2\,\mathrm{i}} \left(\beta^1_\varphi \wedge \beta^2_\varphi - \bar{\beta}^1_\varphi \wedge \bar{\beta}^2_\varphi\right) \ , \quad & \omega_2 = -\tfrac{1}{2} \left(\beta^1_\varphi \wedge \beta^2_\varphi + \bar{\beta}^1_\varphi \wedge \bar{\beta}^2_\varphi\right) \\ \text{and} \quad & \omega_3 = -\tfrac{1}{2\,\mathrm{i}} \left(\beta^1_\varphi \wedge \bar{\beta}^1_\varphi + \beta^2_\varphi \wedge \bar{\beta}^2_\varphi\right) \ . \end{aligned} \tag{8.17}$$

Note that ω_3 coincides (up to a normalisation factor) with the Kähler form on $\mathbb{C}P^2$, cf. App. B.1. The exterior derivatives of β^i_φ and $\bar{\beta}^i_\varphi$ are given as follows

$$\mathrm{d}\beta^i_\varphi = \mathrm{e}^{\,3\,\mathrm{i}\,\varphi}\, \mathrm{d}\beta^i - 3\,\mathrm{i}\, \beta^i_\varphi \wedge \mathrm{d}\varphi \quad \text{and} \quad \mathrm{d}\bar{\beta}^i_\varphi = \mathrm{e}^{\,-3\,\mathrm{i}\,\varphi}\, \mathrm{d}\bar{\beta}^i + 3\,\mathrm{i}\, \bar{\beta}^i_\varphi \wedge \mathrm{d}\varphi \ . \tag{8.18}$$

The distinguished 1-form η is taken to be the contact 1-form dual to the Reeb vector field of the Sasaki-Einstein structure, cf. Sec. 3.1. At this stage, the choice of the quadruple $(\eta, \omega_1, \omega_2, \omega_3)$ defines an $\mathrm{SU}(2)$-structure on the 5-sphere, recall Sec. 3.3. For it to be Sasaki-Einstein, one needs the relations

$$\mathrm{d}\omega_1 = 3\eta \wedge \omega_2 \ , \quad \mathrm{d}\omega_2 = -3\eta \wedge \omega_1 \quad \text{and} \quad \mathrm{d}\eta = -2\omega_3 \tag{8.19}$$

to hold, following (3.11). Employing (8.10) one arrives at

$$\mathrm{d}\omega_1 = 6\,\mathrm{i}\,\kappa\,\eta \wedge \omega_2 \quad \text{and} \quad \mathrm{d}\omega_2 = -6\,\mathrm{i}\,\kappa\,\eta \wedge \omega_1 \ , \tag{8.20a}$$

$$\mathrm{d}\eta = \tfrac{\mathrm{i}}{\kappa}\,\omega_3 \qquad \text{and} \quad \mathrm{d}\omega_3 = 0 \ . \tag{8.20b}$$

Consequently, the coframe $\{\eta, \beta^1_\varphi, \beta^2_\varphi\}$ yields a Sasaki-Einstein structure on S^5 if and only if $\kappa = -\frac{\mathrm{i}}{2}$, and from now on this will be the case.

8.2 Orbifold $S^5/\mathbb{Z}_{q+1}$

Next, our aim is to construct a principal V-bundle over the orbifold $S^5/\mathbb{Z}_{q+1}$ by the following steps: Take the principal SU(2)-bundle $\pi : \mathrm{G} = \mathrm{SU}(3) \to \mathrm{SU}(3)/\mathrm{SU}(2) \cong S^5$, which is SU(2)-equivariant. Embed $\mathbb{Z}_{q+1} \hookrightarrow \mathrm{U}(1) \subset \mathrm{SU}(3)$ such that U(1) commutes with $\mathrm{SU}(2) \subset \mathrm{SU}(3)$, and define a $\mathbb{Z}_{q+1}$-action γ on S^5. The action $\gamma : \mathbb{Z}_{q+1} \times S^5 \to S^5$ can be lifted to an action $\widetilde{\gamma} : \mathbb{Z}_{q+1} \times \mathrm{G} \to \mathrm{G}$ with an isomorphism on the SU(2) fibres induced by this action. The crucial point is that the fibre isomorphism is trivial as SU(2) commutes with $\mathbb{Z}_{q+1}$ by construction. Hence, one can consider the $\mathbb{Z}_{q+1}$-projection of G to the principal SU(2) V-bundle $\widetilde{\mathrm{G}}$, which is schematically given as

$$\begin{tikzcd} \mathrm{G} \arrow[r, "\widetilde{\gamma}"] \arrow[d, "\pi"'] & \widetilde{\mathrm{G}} \arrow[d, "\widetilde{\pi}"] \\ S^5 \arrow[r, "\gamma"'] & S^5/\mathbb{Z}_{q+1} \end{tikzcd} \tag{8.21}$$

With an abuse of notation, we will denote the V-bundles obtained via $\mathbb{Z}_{q+1}$-projection by the same symbols as the fibre bundles they originate from; only $\mathbb{Z}_{q+1}$-equivariant field configurations survive this orbifold projection.

A section $\widetilde{U}$ of the principal V-bundle (8.21) is obtained by a (further) modification of the section (8.5) as

$$\begin{aligned} &\widetilde{U} : \mathcal{U}_0 \times [0, \tfrac{2\pi}{q+1}) \longrightarrow \mathrm{SU}(3) \\ &\qquad (Y, \tfrac{\varphi}{q+1}) \longmapsto \widetilde{U}(Y, \tfrac{\varphi}{q+1}) := U(Y) \begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}\,\varphi}{q+1}}\,\mathbb{1}_2 & 0 \\ 0 & \mathrm{e}^{-2\,\frac{\mathrm{i}\,\varphi}{q+1}} \end{pmatrix} \equiv U(Y)\, Z_{q+1}(\varphi) \ . \end{aligned} \tag{8.22}$$

Here $\varphi \in [0, 2\pi)$ is again the local coordinate on the S^1-fibration $S^5 \xrightarrow{\mathrm{U}(1)} \mathbb{C}P^2$; hence, it holds $\mathrm{e}^{\frac{\mathrm{i}\,\varphi}{q+1}} \in S^1/\mathbb{Z}_{q+1}$. Analogously to the $q=0$ case of S^5 above, one can prove that Z_{q+1} realises the embedding $S^1/\mathbb{Z}_{q+1} \hookrightarrow \mathrm{U}(1) \subset \mathrm{SU}(3)$, and $\widetilde{U}_{|(Y,\bar{Y},\varphi/(q+1))} \in \mathrm{SU}(3)$. As before, one computes the connection 1-form $\widetilde{A}$ and the curvature $\widetilde{F}$ of the flat connection on the V-bundle (8.21). This yields

$$\widetilde{A} := \widetilde{U}^\dagger\, \mathrm{d}\widetilde{U} = \mathrm{Ad}(Z_{q+1}^{-1}) A_0 + Z_{q+1}^\dagger\, \mathrm{d}Z_{q+1} = \begin{pmatrix} B + \mathbb{1}_2\, \frac{\mathrm{i}\,\mathrm{d}\varphi}{q+1} & \bar{\beta}\ \mathrm{e}^{-3\,\frac{\mathrm{i}\,\varphi}{q+1}} \\ -\beta^\top\ \mathrm{e}^{3\,\frac{\mathrm{i}\,\varphi}{q+1}} & -(a + 2\,\frac{\mathrm{i}\,\mathrm{d}\varphi}{q+1}) \end{pmatrix} \ , \tag{8.23a}$$

$$\widetilde{F} := \mathrm{d}\widetilde{A} + \widetilde{A} \wedge \widetilde{A} = \mathrm{Ad}(Z_{q+1}^{-1}) F_0$$

$$= \begin{pmatrix} \mathrm{d}B + B \wedge B - \bar{\beta} \wedge \beta^\top & \left(\mathrm{d}\bar{\beta} + B \wedge \bar{\beta} - \bar{\beta} \wedge a\right) \, \mathrm{e}^{-3 \frac{\mathrm{i}\varphi}{q+1}} \\ -\left(\mathrm{d}\beta^\top + \beta^\top \wedge B - a \wedge \beta^\top\right) \, \mathrm{e}^{3 \frac{\mathrm{i}\varphi}{q+1}} & -\mathrm{d}a - \beta^\top \wedge \bar{\beta} \end{pmatrix} = 0 \ . \qquad (8.23\text{b})$$

Again, the flatness of the connection $\widetilde{A}$ yields the very same relations (8.10).

8.2.1 Local coordinates

Our description of the orbifold $S^5/\mathbb{Z}_{q+1}$ follows the treatment of the 3-sphere orbifolds S^3/Γ of [147]. The key idea is the embedding $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2) \hookrightarrow \mathbb{R}^6 \cong \mathbb{C}^3$ via the relation

$$r^2 = \delta_{\hat{\mu}\hat{\nu}} \, x^{\hat{\mu}} \, x^{\hat{\nu}} = |z^1|^2 + |z^2|^2 + |z^3|^2 \qquad (8.24)$$

where $x^{\hat{\mu}}$ ($\hat{\mu} = 1, \dots, 6$) are coordinates of $\mathbb{R}^6$ and z^α ($\alpha = 1, 2, 3$) are coordinates of $\mathbb{C}^3$; here $r \in \mathbb{R}_{>0}$ gives the radius of the embedded 5-sphere. In general, on the coordinates z^α the $\mathbb{Z}_{q+1}$-action is realised linearly by a representation $h \mapsto (h^\alpha{}_\beta)$ such that

$$z^\alpha \longmapsto h^\alpha{}_\beta \, z^\beta \quad \text{and} \quad \bar{z}^\alpha \longmapsto \bar{h}^\alpha{}_\beta \, \bar{z}^\beta = (h^{-1})^\alpha{}_\beta \, \bar{z}^\beta \ , \qquad (8.25)$$

where h is the generator of the cyclic group $\mathbb{Z}_{q+1}$. In this thesis the action of the finite group $\mathbb{Z}_{q+1}$ is chosen to be realised by the embedding of $\mathbb{Z}_{q+1}$ in the fundamental 3-dimensional complex representation $\underline{C}^{1,0}$ of SU(3) given by

$$(h^\alpha{}_\beta) = \begin{pmatrix} \zeta_{q+1} & 0 & 0 \\ 0 & \zeta_{q+1} & 0 \\ 0 & 0 & \zeta_{q+1}^{-2} \end{pmatrix} \in \mathrm{SU}(3) \qquad \text{with} \quad \zeta_{q+1}^l := \mathrm{e}^{\frac{2\pi\,\mathrm{i}}{q+1} \, l} \ . \qquad (8.26)$$

Since $\mathbb{C}P^2$ is naturally defined via a quotient of $\mathbb{C}^3$, see (8.3), one can deduce the $\mathbb{Z}_{q+1}$-action on the local coordinates (y^1, y^2) of the patch $\mathcal{U}_0$ to be

$$y^\alpha \longmapsto \frac{\zeta_{q+1} \, z^\alpha}{\zeta_{q+1}^{-2} \, z^3} = \zeta_{q+1}^3 \, y^\alpha \quad \text{and} \quad \bar{y}^\alpha \longmapsto \frac{\zeta_{q+1}^{-1} \, \bar{z}^\alpha}{\zeta_{q+1}^{2} \, \bar{z}^3} = \zeta_{q+1}^{-3} \, \bar{y}^\alpha \qquad \text{for} \quad \alpha = 1, 2 \ . \qquad (8.27)$$

Next, we consider the action of $\mathbb{Z}_{q+1}$ on the S^1 coordinate φ. By (8.26) one naturally has

$$\mathrm{e}^{\mathrm{i} \frac{\varphi}{q+1}} \xrightarrow{\mathbb{Z}_{q+1}} \mathrm{e}^{\mathrm{i} \left(\frac{\varphi}{q+1} + \frac{2\pi\, l}{q+1}\right)} = \mathrm{e}^{\mathrm{i} \frac{\varphi}{q+1}} \, \zeta_{q+1}^l \qquad \text{for} \quad l \in \{0, 1, \dots, q\} \ , \qquad (8.28)$$

i.e. the transformed coordinate $\mathrm{e}^{\mathrm{i} \left(\frac{\varphi}{q+1} + \frac{2\pi\, l}{q+1}\right)}$ lies in the $\mathbb{Z}_{q+1}$-orbit $\left[\mathrm{e}^{\mathrm{i} \frac{\varphi}{q+1}}\right]$ of $\mathrm{e}^{\mathrm{i} \frac{\varphi}{q+1}}$.

8.2.2 Lens spaces

The spaces $S^5/\mathbb{Z}_{q+1}$ are known as lens spaces, see for instance [120]. For this, one usually embeds S^5 into $\mathbb{C}^3$ and chooses the action of $p \in \{0, 1, \dots, q\}$ as

$$\begin{aligned} \mathbb{Z}_{q+1} \times \mathbb{C}^3 &\longrightarrow \mathbb{C}^3 \\ \left(p \, , \, (z^1, z^2, z^3)\right) &\longmapsto p \cdot (z^1, z^2, z^3) := \left(\mathrm{e}^{\frac{2\pi\,\mathrm{i}\,p}{q+1}} \, z^1, \, \mathrm{e}^{\frac{2\pi\,\mathrm{i}\,p}{q+1} \, r_1} \, z^2, \, \mathrm{e}^{\frac{2\pi\,\mathrm{i}\,p}{q+1} \, r_2} \, z^3 \right) \ , \end{aligned} \qquad (8.29)$$

where the integers r_1 and r_2 are chosen to be coprime to $q+1$. The coprime condition is necessary for the $\mathbb{Z}_{q+1}$-action to be free away from the origin of $\mathbb{C}^3$. The quotient space $S^5/\mathbb{Z}_{q+1}$ with the action (8.29) is called the lens space $L(q+1, r_1, r_2)$ or $L^2(q+1, r_1, r_2)$. It is a 5-dimensional orbifold with fundamental group $\mathbb{Z}_{q+1}$.

We choose the $\mathbb{Z}_{q+1}$-action to be given by (8.26), i.e. $r_1 = 1$ and $r_2 = -2$. Then r_1 is always coprime to $q+1$, but r_2 is coprime to $q+1$ only if q is even. Thus for $q+1 \in 2\mathbb{N}+1$ the only singular point in $\mathbb{C}^3/\mathbb{Z}_{q+1}$ is the origin, and its isotropy group is $\mathbb{Z}_{q+1}$. However, for $q+1 \in 2\mathbb{N}$ there is a singularity at the origin and also along the circle $\{z^1 = z^2 = 0\ , |z^3| = 1\} \subset S^5$ of singularities with isotropy group $\{0, \frac{q+1}{2}\} \cong \mathbb{Z}_2 \subset \mathbb{Z}_{q+1}$. Hence for the chosen action (8.26) we are forced to take $q \in 2\mathbb{N}$ in all considerations.

8.2.3 Differential forms

Similarly to the previous case, one can construct locally a basis of differential forms. However, one has to work with a *uniformising system* of local charts on the orbifold $S^5/\mathbb{Z}_{q+1}$ instead of local charts for the manifold S^5, cf. [120]. Choosing the identifications

$$\beta^1_{q+1} := \beta^1\ \mathrm{e}^{\frac{3\,\mathrm{i}\,\varphi}{q+1}} \equiv e^1 + \mathrm{i}\, e^2\ , \quad \beta^2_{q+1} := \beta^2\ \mathrm{e}^{\frac{3\,\mathrm{i}\,\varphi}{q+1}} \equiv e^3 + \mathrm{i}\, e^4 \quad \text{and} \quad \eta := e^5 \equiv \mathrm{i}\, a - \tfrac{2\,\mathrm{d}\varphi}{q+1}, \tag{8.30}$$

and by means of the relations imposed by the flatness of (8.23a), one can study the geometry of $S^5/\mathbb{Z}_{q+1}$. Defining the three 2-forms

$$\begin{aligned} \omega_1 &:= \tfrac{1}{2\,\mathrm{i}} \left(\beta^1_{q+1} \wedge \beta^2_{q+1} - \bar\beta^1_{q+1} \wedge \bar\beta^2_{q+1}\right)\ , \qquad \omega_2 := -\tfrac{1}{2} \left(\beta^1_{q+1} \wedge \beta^2_{q+1} + \bar\beta^1_{q+1} \wedge \bar\beta^2_{q+1}\right) \\ &\text{and} \quad \omega_3 := -\tfrac{1}{2\,\mathrm{i}} \left(\beta^1_{q+1} \wedge \bar\beta^1_{q+1} + \beta^2_{q+1} \wedge \bar\beta^2_{q+1}\right) \end{aligned} \tag{8.31}$$

and employing (8.10) implied by the flatness of $\widetilde{A}$, one obtains the correct Sasaki-Einstein relations (8.19).

8.2.4 $\mathbb{Z}_{q+1}$-action on 1-forms

Consider the $\mathbb{Z}_{q+1}$-action on the forms β^i_{q+1}, $\bar\beta^i_{q+1}$, and η. Firstly, recall the definitions (8.30) and (B.1), from which one sees that

$$\beta^i_{q+1} \xrightarrow{\mathbb{Z}_{q+1}} \zeta^3_{q+1}\, \beta^i_{q+1} \quad \text{and} \quad \bar\beta^i_{q+1} \xrightarrow{\mathbb{Z}_{q+1}} \zeta^{-3}_{q+1}\, \bar\beta^i_{q+1}\ . \tag{8.32}$$

This follows directly from the transformation (8.27). Moreover, it agrees with the monodromy of β^i_{q+1} and $\bar\beta^i_{q+1}$ along the S^1 fibres, i.e.

$$\beta^i_{q+1} = \beta^i\ \mathrm{e}^{3\,\frac{\mathrm{i}\,\varphi}{q+1}} \xrightarrow{\varphi \mapsto \varphi + 2\pi} \beta^i_{q+1}\, \zeta^3_{q+1}\ . \tag{8.33}$$

Secondly, for the 1-form η from (8.30) we know that a is a U(1) connection. As any U(1) connection is automatically U(1)-invariant, due to the embedding $\mathbb{Z}_{q+1} \hookrightarrow \mathrm{U}(1)$ one also has $\mathbb{Z}_{q+1}$-invariance[22] of a. We conclude that

$$\eta \xrightarrow{\mathbb{Z}_{q+1}} \eta\ . \tag{8.34}$$

From the definition (8.24) of the radial coordinate, one observes that r is invariant under $\mathbb{Z}_{q+1}$. The same is true for the corresponding 1-form, so that

$$\mathrm{d}r \xrightarrow{\mathbb{Z}_{q+1}} \mathrm{d}r\ . \tag{8.35}$$

Following [147], let T be a $\mathbb{Z}_{q+1}$-invariant 1-form on the metric cone $C(S^5/\mathbb{Z}_{q+1})$ which is locally

[22] Alternatively, one can work out the transformation behaviour of a directly from the explicit expression (8.9c).

expressed as

$$T = T_\mu\, e^\mu + T_r\, \mathrm{d}r \equiv W_i\, \beta^i_{q+1} + \overline{W}_i\, \bar{\beta}^i_{q+1} + W_5\, e^5 + W_r\, \mathrm{d}r \tag{8.36}$$

with $i = 1, 2$ and $\mu = 1, \ldots, 5$, where $W_1 = \frac{1}{2}\,(T_1 - \mathrm{i}\,T_2)$, $W_2 = \frac{1}{2}\,(T_3 - \mathrm{i}\,T_4)$, $W_5 = T_5$ and $W_r = T_r$. This induces a representation π of $\mathbb{Z}_{q+1}$ in $\Omega^1\big(C(S^5)\big)$ as

$$W_i \longmapsto \pi(h)(W_i) = \zeta^{-3}_{q+1}\, W_i \ , \qquad \overline{W}_i \longmapsto \pi(h)(\,\overline{W}_i) = \zeta^3_{q+1}\, \overline{W}_i \ , \tag{8.37a}$$

$$W_5 \longmapsto \pi(h)(W_5) = W_5 \ , \qquad W_r \longmapsto \pi(h)(W_r) = W_r \ . \tag{8.37b}$$

This completes the discussion of the orbifold group action on all relevant geometric objects.

9 Quiver gauge theory

In this chapter we construct quiver bundles over a d-dimensional Riemannian manifold M^d via equivariant dimensional reduction over $M^d \times S^5/\mathbb{Z}_{q+1}$, and derive the generic form of a G-equivariant connection. For this, we recall some aspects from the representation theory of $\mathrm{G} = \mathrm{SU}(3)$, and exemplify the relation between quiver representations and homogeneous bundles over $S^5/\mathbb{Z}_{q+1}$. Then we extend our constructions to G-equivariant bundles over $M^d \times S^5/\mathbb{Z}_{q+1}$, which will furnish a quiver representation in the category of vector bundles instead of vector spaces. We shall also derive the dimensional reduction of the pure Yang-Mills action on $M^d \times S^5$ to obtain a Yang-Mills-Higgs theory on M^d from our twisted reduction procedure (for the special case $q = 0$).

9.1 Preliminaries

To begin with, we recall the necessary facts from Lie algebra and representation theory for SU(3) and a suitably chosen subgroup SU(2).

9.1.1 Cartan-Weyl basis of $\mathfrak{su}(3)$

Our considerations are based on certain irreducible representations of the Lie group $\mathrm{G} = \mathrm{SU}(3)$, which are decomposed into irreducible representations of the subgroup $\mathrm{H}=\mathrm{SU}(2)\times\mathrm{U}(1) \subset \mathrm{SU}(3)$. For this, we recall the root decomposition of the Lie algebra $\mathfrak{su}(3)$. There is a pair of simple roots α_1 and α_2, and the non-null roots are given by $\pm\,\alpha_1$, $\pm\,\alpha_2$, and $\pm\,(\alpha_1+\alpha_2)$. The Lie algebra $\mathfrak{su}(3)$ is 8-dimensional and has a 2-dimensional Cartan subalgebra spanned by H_{α_1} and H_{α_2}. We distinguish one $\mathfrak{su}(2)$ subalgebra, which is spanned by H_{α_1} and $E_{\pm\,\alpha_1}$ with the commutation relations

$$[H_{\alpha_1}, E_{\pm\,\alpha_1}] = \pm\,2E_{\pm\,\alpha_1} \quad \text{and} \quad [E_{\alpha_1}, E_{-\alpha_1}] = H_{\alpha_1} \ . \tag{9.1a}$$

The element H_{α_2} generates a $\mathfrak{u}(1)$ subalgebra that commutes with this $\mathfrak{su}(2)$ subalgebra, i.e.

$$[H_{\alpha_2}, H_{\alpha_1}] = 0 \quad \text{and} \quad [H_{\alpha_2}, E_{\pm\,\alpha_1}] = 0 \ . \tag{9.1b}$$

In the *Cartan-Weyl basis*, the remaining generators $E_{\pm\,\alpha_2}$ and $E_{\pm\,(\alpha_1+\alpha_2)}$ satisfy the following non-vanishing commutation relations with the $\mathfrak{su}(2)$ generators

$$[H_{\alpha_1}, E_{\pm\,\alpha_2}] = \mp\,E_{\pm\,\alpha_2} \qquad \text{and} \qquad [E_{\pm\,\alpha_1}, E_{\pm\,\alpha_2}] = \pm\,E_{\pm\,(\alpha_1+\alpha_2)} \ , \tag{9.1c}$$

$$\Big[H_{\alpha_1}, E_{\pm\,(\alpha_1+\alpha_2)}\Big] = \pm\,E_{\pm\,(\alpha_1+\alpha_2)} \quad \text{and} \quad \Big[E_{\pm\,\alpha_1}, E_{\mp\,(\alpha_1+\alpha_2)}\Big] = \mp\,E_{\mp\,\alpha_2} \ , \tag{9.1d}$$

with the $\mathfrak{u}(1)$ generator

$$[H_{\alpha_2}, E_{\pm\,\alpha_2}] = \pm\,3E_{\pm\,\alpha_2} \quad \text{and} \quad \Big[H_{\alpha_2}, E_{\pm\,(\alpha_1+\alpha_2)}\Big] = \pm\,3E_{\pm\,(\alpha_1+\alpha_2)} \ , \tag{9.1e}$$

and amongst each other

$$[E_{\alpha_2}, E_{-\alpha_2}] = \tfrac{1}{2}\,(H_{\alpha_2} - H_{\alpha_1}) \quad \text{and} \quad [E_{\alpha_1+\alpha_2}, E_{-\alpha_1-\alpha_2}] = \tfrac{1}{2}\,(H_{\alpha_1} + H_{\alpha_2}) \ , \tag{9.1f}$$

$$\left[E_{\pm\,\alpha_2}, E_{\mp\,(\alpha_1+\alpha_2)}\right] = \pm\, E_{\mp\,\alpha_1} \ . \tag{9.1g}$$

9.1.2 Skew-Hermitian basis of $\mathfrak{sl}(3,\mathbb{C})$

Equivalently, we introduce the complex basis given by

$$I_1 \coloneqq E_{\alpha_1+\alpha_2} - E_{-\alpha_1-\alpha_2} \ , \qquad I_2 \coloneqq -\mathrm{i}\ \left(E_{\alpha_1+\alpha_2} + E_{-\alpha_1-\alpha_2}\right) \ , \tag{9.2a}$$

$$I_3 \coloneqq E_{\alpha_2} - E_{-\alpha_2} \ , \qquad I_4 \coloneqq -\mathrm{i}\ \left(E_{\alpha_2} + E_{-\alpha_2}\right) \ , \tag{9.2b}$$

$$I_5 \coloneqq -\tfrac{\mathrm{i}}{2}\, H_{\alpha_2} \ , \tag{9.2c}$$

$$I_6 \coloneqq E_{\alpha_1} - E_{-\alpha_1} \ , \qquad I_7 \coloneqq -\mathrm{i}\ \left(E_{\alpha_1} + E_{-\alpha_1}\right) \ , \tag{9.2d}$$

$$I_8 \coloneqq \mathrm{i}\, H_{\alpha_1} \ , \tag{9.2e}$$

which reflects the splitting $\mathfrak{su}(3) = \mathfrak{su}(2) \oplus \mathfrak{m}$ in which

$$I_i \in \mathfrak{su}(2) \qquad \text{for} \quad i = 6,7,8 \quad \text{and} \quad I_\mu \in \mathfrak{m} \qquad \text{for} \quad \mu = 1,\dots,5 \ . \tag{9.3}$$

This representation of generators is skew-Hermitian, i.e. $I_\mu = -I_\mu^\dagger$ for $\mu = 1,\dots,5$ and $I_i = -I_i^\dagger$ for $i = 6,7,8$, in contrast to the Cartan-Weyl basis. The chosen Cartan subalgebra is spanned by I_5 and I_8, and $[I_5, I_i] = 0$. From the commutation relations (9.1) one can infer the non-vanishing structure constants of these generators as

$$f_{67}{}^{8} = -2 \qquad \text{plus cyclic} \ , \tag{9.4a}$$

$$f_{63}{}^{1} = f_{64}{}^{2} = f_{71}{}^{4} = f_{73}{}^{2} = f_{82}{}^{1} = f_{83}{}^{4} = 1 \qquad \text{plus cyclic} \ , \tag{9.4b}$$

$$f_{12}{}^{5} = f_{34}{}^{5} = 2 \ , \tag{9.4c}$$

$$f_{25}{}^{1} = -f_{15}{}^{2} = f_{45}{}^{3} = -f_{35}{}^{4} = \tfrac{3}{2} \ . \tag{9.4d}$$

The Killing form $K_{AB} \coloneqq f_{AC}{}^{D} f_{DB}{}^{C}$ (with $A,B,\ldots = 1,\dots,8$) associated to this basis is diagonal, but not proportional to the identity, and is given by

$$\begin{aligned} K_{ab} &= 12\,\delta_{ab} \qquad \text{for} \quad a,b = 1,2,3,4 \ , \quad K_{55} = 9 \quad \text{and} \\ K_{ij} &= 12\,\delta_{ij} \qquad \text{for} \quad i,j = 6,7,8 \ . \end{aligned} \tag{9.5}$$

Introducing the 't Hooft tensors η^α_{ab} for $a,b = 1,2,3,4$ and $\alpha = 1,2,3$ one has

$$f_{ab}{}^{5} = 2\,\eta^3_{ab} \quad \text{and} \quad f_{a5}{}^{b} = -\tfrac{3}{2}\,\eta^3_{ab} \ . \tag{9.6}$$

Comparing this to (5.15), one notices that the sign difference $\eta = \pm e^5$ of (8.16) and (3.9) translates into different signs for the structure constants.

9.1.3 Biedenharn basis

The irreducible SU(3)-representations $\underline{C}^{k,l}$ are labelled by a pair of non-negative integers (k,l) and have (complex) dimension

$$p_0 \coloneqq \dim\left(\underline{C}^{k,l}\right) = \tfrac{1}{2}\,(k+l+2)\,(k+1)\,(l+1) \ . \tag{9.7}$$

We decompose $\underline{C}^{k,l}$ with respect to the subgroup $\mathrm{H} = \mathrm{SU}(2)\times\mathrm{U}(1) \subset \mathrm{G}$, just as in [155]. A particularly convenient choice of basis for the vector space $\underline{C}^{k,l}$ is the *Biedenharn basis* [170–172], which is defined to be the eigenvector basis given by

$$H_{\alpha_1} \left| \begin{matrix} n \\ q \end{matrix} m \right\rangle = q \left| \begin{matrix} n \\ q \end{matrix} m \right\rangle \ , \quad L^2 \left| \begin{matrix} n \\ q \end{matrix} m \right\rangle = n\,(n+2) \left| \begin{matrix} n \\ q \end{matrix} m \right\rangle \quad \text{and} \quad H_{\alpha_2} \left| \begin{matrix} n \\ q \end{matrix} m \right\rangle = m \left| \begin{matrix} n \\ q \end{matrix} m \right\rangle \ , \tag{9.8}$$

where $L^2 := 2\,(E_{\alpha_1}\,E_{-\alpha_1} + E_{-\alpha_1}\,E_{\alpha_1}) + H_{\alpha_1}^2$ is the isospin operator of $\mathfrak{su}(2)$. Define the representation space $\underline{(n,m)}$ as the eigenspace with definite isospin $n \in \mathbb{Z}_{\geq 0}$ and magnetic monopole charge $\frac{m}{2}$ for $m \in \mathbb{Z}$. Then the SU(3)-representation $\underline{C}^{k,l}$ decomposes into irreducible $\mathrm{SU}(2) \times \mathrm{U}(1)$-representations $\underline{(n,m)}$ as

$$\underline{C}^{k,l} = \bigoplus_{(n,m) \in Q_0(k,l)} \underline{(n,m)} \ , \tag{9.9}$$

where $Q_0(k,l)$ parametrises the set of all occurring representations $\underline{(n,m)}$. In App. B.2.1 we summarise the matrix elements of all generators in the Biedenharn basis.

9.1.4 Representations of $\mathbb{Z}_{q+1}$

As the cyclic group $\mathbb{Z}_{q+1}$ is abelian, each of its irreducible representations is 1-dimensional. There are exactly $q+1$ inequivalent irreducible unitary representations ρ_l given by

$$\rho_l \ : \ \begin{matrix} \mathbb{Z}_{q+1} & \longrightarrow & S^1 \subset \mathbb{C}^* \\ p & \longmapsto & \mathrm{e}^{\frac{2\pi\,\mathrm{i}\,(p+l)}{q+1}} \end{matrix} \qquad \text{for} \quad l = 0, 1, \dots, q \ . \tag{9.10}$$

9.2 Homogeneous bundles and quiver representations

Consider the groups $\mathrm{G} = \mathrm{SU}(3)$, $\mathrm{H} = \mathrm{SU}(2) \times \mathrm{U}(1)$, $\mathrm{K} = \mathrm{SU}(2)$, $\widetilde{\mathrm{K}} = \mathrm{SU}(2) \times \mathbb{Z}_{q+1} \subset \mathrm{H}$ and a finite-dimensional K-representation $\underline{R}$ which descends from a G-representation. Associate to the principal bundle (8.12) the K-equivariant vector bundle $\mathcal{V}_{\underline{R}} \coloneqq \mathrm{G} \times_{\mathrm{K}} \underline{R}$. Due to the embedding $\mathbb{Z}_{q+1} \hookrightarrow \mathrm{U}(1) \subset \mathrm{SU}(3)$ and the origin of $\underline{R}$ from a G-representation, it follows that $\underline{R}$ is also a $\mathbb{Z}_{q+1}$-representation. Consequently, as in Sec. 8.2, the $\mathbb{Z}_{q+1}$-action $\gamma : \mathbb{Z}_{q+1} \times S^5 \to S^5$ can be lifted to a $\mathbb{Z}_{q+1}$-action $\widetilde{\gamma} : \mathbb{Z}_{q+1} \times \mathcal{V}_{\underline{R}} \to \mathcal{V}_{\underline{R}}$, wherein the linear $\mathbb{Z}_{q+1}$-action on the fibres is trivial. Thus, one can define the corresponding $\widetilde{\mathrm{K}}$-equivariant vector V-bundle $\widetilde{\mathcal{V}}_{\underline{R}}$ by suitable $\mathbb{Z}_{q+1}$-projection as[23]

$$\begin{tikzcd} \mathcal{V}_{\underline{R}} \arrow[r, "\widetilde{\gamma}"] \arrow[d, "\pi"'] & \widetilde{\mathcal{V}}_{\underline{R}} \arrow[d, "\widetilde{\pi}"] \\ S^5 \arrow[r, "\gamma"'] & S^5/\mathbb{Z}_{q+1} \end{tikzcd} \tag{9.11}$$

and again we denote the vector V-bundle $\widetilde{\mathcal{V}}_{\underline{R}}$ by the same symbol $\mathcal{V}_{\underline{R}}$ whenever the context is clear.

According to [54], the category of such homogeneous vector bundles $\mathcal{V}_{\underline{R}}$ is equivalent to the category of finite-dimensional representations of certain quivers with relations. We use this equivalence to associate quivers to homogeneous bundles related to an irreducible SU(3)-representation $\underline{R} = \underline{C}^{k,l}$, which is evidently a finite-dimensional (and usually reducible) representation of $\mathrm{SU}(2) \times \mathbb{Z}_{q+1} \hookrightarrow \mathrm{SU}(2) \times \mathrm{U}(1)$.

[23]See also the treatment in [147].

9.2.1 Flat connections

Inspired by the structure of the flat connection (8.23a) on the V-bundle (8.21), one observes that it can be written as[24]

$$\begin{aligned}\mathcal{A}_0 = \left[B_{11}\, H_{\alpha_1} + B_{12}\, E_{\alpha_1} - (B_{12}\, E_{\alpha_1})^\dagger\,\right] - \tfrac{\mathrm{i}}{2}\,\eta\, H_{\alpha_2} + \bar{\beta}^1_{q+1}\, E_{\alpha_1+\alpha_2} + \bar{\beta}^2_{q+1}\, E_{\alpha_2} \\ - \beta^1_{q+1}\, E_{-\alpha_1-\alpha_2} - \beta^2_{q+1}\, E_{-\alpha_2}\ ,\end{aligned} \tag{9.12a}$$

or equivalently

$$\mathcal{A}_0 = \Gamma + I_\mu\, e^\mu \tag{9.12b}$$

with the coframe $\{e^\mu\}_{\mu=1,\dots,5}$ defined in (8.30) and the definition

$$\Gamma \coloneqq \Gamma^i\, I_i \qquad \text{with} \quad \Gamma^6 = \tfrac{\mathrm{i}}{2}\,\left(B_{12} - \bar{B}_{12}\right)\ , \quad \Gamma^7 = \tfrac{1}{2}\,\left(B_{12} + \bar{B}_{12}\right)\ , \quad \Gamma^8 = -\mathrm{i}\, B_{11}\,. \tag{9.13}$$

Note that Γ is an $\mathfrak{su}(2)$-valued connection 1-form. The flatness of $\mathcal{A}_0$ is encoded in the relation

$$\mathcal{F}_0 = F_\Gamma + I_\mu\, \mathrm{d}e^\mu + \Gamma^i\, [I_i, I_\mu] \wedge e^\mu + \tfrac{1}{2}\, [I_\mu, I_\nu]\, e^{\mu\nu} = 0\ , \tag{9.14a}$$

$$\mathcal{F}_0\big|_{\mathfrak{su}(2)} = 0\ : \qquad F_\Gamma = -\tfrac{1}{2}\, f_{\mu\nu}{}^{i}\, I_i\, e^{\mu\nu}\ , \tag{9.14b}$$

$$\mathcal{F}_0\big|_{\mathfrak{m}} = 0\ : \qquad \mathrm{d}e^\mu = -\Gamma^i\, f_{i\nu}{}^{\mu} \wedge e^\nu - \tfrac{1}{2}\, f_{\rho\sigma}{}^{\mu}\, e^{\rho\sigma}\ , \tag{9.14c}$$

where $F_\Gamma = \mathrm{d}\Gamma + \Gamma\wedge\Gamma$. The equivalent information can be cast in a set of relations starting from (9.12a) and using the Biedenharn basis; we refer to App. B.2.2 for details.

9.2.2 $\mathbb{Z}_{q+1}$-equivariance

Consider the principal V-bundle (8.21), where the $\mathbb{Z}_{q+1}$-action is defined on S^5 as in Sec. 8.2. The connection (9.12) is SU(3)-equivariant by construction, but one can also check its $\mathbb{Z}_{q+1}$-equivariance explicitly. For this, one needs to specify an action of $\mathbb{Z}_{q+1}$ on the fibre $\underline{C}^{k,l}$, which decomposes as an SU(2)-representation via (9.9). Demanding that the $\mathbb{Z}_{q+1}$-action commutes with the SU(2)-action on $\underline{C}^{k,l}$ forces it to act as a multiple of the identity on each irreducible SU(2)-representation by Schur's lemma. Hence, we choose a representation $\gamma : \mathbb{Z}_{q+1} \to \mathrm{U}(p_0)$ of $\mathbb{Z}_{q+1}$ on $\underline{C}^{k,l}$ such that $\mathbb{Z}_{q+1}$ acts on $\underline{(n,m)}$ as

$$\gamma(h)\big|_{\underline{(n,m)}} = \zeta^m_{q+1}\, \mathbb{1}_{n+1} \in \mathrm{U}(1)\ . \tag{9.15}$$

Consider the two parts of the connection (9.12): The connection Γ and the endomorphism-valued 1-form $I_\mu\, e^\mu$. In terms of matrix elements, Γ is completely determined by the 1-forms $B_{(n,m)} \in \Omega^1\left(\mathrm{SU}(2), \mathrm{End}(\underline{\,n,m\,})\right)$ which are instanton connections on the $\widetilde{\mathrm{K}}$-equivariant vector V-bundle

$$\widetilde{\mathcal{V}}_{\underline{(n,m)}} \xrightarrow{\underline{(n,m)}} \mathrm{G}/\widetilde{\mathrm{K}} \cong S^5/\mathbb{Z}_{q+1} \qquad \text{with} \quad \mathcal{V}_{\underline{(n,m)}} \coloneqq \mathrm{G}\times_{\mathrm{K}} \underline{(n,m)}\ , \tag{9.16}$$

simply because they are K-equivariant by construction and $\mathbb{Z}_{q+1} \hookrightarrow \mathrm{U}(1) \subset \mathrm{SU}(3)$ commutes with this particular SU(2) subgroup (see also App. B.1). More explicitly, taking (9.15) one observes that $\mathbb{Z}_{q+1}$ acts trivially on the endomorphism part,

$$\gamma(h)\, B_{(n,m)}\, \gamma(h)^{-1} = B_{(n,m)}\ , \tag{9.17}$$

[24] Note that (8.23a) implicitly uses the fundamental representation $\underline{C}^{1,0}$ of SU(3).

as well as on the 1-form parts Γ_i because they are horizontal in the V-bundle (8.21). For $\mathbb{Z}_{q+1}$-equivariance of the second term $I_\mu\, e^\mu$, from (9.12a) and the representation π defined in (8.37) one demands the conditions

$$\gamma(h)\, E_{\mathfrak{w}}\, \gamma(h)^{-1} = \pi(h)^{-1}(E_{\mathfrak{w}}) = \zeta_{q+1}^{3}\, E_{\mathfrak{w}} \qquad \text{for} \quad \mathfrak{w} = \alpha_2, \alpha_1+\alpha_2\ , \tag{9.18a}$$

$$\gamma(h)\, E_{-\mathfrak{w}}\, \gamma(h)^{-1} = \pi(h)^{-1}(E_{-\mathfrak{w}}) = \zeta_{q+1}^{-3}\, E_{-\mathfrak{w}} \qquad \text{for} \quad \mathfrak{w} = \alpha_2, \alpha_1+\alpha_2\ , \tag{9.18b}$$

$$\gamma(h)\, H_{\alpha_2}\, \gamma(h)^{-1} = \pi(h)^{-1}(H_{\alpha_2}) = H_{\alpha_2}\ . \tag{9.18c}$$

One can check that these conditions are satisfied by our choice of representation (9.15), due to the explicit components of the generators (B.10). We conclude that, due to our ansatz for the connection (9.12) on the principal V-bundle (8.21) and the embedding $\mathbb{Z}_{q+1} \hookrightarrow \mathrm{U}(1) \subset \mathrm{SU}(3)$, the 1-form $\mathcal{A}_0$ is indeed $\mathbb{Z}_{q+1}$-equivariant.

9.2.3 Quiver representations

Recall from [155] that one can interpret the decomposition (9.9) and the structure of the connection (9.12) as a quiver associated to $\underline{C}^{k,l}$ as follows: The appearing H-representations $\underline{(n,m)}$ form a set $Q_0(k,l)$ of vertices, whereas the actions of the generators E_{α_2} and $E_{\alpha_1+\alpha_2}$ intertwine the H-representations. These H-morphisms, together with H_{α_2}, constitute a set $Q_1(k,l)$ of arrows $(n,m) \to (n',m')$ between the vertices. The quiver $\mathcal{Q}^{k,l}$ is then given by the pair $\mathcal{Q}^{k,l} = \big(Q_0(k,l)\,,\, Q_1(k,l)\big)$; the underlying graph of this quiver is obtained from the weight diagram of the representation $\underline{C}^{k,l}$ by collapsing all horizontal edges to vertices, cf. [155]. See App. B.3 for an explicit treatment of the examples $\underline{C}^{1,0}$, $\underline{C}^{2,0}$ and $\underline{C}^{1,1}$.

9.3 Quiver bundles and connections

In the following, we consider representations of quivers not in the category of vector spaces, but rather in the category of vector bundles. We shall construct a G-equivariant gauge theory on the product space

$$M^d \times_{\widetilde{\mathrm{K}}} \mathrm{G} := M^d \times \mathrm{G}/\widetilde{\mathrm{K}} = M^d \times S^5/\mathbb{Z}_{q+1}\ , \tag{9.19}$$

where G and all of its subgroups act trivially on a d-dimensional Riemannian manifold M^d. The equivariant dimensional reduction compensates isometries on $\mathrm{G}/\widetilde{\mathrm{K}}$ with gauge transformations, thus leading to quiver gauge theories on the manifold M^d.

Roughly speaking, the reduction is achieved by extending the homogeneous V-bundles (9.11) by $\widetilde{\mathrm{K}}$-equivariant bundles $E \to M^d$, which furnish a representation of the corresponding quiver in the category of complex vector bundles over M^d. Such a representation is called a *quiver bundle* and it originates from the one-to-one correspondence between G-equivariant Hermitian vector V-bundles over $M^d \times \mathrm{G}/\widetilde{\mathrm{K}}$ and $\widetilde{\mathrm{K}}$-equivariant Hermitian vector bundles over M^d, where $\widetilde{\mathrm{K}}$ acts trivially on the base space M^d [54].

9.3.1 Equivariant bundles

For each irreducible H-representation $\underline{(n,m)}$ in the decomposition of $\underline{C}^{k,l}$, construct the (trivial) vector bundle

$$\underline{(n,m)}_{M^d} := M^d \times_{\widetilde{\mathrm{K}}} \underline{(n,m)} \xrightarrow{(n,m)} M^d \tag{9.20}$$

of rank $n{+}1$, which is $\widetilde{\mathrm{K}}$-equivariant due to the trivial $\widetilde{\mathrm{K}}$-action on M^d and the linear action on the fibres. For each representation $\underline{(n,m)}$ we introduce also a Hermitian vector bundle

$$E_{p_{(n,m)}} \xrightarrow{\mathbb{C}^{p_{(n,m)}}} M^d \qquad \text{with} \quad \mathrm{rk}(E_{p_{(n,m)}}) = p_{(n,m)} \tag{9.21}$$

with structure group $\mathrm{U}(p_{(n,m)})$ and a $\mathfrak{u}(p_{(n,m)})$-valued connection $A_{(n,m)}$, and with trivial $\widetilde{\mathrm{K}}$-action. Denote the identity endomorphism on the fibres of $E_{p_{(n,m)}}$ by $\pi_{(n,m)}$. With these data one constructs a $\widetilde{\mathrm{K}}$-equivariant bundle

$$E^{k,l} \cong \bigoplus_{(n,m)\in Q_0(k,l)} E_{p_{(n,m)}} \otimes \underline{(n,m)}_{M^d} \xrightarrow{\mathbb{C}^p} M^d \tag{9.22}$$

whose rank p is given by

$$p = \sum_{(n,m)\in Q_0(k,l)} p_{(n,m)} \ \dim \underline{(n,m)} = \sum_{(n,m)\in Q_0(k,l)} p_{(n,m)} \, (n+1) \ . \tag{9.23}$$

Following [155], the bundle $E^{k,l}$ is the $\widetilde{\mathrm{K}}$-equivariant vector bundle of rank p associated to the representation $\underline{C}^{k,l}|_{\widetilde{\mathrm{K}}}$ of $\widetilde{\mathrm{K}}$, and (9.22) is its isotopical decomposition. This construction breaks the structure group $\mathrm{U}(p)$ of $E^{k,l}$ via the Higgs effect to the subgroup

$$\mathcal{G}^{k,l} := \prod_{(n,m)\in Q_0(k,l)} \mathrm{U}\big(p_{(n,m)}\big)^{n+1} \tag{9.24}$$

which commutes with the $\mathrm{SU}(2)$-action on the fibres of (9.22).

On the other hand, one can introduce $\widetilde{\mathrm{K}}$-equivariant V-bundles over $S^5/\mathbb{Z}_{q+1}$ by (9.16). On $\mathcal{V}_{\underline{(n,m)}}$ one has the $\mathfrak{su}(2)$-valued 1-instanton connection $B_{(n,m)}$ in the $(n{+}1)$-dimensional irreducible representation. The aim is to establish a G-equivariant V-bundle $\mathcal{E}^{k,l}$ over $M^d{\times}S^5/\mathbb{Z}_{q+1}$ as an extension of the $\widetilde{\mathrm{K}}$-equivariant bundle $E^{k,l}$. By the results of [54] such a V-bundle $\mathcal{E}^{k,l}$ exists and according to [155] it is realised as

$$\mathcal{E}^{k,l} := \mathrm{G} \times_{\widetilde{\mathrm{K}}} E^{k,l} = \bigoplus_{(n,m)\in Q_0(k,l)} E_{p_{(n,m)}} \boxtimes \mathcal{V}_{\underline{(n,m)}} \xrightarrow{V^{k,l}} M^d \times S^5/\mathbb{Z}_{q+1} \ , \tag{9.25}$$

where

$$V^{k,l} = \bigoplus_{(n,m)\in Q_0(k,l)} \mathbb{C}^{p_{(n,m)}} \otimes \underline{(n,m)} \tag{9.26}$$

is the typical fibre of (9.25).

9.3.2 Generic G-equivariant connection

The task now is to determine the generic form of a G-equivariant connection on (9.25). Since the space of connections on $\mathcal{E}^{k,l}$ is an affine space modelled over $\Omega^1(\mathrm{End}(\mathcal{E}^{k,l}))^{\mathrm{G}}$, one has to study the G-representations on this vector space. Recall from [155] that the decomposition of $\Omega^1(\mathrm{End}(\mathcal{E}^{k,l}))^{\mathrm{G}}$ with respect to G yields a *diagonal* subspace which accommodates the connections $A_{(n,m)}$ on (9.21) twisted by G-equivariant connections on (9.16), and an *off-diagonal* subspace which gives rise to bundle morphisms.

In other words, K-equivariance alone introduces only the connections $A_{(n,m)}$ on each bundle (9.21) as well as the SU(2)-connections $B_{(n,m)}$ on the V-bundles (9.16). On the other hand, G-equivariance additionally requires one to introduce a set of bundle morphisms

$$\phi^{\pm}_{(n,m)} \in \mathrm{Hom}\big(E_{p_{(n,m)}}, E_{p_{(n\pm1,m+3)}}\big) \tag{9.27a}$$

and their adjoint maps

$$(\phi^{\pm})^{\dagger}_{(n,m)} \in \mathrm{Hom}\big(E_{p_{(n\pm1,m+3)}}, E_{p_{(n,m)}}\big)\ , \tag{9.27b}$$

for all $(n,m) \in Q_0(k,l)$. One further introduces the bundle endomorphisms

$$\psi_{(n,m)} \in \mathrm{End}\big(E_{p_{(n,m)}}\big) \tag{9.27c}$$

at each vertex $(n,m) \in Q_0(k,l)$ with $m \neq 0$. The morphisms $\phi^{\pm}_{(n,m)}$ and $\psi_{(n,m)}$ are collectively called *Higgs fields*, and they realise the G-action in the same way that the generators I_μ (or more precisely the 1-forms $\bar{\beta}^{\pm}_{(n,m)}$ and $\frac{\mathrm{i}\, m}{2}\, \eta\, \Pi_{(n,m)}$ of App. B.2.1) do in the case of the flat connection (9.12). The *new* Higgs fields $\psi_{(n,m)}$, implementing the vertical connection components on the (orbifold of the) Hopf bundle $S^5 \to \mathbb{C}P^2$, have to be Hermitian, i.e. $\psi_{(n,m)} = \psi^{\dagger}_{(n,m)}$, by construction in order for the connection to be $\mathfrak{u}(p)$-valued.

9.3.3 Ansatz for the connection

The ansatz for a G-equivariant connection on the equivariant V-bundle (9.25) is given by

$$\mathcal{A} = \widehat{A} + \widehat{\Gamma} + X_\mu\, e^\mu \tag{9.28}$$

wherein the $\mathfrak{u}(p_{(n,m)})$-valued connections $A_{(n,m)}$ and the $\mathfrak{su}(2)$-valued connection Γ are extended as

$$\widehat{A} \coloneqq \bigoplus_{(n,m)} A_{(n,m)} \otimes \Pi_{(n,m)} \equiv A \otimes \mathbb{1} \quad \text{and} \quad \widehat{\Gamma} \coloneqq \bigoplus_{(n,m)} \pi_{(n,m)} \otimes \Gamma^i\, I_i^{(n,m)} = \Gamma^i\, \widehat{I}_i \equiv \mathbb{1} \otimes \Gamma\ , \tag{9.29}$$

together with $\widehat{I}_i = \bigoplus_{(n,m)} \pi_{(n,m)} \otimes I_i^{(n,m)}$. Analogous to Sec. 4.3, the matrices X_μ are required to satisfy the equivariance condition

$$\big[\, \widehat{I}_i, X_\mu \big] = f_{i\mu}{}^{\nu}\, X_\nu \qquad \text{for} \quad i = 6,7,8 \quad \text{and} \quad \mu = 1,\dots,5\ . \tag{9.30}$$

Again, the equivariance condition ensures that X_μ are frame-independently defined endomorphisms that are the components of an endomorphism-valued 1-form, which is here given as the difference $\mathcal{A} - (\widehat{A} + \widehat{\Gamma}\,)$.

The general solution to (9.30) expresses X_μ in terms of Higgs fields and generators as

$$\tfrac{1}{2}\,(X_1 + \mathrm{i}\, X_2) = \bigoplus_{\pm,(n,m)} \phi^{\pm}_{(n,m)} \otimes E^{\pm\,(n,m)}_{\alpha_1+\alpha_2}\ , \tag{9.31a}$$

$$\tfrac{1}{2}\,(X_1 - \mathrm{i}\, X_2) = -\bigoplus_{\pm,(n,m)} (\phi^{\pm})^{\dagger}_{(n,m)} \otimes E^{\pm\,(n,m)}_{-\alpha_1-\alpha_2}\ , \tag{9.31b}$$

$$\tfrac{1}{2}\,(X_3 + \mathrm{i}\, X_4) = \bigoplus_{\pm,(n,m)} \phi^{\pm}_{(n,m)} \otimes E^{\pm\,(n,m)}_{\alpha_2}\ , \tag{9.31c}$$

$$\tfrac{1}{2}\,(X_3 - \mathrm{i} X_4) = -\bigoplus_{\pm,(n,m)} (\phi^{\pm})^{\dagger}_{(n,m)} \otimes E^{\pm\,(n,m)}_{-\alpha_2}\ , \tag{9.31d}$$

$$X_5 = -\frac{\mathrm{i}}{2} \bigoplus_{(n,m)} \psi_{(n,m)} \otimes H_{\alpha_2}^{(n,m)} \ . \tag{9.31e}$$

Altogether the G-equivariant connection takes the form

$$\begin{aligned} \mathcal{A} = \bigoplus_{(n,m)\in Q_0(k,l)} \Big(& A_{(n,m)} \otimes \Pi_{(n,m)} + \pi_{(n,m)} \otimes B_{(n,m)} - \psi_{(n,m)} \otimes \tfrac{\mathrm{i}\,m}{2}\, \eta\, \Pi_{(n,m)} \\ & + \phi^+_{(n,m)} \otimes \bar{\beta}^+_{(n,m)} + \phi^-_{(n,m)} \otimes \bar{\beta}^-_{(n,m)} - (\phi^+)^\dagger_{(n,m)} \otimes \beta^+_{(n,m)} - (\phi^-)^\dagger_{(n,m)} \otimes \beta^-_{(n,m)} \Big) \ . \end{aligned} \tag{9.32}$$

9.3.4 $\mathbb{Z}_{q+1}$-equivariance

One needs to extend the $\mathbb{Z}_{q+1}$-representation γ of (9.15) to act on the fibres (9.26) of the equivariant V-bundle (9.25). Since by construction $\widetilde{\mathrm{K}} = \mathrm{SU}(2) \times \mathbb{Z}_{q+1}$ acts trivially on the fibres of the bundles (9.21), one ends up with the representation $\gamma : \mathbb{Z}_{q+1} \to \mathrm{U}(p)$ given by

$$\gamma(h) = \bigoplus_{(n,m)\in Q_0(k,l)} \mathbb{1}_{p_{(n,m)}} \otimes \gamma(h)\big|_{\underline{(n,m)}} = \bigoplus_{(n,m)\in Q_0(k,l)} \mathbb{1}_{p_{(n,m)}} \otimes \zeta_{q+1}^m\, \mathbb{1}_{n+1} \ . \tag{9.33}$$

To prove $\mathbb{Z}_{q+1}$-equivariance of (9.28) one has to show two things. Firstly, the connections $A \otimes \mathbb{1}$ and $\mathbb{1} \otimes \Gamma$ have to be $\mathbb{Z}_{q+1}$-equivariant. This can be seen as follows: For $A \otimes \mathbb{1}$ the representation γ of (9.33) acts trivially on each bundle $E_{p_{(n,m)}}$, and thus

$$\gamma(h)\ (A \otimes \mathbb{1})\ \gamma(h)^{-1} = A \otimes \mathbb{1} \ . \tag{9.34}$$

Furthermore, $\mathbb{1} \otimes \Gamma$ is $\mathbb{Z}_{q+1}$-equivariant because Γ obeys (9.17), and hence the connection $A \otimes \mathbb{1} + \mathbb{1} \otimes \Gamma$ satisfies the equivariance conditions.

Secondly, the endomorphism-valued 1-form $X_\mu\, e^\mu = \mathcal{A} - \widehat{A} - \widehat{\Gamma}$ needs to be $\mathbb{Z}_{q+1}$-equivariant as well. Due to its structure, one needs to consider a combination of the adjoint action of γ from (9.33) and the $\mathbb{Z}_{q+1}$-action on forms from (8.37). As γ acts trivially on each bundle $E_{p_{(n,m)}}$, the $\mathbb{Z}_{q+1}$-equivariance conditions

$$\gamma(h)\, X_\mu\, \gamma(h)^{-1} = \pi(h)^{-1}(X_\mu) \qquad \text{for} \quad \mu = 1, \dots, 5 \tag{9.35}$$

hold also for the quiver connection $\mathcal{A}$ just as they hold for the flat connection $\mathcal{A}_0$ by (9.18).

Thus, the chosen representations (8.37) and (9.33) render the quiver connection (9.28) equivariant with respect to the action of $\mathbb{Z}_{q+1}$. On each irreducible representation $\underline{(n,m)}$ the generator h of $\mathbb{Z}_{q+1}$ is represented by $\zeta_{q+1}^m\, \mathbb{1}_{n+1}$ which depends on the U(1) monopole charge, but not on the SU(2) isospin. This comes about as follows: The bundle morphisms associated to β^i_{q+1} map between bundles $E_{p_{(n,m)}} \otimes \underline{(n,m)}_{M^d}$ that differ in m by -3 (from source to target vertex), but differ in n by either $+1$ or -1. Thus the representation γ should only be sensitive to m and not to n. We shall elucidate this point further in Sec. 12.1.

9.3.5 Curvature

The curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ of the connection (9.28) is given by

$$\mathcal{F} = F_A \otimes \mathbb{1} + \mathbb{1} \otimes F_\Gamma + \big(\mathrm{d}X_\mu + [\,\widehat{A}, X_\mu]\big) \wedge e^\mu + X_\mu\, \mathrm{d}e^\mu + [\,\widehat{\Gamma}, X_\mu] \wedge e^\mu + \tfrac{1}{2}\ [X_\mu, X_\nu]\ e^{\mu\nu} \ , \tag{9.36a}$$

where $F_A = \mathrm{d}A + A \wedge A$. Employing the relations (9.14) then yields

$$\begin{aligned}\mathcal{F} = F_A \otimes \mathbb{1} + \left(\mathrm{d}X_\mu + [\,\widehat{A}, X_\mu]\right) \wedge e^\mu + \Gamma^i \left([\,\widehat{I}_i, X_\mu] - f_{i\mu}{}^{\nu}\, X_\nu\right) \wedge e^\mu \\ + \tfrac{1}{2}\left([X_\mu, X_\nu] - f_{\mu\nu}{}^{\rho}\, X_\rho - f_{\mu\nu}{}^{i}\, \widehat{I}_i\right) e^{\mu\nu}\ .\end{aligned} \tag{9.36b}$$

Since the matrices X_μ satisfy the equivariance relation (9.30), the final form of the curvature reads

$$\mathcal{F} = F_A \otimes \mathbb{1} + (DX)_\mu \wedge e^\mu + \tfrac{1}{2}\left([X_\mu, X_\nu] - f_{\mu\nu}{}^{\rho}\, X_\rho - f_{\mu\nu}{}^{i}\, \widehat{I}_i\right) e^{\mu\nu}\ , \tag{9.36c}$$

where we defined the bifundamental covariant derivatives as

$$(DX)_\mu := \mathrm{d}X_\mu + [\,\widehat{A}, X_\mu]\ . \tag{9.36d}$$

Inserting the explicit form (9.31) for the scalar fields X_μ leads to the curvature components in the Biedenharn basis; the detailed expressions are summarised in App. B.2.3.

9.3.6 Quiver bundles

Let us now exemplify and clarify how the equivariant bundle $E^{k,l} \to M^d$ from (9.22) realises a quiver bundle from our constructions above. Recall that the quiver $\mathcal{Q}^{k,l}$ consists of the pair $(Q_0(k,l)\,,\, Q_1(k,l))$, with vertices $(n,m) \in Q_0(k,l)$ and arrows $(n,m) \to (n',m') \in Q_1(k,l)$ between certain pairs of vertices which are here determined by the decomposition (9.9). We consider a representation $\widetilde{\mathcal{Q}}^{\,k,l} = \left(\widetilde{Q}_0(k,l)\,,\, \widetilde{Q}_1(k,l)\right)$ of this quiver in the category of complex vector bundles. The set of vertices is

$$\widetilde{Q}_0(k,l) = \left\{E_{p_{(n,m)}} \longrightarrow M^d \,\middle|\quad (n,m) \in Q_0(k,l)\right\}\ , \tag{9.37}$$

i.e. the set of Hermitian vector bundles, each equipped with a unitary connection $A_{(n,m)}$. The set of arrows is

$$\begin{aligned}\widetilde{Q}_1(k,l) = \left\{\phi^{\pm}_{(n,m)} \in \mathrm{Hom}\big(E_{p_{(n,m)}}, E_{p_{(n\pm1,m+3)}}\big) \,\middle|\quad (n,m) \in Q_0(k,l)\right\} \\ \cup\ \left\{\psi_{(n,m)} \in \mathrm{End}\big(E_{p_{(n,m)}}\big) \,\middle|\quad (n,m) \in Q_0(k,l)\ ,\ m \neq 0\right\}\ ,\end{aligned} \tag{9.38}$$

which is precisely the set of bundle morphisms, i.e. the Higgs fields. These quivers differ from those considered in [155] by the appearance of vertex loops corresponding to the endomorphisms $\psi_{(n,m)}$. See App. B.3 for details of the quiver bundles based on the representations $\underline{C}^{1,0}$, $\underline{C}^{2,0}$ and $\underline{C}^{1,1}$.

These constructions yield representations of quivers without any relations. We will see later on that relations can arise by minimising the scalar potential of the quiver gauge theory (see Sec. 9.4) or by imposing a generalised instanton equation on the connection $\mathcal{A}$ (see Ch. 10).

9.4 Dimensional reduction of the Yang-Mills action

Consider the reduction of the pure Yang-Mills action from $M^d \times S^5$ to M^d. On S^5 we take as basis of coframes $\{\beta^j_\varphi, \bar\beta^j_\varphi\}_{j=1,2}$ and $e^5 = \eta$, and as metric

$$\mathrm{d}s^2_{S^5} = R^2\left(\beta^1_\varphi \otimes \bar\beta^1_\varphi + \bar\beta^1_\varphi \otimes \beta^1_\varphi + \beta^2_\varphi \otimes \bar\beta^2_\varphi + \bar\beta^2_\varphi \otimes \beta^2_\varphi\right) + r^2\, \eta \otimes \eta\ . \tag{9.39}$$

The Yang-Mills action is given by

$$S = -\frac{1}{4\tilde{g}^2} \int_{M^d \times S^5} \operatorname{tr} \mathcal{F} \wedge \star \mathcal{F} \ , \tag{9.40}$$

with coupling constant $\tilde{g}$ and $\star$ the Hodge duality operator corresponding to the metric on $M^d \times S^5$ given by

$$\mathrm{d}s^2 = \mathrm{d}s^2_{M^d} + \mathrm{d}s^2_{S^5} \ . \tag{9.41}$$

We denote the Hodge operator corresponding to the metric $\mathrm{d}s^2_{M^d}$ on M^d by $\star_{M^d}$. The reduction of (9.40) proceeds by inserting the curvature (9.36c) and performing the integrals over S^5, which can be evaluated by using (9.39) and the identities of App. B.4.2. One finally obtains for the reduced action

$$\begin{aligned} S = -\frac{2\pi^3\, r\, R^4}{\tilde{g}^2} \Bigg(& \int_{M^d} \operatorname{tr} \left(F_A \wedge \star_{M^d} F_A \right) \otimes \mathbb{1} \\ & + \frac{1}{2R^2} \int_{M^d} \sum_{a=1}^{4} \operatorname{tr} \left(DX \right)_a \wedge \star_{M^d} (DX)_a + \frac{1}{r^2} \int_{M^d} \operatorname{tr} \left(DX \right)_5 \wedge \star_{M^d} (DX)_5 \\ & + \frac{1}{8R^4} \int_{M^d} \star_{M^d} \sum_{a,b=1}^{4} \operatorname{tr} \Big([X_a, X_b] - f_{ab}{}^5\, X_5 - f_{ab}{}^i\, \widehat{I}_i \Big)^2 \\ & + \frac{1}{8R^2\, r^2} \int_{M^d} \star_{M^d} \sum_{a=1}^{4} \operatorname{tr} \Big([X_a, X_5] - f_{a5}{}^b\, X_b \Big)^2 \Bigg) \ . \end{aligned} \tag{9.42}$$

Here the explicit structure constants (9.4), i.e. $f_{ab}{}^c = f_{a5}{}^5 = f_{a5}{}^i = 0$, have been used. One may detail this action further by inserting the G-equivariant solution (9.31) for the scalar fields X_μ in the Biedenharn basis, which allows one to perform the trace over the $\mathrm{SU}(2)\times\mathrm{U}(1)$-representations $\underline{(n,m)}$. The explicit, but lengthy formulas are given in App. B.4.3.

9.4.1 Higgs branch

On the Higgs branch of the quiver gauge theory, where all connections $A_{(n,m)}$ are trivial and the Higgs fields are constant, the vacuum is solely determined by the vanishing locus of the scalar potential. The vanishing of the potential gives rise to holomorphic F-term constraints as well as non-holomorphic D-term constraints which read as

$$[X_a, X_b] = f_{ab}{}^5\, X_5 + f_{ab}{}^i\, \widehat{I}_i \quad \text{and} \quad [X_a, X_5] = f_{a5}{}^b\, X_b \ , \tag{9.43}$$

for $a, b = 1, 2, 3, 4$. The equivariance condition (9.30) implies that X_μ lie in a representation of the $\mathfrak{su}(2)$ Lie algebra. Hence, the BPS configurations of the gauge theory X_μ, together with $\widehat{I}_i$, furnish a representation of the Lie algebra $\mathfrak{su}(3)$ in the representation space of the quiver in $\mathfrak{u}(p)$. These constraints respectively give rise to a set of relations and a set of stability conditions for the corresponding quiver representation. The details can be read off from the explicit expressions in App. B.4.3.

10 Spherically symmetric instantons

In this chapter we specialise to the case of a 1-dimensional Riemannian manifold $M^d = M^1$. We investigate the Hermitian Yang-Mills equations on the product $M^1 \times S^5/\mathbb{Z}_{q+1}$ for the generic form of G-equivariant connections derived in Sec. 9.3.

10.1 Preliminaries

Consider the product manifold $M^1 \times S^5/\mathbb{Z}_{q+1}$ for $M^1 = \mathbb{R}$ such that $M^1 \times S^5/\mathbb{Z}_{q+1} \cong C(S^5/\mathbb{Z}_{q+1})$ is the metric cone over the Sasaki-Einstein space $S^5/\mathbb{Z}_{q+1}$, which is an orbifold of the Calabi-Yau manifold $\mathbb{C}^3 \cong C(S^5)$. As discussed in Ch. 3, the Calabi-Yau space $C(S^5)$ is conformally equivalent to the cylinder $\mathbb{R} \times S^5$ with the metric (3.4) and fundamental $(1,1)$-form (3.5).

Connections As $\mathbb{R}$ is contractible, each bundle $E_{p_{(n,m)}} \to \mathbb{R}$ is necessarily trivial, and hence one can gauge away the (global) connection 1-forms $A_{(n,m)} = A_{(n,m)}(t)\, \mathrm{d}t$. Explicitly, there is a gauge transformation $g : \mathbb{R} \to \mathcal{G}^{k,l}$ such that

$$\tilde{A}_{(n,m)} = \mathrm{Ad}(g^{-1}) A_{(n,m)} + g^{-1} \frac{\mathrm{d}g}{\mathrm{d}t} = 0 \qquad \text{with} \quad g = \exp\Big(- \int A_{(n,m)}(t)\, \mathrm{d}t \Big) \ . \tag{10.1}$$

The ansatz for the connection on the equivariant V-bundle then reads

$$\mathcal{A} = \mathbb{1} \otimes \Gamma + X_\mu \, e^\mu \ , \tag{10.2}$$

where the Higgs fields $\phi^{\pm}_{(n,m)}$ and $\psi_{(n,m)}$ depend only on the cylinder coordinate t. The curvature of this connection can be read off from (9.36c) and is evaluated to

$$\mathcal{F} = \frac{\mathrm{d}X_\mu}{\mathrm{d}t} \, \mathrm{d}t \wedge e^\mu + \frac{1}{2} \Big([X_\mu, X_\nu] - f_{\mu\nu}{}^{\rho} \, X_\rho - f_{\mu\nu}{}^{i} \, \widehat{I}_i \Big) \, e^{\mu\nu} \ . \tag{10.3}$$

Generalised instanton equations By construction, the ansatz (10.2) restricts the space of all connections on the SU(3)-equivariant vector V-bundle over $C(S^5/\mathbb{Z}_{q+1})$ to SU(3)-equivariant and $\mathbb{Z}_{q+1}$-equivariant connections. However, we saw that $\mathbb{Z}_{q+1}$-equivariance was automatic due to the choice of $\mathbb{Z}_{q+1}$-action (8.37) and (9.33). Consequently, the situation is a special case of the equivariant instantons discussed in Sec. 4.3.

As such, we conceptually split the HYM-instanton equations (4.29) into *quiver relations* and *stability-like conditions*. In detail, the holomorphicity condition (4.10) can be specialised to a $d = 5$ version of the first four equations in (4.29) and reads

$$[X_1, X_4] + [X_2, X_3] = 0 = [X_1, X_3] - [X_2, X_4] \quad \text{and} \quad [X_a, X_5] = f_{a5}{}^{b} \Big(X_b + \frac{2}{3} \frac{\mathrm{d}X_b}{\mathrm{d}t} \Big) \tag{10.4}$$

for $a = 1, 2, 3, 4$. These conditions will be understood as relations on the quiver bundle itself.

On the other hand, the stability-like condition (4.19) can be obtained from (4.29e) and reads in $d = 5$ as follows:

$$[X_1, X_2] + [X_3, X_4] = 4 X_5 + \frac{\mathrm{d}X_5}{\mathrm{d}t} \ . \tag{10.5}$$

This condition will sometimes just be denoted as *stability condition*. In what follows we will employ the geometric understanding of (10.4) and (10.5) gained in Ch. 4, and return to this interpretation in Sec. 12.2.1.

10.2 Examples

We shall now apply these considerations to the three simplest examples: The quivers based on the representations $\underline{C}^{1,0}$, $\underline{C}^{2,0}$ and $\underline{C}^{1,1}$. For each example we explicitly provide the representation of the generators and the form of the matrices X_μ, followed by the quiver relations and the stability conditions.

10.2.1 $C^{1,0}$-quiver

The generators in the fundamental representation $\underline{C}^{1,0}$, which splits as in (B.24), are given as

$$I_a = \begin{pmatrix} 0_2 & I_a^{(0,-2)} \\ -\big(I_a^{(0,-2)}\big)^\dagger & 0 \end{pmatrix} \quad \text{and} \quad I_5 = \begin{pmatrix} I_5^{(1,1)} & 0 \\ 0 & I_5^{(0,-2)} \end{pmatrix} \tag{10.6a}$$

for $a = 1, 2, 3, 4$, with components

$$I_1^{(0,-2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathrm{i}\, I_2^{(0,-2)} \quad \text{and} \quad I_3^{(0,-2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathrm{i}\, I_4^{(0,-2)} \ , \tag{10.6b}$$

$$I_5^{(0,-2)} = \mathrm{i}\, \mathbb{1}_2 \quad \text{and} \quad I_5^{(1,1)} = -\tfrac{\mathrm{i}}{2} \ . \tag{10.6c}$$

The endomorphisms X_μ read as

$$X_a = \begin{pmatrix} 0_2 & \phi \otimes I_a^{(0,-2)} \\ -\phi^\dagger \otimes \big(I_a^{(0,-2)}\big)^\dagger & 0 \end{pmatrix} \quad \text{and} \quad X_5 = \begin{pmatrix} \psi_1 \otimes I_5^{(1,1)} & 0 \\ 0 & \psi_0 \otimes I_5^{(0,-2)} \end{pmatrix} \ , \tag{10.7}$$

where the Higgs fields from App. B.3 give a representation of the quiver

$$\begin{tikzcd} (0,-2) \arrow[loop above, "\psi_0"] \arrow[rr, "\phi"] & & (1,1) \arrow[loop above, "\psi_1"] \end{tikzcd} \tag{10.8}$$

The $\mathbb{Z}_{q+1}$-representation (9.33) reads

$$\gamma : h \longmapsto \begin{pmatrix} \mathbb{1}_{p_{(1,1)}} \otimes \mathbb{1}_2\ \zeta_{q+1} & 0 \\ 0 & \mathbb{1}_{p_{(0,-2)}} \otimes \zeta_{q+1}^{-2} \end{pmatrix} \ , \tag{10.9}$$

where h is the generator of the cyclic group $\mathbb{Z}_{q+1}$.

Quiver relations The first two equations from (10.4) are trivially satisfied without any further constraints. The second set of equations all have the same non-trivial off-diagonal component (and its adjoint) which yields

$$2\, \frac{\mathrm{d}\phi}{\mathrm{d}t} = -3\, \phi + 2\, \phi\, \psi_0 + \psi_1\, \phi \ . \tag{10.10}$$

Thus for the $C^{1,0}$-quiver there are no purely algebraic quiver relations.

Stability conditions From (10.5) we read off the two non-trivial diagonal components which yield

$$\frac{1}{4}\,\frac{\mathrm{d}\psi_0}{\mathrm{d}t} = -\psi_0 + \phi^\dagger\,\phi \quad\text{and}\quad \frac{1}{4}\,\frac{\mathrm{d}\psi_1}{\mathrm{d}t} = -\psi_1 + \phi\,\phi^\dagger\ . \tag{10.11}$$

By taking ψ_0 and ψ_1 to be identity endomorphisms, we recover the Higgs branch BPS equations from equivariant dimensional reduction over $\mathbb{C}P^2$: In this limit (10.10) implies that the scalar field ϕ is independent of t, while (10.11) correctly reproduces the D-term constraints of the quiver gauge theory for constant matrices [155, 156].

10.2.2 $C^{2,0}$-quiver

The generators in the 6-dimensional representation $\underline{C}^{2,0}$, which splits as in (B.26), are given by

$$I_a = \begin{pmatrix} 0_3 & I_a^{(1,-1)} & 0 \\ -\big(I_a^{(1,-1)}\big)^\dagger & 0_2 & I_a^{(0,-4)} \\ 0 & -\big(I_a^{(0,-4)}\big)^\dagger & 0 \end{pmatrix} \quad\text{and}\quad I_5 = \begin{pmatrix} I_5^{(2,2)} & 0 & 0 \\ 0 & I_5^{(1,-1)} & 0 \\ 0 & 0 & I_5^{(0,-4)} \end{pmatrix} \tag{10.12a}$$

for $a = 1, 2, 3, 4$, with components

$$I_1^{(1,-1)} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{i}\, I_2^{(1,-1)} \quad\text{and}\quad I_1^{(0,-4)} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} = \mathrm{i}\, I_2^{(0,-4)}\ , \tag{10.12b}$$

$$I_3^{(1,-1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \mathrm{i}\, I_4^{(1,-1)} \quad\text{and}\quad I_3^{(0,-4)} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} = \mathrm{i}\, I_4^{(0,-4)}\ , \tag{10.12c}$$

$$I_5^{(2,2)} = -\mathrm{i}\,\mathbb{1}_3\ , \quad I_5^{(1,-1)} = \tfrac{\mathrm{i}}{2}\,\mathbb{1}_2 \quad\text{and}\quad I_5^{(0,-4)} = 2\,\mathrm{i}\ . \tag{10.12d}$$

The endomorphisms X_μ read

$$\begin{aligned} X_a &= \begin{pmatrix} 0_3 & \phi_1 \otimes I_a^{(1,-1)} & 0 \\ -\phi_1^\dagger \otimes \big(I_a^{(1,-1)}\big)^\dagger & 0_2 & \phi_0 \otimes I_a^{(0,-4)} \\ 0 & -\phi_0^\dagger \otimes \big(I_a^{(0,-4)}\big)^\dagger & 0 \end{pmatrix}\ , \\ X_5 &= \begin{pmatrix} \psi_2 \otimes I_5^{(2,2)} & 0 & 0 \\ 0 & \psi_1 \otimes I_5^{(1,-1)} & 0 \\ 0 & 0 & \psi_0 \otimes I_5^{(0,-4)} \end{pmatrix}\ , \end{aligned} \tag{10.13}$$

with the Higgs field content from App. B.3 that furnishes a representation of the quiver

$$\begin{tikzcd} \boxed{(0,-4)} \arrow[loop above, "\psi_0"] \arrow[r, "\phi_0"] & \boxed{(1,-1)} \arrow[loop above, "\psi_1"] \arrow[r, "\phi_1"] & \boxed{(2,2)} \arrow[loop above, "\psi_2"] \end{tikzcd} \tag{10.14}$$

The representation (9.33) in this case reads

$$\gamma : h \longmapsto \begin{pmatrix} \mathbb{1}_{p_{(2,2)}} \otimes \mathbb{1}_3\ \zeta_{q+1}^2 & 0 & 0 \\ 0 & \mathbb{1}_{p_{(1,-1)}} \otimes \mathbb{1}_2\ \zeta_{q+1}^{-1} & 0 \\ 0 & 0 & \mathbb{1}_{p_{(0,-4)}} \otimes \zeta_{q+1}^{-4} \end{pmatrix}\ . \tag{10.15}$$

Quiver relations Again the first two equations of (10.4) turn out to be trivial, while the second set of equations have two non-vanishing off-diagonal components (plus their conjugates) which yield

$$2\,\frac{\mathrm{d}\phi_0}{\mathrm{d}t} = -3\,\phi_0 - \psi_1\,\phi_0 + 4\,\phi_0\,\psi_0 \quad \text{and} \quad 2\,\frac{\mathrm{d}\phi_1}{\mathrm{d}t} = -3\,\phi_1 + \phi_1\,\psi_1 + 2\,\psi_2\,\phi_1\ , \tag{10.16}$$

and the $C^{2,0}$-quiver has no purely algebraic quiver relations either.

Stability conditions From (10.5) one obtains three non-trivial diagonal components, which are given by

$$\frac{1}{4}\,\frac{\mathrm{d}\psi_0}{\mathrm{d}t} = -\psi_0 + \phi_0^\dagger\,\phi_0\ , \tag{10.17a}$$

$$\frac{1}{4}\,\frac{\mathrm{d}\psi_1}{\mathrm{d}t} = -\psi_1 - 2\,\phi_0\,\phi_0^\dagger + 3\,\phi_1^\dagger\,\phi_1\ , \tag{10.17b}$$

$$\frac{1}{4}\,\frac{\mathrm{d}\psi_2}{\mathrm{d}t} = -\psi_2 + \phi_1\,\phi_1^\dagger\ . \tag{10.17c}$$

Taking ψ_0, ψ_1 and ψ_2 again to be identity morphisms, from (10.16) we obtain constant matrices ϕ_0 and ϕ_1 which by (10.17) obey the expected D-term constraints from equivariant dimensional reduction over $\mathbb{C}P^2$ [155, 156].

10.2.3 $C^{1,1}$-quiver

The decomposition of the adjoint representation $\underline{C}^{1,1}$, which splits as given in (B.28), yields

$$I_a = \begin{pmatrix} 0_2 & I_a^{(0,0)} & I_a^{(2,0)} & 0 \\ -\big(I_a^{(0,0)}\big)^\dagger & 0 & 0 & I_a^{-\,(1,-3)} \\ -\big(I_a^{(2,0)}\big)^\dagger & 0 & 0_3 & I_a^{+\,(1,-3)} \\ 0 & -\big(I_a^{-\,(1,-3)}\big)^\dagger & -\big(I_a^{+\,(1,-3)}\big)^\dagger & 0_2 \end{pmatrix}\ , \tag{10.18a}$$

$$I_5 = \begin{pmatrix} I_5^{(1,3)} & 0 & 0 & 0 \\ 0 & I_5^{(0,0)} & 0 & 0 \\ 0 & 0 & I_5^{(2,0)} & 0 \\ 0 & 0 & 0 & I_5^{(1,-3)} \end{pmatrix} \tag{10.18b}$$

for $a = 1,2,3,4$. The components read as

$$I_1^{(0,0)} = \begin{pmatrix} \sqrt{\frac{3}{2}} \\ 0 \end{pmatrix} = \mathrm{i}\,I_2^{(0,0)} \quad \text{and} \quad I_1^{(2,0)} = \begin{pmatrix} 0 & -\sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & -1 \end{pmatrix} = \mathrm{i}\,I_2^{(2,0)}\ , \tag{10.18c}$$

$$I_1^{-\,(1,-3)} = \begin{pmatrix} 0 & -\sqrt{\frac{3}{2}} \end{pmatrix} = \mathrm{i}\,I_2^{-\,(1,-3)} \quad \text{and} \quad I_1^{+\,(1,-3)} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} = \mathrm{i}\,I_2^{+\,(1,-3)}\ , \tag{10.18d}$$

$$I_3^{(0,0)} = \begin{pmatrix} 0 \\ \sqrt{\frac{3}{2}} \end{pmatrix} = \mathrm{i}\,I_4^{(0,0)} \quad \text{and} \quad I_3^{(2,0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & 0 \end{pmatrix} = \mathrm{i}\,I_4^{(2,0)}\ , \tag{10.18e}$$

$$
I_3^{-\,(1,-3)} = \begin{pmatrix} \sqrt{\tfrac{3}{2}} & 0 \end{pmatrix} = \mathrm{i}\, I_4^{-\,(1,-3)} \quad \text{and} \quad I_3^{+\,(1,-3)} = \begin{pmatrix} 0 & 0 \\ \sqrt{\tfrac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{i}\, I_4^{+\,(1,-3)} \ , \tag{10.18f}
$$

$$
I_5^{(0,0)} = 0 \ , \quad I_5^{(2,0)} = 0_3 \quad \text{and} \quad I_5^{(1,\pm 3)} = \mp \tfrac{3\,\mathrm{i}}{2}\, \mathbb{1}_2 \ . \tag{10.18g}
$$

The matrices X_μ are given by

$$
X_a = \begin{pmatrix} 0_2 & \phi_0^+ \otimes I_a^{(0,0)} & \phi_0^- \otimes I_a^{(2,0)} & 0 \\ -(\phi_0^+)^\dagger \otimes \big(I_a^{(0,0)}\big)^\dagger & 0 & 0 & \phi_1^- \otimes I_a^{-\,(1,-3)} \\ -(\phi_0^-)^\dagger \otimes \big(I_a^{(2,0)}\big)^\dagger & 0 & 0_3 & \phi_1^+ \otimes I_a^{+\,(1,-3)} \\ 0 & -(\phi_1^-)^\dagger \otimes \big(I_a^{-\,(1,-3)}\big)^\dagger & -(\phi_1^+)^\dagger \otimes \big(I_a^{+\,(1,-3)}\big)^\dagger & 0_2 \end{pmatrix} \ , \tag{10.19a}
$$

$$
X_5 = \begin{pmatrix} \psi^+ \otimes I_5^{(1,3)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0_3 & 0 \\ 0 & 0 & 0 & \psi^- \otimes I_5^{(1,-3)} \end{pmatrix} \ . \tag{10.19b}
$$

This example involves the collection of Higgs fields from App. B.3 which furnish a representation of the quiver

$$
\begin{tikzcd}
 & (1,+3) \arrow[loop above, "\psi^+"] & \\
(0,0) \arrow[ur, bend left, "\phi_0^+"] & & (2,0) \arrow[ul, bend right, "\phi_0^-"'] \\
 & (1,-3) \arrow[ul, bend left, "\phi_1^-"] \arrow[ur, bend right, "\phi_1^+"'] \arrow[loop below, "\psi^-"] &
\end{tikzcd} \tag{10.20}
$$

In this case the $\mathbb{Z}_{q+1}$-representation (9.33) has the form

$$
\gamma : h \longmapsto \begin{pmatrix} \mathbb{1}_{p_{(1,3)}} \otimes \mathbb{1}_2\ \zeta_{q+1}^{3} & 0 & 0 & 0 \\ 0 & \mathbb{1}_{p_{(0,0)}} \otimes 1 & 0 & 0 \\ 0 & 0 & \mathbb{1}_{p_{(2,0)}} \otimes \mathbb{1}_3 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{p_{(1,-3)}} \otimes \mathbb{1}_2\ \zeta_{q+1}^{-3} \end{pmatrix} \ . \tag{10.21}
$$

Quiver relations For this 8-dimensional example, one finds that the first two equations of (10.4) have the same single non-trivial off-diagonal component (plus its adjoint) which yields

$$
\phi_0^+\, \phi_1^- = \phi_0^-\, \phi_1^+ \ . \tag{10.22}
$$

This equation is precisely the anticipated algebraic relation for the $C^{1,1}$-quiver expressing equality of paths between the vertices $(1,\pm 3)$, cf. [155]. The second set of equations have four non-trivial off-diagonal components (plus their conjugates) which yield

$$
\frac{2}{3}\, \frac{\mathrm{d}\phi_0^\pm}{\mathrm{d}t} = -\phi_0^\pm + \psi^+\, \phi_0^\pm \quad \text{and} \quad \frac{2}{3}\, \frac{\mathrm{d}\phi_1^\pm}{\mathrm{d}t} = -\phi_1^\pm + \phi_1^\pm\, \psi^- \ . \tag{10.23}
$$

Stability conditions From (10.5) one computes four non-vanishing diagonal components that yield

$$(\phi_0^{\pm})^\dagger\,\phi_0^{\pm} = \phi_1^{\mp}\,(\phi_1^{\mp})^\dagger\ , \tag{10.24a}$$

$$\frac{1}{4}\,\frac{\mathrm{d}\psi^+}{\mathrm{d}t} = -\psi^+ + \frac{1}{2}\,\Big(\phi_0^+\,(\phi_0^+)^\dagger + \phi_0^-\,(\phi_0^-)^\dagger\Big)\ , \tag{10.24b}$$

$$\frac{1}{4}\,\frac{\mathrm{d}\psi^-}{\mathrm{d}t} = -\psi^- + \frac{1}{2}\,\Big((\phi_1^-)^\dagger\,\phi_1^- + (\phi_1^+)^\dagger\,\phi_1^+\Big)\ . \tag{10.24c}$$

We thus obtain two non-holomorphic purely algebraic conditions, which coincide with D-term constraints of the quiver gauge theory for the $C^{1,1}$-quiver, and two further differential equations which for identity endomorphisms $\psi^{\pm}$ reproduce the remaining stability equations for constant matrices $\phi_0^{\pm}$ and $\phi_1^{\pm}$ in equivariant dimensional reduction over $\mathbb{C}P^2$ [155, 156].

11 Translationally invariant instantons

In this chapter we study translationally-invariant instantons on a trivial vector V-bundle over the orbifold $\mathbb{C}^3/\mathbb{Z}_{q+1}$. In contrast to the G-equivariant Hermitian Yang-Mills instantons of Ch. 10, the generic form of a translationally-invariant connection is determined by $\mathbb{Z}_{q+1}$-equivariance alone and is associated with a different quiver.

11.1 Preliminaries

Consider the cone $C(S^5)/\mathbb{Z}_{q+1} \cong \mathbb{C}^3/\mathbb{Z}_{q+1}$, with the $\mathbb{Z}_{q+1}$-action given by (8.26), and the (trivial) vector V-bundle

$$\mathfrak{E}^{k,l} \xrightarrow{V^{k,l}} \mathbb{C}^3/\mathbb{Z}_{q+1} \tag{11.1}$$

of rank p. This V-bundle is obtained by a suitable $\mathbb{Z}_{q+1}$-projection from the trivial vector bundle $\mathbb{C}^3 \times V^{k,l} \to \mathbb{C}^3$. The fibres of (11.1) can be regarded as representation spaces

$$V^{k,l} = \bigoplus_{(n,m)\in Q_0(k,l)} \mathbb{C}^{p_{(n,m)}} \otimes \underline{(n,m)} \cong \bigoplus_{(n,m)\in Q_0(k,l)} \left(\mathbb{C}^{p_{(n,m)}} \otimes \mathbb{C}^{n+1}\right) \otimes V_m \ . \tag{11.2}$$

Here, V_m is the $[m]$-th irreducible representation $\rho_{[m]}$ of $\mathbb{Z}_{q+1}$ (cf. (9.10)), with $[m] \in \{0,1,\dots,q\}$ the congruence class of $m \in \mathbb{Z}$ modulo $q+1$, and the vector space $\mathbb{C}^{p_{(n,m)}} \otimes \mathbb{C}^{n+1}$ serves as the multiplicity space of this representation. The structure group of the bundle $\mathfrak{E}^{k,l}$ is

$$\mathfrak{G}^{k,l} := \prod_{(n,m)\in Q_0(k,l)} \mathrm{U}\big(p_{(n,m)}\,(n+1)\big) \ , \tag{11.3}$$

because the fibres are isomorphic to (11.2) and hence the bundle carries a natural complex structure J. This complex structure is simply multiplication with i on each factor V_m. Consequently, the structure group is reduced to the stabiliser of J.

On the base space the canonical Kähler form of $\mathbb{C}^3$ is given by

$$\omega_{\mathbb{C}^3} = \tfrac{\mathrm{i}}{2}\,\delta_{\alpha\beta}\,\mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^\beta \ . \tag{11.4}$$

This Kähler form is induced by the standard metric $\mathrm{d}s^2_{\mathbb{C}^3} = \frac{1}{2}\,\delta_{\alpha\beta}\,(\mathrm{d}z^\alpha \otimes \mathrm{d}\bar{z}^\beta + \mathrm{d}\bar{z}^\alpha \otimes \mathrm{d}z^\beta)$ and the complex structure $J(\mathrm{d}z^\alpha) = \mathrm{i}\,\mathrm{d}z^\alpha$, $J(\mathrm{d}\bar{z}^\alpha) = -\mathrm{i}\,\mathrm{d}\bar{z}^\alpha$.

11.1.1 Connections

Consider a connection 1-form

$$\mathcal{A} = W_\alpha\,\mathrm{d}z^\alpha + \overline{W}_\alpha\,\mathrm{d}\bar{z}^\alpha \tag{11.5}$$

on $\mathfrak{E}^{k,l}$, where $\overline{W}_\alpha = -W_\alpha^\dagger$ holds. Now, we impose translational invariance along the space $\mathbb{C}^3$. For the coordinate basis $\{\mathrm{d}z^\alpha, \mathrm{d}\bar{z}^\alpha\}$ of $T^*_{(z,\bar{z})}\mathbb{C}^3$ at any point $(z,\bar{z}) \in \mathbb{C}^3$, this translates into the condition

$$\mathrm{d}W_\alpha = 0 = \mathrm{d}\overline{W}_\alpha \qquad \text{for} \quad \alpha = 1,2,3 \ . \tag{11.6}$$

Thus, the curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ simplifies to

$$\mathcal{F} = \mathcal{A} \wedge \mathcal{A} = \tfrac{1}{2} \left[W_\alpha, W_\beta \right] \mathrm{d}z^\alpha \wedge \mathrm{d}z^\beta + \left[W_\alpha, \overline{W}_\beta \right] \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^\beta + \tfrac{1}{2} \left[\overline{W}_\alpha, \overline{W}_\beta \right] \mathrm{d}\bar{z}^\alpha \wedge \mathrm{d}\bar{z}^\beta \ . \quad (11.7)$$

11.1.2 $\mathbb{Z}_{q+1}$-Action

Similar to the previous construction, we impose $\mathbb{Z}_{q+1}$-invariance by to the projection from the trivial vector bundle $\mathbb{C}^3 \times V^{k,l} \to \mathbb{C}^3$ to the trivial V-bundle $\mathfrak{E}^{k,l} \to \mathbb{C}^3/\mathbb{Z}_{q+1}$. Again one needs to choose a representation of $\mathbb{Z}_{q+1}$ on the fibres (11.2). For reasons that will become clear later on (see Sec. 12.1), this time one chooses

$$\gamma(h) = \bigoplus_{(n,m)\in Q_0(k,l)} \mathbb{1}_{p_{(n,m)}} \otimes \zeta_{q+1}^{n} \, \mathbb{1}_{n+1} \ \in \ \mathrm{Centre}\big(\mathfrak{G}^{k,l}\big) \ . \quad (11.8)$$

It is immediate that all elements of $\mathfrak{G}^{k,l}$ commute with the action of $\mathbb{Z}_{q+1}$ given by (11.8), i.e. $\gamma(\mathbb{Z}_{q+1}) \subset \mathrm{Centre}(\mathfrak{G}^{k,l})$. The action of $\mathbb{Z}_{q+1}$ on the coordinates z^α defined in (8.26) induces a representation π of $\mathbb{Z}_{q+1}$ in $\Omega^1(\mathbb{C}^3)$, which on the generator h of $\mathbb{Z}_{q+1}$ is given by

$$\pi(h)(W_\alpha) = \begin{cases} \zeta_{q+1}^{-1} \, W_i \ , & i = 1,2 \\ \zeta_{q+1}^{2} \, W_3 & \end{cases} \quad \text{and} \quad \pi(h)(\overline{W}_\alpha) = \begin{cases} \zeta_{q+1} \, W_i \ , & i = 1,2 \\ \zeta_{q+1}^{-2} \, W_3 & \end{cases} \ . \quad (11.9)$$

Requiring $\mathbb{Z}_{q+1}$-equivariance of the connection $\mathcal{A}$ reduces to conditions similar to (9.35), i.e. the equivariance conditions read as

$$\gamma(h) \, W_\alpha \, \gamma(h)^{-1} = \pi(h)^{-1}(W_\alpha) \quad \text{and} \quad \gamma(h) \, \overline{W}_\alpha \, \gamma(h)^{-1} = \pi(h)^{-1}(\overline{W}_\alpha) \quad (11.10)$$

for $\alpha = 1,2,3$, but this time with different $\mathbb{Z}_{q+1}$-actions γ and π.

11.1.3 Quiver representations

For a decomposition of the endomorphisms

$$\begin{aligned} W_\alpha &= \bigoplus_{(n,m),(n',m')} (W_\alpha)_{(n,m),(n',m')} \ , \\ \text{with} \quad & (W_\alpha)_{(n,m),(n',m')} \in \mathrm{Hom}\big(\mathbb{C}^{p_{(n,m)}} \otimes \underline{(n,m)} \, , \, \mathbb{C}^{p_{(n',m')}} \otimes \underline{(n',m')}\big) \end{aligned} \quad (11.11)$$

as before, the equivariance conditions imply that the allowed non-vanishing components are given by

$$\Phi^i_{(n,m)} \coloneqq (W_i)_{(n,m),(n',m')} \qquad \text{for} \quad n' - n = 1 \pmod{q+1} \ , \quad (11.12a)$$

$$\Psi_{(n,m)} \coloneqq (W_3)_{(n,m),(n',m')} \qquad \text{for} \quad n' - n = -2 \pmod{q+1} \ , \quad (11.12b)$$

for $i = 1,2$, together with the analogous conjugate decomposition for $\overline{W}_\alpha$; in each instance m' is implicitly determined by n and m via the requirement $(n',m') \in Q_0(k,l)$. The structure of these endomorphisms thus determines a representation of another quiver $\mathfrak{Q}^{k,l}$ with the same vertex set $Q_0(k,l)$ as before for the quiver $\mathcal{Q}^{k,l}$, but with a new arrow set consisting of allowed components $(n,m) \to (n',m')$.

11.2 Generalised instanton equations

Similar to Ch. 4, the Hermitian Yang-Mills equations on the complex 3-space $\mathbb{C}^3/\mathbb{Z}_{q+1}$ can be regarded in terms of holomorphicity and stability-like conditions.

11.2.1 Quiver relations

The condition that the connection $\mathcal{A}$ defines an integrable holomorphic structure on the bundle (11.1) is, as before, equivalent to the vanishing of the $(2,0)$- and $(0,2)$-parts of the curvature $\mathcal{F}$, i.e. $\mathcal{F}^{0,2} = 0 = \mathcal{F}^{2,0}$, which in the present case is equivalent to

$$[W_\alpha, W_\beta] = 0 \quad \text{and} \quad [\overline{W}_\alpha, \overline{W}_\beta] = 0 \ . \tag{11.13}$$

The general solutions (11.12) to the equivariance conditions allow for a decomposition of the generalised instanton equations (11.13) into components given by

$$(W_1)_{(n,m),(n+1,m')}\,(W_2)_{(n-1,m''),(n,m)} = (W_2)_{(n,m),(n+1,m')}\,(W_1)_{(n-1,m''),(n,m)} \ , \tag{11.14a}$$

$$(W_i)_{(n,m),(n+1,m')}\,(W_3)_{(n+2,m''),(n,m)} = 0 = (W_3)_{(n,m),(n-2,m')}\,(W_i)_{(n-1,m''),(n,m)} \ , \tag{11.14b}$$

for $(n,m) \in Q_0(k,l)$ and $i = 1,2$, together with their conjugate equations. Note that in (11.14a) both combinations are morphisms between the same representation spaces and hence the commutation relation $[W_1, W_2] = 0$ requires only that their difference vanish. On the other hand, in (11.14b) the two terms are morphisms between different spaces and so the relation $[W_i, W_3] = 0$ implies that they each vanish individually; in particular, in the generic case the solution has $W_3 = 0$.

11.2.2 Stability conditions

For invariant connections there is a peculiarity involved in formulating the moment map condition, see for example [166]. Recalling Ch. 4, the stability-like condition equals $\omega \lrcorner \mathcal{F} \in \mathrm{Centre}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of the structure group. For generic connections the centre of $\mathfrak{g}$ is trivial and the usual condition $\omega \lrcorner \mathcal{F} = 0$ follows. However, for invariant connections the structure group is smaller and the centre can be non-trivial. This implies that there are *several* moduli spaces of translationally-invariant (and $\mathbb{Z}_{q+1}$-equivariant) instantons depending on a choice of element in $\mathrm{Centre}(\mathfrak{g})$.

In this case one can use any gauge-invariant element

$$\Xi := \bigoplus_{(n,m)\in Q_0(k,l)} \mathbb{1}_{p_{(n,m)}} \otimes \mathrm{i}\,\xi_{(n,m)}\,\mathbb{1}_{n+1} \ \in \ \mathrm{Centre}\big(\mathfrak{g}^{k,l}\big) \tag{11.15}$$

from the centre of the Lie algebra

$$\mathfrak{g}^{k,l} := \bigoplus_{(n,m)\in Q_0(k,l)} \mathfrak{u}\big(p_{(n,m)}\,(n+1)\big) \ , \tag{11.16}$$

where $\xi_{(n,m)} \in \mathbb{R}$ are called *Fayet-Iliopoulos parameters*. Thus the remaining instanton equations $\omega_{\mathbb{C}^3} \lrcorner \mathcal{F} = -\mathrm{i}\,\Xi$ read

$$[W_1, \overline{W}_1] + [W_2, \overline{W}_2] + [W_3, \overline{W}_3] = -\mathrm{i}\,\Xi \ . \tag{11.17}$$

By substituting the general solutions (11.12a) and (11.12b) to the equivariance conditions we can decompose the generalised instanton equation (11.17) explicitly into component equations

$$\sum_{i=1}^{2}\Big((W_i)_{(n,m),(n+1,m')}\,(\overline{W}_i)_{(n+1,m'),(n,m)} - (\overline{W}_i)_{(n,m),(n-1,m')}\,(W_i)_{(n-1,m'),(n,m)}\Big)$$
$$+\,(W_3)_{(n,m),(n-2,m')}\,(\overline{W}_3)_{(n-2,m'),(n,m)} - (\overline{W}_3)_{(n,m),(n+2,m')}\,(W_3)_{(n+2,m'),(n,m)}$$
$$= \mathbb{1}_{p_{(n,m)}} \otimes \mathbb{1}_{n+1}\,\xi_{(n,m)} \tag{11.18}$$

for $(n,m) \in Q_0(k,l)$.

11.3 Examples

We shall now elucidate this general construction for the three examples $\underline{C}^{1,0}$, $\underline{C}^{2,0}$ and $\underline{C}^{1,1}$. In each case we highlight the non-vanishing components of the matrices W_α and the representation (11.8).

11.3.1 $C^{1,0}$-quiver

The decomposition of the fundamental representation $\underline{C}^{1,0}$ into irreducible SU(2)-representations is given by (B.24). The non-vanishing components can be read off to be $(W_i)_{(0,-2),(1,1)}$ and their adjoints $(\overline{W}_i)_{(1,1),(0,-2)}$. Thus, there are two complex Higgs fields

$$\Phi_i \coloneqq (W_i)_{(0,-2),(1,1)} \qquad \text{for} \quad i=1,2\ , \tag{11.19}$$

which determine a representation of the 2-Kronecker quiver

$$\begin{tikzcd} \boxed{(0,-2)} \arrow[r, shift left, "\Phi_1"] \arrow[r, shift right, "\Phi_2"'] & \boxed{(1,1)} \end{tikzcd} \tag{11.20}$$

By (11.8) the representation of the generator h is given by

$$\gamma : h \longmapsto \begin{pmatrix} \mathbb{1}_{p_{(1,1)}} \otimes \mathbb{1}_2\ \zeta_{q+1} & 0 \\ 0 & \mathbb{1}_{p_{(0,-2)}} \otimes 1 \end{pmatrix}\ . \tag{11.21}$$

Quiver relations The mutual commutativity of the matrices W_α is trivial in this case, and thus there are no quiver relations among the arrows of (11.20).

Stability conditions Choosing Fayet-Iliopoulos parameters $\xi_0, \xi_1 \in \mathbb{R}$, the requirement of a stable quiver bundle yields non-holomorphic matrix equations given by

$$\Phi_1\,\Phi_1^\dagger + \Phi_2\,\Phi_2^\dagger = \mathbb{1}_{p_{(1,1)}} \otimes \xi_0 \quad \text{and} \quad \Phi_1^\dagger\,\Phi_1 + \Phi_2^\dagger\,\Phi_2 = \mathbb{1}_{p_{(0,-2)}} \otimes \mathbb{1}_2\ \xi_1\ . \tag{11.22}$$

11.3.2 $C^{2,0}$-quiver

The representation $\underline{C}^{2,0}$ is decomposed according to (B.26). The non-vanishing components can be determined as before to be $(W_i)_{(0,-4),(1,-1)}$, $(W_i)_{(1,-1),(2,2)}$ and $(W_3)_{(2,2),(0,-4)}$, together with their adjoints $(\overline{W}_i)_{(1,-1),(0,-4)}$, $(\overline{W}_i)_{(2,2),(1,-1)}$ and $(\overline{W}_3)_{(0,-4),(2,2)}$. Thus, there are five complex Higgs fields

$$\Phi_i \coloneqq (W_i)_{(0,-4),(1,-1)}\ , \quad \Phi_{i+2} \coloneqq (W_i)_{(1,-1),(2,2)} \quad \text{and} \quad \Psi \coloneqq (W_3)_{(2,2),(0,-4)}\ , \tag{11.23a}$$

for $i = 1, 2$, which can be encoded in a representation of the quiver

$$
\begin{tikzcd}
\boxed{(0,-4)} \arrow[r, shift left, "\Phi_1"] \arrow[r, shift right, "\Phi_2"'] & \boxed{(1,-1)} \arrow[r, shift left, "\Phi_3"] \arrow[r, shift right, "\Phi_4"'] & \boxed{(2,2)} \arrow[ll, bend right, "\Psi"']
\end{tikzcd}
\tag{11.24}
$$

Lastly, the representation (11.8) for this example is

$$
\gamma : h \longmapsto \begin{pmatrix} \mathbb{1}_{p_{(2,2)}} \otimes \mathbb{1}_3 \ \zeta_{q+1}^2 & 0 & 0 \\ 0 & \mathbb{1}_{p_{(1,-1)}} \otimes \mathbb{1}_2 \ \zeta_{q+1} & 0 \\ 0 & 0 & \mathbb{1}_{p_{(0,-4)}} \otimes 1 \end{pmatrix} . \tag{11.25}
$$

Quiver relations The holomorphicity condition yields

$$
\Phi_i \, \Psi = 0 \ , \quad \Psi \, \Phi_{i+2} = 0 \quad \text{and} \quad \Phi_3 \, \Phi_2 = \Phi_4 \, \Phi_1 \tag{11.26}
$$

for $i = 1, 2$, plus the conjugate equations. The first two sets of quiver relations of (11.26) each describe the vanishing of a path of the quiver (11.24); an obvious trivial solution of these equations is $\Psi = 0$. The last relation expresses equality of two paths with source vertex $(0, -4)$ and target vertex $(2, 2)$.

Stability conditions Choosing Fayet-Iliopoulos parameters $\xi_0, \xi_1, \xi_2 \in \mathbb{R}$, the stability conditions yield

$$
\Phi_1^\dagger \, \Phi_1 + \Phi_2^\dagger \, \Phi_2 - \Psi \, \Psi^\dagger = \mathbb{1}_{p_{(0,-4)}} \otimes \xi_0 \ , \tag{11.27a}
$$

$$
\Phi_1 \, \Phi_1^\dagger + \Phi_2 \, \Phi_2^\dagger - \Phi_3^\dagger \, \Phi_3 - \Phi_4^\dagger \, \Phi_4 = \mathbb{1}_{p_{(1,-1)}} \otimes \mathbb{1}_2 \ \xi_1 \ , \tag{11.27b}
$$

$$
\Phi_3 \, \Phi_3^\dagger + \Phi_4 \, \Phi_4^\dagger - \Psi^\dagger \, \Psi = \mathbb{1}_{p_{(2,2)}} \otimes \mathbb{1}_3 \ \xi_2 \ . \tag{11.27c}
$$

11.3.3 $C^{1,1}$-quiver

The decomposition of the adjoint representation $\underline{C}^{1,1}$ is given by (B.28). The non-vanishing components are $(W_i)_{(0,0),(1,3)}$, $(W_i)_{(0,0),(1,-3)}$, $(W_i)_{(1,3),(2,0)}$, $(W_i)_{(1,-3),(2,0)}$ and $(W_3)_{(2,0),(0,0)}$, together with their adjoint maps $(\overline{W}_i)_{(1,3),(0,0)}$, $(\overline{W}_i)_{(1,-3),(0,0)}$, $(\overline{W}_i)_{(2,0),(1,3)}$, $(\overline{W}_i)_{(2,0),(1,-3)}$ and $(\overline{W}_3)_{(0,0),(2,0)}$. Thus there are nine complex Higgs fields

$$
\Phi_i^\pm \coloneqq (W_i)_{(0,0),(1,\pm 3)} \ , \quad \Phi_{i+2}^\pm \coloneqq (W_i)_{(1,\pm 3),(2,0)} \quad \text{and} \quad \Psi \coloneqq (W_3)_{(2,0),(0,0)} \ , \tag{11.28}
$$

for $i = 1, 2$, which can be assembled into a representation of the quiver

$$
\begin{tikzcd}
 & \boxed{(1,+3)} \arrow[dr, shift left] \arrow[dr, shift right, "\Phi_3^+ , \Phi_4^+"] & \\
\boxed{(0,0)} \arrow[ur, shift left, "\Phi_1^+ , \Phi_2^+"] \arrow[ur, shift right] \arrow[dr, shift left] \arrow[dr, shift right, "\Phi_1^- , \Phi_2^-"'] & & \boxed{(2,0)} \arrow[ll, "\Psi"'] \\
 & \boxed{(1,-3)} \arrow[ur, shift left] \arrow[ur, shift right, "\Phi_3^- , \Phi_4^-"'] &
\end{tikzcd}
\tag{11.29}
$$

In this example the generator h of $\mathbb{Z}_{q+1}$ has the representation

$$\gamma : h \longmapsto \begin{pmatrix} \mathbb{1}_{p_{(1,3)}} \otimes \mathbb{1}_2 \ \zeta_{q+1} & 0 & 0 & 0 \\ 0 & \mathbb{1}_{p_{(0,0)}} \otimes 1 & 0 & 0 \\ 0 & 0 & \mathbb{1}_{p_{(2,0)}} \otimes \mathbb{1}_3 \ \zeta_{q+1}^2 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{p_{(1,-3)}} \otimes \mathbb{1}_2 \ \zeta_{q+1} \end{pmatrix} . \tag{11.30}$$

Quiver relations In this case the holomorphicity condition yields the relations

$$\Phi_i^{\pm}\,\Psi = 0 \ , \quad \Psi\,\Phi_{i+2}^{\pm} = 0 \quad \text{and} \quad \Phi_3^{+}\,\Phi_2^{+} + \Phi_3^{-}\,\Phi_2^{-} = \Phi_4^{+}\,\Phi_1^{+} + \Phi_4^{-}\,\Phi_1^{-} \tag{11.31}$$

for $i = 1, 2$. Again the first two sets of relations of (11.31) each describe the vanishing of a path in the associated quiver (11.29) (with the obvious trivial solution $\Psi = 0$), while the last relation equates two sums of paths.

Stability conditions Introducing Fayet-Iliopoulos parameters $\xi_1^{\pm}, \xi_2, \xi_3 \in \mathbb{R}$, from the stability conditions one obtains

$$(\Phi_1^{+})^{\dagger}\,\Phi_1^{+} + (\Phi_2^{+})^{\dagger}\,\Phi_2^{+} + (\Phi_1^{-})^{\dagger}\,\Phi_1^{-} + (\Phi_2^{-})^{\dagger}\,\Phi_2^{-} - \Psi\,\Psi^{\dagger} = \mathbb{1}_{p_{(0,0)}} \otimes \xi_0 \ , \tag{11.32a}$$

$$\Phi_1^{\pm}\,(\Phi_1^{\pm})^{\dagger} + \Phi_2^{\pm}\,(\Phi_2^{\pm})^{\dagger} - (\Phi_3^{\pm})^{\dagger}\,\Phi_3^{\pm} - (\Phi_4^{\pm})^{\dagger}\,\Phi_4^{\pm} = \mathbb{1}_{p_{(1,\pm 3)}} \otimes \mathbb{1}_2\ \xi_1^{\pm} \ , \tag{11.32b}$$

$$\Phi_3^{+}\,(\Phi_3^{+})^{\dagger} + \Phi_4^{+}\,(\Phi_4^{+})^{\dagger} + \Phi_3^{-}\,(\Phi_3^{-})^{\dagger} + \Phi_4^{-}\,(\Phi_4^{-})^{\dagger} - \Psi^{\dagger}\,\Psi = \mathbb{1}_{p_{(2,0)}} \otimes \mathbb{1}_3\ \xi_2 \ . \tag{11.32c}$$

12 Quiver gauge theories on Calabi-Yau 3-orbifolds: a comparison

In Ch. 10 and 11 we defined Higgs branch moduli spaces of vacua of two distinct quiver gauge theories on the Calabi-Yau cone over the orbifold $S^5/\mathbb{Z}_{q+1}$. In this chapter we shall explore their constructions in more detail, and describe their similarities and differences.

12.1 Quiver bundles

Having constructed two classes of quiver gauge theories on the conical Calabi-Yau orbifold $C(S^5/\mathbb{Z}_{q+1})$, we now compare the arising quiver graphs and subsequently track down the reason for the choices made in the $\mathbb{Z}_{q+1}$-action.

12.1.1 General observations

The examples covered Ch. 10 and 11 exhibit different quiver graphs despite both quiver constructions being based on the same representation space $\underline{C}^{k,l}$ and its decomposition. In the following, we analyse the resulting quiver bundles.

SU(3)**-equivariance** Consider the quiver bundle $\mathcal{E}^{k,l}$ over $\mathbb{R} \times S^5/\mathbb{Z}_{q+1}$ (as a special case of (9.25)). By construction the space of all connections is restricted to those which are both SU(3)-equivariant and $\mathbb{Z}_{q+1}$-equivariant. For holomorphic quiver bundles, one additionally imposes the holomorphicity condition on the allowed connections. The general solution to these constraints (up to gauge equivalence) is given by the ansatz (10.2), where the matrices X_μ satisfy the equivariance conditions (9.30) and (9.35) as well as the quiver relations (10.4). The induced quiver bundles have the following structure:

- A single homomorphism (arrow) $\phi^{\pm}_{(n,m)}$ between two Hermitian bundles (vertices) $E_{p_{(n,m)}}$ and $E_{p_{(n',m')}}$ if $n - n' = \pm 1$ and $m - m' = \pm 3$.
- An endomorphism (vertex loop) $\psi_{(n,m)}$ at each Hermitian bundle (vertex) $E_{p_{(n,m)}}$ with non-trivial monopole charge $\frac{m}{2}$.

The reason why there is precisely one arrow between any two adjacent vertices is SU(3)-equivariance, which forces the horizontal component matrices X_a for $a = 1, 2, 3, 4$ to have exactly the same Higgs fields $\phi^{\pm}_{(n,m)}$, i.e. SU(3)-equivariance intertwines the horizontal components. The vertical component X_5 can be chosen independently as it originates from the Hopf fibration $S^5 \to \mathbb{C}P^2$. No further constraints arise from $\mathbb{Z}_{q+1}$-equivariance because we embed $\mathbb{Z}_{q+1} \hookrightarrow \mathrm{U}(1) \subset \mathrm{SU}(3)$. These quivers are a simple extension of the quivers obtained by [155, 156] from dimensional reduction over $\mathbb{C}P^2$, because the additional vertical components only contribute loops on vertices with $m \neq 0$. This structure is reminiscent of that of the quivers of [147] which arise from reduction over 3-dimensional Sasaki-Einstein manifolds.

We recall from Sec. 4.2 that the HYM equations are the intersection of the holomorphicity condition (10.4) and the stability-like condition (10.5). From the field theory point of view, they give rise to holomorphic F-term and non-holomorohic D-term constraints. Together they combine into Nahm-type equations considered in Sec. 4.3 and we will come back to this point in Sec. 12.2.1.

$\mathbb{C}^3$-invariance Consider the V-bundle $\mathfrak{C}^{k,l}$ over $\mathbb{C}^3/\mathbb{Z}_{q+1}$ from (11.1). Recall that $C(S^5) \cong \mathbb{C}^3$. In contrast to the former case, we now impose invariance under the translation group $\mathbb{C}^3$ acting on the base as well as $\mathbb{Z}_{q+1}$-equivariance. We additionally require that these invariant connections induce a holomorphic structure as previously. The general solution to these constraints is given by the ansatz (11.5), where the matrices W_α are constant along the base by (11.6), they commute with each other, and they solve the $\mathbb{Z}_{q+1}$-equivariance conditions (11.12). The induced quiver representations have the following characteristic structure:

- Two homomorphisms (arrows) $\Phi^i_{(n,m)}$ $(i = 1, 2)$ between each pair of $\mathbb{Z}_{q+1}$-representations (vertices) $\mathbb{C}^{p_{(n,m)}} \otimes \underline{(n,m)}$ and $\mathbb{C}^{p_{(n',m')}} \otimes \underline{(n',m')}$ if $n - n' = \pm 1$ in $\mathbb{Z}_{q+1}$.
- One homomorphism (arrow) $\Psi_{(n,m)}$ between each pair of $\mathbb{Z}_{q+1}$-representations (vertices) $\mathbb{C}^{p_{(n,m)}} \otimes \underline{(n,m)}$ and $\mathbb{C}^{p_{(n',m')}} \otimes \underline{(n',m')}$ if $n - n' = \pm 2$ in $\mathbb{Z}_{q+1}$.

The reason why there are exactly two arrows between adjacent vertices is that the chosen representation (11.8) does not intertwine W_1, W_2 and acts in the same way on both of them. Thus, both endomorphisms have the same allowed non-vanishing components independently of one another, which gives rise to two independent sets of Higgs fields. The next novelty, compared to the former case, is the additional arrow associated to W_3; its existence is again due to the chosen $\mathbb{Z}_{q+1}$-action. Translational invariance plus $\mathbb{Z}_{q+1}$-equivariance are (in some sense) weaker constraints than SU(3)-equivariance plus $\mathbb{Z}_{q+1}$-equivariance, and consequently the allowed number of Higgs fields is larger. On the other hand, holomorphicity seems to impose the constraint $W_3 = 0$ for generic non-trivial endomorphisms W_1 and W_2 as discussed in Ch. 11. Hence, there are two arrows between adjacent vertices, i.e. with $n - n' = \pm 1$, but no vertex loops as in the former case.

It follows that the generalised instanton equations (11.13) and (11.17) give rise to non-linear matrix equations similar to those considered in [166] for moduli spaces of Hermitian Yang-Mills-type generalised instantons and in [147] for instantons on cones over 3-dimensional Sasaki-Einstein orbifolds. We will analyse these equations further in Sec. 12.2.2.

12.1.2 Fibrewise $\mathbb{Z}_{q+1}$-actions

Next, we explain the origin of the difference between the choices of $\mathbb{Z}_{q+1}$-representations (9.33) and (11.8). Consider the generic linear $\mathbb{Z}_{q+1}$-action on $\mathbb{C}^3$: Denoting by h the generator of the cyclic group $\mathbb{Z}_{q+1}$, and choosing $(\theta^\alpha) = (\theta^1, \theta^2, \theta^3) \in \mathbb{Z}^3$ and $(z^\alpha) = (z^1, z^2, z^3) \in \mathbb{C}^3$, one has

$$h \cdot (z^\alpha) = \left(h^\alpha{}_\beta \, z^\beta\right) \qquad \text{with} \quad \left(h^\alpha{}_\beta\right) = \begin{pmatrix} \zeta_{q+1}^{\theta^1} & 0 & 0 \\ 0 & \zeta_{q+1}^{\theta^2} & 0 \\ 0 & 0 & \zeta_{q+1}^{\theta^3} \end{pmatrix} . \tag{12.1}$$

This defines an embedding of $\mathbb{Z}_{q+1}$ into SU(3) if and only if $\theta^1 + \theta^2 + \theta^3 = 0 \bmod q + 1$.

However, we also have to account for the representation γ of $\mathbb{Z}_{q+1}$ in the fibres of the bundles (9.25) and (11.1). These bundles are explicitly constructed from SU(3)-representations $\underline{C}^{k,l}$ which decompose under $\mathrm{SU}(2) \times \mathrm{U}(1)$ into a sum of irreducible representations $\underline{(n,m)}$ from (9.9). If $\underline{(n,m)}$ and $\underline{(n',m')}$ both appear in the decomposition (9.9), then there exists $(r, s) \in \mathbb{Z}^2_{\geq 0}$ such that $n - n' = \pm r$ and $m - m' = \pm 3s$.

SU(3)-equivariance The 1-forms β^i_{q+1} transform under the generic $\mathbb{Z}_{q+1}$-action (12.1) as

$$\beta^i_{q+1} \longmapsto \zeta_{q+1}^{\theta^i-\theta^3}\, \beta^i_{q+1} \qquad \text{for} \quad i=1,2\ , \tag{12.2}$$

while η and $\mathrm{d}\tau$ are invariant. Thus, the equivariance condition for the connection (9.28) becomes

$$\gamma(h)\ (X_{2i-1}-\mathrm{i}\,X_{2i})\ \gamma(h)^{-1} = \zeta_{q+1}^{-\theta^i+\theta^3}\ (X_{2i-1}-\mathrm{i}\,X_{2i}) \qquad \text{for} \quad i=1,2\ , \tag{12.3a}$$

$$\gamma(h)\ (X_{2i-1}+\mathrm{i}\,X_{2i})\ \gamma(h)^{-1} = \zeta_{q+1}^{\theta^i-\theta^3}\ (X_{2i-1}+\mathrm{i}\,X_{2i}) \qquad \text{for} \quad i=1,2\ , \tag{12.3b}$$

$$\gamma(h)\, X_5\, \gamma(h)^{-1} = X_5\ . \tag{12.3c}$$

In this case the aim is to embed $\mathbb{Z}_{q+1}$ in such a way that the entire quiver decomposition (9.25) is automatically $\mathbb{Z}_{q+1}$-equivariant; hence the non-vanishing components of the matrices X_a and X_5 are already prescribed by SU(3)-equivariance. For generic θ^α it seems quite difficult to realise this embedding, because if one assumes a diagonal $\mathbb{Z}_{q+1}$-action on the fibre of the form

$$\gamma(h) = \bigoplus_{(n,m)\in Q_0(k,l)} \mathbb{1}_{p_{(n,m)}} \otimes \zeta_{q+1}^{\gamma(n,m)}\, \mathbb{1}_{n+1} \qquad \text{with} \quad \gamma(n,m)\in\mathbb{Z}\ , \tag{12.4}$$

then these equivariance conditions translate into

$$\gamma(n\pm1,m+3)-\gamma(n,m) = \theta^i-\theta^3 \bmod q+1 \qquad \text{for} \quad i=1,2 \tag{12.5}$$

on the non-vanishing components of X_a, $a=1,2,3,4$.

In this thesis we specialise to the weights $(\theta^\alpha)=(1,1,-2)$ and obtain (8.32) for the $\mathbb{Z}_{q+1}$-action on SU(3)-equivariant 1-forms. From this action we naturally obtain factors $\zeta_{q+1}^{\pm3}$ for the induced representation $\pi(h)$. This justifies the choice of γ in (9.33), as m changes by integer multiples of 3 while n in (12.5) does not have such uniform behaviour.

$\mathbb{C}^3$-invariance The modified equivariance condition under (12.1) is readily read off to be

$$\gamma(h)\, W_\alpha\, \gamma(h)^{-1} = \zeta_{q+1}^{\theta^\alpha}\, W_\alpha \qquad \text{for} \quad \alpha=1,2,3\ . \tag{12.6}$$

In contrast to the SU(3)-equivariant case above, no particular form of the matrices W_α is fixed yet, i.e. here the choice of realisation of the $\mathbb{Z}_{q+1}$-action on the fibres determines the field content. By the same argument as above, a representation of $\mathbb{Z}_{q+1}$ on the fibres of the form (12.4) allows the component $(W_\alpha)_{(n,m),(n',m')}$ to be non-trivial if and only if

$$\gamma(n',m')-\gamma(n,m) = \theta^\alpha \bmod q+1 \qquad \text{for} \quad \alpha=1,2,3\ . \tag{12.7}$$

For the weights $(\theta^\alpha)=(1,1,-2)$ we then pick up factors of $\zeta_{q+1}^{\pm1}$ or $\zeta_{q+1}^{\pm2}$, which excludes the choice (9.33). However, the modification to (11.8) is allowed as n changes in integer increments.

McKay quiver In [147, 167] the correspondence between the HYM moduli space for translationally-invariant and $\mathbb{Z}_{q+1}$-equivariant connections and the representation moduli of the McKay quiver is employed. The McKay quiver associated to the orbifold singularity $\mathbb{C}^3/\mathbb{Z}_{q+1}$ and the weights $(\theta^\alpha)=(1,1,-2)$ is constructed in exactly the same way as the $C^{k,l}$-quivers from Sec. 11, except that it is based on the regular representation of $\mathbb{Z}_{q+1}$ rather than the representations $\underline{C}^{k,l}$ considered here. It is a cyclic quiver with $q+1$ vertices labelled by the irreducible representations of $\mathbb{Z}_{q+1}$, whose underlying graph is the affine extended Dynkin diagram of type $\widehat{A}_q$, and whose arrow set coincides with those of the $C^{k,l}$-quivers. See [173–175] for explicit constructions of instanton moduli on $\mathbb{C}^3/\mathbb{Z}_{q+1}$ in this context. Consequently, our considerations do not enjoy a straightforward comparison to the McKay quiver, which is the reason for the doted arrow labelled *not quite* in Fig. 7.2.

12.2 Moduli spaces

We shall now formalise the treatment of the instanton moduli spaces. Since both scenarios are subsets of HYM-instantons on Calabi-Yau cones, the expositions of Sec. 4.2 apply. Moreover, the SU(3)-equivariant instantons are really a special case of Sec. 4.3; thus, we will only remark the specialisations, but do not need to go over the details again. However, for the translationally-invariant instantons we will discuss the geometric properties in more detail.

12.2.1 SU(3)-equivariance

Consider the space of SU(3)-equivariant connections $\mathbb{A}(\mathcal{E}^{k,l})$ on the bundle (9.25) (for $d=1$), which is an affine space modelled on $\Omega^1\big(C(S^5/\mathbb{Z}_{q+1}),\mathrm{End}_{\mathrm{U}(1)}(V^{k,l})\big)$. The structure group $\mathcal{G}^{k,l}$ of the bundle (9.25) is given by (9.24). An element $X\in\Omega^1\big(C(S^5/\mathbb{Z}_{q+1}),\mathrm{End}_{\mathrm{U}(1)}(V^{k,l})\big)$ can be expressed as

$$X = X_\mu\, e^\mu + X_6\,\mathrm{d}t \equiv Y_j\,\theta^j + Y_{\bar{j}}\,\bar{\theta}^j\ , \tag{12.8}$$

once one has chosen the coframe $\{e^\mu,\mathrm{d}t\}$ of the conformally equivalent cylinder $\mathbb{R}\times S^5/\mathbb{Z}_{q+1}$ with $r=\exp t$. The complexified 1-forms $\{\theta^j,\bar{\theta}^j\}$ are defined in (3.6).

Let us now summarise the main ingredients, which follow by direct application of Sec. 4.3. After transition to the complexified equations (4.32) and rescaling (4.42), we obtain the holomorphicity conditions

$$[\mathcal{Y}_1,\mathcal{Y}_2]=0 \quad\text{and}\quad \frac{\mathrm{d}\mathcal{Y}_j}{\mathrm{d}s} = 2\,[\mathcal{Y}_j,\mathcal{Z}] \qquad\text{for}\quad j=1,2\ , \tag{12.9a}$$

and the stability-like condition

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{Z}+\mathcal{Z}^\dagger\Big) + 2\,[\mathcal{Z},\mathcal{Z}^\dagger] + 2\,\lambda_2(s)\sum_{j=1}^{2}[\mathcal{Y}_j,\mathcal{Y}_j^\dagger] = 0\ . \tag{12.9b}$$

As before, the matrix-valued functions are subject to boundary conditions (4.50). In addition, the real gauge group and its complexification are

$$\widehat{\mathcal{G}}^{\,k,l} := \Omega^0(\mathbb{R}_{>0},\mathcal{G}^{k,l}) \quad\text{and}\quad \big(\widehat{\mathcal{G}}^{\,k,l}\big)^{\mathbb{C}} := \Omega^0(\mathbb{R}_{>0},(\mathcal{G}^{k,l})^{\mathbb{C}})\ . \tag{12.10}$$

The holomorphicity conditions (12.9a) do define the Kähler space $\mathbb{A}^{1,1}(\mathcal{E}^{k,l})$, on which in turn equation (12.9b) can be interpreted as moment map

$$\begin{aligned}\mu:\mathbb{A}^{1,1}(\mathcal{E}^{k,l}) &\longrightarrow \widehat{\mathrm{End}}_{\mathrm{U}(1)}(V^{k,l})\\ (\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Z}) &\longmapsto \frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{Z}+\mathcal{Z}^\dagger\Big) + 2\,[\mathcal{Z},\mathcal{Z}^\dagger] + 2\,\lambda_2(s)\sum_{j=1}^{2}[\mathcal{Y}_j,\mathcal{Y}_j^\dagger]\ ,\end{aligned} \tag{12.11}$$

for the real *framed gauge group*.

Following the discussion from Sec. 4.3, provided we have chosen the quintuple $\{T_\mu\}_{\mu=1,\dots,5}$ as *regular* in the Cartan subalgebra of $\mathrm{End}_{\mathrm{U}(1)}(V^{k,l})$, i.e. the intersection of the centraliser of the T_μ is only the Cartan subalgebra, then the model solution is solely determined by the T_μ (recall the exposition around (4.50)). Further assuming each of the two complex linear combinations

$\mathcal{T}_j = \frac{1}{2}(T_{2j} - \mathrm{i}T_{2j-1})$ for $j = 1, 2$, to be a *regular pair*, i.e. the centraliser of each $\mathcal{T}_j$ in the complexified Lie algebra is the complexified Cartan subalgebra, we have an embedding

$$\begin{aligned} \mathcal{M}_{k,l}^{\mathrm{SU}(3)} &\longrightarrow \mathcal{O}_{\mathrm{diag}}\left(\mathcal{Y}_1(0), \mathcal{Y}_2(0)\right) \\ \left(\mathcal{Y}_1(s), \mathcal{Y}_2(s), \mathcal{Z}(s)\right) &\longmapsto \left(\mathcal{Y}_1(0), \mathcal{Y}_2(0)\right) \end{aligned} \tag{12.12}$$

from the moduli space of solutions, satisfying the boundary conditions (4.50) together with the equivariance condition imposed by our construction, in the *diagonal* orbit $\mathcal{O}_{\mathrm{diag}}\left(\mathcal{Y}_1(0), \mathcal{Y}_2(0)\right)$. The complex dimension of the diagonal orbit can be computed by employing the regularity of the boundary conditions. Similarly to Sec. 4.3.5 we obtain

$$\begin{aligned} \dim_{\mathbb{C}}\left(\mathcal{O}_{\mathrm{diag}}\left(\mathcal{Y}_1(0), \mathcal{Y}_2(0)\right)\right) &= \dim_{\mathbb{R}}\left(\mathcal{G}^{k,l}\right) - \dim_{\mathbb{R}}\left(\mathrm{U}(1)^p\right) \\ &= \sum_{(n,m)\in\mathcal{Q}_0(k,l)} (n+1)p_{(n,m)}\left(p_{(n,m)} - 1\right) \ . \end{aligned} \tag{12.13}$$

As such, the diagonal orbit is a Kähler subspace of the product $\mathcal{O}_{\mathcal{T}_1} \times \mathcal{O}_{\mathcal{T}_2}$ of two regular semi-simple orbits. The latter is naturally a complex symplectic manifold with the product of the standard Kirillov-Kostant-Souriau symplectic forms on the orbits [146].

12.2.2 $\mathbb{C}^3$-invariance

Now we turn our attention to the space of translationally-invariant connections $\mathbb{A}(\mathfrak{E}^{k,l})$ on the bundle (11.1). The structure group $\mathfrak{G}^{k,l}$ of (11.1) (which in this case coincides with the gauge group) is given by (11.3) and its Lie algebra $\mathfrak{g}^{k,l}$ by (11.16). A generic element of the tangent space $T_{\mathcal{A}}\mathbb{A}(\mathfrak{E}^{k,l})$ at a point $\mathcal{A} \in \mathbb{A}(\mathfrak{E}^{k,l})$ is given by

$$W = W_\alpha \, \mathrm{d}z^\alpha + \overline{W}_\alpha \, \mathrm{d}\bar{z}^\alpha \ \in \ \Omega^1\left(\mathbb{C}^3/\mathbb{Z}_{q+1}, \mathfrak{g}^{k,l}\right) \ , \tag{12.14}$$

with constant $W_\alpha, \overline{W}_\alpha$ for $\alpha = 1, 2, 3$. Analogous to Sec. 4.2, let us define a metric $\boldsymbol{g}$ on $\mathbb{A}(\mathfrak{E}^{k,l})$. Gauge transformations of tangent vectors $\boldsymbol{w} = \boldsymbol{w}_\alpha \, \mathrm{d}z^\alpha + \overline{\boldsymbol{w}}_\alpha \, \mathrm{d}\bar{z}^\alpha$ are given by $\overline{\boldsymbol{w}}_\alpha \mapsto \mathrm{Ad}(g)\overline{\boldsymbol{w}}_\alpha$ for $\alpha = 1, 2, 3$. We deduce the metric to be

$$\boldsymbol{g}_{|\mathcal{A}}(\boldsymbol{w}, \boldsymbol{v}) := \frac{1}{2} \sum_{\alpha=1}^{3} \mathrm{tr}\big(\boldsymbol{w}_\alpha^\dagger \, \boldsymbol{v}_\alpha + \boldsymbol{w}_\alpha \, \boldsymbol{v}_\alpha^\dagger\big) \ , \tag{12.15}$$

and a symplectic form via

$$\boldsymbol{\omega}_{|\mathcal{A}}(\boldsymbol{w}, \boldsymbol{v}) := \frac{\mathrm{i}}{2} \sum_{\alpha=1}^{3} \mathrm{tr}\big(\boldsymbol{w}_\alpha^\dagger \, \boldsymbol{v}_\alpha - \boldsymbol{w}_\alpha \, \boldsymbol{v}_\alpha^\dagger\big) \ . \tag{12.16}$$

These definitions follow directly from the translationally-invariant limit of (4.8) and (4.9) (and agree with those of [166]). Evidently the metric and symplectic structure are gauge-invariant.

Next, we define the subspace of invariant connections that satisfy the holomorphicity conditions (11.13) as

$$\mathbb{A}^{1,1}(\mathfrak{E}^{k,l}) = \left\{ \left(\{W_\alpha\}, \{\overline{W}_\alpha\}\right) \in \mathbb{A}(\mathfrak{E}^{k,l}) \,\middle|\, [\overline{W}_\alpha, \overline{W}_\beta] = 0 \qquad \text{for} \quad \alpha, \beta = 1, 2, 3 \right\} \ , \tag{12.17}$$

which is a finite-dimensional Kähler space by the general considerations of Sec. 4.2.

Moment map The corresponding moment map can be introduced as before via

$$\begin{aligned}\mu : \mathbb{A}^{1,1}(\mathfrak{E}^{k,l}) &\longrightarrow \mathfrak{g}^{k,l} \\ (\{W_\alpha\},\{\overline{W}_\alpha\}) &\longmapsto \mathrm{i}\sum_{\alpha=1}^{3}\,[W_\alpha,\overline{W}_\alpha]\ ,\end{aligned} \tag{12.18}$$

but in this case it is possible to choose various gauge-invariant levels Ξ from (11.15) and consequently define different moduli spaces

$$\mathcal{M}^{\mathbb{C}^3}_{k,l}(\Xi) = \mu^{-1}(\Xi)\,/\,\mathfrak{G}^{k,l}\ . \tag{12.19}$$

Gauge group The complete set of instanton equations (11.13) and (11.17) is invariant under the action of the gauge group (11.3) with the usual transformations

$$\overline{W}_\alpha \longmapsto \mathrm{Ad}(g)\overline{W}_\alpha \qquad \text{for} \quad \alpha = 1,2,3 \tag{12.20}$$

for $g \in \mathfrak{G}^{k,l} \hookrightarrow \mathrm{U}(p)$.

In contrast, only the holomorphicity condition (11.13) is invariant under $(\mathfrak{G}^{k,l})^{\mathbb{C}}$ gauge transformations; whereas, the equation (11.17) is not invariant under the action of the complex gauge group.

Stable points The set of stable points is defined, as in Sec. 4.2, to be

$$\mathbb{A}^{1,1}_{\mathrm{st}}(\mathfrak{E}^{k,l};\Xi) := \Big\{(\{W_\alpha\},\{\overline{W}_\alpha\}) \in \mathbb{A}^{1,1}(\mathfrak{E}^{k,l})\ :\ (\mathfrak{G}^{k,l})^{\mathbb{C}}_{(\{W_\alpha\},\{\overline{W}_\alpha\})} \cap \mu^{-1}(\Xi) \neq \emptyset\Big\}\ , \tag{12.21}$$

and by taking the GIT quotient one obtains the Ξ-dependent moduli spaces[25]

$$\mathcal{M}^{\mathbb{C}^3}_{k,l}(\Xi) \cong \mathbb{A}^{1,1}_{\mathrm{st}}(\mathfrak{E}^{k,l};\Xi)\,/\,(\mathfrak{G}^{k,l})^{\mathbb{C}}\ . \tag{12.22}$$

The moment map (12.18) transforms under $g \in (\mathfrak{G}^{k,l})^{\mathbb{C}}$ as

$$\mu(\{W_\alpha\},\{\overline{W}_\alpha\}) = \mathrm{i}\sum_{\alpha=1}^{3}\,[W_\alpha,\overline{W}_\alpha] \longmapsto \mathrm{i}\,\mathrm{Ad}(g)\sum_{\alpha=1}^{3}\,[h^{-1}\,W_\alpha\,h,\overline{W}_\alpha]\ , \tag{12.23}$$

where we introduced $h = h(g) = g^\dagger\, g \in (\mathfrak{G}^{k,l})^{\mathbb{C}}/\mathfrak{G}^{k,l}$. Thus, h is a positive Hermitian $p{\times}p$ matrix. Moreover, $\mathrm{Ad}(g'\,)\Xi = \Xi$ for any $g' \in \mathfrak{G}^{k,l}$. By the embedding $\mathfrak{G}^{k,l} \hookrightarrow \mathrm{U}(p)$ and the polar decomposition of an element $g \in (\mathfrak{G}^{k,l})^{\mathbb{C}}$ into $g = h'\ \exp(\mathrm{i}\,X)$ for Hermitian $h' \in \mathfrak{G}^{k,l}$ and skew-adjoint $X \in \mathfrak{g}^{k,l}$, we have

$$\mathrm{Ad}(g)\Xi = \mathrm{Ad}(h'\,)\big(\mathrm{Ad}(\exp{(\mathrm{i}\,X)})\Xi\big) = \mathrm{Ad}(h'\,)\big(\Xi + \mathrm{i}\,[X,\Xi]\,\big) = \mathrm{Ad}(h'\,)\Xi = \Xi\ , \tag{12.24}$$

where we used the Baker-Campbell-Hausdorff formula and the fact that Ξ is central in $\mathfrak{g}^{k,l}$. It follows that $\mathrm{Centre}(\mathfrak{g}^{k,l}) \subset \mathrm{Centre}\big((\mathfrak{g}^{k,l})^{\mathbb{C}}\big)$. Hence, a point $(\{W_\alpha\},\{\overline{W}_\alpha\}) \in \mathbb{A}^{1,1}(\mathfrak{E}^{k,l})$ is stable if and only if there exists a positive Hermitian matrix h (modulo unitary transformations) satisfying the equation

$$\sum_{\alpha=1}^{3}\,[h^{-1}\,W_\alpha\,h,\overline{W}_\alpha] = -\mathrm{i}\,\Xi\ . \tag{12.25}$$

[25] This description is analogous to the quiver GIT quotients used by [167,173] to describe instanton moduli on $\mathbb{C}^3/\mathbb{Z}_{q+1}$ as representation moduli of the McKay quiver.

By our general constructions the moduli spaces $\mathcal{M}^{\mathbb{C}^3}_{k,l}(\Xi)$ are Kähler spaces; however, there is no reason to expect them to be smooth manifolds. Generally, comparing to [166], the canonical map $\mathcal{M}^{\mathbb{C}^3}_{k,l}(\Xi) \to \mathcal{M}^{\mathbb{C}^3}_{k,l}(0)$ is then a partial resolution of singularities for generic Ξ.

13 Conclusions and outlook

In the second part of this thesis, we explored the Sasakian quiver gauge theories associated to the Sasaki-Einstein orbifold $S^5/\mathbb{Z}_{q+1}$.

The construction of the quiver bundle, based on [54], has been extensively discussed in Ch. 9. The resulting quiver graphs differ from the quivers associated to the homogeneous Kähler space $\mathrm{SU}(3)/S(\mathrm{U}(2)\times\mathrm{U}(1))$ of [155] in the additional endomorphisms $\psi_{(n,m)}$, which originate from the U(1)-fibration $S^5 \to \mathbb{C}P^2$. The resulting quiver gauge theories are new and have not been discussed in the literature before. By dimensional reduction over S^5, we provided the explicit expressions for the Yang-Mills-Higgs theory on M^d, and discussed the representation-theoretic implications of the Higgs-branch.

We then first considered the SU(3)-equivariant instantons over $C(S^5/\mathbb{Z}_{q+1})$ in order to describe the vacua of the Sasakian quiver gauge theories. Employing the results of Ch. 4, we described the Kähler structure of the framed moduli space. Moreover, this space is a finite dimensional Kähler space that is injectively mapped into a *diagonal* orbit, which is a finite-dimensional Kähler subspace of the product of two complex coadjoint orbits, provided the boundary conditions are regular.

For non-regular boundary conditions, we speculated in [149] that the coadjoint orbits have to be replaced by the singular nilpotent cone such that the moduli space might exhibit singularities. However, we have not yet preformed a precise analysis of this case and leave this for future work.

The construction of the quiver bundle provides quivers without relations, but there are two ways to impose such, which we subsequently explored. On the one hand, we imposed the vacuum condition, i.e. vanishing of the scalar potential as in Sec. 9.4; on the other hand, we enforced a generalised instanton condition as Ch. 10. In the limit of constant Higgs fields, the generalised instanton equations correctly reproduced the defining equations of the Higgs branch moduli space. In a different limit, reducing the new Higgs fields, associated to vertex edge loops of the Sasakian quiver, to the identity, transforms the quiver theory consistently to the SU(3)-equivariant quiver gauge theory on $\mathbb{C}P^2$.

Secondly, we considered translationally invariant and $\mathbb{Z}_{q+1}$-equivariant instantons. The constructed quiver theories differ in two important aspects: (i) the entire set-up concerns constant matrices, in contrast to the equivariant scenario; and (ii) the resulting quiver graphs have changed. Moreover, translationally invariant instantons allow to introduce non-trivial FI-parameters, resembling SYM theories on a D-brane world volume. In addition, each instanton moduli space, i.e. each allowed level-set of the moment map modulo gauge group action, is a finite-dimensional Kähler space, for which we explicitly constructed the Kähler structure.

Consequently, we constructed two sets of new quiver gauge theories on the Calabi-Yau orbifold $C(S^5/\mathbb{Z}_{q+1})$, with manifestly distinct quiver diagrams and instanton moduli space descriptions. The distinctions can be traced back to the different invariance conditions imposed. The more surprising is that we suspect both moduli spaces to have the same orbifold singularities, as speculated in [149]. Again, the precise statement has yet to be determined and we hope to report on this in the future.

Lastly, we would like to emphasise that the quiver structure of the Sasakian quiver crucially depends on the coset G/H under consideration, which follows directly from the construction of [54]. In contrast to the 3-dimensional case [147], in which S^3/Γ accounts for all 3-dimensional

Sasaki-Einstein orbifolds, the 5-sphere does not exhaust all Sasaki-Einstein 5-manifolds, not even all cosets. For example, the 5-dimensional Sasaki-Einstein coset $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathrm{U}(1)$, known as T^{11}, has different quiver diagrams and does not allow for a comparison with translational invariant instantons on its cone, which is the conifold. The details of this case can be found in [148].

B Appendix: Sasakian quiver gauge theories

B.1 Bundles on $\mathbb{C}P^2$

B.1.1 Geometry of $\mathbb{C}P^2$

$\mathrm{SU}(3)$-equivariant 1-forms Consider the row vector $\beta^\top = (\beta^1, \beta^2)$. The relations (8.9) and (8.10) dictate the explicit form of the 1-forms β^i and their exterior derivatives as

$$\beta^i = \frac{1}{\gamma}\,\mathrm{d}y^i - \frac{1}{\gamma^2\,(\gamma+1)}\,y^i\sum_{j=1}^{2}\bar{y}^j\,\mathrm{d}y^j\ , \qquad \bar{\beta}^i = \frac{1}{\gamma}\,\mathrm{d}\bar{y}^i - \frac{1}{\gamma^2\,(\gamma+1)}\,\bar{y}^i\sum_{j=1}^{2}y^j\,\mathrm{d}\bar{y}^j\ , \tag{B.1a}$$

$$\mathrm{d}\beta^1 = -\beta^1\wedge\left(B_{11}+\tfrac{3}{2}\,a\right)+\beta^2\wedge\bar{B}_{12}\ , \qquad \mathrm{d}\beta^2 = -\beta^1\wedge B_{12}+\beta^2\wedge\left(B_{11}-\tfrac{3}{2}\,a\right)\ , \tag{B.1b}$$

$$\mathrm{d}\bar{\beta}^1 = -\left(B_{11}+\tfrac{3}{2}\,a\right)\wedge\bar{\beta}^1 - B_{12}\wedge\bar{\beta}^2\ , \qquad \mathrm{d}\bar{\beta}^2 = \bar{B}_{12}\wedge\bar{\beta}^1+\left(B_{11}-\tfrac{3}{2}\,a\right)\wedge\bar{\beta}^2\ . \tag{B.1c}$$

One can regard β^i as a basis for the $(1,0)$-forms and $\bar{\beta}^i$ as a basis for the $(0,1)$-forms of the complex cotangent bundle over the patch $\mathcal{U}_0$ of $\mathbb{C}P^2$ with respect to an almost complex structure J. The canonical 1-forms $\mathrm{d}y^i$ and $\mathrm{d}\bar{y}^i$ could equally well be used for a holomorphic decomposition with respect to J, but the forms β^i and $\bar{\beta}^i$ are $\mathrm{SU}(3)$-equivariant.

Hermitian Yang-Mills equations The canonical Kähler 2-form on the patch $\mathcal{U}_0$ is given by

$$\omega_{\mathbb{C}P^2} = -\mathrm{i}\,R^2\,\beta^\top\wedge\bar{\beta} = \mathrm{i}\,R^2\,\left(\beta^1\wedge\bar{\beta}^1+\beta^2\wedge\bar{\beta}^2\right)\ , \tag{B.2}$$

where R is the radius of the linearly embedded projective line $\mathbb{C}P^1\subset\mathbb{C}P^2$. The 1-form $B_{(1)}$ of (8.11) is then an instanton connection by the following argument: Locally, one can define a $(2,0)$-form Ω proportional to $\beta^1\wedge\beta^2$. The HYM equations for a curvature 2-form F are

$$\Omega\wedge F = 0 \quad\text{and}\quad \omega_{\mathbb{C}P^2}\,\lrcorner\,F = 0\ , \tag{B.3}$$

which translate to $F = F^{1,1}$ being a $(1,1)$-form for which $\mathrm{tr}(F^{1,1}) = 0$. As before, the contraction $\lrcorner$ between two forms η and η' is defined as $\eta\,\lrcorner\,\eta' := \star\,(\eta\wedge\star\,\eta')$. The curvature $F_B = \bar{\beta}\wedge\beta^\top$ is a $(1,1)$-form which is $\mathfrak{u}(2)$-valued, i.e. $\mathrm{tr}(F_B) = 2a \neq 0$. However, $F_a = \beta^\dagger\wedge\beta$ is also a $(1,1)$-form. Thus the curvature of the connection $B_{(1)} = B - \frac{1}{2}\,a\,\mathbb{1}_2$ given by $F_{B_{(1)}} = F_B - \frac{1}{2}\,F_a\,\mathbb{1}_2$ is a $(1,1)$-form and by construction traceless; hence $B_{(1)}$ is an $\mathfrak{su}(2)$-valued connection satisfying the HYM equations, i.e. it is an instanton connection.

B.1.2 Hopf fibration and associated bundles

Consider the principal $\mathrm{U}(1)$-bundle $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)\to\mathbb{C}P^2$. One can associate a complex vector bundle whose fibres carry any representation of the structure group $\mathrm{U}(1)$, i.e. a complex vector space V together with a group homomorphism $\rho:\mathrm{U}(1)\to\mathrm{GL}(V)$. Then the associated vector bundle E is given as $E := S^5\times_\rho V\to\mathbb{C}P^2$. In particular, one can choose $V = \underline{m}$ to be the 1-dimensional irreducible representation of highest weight $m \in \mathbb{Z}$. Following [156], one then generates associated complex line bundles $L_{\frac{m}{2}} := (L^{\otimes m})^{\frac{1}{2}}$.

Chern classes and monopole charges Using the conventions of [156] for $\mathbb{C}P^2$, there is a normalised volume form

$$\beta_{\mathrm{vol}} := \frac{1}{2\pi^2}\,\beta^1 \wedge \bar{\beta}^1 \wedge \beta^2 \wedge \bar{\beta}^2 \qquad \text{with} \quad \int_{\mathbb{C}P^2} \beta_{\mathrm{vol}} = 1 \ , \tag{B.4}$$

and the canonical Kähler 2-form (B.2) with

$$\omega_{\mathbb{C}P^2} \wedge \omega_{\mathbb{C}P^2} = -\big(2\pi\, R^2\big)^2\, \beta_{\mathrm{vol}} \ . \tag{B.5}$$

Consider the connection a from (8.9c) on the line bundle L associated to the Hopf bundle $S^5 \to \mathbb{C}P^2$ and the fundamental representation. Since its curvature is $F_a = \frac{\mathrm{i}}{R^2}\,\omega_{\mathbb{C}P^2}$, the total Chern character of the monopole bundle L is

$$\mathrm{ch}(L) = \exp\big(\tfrac{\mathrm{i}}{2\pi}\,F_a\big) = \exp(\xi) \tag{B.6}$$

where $\xi := -\frac{1}{2\pi\,R^2}\,\omega_{\mathbb{C}P^2}$. Then one immediately reads off the first Chern class

$$c_1(L) = \xi \qquad \text{with} \quad \int_{\mathbb{C}P^2} \xi \wedge \xi = -1 \ . \tag{B.7}$$

Since $[\xi] = [c_1(L)]$ generates $H^2(\mathbb{C}P^2, \mathbb{Z}) \cong \mathbb{Z}$ [155], this identifies the first Chern number of L as -1. Thus $L \equiv L_1$ exists globally, and the dual bundle L_{-1} has first Chern class $c_1(L_{-1}) = -c_1(L)$ and hence first Chern number $+1$. For all other bundles $L_{\frac{m}{2}}$ one takes the connection to be $\frac{m}{2}\,a$, which changes the first Chern class accordingly to

$$c_1\big(L_{\frac{m}{2}}\big) = \tfrac{m}{2}\,\xi \ , \tag{B.8}$$

and the first Chern number to $-\frac{m}{2}$. Hence, only for even values of m do the line bundles $L_{\frac{m}{2}}$ exist globally in the sense of conventional bundles. On the other hand, for odd values of m the line bundles $L_{\frac{m}{2}}$ (and also the instanton bundles I_n for odd values of the isospin n [156]) are examples of *twisted bundles*. The obstruction to the global existence of these bundles is the failure of the cocycle condition for transition functions on triple overlaps of patches, which is given by a non-trivial integral 3-cocycle representing the Dixmier-Douady class of an abelian gerbe; see for example [176] for more details. As argued in [156], the Chern number $\frac{m}{2}$ of the line bundle $L_{-\frac{m}{2}}$ should be taken as the monopole charge rather than the H_{α_2}-eigenvalue m in the Biedenharn basis.

B.2 Representations

B.2.1 Biedenharn basis

Let us summarise the relevant details concerning the Biedenharn basis [170–172], which is defined as the basis of eigenvectors according to (9.8); we follow [155, 156] for the presentation and notation.

Generators The remaining generators of $\mathfrak{su}(3)$ act on this eigenvector basis as

$$E_{\pm\alpha_1}\left|\begin{matrix}n\\q\end{matrix}\,m\right\rangle = \frac{1}{2}\sqrt{(n\mp q)\,(n\pm q+2)}\,\left|\begin{matrix}n\\q\pm 2\end{matrix}\,m\right\rangle\ , \tag{B.9a}$$

$$\begin{aligned} E_{\alpha_2}\left|\begin{matrix}n\\q\end{matrix}\,m\right\rangle &= \sqrt{\tfrac{n-q-2}{2(n+1)}}\,\Lambda^+_{k,l}(n,m)\,\left|\begin{matrix}n+1\\q-1\end{matrix}\,m+3\right\rangle \\ &\quad + \sqrt{\tfrac{n+q}{2(n+1)}}\,\Lambda^-_{k,l}(n,m)\,\left|\begin{matrix}n-1\\q-1\end{matrix}\,m+3\right\rangle\ , \end{aligned} \tag{B.9b}$$

$$\begin{aligned} E_{\alpha_1+\alpha_2}\left|\begin{matrix}n\\q\end{matrix}\,m\right\rangle &= \sqrt{\tfrac{n+q+2}{2(n+1)}}\,\Lambda^+_{k,l}(n,m)\,\left|\begin{matrix}n+1\\q+1\end{matrix}\,m+3\right\rangle \\ &\quad + \sqrt{\tfrac{n-q}{2(n+1)}}\,\Lambda^-_{k,l}(n,m)\,\left|\begin{matrix}n-1\\q+1\end{matrix}\,m+3\right\rangle\ , \end{aligned} \tag{B.9c}$$

with $E^\dagger_{\alpha_2} = E^\top_{\alpha_2} = E_{-\alpha_2}$ and $E^\dagger_{\alpha_1+\alpha_2} = E^\top_{\alpha_1+\alpha_2} = E_{-(\alpha_1+\alpha_2)}$. It is convenient to express the generators as

$$E^{+\,(n,m)}_{\alpha_1+\alpha_2} = \sum_{q\in Q_n} \sqrt{\tfrac{n+q+2}{2(n+1)}}\,\Lambda^+_{k,l}(n,m)\,\left|\begin{matrix}n+1\\q+1\end{matrix}\,m+3\right\rangle\left\langle\begin{matrix}n\\q\end{matrix}\,m\right|\ , \tag{B.10a}$$

$$E^{-\,(n,m)}_{\alpha_1+\alpha_2} = \sum_{q\in Q_n} \sqrt{\tfrac{n-q}{2(n+1)}}\,\Lambda^-_{k,l}(n,m)\,\left|\begin{matrix}n-1\\q+1\end{matrix}\,m+3\right\rangle\left\langle\begin{matrix}n\\q\end{matrix}\,m\right|\ , \tag{B.10b}$$

$$E^{+\,(n,m)}_{\alpha_2} = \sum_{q\in Q_n} \sqrt{\tfrac{n-q-2}{2(n+1)}}\,\Lambda^+_{k,l}(n,m)\,\left|\begin{matrix}n+1\\q-1\end{matrix}\,m+3\right\rangle\left\langle\begin{matrix}n\\q\end{matrix}\,m\right|\ , \tag{B.10c}$$

$$E^{-\,(n,m)}_{\alpha_2} = \sum_{q\in Q_n} \sqrt{\tfrac{n+q}{2(n+1)}}\,\Lambda^-_{k,l}(n,m)\,\left|\begin{matrix}n-1\\q-1\end{matrix}\,m+3\right\rangle\left\langle\begin{matrix}n\\q\end{matrix}\,m\right|\ , \tag{B.10d}$$

where $Q_n := \{-n, -n+2, \dots, n-2, n\}$ and

$$\Lambda^+_{k,l}(n,m) = \frac{1}{\sqrt{n+2}}\sqrt{\left(\tfrac{k+2l}{3}+\tfrac{n}{2}+\tfrac{m}{6}+2\right)\left(\tfrac{k-l}{3}+\tfrac{n}{2}+\tfrac{m}{6}+1\right)\left(\tfrac{2k+l}{3}-\tfrac{n}{2}-\tfrac{m}{6}\right)}\ , \tag{B.11a}$$

$$\Lambda^-_{k,l}(n,m) = \frac{1}{\sqrt{n}}\sqrt{\left(\tfrac{k+2l}{3}-\tfrac{n}{2}+\tfrac{m}{6}+1\right)\left(\tfrac{l-k}{3}+\tfrac{n}{2}-\tfrac{m}{6}\right)\left(\tfrac{2k+l}{3}+\tfrac{n}{2}-\tfrac{m}{6}+1\right)}\ , \tag{B.11b}$$

with $\Lambda^-_{k,l}(0,m) := 0$ [156]. The identity operator $\Pi_{(n,m)}$ of the representation $\underline{(n,m)}$ is given by

$$\Pi_{(n,m)} = \sum_{q\in Q_n}\left|\begin{matrix}n\\q\end{matrix}\,m\right\rangle\left\langle\begin{matrix}n\\q\end{matrix}\,m\right|\ . \tag{B.12}$$

Fields The 1-instanton connection (8.11) is represented in the Biedenharn basis by

$$
\begin{aligned}
B_{(1)} &= B_{11}\,H_{\alpha_1} + B_{12}\,E_{\alpha_1} - \left(B_{12}\,E_{\alpha_1}\right)^\dagger \\
&= \sum_{n,q,m} \Bigg(B_{11}\,q \left|\begin{matrix} n \\ q \end{matrix}\, m\right\rangle \left\langle \begin{matrix} n \\ q \end{matrix}\, m \right| + \frac{1}{2}\,B_{12}\,\sqrt{(n-q)\,(n+q+2)} \left|\begin{matrix} n \\ q+2 \end{matrix}\, m\right\rangle \left\langle \begin{matrix} n \\ q \end{matrix}\, m \right| \\
&\qquad\qquad - \frac{1}{2}\,\bar{B}_{12}\,\sqrt{(n+q)\,(n-q+2)} \left|\begin{matrix} n \\ q-2 \end{matrix}\, m\right\rangle \left\langle \begin{matrix} n \\ q \end{matrix}\, m \right| \Bigg) \\
&\equiv \bigoplus_{(n,m)\in Q_0(k,l)} B_{(n,m)} \ ,
\end{aligned}
\tag{B.13}
$$

where $B_{(n,m)} \in \Omega^1\big(\mathrm{SU}(2), \mathrm{End}(\,\underline{(n,m)}\,)\big)$. One further introduces matrix-valued 1-forms given by

$$
\bar{\beta}_{q+1} = \bar{\beta}^1_{q+1}\,E_{\alpha_1+\alpha_2} + \bar{\beta}^2_{q+1}\,E_{\alpha_2} \equiv \bigoplus_{(n,m)\in Q_0(k,l)} \left(\bar{\beta}^+_{(n,m)} + \bar{\beta}^-_{(n,m)}\right) \ , \tag{B.14a}
$$

with the morphism-valued 1-forms

$$
\bar{\beta}^\pm_{(n,m)} \in \Omega^1\Big(S^5/\mathbb{Z}_{q+1}\,,\, \mathrm{Hom}\big(\,\underline{(n,m)}\,,\, \underline{(n\pm 1, m+3)}\,\big)\Big) \ , \tag{B.14b}
$$

and the corresponding adjoint maps

$$
\beta^\pm_{(n,m)} \in \Omega^1\Big(S^5/\mathbb{Z}_{q+1}\,,\, \mathrm{Hom}\big(\,\underline{(n\pm 1, m+3)}\,,\, \underline{(n,m)}\,\big)\Big) \ . \tag{B.14c}
$$

They have the explicit form

$$
\begin{aligned}
\bar{\beta}^\pm_{(n,m)} = \frac{\Lambda^\pm_{k,l}(n,m)}{\sqrt{2(n+1)}} \sum_{q\in Q_n} \Bigg(&\sqrt{n\pm q+1\pm 1}\ \bar{\beta}^1_{q+1} \left|\begin{matrix} n\pm 1 \\ q+1 \end{matrix}\, m+3\right\rangle \left\langle \begin{matrix} n \\ q \end{matrix}\, m \right| \\
&+ \sqrt{n\mp q+1\pm 1}\ \bar{\beta}^2_{q+1} \left|\begin{matrix} n\pm 1 \\ q-1 \end{matrix}\, m+3\right\rangle \left\langle \begin{matrix} n \\ q \end{matrix}\, m \right| \Bigg) \ .
\end{aligned}
\tag{B.15}
$$

Skew-Hermitian basis Similarly to [141], for a given representation $\underline{C}^{k,l}$ of the generators I_i and I_μ defined in (9.2) the decomposition into the Biedenharn basis yields

$$
I_1 = \bigoplus_{(n,m)} I_1^{(n,m)} = \bigoplus_{\pm\,,\,(n,m)} \left(E^{\pm\,(n,m)}_{\alpha_1+\alpha_2} - E^{\pm\,(n,m)}_{-\alpha_1-\alpha_2}\right) \ , \tag{B.16a}
$$

$$
I_2 = \bigoplus_{(n,m)} I_2^{(n,m)} = -\mathrm{i} \bigoplus_{\pm\,,\,(n,m)} \left(E^{\pm\,(n,m)}_{\alpha_1+\alpha_2} + E^{\pm\,(n,m)}_{-\alpha_1-\alpha_2}\right) \ , \tag{B.16b}
$$

$$
I_3 = \bigoplus_{(n,m)} I_3^{(n,m)} = \bigoplus_{\pm\,,\,(n,m)} \left(E^{\pm\,(n,m)}_{\alpha_2} - E^{\pm\,(n,m)}_{-\alpha_2}\right) \ , \tag{B.16c}
$$

$$
I_4 = \bigoplus_{(n,m)} I_4^{(n,m)} = -\mathrm{i} \bigoplus_{\pm\,,\,(n,m)} \left(E^{\pm\,(n,m)}_{\alpha_2} + E^{\pm\,(n,m)}_{-\alpha_2}\right) \ , \tag{B.16d}
$$

$$
I_5 = \bigoplus_{(n,m)} I_5^{(n,m)} = -\frac{\mathrm{i}}{2} \bigoplus_{(n,m)} H^{(n,m)}_{\alpha_2} \ . \tag{B.16e}
$$

The commutation relations $[I_i, I_a] = f_{ia}{}^b\, I_b$ and $[I_i, I_5] = 0$ induced by (9.4) respectively imply relations among the components given by

$$I_i^{(n',m')}\, I_a^{(n,m)} = I_a^{(n,m)}\, I_i^{(n,m)} + f_{ia}{}^b\, I_b^{(n,m)} \ , \tag{B.17a}$$

$$I_i^{(n,m)}\, I_5^{(n,m)} = I_5^{(n,m)}\, I_i^{(n,m)} \ , \tag{B.17b}$$

where $i \in \{6,7,8\}$, $a \in \{1,2,3,4\}$, $I_i = \bigoplus_{(n,m)} I_i^{(n,m)}$ and $(n', m') = (n \pm 1, m+3)$.

B.2.2 Flat connections

One can compute the matrix elements of $\mathcal{A}_0$ from (9.12) with respect to the Biedenharn basis. By choosing an SU(3)-representation $\underline{C}^{k,l}$, which yields an SU(2)-representation by restriction, one induces a connection $\mathcal{A}_0$ on the vector V-bundle

$$\widetilde{\mathcal{V}}_{\underline{C}^{k,l}} \xrightarrow{\underline{C}^{k,l}} \mathrm{G}/\widetilde{\mathrm{K}} \qquad \text{with} \quad \mathcal{V}_{\underline{C}^{k,l}} := \mathrm{G} \times_{\mathrm{K}} \underline{C}^{k,l} \tag{B.18}$$

associated to the principal V-bundle (8.21). Then the connection $\mathcal{A}_0$ can be decomposed into morphism-valued 1-forms

$$\mathcal{A}_0 = \bigoplus_{(n,m)\in Q_0(k,l)} \Big(B_{(n,m)} - \frac{\mathrm{i}\, m}{2}\, \eta\, \Pi_{(n,m)} + \bar\beta^+_{(n,m)} + \bar\beta^-_{(n,m)} - \beta^+_{(n,m)} - \beta^-_{(n,m)} \Big) \tag{B.19}$$

with respect to this basis. The computation of the vanishing curvature $\mathcal{F}_0 = 0$ yields relations between the different matrix elements given by

$$\begin{aligned} \mathrm{d}B_{(n,m)} + B_{(n,m)} \wedge B_{(n,m)} - \tfrac{\mathrm{i}\, m}{2}\, \mathrm{d}\eta\, \Pi_{(n,m)} = \ & \bar\beta^+_{(n-1,m-3)} \wedge \beta^+_{(n-1,m-3)} + \bar\beta^-_{(n+1,m-3)} \wedge \beta^-_{(n+1,m-3)} \\ & + \beta^+_{(n,m)} \wedge \bar\beta^+_{(n,m)} + \beta^-_{(n,m)} \wedge \bar\beta^-_{(n,m)} \ , \end{aligned} \tag{B.20a}$$

$$0 = \mathrm{d}\bar\beta^\pm_{(n,m)} + B_{(n+1,m+3)} \wedge \bar\beta^\pm_{(n,m)} + \bar\beta^\pm_{(n,m)} \wedge B_{(n,m)} - \tfrac{3\,\mathrm{i}}{2}\, \eta\, \Pi_{(n\pm1,m+3)} \wedge \bar\beta^\pm_{(n,m)} \ , \tag{B.20b}$$

$$0 = \bar\beta^+_{(n,m)} \wedge \bar\beta^-_{(n+1,m-3)} + \bar\beta^-_{(n+2,m)} \wedge \bar\beta^+_{(n+1,m-3)} \ , \tag{B.20c}$$

$$0 = \bar\beta^+_{(n,m)} \wedge \beta^-_{(n,m)} + \beta^-_{(n+1,m+3)} \wedge \bar\beta^+_{(n-1,m+3)} \ , \tag{B.20d}$$

$$0 = \bar\beta^\pm_{(n,m)} \wedge \bar\beta^\pm_{(n\mp1,m-3)} \ , \tag{B.20e}$$

plus their conjugate equations.

B.2.3 Quiver connections

Alternatively, one can compute the matrix elements of the curvature (9.36c) in the Biedenharn basis. For this, the curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is arranged into components

$$(\mathcal{F})_{(n,m),(n',m')} \in \Omega^2\Big(\mathcal{E}^{k,l} \, , \, \mathrm{End}\big(E_{p_{(n,m)}}, E_{p_{(n',m')}}\big) \otimes \mathrm{End}\big(\underline{(n,m)} \, , \, \underline{(n',m')} \big) \Big) \ , \tag{B.21}$$

which can be simplified by using the relations (B.20). We denote the curvature of the connection $A_{(n,m)}$ on the bundle (9.21) by

$$F_{(n,m)} := \mathrm{d}A_{(n,m)} + A_{(n,m)} \wedge A_{(n,m)} \tag{B.22a}$$

and the bifundamental covariant derivatives of the Higgs fields as

$$D\phi^{\pm}_{(n,m)} \coloneqq \mathrm{d}\phi^{\pm}_{(n,m)} + A_{(n\pm1,m+3)}\,\phi^{\pm}_{(n,m)} - \phi^{\pm}_{(n,m)}\,A_{(n,m)}\ , \tag{B.22b}$$

$$D\psi_{(n,m)} \coloneqq \mathrm{d}\psi_{(n,m)} + A_{(n,m)}\,\psi_{(n,m)} - \psi_{(n,m)}\,A_{(n,m)}\ . \tag{B.22c}$$

Then the non-zero curvature components read as

$$\begin{aligned}
(\mathcal{F})_{(n,m),(n,m)} = {} & F_{(n,m)} \otimes \Pi_{(n,m)} - D\psi_{(n,m)} \wedge \tfrac{\mathrm{i}\,m}{2}\,\eta\,\Pi_{(n,m)} \\
& - \big(\psi_{(n,m)} - \mathbb{1}_{p_{(n,m)}}\big) \otimes \tfrac{\mathrm{i}\,m}{2}\,\mathrm{d}\eta\,\Pi_{(n,m)} \\
& + \big(\mathbb{1}_{p_{(n,m)}} - \phi^{+}_{(n-1,m-3)}\,(\phi^{+})^{\dagger}_{(n-1,m-3)}\big) \otimes \bar\beta^{+}_{(n-1,m-3)} \wedge \beta^{+}_{(n-1,m-3)} \\
& + \big(\mathbb{1}_{p_{(n,m)}} - \phi^{-}_{(n+1,m-3)}\,(\phi^{-})^{\dagger}_{(n+1,m-3)}\big) \otimes \bar\beta^{-}_{(n+1,m-3)} \wedge \beta^{-}_{(n+1,m-3)} \\
& + \big(\mathbb{1}_{p_{(n,m)}} - (\phi^{+})^{\dagger}_{(n,m)}\,\phi^{+}_{(n,m)}\big) \otimes \beta^{+}_{(n,m)} \wedge \bar\beta^{+}_{(n,m)} \\
& + \big(\mathbb{1}_{p_{(n,m)}} - (\phi^{-})^{\dagger}_{(n,m)}\,\phi^{-}_{(n,m)}\big) \otimes \beta^{-}_{(n,m)} \wedge \bar\beta^{-}_{(n,m)}\ ,
\end{aligned} \tag{B.23a}$$

$$\begin{aligned}
(\mathcal{F})_{(n,m),(n\pm1,m+3)} = {} & D\phi^{\pm}_{(n,m)} \wedge \bar\beta^{\pm}_{(n,m)} - \Big((m+3)\,\psi_{(n\pm1,m+3)}\,\phi^{\pm}_{(n,m)} \\
& - m\,\phi^{\pm}_{(n,m)}\,\psi_{(n,m)} - 3\phi^{\pm}_{(n,m)}\Big) \otimes \tfrac{\mathrm{i}}{2}\,\eta\,\Pi_{(n\pm1,m+3)} \wedge \bar\beta^{\pm}_{(n,m)}\ ,
\end{aligned} \tag{B.23b}$$

$$\begin{aligned}
(\mathcal{F})_{(n+1,m-3),(n+1,m+3)} = {} & \Big(\phi^{+}_{(n,m)}\,\phi^{-}_{(n+1,m-3)} - \phi^{-}_{(n+2,m)}\,\phi^{+}_{(n+1,m-3)}\Big) \\
& \otimes \bar\beta^{+}_{(n,m)} \wedge \bar\beta^{-}_{(n+1,m-3)}\ ,
\end{aligned} \tag{B.23c}$$

$$\begin{aligned}
(\mathcal{F})_{(n-1,m+3),(n+1,m+3)} = {} & -\Big(\phi^{+}_{(n,m)}\,(\phi^{-})^{\dagger}_{(n,m)} - (\phi^{-})^{\dagger}_{(n+1,m+3)}\,\phi^{+}_{(n-1,m+3)}\Big) \\
& \otimes \bar\beta^{+}_{(n,m)} \wedge \beta^{-}_{(n,m)}\ ,
\end{aligned} \tag{B.23d}$$

which are accompanied by the anti-Hermiticity conditions

$$(\mathcal{F})_{(n',m'),(n,m)} = -\Big((\mathcal{F})_{(n,m),(n',m')}\Big)^{\dagger}\ . \tag{B.23e}$$

By setting $\psi_{(n,m)} = \mathbb{1}_{p_{(n,m)}}$ for all $(n,m) \in Q_0(k,l)$, these curvature matrix elements correctly reproduce those computed in [156] for equivariant dimensional reduction over $\mathbb{C}P^2$.

B.3 Quiver bundle examples

B.3.1 $C^{1,0}$-quiver

Consider the fundamental 3-dimensional representation $\underline{C}^{1,0}$ of $\mathrm{G} = \mathrm{SU}(3)$. Its decomposition into irreducible SU(2)-representations is given by

$$\underline{C}^{1,0}\big|_{\mathrm{SU}(2)} = \underline{(0,-2)}\ \oplus\ \underline{(1,1)}\ , \tag{B.24}$$

wherein $\underline{(0,-2)}$ is the SU(2)-singlet and $\underline{(1,1)}$ is the SU(2)-doublet. Using the general quiver construction of Sec. 9.3, the G-action dictates the existence of bundle morphisms

$$\phi := \phi^{+}_{(0,-2)} \in \mathrm{Hom}\big(E_{p_{(0,-2)}}, E_{p_{(1,1)}}\big)\ , \quad \phi^{\dagger} := (\phi^{+})^{\dagger}_{(0,-2)} \in \mathrm{Hom}\big(E_{p_{(1,1)}}, E_{p_{(0,-2)}}\big)\ , \tag{B.25a}$$

$$\psi_0 := \psi_{(0,-2)} \in \mathrm{End}\big(E_{p_{(0,-2)}}\big)\ , \qquad \psi_1 := \psi_{(1,1)} \in \mathrm{End}\big(E_{p_{(1,1)}}\big)\ . \tag{B.25b}$$

B.3.2 $C^{2,0}$-quiver

The 6-dimensional representation $\underline{C}^{2,0}$ of SU(3) splits under restriction to SU(2) as

$$\underline{C}^{2,0}|_{\mathrm{SU}(2)} = \underline{(2,2)} \ \oplus \ \underline{(1,-1)} \ \oplus \ \underline{(0,-4)} \ . \tag{B.26}$$

The SU(3)-action intertwines the irreducible SU(2)-modules and the corresponding bundles. The actions of $E_{\alpha_1+\alpha_2}$ and E_{α_2} respectively yield Higgs fields

$$\phi_0 := \phi^+_{(0,-4)} \in \mathrm{Hom}\big(E_{p_{(0,-4)}}, E_{p_{(1,-1)}}\big) \ , \quad \phi_1 := \phi^+_{(1,-1)} \in \mathrm{Hom}\big(E_{p_{(1,-1)}}, E_{p_{(2,2)}}\big) \ . \tag{B.27a}$$

Due to the non-zero restrictions of H_{α_2} to its eigenspaces $\underline{(0,-4)}$, $\underline{(1,-1)}$ and $\underline{(2,2)}$, one further has three bundle endomorphisms

$$\begin{aligned} &\psi_0 := \psi_{(0,-4)} \in \mathrm{End}\big(E_{p_{(0,-4)}}\big) \ , \quad \psi_1 := \psi_{(1,-1)} \in \mathrm{End}\big(E_{p_{(1,-1)}}\big) \ , \\ &\psi_2 := \psi_{(2,2)} \in \mathrm{End}\big(E_{p_{(2,2)}}\big) \ . \end{aligned} \tag{B.27b}$$

B.3.3 $C^{1,1}$-quiver

The 8-dimensional adjoint representation of SU(3) splits under restriction to SU(2) as

$$\underline{C}^{1,1}|_{\mathrm{SU}(2)} = \underline{(1,3)} \ \oplus \ \underline{(0,0)} \ \oplus \ \underline{(2,0)} \ \oplus \ \underline{(1,-3)} \ . \tag{B.28}$$

The action of SU(3) implies the existence of the following bundle morphisms: The actions of $E_{\alpha_1+\alpha_2}$ and E_{α_2} translate into the Higgs fields

$$\phi_1^+ := \phi^+_{(1,-3)} \in \mathrm{Hom}\big(E_{p_{(1,-3)}}, E_{p_{(2,0)}}\big) \ , \quad \phi_1^- := \phi^-_{(1,-3)} \in \mathrm{Hom}\big(E_{p_{(1,-3)}}, E_{p_{(0,0)}}\big) \ , \tag{B.29a}$$

$$\phi_0^+ := \phi^+_{(0,0)} \in \mathrm{Hom}\big(E_{p_{(0,0)}}, E_{p_{(1,3)}}\big) \ , \qquad \phi_0^- := \phi^-_{(2,0)} \in \mathrm{Hom}\big(E_{p_{(2,0)}}, E_{p_{(1,3)}}\big) \ , \tag{B.29b}$$

whereas the action of H_{α_2} generates

$$\psi^\pm := \psi_{(1,\pm 3)} \in \mathrm{End}\big(E_{p_{(1,\pm3)}}\big) \ . \tag{B.29c}$$

Note that H_{α_2} neither introduces endomorphisms on $\underline{(0,0)}$ and $\underline{(2,0)}$ nor does it intertwine these SU(2)-multiplets. This follows from the fact that these representations are subspaces of the kernel of H_{α_2}, and that H_{α_2} commutes with the entire Lie algebra $\mathfrak{su}(2)$.

B.4 Equivariant dimensional reduction details

B.4.1 1-form products on $\mathbb{C}P^2$

The metric on $M^d \times \mathbb{C}P^2$ is given as

$$\mathrm{d}s^2 = \mathrm{d}s^2_{M^d} + \mathrm{d}s^2_{\mathbb{C}P^2} \ , \tag{B.30}$$

where

$$\mathrm{d}s^2_{M^d} = G_{\mu'\nu'}\,\mathrm{d}x^{\mu'} \otimes \mathrm{d}x^{\nu'} \tag{B.31}$$

with $(x^{\mu'})$ local real coordinates on the manifold M^d and $\mu', \nu', \ldots = 1, \ldots, d$. The metric on $\mathbb{C}P^2$ is written as

$$g_{\mathbb{C}P^2} := \mathrm{d}s^2_{\mathbb{C}P^2} = R^2 \left(\beta^1 \otimes \bar\beta^1 + \bar\beta^1 \otimes \beta^1 + \beta^2 \otimes \bar\beta^2 + \bar\beta^2 \otimes \beta^2\right) \ . \tag{B.32}$$

This metric is compatible with the Kähler form (B.2), and by defining the complex structure J via $\omega_{\mathbb{C}P^2}(\cdot,\cdot) = g_{\mathbb{C}P^2}(\cdot, J\,\cdot)$ on the cotangent bundle of $\mathbb{C}P^2$ one obtains $J\beta^i = \mathrm{i}\,\beta^i$ and $J\bar\beta^i = -\mathrm{i}\,\bar\beta^i$ for $i=1,2$. The corresponding Hodge operator is denoted $\star_{\mathbb{C}P^2}$, with the non-vanishing 1-form products

$$\star_{\mathbb{C}P^2} 1 = R^4\, \beta^1 \wedge \bar\beta^1 \wedge \beta^2 \wedge \bar\beta^2 = 2\big(\pi\, R^2\big)^2\, \beta_{\rm vol}\ , \tag{B.33a}$$

$$\bar\beta^1 \wedge \star_{\mathbb{C}P^2}\beta^1 = \bar\beta^2 \wedge \star_{\mathbb{C}P^2}\beta^2 = \beta^1 \wedge \star_{\mathbb{C}P^2}\bar\beta^1 = \beta^2 \wedge \star_{\mathbb{C}P^2}\bar\beta^2 = 2\pi^2\, R^2\, \beta_{\rm vol}\ , \tag{B.33b}$$

$$\star_{\mathbb{C}P^2}\bar\beta^1 \wedge \beta^1 = \beta^2 \wedge \bar\beta^2\ , \qquad \star_{\mathbb{C}P^2}\bar\beta^2 \wedge \beta^2 = \beta^1 \wedge \bar\beta^1\ , \tag{B.33c}$$

$$\star_{\mathbb{C}P^2}\bar\beta^1 \wedge \beta^2 = \bar\beta^1 \wedge \beta^2\ , \qquad \star_{\mathbb{C}P^2}\bar\beta^2 \wedge \beta^1 = \bar\beta^2 \wedge \beta^1\ . \tag{B.33d}$$

For later use we shall also need to compute various products involving matrix-valued 1-forms. Firstly, we have[26]

$$\mathrm{tr}\, \frac{\beta^\pm_{(n,m)} \wedge \star_{\mathbb{C}P^2}\bar\beta^\pm_{(n,m)}}{\Lambda^\pm_{k,l}(n,m)^2} = 2\pi^2\, R^2\, (n+1\pm 1)\, \beta_{\rm vol}\ , \tag{B.34a}$$

$$\mathrm{tr}\, \frac{\beta^\pm_{(n,m)} \wedge \bar\beta^\pm_{(n,m)} \wedge \star_{\mathbb{C}P^2}\big(\beta^\pm_{(n,m)} \wedge \bar\beta^\pm_{(n,m)}\big)^\dagger}{\Lambda^\pm_{k,l}(n,m)^4} = 2\pi^2\, (n+1\pm 1)\, \beta_{\rm vol}\ , \tag{B.34b}$$

$$\mathrm{tr}\, \frac{\bar\beta^\pm_{(n,m)} \wedge \beta^\pm_{(n,m)} \wedge \star_{\mathbb{C}P^2}\big(\bar\beta^\pm_{(n,m)} \wedge \beta^\pm_{(n,m)}\big)^\dagger}{\Lambda^\pm_{k,l}(n,m)^4} = 2\pi^2\, \frac{(n+1\pm 1)^2}{n+1}\, \beta_{\rm vol}\ , \tag{B.34c}$$

$$\mathrm{tr}\, \frac{\bar\beta^+_{(n,m)} \wedge \bar\beta^-_{(n+1,m-3)} \wedge \star_{\mathbb{C}P^2}\big(\bar\beta^+_{(n,m)} \wedge \bar\beta^-_{(n+1,m-3)}\big)^\dagger}{\Lambda^+_{k,l}(n,m)^2\, \Lambda^-_{k,l}(n+1,m-3)^2} = 2\pi^2\, \frac{n+1}{3}\, \beta_{\rm vol}\ , \tag{B.34d}$$

$$\mathrm{tr}\, \frac{\bar\beta^+_{(n,m)} \wedge \beta^-_{(n,m)} \wedge \star_{\mathbb{C}P^2}\big(\bar\beta^+_{(n,m)} \wedge \beta^-_{(n,m)}\big)^\dagger}{\Lambda^+_{k,l}(n,m)^2\, \Lambda^-_{k,l}(n,m)^2} = 2\pi^2\, \frac{n\,(n+2)}{n+1}\, \beta_{\rm vol}\ . \tag{B.34e}$$

The trace formulas (B.34) will have to be supplemented by

$$\mathrm{tr}\, \frac{\beta^+_{(n,m)} \wedge \bar\beta^+_{(n,m)} \wedge \star_{\mathbb{C}P^2}\big(\beta^-_{(n,m)} \wedge \bar\beta^-_{(n,m)}\big)^\dagger}{\Lambda^+_{k,l}(n,m)^2\, \Lambda^-_{k,l}(n,m)^2} = 2\pi^2\, \frac{2n\,(n+2)}{3(n+1)}\, \beta_{\rm vol}\ , \tag{B.35a}$$

$$\mathrm{tr}\, \frac{\bar\beta^+_{(n-1,m-3)} \wedge \beta^+_{(n-1,m-3)} \wedge \star_{\mathbb{C}P^2}\big(\bar\beta^-_{(n+1,m-3)} \wedge \beta^-_{(n+1,m-3)}\big)^\dagger}{\Lambda^+_{k,l}(n-1,m-3)^2\, \Lambda^-_{k,l}(n+1,m-3)^2} = 2\pi^2\, \frac{2(n+1)}{3}\, \beta_{\rm vol}\ , \tag{B.35b}$$

$$\mathrm{tr}\, \frac{\beta^\pm_{(n,m)} \wedge \bar\beta^\pm_{(n,m)} \wedge \star_{\mathbb{C}P^2}\big(\bar\beta^\pm_{(n\mp1,m-3)} \wedge \beta^\pm_{(n\mp1,m-3)}\big)^\dagger}{\Lambda^\pm_{k,l}(n,m)^2\, \Lambda^\pm_{k,l}(n\mp1,m-3)^2} = -2\pi^2\, \frac{n\,(n+2)}{n+1\mp1}\, \beta_{\rm vol}\ , \tag{B.35c}$$

$$\mathrm{tr}\, \frac{\beta^\pm_{(n,m)} \wedge \bar\beta^\pm_{(n,m)} \wedge \star_{\mathbb{C}P^2}\big(\bar\beta^\mp_{(n\pm1,m-3)} \wedge \beta^\mp_{(n\pm1,m-3)}\big)^\dagger}{\Lambda^\pm_{k,l}(n,m)^2\, \Lambda^\mp_{k,l}(n\pm1,m-3)^2} = 2\pi^2\, \bigg(\frac{n\,(n+2)}{3(n+1\pm1)} - (n+1)\bigg)\, \beta_{\rm vol} \tag{B.35d}$$

[26] The expressions (B.34) correct the trace formulas from [156, Eq. (B.7)].

and one additionally needs the traces

$$\operatorname{tr} \frac{\beta^{\pm}_{(n,m)} \wedge \bar{\beta}^{\pm}_{(n,m)}}{\Lambda^{\pm}_{k,l}(n,m)^2} = -\frac{\mathrm{i}}{2R^2}\,(n+1\pm 1)\,\omega_{\mathbb{C}P^2} = \star_{\mathbb{C}P^2} \operatorname{tr} \frac{\big(\beta^{\pm}_{(n,m)} \wedge \bar{\beta}^{\pm}_{(n,m)}\big)^\dagger}{\Lambda^{\pm}_{k,l}(n,m)^2} \ , \tag{B.36a}$$

$$\operatorname{tr} \frac{\bar{\beta}^{\pm}_{(n\mp1,m-3)} \wedge \beta^{\pm}_{(n\mp1,m-3)}}{\Lambda^{\pm}_{k,l}(n\mp1,m-3)^2} = \frac{\mathrm{i}}{2R^2}\,(n{+}1)\,\omega_{\mathbb{C}P^2} = \star_{\mathbb{C}P^2} \operatorname{tr} \frac{\big(\bar{\beta}^{\pm}_{(n\mp1,m-3)} \wedge \beta^{\pm}_{(n\mp1,m-3)}\big)^\dagger}{\Lambda^{\pm}_{k,l}(n\mp1,m-3)^2} \ . \tag{B.36b}$$

B.4.2 1-form products on S^5

Let us write the metric (9.39) in the form

$$\mathrm{d}s^2_{S^5} = g_{ij}\,\Big(\beta^i_\varphi \otimes \bar{\beta}^j_\varphi + \bar{\beta}^j_\varphi \otimes \beta^i_\varphi\Big) + g_{55}\,\eta\otimes\eta = 2R^2\,\delta_{ab}\,e^a\otimes e^b + r^2\,e^5\otimes e^5 \ , \tag{B.37}$$

for $i,j = 1,2$ and $a,b = 1,2,3,4$, where r is the radius of the S^1-fibre of the Hopf bundle $S^5 \to \mathbb{C}P^2$. The corresponding Hodge operator is denoted $\star_{S^5}$. Define the normalised volume form η_{vol} on S^5 as

$$\begin{aligned} \star_{S^5} 1 = -(2\pi)^3\, r\, R^4\ \eta_{\mathrm{vol}} \qquad &\text{with} \quad \beta_{\mathrm{vol}} \wedge \eta = -4\pi\ \eta_{\mathrm{vol}} = -\frac{2}{\pi^2}\, e^{12345} \\ &\text{and} \quad \int_{S^5} \eta_{\mathrm{vol}} = 1 \ . \end{aligned} \tag{B.38}$$

In the computation of the reduced action (9.42) we employ the identities

$$e^\mu \wedge \star_{S^5} e^\nu = \sqrt{g}\, g^{\mu\nu}\, e^{12345} = \begin{cases} 4\pi^3\, r\, R^2\ \eta_{\mathrm{vol}} \ , & \mu=\nu=a \ , \\ \frac{(2\pi)^3\, R^4}{r}\ \eta_{\mathrm{vol}} \ , & \mu=\nu=5 \ , \\ 0 \ , & \mu\neq\nu \ , \end{cases} \tag{B.39a}$$

$$e^{\mu\nu} \wedge \star_{S^5} e^{\rho\sigma} = \begin{cases} \sqrt{g}\, g^{\mu\rho}\, g^{\nu\sigma}\, e^{12345} \ , & \mu=\rho,\ \nu=\sigma \ , \\ -\sqrt{g}\, g^{\mu\sigma}\, g^{\nu\rho}\, e^{12345} \ , & \mu=\sigma,\ \nu=\rho \ , \\ 0 \ , & \text{otherwise} \ , \end{cases} \tag{B.39b}$$

$$e^{ab} \wedge \star_{S^5} e^{ab} = 2\pi^3\, r\ \eta_{\mathrm{vol}} \quad \text{and} \quad e^{a5} \wedge \star_{S^5} e^{a5} = \tfrac{4\pi^3\, R^2}{r}\ \eta_{\mathrm{vol}} \ . \tag{B.39c}$$

We can reduce the action of the Hodge operator in 5 dimensions to the action of $\star_{\mathbb{C}P^2}$ from App. B.4.1 to get

$$\star_{S^5} \beta^i_\varphi = r\,\Big(\star_{\mathbb{C}P^2} \beta^i_\varphi\Big) \wedge \eta \ , \qquad \star_{S^5} \bar{\beta}^i_\varphi = r\,\Big(\star_{\mathbb{C}P^2} \bar{\beta}^i_\varphi\Big) \wedge \eta \ , \tag{B.40a}$$

$$\star_{S^5}\Big(\beta^i_\varphi \wedge \bar{\beta}^j_\varphi\Big) = r\,\Big(\star_{\mathbb{C}P^2} \beta^i_\varphi \wedge \bar{\beta}^j_\varphi\Big) \wedge \eta \ , \quad \star_{S^5}\Big(\beta^i_\varphi \wedge \beta^j_\varphi\Big) = r\,\Big(\star_{\mathbb{C}P^2} \beta^i_\varphi \wedge \beta^j_\varphi\Big) \wedge \eta \ , \tag{B.40b}$$

$$\star_{S^5}\Big(\eta \wedge \beta^i_\varphi\Big) = \tfrac{1}{r}\ \star_{\mathbb{C}P^2}\ \beta^i_\varphi \ , \qquad \star_{S^5}\Big(\eta \wedge \bar{\beta}^i_\varphi\Big) = \tfrac{1}{r}\ \star_{\mathbb{C}P^2}\ \bar{\beta}^i_\varphi \ , \tag{B.40c}$$

$$\star_{S^5} \eta = \tfrac{2(\pi R^2)^2}{r}\ \beta_{\mathrm{vol}} \ , \qquad \eta \wedge \star_{S^5} \eta = -\tfrac{(2\pi)^3\, R^4}{r}\ \eta_{\mathrm{vol}} \ . \tag{B.40d}$$

We can additionally compute

$$\mathrm{d}\eta = -2\omega_3 = \mathrm{i}\,\Big(\beta^1_\varphi \wedge \bar{\beta}^1_\varphi + \beta^2_\varphi \wedge \bar{\beta}^2_\varphi\Big) = -\tfrac{1}{R^2}\,\omega_{\mathbb{C}P^2} \ , \tag{B.41a}$$

$$\star_{S^5} \mathrm{d}\eta = -\tfrac{1}{R^2}\ \star_{S^5}\ \omega_{\mathbb{C}P^2} = -\tfrac{r}{R^2}\,\big(\star_{\mathbb{C}P^2}\omega_{\mathbb{C}P^2}\big) \wedge \eta = \tfrac{r}{R^2}\,\omega_{\mathbb{C}P^2} \wedge \eta \ , \tag{B.41b}$$

$$\mathrm{d}\eta \wedge \star_{S^5} \mathrm{d}\eta = -2(2\pi)^3\, r\ \eta_{\mathrm{vol}}\ , \tag{B.41c}$$

wherein we used $\star_{\mathbb{C}P^2}\omega_{\mathbb{C}P^2} = -\omega_{\mathbb{C}P^2}$ and (B.5). Note that due to the structure of the extension from $\mathbb{C}P^2$ to S^5, the matrices accompanying contributions from η or $\mathrm{d}\eta$ are always proportional to the identity operators $\Pi_{(n,m)}$; thus their inclusion does not alter the trace formulas of App. B.4.1.

B.4.3 Yang-Mills action

The reduction of (9.40) proceeds by expanding

$$\mathrm{tr}\, \mathcal{F} \wedge \star \mathcal{F} = - \sum_{(n,m)\in Q_0(k,l)} \mathrm{tr}\big(\mathcal{F} \wedge \star \mathcal{F}^\dagger\,\big)_{(n,m),(n,m)}\ . \tag{B.42}$$

We insert the explicit non-vanishing components (B.23), rescale the horizontal Higgs fields

$$\phi^{\pm}_{(n,m)} \longrightarrow \tfrac{1}{\Lambda^{\pm}_{k,l}(n,m)}\, \phi^{\pm}_{(n,m)} \tag{B.43}$$

as in [156] (but *not* the vertical Higgs fields $\psi_{(n,m)}$), and evaluate the traces over the representation spaces $\underline{(n,m)}$ for each weight $(n,m)\in Q_0(k,l)$ using the matrix products from App. B.4.1 and the relations of App. B.4.2. Finally, one then integrates over S^5 using the unit volume form η_{vol} introduced in App. B.4.2. The dimensionally reduced Yang-Mills action on M^d then reads as[27]

$$\begin{aligned}
S = \frac{2\pi^3\, r\, R^4}{\tilde g^2} \int_{M^d} \mathrm{d}^d x\ \sqrt{G} \sum_{(n,m)\in Q_0(k,l)} \mathrm{tr}\bigg(& (n+1)\, \big(F_{(n,m)}\big)^\dagger_{\mu'\nu'}\, \big(F_{(n,m)}\big)^{\mu'\nu'} \\
& + \frac{n+2}{R^2}\, \big(D_{\mu'}\phi^+_{(n,m)}\big)^\dagger\, D^{\mu'}\phi^+_{(n,m)} + \frac{n+1}{R^2}\, D_{\mu'}\phi^+_{(n-1,m-3)}\, \big(D^{\mu'}\phi^+_{(n-1,m-3)}\big)^\dagger \\
& + \frac{n}{R^2}\, \big(D_{\mu'}\phi^-_{(n,m)}\big)^\dagger\, D^{\mu'}\phi^-_{(n,m)} + \frac{n+1}{R^2}\, D_{\mu'}\phi^-_{(n+1,m-3)}\, \big(D^{\mu'}\phi^-_{(n+1,m-3)}\big)^\dagger \\
& + \frac{n+2}{R^4}\, \Big(\Lambda^+_{k,l}(n,m)^2\, \mathbb{1}_{p_{(n,m)}} - (\phi^+)^\dagger_{(n,m)}\, \phi^+_{(n,m)}\Big)^2 \\
& + \frac{n}{R^4}\, \Big(\Lambda^-_{k,l}(n,m)^2\, \mathbb{1}_{p_{(n,m)}} - (\phi^-)^\dagger_{(n,m)}\, \phi^-_{(n,m)}\Big)^2 \\
& + \frac{n+1}{n\, R^4}\, \Big(\Lambda^+_{k,l}(n-1,m-3)^2\, \mathbb{1}_{p_{(n,m)}} - \phi^+_{(n-1,m-3)}\, (\phi^+)^\dagger_{(n-1,m-3)}\Big)^2 \\
& + \frac{(n+1)^2}{(n+2)\, R^4}\, \Big(\Lambda^-_{k,l}(n+1,m-3)^2\, \mathbb{1}_{p_{(n,m)}} - \phi^-_{(n+1,m-3)}\, (\phi^-)^\dagger_{(n+1,m-3)}\Big)^2 \\
& + \frac{2(n+3)}{3R^4}\, \Big|\phi^+_{(n,m)}\, \phi^-_{(n+1,m-3)} - \frac{\Lambda^+_{k,l}(n,m)\, \Lambda^-_{k,l}(n+1,m-3)}{\Lambda^+_{k,l}(n+1,m-3)\, \Lambda^-_{k,l}(n+2,m)}\, \phi^-_{(n+2,m)}\, \phi^+_{(n+1,m-3)}\Big|^2 \\
& + \frac{2n\,(n+2)}{(n+1)\, R^4}\, \Big|\phi^+_{(n,m)}\, (\phi^-)^\dagger_{(n,m)} \\
& \qquad - \frac{\Lambda^+_{k,l}(n,m)\, \Lambda^-_{k,l}(n,m)}{\Lambda^+_{k,l}(n-1,m+3)\, \Lambda^-_{k,l}(n+1,m+3)}\, (\phi^-)^\dagger_{(n+1,m+3)}\, \phi^+_{(n-1,m+3)}\Big|^2 \\
& + \frac{4n\,(n+2)}{3(n+1)\, R^4}\, \Big(\big(\Lambda^+_{k,l}(n,m)^2\, \mathbb{1}_{p_{(n,m)}} - (\phi^+)^\dagger_{(n,m)}\, \phi^+_{(n,m)}\big)
\end{aligned}$$

[27]By setting $\psi_{(n,m)} = \mathbb{1}_{p_{(n,m)}}$ for all $(n,m)\in Q_0(k,l)$ and $r=\frac{1}{4\pi}$ in (B.44) we obtain the quiver gauge theory action for equivariant dimensional reduction over the complex projective plane $\mathbb{C}P^2$; this reduction eliminates the last nine lines of (B.44) and the resulting expression corrects [156, Eq. (3.5)].

$$
\begin{aligned}
& \qquad\qquad\qquad \times \ \left(\Lambda^-_{k,l}(n,m)^2\,\mathbb{1}_{p_{(n,m)}} - (\phi^-)^\dagger_{(n,m)}\,\phi^-_{(n,m)}\right)\Big) \\
& - \frac{2(n+2)}{R^4}\Big(\left(\Lambda^+_{k,l}(n,m)^2\,\mathbb{1}_{p_{(n,m)}} - (\phi^+)^\dagger_{(n,m)}\,\phi^+_{(n,m)}\right) \\
& \qquad\qquad\qquad \times \ \left(\Lambda^+_{k,l}(n-1,m-3)^2\,\mathbb{1}_{p_{(n,m)}} - \phi^+_{(n-1,m-3)}\,(\phi^+)^\dagger_{(n-1,m-3)}\right)\Big) \\
& + \frac{2}{R^4}\left(\frac{n}{3} - n - 1\right)\Big(\left(\Lambda^+_{k,l}(n,m)^2\,\mathbb{1}_{p_{(n,m)}} - (\phi^+)^\dagger_{(n,m)}\,\phi^+_{(n,m)}\right) \\
& \qquad\qquad\qquad \times \ \left(\Lambda^-_{k,l}(n+1,m-3)^2\,\mathbb{1}_{p_{(n,m)}} - \phi^-_{(n+1,m-3)}\,(\phi^-)^\dagger_{(n+1,m-3)}\right)\Big) \\
& + \frac{2}{R^4}\left(\frac{n+2}{3} - n - 1\right)\Big(\left(\Lambda^-_{k,l}(n,m)^2\,\mathbb{1}_{p_{(n,m)}} - (\phi^-)^\dagger_{(n,m)}\,\phi^-_{(n,m)}\right) \\
& \qquad\qquad\qquad \times \ \left(\Lambda^+_{k,l}(n-1,m-3)^2\,\mathbb{1}_{p_{(n,m)}} - \phi^+_{(n-1,m-3)}\,(\phi^+)^\dagger_{(n-1,m-3)}\right)\Big) \\
& - \frac{2n}{R^4}\Big(\left(\Lambda^-_{k,l}(n,m)^2\,\mathbb{1}_{p_{(n,m)}} - (\phi^-)^\dagger_{(n,m)}\,\phi^-_{(n,m)}\right) \\
& \qquad\qquad\qquad \times \ \left(\Lambda^-_{k,l}(n+1,m-3)^2\,\mathbb{1}_{p_{(n,m)}} - \phi^-_{(n+1,m-3)}\,(\phi^-)^\dagger_{(n+1,m-3)}\right)\Big) \\
& + \frac{4(n+1)}{3R^4}\Big(\left(\Lambda^+_{k,l}(n-1,m-3)^2\,\mathbb{1}_{p_{(n,m)}} - \phi^+_{(n-1,m-3)}\,(\phi^+)^\dagger_{(n-1,m-3)}\right) \\
& \qquad\qquad\qquad \times \ \left(\Lambda^-_{k,l}(n+1,m-3)^2\,\mathbb{1}_{p_{(n,m)}} - \phi^-_{(n+1,m-3)}\,(\phi^-)^\dagger_{(n+1,m-3)}\right)\Big) \\
& + \frac{(n+1)\,m^2}{4r^2}\,D_{\mu'}\psi_{(n,m)}\,(D^{\mu'}\psi_{(n,m)})^\dagger + \frac{2(n+1)\,m^2}{R^4}\left(\psi_{(n,m)} - \mathbb{1}_{p_{(n,m)}}\right)^2 \\
& - \frac{m\,(n+2)}{R^4}\Big(\left(\Lambda^+_{k,l}(n,m)^2\,\mathbb{1}_{p_{(n,m)}} - (\phi^+)^\dagger_{(n,m)}\,\phi^+_{(n,m)}\right)\left(\psi_{(n,m)} - \mathbb{1}_{p_{(n,m)}}\right)\Big) \\
& - \frac{m\,n}{R^4}\Big(\left(\Lambda^-_{k,l}(n,m)^2\,\mathbb{1}_{p_{(n,m)}} - (\phi^-)^\dagger_{(n,m)}\,\phi^-_{(n,m)}\right)\left(\psi_{(n,m)} - \mathbb{1}_{p_{(n,m)}}\right)\Big) \\
& + \frac{m\,(n+1)}{R^4}\Big(\left(\Lambda^+_{k,l}(n-1,m-3)^2\,\mathbb{1}_{p_{(n,m)}} - \phi^+_{(n-1,m-3)}\,(\phi^+)^\dagger_{(n-1,m-3)}\right)\left(\psi_{(n,m)} - \mathbb{1}_{p_{(n,m)}}\right)\Big) \\
& + \frac{m\,(n+1)}{R^4}\Big(\left(\Lambda^-_{k,l}(n+1,m-3)^2\,\mathbb{1}_{p_{(n,m)}} - \phi^-_{(n+1,m-3)}\,(\phi^-)^\dagger_{(n+1,m-3)}\right)\left(\psi_{(n,m)} - \mathbb{1}_{p_{(n,m)}}\right)\Big) \\
& + \frac{n+1}{4R^2\,r^2}\left|m\,\psi_{(n,m)}\,\phi^+_{(n-1,m-3)} - (m-3)\,\phi^+_{(n-1,m-3)}\,\psi_{(n-1,m-3)} - 3\phi^+_{(n-1,m-3)}\right|^2 \\
& + \frac{n+1}{4R^2\,r^2}\left|m\,\psi_{(n,m)}\,\phi^-_{(n+1,m-3)} - (m-3)\,\phi^-_{(n+1,m-3)}\,\psi_{(n+1,m-3)} - 3\phi^-_{(n+1,m-3)}\right|^2 \\
& + \frac{n+2}{4R^2\,r^2}\left|(m+3)\,\psi_{(n+1,m+3)}\,\phi^+_{(n,m)} - m\,\phi^+_{(n,m)}\psi_{(n,m)} - 3\phi^+_{(n,m)}\right|^2 \\
& + \frac{n}{4R^2\,r^2}\left|(m+3)\,\psi_{(n-1,m+3)}\,\phi^-_{(n,m)} - m\,\phi^-_{(n,m)}\,\psi_{(n,m)} - 3\phi^-_{(n,m)}\right|^2\Big)\ .
\end{aligned}
\tag{B.44}
$$

Note that while the trace in (9.42) is taken over the full fibre space $V^{k,l}$ of the equivariant vector bundle (9.22), in (B.44) the trace over the $\mathrm{SU}(2)\times\mathrm{U}(1)$-representations $\underline{(n,m)}$ has already been evaluated.

Part III

Coulomb branch for rank two gauge groups in 3-dimensional $\mathcal{N} = 4$ gauge theories

Contents

14 Introduction and motivation

The moduli spaces of supersymmetric gauge theories with 8 supercharges have generically two branches: the Higgs and the Coulomb branch. In this third part of the thesis we focus on 3-dimensional $\mathcal{N} = 4$ gauge theories, for which both branches are hyper-Kähler spaces. Despite this fact, the branches are fundamentally different.

14.1 3-dimensional gauge theories with 8 supercharges

A 3-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) is defined by two choices: (i) a gauge group G, which is a compact Lie group, and (ii) a matter content, which corresponds to a quaternionic representation $\mathcal{R}$ of G. The field theory then comprises a vector multiplet, transforming in the adjoint representation of G. The multiplet contains the Yang-Mills field A and a triplet of real scalar fields (ϕ_1, ϕ_2, ϕ_3) taking values in $\mathfrak{g} = \mathrm{Lie(G)}$ at a given point. In addition, there are hypermultiplets, describing the matter content, transforming in the chosen representation $\mathcal{R}$. The hypermultiplets contain a total of $4N$ real scalars for some $N \geq 0$. To completely determine the Lagrangian, one needs to add gauge couplings for each factor in G as well as masses and FI-parameters, where the latter two serve as deformation parameters.

The R-symmetry group is $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, wherein the three scalars of the vector multiplet transform as triplet under $\mathrm{SU}(2)_L$, but are trivial under $\mathrm{SU}(2)_R$. In addition, there is a global symmetry group $\mathrm{G}_H \times \mathrm{G}_C$. The Higgs branch global symmetry G_H is given by the normaliser of G inside the hyper-Kähler isometry group $\mathrm{USp}(N)$ modulo the action of G. More interestingly here is the Coulomb branch global symmetry G_C that arises due to abelian factors in the gauge group, i.e. in the ultraviolet regime $\mathrm{G}_C = \mathrm{U}(1)^{\#\mathrm{U}(1) \text{ factors in G}}$. The infrared G_C may be enhanced to a non-abelian group whose maximal torus is precisely the UV group, see for instance [177] or the analyis of [178, 179].

14.1.1 Higgs branch

The Higgs branch $\mathcal{M}_H$ is understood as hyper-Kähler quotient

$$\mathcal{M}_H = \mathbb{R}^{4N} /\!/\!/ \, \mathrm{G} \tag{14.1}$$

in which the vanishing locus of the $\mathcal{N} = 4$ F-terms is quotient by the complexified gauge group. The F-term equations play the role of complex hyper-Kähler moment maps, while the transition to the complexified gauge group eliminates the necessity to impose the D-term constraints. The quaternionic dimension of the quotient equals $N - \dim(\mathrm{G})$. Moreover, this classical description is sufficient as the Higgs branch is protected from quantum corrections. The explicit quotient construction can be supplemented by the study of the Hilbert series, which allows to gain further understanding of $\mathcal{M}_H$ as a complex space.

14.1.2 Coulomb branch

Classically, the Coulomb branch $\mathcal{M}_C$ is the hyper-Kähler space

$$\mathcal{M}_C^{\mathrm{class}} \approx \left(\mathbb{R}^3 \times S^1\right)^{\mathrm{rk(G)}} / \mathcal{W}_{\mathrm{G}} \tag{14.2}$$

where $\mathcal{W}_{\mathrm{G}}$ is the Weyl group of G and rk(G) denotes the rank of G, which coincides with the quaternionic dimension of $\mathcal{M}_C$. This conclusion is based on the following arguments: in the vacuum, the scalar fields in the vector multiplet may acquire non-trivial vacuum expectation values (VEVs) and the terms in the scalar potential $\sim |[\phi_i, \phi_j]|^2$ enforce that all VEVs lie in the same Cartan subalgebra. Thus, the scalar VEVs parametrise a $\mathbb{R}^{3\cdot \mathrm{rk(G)}}$. However, this is an insufficient description and one needs to consider the VEVs of so-called *dual photons*. In 3 dimensions, one can (Hodge-)dualise an abelian field strength $F = \mathrm{d}A$ to a periodic scalar γ via $\mathrm{d}\gamma = \star F$. Assuming that the choice of VEVs breaks G to its maximal torus $\mathrm{T} \subset \mathrm{G}$ (or G being abelian), one introduces rk(G) dual photons, each parametrising an S^1. Lastly, the remaining gauge group acting after the choice of a Cartan subalgebra is precisely the Weyl group, which then completes the classical description of $\mathcal{M}_C$.

Nonetheless, there are two remaining points to be addressed: firstly, the geometry and topology of $\mathcal{M}_C$ are affected by quantum corrections. Secondly, the above description only holds for abelian groups or complete breaking of the gauge group. Following [180, 181], an accurate description of the classical Coulomb branch is

$$\mathcal{M}_C^{\mathrm{class}} \approx \left[\left(\mathbb{R}^{3\cdot \mathrm{rk(G)}} \setminus \Upsilon \right) \times \left(S^1 \right)^{\mathrm{rk(G)}} \right] / \mathcal{W}_{\mathrm{G}} \,, \tag{14.3}$$

with Υ the locus where the set of vector multiplet scalars ϕ_i does not fully break G to its maximal torus. In terms of W-boson masses M_j^W, one could characterise $\Upsilon = \left\{ \prod_j |M_j^W| = 0 \right\}$. Thus, in the complement of Υ the (classical) Coulomb branch of an non-abelian theory coincides with an abelian moduli space.

The quantum corrections comprise perturbative and non-perturbative contributions. For an abelian theory there are no non-perturbative corrections, and the perturbative expansion is exhaustive at 1-loop level [181]. In addition, the 1-loop corrections to the Coulomb branch can be computed [177, 182, 183]. It turns out that the metric on $\mathcal{M}_C$ is schematically given by

$$\mathrm{d}s^2 = U^{ab} \mathrm{d}\vec{\phi}_a \cdot \mathrm{d}\vec{\phi}_b + (U^{-1})_{ab} \left(g_a^{-2} \mathrm{d}\gamma_a + \vec{\omega}^{ac} \cdot \mathrm{d}\vec{\phi}_c \right) \left(g_b^{-2} \mathrm{d}\gamma_b + \vec{\omega}^{bd} \cdot \mathrm{d}\vec{\phi}_d \right) \,, \tag{14.4}$$

which is the hyper-Kähler metric on a generalised Taub-NUT space. It describes an $(S^1)^{\mathrm{rk(G)}}$-fibration over $\mathbb{R}^{3\cdot \mathrm{rk(G)}}$, in which an S^1 shrinks to a point whenever a hypermultiplet becomes massless. Interestingly, the perturbative metric at one-loop level for a non-abelian Coulomb branch [180, 184–187] can be cast in the same form as (14.4), but the form of U^{ab} and, consequently, the Dirac connection $\vec{\omega}$ differ. We can sketch the contributions to U^{ab} as follows:

$$U^{ab} \sim \frac{1}{g_a^2} \delta^{ab} + \underbrace{\sum_i \frac{\text{pre-factor}}{|M_i|}}_{\text{hypermultiplets}} - \underbrace{\sum_{j=1}^{\dim(\mathrm{G})-\mathrm{rk}(\mathrm{G})} \frac{\text{pre-factor}}{|M_j^W|}}_{\text{vector multiplets}} \,, \tag{14.5}$$

it is worth pointing out that hypermultiplets and vector multiplets contribute with opposite signs. Moreover, the W-boson masses are a true non-abelian effect. In contrast, non-perturbative corrections are more delicate to handle, but one can employ a non-renormalisation theorem, which states that the chiral ring is independent of the gauge couplings g_a. As non-perturbative corrections are dominated by $\exp(-C\,|M_j^W|/g_a^2)$, due to the instanton action, one could suppress the non-perturbative effects in the limit $\min_j |M_j^W| \gg \max_a g_a$. Evidently, this line of thought runs into trouble if the gauge group is not completely broken, i.e. a configuration in Υ, or if the theory is strongly coupled. In addition, Coulomb branches of non-abelian gauge theories are still not fully understood as there is no (Hodge-)dualisation of non-abelian gauge fields known.

Recently, the understanding of the Coulomb branch has been subject of active research from various viewpoints: the authors of [181] aim to provide a description for the quantum-corrected Coulomb branch of any $3d$ $\mathcal{N}=4$ gauge theory, with particular emphasis on the full Poisson algebra of the chiral ring $\mathbb{C}[\mathcal{M}_C]$. In particular, by the understanding of abelian Coulomb branches it is hoped that an *abelianisation map* can be constructed, which would allow to map any non-abelian Coulomb branch into an abelian problem. In contrast, a rigorous mathematical definition of the Coulomb branch itself lies at the heart of the attempts presented in [188–190]. In this part, we take the perspective centred around the *monopole formula* proposed in [191]; that is, the computation of the Hilbert series for the Coulomb branch allows to gain information on $\mathcal{M}_C$ as a complex space.

14.1.3 3-dimensional mirror symmetry

Intriligator and Seiberg [177] introduced a non-perturbative duality known as $3d$ mirror symmetry. This duality exchanges three pairs of objects for a pair of dual theories, these are (i) the $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$, (ii) the Higgs and Coulomb branch, and (ii) the masses and FI-parameters. The original examples concerned the class of Kronheimer gauge theories, constructed in [163], and revealed a connection between ALE spaces and the moduli space of instantons for the corresponding ADE gauge group. Since the Higgs branch is classically exact, while the Coulomb branch is not, quantum effects in one theory arise classically in the dual and vice versa.

In particular, $3d$ mirror symmetry was crucial in the understanding of the quantum corrected Coulomb branch [47,48], because (i) this duality applies also for abelian theories, (ii) it is known how to construct mirror duals for any abelian theory, (iii) all (abelian) mirror pairs can be derived from some *basic* mirror pairs, and (iv) the relevant topological soliton is known.

The last point is particularly interesting, as the belief was that the identification of soliton-like solutions in one theory and re-writing the theory in these *topological variables* might yield the mirror dual theory.

14.2 Monopole formula

The modern treatment of Coulomb branches in 3-dimensional gauge theories with 8 supercharges relies on the use of so-called *monopole operators*. Before exploring the details, let us briefly recall the set-up. Select an $\mathcal{N}=2$ subalgebra in the $\mathcal{N}=4$ algebra, which implies a decomposition of the $\mathcal{N}=4$ vector multiplet into an $\mathcal{N}=2$ vector multiplet (the bosonic sector contains a gauge field A and a real adjoint scalar σ) and an $\mathcal{N}=2$ chiral multiplet (containing a complex adjoint scalar Φ) which transforms in the adjoint representation of the gauge group G. In addition, the selection of an $\mathcal{N}=2$ subalgebra is equivalent to the choice of a complex structure on $\mathcal{M}_C$ and $\mathcal{M}_H$, which is the reason why one studies the branches only as complex and not as hyper-Kähler spaces. This is intuitively clear as the selection of the $\mathcal{N}=2$ subalgebra determines which two out of the three adjoint scalars (ϕ_1, ϕ_2, ϕ_3) are combined into the holomorphic (or chiral) combination $\Phi = \phi_1 + \mathrm{i}\phi_2$, while the remaining real scalar field is $\sigma = \phi_3$, for example. By the same reasoning, the $4N$ real scalars from the $\mathcal{N}=4$ hypermultiplets are paired up to complex linear combinations by the $\mathcal{N}=2$ choice to fit into the complex scalar fields of the chiral multiplets.

14.2.1 Monopole operators

The description of the Coulomb branch relies on 't Hooft monopole operators [192], which are local disorder operators [47] defined by specifying a Dirac monopole singularity

$$A_\pm \sim \frac{m}{2}\left(\pm 1 - \cos\theta\right)\mathrm{d}\varphi \tag{14.6}$$

for the gauge field, where $m \in \mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ and (θ, φ) are coordinates on the 2-sphere around the insertion point. An important consequence is that the *generalised Dirac quantisation condition* [193]

$$\exp(2\pi \mathrm{i} m) = \mathbb{1}_{\mathrm{G}} \tag{14.7}$$

has to hold. As proven in [194], the set of solutions to (14.7) equals the weight lattice $\Lambda_w(\widehat{\mathrm{G}})$ of the GNO (or Langlands) dual group $\widehat{\mathrm{G}}$, which is uniquely associated to the gauge group G.

For Coulomb branches of supersymmetric gauge theories, the monopole operators need to be supersymmetric as well, see for instance [48]. In a pure $\mathcal{N}=2$ theory, the supersymmetry condition amounts to the singular boundary condition

$$\sigma \sim \frac{m}{2r} \qquad \text{for} \quad r \to \infty \,, \tag{14.8}$$

for the real adjoint scalar in the $\mathcal{N}=2$ vector multiplet. Moreover, an $\mathcal{N}=4$ theory also allows for a non-vanishing vacuum expectation value of the complex adjoint scalar Φ of the adjoint-valued chiral multiplet. Compatibility with supersymmetry requires the real and imaginary part of Φ to take values in the centraliser $\mathfrak{h}_m$ of the *magnetic weight* m in $\mathfrak{g}$, i.e. in the Lie algebra of the residual gauge group $\mathrm{H}_m = \mathrm{Stab}_{\mathrm{G}}(m)$. This phenomenon gives rise to the distinction between two sets of monopole operators. Following [191], an $\mathcal{N}=2$ BPS monopole operator, i.e. an operator with the singularity (14.6) of the gauge field and the boundary condition (14.8) for the real scalar, is denoted as *bare monopole operator* if the VEV of Φ vanishes, and is denoted as *dressed monopole operator* if the VEV takes a non-trivial value in $\mathfrak{h}_m$.

14.2.2 Hilbert series

Studying the hyper-Kähler metric is a reasonable endeavour for the Higgs branch, as it is classically correct; in contrast, the (semi-classical) hyper-Kähler metric on the Coulomb branch is only reliable at weak coupling and if the corrections exhaust at 1-loop. It is assumed that all hypermultiplets and all W-bosons are massive. An alternative, algebraic perspective can be taken by the studying the chiral ring, i.e. the ring of holomorphic functions (in the superfields) subject to constraints imposed by the vacuum conditions.

On any finitely generated, graded, commutative algebra $\boldsymbol{A} = \oplus_{i\in\mathbb{N}} \boldsymbol{A}_i$ over a field $\mathbb{K}$, such as a polynomial ring in finitely many variables, one can introduce the Hilbert function on each graded piece $\boldsymbol{A}_i$ via

$$\mathrm{HF} : \boldsymbol{A}_i \mapsto \dim_{\mathbb{K}}(\boldsymbol{A}_i) \,. \tag{14.9a}$$

One then defines a formal power series in a dummy variable t (in physics literature *fugacity*) as generating function

$$\mathrm{HS}_{\boldsymbol{A}}(t) \coloneqq \sum_{i\in\mathbb{N}} \mathrm{HF}(\boldsymbol{A}_i)\, t^i \,, \tag{14.9b}$$

which is then called Hilbert series.

Suppose our moduli space is finitely generated and the chiral ring is denoted by R over the complex field. The vacuum conditions provide a set of relations which translate into an ideal I of R, such that the moduli space is then described by the R-module R/I of chiral functions. It is a well-known result, see for instance [195], that the Hilbert series of R/I is a rational function of the form

$$\mathrm{HS}_{R/I}(t) = \frac{K(t)}{\prod_{l=1}^{N}(1-t^{d_l})} \tag{14.10}$$

where d_l denotes the degrees for the N generators and $K(t)$ is an integer polynomial, sometimes denoted K-polynomial. The Hilbert series is not a topological invariant of the variety in question, but does depend on the embedding. However, one can extract various properties from the Hilbert series such as the (Krull) dimension of R/I which equals the order of the pole of HS at $t=1$. Moreover, by Stanley's Gorenstein criterion [196] the symmetry properties of the Hilbert series can be used to infer properties of the considered algebra. In [197,198] this palindromic symmetry of the numerator indicated the Calabi-Yau property of the (affine) variety under consideration.

The application of Hilbert series to supersymmetric gauge theories has been pioneered by [199–202] in the context of counting BPS operators and gauge invariants. In particular, for the Higgs branch of $3d$ $\mathcal{N}=4$ SYM this perspective amounts to using the Molien formula by integrating rational functions with the Haar measure, see for instance [203]. In the physical picture, the Hilbert series counts gauge invariant chiral operators, graded according to their dimension and quantum numbers under global symmetries. Moreover, the Hilbert series contains the full information of the quantum numbers of the chiral operators and all the relations among them.

For the Coulomb branch, the algebraic approach is less straightforward as the classical moduli space receives quantum corrections. Some hints were available by 3-dimensional mirror symmetry, but it was the realisation of the authors of [191] that counting dressed monopole operators is the key.

Particularly useful tools for studying commutative algebras and their Hilbert series are the *plethystic exponential* and the *plethystic logarithm*. These provide a systematic method for the study of generators and relations in the chiral ring of gauge theories. We refer to App. C for further details.

14.2.3 The formula

Dressed monopole operators and G-invariant functions of Φ are believed to generate the entire chiral ring $\mathbb{C}[\mathcal{M}_C]$. The corresponding Hilbert series allows for two points of view: seen via the *monopole formula*, each operator is precisely counted once in the Hilbert series — no over-counting appears. Evaluating the Hilbert series as rational function, however, provides an over-complete set of generators that, in general, satisfies relations. In order to count polynomials in the chiral ring, a notion of degree or dimension is required. Fortunately, in a CFT one employs the conformal dimension Δ, which for BPS states agrees with the $\mathrm{SU}(2)_R$ highest weight. Following [48,178,179,204], the conformal dimension of a BPS bare monopole operator of GNO-charge m is given by

$$\Delta(m) = \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho\in\mathcal{R}_i}|\rho(m)| - \sum_{\alpha\in\Phi_+}|\alpha(m)| \ , \tag{14.11}$$

where $\mathcal{R}_i$ denotes the set of all weights ρ of the G-representation in which the i-th flavour of $\mathcal{N}=4$ hypermultiplets transform. Moreover, Φ_+ denotes the set of positive roots α of the Lie algebra $\mathfrak{g}$ and provides the contribution of the $\mathcal{N}=4$ vector multiplet. The pairing $\rho(m)$ of weights ρ (which includes roots α) with magnetic weights m is defined by the duality pairing between weights and coweights of G (explained in Sec. 15.1.3 and 15.2.1). Bearing in mind the proposed classification of $3d$ $\mathcal{N}=4$ theories by [178], we restrict ourselves to *good* theories (i.e. $\Delta > \frac{1}{2}$ for all BPS monopoles).

If the centre $\mathcal{Z}(\widehat{\mathrm{G}})$ is non-trivial, then the monopole operators can be charged under this topological symmetry group and one can refine the counting on the chiral ring.

Putting all the pieces together, the by now well-established *monopole formula* of [191] reads

$$\mathrm{HS}_{\mathrm{G}}(t,z) = \sum_{m\in\Lambda_w\left(\widehat{\mathrm{G}}\right)/\mathcal{W}_{\widehat{\mathrm{G}}}} z^{J(m)} t^{\Delta(m)} P_{\mathrm{G}}(t,m) \ . \tag{14.12}$$

Here, the fugacity t counts the $\mathrm{SU}(2)_R$-spin, while the (multi-)fugacity z counts the quantum numbers $J(m)$ of the topological symmetry $\mathcal{Z}(\widehat{\mathrm{G}})$.

14.3 Outline

This part serves three purposes: firstly, we provide a geometric derivation of a sufficient set of monopole operators, called the *Hilbert basis*, that generates the entire chiral ring. Secondly, employing the Hilbert basis allows an explicit summation of (14.12), which we demonstrate for $\mathrm{rk(G)} = 2$ explicitly. Thirdly, we provide various examples for all rank two gauge groups and display how the knowledge of the Hilbert basis completely determines the Hilbert series.

The remainder of this third part is organised as follows: Ch. 15 is devoted to the exposition of our main points: after recapitulating basics on (root and weight) lattices and rational polyhedral cones in Sec. 15.1, we explain in Sec. 15.2 how the conformal dimension decomposes the Weyl chamber of $\widehat{\mathrm{G}}$ into a fan. Intersecting the fan with the weight lattice $\Lambda_w(\widehat{\mathrm{G}})$ introduces affine semi-groups, which are finitely generated by a unique set of irreducible elements — the Hilbert basis. Moving on to Sec. 15.3, we collect mathematical results that interpret the dressing factors $P_{\mathrm{G}}(t, m)$ as Poincaré series for the set of H_m-invariant polynomials on the Lie algebra $\mathfrak{h}_m$. Finally, we explicitly sum the unrefined Hilbert series in Sec. 15.4 and the refined Hilbert series in 15.5 utilising the knowledge about the Hilbert basis. After establishing the generic results, we provide a comprehensive collection of examples for all rank two gauge groups in Ch. 16-21. Lastly, Ch. 22 concludes.

The contents of this part originate from a collaboration [205] with A. Hanany.

15 Hilbert basis for monopole operators

The monopole formula allows to compute the Hilbert series for the Coulomb branch as an algebraic variety. Although the physical picture is quite clear, the actual computation can be very cumbersome and the extraction of the chiral ring generators is not at all clear. Here, we introduce the concept of Hilbert bases for affine semi-groups by examining the structure of the monopole formula. In particular, the interplay between the Weyl chamber of the GNO-dual group, which provides the summation range, and the conformal dimension, as the employed grading, is a vital ingredient for our consideration. This will partly resolve the tedious nature of the summation and identifies the set of generators.

15.1 Preliminaries

Let us recall some basic properties of Lie algebras, cf. [206], and combine them with the description of *strongly convex rational polyhedral cones* and *affine semi-groups*, cf. [207]. Moreover, we recapitulate the definition and properties of the GNO-dual group, which can be found in [194, 208].

15.1.1 Root and weight lattices of $\mathfrak{g}$

Let G be a Lie group with semi-simple Lie algebra $\mathfrak{g}$ and $\mathrm{rk}(\mathrm{G}) = r$. Moreover, $\widetilde{\mathrm{G}}$ is the universal covering group of G, i.e. the unique simply connected Lie group with Lie algebra $\mathfrak{g}$. Choose a maximal torus $\mathrm{T} \subset \mathrm{G}$ and the corresponding Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. Denote by $\mathbf{\Phi}$ the set of all roots $\alpha \in \mathfrak{t}^*$. By the choice of a hyperplane, one divides the root space into positive $\mathbf{\Phi}_+$ and negative roots $\mathbf{\Phi}_-$. In the half-space of positive roots one introduces the simple positive roots as irreducible basis elements and denotes their set by $\mathbf{\Phi}_s$. The roots span a lattice $\Lambda_r(\mathfrak{g}) \subset \mathfrak{t}^*$, the *root lattice*, with basis $\mathbf{\Phi}_s$.

Besides roots, one can always choose a basis in the complexified Lie algebra that gives rise to the notion of coroots $\alpha^\vee \in \mathfrak{t}$ which satisfy $\alpha\left(\beta^\vee\right) \in \mathbb{Z}$ for any $\alpha, \beta \in \mathbf{\Phi}$. Define $\alpha^\vee$ to be a simple coroot if and only if α is a simple root. Then the coroots span a lattice $\Lambda_r^\vee(\mathfrak{g})$ in $\mathfrak{t}$, called the *coroot lattice* of $\mathfrak{g}$.

The dual lattice $\Lambda_w(\mathfrak{g})$ of the coroot lattice is the set of points $\mu \in \mathfrak{t}^*$ for which $\mu(\alpha^\vee) \in \mathbb{Z}$ for all $\alpha \in \mathbf{\Phi}$. This lattice is called *weight lattice* of $\mathfrak{g}$. Choosing a basis $\boldsymbol{B}$ of simple coroots

$$\boldsymbol{B} \coloneqq \left\{\alpha^\vee \mid \alpha \in \mathbf{\Phi}_s\right\} \subset \mathfrak{t}\,, \tag{15.1}$$

one readily defines a basis for the dual space via

$$\boldsymbol{B}^* \coloneqq \{\lambda_\alpha \mid \alpha \in \mathbf{\Phi}_s\} \subset \mathfrak{t}^* \qquad \text{for} \quad \lambda_\alpha\left(\beta^\vee\right) = \delta_{\alpha,\beta}\,,\ \forall \alpha, \beta \in \mathbf{\Phi}_s\,. \tag{15.2}$$

The basis elements λ_α are precisely the fundamental weights of $\mathfrak{g}$ (or $\widetilde{\mathrm{G}}$) and they are a basis for the weight lattice.

Analogous, the dual lattice $\Lambda_{mw}(\mathfrak{g}) \subset \mathfrak{t}$ of the root lattice is the set of points $m \in \mathfrak{t}$ such that $\alpha(m) \in \mathbb{Z}$ for all $\alpha \in \mathbf{\Phi}$. In particular, the coroot lattice is a sublattice of $\Lambda_{mw}(\mathfrak{g})$.

As a remark, the lattices defined so far solely depend on the Lie algebra $\mathfrak{g}$, or equivalently on $\widetilde{G}$, but not on G. Because any group defined via $\widetilde{\mathrm{G}}/\Gamma$ for $\Gamma \subset \mathcal{Z}(\widetilde{\mathrm{G}})$ has the same Lie algebra.

15.1.2 Weight and coweight lattice of G

The weight lattice of the group G is the lattice of the infinitesimal characters, i.e. a character $\chi : \mathrm{T} \to \mathrm{U}(1)$ is a homomorphism, which is then uniquely determined by the derivative at the identity. Let $X \in \mathfrak{t}$ then $\chi(\exp(X)) = \exp(i\mu(X))$, wherein $\mu \in \mathfrak{t}^*$ is an *infinitesimal character* or weight of G. The weights form then a lattice $\Lambda_w(\mathrm{G}) \subset \mathfrak{t}^*$, because the exponential map translates the multiplicative structure of the character group into an additive structure. Most importantly, the following inclusion of lattices holds:

$$\Lambda_r(\mathfrak{g}) \subset \Lambda_w(\mathrm{G}) \subset \Lambda_w(\mathfrak{g}) \ . \tag{15.3}$$

Note that the weight lattice Λ_w of $\mathfrak{g}$ equals the weight lattice of the universal cover $\widetilde{\mathrm{G}}$.

As before, the dual lattice for $\Lambda_w(\mathrm{G})$ in $\mathfrak{t}$ is readily defined

$$\Lambda_w^*(\mathrm{G}) \coloneqq \mathrm{Hom}\left(\Lambda_w(\mathrm{G}), \mathbb{Z}\right) = \ker \left\{ \begin{matrix} \mathfrak{t} & \to & \mathrm{T} \\ X & \mapsto & \exp(2\pi \mathrm{i} X) \end{matrix} \right\} \ . \tag{15.4}$$

As we see, the *coweight lattice* $\Lambda_w^*(\mathrm{G})$ is precisely the set of solutions to the generalised Dirac quantisation condition (14.7) for G. In addition, an inclusion of lattices holds

$$\Lambda_r^\vee(\mathfrak{g}) \subset \Lambda_w^*(\mathrm{G}) \subset \Lambda_{mw}(\mathfrak{g}) \ , \tag{15.5}$$

which follows from dualising (15.3).

15.1.3 GNO-dual group and algebra

Following [194,208], a Lie algebra $\widehat{\mathfrak{g}}$ is the *magentic dual* of $\mathfrak{g}$ if its roots coincide with the coroots of $\mathfrak{g}$. Hence, the Weyl groups of $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ agree. The *magnetic dual group* $\widehat{\mathrm{G}}$ is, by definition, the unique Lie group with Lie algebra $\widehat{\mathfrak{g}}$ and weight lattice $\Lambda_w(\widehat{\mathrm{G}})$ equal to $\Lambda_w^*(\mathrm{G})$. In physics, $\widehat{\mathrm{G}}$ is called the GNO-dual group; while in mathematics, it is known under Langlands dual group.

15.1.4 Polyhedral cones

A (particular) *rational convex polyhedral cone* in $\mathfrak{t}$ is a set σ_B of the form

$$\sigma_B \equiv \mathrm{Cone}(\boldsymbol{B}) = \left\{ \sum_{\alpha^\vee \in \boldsymbol{B}} f_{\alpha^\vee} \ \alpha^\vee \mid f_{\alpha^\vee} \geq 0 \right\} \subseteq \mathfrak{t} \tag{15.6}$$

where $\boldsymbol{B} \subseteq \Lambda_r^\vee$, the basis of simple coroots, is finite. Moreover, we note that σ_B is a *strongly convex* cone, i.e. $\{0\}$ is a face of the cone, and of *maximal dimension*, i.e. $\dim(\sigma_B) = r$. Following [207], such cones σ_B are generated by the *ray generators* of their edges, where the ray generators in this case are precisely the simple coroots of $\mathfrak{g}$.

For a polyhedral cone $\sigma_B \subseteq \mathfrak{t}$ one naturally defines the *dual cone*

$$\sigma_B^\vee = \{ m \in \mathfrak{t}^* \mid m(u) \geq 0 \text{ for all } u \in \sigma_B \} \subseteq \mathfrak{t}^* \ . \tag{15.7}$$

One can prove that $\sigma_B^\vee$ equals the rational convex polyhedral cone generated by $\boldsymbol{B}^*$, i.e.

$$\sigma_B^\vee = \sigma_{B^*} = \mathrm{Cone}(\boldsymbol{B}^*) = \left\{ \sum_{\lambda \in \boldsymbol{B}^*} g_\lambda \ \lambda \mid g_\lambda \geq 0 \right\} \subseteq \mathfrak{t}^* \ , \tag{15.8}$$

which is well-known under the name *(closed) principal Weyl chamber*. By the very same arguments as above, the cone σ_{B^*} is generated by its ray generators, which are the fundamental weights of $\mathfrak{g}$.

For any $m \in \mathfrak{t}$ and $d \geq 0$, let us define an *affine hyperplane* $H_{m,d}$ and *closed linear half-spaces* $H^{\pm}_{m,d}$ in $\mathfrak{t}^*$ via

$$H_{m,d} \coloneqq \{\mu \in \mathfrak{t}^* \mid \mu(m) = d\} \subseteq \mathfrak{t}^* \,, \tag{15.9a}$$

$$H^{\pm}_{m,d} \coloneqq \{\mu \in \mathfrak{t}^* \mid \mu(m) \geq \pm d\} \subseteq \mathfrak{t}^* \,. \tag{15.9b}$$

If $d = 0$ then $H_{m,0}$ is hyperplane through the origin, sometimes denoted as *central* affine hyperplane. Following [209, Thm. 1.3], a cone $\sigma \subset \mathbb{R}^n$ is finitely generated if and only if it is the finite intersection of central closed linear half spaces.

This result allows to make contact with the usual definition of the Weyl chamber. Since we know that σ_{B^*} is finitely generated by the fundamental weights $\{\lambda_\alpha\}$ and the dual basis is $\{\alpha^\vee\}$, one arrives at $\sigma_{B^*} = \cap_{\alpha \in \mathbf{\Phi}_s} H^+_{\alpha^\vee,0}$; thus, the dominant Weyl chamber is obtained by cutting the root space along the hyperplanes orthogonal to some root and selecting the cone which has only positive entries.

Remark Consider the group SU(2), then the fundamental weight is simply $\frac{1}{2}$ such that $\Lambda_w^{\mathrm{SU}(2)} = \mathrm{Span}_{\mathbb{Z}}(\frac{1}{2}) = \mathbb{Z} \cup \{\mathbb{Z} + \frac{1}{2}\}$. Moreover, the corresponding cone (Weyl chamber) will be denoted by $\sigma_{\boldsymbol{B^*}}^{\mathrm{SU}(2)} = \mathrm{Cone}(\frac{1}{2})$.

15.2 Effect of conformal dimension

Next, while considering the conformal dimension $\Delta(m)$ as map between two Weyl chambers we will stumble across the notion of *affine semi-groups*, which are known to constitute the combinatorial background for toric varieties [207].

15.2.1 Conformal dimensions — revisited

Recalling the conformal dimension Δ to be interpreted as the highest weight under $\mathrm{SU}(2)_R$, it can be understood as the following map[28]

$$\Delta : \begin{array}{ccc} \sigma_{B^*}^{\widehat{\mathrm{G}}} \cap \Lambda_w(\widehat{\mathrm{G}}) & \to & \sigma_{\boldsymbol{B^*}}^{\mathrm{SU}(2)} \cap \Lambda_w(\mathrm{SU}(2)) \\ m & \mapsto & \Delta(m) \end{array} . \tag{15.10}$$

Where $\sigma_{B^*}^{\widehat{\mathrm{G}}}$ is the cone spanned by the fundamental weights of $\widehat{\mathfrak{g}}$, i.e. the dual basis of the simple roots $\mathbf{\Phi}_s$ of $\mathfrak{g}$. Likewise, $\sigma_{\boldsymbol{B^*}}^{\mathrm{SU}(2)}$ is the Weyl chamber for $\mathrm{SU}(2)_R$. Upon continuation, Δ becomes a map between the dominant Weyl chamber of $\widehat{\mathrm{G}}$ and $\mathrm{SU}(2)_R$

$$\Delta : \begin{array}{ccc} \sigma_{B^*}^{\widehat{\mathrm{G}}} & \to & \sigma_{\boldsymbol{B^*}}^{\mathrm{SU}(2)} \\ m & \mapsto & \Delta(m) \end{array} . \tag{15.11}$$

By definition, the conformal dimension (14.11) has two types of contributions: firstly, a positive contribution $|\rho(m)|$ for a weight $\rho \in \Lambda_w(\mathrm{G}) \subset \mathfrak{t}^*$ and a magnetic weight $m \in \Lambda_w(\widehat{\mathrm{G}}) \subset \widehat{\mathfrak{t}}^*$. By definition $\Lambda_w(\widehat{\mathrm{G}}) = \Lambda_w^*(\mathrm{G})$; thus, m is a coweight of G and $\rho(m)$ is the duality paring. Secondly, a negative contribution $-|\alpha(m)|$ for a positive root $\alpha \in \mathbf{\Phi}_+$ of $\mathfrak{g}$. By the same arguments, $\alpha(m)$

[28]With respect to the use in the monopole formula.

is the duality pairing of weights and coweights. The paring is also well-defined on the entire the cone.

15.2.2 Fan generated by conformal dimension

The individual absolute values in Δ allow for another interpretation: we use them to associate a collection of affine central hyperplanes and closed linear half-spaces

$$H_{\mu,0}^{\pm} = \{m \in \mathfrak{t} \mid \pm\mu(m) \geq 0\} \subset \mathfrak{t} \qquad \text{and} \qquad H_{\mu,0} = \{m \in \mathfrak{t} \mid \mu(m) = 0\} \subset \mathfrak{t}\,. \tag{15.12}$$

Here, μ ranges over all weights ρ and all positive roots α appearing in the theory. If two weights μ_1, μ_2 are (integer) multiples of each other, then $H_{\mu_1,0} = H_{\mu_2,0}$ and we can reduce the number of relevant weights. From now on, denote by Γ the set of weights ρ and positive roots α which are not multiples of one another. Then the conformal dimension contains $Q := |\Gamma| \in \mathbb{N}$ distinct hyperplanes such that there exist 2^Q different finitely generates cones

$$\sigma_{\epsilon_1,\epsilon_2,\dots,\epsilon_Q} := H_{\mu_1,0}^{\epsilon_1} \cap H_{\mu_2,0}^{\epsilon_2} \cap \dots \cap H_{\mu_Q,0}^{\epsilon_Q} \subset \mathfrak{t} \qquad \text{with} \quad \epsilon_i = \pm \qquad \text{for} \quad i = 1,\dots,Q\,. \tag{15.13}$$

By construction, each cone $\sigma_{\epsilon_1,\epsilon_2,\dots,\epsilon_Q}$ is a strongly convex[29] rational polyhedral cone, which is of dimension r whenever $\sigma_{\epsilon_1,\epsilon_2,\dots,\epsilon_Q} \neq \{0\}$. Consequently, each non-trivial cone is generated by its ray generators and these can be chosen to be lattice points of $\Lambda_w(\widehat{\mathrm{G}})$. Moreover, the restriction of Δ to any $\sigma_{\epsilon_1,\epsilon_2,\dots,\epsilon_Q}$ yields a linear map, because we effectively resolved the absolute values by defining these cones.

It is, however, sufficient to restrict the considerations to the Weyl chamber of $\widehat{\mathrm{G}}$; hence, we simply intersect the cones with the hyperplanes defining $\sigma_{B^*}^{\widehat{\mathrm{G}}}$, i.e.

$$C_p \equiv C_{\epsilon_1,\epsilon_2,\dots,\epsilon_Q} := \sigma_{\epsilon_1,\epsilon_2,\dots,\epsilon_Q} \cap \sigma_{B^*}^{\widehat{\mathrm{G}}} \qquad \text{with} \quad p = (\epsilon_1,\epsilon_2,\dots,\epsilon_Q)\,. \tag{15.14}$$

Naturally, we would like to know for which $\mu \in \Lambda_w(\mathrm{G})$ the hyperplane $H_{\mu,0}$ intersects the Weyl chamber $\sigma_{B^*}^{\widehat{\mathrm{G}}}$ non-trivially, i.e. not only in the origin. Let us emphasis the differences of the Weyl chamber (and their dual cones) of G and $\widehat{\mathrm{G}}$:

$$\sigma_{B^*}^{\mathrm{G}}=\mathrm{Cone}\left(\lambda_\alpha \mid \lambda_\alpha(\beta^\vee)=\delta_{\alpha,\beta}\ ,\forall\alpha,\beta\in\mathbf{\Phi}_s\right)\subset\mathfrak{t}^* \xleftrightarrow{\;*\;} \sigma_B^{\mathrm{G}}=\mathrm{Cone}\left(\alpha^\vee \mid \forall\alpha\in\mathbf{\Phi}_s\right)\subset\mathfrak{t}, \tag{15.15a}$$
$$\sigma_{B^*}^{\widehat{\mathrm{G}}}=\mathrm{Cone}\left(m_\alpha \mid \beta(m_\alpha)=\delta_{\alpha,\beta}\ ,\forall\alpha,\beta\in\mathbf{\Phi}_s\right)\subset\mathfrak{t} \xleftrightarrow{\;*\;} \sigma_B^{\widehat{\mathrm{G}}}=\mathrm{Cone}\left(\alpha \mid \forall\alpha\in\mathbf{\Phi}_s\right)\subset\mathfrak{t}^*. \tag{15.15b}$$

It is possible to prove the following statements:

1. If $\mu \in \mathrm{Int}\left(\sigma_B^{\widehat{\mathrm{G}}} \cup (-\sigma_B^{\widehat{\mathrm{G}}})\right)$, i.e. $\mu = \sum_{\alpha\in\mathbf{\Phi}_s} g_\alpha \alpha$ where either all $g_\alpha > 0$ *or* all $g_\alpha < 0$, then $H_{\mu,0} \cap \sigma_{B^*}^{\widehat{\mathrm{G}}} = \{0\}$.

2. If $\mu \in \partial\left(\sigma_B^{\widehat{\mathrm{G}}} \cup (-\sigma_B^{\widehat{\mathrm{G}}})\right)$ and $\mu \neq 0$, i.e. $\mu = \sum_{\alpha\in\mathbf{\Phi}_s} g_\alpha \alpha$ where at least one $g_\alpha = 0$, then $H_{\mu,0}$ intersects $\sigma_{B^*}^{\widehat{\mathrm{G}}}$ at one of its boundary faces.

3. If $\mu \notin \sigma_B^{\widehat{\mathrm{G}}} \cup (-\sigma_B^{\widehat{\mathrm{G}}})$, i.e. $\mu = \sum_{\alpha\in\mathbf{\Phi}_s} g_\alpha \alpha$ with at least one $g_\alpha > 0$ *and* at least one $g_\beta < 0$, then $\left(H_{\mu,0} \cap \sigma_{B^*}^{\widehat{\mathrm{G}}}\right) \setminus \{0\} \neq \emptyset$.

[29]The set of weights Γ contains always the simple roots, whose associated hyperplanes give a Weyl chamber of the GNO dual group. Thus, the cones are subcones of strongly convex cones.

Consequently, a weight $\boldsymbol{\mu} \in \boldsymbol{\Lambda}_w(\mathbf{G})$ appearing in $\boldsymbol{\Delta}$ leads to a hyperplane intersecting the Weyl chamber of $\widehat{\mathrm{G}}$ non-trivially if and only if neither $\boldsymbol{\mu}$ nor $-\boldsymbol{\mu}$ lies in the rational cone spanned by the simple roots $\boldsymbol{\Phi}_s$ of G.

Therefore, the contributions $-|\alpha(m)|$, for $\alpha \in \boldsymbol{\Phi}_+$, of the vector multiplet never yield a relevant hyperplane. From now on, assume that trivial cones C_p are omitted in the index set I for p. The appropriate geometric object to consider is then the *fan* $F_\Delta \subset \mathfrak{t}$ defined by the family $F_\Delta = \{C_p \ , \ p \in I\}$ in $\mathfrak{t}$. A fan F is a family of non-empty polyhedral cones such that (i) every non-empty face of a cone in F is a cone in F and (ii) the intersection of any two cones in F is a face of both. In addition, the fan F_Δ defined above is a *pointed fan*, because $\{0\}$ is a cone in F_Δ (called the trivial cone).

15.2.3 Semi-groups

Although we already know the cone generators for the fan F_Δ, we have to distinguish them from the generators of $F_\Delta \cap \Lambda_w(\widehat{\mathrm{G}})$, i.e. we need to restrict to the weight lattice of $\widehat{\mathrm{G}}$. The first observation is that

$$S_p \coloneqq C_p \cap \Lambda_w(\widehat{\mathrm{G}}) \qquad \text{for} \quad p \in I \tag{15.16}$$

are semi-groups, i.e. sets with an associative binary operation. This is because the addition of elements is commutative, but there is no inverse defined as *subtraction* would lead out of the cone. Moreover, the S_p satisfy further properties, which we now simply collect, see for instance [209]. Firstly, the S_p are affine semi-groups, which are semi-groups that can be embedded in $\mathbb{Z}^n$ for some n. Secondly, every S_p possesses an identity element, here $m = 0$, and such semi-groups are called *monoids*. Thirdly, the S_p are *positive* because the only invertible element is $m = 0$.

Now, according to *Gordan's Lemma* [207, 209], we know that every S_p is finitely generated, because all C_p's are finitely generated, rational polyhedral cones. Even more is true, since the division into the C_p is realised via affine hyperplanes $H_{\mu_i,0}$ passing through the origin, the C_p are strongly convex rational cones of maximal dimension. Then [207, Prop. 1.2.22.] holds and we know that there exist a unique minimal generating set for S_p, which is called *Hilbert basis*.

The Hilbert basis $\mathcal{H}(S_p)$ is defined via

$$\mathcal{H}(S_p) \coloneqq \{m \in S_p \mid m \text{ is irreducible}\} \ , \tag{15.17}$$

where an element is called *irreducible* if and only if $m = x + y$ for $x, y \in S_p$ implies $x = 0$ or $y = 0$. The importance of the Hilbert basis is that it is a unique, finite, minimal set of irreducible elements that generate S_p. Moreover, $\mathcal{H}(S_p)$ always contains the ray generators of the edges of C_p. The elements of $\mathcal{H}(S_p)$ are sometimes called *minimal generators*.

As a remark, there exist various algorithms for computing the Hilbert basis, which are, for example, discussed in [195, 210]. For the computations presented in this part, we used the `Sage` module *Toric varieties* programmed by A. Novoseltsev and V. Braun.

After the exposition of the idea to employ the conformal dimension to define a fan in the Weyl chamber of $\widehat{\mathrm{G}}$, for which the intersection with the weight lattice leads to affine semi-groups, we now state the main consequence:

The collection $\{\boldsymbol{\mathcal{H}(S_p)} \ , \boldsymbol{p \in I}\}$ of all Hilbert bases is the set of necessary (bare) monopole operators for a theory with conformal dimension $\boldsymbol{\Delta}$.

At this stage we did not include the Casimir invariance described by the dressing factors $P_G(t, m)$. For a generic situation, the bare and dressed monopole operators for a GNO-charge $m \in \mathcal{H}(S_p)$ for some p are all necessary generators for the chiral ring $\mathbb{C}[\mathcal{M}_C]$. However, there will be scenarios for which there exists a further reduction of the number of generators. For those cases, we will comment and explain the cancellations.

15.3 Dressing of monopole operators

One crucial ingredient of the monopole formula of [191] are the dressing factors $P_{\mathrm{G}}(t,m)$ and this section provides an algebraic understanding. We refer to [206, 211, 212] for the exposition of the mathematical details used here.

It is known that in $\mathcal{N}=4$ the $\mathcal{N}=2$ BPS-monopole operator V_m is compatible with a constant background of the $\mathcal{N}=2$ adjoint complex scalar Φ, provided $\mathrm{Re}(\Phi)$ and $\mathrm{Im}(\Phi)$ take values on the Lie algebra $\mathfrak{h}_m$ of the residual gauge group $\mathrm{H}_m \subset \mathrm{G}$, i.e. the stabiliser of m in G. Consequently, each bare monopole operator V_m is compatible with *any* H_m-invariant polynomial on $\mathfrak{h}_m$. We will now argue that the dressing factors $P_{\mathrm{G}}(t,m)$ are to be understood as Hilbert (or Poincaré) series for this so-called *Casimir-invariance.*

15.3.1 Chevalley-Restriction theorem

Let G be a Lie group of rank l with a semi-simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and G acts via the adjoint representation on $\mathfrak{g}$. Denote by $\mathfrak{P}(\mathfrak{g})$ the algebra of all polynomial functions on $\mathfrak{g}$. The action of G extends to $\mathfrak{P}(\mathfrak{g})$ and $\mathfrak{I}(\mathfrak{g})^{\mathrm{G}}$ denotes the set of G-invariant polynomials in $\mathfrak{P}(\mathfrak{g})$. In addition, denote by $\mathfrak{P}(\mathfrak{h})$ the algebra of all polynomial functions on $\mathfrak{h}$. The Weyl group $\mathcal{W}_{\mathrm{G}}$, which acts naturally on $\mathfrak{h}$, acts also on $\mathfrak{P}(\mathfrak{h})$, and $\mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}}$ denotes the Weyl-invariant polynomials on $\mathfrak{h}$. The *Chevalley-Restriction Theorem* now states

$$\mathfrak{I}(\mathfrak{g})^{\mathrm{G}} \cong \mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}} \; , \tag{15.18}$$

where the isomorphism is given by the restriction map $p \mapsto p|_{\mathfrak{h}}$ for $p \in \mathfrak{I}(\mathfrak{g})^{\mathrm{G}}$.

Therefore, the study of H_m-invariant polynomials on $\mathfrak{h}_m$ is reduced to $\mathcal{W}_{\mathrm{H}_m}$-invariant polynomials on a Cartan subalgebra $\mathfrak{t}_m \subset \mathfrak{h}_m$.

15.3.2 Finite reflection groups

It is due to a theorem by Chevalley [213], in the context of *finite reflection groups*, that there exist l algebraically independent homogeneous elements $p_1,\ldots,p_l$ of positive degrees d_i, for $i=1,\ldots,l$, such that

$$\mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}} = \mathbb{C}\left[p_1,\ldots,p_l\right] \; . \tag{15.19}$$

In addition, the degrees d_i satisfy

$$|\mathcal{W}_{\mathrm{G}}| = \prod_{i=1}^{l} d_i \qquad \text{and} \qquad \sum_{i=1}^{d}(d_i-1) = \text{number of reflections in } \mathcal{W}_{\mathrm{G}} \; . \tag{15.20}$$

The degrees d_i are unique [212] and tabulated for all Weyl groups, see for instance [212, Sec. 3.7]. However, the generators p_i themselves are not uniquely determined.

15.3.3 Poincaré or Molien series

On the one hand, the Poincaré series for the $\mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}}$ is simply given by

$$P_{\mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}}}(t) = \prod_{i=1}^{l} \frac{1}{1-t^{d_i}} \; . \tag{15.21}$$

On the other hand, since $\mathfrak{h}$ is a l-dimensional complex vector space and $\mathcal{W}_{\mathrm{G}}$ a finite group, the generating function for the invariant polynomials is known as Molien series [214]

$$P_{\mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}}}(t) = \frac{1}{|\mathcal{W}_{\mathrm{G}}|} \sum_{g\in\mathcal{W}_{\mathrm{G}}} \frac{1}{\det\left(\mathbb{1} - t\ g\right)} \ . \tag{15.22}$$

Therefore, the dressing factors $P_{\mathrm{G}}(t,m)$ in the Hilbert series (14.12) for the Coulomb branch are the Poincaré series for graded algebra of H_m-invariant polynomials on $\mathfrak{h}_m$.

15.3.4 Harish-Chandra isomorphism

In [191], the construction of the $P_{\mathrm{G}}(t,m)$ is based on Casimir invariants of G and H_m; hence, we need to make contact with that idea. Casimir invariants live in the centre $\mathcal{Z}(\mathfrak{U}(\mathfrak{g}))$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$. Fortunately, the Harish-Candra isomorphism [215] provides us with

$$\mathcal{Z}(\mathfrak{U}(\mathfrak{g})) \cong \mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}} \ . \tag{15.23}$$

Consequently, $\mathcal{Z}(\mathfrak{U}(\mathfrak{g}))$ is a polynomial algebra with l algebraically independent homogeneous elements that have the same positive degrees d_i as the generators of $\mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}}$. It is known that for semi-simple groups G these generators can be chosen to be the rk(G) Casimir invariants; i.e. the space of Casimir-invariants is freely generated by l generators (together with the unity).

15.3.5 Conclusions

So far, G (and H_m) had been restricted to be semi-simple. However, in most cases H_m is a direct product group of semi-simple Lie groups and U(1)-factors. We proceed in two steps: firstly, U(1) acts trivially on its Lie-algebra $\cong \mathbb{R}$, thus all polynomials are invariant and we obtain

$$\mathfrak{I}(\mathbb{R})^{\mathrm{U}(1)} = \mathbb{R}[x] \qquad \text{and} \qquad P_{\mathrm{U}(1)}(t) = \frac{1}{1-t} \ . \tag{15.24}$$

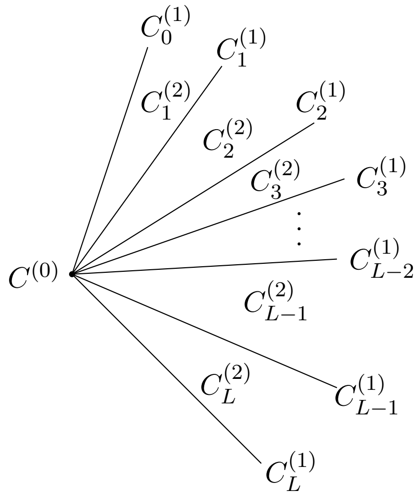
Secondly, each factor G_i of a direct product $\mathrm{G}_1 \times \cdots \times \mathrm{G}_M$ acts via the adjoint representation on its own Lie algebra $\mathfrak{g}_i$ and trivially on all other $\mathfrak{g}_j$ for $j \neq i$. Hence, the space of $\mathrm{G}_1 \times \cdots \times \mathrm{G}_M$-invariant polynomials on $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_M$ factorises into the product of the $\mathfrak{I}(\mathfrak{g}_i)^{\mathrm{G}_i}$ such that

$$\mathfrak{I}(\oplus_i \mathfrak{g}_i)^{\prod_i \mathrm{G}_i} = \prod_i \mathfrak{I}(\mathfrak{g}_i)^{\mathrm{G}_i} \qquad \text{and} \qquad P_{\mathfrak{I}(\oplus_i \mathfrak{g}_i)^{\prod_i \mathrm{G}_i}}(t) = \prod_i P_{\mathfrak{I}(\mathfrak{g}_i)^{\mathrm{G}_i}}(t) \ . \tag{15.25}$$

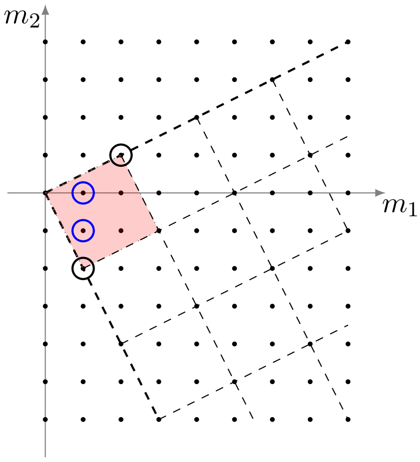
For abelian groups G, the Hilbert series for the Coulomb branch factorises in the Poincaré series G-invariant polynomials on $\mathfrak{g}$ times the contribution of the (bare) monopole operators. In contrast, the Hilbert series does not factorise for non-abelian groups G as the stabiliser $\mathrm{H}_m \subset \mathrm{G}$ depends on m.

15.4 Consequences for unrefined Hilbert series

The aforementioned partition of the Weyl chamber $\sigma_{B^*}^{\widehat{\mathrm{G}}}$ into a fan, induced by the conformal dimension Δ, and the subsequent collection of semi-groups in $\Lambda_w(\widehat{\mathrm{G}})/\mathcal{W}_{\widehat{\mathrm{G}}}$ provides an immediate consequence for the unrefined Hilbert series. For simplicity, we illustrate the consequences for a rank two example. Assume that the Weyl chamber is divided into a fan generated the 2-dimensional cones $C_p^{(2)}$ for $p = 1, \ldots, L$, as sketched in Fig. 15.1b. For each cone, one has two 1-dimensional cones $C_{p-1}^{(1)}$, $C_p^{(1)}$ and the trivial cone $C^{(0)} = \{0\}$ as boundary, i.e. $\partial C_p^{(2)} = C_{p-1}^{(1)} \cup C_p^{(1)}$, where $C_{p-1}^{(1)} \cap C_p^{(1)} = C^{(0)}$.



(a)



(b)

Figure 15.1: *A representative fan, which is spanned by the 2-dimensional cones $C_p^{(2)}$ for $p = 1, \ldots, L$, is displayed in 15.1a. In addition, 15.1b contains a 2-dimensional cone with a Hilbert basis of the two ray generators (black) and two additional minimal generators (blue). The ray generators span the fundamental parallelotope (red region).*

The Hilbert basis $\mathcal{H}(S_p^{(2)})$ for $S_p^{(2)} \coloneqq C_p^{(2)} \cap \Lambda_w^{\widehat{\mathrm{G}}}$ contains the ray generators $\{x_{p-1}, x_p\}$, such that $\mathcal{H}(S_p^{(1)}) = \{x_p\}$, and potentially other minimal generators u_κ^p for κ in some *finite* index set. Although any element $s \in S_p^{(2)}$ can be generated by $\{x_{p-1}, x_p, \{u_\kappa^p\}_\kappa\}$, the representation $s = a_0 x_{p-1} + a_1 x_p + \sum_\kappa b_\kappa u_\kappa^p$ is *not* unique. Therefore, great care needs to be taken if one would like to sum over all elements in $S_p^{(2)}$. A possible realisation employs the *fundamental parallelotope*, also known as (the closure of) the fundamental mesh:

$$\mathcal{P}(C_p^{(2)}) \coloneqq \{a_0 x_{p-1} + a_1 x_p \mid 0 \leq a_0, a_1 \leq 1\} , \tag{15.26}$$

see also Fig. 15.1b. The number of points contained in $\mathcal{P}(C_p^{(2)})$ is computed by the discriminant

$$d(C_p^{(2)}) \coloneqq |\det(x_{p-1}, x_p)| . \tag{15.27}$$

However, as known from solid state physics, the discriminant counts each of the four boundary lattice points by $\frac{1}{4}$; thus, there are $d(C_p^{(2)}) - 1$ points in the interior. Remarkably, each point $s \in \mathrm{Int}(\mathcal{P}(C_p^{(2)}))$ is given by positive integer combinations of the $\{u_\kappa^p\}_\kappa$ alone. A translation of $\mathcal{P}(C_p^{(2)})$ by non-negative integer combinations of the ray-generators $\{x_{p-1}, x_p\}$ fills the entire semi-group $S_p^{(2)}$ and each point is only realised once.

Now, we employ this fact to evaluate the un-refined Hilbert series explicitly.

$$\begin{aligned}
\mathrm{HS}_{\mathrm{G}}(t) &= \sum_{m \in \Lambda_w(\widehat{\mathrm{G}})/\mathcal{W}_{\widehat{\mathrm{G}}}} t^{\Delta(m)} P_{\mathrm{G}}(t,m) \\
&= P_{\mathrm{G}}(t,0) + \sum_{p=0}^{L} P_{\mathrm{G}}(t,x_p) \sum_{n_p>0} t^{n_p \Delta(x_p)} + \sum_{p=1}^{L} \sum_{n_{p-1},n_p>0} P_{\mathrm{G}}(t, x_{p-1}+x_p) t^{\Delta(n_{p-1}x_{p-1}+n_p x_p)} \\
&\qquad + \sum_{p=1}^{L} \sum_{s \in \mathrm{Int}(\mathcal{P}(C_p^{(2)}))} \sum_{n_{p-1},n_p \geq 0} P_{\mathrm{G}}(t,s) t^{\Delta(s+n_{p-1}x_{p-1}+n_p x_p)}
\end{aligned}$$

$$
\begin{aligned}
&= P_{\mathrm{G}}(t,0) + \sum_{p=0}^{L} P_{\mathrm{G}}(t,x_p) \frac{t^{\Delta(x_p)}}{1-t^{\Delta(x_p)}} + \sum_{p=1}^{L} \frac{P_{\mathrm{G}}(t,x_{p-1}+x_p)\ t^{\Delta(x_{p-1})+\Delta(x_p)}}{\left(1-t^{\Delta(x_{p-1})}\right)\left(1-t^{\Delta(x_p)}\right)} \\
&\quad + \sum_{p=1}^{L} \sum_{s\in \mathrm{Int}(\mathcal{P}(C_p^{(2)}))} \frac{P_{\mathrm{G}}(t,s)\ t^{\Delta(s)}}{\left(1-t^{\Delta(x_{p-1})}\right)\left(1-t^{\Delta(x_p)}\right)} \\
&= \frac{P_{\mathrm{G}}(t,0)}{\prod_{p=0}^{L}\left(1-t^{\Delta(x_p)}\right)} \left\{ \prod_{q=0}^{L}\left(1-t^{\Delta(x_q)}\right) + \sum_{q=0}^{L} \frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)} t^{\Delta(x_q)} \prod_{\substack{r=0\\ r\neq q}}^{L}\left(1-t^{\Delta(x_r)}\right) \right. \qquad (15.28) \\
&\quad \left. + \sum_{q=1}^{L} \frac{P_{\mathrm{G}}(t,C_q^{(2)})}{P_{\mathrm{G}}(t,0)} \left[t^{\Delta(x_{q-1})+\Delta(x_q)} + \sum_{s\in \mathrm{Int}(\mathcal{P}(C_q^{(2)}))} t^{\Delta(s)} \right] \prod_{\substack{r=0\\ r\neq q-1,q}}^{L} \left(1-t^{\Delta(x_r)}\right) \right\} .
\end{aligned}
$$

Next, we utilise that the classical dressing factors, for rank two examples, only have three different values: in the 2-dimensional interior of the Weyl chamber W, the residual gauge group is the maximal torus T and $P_{\mathrm{G}}(t,\mathrm{Int}W) \equiv P_2(t) = \prod_{i=1}^{2} \frac{1}{(1-t)}$. Along the 1-dimensional boundaries, the residual gauge group is a non-abelian subgroup H such that $\mathrm{T} \subset \mathrm{H} \subset \mathrm{G}$ and the $P_{\mathrm{G}}(t,\partial W \setminus \{0\}) \equiv P_1(t) = \prod_{i=1}^{2} \frac{1}{(1-t^{b_i})}$, for the two degree b_i Casimirs of H. At the 0-dimensional boundary of the boundary, the group is unbroken and $P_{\mathrm{G}}(t,0) \equiv P_0(t) = \prod_{i=1}^{2} \frac{1}{(1-t^{d_i})}$ contains the Casimir invariants of G of degree d_i. Thus, there are a few observations to be addressed.

1. The numerator of (15.28), which is everything in the curly brackets $\{\ldots\}$, starts with a one and is a polynomial with integer coefficients, which is required for consistency.

2. The denominator of (15.28) is given by $P_{\mathrm{G}}(t,0)/\prod_{p=0}^{L}(1-t^{\Delta(x_p)})$ and describes the poles due to the Casimir invariants of G and the bare monopole $(x_p, \Delta(x_p))$ which originate from ray generators x_p.

3. The numerator has contributions $\sim t^{\Delta(x_p)}$ for the ray generators with pre-factors $\frac{P_1(t)}{P_0(t)} - 1$ for the two outermost rays $p=0$, $p=L$ and pre-factors $\frac{P_2(t)}{P_0(t)} - 1$ for the remaining ray generators. None of the two pre-factors has a constant term as $P_i(t\to 0) = 1$ for each $i = 0,1,2$. Also $\deg(1/P_0(t)) \geq \deg(1/P_1(t)) \geq \deg(1/P_2(t)) = 2$ and

$$
\frac{P_2(t)}{P_0(t)} = \frac{(1-t^{d_1})(1-t^{d_2})}{(1-t)(1-t)} = \sum_{i=0}^{d_1-1}\sum_{j=0}^{d_2-1} t^{i+j} \qquad (15.29)
$$

is a polynomial for any rank two group. For the examples considered here, we also obtain

$$
\frac{P_1(t)}{P_0(t)} = \frac{(1-t^{d_1})(1-t^{d_2})}{(1-t^{b_1})(1-t^{b_2})} = \frac{(1-t^{k_1 b_1})(1-t^{k_2 b_2})}{(1-t^{b_1})(1-t^{b_2})} = \sum_{i=0}^{b_1-1}\sum_{j=0}^{b_2-1} t^{i\cdot k_1 + j\cdot k_2} \qquad (15.30)
$$

for some $k_1, k_2 \in \mathbb{N}$. In summary, $(\frac{P_{\mathrm{G}}(t,x_p)}{P_{\mathrm{G}}(t,0)} - 1)t^{\Delta(x_p)}$ describes the dressed monopole operators corresponding to the ray generators x_p.

4. The finite sums $\sum_{s\in \mathrm{Int}(\mathcal{P}(C_p^{(2)}))} t^{\Delta(s)}$ are entirely determined by the conformal dimensions of the minimal generators u_κ^p.

5. The first contributions for the minimal generators u_κ^p are of the form

$$\frac{P_2(t)}{P_0(t)} t^{\Delta(u_\kappa^p)} = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_2-1} t^{i+j+\Delta(u_\kappa^p)} \,, \tag{15.31}$$

which then comprise the bare and the dressed monopole operators simultaneously.

6. If $C_p^{(2)}$ is simplicial, i.e. $\mathcal{H}(S_p^{(2)}) = \{x_{p-1}, x_p\}$, then the sum over $s \in \mathrm{Int}(\mathcal{P}(C_p^{(2)}))$ in (15.28) is zero, as the interior is empty. Also indicated by $d(C_p^{(2)}) = 1$.

In conclusion, the Hilbert series (15.28) suggests that *ray generators* are to be expected in the denominator, while other minimal generators are manifest in the numerator. Moreover, the entire Hilbert series is determined by a *finite* set of numbers: the conformal dimensions of the minimal generators $\{\Delta(x_p) \mid p = 0, 1, \ldots, L\}$ and $\{\{\Delta(u_\kappa^{(p)}) \mid \kappa = 1, \ldots, d(C_p^{(2)}) - 1\} \mid p = 1, \ldots, L\}$ as well as the classical dressing factors.

Moreover, the dressing behaviour, i.e. number and degree, of a minimal generator m is described by the quotient $P_{\mathrm{G}}(t, m)/P_{\mathrm{G}}(t, 0)$. Consolidating evidence for this statement comes from the analysis of the plethystic logarithm, which we present in App. C. Together, the Hilbert series and the plethystic logarithm allow a better understanding of the chiral ring.

We illustrate the formula (15.28) for the two simplest cases in order to hint on the differences that arise if $d(C_p^{(2)}) > 1$ for cones within the fan.

15.4.1 Example: one simplicial cone

Adapting the result (15.28) to one cone $C_1^{(2)}$ with Hilbert basis $\{x_0, x_1\}$, we find

$$\mathrm{HS} = \frac{1 + \left(\frac{P_1(t)}{P_0(t)} - 1\right)\left(t^{\Delta(x_0)} + t^{\Delta(x_1)}\right) + \left(1 - 2\frac{P_1(t)}{P_0(t)} + \frac{P_2(t)}{P_0(t)}\right) t^{\Delta(x_0)+\Delta(x_1)}}{\prod_{i=1}^{2}(1 - t^{d_i}) \prod_{p=0}^{1}(1 - t^{\Delta(x_p)})} \,. \tag{15.32}$$

Examples treated in this part are as follows: firstly, the representation $[2, 0]$ for the quotients $\mathrm{Spin}(4)$, $\mathrm{SO}(3) \times \mathrm{SU}(2)$, $\mathrm{SU}(2) \times \mathrm{SO}(3)$, $\mathrm{PSO}(4)$ of Sec. 18.2; secondly, $\mathrm{USp}(4)$ for the case $N_3 = 0$ of Sec. 19.5; thirdly, G_2 in the representations $[1, 0]$, $[0, 1]$ and $[2, 0]$ of Sec. 20.2. The corresponding expression for the plethystic logarithm is provided in (C.16).

15.4.2 Example: one non-simplicial cone

Adapting the result (15.28) to one cone $C_1^{(2)}$ with Hilbert basis $\{x_0, x_1, \{u_\kappa\}\}$, fundamental parallelotope $\mathcal{P}$, and discriminant $d > 1$, we find

$$\mathrm{HS} = \frac{1 + \left(\frac{P_1(t)}{P_0(t)} - 1\right)\left(t^{\Delta(x_0)} + t^{\Delta(x_1)}\right) + \left(1 - 2\frac{P_1(t)}{P_0(t)} + \frac{P_2(t)}{P_0(t)}\right) t^{\Delta(x_0)+\Delta(x_1)} + \frac{P_2(t)}{P_0(t)} \sum_{s \in \mathrm{Int}(\mathcal{P})} t^{\Delta(s)}}{\prod_{i=1}^{2}(1 - t^{d_i}) \prod_{p=0}^{1}(1 - t^{\Delta(x_p)})} \,. \tag{15.33}$$

An example for this case is $\mathrm{SO}(4)$ with representation $[2, 0]$ treated in Sec. 18.2. For the plethystic logarithm we refer to (C.17).

The difference between (15.32) and (15.33) lies in the finite sum added in the numerator which accounts for the minimal generators that are not ray generators.

15.5 Consequences for refined Hilbert series

If the centre $\mathcal{Z}(\widehat{\mathrm{G}})$ of the GNO-dual group $\widehat{\mathrm{G}}$ is a non-trivial Lie-group of rank $\mathrm{rk}(\mathcal{Z}(\widehat{\mathrm{G}})) = \rho$, one introduces additional fugacities $\vec{z} \equiv (z_i)$ for $i = 1, \ldots, \rho$ such that the Hilbert series counts operators according to $\mathrm{SU}(2)_R$-spin $\Delta(m)$ and topological charges $\vec{J}(m) \equiv (J_i(m))$ for $i = 1, \ldots, \rho$. Let us introduce the notation

$$\vec{z}^{\vec{J}(m)} := \prod_{i=1}^{\rho} z_i^{J_i(m)} \quad \text{suchthat} \quad \vec{z}^{\vec{J}(m_1+m_2)} = \vec{z}^{\vec{J}(m_1)+\vec{J}(m_2)} = \vec{z}^{\vec{J}(m_1)} \cdot \vec{z}^{\vec{J}(m_2)} \,, \tag{15.34}$$

where we *assumed* each component $J_i(m)$ to be a linear function in m. By the very same arguments as in (15.28), one can evaluate the refined Hilbert series explicitly and obtains

$$\begin{aligned}
\mathrm{HS}_{\mathrm{G}}(t, \vec{z}) &= \sum_{m \in \Lambda_w^{\widehat{\mathrm{G}}}/\mathcal{W}_{\widehat{\mathrm{G}}}} \vec{z}^{\vec{J}(m)} t^{\Delta(m)} P_{\mathrm{G}}(t,m) \\
&= \frac{P_{\mathrm{G}}(t,0)}{\prod_{p=0}^{L} \left(1 - \vec{z}^{\vec{J}(x_p)} t^{\Delta(x_p)}\right)} \Bigg\{ \prod_{q=0}^{L} \left(1 - \vec{z}^{\vec{J}(x_q)} t^{\Delta(x_q)}\right) \\
&\qquad + \sum_{q=0}^{L} \frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)} \vec{z}^{\vec{J}(x_q)} t^{\Delta(x_q)} \prod_{\substack{r=0 \\ r \neq q}}^{L} \left(1 - \vec{z}^{\vec{J}(x_r)} t^{\Delta(x_r)}\right) \\
&\qquad + \sum_{q=1}^{L} \frac{P_{\mathrm{G}}(t,C_q^{(2)})}{P_{\mathrm{G}}(t,0)} \Bigg[\vec{z}^{\vec{J}(x_{q-1})+\vec{J}(x_q)} t^{\Delta(x_{q-1})+\Delta(x_q)} \\
&\qquad\quad + \sum_{s \in \mathrm{Int}(\mathcal{P}(C_q^{(2)}))} \vec{z}^{\vec{J}(s)} t^{\Delta(s)} \Bigg] \prod_{\substack{r=0 \\ r \neq q-1,q}}^{L} \left(1 - \vec{z}^{\vec{J}(x_r)} t^{\Delta(x_r)}\right) \Bigg\} \,.
\end{aligned} \tag{15.35}$$

The interpretation of the refined Hilbert series (15.35) remains the same as before: the minimal generators, i.e. their GNO-charge, $\mathrm{SU}(2)_R$-spin, topological charges $\vec{J}$, and their dressing factors, completely determine the Hilbert series. In principle, this data makes the (sometimes cumbersome) explicit summation of (14.12) obsolete.

16 Case: $\mathrm{U}(1) \times \mathrm{U}(1)$

In this chapter we analyse the abelian product $\mathrm{U}(1) \times \mathrm{U}(1)$. By construction, the Hilbert series simplifies as the dressing factors are constant throughout the lattice of magnetic weights. Consequently, abelian theories do not exhibit dressed monopole operators.

16.1 Set-up

The weight lattice of the GNO-dual of U(1) is simply $\mathbb{Z}$ and no Weyl-group exists due the abelian character; thus, $\Lambda_w(\widehat{\mathrm{U}(1) \times \mathrm{U}(1)}) = \mathbb{Z}^2$. Moreover, since $\mathrm{U}(1) \times \mathrm{U}(1)$ is abelian the classical dressing factors are the same for any magnetic weight (m_1, m_2), i.e.

$$P_{\mathrm{U}(1)\times\mathrm{U}(1)}(t, m_1, m_2) = \frac{1}{(1-t)^2} \;, \tag{16.1}$$

which reflects the two degree one Casimir invariants.

16.2 Two types of hypermultiplets

Set-up To consider a rank 2 abelian gauge group of the form $\mathrm{U}(1) \times \mathrm{U}(1)$ requires a delicate choice of matter content. If one considers N_1 hypermultiplets with charges $(a_1, b_1) \in \mathbb{N}^2$ under $\mathrm{U}(1) \times \mathrm{U}(1)$, then the conformal dimension reads

$$\Delta_{\text{1h-plet}}(m_1, m_2) = \frac{N_1}{2} |a_1 m_1 + b_1 m_2| \qquad \text{for} \quad (m_1, m_2) \in \mathbb{Z}^2 \;. \tag{16.2a}$$

However, there exists an infinite number of points $\{m_1 = b_1 k, m_2 = -a_1 k, k \in \mathbb{Z}\}$ with zero conformal dimension, i.e. the Hilbert series does not converge due to a decoupled U(1). Fixing this symmetry would reduce the rank to one.

Fortunately, we can circumvent this problem by introducing a second set of N_2 hypermultiplets with charges $(a_2, b_2) \in \mathbb{N}^2$, such that the matrix

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \tag{16.2b}$$

has maximal rank. The relevant conformal dimension then reads

$$\Delta_{\text{2h-plet}}(m_1, m_2) = \sum_{j=1}^{2} \frac{N_j}{2} |a_j m_1 + b_j m_2| \qquad \text{for} \quad (m_1, m_2) \in \mathbb{Z}^2 \;. \tag{16.2c}$$

Nevertheless, this set-up would introduce four charges and the summation of the Hilbert series becomes tricky. We evade the difficulties by the *choice* $a_2 = b_1$ and $b_2 = -a_1$. Dealing with such a scenario leads to summation bounds such as

$$a\, m_1 \geq b\, m_2 \quad \Leftrightarrow \quad m_1 \geq \tfrac{b}{a} m_2 \quad \Leftrightarrow \quad m_1 \geq \lceil \tfrac{b}{a} m_2 \rceil \;, \tag{16.2d}$$

$$a\, m_1 < b\, m_2 \quad \Leftrightarrow \quad m_1 < \tfrac{b}{a} m_2 \quad \Leftrightarrow \quad m_1 < \lceil \tfrac{b}{a} m_2 \rceil - 1 \;. \tag{16.2e}$$

Having the summation variable within a floor or ceiling function seems to be an elaborate task with `Mathematica`. Therefore, we simplify the setting by *assuming* $\exists\, k \in \mathbb{N}$ such that $b_1 = k a_1$. Then we arrive at

$$\Delta_{\text{2h-plet}}(m_1, m_2) = \frac{a_1}{2}\left(N_1 \left|m_1 + k m_2\right| + N_2 \left|k m_1 - m_2\right|\right) \qquad \text{for} \quad (m_1, m_2) \in \mathbb{Z}^2 \,. \tag{16.2f}$$

For this conformal dimension, there exists exactly one point (m_1, m_2) with zero conformal dimension — the trivial solution. Further, by a redefinition of N_1 and N_2 we can consider $a_1 = 1$.

Hilbert basis Consider the conformal dimension (16.2f) for $a_1 = 1$. By resolving the absolute

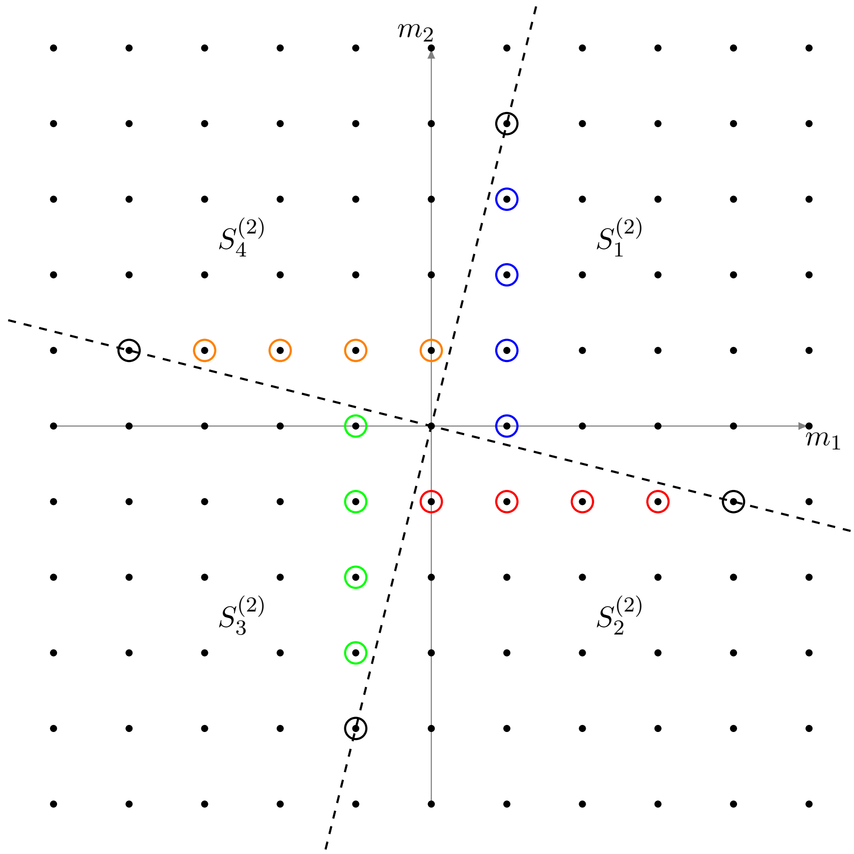


Figure 16.1: *The dashed lines correspond the* $k\,m_1 = m_2$ *and* $m_1 = -k\,m_2$ *and divide the lattice* $\mathbb{Z}^2$ *into four semi-groups* $S_j^{(2)}$ *for* $j = 1, 2, 3, 4$*. The black circles denote the ray generators, while the blue circles complete the Hilbert basis for* $S_1^{(2)}$*, red circled points complete the basis for* $S_2^{(2)}$*. Green circles correspond to the remaining minimal generators of* $S_3^{(2)}$ *and orange circled points are the analogue for* $S_4^{(2)}$*. (Here, the example is* $k = 4$*.)*

values, we divide $\mathbb{Z}^2$ into four semi-groups

$$S_1^{(2)} = \left\{(m_1, m_2) \in \mathbb{Z}^2 \middle| \, (k\,m_1 \geq m_2) \wedge (m_1 \geq -k\,m_2)\right\} \,, \tag{16.3a}$$

$$S_2^{(2)} = \left\{(m_1, m_2) \in \mathbb{Z}^2 \middle| \, (k\,m_1 \geq m_2) \wedge (m_1 \leq -k\,m_2)\right\} \,, \tag{16.3b}$$

$$S_3^{(2)} = \left\{(m_1, m_2) \in \mathbb{Z}^2 \middle| \, (k\,m_1 \leq m_2) \wedge (m_1 \geq -k\,m_2)\right\} \,, \tag{16.3c}$$

$$S_4^{(2)} = \left\{(m_1, m_2) \in \mathbb{Z}^2 \middle| \, (k\,m_1 \leq m_2) \wedge (m_1 \leq -k\,m_2)\right\} \,, \tag{16.3d}$$

which all descend from 2-dimensional rational polyhedral cones. The situation is depicted in Fig. 16.1.

Next, one needs to compute the Hilbert basis $\mathcal{H}(S)$ for each semi-group S. In this example, it follows from the drawing that

$$\mathcal{H}(S_1^{(2)}) = \Big\{(k,-1), \{(1,l) \mid l=0,1,\ldots,k\}\Big\}\,, \tag{16.4a}$$

$$\mathcal{H}(S_2^{(2)}) = \Big\{(-1,-k), \{(l,-1) \mid l=0,1,\ldots,k\}\Big\}\,, \tag{16.4b}$$

$$\mathcal{H}(S_3^{(2)}) = \Big\{(-k,1), \{(-1,-l) \mid l=0,1,\ldots,k\}\Big\}\,, \tag{16.4c}$$

$$\mathcal{H}(S_4^{(2)}) = \Big\{(1,k), \{(-l,1) \mid l=0,1,\ldots,k\}\Big\}\,. \tag{16.4d}$$

For a fixed $k \geq 1$ we obtain $4(k+1)$ basis elements.

Hilbert series We then compute the following Hilbert series

$$\mathrm{HS}^k_{\mathrm{U}(1)\times\mathrm{U}(1)}(t,z_1,z_2) = \frac{1}{(1-t)^2} \sum_{m_1,m_2\in\mathbb{Z}} z_1^{m_1} z_2^{m_2} t^{\Delta_{\text{2h-plet}}(m_1,m_2)}\,, \tag{16.5}$$

for which we obtain

$$\mathrm{HS}^k_{\mathrm{U}(1)\times\mathrm{U}(1)}(t,z_1,z_2) = \frac{R(t,z_1,z_2)}{P(t,z_1,z_2)}\,, \tag{16.6a}$$

with denominator

$$\begin{aligned} P(t,z_1,z_2) = (1-t)^2 &\left(1-\frac{1}{z_1}t^{\frac{kN_2-N_1}{2}}\right)\left(1-z_1 t^{\frac{kN_2-N_1}{2}}\right)\left(1-\frac{1}{z_2}t^{\frac{kN_1-N_2}{2}}\right)\left(1-z_2 t^{\frac{kN_1-N_2}{2}}\right) \\ \times &\left(1-\frac{1}{z_1}t^{\frac{kN_2+N_1}{2}}\right)\left(1-z_1 t^{\frac{kN_2+N_1}{2}}\right)\left(1-\frac{1}{z_2}t^{\frac{kN_1+N_2}{2}}\right)\left(1-z_2 t^{\frac{kN_1+N_2}{2}}\right) \\ \times &\left(1-\frac{1}{z_1 z_2^k}t^{\frac{1}{2}(k^2+1)N_1}\right)\left(1-z_1 z_2^k t^{\frac{1}{2}(k^2+1)N_1}\right) \\ \times &\left(1-\frac{z_1^k}{z_2}t^{\frac{1}{2}(k^2+1)N_2}\right)\left(1-\frac{z_2}{z_1^k}t^{\frac{1}{2}(k^2+1)N_2}\right)\,, \end{aligned} \tag{16.6b}$$

while the numerator $R(t,z_1,z_2)$ is too long to be displayed, as it contains 1936 monomials. Nonetheless, one can explicitly verify a few properties of the Hilbert series. For example, the Hilbert series (16.6) has a pole of order 4 at $t \to 1$, because $R(1,z_1,z_2)=0$ and the derivatives $\frac{\mathrm{d}^n}{\mathrm{d}t^n}R(t,z_1,z_2)|_{t=1} = 0$ for $n=1,2,\ldots 9$ (at least for $z_1 = z_2 = 1$). Moreover, the degrees of numerator and denominator depend on the relations between N_1, N_2, and k; however, one can show that the difference in degrees is precisely 2, i.e. it matches the quaternionic dimension of the moduli space.

Discussion Analysing the plethystic logarithm and the Hilbert series, the monopole operators corresponding to the Hilbert basis can be identified as follows: Eight poles of the Hilbert series (16.6) can be identified with monopole generators as shown in Tab. 16.1a. Studying the plethystic logarithm clearly displays the remaining set, which is displayed in Tab. 16.1b.

Remark A rather special case of (16.2c) is $a_2 = 0 = b_1$, for which the theory becomes the product of two U(1)-theories with N_1 or N_2 electrons of charge a or b, respectively. In detail,

(m_1,m_2)	$\Delta(m_1,m_2)$	(m_1,m_2)	$\Delta(m_1,m_2)$
$(1,0)$, $(-1,0)$	$\frac{1}{2}(N_1+kN_2)$	$(0,1)$, $(0,-1)$	$\frac{1}{2}(kN_1+N_2)$
$(1,k)$, $(-1,-k)$	$\frac{1}{2}(1+k^2)N_1$	$(-k,1)$, $(k,-1)$	$\frac{1}{2}(1+k^2)N_2$

(a) *The minimal generators which are poles of the Hilbert series. The second row comprises the ray generators.*

(m_1,m_2)	$\Delta(m_1,m_2)$	(m_1,m_2)	$\Delta(m_1,m_2)$
$(1,l)$, $(-1,-l)$	$\frac{1}{2}N_1(kl+1)+\frac{1}{2}N_2(k-l)$	$(-l,1)$, $(l,-1)$	$\frac{1}{2}N_1(k-l)+\frac{1}{2}N_2(kl+1)$

(b) *The minimal generators, labelled by $l=1,2,\ldots,k-1$, which are neither poles of the Hilbert series nor ray generators.*

Table 16.1: *The set of bare monopole operators for a* U(1) × U(1) *theory with conformal dimension* (16.2f).

the conformal dimension is simply

$$\Delta_{\text{2h-plet}}(m_1,m_2) \overset{a_2=0=b_1}{=} \frac{N_1}{2}\,|a\,m_1| + \frac{N_2}{2}\,|b\,m_2| \qquad \text{for} \quad (m_1,m_2)\in\mathbb{Z}^2 \;, \tag{16.7}$$

such that the Hilbert series becomes

$$\begin{aligned}\mathrm{HS}^{a,b}_{\mathrm{U}(1)^2}(t,z_1,z_2) &= \frac{1-t^{aN_1}}{(1-t)\left(1-z_1 t^{\frac{aN_1}{2}}\right)\left(1-\frac{1}{z_1}t^{\frac{aN_1}{2}}\right)} \times \frac{1-t^{bN_2}}{(1-t)\left(1-z_2 t^{\frac{bN_2}{2}}\right)\left(1-\frac{1}{z_2}t^{\frac{bN_2}{2}}\right)} \\ &= \mathrm{HS}^{a}_{\mathrm{U}(1)}(t,z_1,N_1)\times \mathrm{HS}^{b}_{\mathrm{U}(1)}(t,z_2,N_2)\;. \end{aligned} \tag{16.8}$$

For the unrefined Hilbert series, that is $z_1=1=z_2$, the rational function $\mathrm{HS}^a_{\mathrm{U}(1)}(t,N)$ equals the Hilbert series of the (abelian) ADE-orbifold $\mathbb{C}^2/\mathbb{Z}_{a\cdot N}$, see for instance [31]. Thus, the U(1)×U(1) Coulomb branch is the product of two A-type singularities.

Quite intuitively, taking the corresponding limit $k\to 0$ in (16.6) yields the product

$$\lim_{k\to 0}\mathrm{HS}^{k}_{\mathrm{U}(1)\times\mathrm{U}(1)}(t,z_1,z_2) = \mathrm{HS}_{\mathrm{U}(1)}(t,z_1,N_1)\times\mathrm{HS}_{\mathrm{U}(1)}(t,z_2,N_2)\;, \tag{16.9}$$

which are U(1) theories with N_1 and N_2 electrons of unit charge. The unrefined rational functions are the Hilbert series of $\mathbb{Z}_{N_1}$ and $\mathbb{Z}_{N_2}$ singularities in the ADE-classification. From Fig. 16.1 one observes that in the limit $k\to 0$ the relevant rational cones coincide with the four quadrants of $\mathbb{R}^2$ and the Hilbert basis reduces to the cone generators.

16.3 Reduced moduli space of one SO(5)-instanton

Consider the Coulomb branch of the quiver gauge theory depicted in Fig. 16.2 with conformal dimension given by

$$\Delta(m_1,m_2) = \frac{1}{2}\left(|m_1|+|m_1-2m_2|\right)\;. \tag{16.10}$$

Instead of associating (16.10) with the quiver of Fig. 16.2, one could equally well understand it as a special case of a $\mathrm{U}(1)^2$ theory with two different hypermultiplets (16.2c).

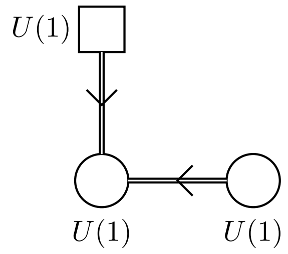


Figure 16.2: *Quiver gauge theory whose Coulomb branch is the reduced moduli space of one* SO(5)*-instanton.*

Hilbert basis Similar to the previous case, the conformal dimensions induces a fan which, in this case, is generated by four 2-dimensional cones

$$C_1^{(2)} = \mathrm{Cone}\left((2,1),(0,1)\right) \ , \qquad C_2^{(2)} = \mathrm{Cone}\left((2,1),(0,-1)\right) \ , \tag{16.11a}$$
$$C_3^{(2)} = \mathrm{Cone}\left((-2,-1),(0,-1)\right) \ , \qquad C_4^{(2)} = \mathrm{Cone}\left((-2,-1),(0,1)\right) \ . \tag{16.11b}$$

The intersection with the $\mathbb{Z}^2$ lattice defines the semi-groups $S_p^{(2)} := C_p^{(2)} \cap \mathbb{Z}^2$ for which we need to compute the Hilbert bases. Fig. 16.3 illustrates the situation and we obtain

$$\mathcal{H}(S_1^{(2)}) = \{(2,1),(1,1),(0,1)\} \ , \qquad \mathcal{H}(S_2^{(2)}) = \{(2,1),(1,0),(0,-1)\} \ , \tag{16.12a}$$
$$\mathcal{H}(S_3^{(2)}) = \{(-2,-1),(-1,-1),(0,-1)\} \ , \qquad \mathcal{H}(S_4^{(2)}) = \{(-2,-1),(-1,0),(0,1)\} \ . \tag{16.12b}$$

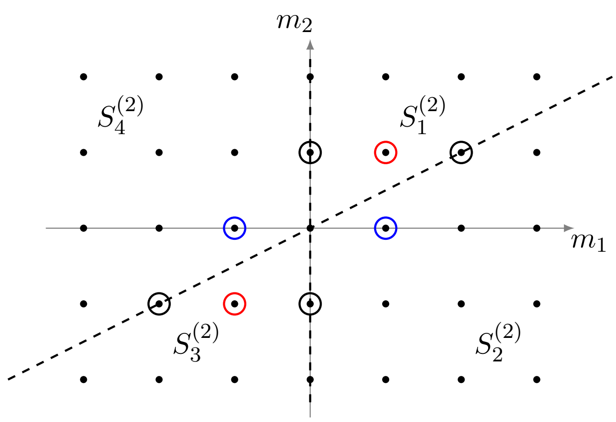


Figure 16.3: *The dashed lines correspond the $m_1 = 2m_2$ and $m_1 = 0$ and divide the lattice $\mathbb{Z}^2$ into four semi-groups $S_j^{(2)}$ for $j = 1,2,3,4$. The black circles denote the ray generators, while the red circles complete the Hilbert bases for $S_1^{(2)}$ and $S_3^{(2)}$. Blue circled lattice points complete the bases for $S_2^{(2)}$ and $S_4^{(2)}$.*

Hilbert series The Hilbert series is evaluated to

$$\mathrm{HS}_{\mathrm{U}(1)^2}^{\mathrm{SO}(5)}(t,z_1,z_2) = \frac{R(t,z_1,z_2)}{(1-t)^2\left(1-\frac{t}{z_2}\right)(1-z_2 t)\left(1-\frac{t}{z_1^2 z_2}\right)(1-z_1^2 z_2 t)} \ , \tag{16.13a}$$
$$R(t,z_1,z_2) = 1 + t\left(z_1 + \frac{1}{z_1} + z_1 z_2 + \frac{1}{z_1 z_2}\right) \tag{16.13b}$$

$$
\begin{aligned}
&- 2t^2 \left(1 + z_1 + \frac{1}{z_1} + z_1 z_2 + \frac{1}{z_1 z_2}\right) \\
&+ t^3 \left(z_1 + \frac{1}{z_1} + z_1 z_2 + \frac{1}{z_1 z_2}\right) + t^4 \,.
\end{aligned}
$$

The Hilbert series (16.13) has a pole of order 4 at $t = 1$, because one can explicitly verify that $R(t = 1, z_1, z_2) = 0$, $\frac{\mathrm{d}}{\mathrm{d}t} R(t, z_1, z_2)|_{t=1} = 0$, but $\frac{\mathrm{d}^2}{\mathrm{d}t^2} R(t, z_1, z_2)|_{t=1} \neq 0$. Thus, the complex dimension of the moduli space is 4. Moreover, the difference in degrees of numerator and denominator is 2, which equals the quaternionic dimension of the Coulomb branch.

Plethystic logarithm The plethystic logarithm for this scenario reads

$$
\begin{aligned}
\mathrm{PL}(\mathrm{HS}^{\mathrm{SO}(5)}_{\mathrm{U}(1)^2}) = &\left(2 + z_1^2 z_2 + \frac{1}{z_1^2 z_2} + z_1 z_2 + \frac{1}{z_1 z_2} + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2}\right) t \\
&- \left(4 + z_1^2 + \frac{1}{z_1^2} + z_2 + \frac{1}{z_2} + z_1^2 z_2^2 + \frac{1}{z_1^2 z_2^2} + z_1^2 z_2 + \frac{1}{z_1^2 z_2} \right. \\
&\left. + 2z_1 + \frac{2}{z_1} + 2 z_1 z_2 + \frac{2}{z_1 z_2}\right) t^2 + \mathcal{O}(t^3) \,.
\end{aligned}
\tag{16.14}
$$

Symmetry enhancement The information conveyed by the Hilbert basis (16.12), the Hilbert series (16.13), and the plethystic logarithm (16.14) is that there are eight minimal generators of conformal dimension one which, together with the two Casimir invariants, span the adjoint representation of SO(5). It is known [31, 33] that (16.13) is the Hilbert series for the reduced moduli space of one SO(5)-instanton over $\mathbb{C}^2$.

16.4 Reduced moduli space of one SU(3)-instanton

The quiver gauge theories associated to the affine Dynkin diagram $\hat{A}_n$ have been studied in [191]. Here, we consider the Coulomb branch of the $\hat{A}_2$ quiver gauge theory as depicted in Fig. (16.4) and with conformal dimension given by

$$
\Delta(m_1, m_2) = \frac{1}{2} \left(|m_1| + |m_2| + |m_1 - m_2|\right) \,. \tag{16.15}
$$

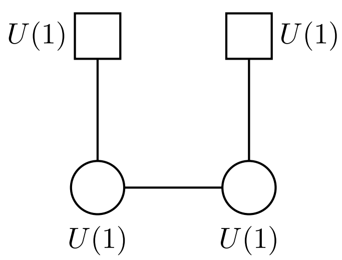


Figure 16.4: *Quiver gauge theory whose Coulomb branch is the reduced moduli space of one* SU(3)*-instanton.*

Hilbert basis Similar to the previous case, the conformal dimensions induces a fan which, in this case, is generated by six 2-dimensional cones

$$
C_1^{(2)} = \mathrm{Cone}\left((0,1),(1,1)\right) \,, \qquad C_2^{(2)} = \mathrm{Cone}\left((1,1),(1,0)\right) \,, \tag{16.16a}
$$

$$C_3^{(2)} = \mathrm{Cone}\left((1,0),(0,-1)\right) \ , \qquad C_4^{(2)} = \mathrm{Cone}\left((0,-1),(-1,-1)\right) \ , \tag{16.16b}$$
$$C_5^{(2)} = \mathrm{Cone}\left((-1,-1),(-1,0)\right) \ , \qquad C_6^{(2)} = \mathrm{Cone}\left((-1,0),(0,1)\right) \ . \tag{16.16c}$$

The intersection with the $\mathbb{Z}^2$ lattice defines the semi-groups $S_p^{(2)} := C_p^{(2)} \cap \mathbb{Z}^2$ for which we need to compute the Hilbert bases. Fig. 16.5 illustrates the situation. We compute the Hilbert bases to read

$$\mathcal{H}(S_1^{(2)}) = \{(0,1),(1,1)\} \qquad \mathcal{H}(S_2^{(2)}) = \{(1,1),(1,0))\} \ , \tag{16.17a}$$
$$\mathcal{H}(S_3^{(2)}) = \{(1,0),(0,-1)\} \qquad \mathcal{H}(S_4^{(2)}) = \{(0,-1),(-1,-1)\} \ , \tag{16.17b}$$
$$\mathcal{H}(S_5^{(2)}) = \{(-1,-1),(-1,0)\} \qquad \mathcal{H}(S_6^{(2)}) = \{(-1,0),(0,1)\} \ . \tag{16.17c}$$

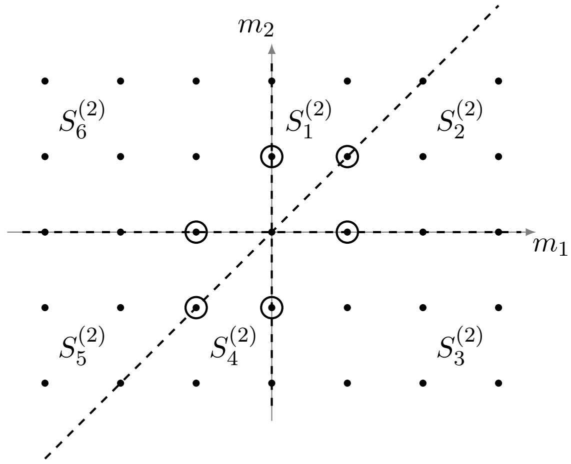


Figure 16.5: *The dashed lines correspond the $m_1 = m_2$, $m_1 = 0$, and $m_2 = 0$ and divide the lattice $\mathbb{Z}^2$ into six semi-groups $S_j^{(2)}$ for $j = 1, \ldots, 6$. The black circled points denote the ray generators, which coincide with the minimal generators.*

Hilbert series The Hilbert series is readily computed and reads

$$\mathrm{HS}_{\mathrm{U}(1)^2}^{\mathrm{SU}(3)}(t,z_1,z_2) = \frac{R(t,z_1,z_2)}{P(t,z_1,z_2)} \ , \tag{16.18a}$$
$$\begin{aligned} P(t,z_1,z_2) &= (1-t)^2 \left(1 - \frac{1}{z_1} t\right)(1-z_1 t)\left(1 - \frac{1}{z_2} t\right)(1 - z_2 t) \\ &\quad \times \left(1 - \frac{1}{z_1 z_2} t\right)(1 - z_1 z_2 t) \ , \end{aligned} \tag{16.18b}$$
$$\begin{aligned} R(t,z_1,z_2) &= 1 - \left(3 + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2 + \frac{1}{z_1 z_2}\right) t^2 \\ &\quad + 2\left(2 + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2 + \frac{1}{z_1 z_2}\right) t^3 \\ &\quad - \left(3 + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2 + \frac{1}{z_1 z_2}\right) t^4 + t^6 \ . \end{aligned} \tag{16.18c}$$

The Hilbert series (16.18) has a pole of order 4 as $t \to 1$, because $R(t{=}1, z_1, z_2) = 0$ and $\frac{\mathrm{d}^n}{\mathrm{d}t^n} R(t,z_1,z_2)|_{t=1,z_1=z_2=1} = 0$ for $n = 1,2,3$. Thus, the Coulomb branch is of complex dimension 4. In addition, the difference in degrees of numerator and denominator is 2, which equals the quaternionic dimension of the moduli space.

Plethystic logarithm

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{\mathrm{SU}(3)}_{\mathrm{U}(1)^2}) = &\left(2 + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2 + \frac{1}{z_1 z_2}\right) t \\ &- \left(3 + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2 + \frac{1}{z_1 z_2}\right) t^2 + \mathcal{O}(t^3)\end{aligned} \tag{16.19}$$

Symmetry enhancement The information conveyed by the Hilbert basis (16.17), the Hilbert series (16.18), and the plethystic logarithm (16.19) is that there are six minimal generators of conformal dimension one which, together with the two Casimir invariants, span the adjoint representation of SU(3). As proven in [191], the Hilbert series (16.18) can be resumed as

$$\mathrm{HS}^{\mathrm{SU}(3)}_{\mathrm{U}(1)^2}(t, z_1, z_2) = \sum_{k=0}^{\infty} \chi_{[k,k]} t^k \tag{16.20}$$

with $\chi_{[k,k]}$ being the character of the SU(3)-representation $[k, k]$. Therefore, this theory has an explicit SU(3)-enhancement in the Coulomb branch. It is known [28] that (16.20) is the reduced instanton moduli space of one SU(3)-instanton over $\mathbb{C}^2$.

17 Case: U(2)

In this chapter we aim to consider two classes of U(2) gauge theories wherein $\mathrm{U}(2) \cong \mathrm{SU}(2)\times\mathrm{U}(1)$, i.e. this is effectively an SU(2) theory with varying U(1)-charge. As a unitary group, U(2) is self-dual under GNO-duality.

17.1 Set-up

To start with, let consider the two view points and elucidate the relation between them.

U(2) view point The GNO-dual group of U(2) is U(2) itself; hence, the weight lattice is $\Lambda_w(\mathrm{U}(2)) \cong \mathbb{Z}^2$. Moreover, the Weyl-group is S_2 and acts via permuting the two Cartan generators; consequently, $\Lambda_w(\mathrm{U}(2))/S_2 = \{(m_1, m_2) \in \mathbb{Z}^2 : m_1 \geq m_2\}$.

U(1) × SU(2) view point Considering $\mathrm{U}(1) \times \mathrm{SU}(2)$, we need to find the weight lattice of the GNO-dual, i.e. find all solutions to the Dirac quantisation condition, see for instance [194]. Since we consider the product, the exponential in (14.7) factorises in $\exp(2\pi\mathrm{i}\ n\ T_{\mathrm{U}(1)})$ and $\exp(2\pi\mathrm{i}\ m\ T_{\mathrm{SU}(2)})$, where the T's are the Cartan generators. Besides the solution

$$(n, m) \in H_0 \coloneqq \mathbb{Z}^2 = \mathbb{Z} \times \Lambda_w(\mathrm{SO}(3)) = \mathbb{Z} \times \Lambda_r(\mathrm{SU}(2)) \tag{17.1a}$$

corresponding to the weight lattice of $\mathrm{U}(1) \times \mathrm{SO}(3)$, there exists also the solution

$$(n, m) \in H_1 \coloneqq \mathbb{Z}^2 + (\tfrac{1}{2}, \tfrac{1}{2}) = \left(\mathbb{Z} + \tfrac{1}{2}\right) \times \left(\Lambda_w(\mathrm{SU}(2)) \setminus \Lambda_r(\mathrm{SU}(2))\right) \ , \tag{17.1b}$$

for which both factors are equal to -1. The action of the Weyl-group S_2 restricts then to non-negative m i.e. $H_0^+ = H_0 \cap \{m \geq 0\}$ and $H_1^+ = H_1 \cap \{m \geq 0\}$.

Relation between both To identify both views with one another, we select the U(1) as diagonally embedded, i.e. identify the charges as follows:

$$\left.\begin{aligned} n &\coloneqq \tfrac{m_1+m_2}{2} \\ m &\coloneqq \tfrac{m_1-m_2}{2} \end{aligned}\right\} \qquad \Leftrightarrow \qquad \begin{cases} m_1 = n + m \\ m_2 = n - m \end{cases} . \tag{17.2}$$

The two classes of U(2)-representations under consideration in this chapter are

$$[1, a] \qquad \text{with} \quad \chi_{[1,a]}^{\mathrm{U}(2)} = y_1^{a+1} y_2^a + y_1^a y_2^{a+1} \ , \tag{17.3a}$$

$$[2, a] \qquad \text{with} \quad \chi_{[2,a]}^{\mathrm{U}(2)} = y_1^{a+2} y_2^a + y_1^{a+1} y_2^{a+1} + y_1^a y_2^{a+2} \ , \tag{17.3b}$$

for $a \in \mathbb{N}_0$ and χ are the group character. Following (17.2), we define the fugacities

$$q \coloneqq \sqrt{y_1\ y_2} \qquad \text{for} \quad \mathrm{U}(1) \qquad \text{and} \qquad x \coloneqq \sqrt{\frac{y_1}{y_2}} \qquad \text{for} \quad \mathrm{SU}(2), \tag{17.4}$$

and consequently observe

$$\chi^{\mathrm{U}(2)}_{[1,a]} = q^{2a+1}\left(x+\tfrac{1}{x}\right) = \chi^{\mathrm{U}(1)}_{2a+1}\cdot\chi^{\mathrm{SU}(2)}_{[1]} \,, \tag{17.5a}$$

$$\chi^{\mathrm{U}(2)}_{[2,a]} = q^{2a+2}\left(x^2+1+\tfrac{1}{x^2}\right) = \chi^{\mathrm{U}(1)}_{2a+2}\cdot\chi^{\mathrm{SU}(2)}_{[2]} \,, \tag{17.5b}$$

where the SU(2)-characters are defined via

$$\chi^{\mathrm{SU}(2)}_{[L]} = \sum_{r=-\frac{L}{2}}^{\frac{L}{2}} x^{2r} \,. \tag{17.5c}$$

Therefore, the family $[1,a]$ corresponds to the fundamental representation of SU(2) with *odd* U(1)-charge $2a+1$; while the family $[2,a]$ represents the adjoint representation of SU(2) with *even* U(1)-charge $2a+2$.

Dressing factors Lastly, the calculation employs the classical dressing function

$$P_{\mathrm{U}(2)}(t^2,m) := \begin{cases} \frac{1}{(1-t^2)^2} & ,m\neq 0 \\ \frac{1}{(1-t^2)(1-t^4)} & ,m=0 \end{cases} \,, \tag{17.6}$$

as presented in [191]. (Note that we rescaled t to be t^2 for later convenience.) Following the discussion of App. C, monopoles with $m\neq 0$ have precisely one dressing by a U(1) Casimir due to $P_{\mathrm{U}(2)}(t^2,m)/P_{\mathrm{U}(2)}(t^2,0) = 1+t^2$. In contrast, there are no dressed monopole operators for $m=0$.

Remark The Lie group U(2) is not semi-simple and, thus, the dominant Weyl-chamber of the GNO-dual U(2) is not a strongly convex cone. Nevertheless, by the affine central hyperplanes induced by Δ, all the rational cones in the resulting fan will be strongly convex, and we can employ the results of Ch. 15.

17.2 N hypermultiplets in fundamental representation of SU(2)

The conformal dimension for a U(2) theory with N hypermultiplets transforming in $[1,a]$ is given as

$$\Delta(n,m) = \frac{N}{2}\left(\,|(2a+1)\cdot n+m| + |(2a+1)\cdot n-m|\,\right) - 2|m| \tag{17.7}$$

such that the Hilbert series is computed via

$$\mathrm{HS}^{[1,a]}_{\mathrm{U}(2)}(t,z) = \sum_{n,m} P_{\mathrm{U}(2)}(t^2,m)\; t^{2\Delta(n,m)} z^{2n} \,, \tag{17.8}$$

where the ranges of n,m have been specified above. Here we use the fugacity t^2 instead of t to avoid half-integer powers.

Hilbert basis The conformal dimension (17.7) divides $\Lambda_w(\mathrm{U}(2))/S_2$ into semi-groups via the absolute values $|m|$, $|(2a+1)n+m|$, and $|(2a+1)n-m|$, which we understand as hyperplanes. Thus, there are three semi-groups

$$S^{(2)}_{+} = \left\{(m,n)\in\Lambda^{\mathrm{U}(2)}_w/S_2 \mid (n\geq 0)\ \wedge\ (0\leq m\leq (2a+1)n)\right\} \,, \tag{17.9a}$$

$$S_0^{(2)} = \left\{ (m,n) \in \Lambda_w^{\mathrm{U}(2)}/S_2 \mid -(2a+1)n \leq m \leq (2a+1)n \right\} , \tag{17.9b}$$
$$S_-^{(2)} = \left\{ (m,n) \in \Lambda_w^{\mathrm{U}(2)}/S_2 \mid (n \leq 0) \ \wedge \ (0 \leq m \leq -(2a+1)n) \right\} \tag{17.9c}$$

originating from 2-dimensional cones, see Fig. 17.1. Since all these semi-groups $S_\pm^{(2)}$, $S_0^{(2)}$ are finitely generated, one can compute the Hilbert basis $\mathcal{H}(S_p)$ for each p and obtains

$$\mathcal{H}(S_\pm^{(2)}) = \left\{ (0,\pm1), \{(l+\tfrac{1}{2}, \pm\tfrac{1}{2}) \mid l=0,1,\ldots,a\} \right\} , \tag{17.10a}$$
$$\mathcal{H}(S_0^{(2)}) = \left\{ (a+\tfrac{1}{2}, \tfrac{1}{2}), (1,0), (a+\tfrac{1}{2}, -\tfrac{1}{2}) \right\} . \tag{17.10b}$$

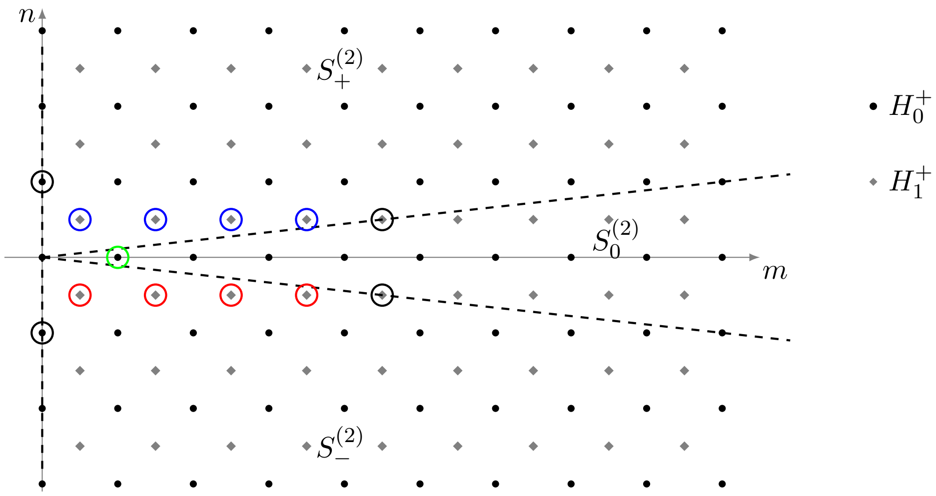


Figure 17.1: *The Weyl-chamber for the example $a = 4$. The black circled lattice points are the ray generators. The blue circled lattice points complete the Hilbert basis (together with two ray generators) for $S_+^{(2)}$; while the red circled points analogously complete the Hilbert basis for $S_-^{(2)}$. The green circled point represents the remaining minimal generator for $S_0^{(2)}$.*

Hilbert series Computing the Hilbert series yields

$$\mathrm{HS}_{\mathrm{U}(2)}^{[1,a]}(t,z,N) = \frac{R(t,z)}{P(t,z)} , \tag{17.11a}$$

$$\begin{aligned} P(t,z) = {} & \left(1-t^2\right)^2 \left(1-t^4\right) \left(1-t^{2N-4}\right) \left(1-\tfrac{1}{z^2} t^{(4a+2)N}\right) \left(1-z^2 t^{(4a+2)N}\right) \\ & \times \left(1-\tfrac{1}{z} t^{(2a+1)(N-2)}\right) \left(1-z t^{(2a+1)(N-2)}\right) , \end{aligned} \tag{17.11b}$$

$$\begin{aligned} R(t,z) = {} & 1 - t^2 + t^{2N-2} - t^{2N} + 2t^{4aN-4a+2N} - t^{4aN-8a+2N-4} - t^{4aN-8a+2N-2} \\ & - 2t^{4aN-4a+4N-4} + t^{4aN-8a+4N-6} + t^{4aN-8a+4N-4} + t^{8aN+4N} + t^{8aN+4N+2} \\ & - 2t^{8aN-4a+4N} - t^{8aN+6N-2} - t^{8aN+6N} + 2t^{8aN-4a+6N-4} - t^{12aN-8a+6N-4} \\ & + t^{12aN-8a+6N-2} - t^{12aN-8a+8N-6} + t^{12aN-8a+8N-4} \\ & + \left(z+\tfrac{1}{z}\right) \left(t^{2aN-4a+N} - t^{2aN+N+2} + t^{2aN+3N-2} - t^{2aN-4a+3N-4} + t^{6aN+3N+2} \right. \\ & \left. - t^{6aN-8a+3N-2} - t^{6aN+5N-2} + t^{6aN-8a+5N-6} - t^{10aN-4a+5N} + t^{10aN-8a+5N-2} \right. \end{aligned} \tag{17.11c}$$

$$
\begin{aligned}
&\quad + t^{10aN-4a+7N-4} - t^{10aN-8a+7N-6}\Big) \\
&+ \left(z^2 + \tfrac{1}{z^2}\right) \big(t^{4aN-4a+2N} - t^{4aN+2N} + t^{4aN+4N} - t^{4aN-4a+4N-4} - t^{8aN-4a+4N} \\
&\quad + t^{8aN-8a+4N-4} + t^{8aN-4a+6N-4} - t^{8aN-8a+6N-4}\big) \,.
\end{aligned}
$$

The Hilbert series (17.11) has a pole of order 4 at $t \to 1$, because $R(t=1,z) = 0$ and $\frac{\mathrm{d}^n}{\mathrm{d}t^n} R(t,z)|_{t=1} = 0$ for $n = 1,2,3$. Hence, the moduli space is of (complex) dimension 4. As a comment, the additional $(1-t^2)$-term in the denominator can be cancelled with a corresponding term in the numerator either explicitly for each $a =$ fixed or for any a, but the resulting expressions are not particularly insightful.

Discussion The four poles of the Hilbert series (17.11), which are graded as $z^{\pm 2}$ and $z^{\pm 1}$, can be identified with the four ray generators $(0,\pm 1)$ and $(a+\frac{1}{2}, \pm\frac{1}{2})$, i.e. they correspond to bare monopole operators. In addition, the bare monopole operator for the minimal generator $(1,0)$ is present in the denominator (17.11b), too.

In contrast, the family of monopoles $\{(l+\frac{1}{2}, \pm\frac{1}{2}), l = 0,1,\ldots,a-1\}$ is not directly visible in the Hilbert series, but can be deduced unambiguously from the plethystic logarithm. These monopole operators correspond the *minimal generators* of $S_{\pm}^{(2)}$ which are not ray generators. Tab. 17.1 provides as summary of the monopole generators and their properties. As a remark,

(m,n)	(m_1,m_2)	$2\Delta(m,n)$	$\mathrm{H}_{(m,n)}$	dressings
$(1,0)$	$(1,-1)$	$2N-4$	$\mathrm{U}(1)^2$	1 by U(1)
$(l+\frac{1}{2},\frac{1}{2})$, for $l=0,1,\ldots,a$	$(l+1,-l)$	$(2a+1)N-2(2l+1)$	$\mathrm{U}(1)^2$	1 by U(1)
$(l+\frac{1}{2},-\frac{1}{2})$, for $l=0,1,\ldots,a$	$(l,-(l+1))$	$(2a+1)N-2(2l+1)$	$\mathrm{U}(1)^2$	1 by U(1)
$(0,\pm 1)$	$\pm(1,1)$	$(4a+2)N$	U(2)	none

Table 17.1: *Bare and dressed monopole operators for the family* $[1,a]$ *of* U(2)*-representations.*

the family of monopole operators $(l+\frac{1}{2}, \pm\frac{1}{2})$ is not always completely present in the plethystic logarithm. We observe that l-th bare operator is a generator if $N \geq 2(a-l+1)$, while the dressing of the l-th object is a generator if $N > 2(a-l+1)$. The reason for the disappearance lies in a relation at degree $\Delta(1,0) + \Delta(a+\frac{1}{2}, \pm\frac{1}{2}) + 2$, which coincides with $\Delta(l+\frac{1}{2}, \pm\frac{1}{2})$ for $N-1 = 2(a-l+1)$, such that the terms cancel in the PL. (See also App. C for the degrees of the first relations.) Thus, for large N all above listed objects are generators.

17.2.1 Case: $a = 0$, complete intersection

For the choice $a = 1$, we obtain the Hilbert series for the 2-dimensional fundamental representation $[1,0]$ of U(2) as

$$
\mathrm{HS}_{\mathrm{U}(2)}^{[1,0]}(t,z,N) = \frac{\left(1-t^{2N}\right)\left(1-t^{2N-2}\right)}{(1-t^2)(1-t^4)\left(1-\frac{1}{z}t^N\right)(1-zt^N)\left(1-\frac{1}{z}t^{N-2}\right)(1-zt^{N-2})} \tag{17.12}
$$

which agrees with the results of [191].

Let us comment on the reduction of generators compared to the Hilbert basis (17.10). The minimal generators have conformal dimensions $2\Delta(\frac{1}{2},\pm\frac{1}{2}) = N-2$, $2\Delta(1,0) = 2N-4$, and $2\Delta(0,\pm 1) = 2N$. Thus, $(1,0)$ is generated by $(\frac{1}{2},\pm\frac{1}{2})$ and $(0,\pm 1)$ are generated by utilising the dressed monopoles of $(\frac{1}{2},\pm\frac{1}{2})$ and suitable elements in their Weyl-orbits.

17.3 N hypermultiplets in adjoint representation of SU(2)

The conformal dimension for a U(2)-theory with N hypermultiplets transforming in the adjoint representation of SU(2) and arbitrary even U(1)-charge is given by

$$\Delta(n,m) = \frac{N}{2}\big(\,|(2a+2)n+2m| + |(2a+2)n| + |(2a+2)n-2m|\,\big) - 2|m| \, . \tag{17.13}$$

Already at this stage, one can define the four semi-groups induced by the conformal dimension, which originate from 2-dimensional cones

$$S^{(2)}_{2,\pm} = \Big\{(m,n) \in \Lambda_w^{\mathrm{U}(2)}/S_2 \mid (m \geq 0) \wedge (m \leq \pm(a+1)n) \wedge (\pm n \geq 0)\Big\} \, , \tag{17.14a}$$

$$S^{(2)}_{1,\pm} = \Big\{(m,n) \in \Lambda_w^{\mathrm{U}(2)}/S_2 \mid (m \geq 0) \wedge (m \geq \pm(a+1)n) \wedge (\pm n \geq 0)\Big\} \, . \tag{17.14b}$$

It turns out that the precise form of the Hilbert basis depends on the divisibility of a by 2; thus, we split the considerations in two cases: $a = 2k-1$ and $a = 2k$.

17.3.1 Case: $a = 1 \mod 2$

Hilbert basis The collection of semi-groups (17.14) is depicted in Fig. 17.2. As before, we compute the Hilbert basis $\mathcal{H}$ for each semi-group:

$$\mathcal{H}(S^{(2)}_{2,\pm}) = \Big\{(0,\pm 1), (2k,\pm 1), \{(j+\tfrac{1}{2}, \pm\tfrac{1}{2}) \mid j = 0,\ldots,k-1\}\Big\} \, , \tag{17.15a}$$

$$\mathcal{H}(S^{(2)}_{1,\pm}) = \Big\{(2k,\pm 1), (k+\tfrac{1}{2}, \pm\tfrac{1}{2}), (1,0)\Big\} \, . \tag{17.15b}$$

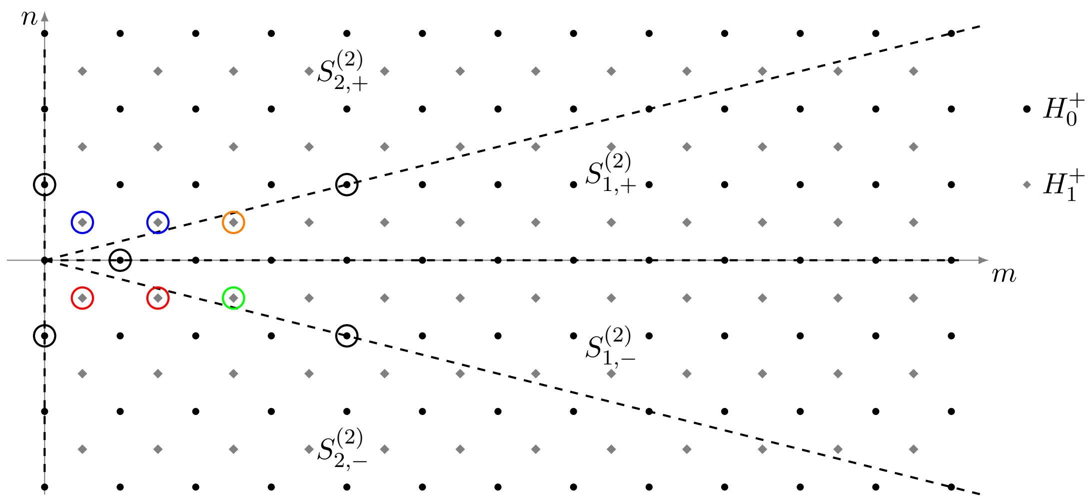


Figure 17.2: The Weyl-chamber for odd a, here with the example $a = 3$. The black circled lattice points correspond to the ray generators originating from the fan. The blue/red circled points are the remaining minimal generators for $S^{(2)}_{2,\pm}$, respectively. Similarly, the orange/green circled point are the generators that complete the Hilbert basis for $S^{(2)}_{1,\pm}$.

Hilbert series The computation of the Hilbert series yields

$$\mathrm{HS}_{\mathrm{U}(2)}^{[2,2k-1]}(t,z,N) = \frac{R(t,z,N)}{P(t,z,N)} \,, \tag{17.16a}$$

$$\begin{aligned} P(t,z,N) = {}& \left(1-t^2\right)^2 \left(1-t^4\right) \left(1-t^{4N-4}\right) \left(1-\tfrac{1}{z^2}t^{12kN}\right) \left(1-z^2 t^{12kN}\right) \\ & \times \left(1-\tfrac{1}{z^2}t^{12kN-8k}\right) \left(1-z^2 t^{12kN-8k}\right) \,, \end{aligned} \tag{17.16b}$$

$$\begin{aligned} R(t,z,N) = {}& 1 - t^2 + t^{4N-2} - t^{4N}t^{24kN} + t^{24kN+2} - t^{24kN-16k} - t^{24kN-16k+2} \\ & - t^{24kN+4N-2} - t^{24kN+4N} + t^{24kN-16k+4N} + t^{24kN-16k+4N-2} - t^{48kN-16k} \\ & + t^{48kN-16k+2} + t^{48kN-16k+4N} - t^{48kN-16k+4N-2} \\ & + \left(z+\tfrac{1}{z}\right)\Big(- t^{6kN+2} + t^{6kN-4k+2} + t^{6kN-4k+2N-2} - t^{6kN-4k+2N+2} + t^{6kN+4N-2} \\ & - t^{6kN-4k+4N-2} + t^{18kN+2} - t^{18kN-4k+2} + t^{18kN-8k+2} - t^{18kN-12k+2} \\ & - t^{18kN-4k+2N-2} + t^{18kN-4k+2N+2} - t^{18kN-12k+2N-2} + t^{18kN-12k+2N+2} \\ & - t^{18kN+4N-2} + t^{18kN-4k+4N-2} - t^{18kN-8k+4N-2} + t^{18kN-12k+4N-2} + t^{30kN-4k+2} \\ & - t^{30kN-8k+2} + t^{30kN-12k+2} - t^{30kN-16k+2} + t^{30kN-4k+2N-2} - t^{30kN-4k+2N+2} \\ & + t^{30kN-12k+2N-2} - t^{30kN-12k+2N+2} - t^{30kN-4k+4N-2} + t^{30kN-8k+4N-2} \\ & - t^{30kN-12k+4N-2} + t^{30kN-16k+4N-2} - t^{42kN-12k+2} + t^{42kN-16k+2} \\ & - t^{42kN-12k+2N-2} + t^{42kN-12k+2N+2} + t^{42kN-12k+4N-2} - t^{42kN-16k+4N-2}\Big) \\ & + \left(z^2+\tfrac{1}{z^2}\right)\Big(- t^{12kN} + t^{12kN-8k+2} + t^{12kN+4N} - t^{12kN-8k+4N-2} + t^{36kN-16k} \\ & - t^{36kN-8k+2} - t^{36kN-16k+4N} + t^{36kN-8k+4N-2}\Big) \\ & + \left(z^3+\tfrac{1}{z^3}\right)\Big(- t^{18kN-4k+2} + t^{18kN-8k+2} - t^{18kN-4k+2N-2} + t^{18kN-4k+2N+2} \\ & + t^{18kN-4k+4N-2} - t^{18kN-8k+4N-2} - t^{30kN-8k+2} + t^{30kN-12k+2} \\ & + t^{30kN-12k+2N-2} - t^{30kN-12k+2N+2} + t^{30kN-8k+4N-2} - t^{30kN-12k+4N-2}\Big) \,. \end{aligned} \tag{17.16c}$$

Inspection of the Hilbert series (17.16) reveals that it has a pole of order 4 as $t \to 1$ because $R(t=1,z,N)=0$, $\frac{\mathrm{d}}{\mathrm{d}t}R(t,z,N)|_{t=1}=0$, and $\frac{\mathrm{d}^n}{\mathrm{d}t^n}R(t,z,N)|_{t=1,z=1}=0$ for $n=2,3$.

Discussion The denominator of the Hilbert series (17.16) displays poles for the five bare monopole operators $(0,\pm1)$, $(2k,\pm1)$, and $(1,0)$, which are ray generators and charged under $\mathrm{U}(1)_J$ as ±2, ±2, and 0, respectively. The remaining operators, corresponding to the minimal generators which are not ray generators, are apparent in the analysis of the plethystic logarithm. The relevant bare and dressed monopole operators are summarised in Tab. 17.2.

The plethystic logarithm, moreover, displays that not always all monopoles of the family $(j+\frac{1}{2},\pm\frac{1}{2})$ are generators (in the sense of the PL). The observation is: if $k-j<N$ then the j-th operator (bare as well as dressed) is truly a generator in the PL. The reason behind lies in a relation at degree $\Delta(k-\frac{1}{2},\pm\frac{1}{2})+\Delta(1,0)$, which coincides with $\Delta(j+\frac{1}{2},\pm\frac{1}{2})$ for $k-j=N$. (See also App. C.) Hence, for large enough N all above listed operators are generators.

(m,n)	(m_1,m_2)	$2\Delta(m,n)$	$\mathrm{H}_{(m,n)}$	dressings
$(1,0)$	$(1,-1)$	$4N-4$	$\mathrm{U}(1)^2$	1 by U(1)
$(j+\frac{1}{2},\frac{1}{2})$, $j{=}0,\ldots,k{-}1$	$(j+1,-j)$	$6kN-4j-2$	$\mathrm{U}(1)^2$	1 by U(1)
$(j+\frac{1}{2},-\frac{1}{2})$, $j{=}0,\ldots,k{-}1$	$(j,-(j+1))$	$6kN-4j-2$	$\mathrm{U}(1)^2$	1 by U(1)
$(k+\frac{1}{2},\frac{1}{2})$	$(k+1,-k)$	$6kN+2N-4k-2$	$\mathrm{U}(1)^2$	1 by U(1)
$(k+\frac{1}{2},-\frac{1}{2})$	$(k,-(k+1))$	$6kN+2N-4k-2$	$\mathrm{U}(1)^2$	1 by U(1)
$(0,\pm1)$	$\pm(1,1)$	$12kN$	$\mathrm{U}(2)$	none
$(2k,1)$	$(2k+1,1-2k)$	$12kN-8k$	$\mathrm{U}(1)^2$	1 by U(1)
$(2k,-1)$	$(2k-1,-(2k+1))$	$12kN-8k$	$\mathrm{U}(1)^2$	1 by U(1)

Table 17.2: *Summary of the monopole operators for odd a.*

17.3.2 Case: $a = 0 \mod 2$

Hilbert basis The diagram for the minimal generators is provided in Fig. 17.3. Again, the appearing (bare) monopoles correspond to the Hilbert basis of the semi-groups:

$$\mathcal{H}(S^{(2)}_{2,\pm}) = \Big\{(0,\pm1), \{(j+\tfrac{1}{2},\pm\tfrac{1}{2})\ ,\ j=0,1,\ldots,k\}\Big\}\ , \tag{17.17a}$$

$$\mathcal{H}(S^{(2)}_{1,\pm}) = \Big\{(k+\tfrac{1}{2},\pm\tfrac{1}{2}),(1,0)\Big\}\ . \tag{17.17b}$$

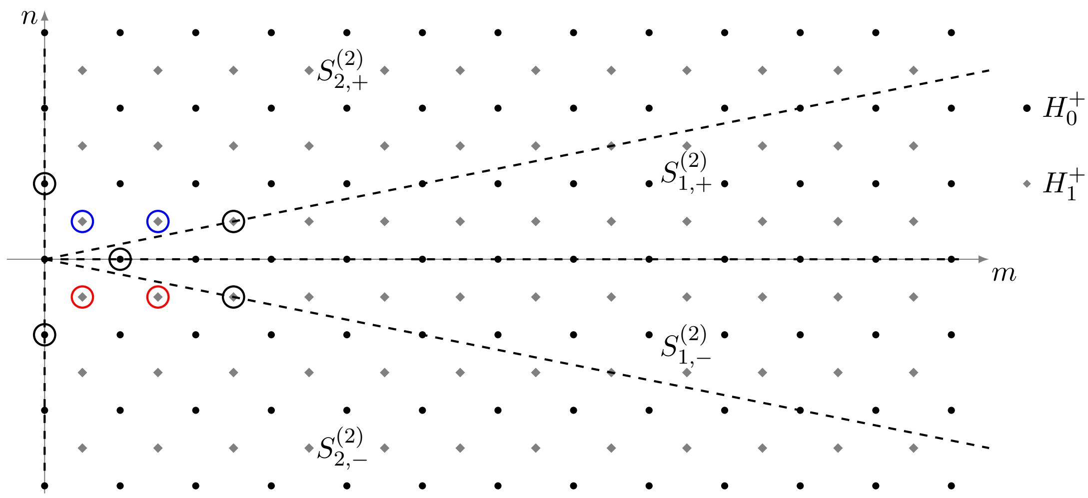


Figure 17.3: *The Weyl-chamber for $a = 0 \mod 2$, here with the example $a = 4$. The black circled lattice points correspond to the ray generators originating from the fan. The blue/red circled points are the remaining minimal generators for $S^{(2)}_{2,\pm}$, respectively.*

Hilbert series The computation of the Hilbert series for this case yields

$$\mathrm{HS}^{[2,2k]}_{\mathrm{U}(2)}(t,z,N) = \frac{R(t,z,N)}{P(t,z,N)}\ , \tag{17.18a}$$

$$P(t,z,N) = \left(1-t^2\right)^2 \left(1-t^4\right) \left(1-t^{4N-4}\right) \left(1-\tfrac{1}{z} t^{6kN-4k+3N-2}\right) \left(1-z t^{6kN-4k+3N-2}\right) \\ \times \left(1-\tfrac{1}{z^2} t^{12kN+6N}\right) \left(1-z^2 t^{12kN+6N}\right) \,, \tag{17.18b}$$

$$\begin{aligned} R(t,z,N) = 1 &- t^2 + t^{4N-2} - t^{4N} + 2t^{12kN-4k+6N} - t^{12kN-8k+6N-4} - t^{12kN-8k+6N-2} \\ &- 2t^{12kN-4k+10N-4} + t^{12kN-8k+10N-6} + t^{12kN-8k+10N-4} + t^{24kN+12N} \\ &+ t^{24kN+12N+2} - 2t^{24kN-4k+12N} - t^{24kN+16N-2} - t^{24kN+16N} + 2t^{24kN-4k+16N-4} \\ &- t^{36kN-8k+18N-4} + t^{36kN-8k+18N-2} - t^{36kN-8k+22N-6} + t^{36kN-8k+22N-4} \\ + \left(z+\tfrac{1}{z}\right) \Big(&- t^{6kN+3N+2} + t^{6kN-4k+3N} + t^{6kN+7N-2} - t^{6kN-4k+7N-4} + t^{18kN+9N+2} \\ &- t^{18kN-8k+9N-2} - t^{18kN+13N-2} + t^{18kN-8k+13N-6} - t^{30kN-4k+15N} \\ &+ t^{30kN-8k+15N-2} + t^{30kN-4k+19N-4} - t^{30kN-8k+19N-6} \Big) \\ + \left(z^2+\tfrac{1}{z^2}\right) \Big(&- t^{12kN+6N} + t^{12kN-4k+6N} + t^{12kN+10N} - t^{12kN-4k+10N-4} \\ &- t^{24kN-4k+12N} + t^{24kN-8k+12N-4} + t^{24kN-4k+16N-4} - t^{24kN-8k+16N-4} \Big) \,. \end{aligned} \tag{17.18c}$$

The Hilbert series (17.18) has a pole of order 4 as $t \to 1$ because one can explicitly verify that $R(t=1,z,N) = 0$, $\frac{\mathrm{d}}{\mathrm{d}t} R(t,z,N)|_{t=1} = 0$, and $\frac{\mathrm{d}^n}{\mathrm{d}t^n} R(t,z,N)|_{t=1,z=1} = 0$ for $n = 2,3$.

Discussion The five monopoles corresponding to the ray generators, i.e. $(0,\pm1)$, $(k+\frac{1}{2},\pm\frac{1}{2})$, and $(1,0)$, appear as poles in the Hilbert series (17.18) and are charged under $\mathrm{U}(1)_J$ as ±2, ±1, and 0, respectively. The remaining minimal generator can be deduced by inspecting the plethystic logarithm. We summarise the monopole generators in Tab. 17.3. Similarly to the

(m,n)	(m_1,m_2)	$2\Delta(m,n)$	$\mathrm{H}_{(m,n)}$	dressings
$(1,0)$	$(1,-1)$	$4N-4$	$\mathrm{U}(1)^2$	1 by $\mathrm{U}(1)$
$(j+\frac{1}{2},\frac{1}{2})$, for $j=0,1,\ldots,k$	$(j+1,-j)$	$6kN+3N-4j-2$	$\mathrm{U}(1)^2$	1 by $\mathrm{U}(1)$
$(j+\frac{1}{2},-\frac{1}{2})$, for $j=0,1,\ldots,k$	$(j,-(j+1))$	$6kN+3N-4j-2$	$\mathrm{U}(1)^2$	1 by $\mathrm{U}(1)$
$(0,\pm1)$	$\pm(1,1)$	$12kN+6N$	$\mathrm{U}(2)$	none

Table 17.3: *Summary of the monopole operators for even a.*

case of odd a, the plethystic logarithm displays that not always all monopoles of the family $(j+\frac{1}{2},\pm\frac{1}{2})$ are generators. The observation is: if $k-j+1 \geq N$ then the j-th bare operator is a generator in the PL, while for $k-j+2 \geq N$ then also the dressing of the j-th monopole is a generator. The reason behind lies, again, in a relation at degree $\Delta(k-\frac{1}{2},\pm\frac{1}{2}) + \Delta(1,0) + 2$, which coincides with $\Delta(j+\frac{1}{2},\pm\frac{1}{2})$ for $k-j=N$. (See also App. C for the degrees of the first relations.) Hence, for large enough N all above listed operators are generators.

17.4 Direct product of SU(2) and U(1)

A rather simple example is obtained by considering the non-interacting product of an SU(2) and a U(1) theory. Nonetheless, it illustrates how the rank two Coulomb branches contain the product of rank one Coulomb branches as subclasses.

As first example, take N_1 fundamentals of SU(2) and N_2 hypermultiplets charged under U(1) with charges $a \in \mathbb{N}$. The conformal dimension is given by

$$\Delta(m,n) = (N_1-2)|m| + \frac{N_2 \cdot a}{2}|n| \qquad \text{for} \quad m \in \mathbb{N} \qquad \text{and} \qquad n \in \mathbb{Z} \tag{17.19}$$

and the dressing factor splits as

$$P_{\mathrm{SU}(2)}(t,m,n) = P_{\mathrm{SU}(2)}(t,m) \times P_{\mathrm{U}(1)}(t,n) \ , \tag{17.20}$$

such that the Hilbert series factorises

$$\mathrm{HS}^{[1],a}_{\mathrm{SU}(2)\times\mathrm{U}(1)}(t,N_1,N_2) = \mathrm{HS}^{[1]}_{\mathrm{SU}(2)}(t,N_1) \times \mathrm{HS}^{a}_{\mathrm{U}(1)}(t,N_2) \ . \tag{17.21}$$

The rank one Hilbert series have been presented in [191]. Moreover, $\mathrm{HS}^{a}_{\mathrm{U}(1)}(t,N_2)$ equals the $A_{a\cdot N_2 -1}$ singularity $\mathbb{C}^2/\mathbb{Z}_{a\cdot N_2}$; whereas $\mathrm{HS}^{[1]}_{\mathrm{SU}(2)}(t,N_1)$ is precisely the D_{N_1} singularity.

The second, follow-up example is simply a theory comprise of N_1 hypermultiplets in the adjoint representation of SU(2) and N_2 hypermultiplets charged under U(1) as above. The conformal dimension is modified to

$$\Delta(m,n) = 2(N_1-1)|m| + \frac{N_2 \cdot a}{2}|n| \qquad \text{for} \quad m \in \mathbb{N} \qquad \text{and} \qquad n \in \mathbb{Z} \tag{17.22}$$

and Hilbert series is obtained as

$$\mathrm{HS}^{[2],a}_{\mathrm{SU}(2)\times\mathrm{U}(1)}(t,N_1,N_2) = \mathrm{HS}^{[2]}_{\mathrm{SU}(2)}(t,N_1) \times \mathrm{HS}^{a}_{\mathrm{U}(1)}(t,N_2) \ . \tag{17.23}$$

Applying the results of [191], $\mathrm{HS}^{[2]}_{\mathrm{SU}(2)}(t,N_1)$ is the Hilbert series of the D_{2N_1}-singularity on $\mathbb{C}^2$.

Summarising, the direct product of these SU(2)-theories with U(1)-theories results in moduli spaces that are products of A and D type singularities, which are complete intersections. Moreover, any non-trivial interactions between these two gauge groups, as discussed in Sec. 17.2 and 17.3, leads to a very elaborate expression for the Hilbert series as rational functions. Also, the Hilbert basis becomes an important concept for understanding the moduli space.

18 Case: $A_1 \times A_1$

This chapter concerns all Lie groups with Lie algebra D_2, which allows to study products of the rank one gauge groups SO(3) and SU(2), but also the proper rank two group SO(4).

18.1 Set-up

Let us consider the Lie algebra $D_2 \cong A_1 \times A_1$. Following [194], there are five different groups with this Lie algebra. The reason is that the universal covering group $\widetilde{\mathrm{SO}}(4)$ of SO(4) has a non-trivial centre $\mathcal{Z}(\widetilde{\mathrm{SO}}(4)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ of order 4. The quotient of $\widetilde{\mathrm{SO}}(4)$ by any of the five different subgroups $\mathcal{Z}(\widetilde{\mathrm{SO}}(4))$ yields a Lie group with the same Lie algebra. Fortunately, working with SO(4) allows to use the isomorphism $\widetilde{\mathrm{SO}}(4) = \mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. We can summarise the setting as displayed in Tab. 18.1. Here, we employed $\widehat{\mathrm{SU}(2)} = \mathrm{SO}(3)$ and that for semi-simple

Quotient	isomorphic group G	GNO-dual $\widehat{\mathrm{G}}$	$\mathcal{Z}(\widehat{\mathrm{G}})$	GNO-charges (m_1, m_2)
$\frac{\widetilde{\mathrm{SO}}(4)}{\{1\}}$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$	$\mathrm{SO}(3) \times \mathrm{SO}(3)$	$\{1\}$	$K^{[0]}$
$\frac{\widetilde{\mathrm{SO}}(4)}{\mathbb{Z}_2 \times \{1\}}$	$\mathrm{SO}(3) \times \mathrm{SU}(2)$	$\mathrm{SU}(2) \times \mathrm{SO}(3)$	$\mathbb{Z}_2 \times \{1\}$	$K^{[0]} \cup K^{[1]}$
$\frac{\widetilde{\mathrm{SO}}(4)}{\mathrm{diag}(\mathbb{Z}_2)}$	$\mathrm{SO}(4)$	$\mathrm{SO}(4)$	$\mathbb{Z}_2$	$K^{[0]} \cup K^{[2]}$
$\frac{\widetilde{\mathrm{SO}}(4)}{\{1\} \times \mathbb{Z}_2}$	$\mathrm{SU}(2) \times \mathrm{SO}(3)$	$\mathrm{SO}(3) \times \mathrm{SU}(2)$	$\{1\} \times \mathbb{Z}_2$	$K^{[0]} \cup K^{[3]}$
$\frac{\widetilde{\mathrm{SO}}(4)}{\mathbb{Z}_2 \times \mathbb{Z}_2}$	$\mathrm{SO}(3) \times \mathrm{SO}(3)$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$K^{[0]} \cup K^{[1]} \cup K^{[2]} \cup K^{[3]}$

Table 18.1: *All the Lie groups that arise taking the quotient of $\widetilde{\mathrm{SO}}(4)$ by a subgroup of its centre; hence, their Lie algebra is D_2.*

groups G_1, G_2

$$\widehat{\mathrm{G}_1 \times \mathrm{G}_2} = \widehat{\mathrm{G}}_1 \times \widehat{\mathrm{G}}_2 \tag{18.1}$$

holds [194]. Moreover, the GNO-charges are defined via the following sublattices of the weight lattice of Spin(4) (see also Fig. 18.1)

$$K^{[0]} = \Big\{ (m_1, m_2) \mid m_i = p_i \in \mathbb{Z}\,,\ p_1 + p_2 = \text{even} \Big\}\,, \tag{18.2a}$$

$$K^{[1]} = \Big\{ (m_1, m_2) \mid m_i = p_i + \tfrac{1}{2}\,,\ p_i \in \mathbb{Z}\,,\ p_1 + p_2 = \text{even} \Big\}\,, \tag{18.2b}$$

$$K^{[2]} = \Big\{ (m_1, m_2) \mid m_i = p_i \in \mathbb{Z}\,,\ p_1 + p_2 = \text{odd} \Big\}\,, \tag{18.2c}$$

$$K^{[3]} = \Big\{ (m_1, m_2) \mid m_i = p_i + \tfrac{1}{2}\,,\ p_i \in \mathbb{Z}\,,\ p_1 + p_2 = \text{odd} \Big\}\,. \tag{18.2d}$$

The important consequence of this set-up is that the fan defined by the conformal dimension will be the same for a given representation in each of the five quotients, but the semi-groups will differ due to the different lattices $\Lambda_w(\widehat{\mathrm{G}})$ used in the intersection. Hence, we will find different

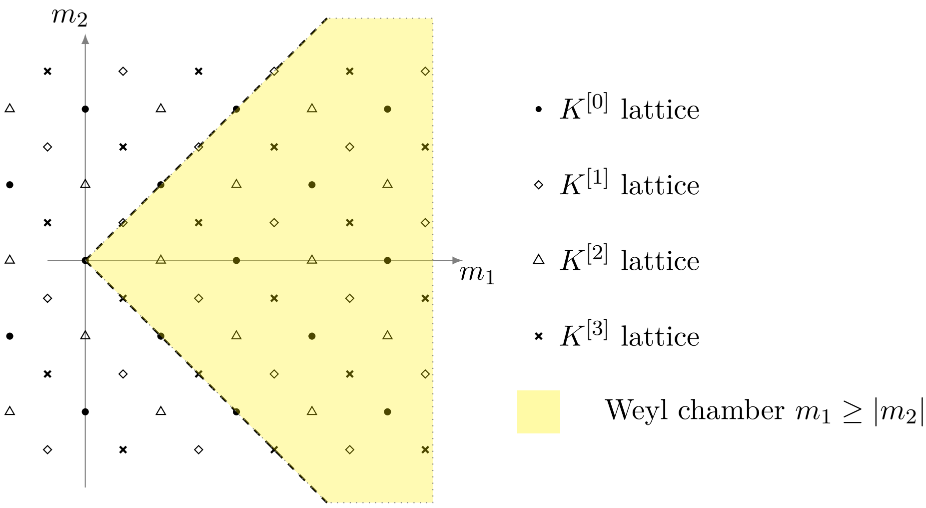


Figure 18.1: *The four different sublattices of the covering group of* SO(4)*. One recognises the root lattice* $\Lambda_r^{\widetilde{\mathrm{SO}(4)}} = K^{[0]}$ *and the weight lattice* $\Lambda_w^{\widetilde{\mathrm{SO}(4)}} = K^{[0]} \cup K^{[1]} \cup K^{[2]} \cup K^{[3]}$.

Hilbert basis in each quotient group. Nevertheless, we are forced to consider representations on the root lattice as we otherwise cannot compare all quotients.

18.1.1 Dressings

In addition, we have chosen to parametrise the principal Weyl chamber via $m_1 \geq |m_2|$ such that the classical dressing factors are given by [191]

$$P_{A_1 \times A_1}(t, m_1, m_2) = \begin{cases} \frac{1}{(1-t^2)^2} & \text{for} \quad m_1 = m_2 = 0 \; , \\ \frac{1}{(1-t)(1-t^2)} & \text{for} \quad m_1 = |m_2| > 0 \; , \\ \frac{1}{(1-t)^2} & \text{for} \quad m_1 > |m_2| \geq 0 \; . \end{cases} \tag{18.3}$$

Regardless of the quotient $\widetilde{\mathrm{SO}(4)}/\Gamma$, the space of Casimir invariance is 2-dimensional. We choose a basis[30] such that the two degree 2 Casimir invariants stem either from SU(2) or SO(3), i.e.

$$\mathrm{diag}(\Phi) = (\phi_1, \phi_2) \quad \longrightarrow \quad \mathcal{C}_2^{(i)} = (\phi_i)^2 \; . \tag{18.4}$$

Next, we can clarify all relevant bare and dressed monopole operators for an (m_1, m_2) that is a minimal generator. There are two cases: On the one hand, for $m_2 = \pm m_1$, i.e. at the boundary of the Weyl chamber, the residual gauge group is either $\mathrm{U}(1)_i \times \mathrm{SU}(2)_j$ or $\mathrm{U}(1)_i \times \mathrm{SO}(3)_j$ (for $i, j = 1, 2$ and $i \neq j$), depending on the quotient under consideration. Thus, only the degree 1 Casimir invariant of the $\mathrm{U}(1)_i$ can be employed for a dressing, as the Casimir invariant of $\mathrm{SU}(2)_j$ or $\mathrm{SO}(3)_j$ belongs to the quotient $\widetilde{\mathrm{SO}(4)}/\Gamma$ itself. Hence, we get

$$V^{\mathrm{dress},0}_{(m_1, \pm m_1)} = (m_1, \pm m_1) \qquad \text{and} \qquad V^{\mathrm{dress},1}_{(m_1, \pm m_1)} = \phi_i \; (m_1, \pm m_1) \; . \tag{18.5a}$$

Alternatively, we can apply the results of App. C and deduce the dressing behaviour at the boundary of the Weyl chamber to be $P_{A_1 \times A_1}(t, m_1, \pm m_1) / P_{A_1 \times A_1}(t, 0, 0) = 1 + t$, i.e. only one dressed monopole arises.

On the other hand, for $m_1 > |m_2| \geq 0$, i.e. in the interior of the Weyl chamber, the residual

[30] In a different basis, the Casimir invariants for SO(4) are the quadratic Casimir and the Pfaffian.

gauge group is $\mathrm{U}(1)^2$. From the resulting two degree 1 Casimir invariants one constructs the following monopole operators:

$$V^{\mathrm{dress},0}_{(m_1,m_2)} = (m_1,m_2) \quad \longrightarrow \quad \begin{cases} V^{\mathrm{dress},1,i}_{(m_1,m_2)} = \phi_i \ (m_1,m_2) \ , & \text{for} \quad i=1,2 \\ V^{\mathrm{dress},2}_{(m_1,m_2)} = \phi_1\phi_2 \ (m_1,m_2) \ . \end{cases} \tag{18.5b}$$

Using App. C, we obtain that monopole operators with GNO-charge in the interior of the Weyl chamber exhibit the following dressings $P_{A_1\times A_1}(t,m_1,m_2)/P_{A_1\times A_1}(t,0,0) = 1+2t+t^2$, which agrees with our discussion above.

18.2 Representation $[2,0]$

The conformal dimension for this case reads

$$\Delta(m_1,m_2) = (N-1)\left(|m_1+m_2| + |m_1-m_2|\right) \ . \tag{18.6}$$

Following the ideas outlined earlier, the conformal dimension (18.6) defines a fan in the dominant Weyl chamber. In this example, Δ is already a linear function on the entire dominant Weyl chamber; thus, we generate a fan which just consists of one 2-dimensional rational cone

$$C^{(2)} = \Big\{ (m_1 \geq m_2) \wedge (m_1 \geq -m_2) \Big\} \ . \tag{18.7}$$

18.2.1 Quotient $\mathrm{Spin}(4)$

Hilbert basis Starting from the fan (18.7) with the cone $C^{(2)}$, the Hilbert basis for the semi-group $S^{(2)} := C^{(2)} \cap K^{[0]}$ is simply given by the ray generators

$$\mathcal{H}(S^{(2)}) = \Big\{ (1,1), (1,-1) \Big\} \ , \tag{18.8}$$

see for instance Fig. 18.2. Both minimal generators exhibit a bare monopole operator and one dressed operators, as explained in (18.5).

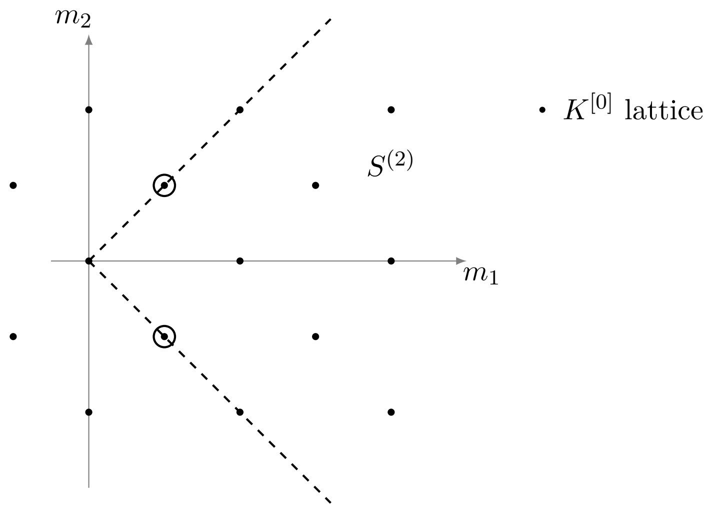


Figure 18.2: *The semi-group $S^{(2)}$ and its ray-generators (black circled points) for the quotient* $\mathrm{Spin}(4)$ *and the representation* $[2,0]$.

Hilbert series We compute the Hilbert series to

$$\mathrm{HS}_{\mathrm{Spin}(4)}^{[2,0]}(t,N) = \frac{\left(1-t^{4N-2}\right)^2}{(1-t^2)^2\,(1-t^{2N-2})^2\,(1-t^{2N-1})^2}\ , \tag{18.9}$$

which is a complete intersection with 6 generators and 2 relations. The generators are given in Tab. 18.2.

object	(m_1,m_2)	lattice	$\Delta(m_1,m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(1,\pm1)$	$K^{[0]}$	$2N-2$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by U(1)

Table 18.2: *Bare and dressed monopole generators for a* Spin(4) *gauge theory with matter transforming in* $[2,0]$.

Remark The Hilbert series (18.9) can be compared to the case of SU(2) with n fundamentals and n_a adjoints such that $2N = n+2n_a$, cf. [191]. One derives at

$$\mathrm{HS}_{\mathrm{Spin}(4)}^{[2,0]}(t,N) = \mathrm{HS}_{\mathrm{SU}(2)}^{[1]+[2]}(t,n,n_a) \times \mathrm{HS}_{\mathrm{SU}(2)}^{[1]+[2]}(t,n,n_a)\ , \tag{18.10}$$

which equals the product of two D_{2N} singularities. As a consequence, the minimal generator $(1,1)$ belongs to one SU(2) Hilbert series with adjoint matter content, while $(1,-1)$ belongs to the other.

18.2.2 Quotient SO(4)

The centre of the GNO-dual SO(4) is a $\mathbb{Z}_2$, which we choose to count if (m_1,m_2) belongs to $K^{[0]}$ or $K^{[2]}$. A realisation is given by

$$z^{m_1+m_2} = \begin{cases} z^{\text{even}} = 1 & \text{for} \quad (m_1,m_2)\in K^{[0]}\ , \\ z^{\text{odd}} = z & \text{for} \quad (m_1,m_2)\in K^{[2]}\ . \end{cases} \tag{18.11}$$

In other words, z is a $\mathbb{Z}_2$-fugacity.

Hilbert basis The semi-group $S^{(2)} := C^{(2)} \cap \left(K^{[0]}\cup K^{[2]}\right)$ has a Hilbert basis as displayed in Fig. 18.3 or explicitly

$$\mathcal{H}(S^{(2)}) = \Big\{(1,\pm1),(1,0)\Big\}\ . \tag{18.12}$$

Hilbert series The Hilbert series for SO(4) is given by

$$\mathrm{HS}_{\mathrm{SO}(4)}^{[2,0]}(t,z,N) = \frac{1+t^{2N-2}+2t^{2N-1}+zt^{2N}+2zt^{2N-1}+zt^{4N-2}}{(1-t^2)^2\,(1-t^{2N-2})\,(1-zt^{2N-2})}\ , \tag{18.13}$$

which is a rational function with a palindromic polynomial of degree $4N-2$ as numerator, while the denominator is of degree $4N$. Hence, the difference in degrees is 2, i.e. the quaternionic dimension of the moduli space. In addition, the denominator (18.13) has a pole of order 4 at $t\to1$, which equals the complex dimension of the moduli space.

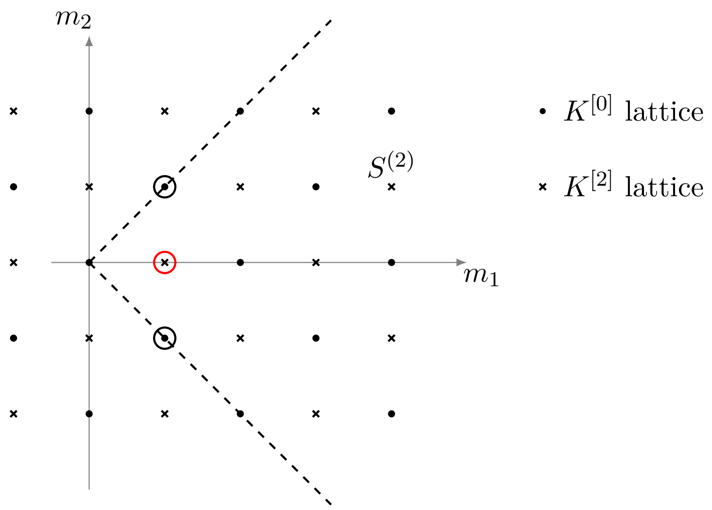


Figure 18.3: *The semi-group $S^{(2)}$ and its ray-generators (black circled points) for the quotient* $\mathrm{SO}(4)$ *and the representation* $[2,0]$*. The red circled lattice point completes the Hilbert basis for* $S^{(2)}$.

Plethystic logarithm Analysing the PL yields for $N \geq 3$

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}_{\mathrm{SO}(4)}^{[2,0]}) = 2t^2 + zt^{\Delta(1,0)}(1+2t^2+t^2) + 2t^{\Delta(1,\pm1)}(1+t) \\ - t^{2\Delta(1,0)}(1+2(1+z)t+(6+4z)t^2+2(1+z)t^3+t^4)+\dots\end{aligned} \tag{18.14}$$

and for $N=2$

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}_{\mathrm{SO}(4)}^{[2,0]}) = 2t^2 + zt^2(1+2t+t^2) + 2t^2(1+t) \\ - t^4(1+2(1+z)t+(6+4z)t^2)+\dots\end{aligned} \tag{18.15}$$

such that we have generators as summarised in Tab. 18.3.

object	(m_1,m_2)	lattice	$\Delta(m_1,m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(1,0)$	$K^{[2]}$	$2N-2$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
monopole	$(1,\pm1)$	$K^{[0]}$	$2N-2$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$

Table 18.3: *Bare and dressed monopole generators for a* $\mathrm{SO}(4)$ *gauge theory with matter transforming in* $[2,0]$.

Gauging a $\mathbb{Z}_2$ Although the Hilbert series (18.13) is not a complete intersection, the gauging of the topological $\mathbb{Z}_2$ reproduces the Spin(4) result (18.9), that is

$$\mathrm{HS}_{\mathrm{Spin}(4)}^{[2,0]}(t,N) = \frac{1}{2}\left(\mathrm{HS}_{\mathrm{SO}(4)}^{[2,0]}(t,z{=}1,N) + \mathrm{HS}_{\mathrm{SO}(4)}^{[2,0]}(t,z{=}-1,N)\right) . \tag{18.16}$$

18.2.3 Quotient $\mathrm{SO}(3)\times\mathrm{SU}(2)$

The dual group is $\mathrm{SU}(2)\times\mathrm{SO}(3)$ and the summation extends over $(m_1,m_2)\in K^{[0]}\cup K^{[1]}$. The non-trivial centre $\mathbb{Z}_2\times\{1\}$ gives rise to a $\mathbb{Z}_2$-action, which we choose to distinguish the two lattices $K^{[0]}$ and $K^{[1]}$ as follows:

$$z_1^{m_1+m_2} = \begin{cases} z_1^{p_1+p_2} = z_1^{\text{even}} = 1 & \text{for} \quad (m_1,m_2) \in K^{[0]} \,, \\ z_1^{p_1+\frac{1}{2}+p_2+\frac{1}{2}} = z_1^{\text{even}+1} = z_1 & \text{for} \quad (m_1,m_2) \in K^{[1]} \,. \end{cases} \tag{18.17}$$

Hilbert basis The semi-group $S^{(2)} := C^{(2)} \cap \left(K^{[0]} \cup K^{[1]}\right)$ has a Hilbert basis comprised of the ray generators. We refer to Fig. 18.4 and provide the minimal generators for completeness:

$$\mathcal{H}(S^{(2)}) = \left\{ (\tfrac{1}{2},\tfrac{1}{2}), (1,-1) \right\} \,. \tag{18.18}$$

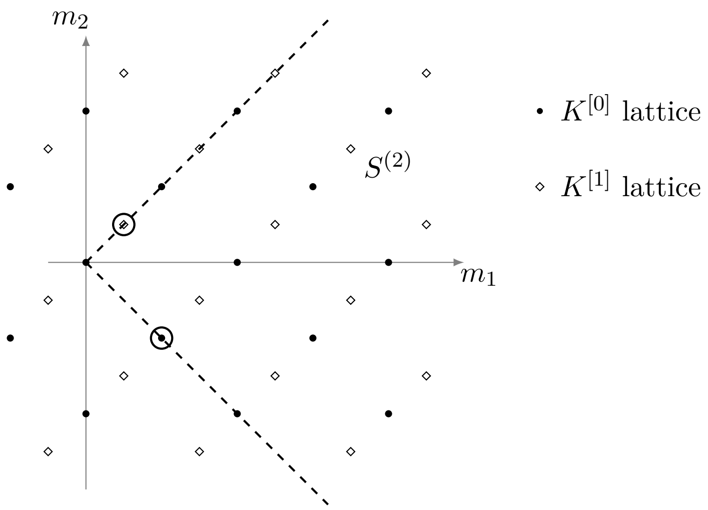


Figure 18.4: *The semi-group $S^{(2)}$ for the quotient $\mathrm{SO}(3) \times \mathrm{SU}(2)$ and the representation $[2,0]$. The black circled points are the ray generators.*

Hilbert series Computing the Hilbert series and using explicitly the $\mathbb{Z}_2$-properties of z_1 yields

$$\mathrm{HS}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}^{[2,0]}(t,z_1,N) = \frac{\left(1-t^{2N}\right)\left(1-t^{4N-2}\right)}{(1-t^2)^2\,(1-t^{2N-2})\,(1-t^{2N-1})\,(1-z_1 t^{N-1})\,(1-z_1 t^N)} \,, \tag{18.19}$$

which is a complete intersection with 6 generators and 2 relations. The generators are displayed in Tab. 18.4.

object	(m_1,m_2)	lattice	$\Delta(m_1,m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(\frac{1}{2},\frac{1}{2})$	$K^{[1]}$	$N-1$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(1,-1)$	$K^{[0]}$	$2N-2$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$

Table 18.4: *Bare and dressed monopole generators for a $\mathrm{SO}(3)\times\mathrm{SU}(2)$ gauge theory with matter transforming in $[2,0]$.*

Remark Comparing to the case of SU(2) with n_a adjoints and SO(3) with n fundamentals presented in [191], we can re-express the Hilbert series (18.19) as the product

$$\mathrm{HS}^{[2,0]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}(t, z_1, N) = \mathrm{HS}^{[1]}_{\mathrm{SO}(3)}(t, z_1, n = N) \times \mathrm{HS}^{[2]}_{\mathrm{SU}(2)}(t, n_a = N) , \tag{18.20}$$

where the z_1-grading belongs to SO(3) with N fundamentals. The minimal generator $(\frac{1}{2}, \frac{1}{2})$ is the minimal generator for SO(3) with N fundamentals, while $(1, -1)$ is the minimal generator for SU(2) with N adjoints.

18.2.4 Quotient SU(2) × SO(3)

The dual group is SO(3) × SU(2) and the summation extends over $(m_1, m_2) \in K^{[0]} \cup K^{[3]}$. The non-trivial centre $\{1\} \times \mathbb{Z}_2$ gives rise to a $\mathbb{Z}_2$-action, which we choose to distinguish the two lattices $K^{[0]}$ and $K^{[3]}$ as follows:

$$z_2^{p_1+p_2} = \begin{cases} z_2^{\mathrm{even}} = 1 & \text{for} \quad (m_1, m_2) \in K^{[0]} , \\ z_2^{\mathrm{odd}} = z_2 & \text{for} \quad (m_1, m_2) \in K^{[3]} . \end{cases} \tag{18.21}$$

Hilbert basis The semi-group $S^{(2)} := C^{(2)} \cap \left(K^{[0]} \cup K^{[3]}\right)$ has as Hilbert basis the set of ray generators

$$\mathcal{H}(S^{(2)}) = \left\{(1, 1), (\tfrac{1}{2}, -\tfrac{1}{2})\right\} . \tag{18.22}$$

Fig. 18.5 depicts the situation. We observe that bases (18.18) and (18.22) are related by reflection along the $m_2 = 0$ axis, which in turn corresponds to the interchange of $K^{[1]}$ and $K^{[3]}$.

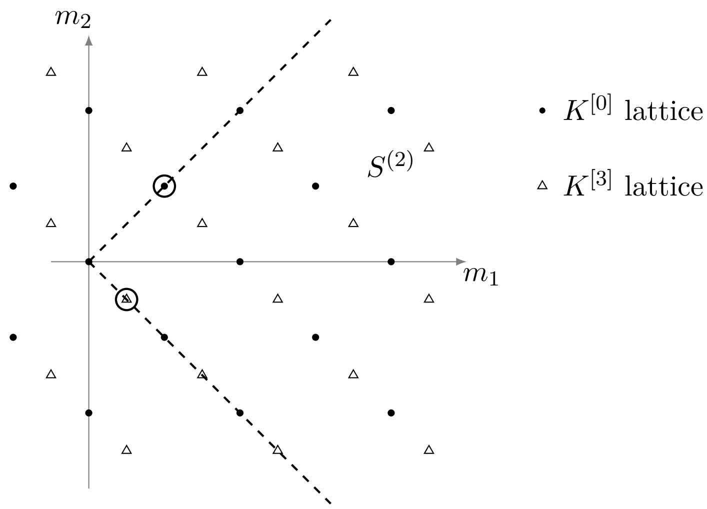


Figure 18.5: *The semi-group $S^{(2)}$ for the quotient* SU(2) × SO(3) *and the representation* $[2, 2]$. *The black circled points are the ray generators.*

Hilbert series Similar to the previous case, employing the $\mathbb{Z}_2$-properties of z_2 we obtain the following Hilbert series:

$$\mathrm{HS}^{[2,0]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t, z_2, N) = \frac{\left(1 - t^{2N}\right)\left(1 - t^{4N-2}\right)}{(1-t^2)^2\,(1 - t^{2N-2})\,(1 - t^{2N-1})\,(1 - z_2 t^{N-1})\,(1 - z_2 t^N)} , \tag{18.23}$$

which is a complete intersection with 6 generators and 2 relations. We summarise the generators in Tab. 18.5.

object	(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(\frac{1}{2}, -\frac{1}{2})$	$K^{[3]}$	$N-1$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(1,1)$	$K^{[0]}$	$2N-2$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$

Table 18.5: *Bare and dressed monopole generators for a* $\mathrm{SU}(2) \times \mathrm{SO}(3)$ *gauge theory with matter transforming in* $[2,0]$.

Remark Also, the equivalence

$$\mathrm{HS}^{[2,0]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}(t, z_1, N) \xleftrightarrow{z_1 \leftrightarrow z_2} \mathrm{HS}^{[2,0]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t, z_2, N) \tag{18.24}$$

holds, which then also implies

$$\mathrm{HS}^{[2,0]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t, z_2, N) = \mathrm{HS}^{[1]}_{\mathrm{SO}(3)}(t, z_2, n = N) \times \mathrm{HS}^{[2]}_{\mathrm{SU}(2)}(t, n_a = N) \ . \tag{18.25}$$

Thus, the moduli space is a product of two complete intersections.

18.2.5 Quotient $\mathrm{PSO}(4)$

Taking the quotient with respect to the entire centre of $\widetilde{\mathrm{SO}(4)}$ yields the projective group $\mathrm{PSO}(4)$, which has as GNO-dual $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. Consequently, the summation extends over the whole weight lattice $K^{[0]} \cup K^{[1]} \cup K^{[2]} \cup K^{[3]}$ and there is an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on this lattice, which is chosen as displayed in Tab. 18.6.

lattice	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\widetilde{\mathbb{Z}}_2 \times \widetilde{\mathbb{Z}}_2$
$K^{[0]}$	$(z_1)^0, (z_2)^0$	$(w_1)^0, (w_2)^0$
$K^{[1]}$	$(z_1)^1, (z_2)^0$	$(w_1)^1, (w_2)^1$
$K^{[2]}$	$(z_1)^0, (z_2)^1$	$(w_1)^0, (w_2)^1$
$K^{[3]}$	$(z_1)^1, (z_2)^1$	$(w_1)^1, (w_2)^0$

Table 18.6: *The* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *distinguishes the four different lattice* $K^{[j]}$, $j = 0,1,2,3$. *The choice of fugacities* z_1, z_2 *is used in the computation, while the second choice* w_1, w_2 *is convenient for gauging* $\mathrm{PSO}(4)$ *to* $\mathrm{SU}(2) \times \mathrm{SO}(3)$.

Hilbert basis The semi-group $S^{(2)} := C^{(2)} \cap \left(K^{[0]} \cup K^{[1]} \cup K^{[2]} \cup K^{[3]} \right)$ has a Hilbert basis that is determined by the ray generators. Fig. 18.6 depicts the situation and the Hilbert basis reads

$$\mathcal{H}(S^{(2)}) = \left\{ (\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, -\tfrac{1}{2}) \right\} \ . \tag{18.26}$$

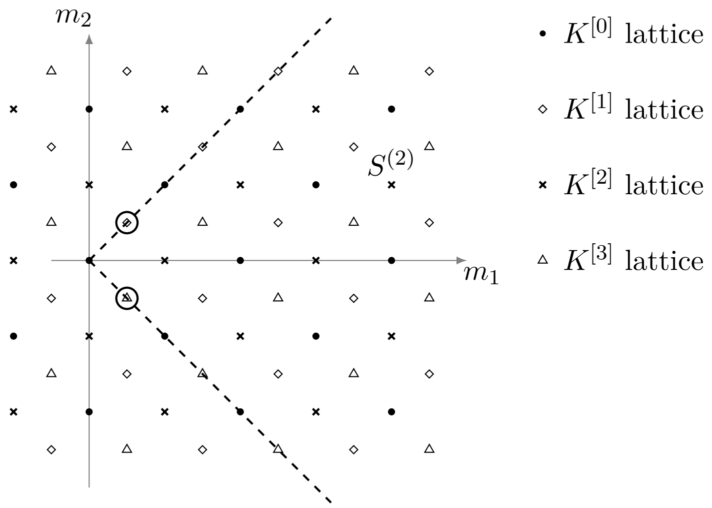


Figure 18.6: *The semi-group $S^{(2)}$ and its ray-generators (black circled points) for the quotient* PSO(4) *and the representation* $[2,0]$.

Hilbert series An evaluation of the Hilbert series yields

$$\mathrm{HS}^{[2,0]}_{\mathrm{PSO}(4)}(t,z_1,z_2,N)=\frac{\left(1-t^{2N}\right)^2}{(1-t^2)^2\,(1-z_1t^{N-1})\,(1-z_1t^N)\,(1-z_1z_2t^{N-1})\,(1-z_1z_2t^N)}\ , \quad (18.27)$$

which is a complete intersection with 6 generators and 2 relations. Tab. 18.7 summarises the generators with their properties.

object	(m_1,m_2)	lattice	$\Delta(m_1,m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(\frac{1}{2},\frac{1}{2})$	$K^{[1]}$	$N-1$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by U(1)
monopole	$(\frac{1}{2},-\frac{1}{2})$	$K^{[3]}$	$N-1$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by U(1)

Table 18.7: *Bare and dressed monopole generators for a* PSO(4) *gauge theory with matter transforming in* $[2,0]$.

Gauging a $\mathbb{Z}_2$ Now, we utilise the $\mathbb{Z}_2\times\mathbb{Z}_2$ global symmetry to recover the Hilbert series for *all* five quotients solely from the PSO(4) result. Firstly, to obtain the SO(4) result, we need to average out the contributions of $K^{[1]}$ and $K^{[3]}$, which is achieved for $z_1\to\pm1$ (we relabel z_2 for consistence), see also Tab. 18.6. This yields

$$\begin{aligned}\mathrm{HS}^{[2,0]}_{\mathrm{SO}(4)}(t,z,N)=\frac{1}{2}\Big(&\mathrm{HS}^{[2,0]}_{\mathrm{PSO}(4)}(t,z_1{=}1,z_2{=}z,N)\\ &+\mathrm{HS}^{[2,0]}_{\mathrm{PSO}(4)}(t,z_1{=}-1,z_2{=}z,N)\Big)\ .\end{aligned} \quad (18.28a)$$

Secondly, a subsequent gauging leads to the Spin(4) result as demonstrated in (18.16), because one averages the $K^{[2]}$ contributions out. Thirdly, one can gauge the other $\mathbb{Z}_2$-factor corresponding to $z_2\to\pm1$, which then eliminates the contributions of $K^{[2]}$ and $K^{[3]}$ due to the choices of

Tab. 18.6. The result then reads

$$\begin{aligned} \mathrm{HS}^{[2,0]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}(t,z_1,N) = \frac{1}{2}\Big(&\mathrm{HS}^{[2,0]}_{\mathrm{PSO}(4)}(t,z_1,z_2{=}1,N) \\ &+ \mathrm{HS}^{[2,0]}_{\mathrm{PSO}(4)}(t,z_1,z_2{=}-1,N)\Big)\ . \end{aligned} \tag{18.28b}$$

Lastly, for obtaining the $\mathrm{SU}(2)\times\mathrm{SO}(3)$ Hilbert series one needs to eliminate the $K^{[1]}$ and $K^{[2]}$ contributions. For that, we have to redefine the $\mathbb{Z}_2$-fugacities conveniently. One choice is

$$z_1 \cdot z_2 \mapsto w_1\ , \qquad z_1 \mapsto w_1 \cdot w_2\ , \qquad \text{and} \qquad z_2 \mapsto w_2\ , \tag{18.28c}$$

which is consistent in $\mathbb{Z}_2\times\mathbb{Z}_2$. The effect on the lattices is summarised in Tab. 18.6. Hence, $w_2 \to \pm 1$ has the desired effect and leads to

$$\begin{aligned} \mathrm{HS}^{[2,0]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t,z_2{=}w_1,N) = \frac{1}{2}\Big(&\mathrm{HS}^{[2,0]}_{\mathrm{PSO}(4)}(t,w_1,w_2{=}1,N) \\ &+ \mathrm{HS}^{[2,0]}_{\mathrm{PSO}(4)}(t,w_1,w_2{=}-1,N)\Big)\ . \end{aligned} \tag{18.28d}$$

Consequently, the Hilbert series for *all* five quotients can be computed from the PSO(4)-result by gauging $\mathbb{Z}_2$-factors.

Remark As for most of the cases in this section, the Hilbert series (18.27) can be written as a product of two complete intersections. Employing the results of [191] for SO(3) with n fundamentals, we obtain

$$\mathrm{HS}^{[2,0]}_{\mathrm{PSO}(4)}(t,z_1,z_2,N) = \mathrm{HS}^{[1]}_{\mathrm{SO}(3)}(t,z_1,n=N)\times \mathrm{HS}^{[1]}_{\mathrm{SO}(3)}(t,z_1 z_2,n=N)\ . \tag{18.29}$$

18.3 Representation $[2,2]$

Let us study the representation $[2,2]$ to further compare the results for the five different quotient groups. The conformal dimension reads

$$\Delta(m_1,m_2) = N(|m_1-m_2| + |m_1+m_2| + 2\,|m_1| + 2\,|m_2|) - |m_1-m_2| - |m_1+m_2|\ . \tag{18.30}$$

As described in the introduction, the conformal dimension (18.30) defines a fan in the dominant Weyl chamber, which is spanned by two 2-dimensional rational cones

$$C^{(2)}_{\pm} = \Big\{(m_1 \geq \pm m_2)\wedge(m_2 \geq \pm 0)\Big\}\ . \tag{18.31}$$

18.3.1 Quotient Spin(4)

Hilbert basis Starting from the fan (18.31) defined by the cones $C^{(2)}_{\pm}$, the Hilbert bases for the semi-groups $S^{(2)}_{\pm} \coloneqq C^{(2)}_{\pm}\cap K^{[0]}$ are simply given by the ray generators, see for instance Fig. 18.7.

$$\mathcal{H}(S^{(2)}_{\pm}) = \Big\{(1,\pm 1),(2,0)\Big\}\ . \tag{18.32}$$

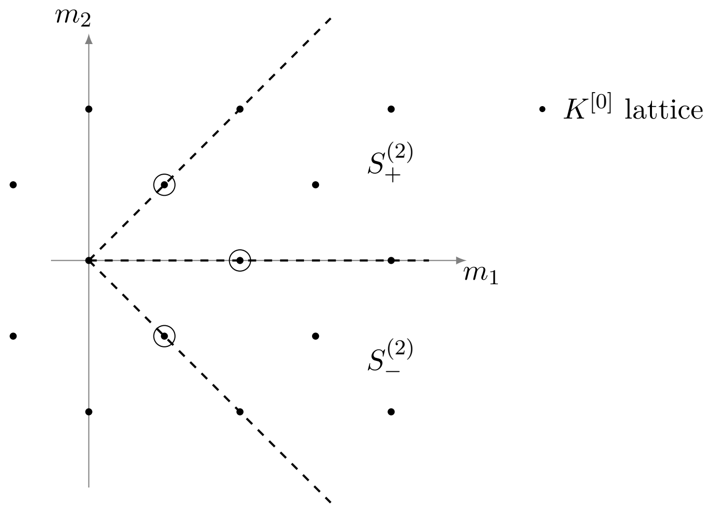


Figure 18.7: *The semi-groups and their ray-generators (black circled points) for the trivial quotient* Spin(4) *and the representation* $[2,2]$.

Hilbert series The GNO-dual $\mathrm{SO}(3)\times\mathrm{SO}(3)$ has a trivial centre and the Hilbert series reads

$$\mathrm{HS}^{[2,2]}_{\mathrm{Spin}(4)}(t,N) = \frac{1+t^{6N-2}+2t^{6N-1}+2t^{8N-3}+t^{8N-2}+t^{14N-4}}{(1-t^2)^2\,(1-t^{6N-2})\,(1-t^{8N-4})}\ . \tag{18.33}$$

The numerator of (18.33) is a palindromic polynomial of degree $14N-4$; while the denominator is a polynomial of degree $14N-2$. Hence, the difference in degree is two, which equals the quaternionic dimension of the moduli space. In addition, denominator of (18.33) has a pole of order four at $t=1$, which equals the complex dimension of the moduli space.

Plethystic logarithm The plethystic logarithm takes the form

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[2,2]}_{\mathrm{Spin}(4)}) &= 2t^2+2t^{\Delta(1,\pm1)}(1+t)+t^{\Delta(2,0)}(1+2t+t^2)\\ &\quad - t^{2\Delta(1,\pm1)}(1+2t+3t^2+2t^3+4t^4+2t^5+3t^6+2t^7+t^8)+\dots\end{aligned} \tag{18.34}$$

The appearing terms agree with the minimal generators of the Hilbert bases (18.7). One has two independent degree two Casimir invariants. Further, there are monopole operators of GNO-charge $(1,1)$ and $(1,-1)$ at conformal dimension $6N-2$ with an independent dressed monopole generator of conformal dimension $6N-1$ for both charges. Moreover, there is a monopole operator of GNO-charge $(2,0)$ at dimension $8N-4$ with two associated dressed monopole operators at dimension $8N-3$ and one at $8N-2$.

18.3.2 Quotient SO(4)

Hilbert basis The semi-groups $S_\pm^{(2)} \coloneqq C_\pm^{(2)}\cap\left(K^{[0]}\cup K^{[2]}\right)$ have Hilbert bases which again equal (the now different) ray generators. The situation is depicted in Fig. 18.8 and the Hilbert bases are as follows:

$$\mathcal{H}(S_\pm^{(2)}) = \Big\{(1,\pm1),(1,0)\Big\}\ . \tag{18.35}$$

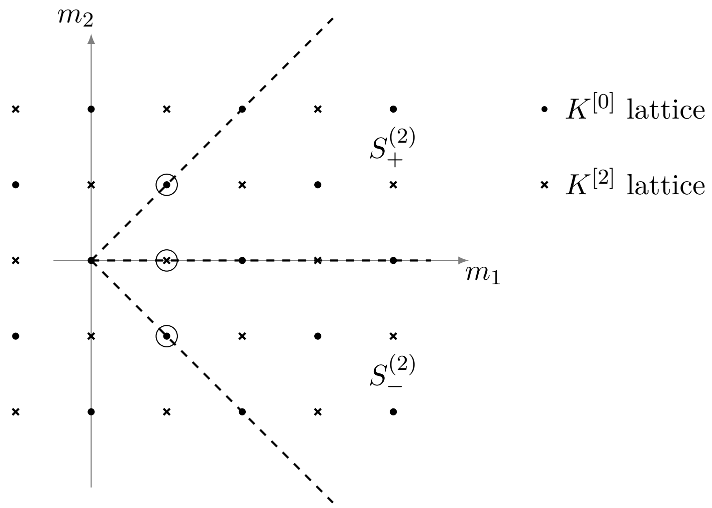


Figure 18.8: *The semi-groups and their ray-generators (black circled points) for the quotient* SO(4) *and the representation* [2, 2].

Hilbert series The Hilbert series reads

$$\mathrm{HS}^{[2,2]}_{\mathrm{SO}(4)}(t,z,N) = \frac{1 + zt^{4N} + 2zt^{4N-1} + t^{6N-2} + 2t^{6N-1} + zt^{10N-2}}{(1-t^2)^2\,(1-zt^{4N-2})\,(1-t^{6N-2})}\ . \tag{18.36}$$

The numerator of (18.36) is a palindromic polynomial of degree $10N-2$ (neglecting the dependence on z); while the denominator is a polynomial of degree $10N$. Hence, the difference in degree is two, which matches the quaternionic dimension of the moduli space. Moreover, the denominator has a pole of order four at $t=1$, which equals the complex dimension of $\mathcal{M}_C$.

Plethystic logarithm Studying the PL, we observe

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[2,2]}_{\mathrm{SO}(4)}) &= 2t^2 + zt^{\Delta(1,0)}(1+2t^2+t) + 2t^{\Delta(1,\pm1)}(1+t) \\ &\quad - t^{2\Delta(1,0)+2}(3+2t^2+t^2+2t^3+4t^4+2t^5+t^6+2t^7+3t^8) + \ldots\end{aligned} \tag{18.37}$$

such that we can associate the generators as follows: two degree two Casimir invariants of SO(4), i.e. the quadratic Casimir and the Pfaffian; a monopole of GNO-charge $(1,0) \in K^{[2]}$ at conformal dimension $4N-2$ with two dressings at dimension $4N-1$ and another dressing at $4N$; and two monopole operators of GNO-charges $(1,1)$, $(1,-1) \in K^{[0]}$ at dimension $6N-2$ with each one dressed monopole at dimension $6N-1$.

Gauging the $\mathbb{Z}_2$ In addition, one can gauge the topological $\mathbb{Z}_2$ in (18.36) and obtains

$$\mathrm{HS}^{[2,2]}_{\mathrm{Spin}(4)}(t,N) = \frac{1}{2}\left(\mathrm{HS}^{[2,2]}_{\mathrm{SO}(4)}(t,z{=}1,N) + \mathrm{HS}^{[2,2]}_{\mathrm{SO}(4)}(t,z{=}-1,N)\right)\ . \tag{18.38}$$

18.3.3 Quotient SO(3) × SU(2)

Hilbert basis The semi-groups $S_\pm^{(2)} := C_\pm^{(2)} \cap \left(K^{[0]} \cup K^{[1]}\right)$ have Hilbert bases that go beyond the set of ray generators. We refer to Fig. 18.9 and the Hilbert bases are obtained as follows:

$$\mathcal{H}(S_+^{(2)}) = \left\{(\tfrac{1}{2},\tfrac{1}{2}),(2,0)\right\} \qquad \text{and} \qquad \mathcal{H}(S_-^{(2)}) = \left\{(1,-1),(\tfrac{3}{2},-\tfrac{1}{2}),(2,0)\right\}\ . \tag{18.39}$$

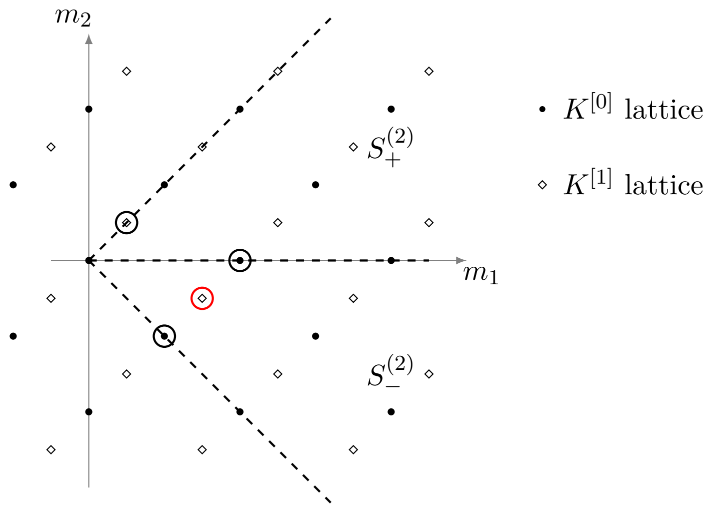


Figure 18.9: *The semi-groups for the quotient* $\mathrm{SO}(3)\times\mathrm{SU}(2)$ *and the representation* $[2,2]$*. The black circled points are the ray generators and the red circled point completes the Hilbert basis for* $S_-^{(2)}$*.*

Hilbert series The Hilbert series is computed to be

$$\mathrm{HS}^{[2,2]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}(t,z_1,N) = \frac{R(t,z_1,N)}{(1-t^2)^2\,(1-t^{6N-2})\,(1-t^{8N-4})}\,, \tag{18.40a}$$

$$\begin{aligned} R(t,z_1,N) &= 1+z_1 t^{3N-1}(1+t)+t^{6N-2}(1+2t)+z_1 t^{7N-3}(1+2t+t^2)\\ &\quad + t^{8N-3}(2+t)+z_1 t^{11N-4}(1+t)+t^{14N-4}\,. \end{aligned} \tag{18.40b}$$

Again, the numerator of (18.40) is a palindromic polynomial of degree $14N-4$; while the denominator is a polynomial of degree $14N-2$. Hence, the difference in degree is two, which matches the quaternionic dimension of the moduli space. Also, the denominator has a pole of order four at $t=1$, which equals the complex dimension of the moduli space.

Plethystic logarithm The inspection of the PL for $N\geq 2$ reveals

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}^{[2,2]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}) &= 2t^2 + z_1 t^{\Delta(\frac{1}{2},\frac{1}{2})}(1+t) + t^{\Delta(1,\pm1)}(1+t-t^2)\\ &\quad + z_1 t^{\Delta(1+\frac{1}{2},-1+\frac{1}{2})}(1+2t+t^2) + t^{\Delta(2,0)}(1+2t+t^2)\\ &\quad - z_1 t^{3\Delta(\frac{1}{2},\frac{1}{2})}(1+2t+t^2) + \ldots\,. \end{aligned} \tag{18.41}$$

We summarise the generators in Tab. 18.8.

18.3.4 Quotient $\mathrm{SU}(2)\times\mathrm{SO}(3)$

Hilbert basis The semi-groups $S_\pm^{(2)} \coloneqq C_\pm^{(2)} \cap \left(K^{[0]}\cup K^{[3]}\right)$ have Hilbert bases that go beyond the set of ray generators. Fig. 18.10 depicts the situation and the Hilbert bases are computed to be

$$\mathcal{H}(S_+^{(2)}) = \left\{(1,1),(\tfrac{3}{2},\tfrac{1}{2}),(2,0)\right\} \qquad \text{and} \qquad \mathcal{H}(S_-^{(2)}) = \left\{(\tfrac{1}{2},-\tfrac{1}{2}),(2,0)\right\}\,. \tag{18.42}$$

(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
$(\frac{1}{2}, \frac{1}{2})$	$K^{[1]}$	$3N-1$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
$(1, -1)$	$K^{[0]}$	$6N-2$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
$(\frac{3}{2}, -\frac{1}{2})$	$K^{[1]}$	$7N-3$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
$(2, 0)$	$K^{[0]}$	$8N-4$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 18.8: *The generators for the chiral ring of a* $\mathrm{SO}(3) \times \mathrm{SU}(2)$ *gauge theory with matter in* $[2, 2]$.

We observe that the bases (18.39) and (18.42) are related by reflection along the $m_2 = 0$ axis, which in turn corresponds to the interchange of $K^{[1]}$ and $K^{[3]}$.

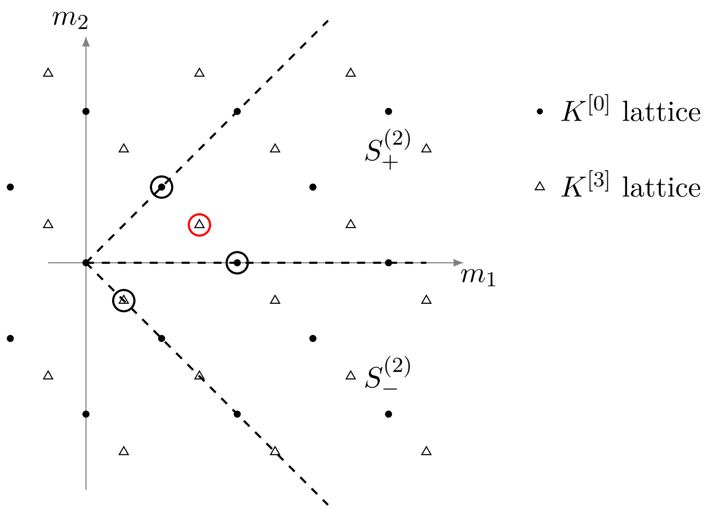


Figure 18.10: *The semi-groups for the quotient* $\mathrm{SU}(2) \times \mathrm{SO}(3)$ *and the representation* $[2, 2]$. *The black circled points are the ray generators and the red circled point completes the Hilbert basis for* $S_+^{(2)}$.

Hilbert series The Hilbert series reads

$$\mathrm{HS}^{[2,2]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t, z_2, N) = \frac{R(t, z_2, N)}{(1-t^2)^2 \, (1-t^{6N-2}) \, (1-t^{8N-4})} \,, \tag{18.43a}$$

$$R(t, z_2, N) = 1 + z_2 t^{3N-1}(1+t) + t^{6N-2}(1+2t) + z_2 t^{7N-3}(1+2t+t^2) + t^{8N-3}(2+t) + z_2 t^{11N-4}(1+t) + t^{14N-4} \,. \tag{18.43b}$$

The numerator of (18.43) is palindromic polynomial of degree $14N-4$; while the denominator is a polynomial of degree $14N-2$. Hence, the difference in degree is two, which equals the quaternionic dimension of the moduli space. In addition, the denominator has a pole of order four at $t = 1$, which matches the complex dimension of the moduli space.

As before, comparing the quotients $\mathrm{SO}(3) \times \mathrm{SU}(2)$ and $\mathrm{SU}(2) \times \mathrm{SO}(3)$ as well as the symmetry of (18.30), it is natural to expect the relationship

$$\mathrm{HS}^{[2,2]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}(t, z_1, N) \xleftrightarrow{z_1 \leftrightarrow z_2} \mathrm{HS}^{[2,2]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t, z_2, N) \,, \tag{18.44}$$

which is verified explicitly for (18.40) and (18.43).

Plethystic logarithm The equivalence to $\mathrm{SO}(3)\times\mathrm{SU}(2)$ is further confirmed by the inspection of the PL for $N\geq 2$

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[2,2]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}) &= 2t^2 + z_2 t^{\Delta(\frac{1}{2},-\frac{1}{2})}(1+t) + t^{\Delta(1,1)}(1+t-t^2) \\ &\quad + z_2 t^{\Delta(\frac{3}{2},\frac{1}{2})}(1+2t+t^2) + t^{\Delta(2,0)}(1+2t+t^2) + \dots\ ,\end{aligned} \tag{18.45}$$

where we can summarise the monopole generators as in Tab. 18.9. Note the change in GNO-

(m_1,m_2)	lattice	$\Delta(m_1,m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
$(\frac{1}{2},-\frac{1}{2})$	$K^{[3]}$	$3N-1$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
$(1,1)$	$K^{[0]}$	$6N-2$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
$(\frac{3}{2},\frac{1}{2})$	$K^{[3]}$	$7N-3$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
$(2,0)$	$K^{[0]}$	$8N-4$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 18.9: *The generators for the chiral ring of a* $\mathrm{SU}(2)\times\mathrm{SO}(3)$ *gauge theory with matter in* $[2,2]$.

charges in accordance with the use of $K^{[3]}$ instead of $K^{[1]}$.

18.3.5 Quotient $\mathrm{PSO}(4)$

Hilbert basis The semi-groups $S^{(2)}_{\pm} \coloneqq C^{(2)}_{\pm} \cap \left(K^{[0]}\cup K^{[1]}\cup K^{[2]}\cup K^{[3]}\right)$ have Hilbert bases that are determined by the ray generators. Fig. 18.11 depicts the situation and the Hilbert bases read

$$\mathcal{H}(S^{(2)}_{\pm}) = \left\{(\tfrac{1}{2},\pm\tfrac{1}{2}),(1,0)\right\}\ . \tag{18.46}$$

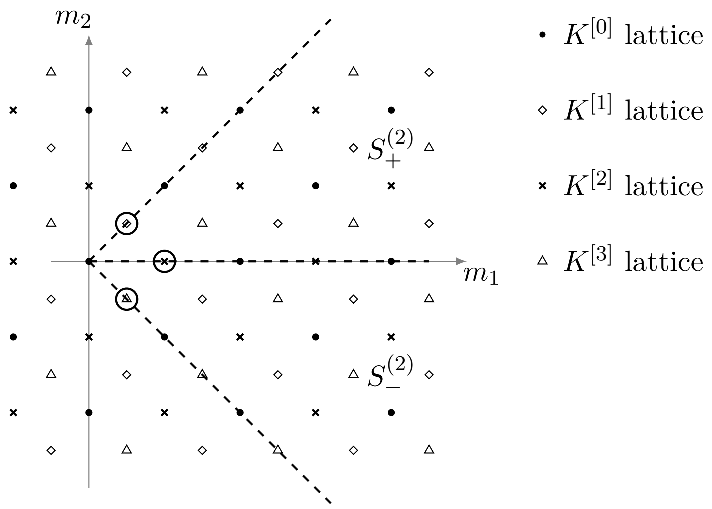


Figure 18.11: *The semi-groups and their ray-generators (black circled points) for the quotient* $\mathrm{PSO}(4)$ *and the representation* $[2,2]$.

Hilbert series The Hilbert series reads

$$\mathrm{HS}^{[2,2]}_{\mathrm{PSO}(4)}(t,z_1,z_2,N) = \frac{R(t,z_1,z_2,N)}{(1-t^2)^2\,(1-t^{6N-2})\,(1-z_2 t^{4N-2})}\ , \tag{18.47a}$$

$$R(t,z_1,z_2,N) = 1 + z_1 t^{3N-1}(1+t) + z_1 z_2 t^{3N-1}(1+t) + z_2 t^{4N-1}(2+t) + t^{6N-2}(1+2t) + z_1 z_2 t^{7N-2}(1+t) + z_1 t^{7N-2}(1+t) + z_2 t^{10N-2} \,. \tag{18.47b}$$

The numerator of (18.47) is a palindromic polynomial of degree $10N-2$; while the denominator is a polynomial of degree $10N$. Hence, the difference in degree is two, which corresponds to the quaternionic dimension of $\mathcal{M}_C$. Similarly to the previous cases, the denominator of (18.47) has a pole of order four at $t=1$, which equals the complex dimension of the moduli space.

Gauging a $\mathbb{Z}_2$ As before, by gauging the $\mathbb{Z}_2$-factor corresponding to z_1 we recover the SO(4)-result

$$\mathrm{HS}^{[2,2]}_{\mathrm{SO}(4)}(t,z,N) = \frac{1}{2}\Big(\mathrm{HS}^{[2,2]}_{\mathrm{PSO}(4)}(t,z_1{=}1,z_2{=}z,N) + \mathrm{HS}^{[2,2]}_{\mathrm{PSO}(4)}(t,z_1{=}-1,z_2{=}z,N)\Big) \,, \tag{18.48a}$$

while gauging the $\mathbb{Z}_2$-factor with fugacity z_2 provides the $\mathrm{SO}(3)\times\mathrm{SU}(2)$-result

$$\mathrm{HS}^{[2,2]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}(t,z_1,N) = \frac{1}{2}\Big(\mathrm{HS}^{[2,2]}_{\mathrm{PSO}(4)}(t,z_1,z_2{=}1,N) + \mathrm{HS}^{[2,2]}_{\mathrm{PSO}(4)}(t,z_1,z_2{=}{-1},N)\Big) \,. \tag{18.48b}$$

Furthermore, employing the redefined fugacities w_1, w_2 of (18.28c) one reproduces the Hilbert series for $\mathrm{SU}(2)\times\mathrm{SO}(3)$ as follows:

$$\mathrm{HS}^{[2,2]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t,z_2{=}w_1,N) = \frac{1}{2}\Big(\mathrm{HS}^{[2,2]}_{\mathrm{PSO}(4)}(t,w_1,w_2{=}1,N) + \mathrm{HS}^{[2,2]}_{\mathrm{PSO}(4)}(t,w_1,w_2{=}{-1},N)\Big) \,. \tag{18.48c}$$

Therefore, one can obtain the Hilbert series for *all* five quotients from the PSO(4)-result (18.47) by employing the $\mathbb{Z}_2$-gaugings (18.38) and (18.48).

Plethystic logarithm Inspecting the PL leads to

$$\mathrm{PL}(\mathrm{HS}^{[2,2]}_{\mathrm{PSO}(4)}) = 2t^2 + z_1 t^{\Delta(\frac{1}{2},\frac{1}{2})}(1+t) + z_1 z_2 t^{\Delta(\frac{1}{2},-\frac{1}{2})}(1+t) + z_2 t^{\Delta(1,0)}(1+2t+t^2) + \dots \tag{18.49}$$

such that we can summarise the monopole generators as in Tab. 18.10.

(m_1,m_2)	lattice	$\Delta(m_1,m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
$(\frac{1}{2},\frac{1}{2})$	$K^{[1]}$	$3N-1$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by U(1)
$(\frac{1}{2},-\frac{1}{2})$	$K^{[3]}$	$3N-1$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by U(1)
$(1,0)$	$K^{[2]}$	$4N-2$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 18.10: *The generators for the chiral ring of a* PSO(4) *gauge theory with matter transforming in the representation* $[2,2]$.

18.4 Representation $[4,2]$

The conformal dimension for this case reads

$$\begin{aligned}\Delta(m_1,m_2) = N\big(&|3m_1 - m_2| + |m_1 - 3m_2| + |m_1+m_2| + 3\,|m_1-m_2| + 2\,|m_1| + 2\,|m_2|\big) \\ &- |m_1+m_2| - |m_1-m_2| \ . \end{aligned} \tag{18.50}$$

The interesting feature of this representation is its asymmetric behaviour under exchange of m_1 and m_2. As before, the conformal dimension (18.50) defines a fan in the dominant Weyl chamber that is spanned by three 2-dimensional cones

$$C_1^{(2)} = \Big\{ (m_1 \geq -m_2) \wedge (m_2 \leq 0) \Big\} \ , \tag{18.51a}$$

$$C_2^{(2)} = \Big\{ (m_1 \geq 3\,m_2) \wedge (m_2 \geq 0) \Big\} \ , \tag{18.51b}$$

$$C_3^{(2)} = \Big\{ (m_1 \geq m_2) \wedge (m_1 \leq 3\,m_2) \Big\} \ . \tag{18.51c}$$

18.4.1 Quotient Spin(4)

Hilbert basis Starting from the fan (18.51) with cones $C_p^{(2)}$ (for $p=1,2,3$), the Hilbert bases for the semi-groups $S_p^{(2)} \coloneqq C_p^{(2)} \cap K^{[0]}$ are simply given by the ray generators:

$$\mathcal{H}(S_1^{(2)}) = \Big\{(2,0),(1,-1)\Big\}, \quad \mathcal{H}(S_2^{(2)}) = \Big\{(3,1),(2,0)\Big\}, \quad \mathcal{H}(S_3^{(2)}) = \Big\{(1,1),(3,1)\Big\}, \tag{18.52}$$

which is apparent from Fig. 18.12.

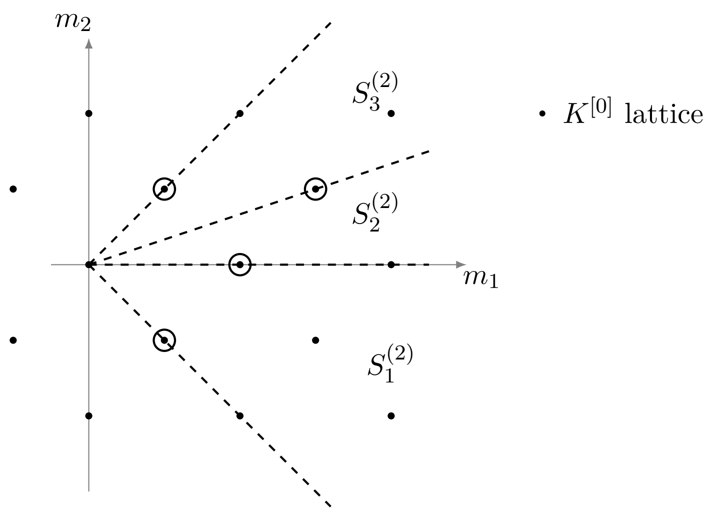


Figure 18.12: *The semi-groups and their ray-generators (black circled points) for the quotient* Spin(4) *and the representation* $[4,2]$.

Hilbert series The Hilbert series reads

$$\mathrm{HS}_{\mathrm{Spin}(4)}^{[4,2]}(t,N) = \frac{R(t,N)}{(1-t^2)^2\,(1-t^{18N-2})\,(1-t^{20N-4})\,(1-t^{26N-6})} \ , \tag{18.53a}$$

$$\begin{aligned} R(t,N) = 1 &+ t^{10N-2}(1+t) + t^{18N-1} + t^{20N-4}(1+3t+t^2) \\ &+ t^{26N-5}(2+t) - t^{28N-4}(1+t) + t^{36N-7}(1+t) \end{aligned} \tag{18.53b}$$

$$- t^{38N-6}(1+2t) - t^{44N-8}(1+3t+t^2) - t^{46N-9} - t^{54N-9}(1+t) - t^{64N-10} \,.$$

The numerator of (18.53) is an anti-palindromic polynomial of degree $64N - 10$, while the denominator is of degree $64N - 8$. Consequently, the difference in degree is two. Moreover, the rational function (18.53) has a pole of order four as $t \to 1$ because $R(t{=}1, N) = 0$, but $\frac{\mathrm{d}}{\mathrm{d}t} R(t,N)|_{t=1} \neq 0$.

Plethystic logarithm Inspecting the PL yields for $N \geq 3$

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[4,2]}_{\mathrm{Spin}(4)}) = 2t^2 &+ t^{\Delta(1,1)}(1+t) + t^{\Delta(1,-1)}(1+t) + t^{\Delta(2,0)}(1+2t) \\ &+ t^{\Delta(3,1)}(1+2t+t^2) - t^{\Delta(1,1)+\Delta(1,-1)}(1+2t+t^2) \\ &- t^{\Delta(1,1)+\Delta(2,0)}(1+3t+3t^2+t^3) + \dots\end{aligned} \tag{18.54}$$

leads to an identification of generators as in Tab. 18.11. We observe that $(2,0)$ has only 2

object	(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	# dressings
Casimirs	—	—	2	—	—
monopole	$(1,1)$	$K^{[0]}$	$10N-2$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(1,-1)$	$K^{[0]}$	$18N-2$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(2,0)$	$K^{[0]}$	$20N-4$	$\mathrm{U}(1) \times \mathrm{U}(1)$	2 by $\mathrm{U}(1)^2$
monopole	$(3,1)$	$K^{[0]}$	$26N-6$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 18.11: *The chiral ring generators for a* $\mathrm{Spin}(4)$ *gauge theory with matter transforming in* $[4,2]$.

dressings, although we would expect 3. We know from other examples that there should be a relation at $2\Delta(1,1) + 2 = 20N - 2$ which is precisely the dimension of the second dressing of $(2,0)$. (See also App. C.)

18.4.2 Quotient $\mathrm{SO}(4)$

Hilbert basis The semi-groups $S_p^{(2)} := C_p^{(2)} \cap \left(K^{[0]} \cup K^{[2]}\right)$ have Hilbert bases as shown in Fig. 18.13 or explicitly:

$$\mathcal{H}(S_1^{(2)}) = \left\{(1,0),(1,-1)\right\}\,, \qquad \mathcal{H}(S_2^{(2)}) = \left\{(3,1),(1,0)\right\}\,, \tag{18.55a}$$

$$\mathcal{H}(S_3^{(2)}) = \left\{(1,1),(2,1),(3,0)\right\}\,. \tag{18.55b}$$

Hilbert series A computation then yields

$$\mathrm{HS}^{[4,2]}_{\mathrm{SO}(4)}(t,z,N) = \frac{R(t,z,N)}{(1-t^2)^2\,(1-t^{10N-2})\,(1-t^{18N-2})\,(1-t^{26N-6})\,(1-zt^{10N-2})}\,, \tag{18.56a}$$

$$\begin{aligned}R(t,z,N) = 1 &+ t^{10N-1} + zt^{10N-1}(2+t) + zt^{18N-4}(1+2t+t^3) + t^{18N-1} \\ &- zt^{20N-4}(1+3t+t^2) + 2t^{26N-5}(2+t) \\ &- t^{28N-6}(1+2t+2t^2+2t^3) - zt^{28N-3}\end{aligned} \tag{18.56b}$$

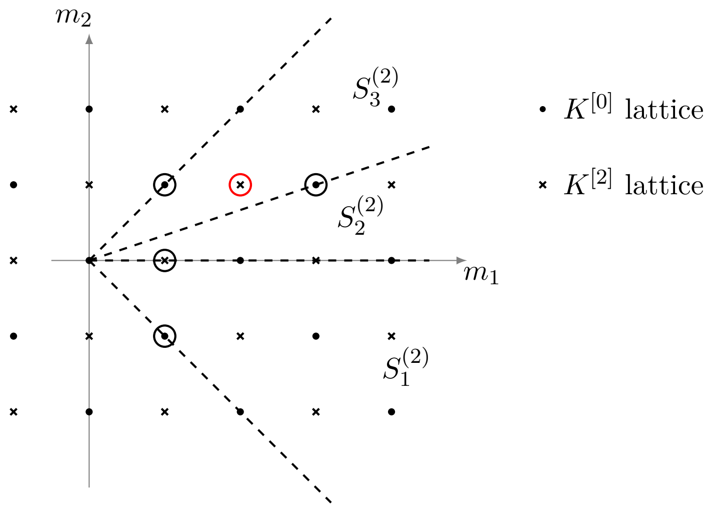


Figure 18.13: *The semi-groups for the quotient* SO(4) *and the representation* [4, 2]. *The black circled points are the ray generators and the red circled point completes the Hilbert basis for* $S_3^{(2)}$.

$$
\begin{aligned}
&- t^{36N-7} - zt^{36N-7}(2+2t+2t^2+t^3) + zt^{38N-6}(1+2t)\\
&- t^{44N-8}(1+3t+t^2) + zt^{46N-9} + t^{46N-8}(1+2t+t^2)\\
&+ t^{54N-10}(1+2t) + zt^{54N-9} + zt^{64N-10}\ .
\end{aligned}
$$

The numerator (18.56b) is a palindromic polynomial of degree $64N-10$, while the denominator is of degree $64N-8$. Consequently, the difference of the degree is two. Also, the Hilbert series (18.56) has a pole of order four as $t \to 1$, because $R(t{=}1, z, N) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}R(t, z, N)|_{t=1} = 0$, but $\frac{\mathrm{d}^2}{\mathrm{d}t^2}R(t, z, N)|_{t=1} \neq 0$.

Plethystic logarithm Inspecting the PL reveals

$$
\begin{aligned}
\mathrm{PL}(\mathrm{HS}_{\mathrm{SO}(4)}^{[4,2]}) = 2t^2 &+ zt^{\Delta(1,0)}(1+2t+t^2) + t^{\Delta(1,1)}(1+t) + zt^{\Delta(2,1)}(1+2t+t^2)\\
&+ t^{\Delta(1,-1)}(1+t) - zt^{2\Delta(1,0)}(1+3t+3t^2+t^3)\\
&- t^{2\Delta(1,1)+2}(4+2t+t^2) + t^{\Delta(3,1)}(1+2t+t^2) + \dots\ ,
\end{aligned}
\tag{18.57}
$$

such that the monopole generators can be summarised as in Tab. 18.12.

object	(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(1, 0)$	$K^{[2]}$	$10N-2$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
monopole	$(1, 1)$	$K^{[0]}$	$10N-2$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(2, 1)$	$K^{[2]}$	$18N-4$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
monopole	$(1, -1)$	$K^{[0]}$	$18N-2$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(3, 1)$	$K^{[0]}$	$26N-6$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 18.12: *The chiral ring generators for a* SO(4) *gauge theory with matter transforming in* [4, 2].

Gauging the $\mathbb{Z}_2$ Again, one can gauge the finite symmetry to recover the Spin(4) Hilbert series

$$\mathrm{HS}^{[4,2]}_{\mathrm{Spin}(4)}(t,N) = \frac{1}{2}\left(\mathrm{HS}^{[4,2]}_{\mathrm{SO}(4)}(t,z{=}1,N) + \mathrm{HS}^{[4,2]}_{\mathrm{SO}(4)}(t,z{=}-1,N)\right) \,. \tag{18.58}$$

18.4.3 Quotient $\mathrm{SO}(3)\times\mathrm{SU}(2)$

Hilbert basis The semi-groups $S_p^{(2)} := C_p^{(2)} \cap \left(K^{[0]} \cup K^{[1]}\right)$ (for $p=1,2,3$) have Hilbert bases that go beyond the set of ray generators. We refer to Fig. 18.14 and the Hilbert bases are obtained as follows:

$$\mathcal{H}(S_1^{(2)}) = \left\{(2,1),(\tfrac{3}{2},-\tfrac{1}{2}),(1,-1)\right\}\,, \qquad \mathcal{H}(S_2^{(2)}) = \left\{(3,1),(\tfrac{5}{2},\tfrac{1}{2}),(2,0)\right\}\,, \tag{18.59a}$$

$$\mathcal{H}(S_3^{(2)}) = \left\{(1,1),(3,1)\right\}\,. \tag{18.59b}$$

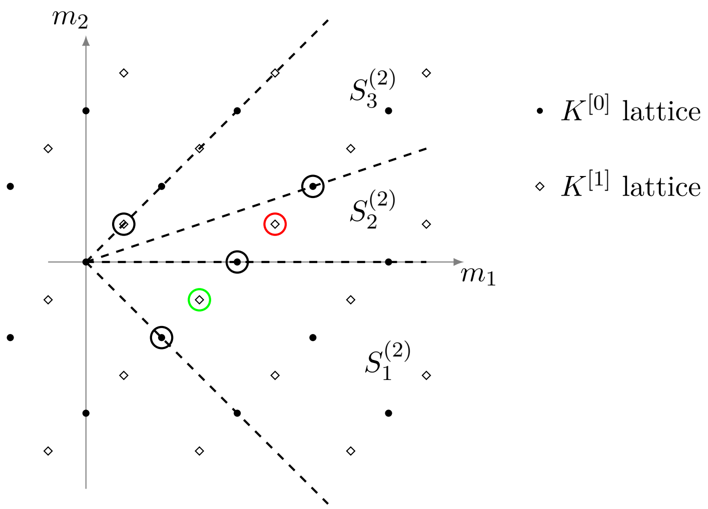


Figure 18.14: *The semi-groups for the quotient* $\mathrm{SO}(3)\times\mathrm{SU}(2)$ *and the representation* $[4,2]$. *The black circled points are the ray generators, the red circled point completes the Hilbert basis for* $S_2^{(2)}$*, while the green circled point completes the Hilbert basis of* $S_1^{(2)}$.

Hilbert series We compute the Hilbert series to

$$\mathrm{HS}^{[4,2]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}(t,z_1,N) = \frac{R(t,z_1,N)}{(1-t^2)^2\,(1-t^{18N-2})\,(1-t^{20N-4})\,(1-t^{26N-6})}\,, \tag{18.60a}$$

$$\begin{aligned} R(t,z_1,N) = {} & 1 + z_1 t^{5N-1}(1+t) + t^{10N-2}(1+t) + z_1 t^{15N-3}(1+t) \\ & + t^{18N-1} + z_1 t^{19N-3}(1+2t+t^3) \\ & + t^{20N-4}(1+3t+t^2) + z_1 t^{23N-5}(1+2t-t^3) \\ & + t^{26N-5}(2+t) - t^{28N-4}(1+t) + z_1 t^{31N-6}(1+t) \\ & - z_1 t^{33N-5}(1+t) + t^{36N-7}(1+t) - t^{38N-6}(1+2t) \\ & + z_1 t^{41N-8}(1-2t^2-t^3) - t^{44N-8}(1+3t+t^2) \\ & - z_1 t^{45N-9}(1+2t+t^2) - t^{46N-9} - z_1 t^{49N-8}(1+t) \\ & - t^{54N-9}(1+t) - z_1 t^{59N-10}(1+t) - t^{64N-10}\,. \end{aligned} \tag{18.60b}$$

The numerator of (18.60) is an anti-palindromic polynomial of degree $64N - 10$, while the denominator is of degree $64N - 8$. Thus, the difference in degrees is again 2. In addition, the Hilbert series (18.60) has a pole of order 4 as $t \to 1$, because $R(t{=}1, z_1, N) = 0$, but $\frac{\mathrm{d}}{\mathrm{d}t} R(t, z_1, N)|_{t=1} \neq 0$.

Plethystic logarithm Analysing the PL yields

$$
\begin{aligned}
PL = 2t^2 &+ z_1 t^{\Delta(\frac{1}{2},\frac{1}{2})}(1+t) - t^{\Delta(\frac{1}{2},\frac{1}{2})+2} + t^{\Delta(1,-1)}(1+t) \\
&+ z_1 t^{\Delta(\frac{3}{2},-\frac{1}{2})}(1+2t+t^2) + t^{\Delta(2,0)}(1+2t+t^2) \\
&+ z_1 t^{\Delta(\frac{5}{2},\frac{1}{2})}(1+2t+1) - z_1 t^{\Delta(\frac{1}{2},\frac{1}{2})+\Delta(1,-1)}(1+2t+t^2) \\
&- t^{\Delta(\frac{1}{2},\frac{1}{2})+\Delta(\frac{3}{2},-\frac{1}{2})}(1+3t+3t^2+t^3) \\
&- z_1 t^{\Delta(\frac{1}{2},\frac{1}{2})+\Delta(2,0)}(1+3t+3t^2+t^3) \\
&+ t^{\Delta(3,1)}(1+2t+t^2) + \dots \,,
\end{aligned}
\tag{18.61}
$$

verifies the set of generators as presented in Tab. 18.13. The coloured term indicates that we

object	(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(\frac{1}{2}, \frac{1}{2})$	$K^{[1]}$	$5N-1$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(1, -1)$	$K^{[0]}$	$18N-2$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(\frac{3}{2}, -\frac{1}{2})$	$K^{[1]}$	$19N-3$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
monopole	$(2, 0)$	$K^{[0]}$	$20N-4$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)$
monopole	$(\frac{5}{2}, \frac{1}{2})$	$K^{[1]}$	$23N-5$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3(2) by $\mathrm{U}(1)^2$
monopole	$(3, 1)$	$K^{[0]}$	$26N-6$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 18.13: *The chiral ring generators for a* $\mathrm{SO}(3) \times \mathrm{SU}(2)$ *gauge theory with matter transforming in* $[4, 2]$.

suspect a cancellation between one dressing of $(\frac{5}{2}, \frac{1}{2})$ and one relation because $\Delta(\frac{5}{2}, \frac{5}{2}) + 2 = 23N - 3 = \Delta(\frac{1}{2}, \frac{1}{2}) + \Delta(1, -1) = 5N - 1 + 18N - 2$.

18.4.4 Quotient $\mathrm{SU}(2) \times \mathrm{SO}(3)$

Hilbert basis The semi-groups $S_p^{(2)} := C_p^{(2)} \cap \left(K^{[0]} \cup K^{[3]}\right)$ (for $p = 1, 2, 3$) have Hilbert bases consisting of the ray generators as shown in Fig. 18.15 and we obtain explicitly

$$
\mathcal{H}(S_1^{(2)}) = \left\{(2,0), (\tfrac{1}{2}, -\tfrac{1}{2})\right\}, \quad \mathcal{H}(S_2^{(2)}) = \left\{(\tfrac{3}{2}, \tfrac{1}{2}), (2,0)\right\}, \quad \mathcal{H}(S_3^{(2)}) = \left\{(1,1), (\tfrac{3}{2}, \tfrac{1}{2})\right\}. \tag{18.62}
$$

Hilbert series We compute the Hilbert series to

$$
\mathrm{HS}^{[4,2]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t, z_2, N) = \frac{R(t, z_2, N)}{(1-t^2)^2 (1-t^{18N-2})(1-t^{20N-4})(1-t^{26N-6})} \,, \tag{18.63a}
$$

$$
\begin{aligned}
R(t, z_2, N) = 1 &+ z_2 t^{9N-1}(1+t) + t^{10N-2}(1+t) + z_2 t^{13N-3}(1+2t+t^2) \\
&+ t^{18N-1} + t^{20N-4}(1+3t+t^2) + z_2 t^{23N-5}(1+2t+t^2)
\end{aligned}
$$

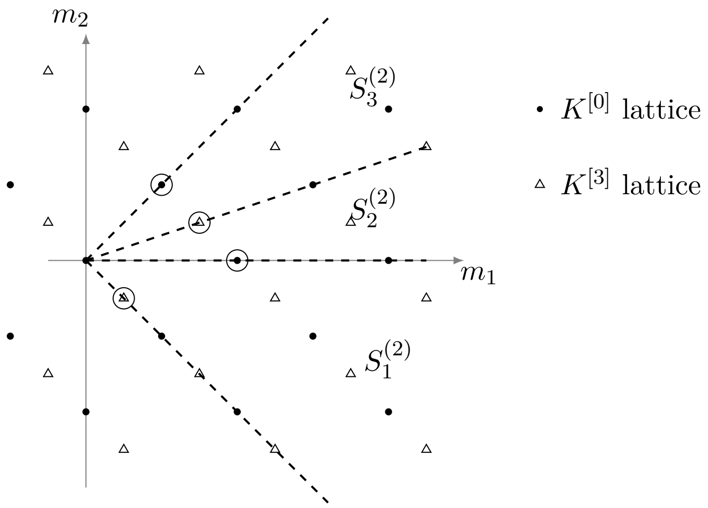


Figure 18.15: *The semi-groups for the quotient* SU(2) × SO(3) *and the representation* [4, 2]. *The black circled points are the ray generators.*

$$\begin{aligned}&+ t^{26N-5}(2+t) - t^{28N-4}(1+t) + z_2 t^{29N-4}(1+t) \\ &- z_2 t^{31N-5}(1+2t+t^2) + z_2 t^{33N-7}(1+2t+t^2) \\ &- z_2 t^{35N-7}(1+t) + t^{36N-7}(1+t) - t^{38N-6}(1+2t) \\ &- z_2 t^{41N-7}(1+2t+t^2) - t^{44N-8}(1+3t+t^2) \\ &- t^{46N-9} - z_2 t^{51N-9}(1+2t+t^2) - t^{54N-9}(1+t) \\ &- z_2 t^{55N-10}(1+t) - t^{64N-10} \ .\end{aligned} \tag{18.63b}$$

As before, we can try to compare the quotients SO(3) × SU(2) and SU(2) × SO(3). However, due to the asymmetry in m_1, m_2 or the asymmetry of the fan in the Weyl chamber, the Hilbert series for the two quotients are *not* related by an exchange of z_1 and z_2. We will comment on the correct gauging of $\mathbb{Z}_2$ subgroups in the next subsection.

Plethystic logarithm Upon analysing the PL we find

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[4,2]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}) = &\, 2t^2 + z_2 t^{\Delta(\frac{1}{2},-\frac{1}{2})}(1+t) + t^{\Delta(1,1)}(1+t) + z_2 t^{\Delta(\frac{3}{2},\frac{1}{2})}(1+2t+t^2) \\ &- t^{2\Delta(\frac{1}{2},-\frac{1}{2})+2} - z_2 t^{\Delta(\frac{1}{2},-\frac{1}{2})+\Delta(1,1)}(1+2t+t^2) \\ &+ t^{\Delta(2,0)}(1+2t) - t^{\Delta(\frac{1}{2},-\frac{1}{2})+\Delta(\frac{3}{2},\frac{1}{2})}(1+3t+3t^2+t^3) + \dots \ ,\end{aligned} \tag{18.64}$$

through which one identifies the generators as given in Tab. 18.14. The terms in the denominator of the Hilbert series can be seen to reproduce these generators

$$(1-t^{18N-2}) = (1-z_2 t^{9N-1})(1+z_2 t^{9N-1}) \ , \tag{18.65a}$$
$$(1-t^{26N-6}) = (1-z_2 t^{13N-3})(1+z_2 t^{13N-3}) \ . \tag{18.65b}$$

Unfortunately, we are unable to reduce the numerator accordingly.

object	(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(\frac{1}{2}, -\frac{1}{2})$	$K^{[3]}$	$9N-1$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(1,1)$	$K^{[0]}$	$10N-2$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(\frac{3}{2}, \frac{1}{2})$	$K^{[3]}$	$13N-3$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
monopole	$(2,0)$	$K^{[0]}$	$20N-4$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 18.14: *The chiral ring generators for a* $\mathrm{SU}(2) \times \mathrm{SO}(3)$ *gauge theory with matter transforming in* $[4, 2]$.

18.4.5 Quotient PSO(4)

Hilbert basis The semi-groups $S_p^{(2)} := C_p^{(2)} \cap \left(K^{[0]} \cup K^{[1]} \cup K^{[2]} \cup K^{[3]}\right)$ (for $p = 1, 2, 3$) have Hilbert bases that are determined by the ray generators. Fig. 18.16 depicts the situation and the Hilbert bases read:

$$\begin{aligned} &\mathcal{H}(S_1^{(2)}) = \left\{(1,0), (\tfrac{1}{2}, -\tfrac{1}{2})\right\} , \quad \mathcal{H}(S_2^{(2)}) = \left\{(\tfrac{3}{2}, \tfrac{1}{2}), (1,0)\right\} , \\ &\mathcal{H}(S_3^{(2)}) = \left\{(\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{3}{2}, \tfrac{1}{2})\right\} . \end{aligned} \tag{18.66}$$

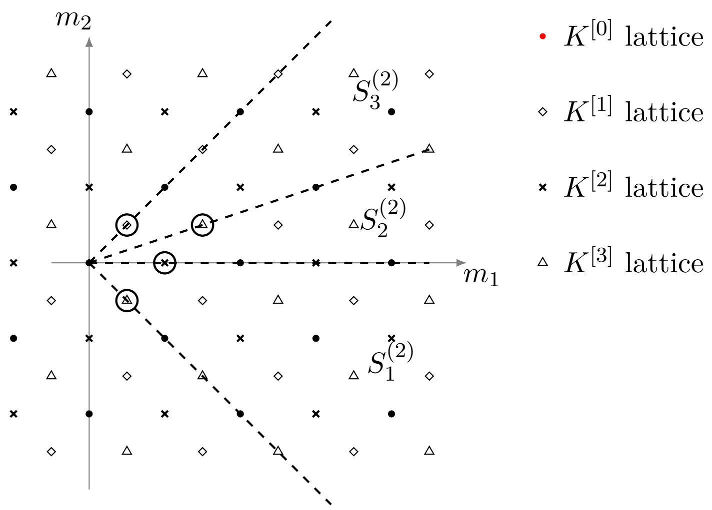


Figure 18.16: *The semi-groups and their ray-generators (black circled points) for the quotient* PSO(4) *and the representation* $[4, 2]$.

Hilbert series We obtain the following Hilbert series

$$\mathrm{HS}_{\mathrm{PSO}(4)}^{[4,2]}(t, z_1, z_2, N) = \frac{R(t, z_1, z_2, N)}{P(t, z_1, z_2, N)} , \tag{18.67a}$$

$$\begin{aligned} P(t, z_1, z_2, N) = &\left(1-t^2\right)^2 \left(1-t^{10N-2}\right) \left(1-z_2 t^{10N-2}\right) \\ &\times \left(1-t^{18N-2}\right) \left(1-t^{26N-6}\right) , \end{aligned} \tag{18.67b}$$

$$\begin{aligned} R(t, z_1, z_2, N) = &1 + z_1 t^{5N-1}(1+t) + z_1 z_2 t^{9N-1}(1+t) + z_1 z_2 t^{9N} + t^{10N-1} \\ &+ z_2 t^{10N-1}(2+t) + z_1 z_2 t^{13N-3}(1+2t+t^2) - z_1 z_2 t^{15N-3}(1+t) \end{aligned} \tag{18.67c}$$

$$
\begin{aligned}
&+ z_2 t^{18N-4}(1+2t+t^2) + t^{18N-1} - z_1 z_2 t^{19N-3}(1+t) \\
&+ z_1 t^{19N-2}(1+t) - z_2 t^{20N-4}(1+3t+t^2) - z_1 t^{23N-3}(1+t) \\
&+ t^{26N-5}(2+t) - t^{28N-6}(1+2t+2t^2+2t^3) - z_2 t^{28N-3} \\
&- z_1 t^{29N-4}(1+t) + z_1 t^{31N-6}(1+t) - z_1 z_2 t^{31N-5}(1+2t+t) \\
&- z_1 t^{33N-7}(1+2t+t^2) + z_1 z_2 t^{33N-5}(1+t) \\
&- z_1 z_2 t^{35N-7}(1+t) - z_2 t^{36N-7}(2+2t+2t^2+t^3) - t^{36N-7} \\
&+ z_2 t^{38N-6}(1+2t) - z_1 z_2 t^{41N-8}(1+t) - t^{44N-8}(1+3t+t^2) \\
&+ z_1 z_2 t^{45N-9}(1+t) - z_1 t^{45N-8}(1+t) \\
&+ z_2 t^{46N-9} + t^{46N-8}(1+2t+t^2) - z_1 t^{49N-8}(1+t) \\
&+ z_1 t^{51N-9}(1+2t+t^2) + t^{54N-10}(1+2t) + z_2 t^{54N-9} \\
&+ z_1 t^{55N-10}(1+t) + z_1 z_2 t^{59N-10}(1+t) + z_2 t^{64N-10} \,.
\end{aligned}
$$

The numerator of (18.67) is a palindromic polynomial of degree $64N-10$, while the denominator is of degree $64N-8$. Hence, the difference in degrees is again 2. Moreover, the Hilbert series (18.67) has a pole of order 4 as $t \to 1$ because $R(1, z_1, z_2, N) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t} R(t, z_1, z_2, N)|_{t\to 1} = 0$, while $\frac{\mathrm{d}^2}{\mathrm{d}t^2} R(t, z_1, z_2, N)|_{t\to 1} \neq 0$.

Plethystic logarithm Working with the PL instead reveals further insights

$$
\begin{aligned}
\mathrm{PL}(\mathrm{HS}^{[4,2]}_{\mathrm{PSO}(4)}) = 2t^2 &+ z_1 t^{\Delta(\frac{1}{2},\frac{1}{2})}(1+t) + z_1 z_2 t^{\Delta(\frac{1}{2},-\frac{1}{2})}(1+t) + z_2 t^{\Delta(1,0)}(1+2t+t^2) \\
&- t^{2\Delta(\frac{1}{2},\frac{1}{2})+2} + z_1 z_2 t^{\Delta(\frac{3}{2},\frac{1}{2})}(1+2t+t^2) \\
&- z_2 t^{\Delta(\frac{1}{2},\frac{1}{2})+\Delta(\frac{1}{2},\frac{1}{2})}(1+2t+t^2) \\
&- z_1 z_2 t^{\Delta(\frac{1}{2},\frac{1}{2})+\Delta(1,0)}(1+3t+3t^2+t^3) + \dots \,.
\end{aligned}
\tag{18.68}
$$

The list of generators, together with their properties, is provided in Tab. 18.15.

object	(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(\frac{1}{2}, \frac{1}{2})$	$K^{[1]}$	$5N-1$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(\frac{1}{2}, -\frac{1}{2})$	$K^{[3]}$	$9N-1$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(1,0)$	$K^{[2]}$	$10N-2$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
monopole	$(\frac{3}{2}, \frac{1}{2})$	$K^{[3]}$	$13N-3$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 18.15: *The chiral ring generators for a* $\mathrm{PSO}(4)$ *gauge theory with matter transforming in* $[4,2]$.

Gauging a $\mathbb{Z}_2$ The global $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry allows us to compute the Hilbert series for all five quotients from the PSO(4) result. We start by gauging the $\mathbb{Z}_2$-factor with fugacity z_1 (and relabel z_2 as z) and recover the SO(4)-result

$$\mathrm{HS}^{[4,2]}_{\mathrm{SO}(4)}(t,z,N) = \frac{1}{2}\Big(\mathrm{HS}^{[4,2]}_{\mathrm{PSO}(4)}(t,z_1{=}1,z_2{=}z,N) + \mathrm{HS}^{[4,2]}_{\mathrm{PSO}(4)}(t,z_1{=}-1,z_2{=}z,N)\Big) \ . \tag{18.69a}$$

In contrast, gauging the other $\mathbb{Z}_2$-factor with fugacity z_1 provides the $\mathrm{SO}(3)\times\mathrm{SU}(2)$-result

$$\mathrm{HS}^{[4,2]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}(t,z_1,N) = \frac{1}{2}\Big(\mathrm{HS}^{[4,2]}_{\mathrm{PSO}(4)}(t,z_1,z_2{=}1,N) + \mathrm{HS}^{[4,2]}_{\mathrm{PSO}(4)}(t,z_1,z_2{=}-1,N)\Big) \ . \tag{18.69b}$$

Lastly, switching to w_1, w_2 fugacities as in (18.28c) allows to recover the Hilbert series for $\mathrm{SU}(2)\times\mathrm{SO}(3)$ as follows:

$$\mathrm{HS}^{[4,2]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t,z_2{=}w_1,N) = \frac{1}{2}\Big(\mathrm{HS}^{[4,2]}_{\mathrm{PSO}(4)}(t,w_1,w_2{=}1,N) + \mathrm{HS}^{[4,2]}_{\mathrm{PSO}(4)}(t,w_1,w_2{=}-1,N)\Big) \ . \tag{18.69c}$$

In conclusion, the PSO(4) result is sufficient to obtain the remaining four quotients by gauging of various $\mathbb{Z}_2$ global symmetries as in (18.69) and (18.58).

18.5 Comparison to O(4)

In this section we explore the orthogonal group O(4), related to SO(4) by $\mathbb{Z}_2$. To begin with, we summarise the set-up as presented in [216, App. A]. The dressing factor $P_{\mathrm{O}(4)}(t)$ and the GNO lattice of O(4) equal those of SO(5). Moreover, the dominant Weyl chamber is parametrised by (m_1,m_2) subject to $m_1\geq m_2\geq 0$. Graphically, the Weyl chamber is the upper half of the yellow-shaded region in Fig. 18.1 with the lattices $K^{[0]}\cup K^{[2]}$ present. Consequently, the dressing function is given as

$$P_{\mathrm{O}(4)}(t,m_1,m_2) = \begin{cases} \frac{1}{(1-t^2)(1-t^4)} \ , & m_1=m_2=0 \ , \\ \frac{1}{(1-t)(1-t^2)} \ , & m_1=m_2>0 \ , \\ \frac{1}{(1-t)(1-t^2)} \ , & m_1>0, \ m_2=0 \ , \\ \frac{1}{(1-t)^2} \ , & m_1>m_2>0 \ . \end{cases} \tag{18.70}$$

It is apparent that O(4) has a different Casimir invariant as SO(4), which comes about as the Levi-Civita tensor ε is not an invariant tensor under O(4). In other words, the Pfaffian of SO(4) is not a invariant of O(4).

Now, we provide the Hilbert series for the three different representations studied above.

18.5.1 Representation $[2,0]$

The conformal dimension is the same as in (18.6) and the rational cone of the Weyl chamber is simply

$$C^{(2)} = \mathrm{Cone}\left((1,0),(1,1)\right) \ , \tag{18.71}$$

such that the cone generators and the Hilbert basis for $S^{(2)} := C^{(2)} \cap \left(K^{[0]} \cup K^{[2]}\right)$ coincide. The upper half-space of Fig. 18.3 depicts the situation.

The Hilbert series is then computed to read

$$\mathrm{HS}_{\mathrm{O}(4)}^{[2,0]}(t,N) = \frac{1+2t^{2N-1}+2t^{2N}+2t^{2N+1}+t^{4N}}{(1-t^2)\,(1-t^4)\,(1-t^{2N-2})^2}\ , \tag{18.72}$$

which clearly displays the palindromic numerator, the order four for $t \to 1$, and the order two pole for $t \to \infty$, i.e. the difference in degrees of denominator and numerator is two. By inspection of (18.72) and use of the plethystic logarithm

$$\mathrm{PL}(\mathrm{HS}_{\mathrm{O}(4)}^{[2,0]}) = t^2 + t^4 + t^{\Delta(1,0)}(1+t+t^2+t^3) + t^{\Delta(1,1)}(1+t+t^2+t^3) - \mathcal{O}(t^{2\Delta(1,0)+2})\ , \tag{18.73}$$

for $N \geq 2$, we can summarise the generators as in Tab. 18.16. The different dressing behaviour

object	(m_1,m_2)	lattice	$\Delta(m_1,m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2, 4	—	—
monopole	$(1,0)$	$K^{[2]}$	$2N-2$	$\mathrm{U}(2)$	3
monopole	$(1,1)$	$K^{[0]}$	$2N-2$	$\mathrm{U}(1)\times\mathrm{O}(2)$	3

Table 18.16: *Bare and dressed monopole generators for an* $\mathrm{O}(4)$ *gauge theory with matter transforming in* $[2,0]$.

of the $\mathrm{O}(4)$ monopole generators $(1,0)$ and $(1,1)$ compared to their $\mathrm{SO}(4)$ counterparts can be deduced from dividing the relevant dressing factor by the trivial one. In detail

$$\frac{P_{\mathrm{O}(4)}(t,\{(1,0)\text{ or }(1,1)\})}{P_{\mathrm{O}(4)}(t,0,0)} = \frac{(1-t^2)(1-t^4)}{(1-t)(1-t^2)} = 1+t+t^2+t^3\ . \tag{18.74}$$

18.5.2 Representation $[2,2]$

The conformal dimension is the same as in (18.30) and the rational cone of the Weyl chamber is still

$$C^{(2)} = \mathrm{Cone}\left((1,0),(1,1)\right)\ , \tag{18.75}$$

such that the cone generators and the Hilbert basis for $S^{(2)} := C^{(2)} \cap \left(K^{[0]} \cup K^{[2]}\right)$ coincide. The upper half-space of Fig. 18.8 depicts the situation. We note that the Weyl chamber for $\mathrm{SO}(4)$ is already divided into a fan by two rational cones, while the Weyl chamber for $\mathrm{O}(4)$ is not.

The computation of the Hilbert series then yields

$$\mathrm{HS}_{\mathrm{O}(4)}^{[2,2]}(t,N) = \frac{1+t^{4N-1}+t^{4N}+t^{4N+1}+t^{6N-1}+t^{6N}+t^{6N+1}+t^{10N}}{(1-t^2)\,(1-t^4)\,(1-t^{4N-2})\,(1-t^{6N-2})}\ . \tag{18.76}$$

Again, the rational function clearly displays a palindromic numerator, an order four pole for $t \to 1$, and an order two pole for $t \to \infty$, i.e. the difference in degrees of denominator and numerator is two. By inspection of (18.76) and use of the plethystic logarithm

$$\mathrm{PL}(\mathrm{HS}_{\mathrm{O}(4)}^{[2,2]}) = t^2 + t^4 + t^{\Delta(1,0)}(1+t+t^2+t^3) + t^{\Delta(1,1)}(1+t+t^2+t^3) - \mathcal{O}(t^{2\Delta(1,0)+2})\ , \tag{18.77}$$

for $N \geq 2$, we can summarise the generators as in Tab. 18.17. The dressings behave as discussed earlier.

object	(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2, 4	—	—
monopole	$(1,0)$	$K^{[2]}$	$4N-2$	U(2)	3
monopole	$(1,1)$	$K^{[0]}$	$6N-2$	$\mathrm{U}(1)\times\mathrm{O}(2)$	3

Table 18.17: *Bare and dressed monopole generators for an* O(4) *gauge theory with matter transforming in* $[2,2]$.

18.5.3 Representation $[4,2]$

The conformal dimension is given in (18.50) and the Weyl chamber is split into a fan generated by two rational cones

$$C_2^{(2)} = \mathrm{Cone}\left((1,0),(3,1)\right) \qquad \text{and} \qquad C_3^{(2)} = \mathrm{Cone}\left((3,1),(1,1)\right) \ , \tag{18.78}$$

where we use the notation of the SO(4) setting, see the upper half plan of Fig. 18.13. The Hilbert bases for $S_p^{(2)} := C_p^{(2)} \cap \left(K^{[0]} \cup K^{[2]}\right)$ differ from the cone generators and are obtained as

$$\mathcal{H}(S_2^{(2)}) = \{(1,0),(3,1)\} \qquad \text{and} \qquad \mathcal{H}(S_3^{(2)}) = \{(3,1),(2,1),(1,1)\} \ . \tag{18.79}$$

The computation of the Hilbert series then yields

$$\mathrm{HS}_{\mathrm{O}(4)}^{[4,2]}(t,N) = \frac{R(t,N)}{(1-t^2)\,(1-t^4)\,(1-t^{10N-2})\,(1-t^{26N-6})} \ , \tag{18.80a}$$

$$\begin{aligned} R(t,N) = 1 &+ t^{10N-2} + 2t^{10N-1} + 2t^{10N} + 2t^{10N+1} \\ &+ t^{18N-4} + 2t^{18N-3} + 2t^{18N-2} + 2t^{18N-1} + t^{18N} \\ &+ 2t^{26N-5} + 2t^{26N-4} + 2t^{26N-3} + t^{26N-2} + t^{36N-4} \end{aligned} \tag{18.80b}$$

As before, the rational function (18.80) clearly displays a palindromic numerator, an order four pole for $t \to 1$, and an order two pole for $t \to \infty$, i.e. the difference in degrees of denominator and numerator is two. By inspection of (18.80) and use of the plethystic logarithm

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}_{\mathrm{O}(4)}^{[4,2]}) = t^2 + t^4 &+ t^{\Delta(1,0)}(1+t+t^2+t^3) + t^{\Delta(1,1)}(1+t+t^2+t^3) \\ &+ t^{\Delta(2,1)}(1+2(t+t^2+t^3)+t^4) \\ &- t^{\Delta(1,0)+\Delta(1,1)}(1+2t+5t^2+6t^3+7t^4+4t^5+3t^6) \\ &+ t^{\Delta(3,1)}(1+2(t+t^2+t^3)+t^4) - \mathcal{O}(t^{\Delta(1,0)+\Delta(2,1)}) \ , \end{aligned} \tag{18.81}$$

for $N \geq 2$, we can summarise the generators as in Tab. 18.18. The dressing behaviour of $(1,0)$,

object	(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2, 4	—	—
monopole	$(1,0)$	$K^{[2]}$	$10N-2$	U(2)	3
monopole	$(1,1)$	$K^{[0]}$	$10N-2$	$\mathrm{U}(1)\times\mathrm{O}(2)$	3
monopole	$(2,1)$	$K^{[2]}$	$18N-4$	$\mathrm{U}(1)^2$	7
monopole	$(3,1)$	$K^{[0]}$	$26N-6$	$\mathrm{U}(1)^2$	7

Table 18.18: *Bare and dressed monopole generators for an* O(4) *gauge theory with matter transforming in* $[4,2]$.

$(1,1)$ is as discussed earlier; however, we need to describe the dressings of $(2,1)$ and $(3,1)$ as it differs from the SO(4) counterparts. Again, we compute the quotient of the dressing factor of the maximal torus divided by the trivial one, i.e.

$$\frac{P_{\mathrm{O}(4)}(t, m_1 > m_2 > 0)}{P_{\mathrm{O}(4)}(t, 0, 0)} = \frac{(1-t^2)(1-t^4)}{(1-t)^2} = 1 + 2(t + t^2 + t^3) + t^4 \; . \tag{18.82}$$

Consequently, each bare monopole $(2,1)$, $(3,1)$ is accompanied by seven dressings, which is in agreement with (18.81).

19 Case: USp(4)

This chapter is devoted to the study of the compact symplectic group USp(4) with corresponding Lie algebra C_2. GNO-duality relates them with the special orthogonal group SO(5) and the Lie algebra B_2.

19.1 Set-up

For studying the non-abelian group USp(4), we start by providing the contributions of $N_{a,b}$ hypermultiplets in various representations $[a, b]$ of USp(4) to the conformal dimension

$$\Delta^{[1,0]}_{\mathrm{h-plet}} = N_{1,0} \sum_i |m_i| \,, \tag{19.1a}$$

$$\Delta^{[0,1]}_{\mathrm{h-plet}} = N_{0,1} \Big(\sum_{i<j} |m_i - m_j| + \sum_{i<j} |m_i + m_j| \Big) \,, \tag{19.1b}$$

$$\Delta^{[2,0]}_{\mathrm{h-plet}} = 2N_{2,0} \sum_i |m_i| + N_{2,0} \Big(\sum_{i<j} |m_i - m_j| + \sum_{i<j} |m_i + m_j| \Big) \,, \tag{19.1c}$$

$$\Delta^{[0,2]}_{\mathrm{h-plet}} = 2N_{0,2} \sum_i |m_i| + 3N_{0,2} \Big(\sum_{i<j} |m_i - m_j| + \sum_{i<j} |m_i + m_j| \Big) \,, \tag{19.1d}$$

$$\begin{aligned} \Delta^{[1,1]}_{\mathrm{h-plet}} = 2N_{1,1} \sum_i |m_i| + N_{1,1} \Big(&\sum_{i<j} \left(|2m_i - m_j| + |m_i - 2m_j| \right) \\ &+ \sum_{i<j} \left(|2m_i + m_j| + |m_i + 2m_j| \right) \Big) \,, \end{aligned} \tag{19.1e}$$

$$\begin{aligned} \Delta^{[3,0]}_{\mathrm{h-plet}} = 5N_{3,0} \sum_i |m_i| + N_{3,0} \Big(&\sum_{i<j} \left(|2m_i - m_j| + |m_i - 2m_j| \right) \\ &+ \sum_{i<j} \left(|2m_i + m_j| + |m_i + 2m_j| \right) \Big) \,, \end{aligned} \tag{19.1f}$$

where $i, j = 1, 2$, and the contribution of the vector multiplet is given by

$$\Delta_{\mathrm{V-plet}} = -2 \sum_i |m_i| - \Big(\sum_{i<j} |m_i - m_j| + \sum_{i<j} |m_i + m_j| \Big) \,. \tag{19.1g}$$

Such that we will consider the following conformal dimension

$$\begin{aligned} \Delta(m_1, m_2) = (N_1 - 2)(|m_1| + |m_2|) &+ (N_2 - 1) \left(|m_1 - m_2| + |m_1 + m_2| \right) \\ &+ N_3 \left(|2m_1 - m_2| + |m_1 - 2m_2| + |2m_1 + m_2| + |m_1 + 2m_2| \right) \end{aligned} \tag{19.2a}$$

and we can vary the representation content via

$$N_1 = N_{1,0} + 2N_{2,0} + 2N_{0,2} + 2N_{1,1} + 5N_{3,0} \,, \tag{19.2b}$$

$$N_2 = N_{0,1} + N_{2,0} + 3N_{0,2} \,, \tag{19.2c}$$

$$N_3 = N_{1,1} + N_{3,0} \,. \tag{19.2d}$$

The Hilbert series is computed as usual

$$\mathrm{HS}_{\mathrm{USp}(4)}(t,N) = \sum_{m_1 \geq m_2 \geq 0} t^{\Delta(m_1,m_2)} P_{\mathrm{USp}(4)}(t,m_1,m_2) \;, \tag{19.3}$$

where the summation for m_1, m_2 has been restricted to the principal Weyl chamber of the GNO-dual group SO(5), whose Weyl group is $S_2 \ltimes (\mathbb{Z}_2)^2$. Thus, we use the reflections to restrict to non-negative $m_i \geq 0$ and the permutations to restrict to a ordering $m_1 \geq m_2$. The classical dressing factor takes the following form [191]:

$$P_{\mathrm{USp}(4)}(t,m_1,m_2) = \begin{cases} \frac{1}{(1-t)^2} \;, & m_1 > m_2 > 0 \;, \\ \frac{1}{(1-t)(1-t^2)} \;, & (m_1 > m_2 = 0) \vee (m_1 = m_2 > 0) \;, \\ \frac{1}{(1-t^2)(1-t^4)} \;, & m_1 = m_2 = 0 \;. \end{cases} \tag{19.4}$$

19.2 Hilbert basis

The conformal dimension (19.2a) divides the dominant Weyl chamber of SO(5) into a fan. The intersection with the corresponding weight lattice $\Lambda_w(\mathrm{SO}(5))$ introduces semi-groups S_p, which are sketched in Fig. 19.1. As displayed, the set of semi-groups (and rational cones that constitute the fan) differ if $N_3 \neq 0$. The Hilbert bases for both case are readily computed, because they coincide with the set of ray generators.

- For $N_3 \neq 0$, which is displayed in Fig. 19.1a, there exists one hyperplane $|m_1 - 2m_2| = 0$ which non-trivially intersects the Weyl chamber. Therefore, $\Lambda_w(\mathrm{SO}(5))/\mathcal{W}_{\mathrm{SO}(5)}$ becomes a fan generated by two 2-dimensional cones. The Hilbert bases of the corresponding semi-groups are computed to

$$\mathcal{H}(S_+^{(2)}) = \Big\{(1,1),(2,1)\Big\} \;, \qquad \mathcal{H}(S_-^{(2)}) = \Big\{(2,1),(1,0)\Big\} \;. \tag{19.5}$$

- For $N_3 = 0$, as shown in Fig. 19.1b, there exists no hyperplane that intersects the dominant Weyl chamber non-trivially. As a consequence, the $\Lambda_w(\mathrm{SO}(5))/\mathcal{W}_{\mathrm{SO}(5)}$ is described by one rational polyhedral cone of dimension 2. The Hilbert basis for the semi-group is given by

$$\mathcal{H}(S^{(2)}) = \Big\{(1,1),(1,0)\Big\} \;. \tag{19.6}$$

19.3 Dressings

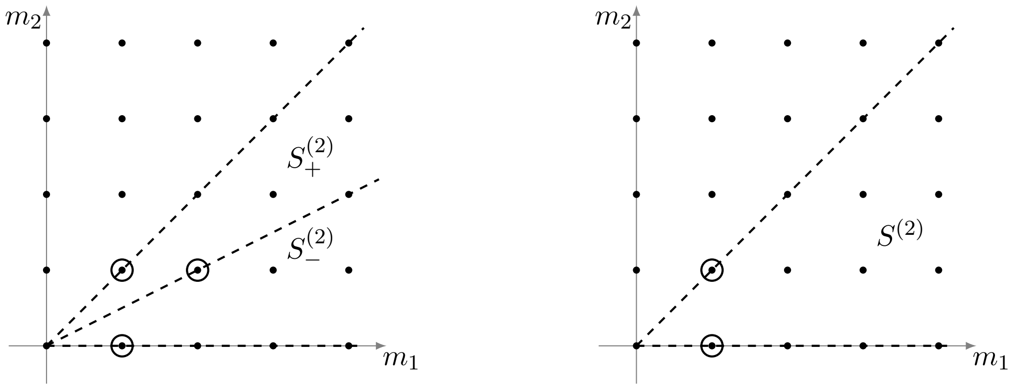
Before evaluating the Hilbert series, let us analyse the classical dressing factors for the minimal generators (19.5) or (19.6). Firstly, the classical Lie group USp(4) has two Casimir invariants of degree 2 and 4 and can they can be written as $\mathrm{tr}(\Phi^2) = \sum_{i=1}^2 (\phi_i)^2$ and $\mathrm{tr}(\Phi^4) = \sum_{i=1}^2 (\phi_i)^4$, respectively. Again, we employed the diagonal form of the adjoint valued scalar field Φ.

Secondly, the bare monopole operator corresponding to GNO-charge $(1,0)$ has conformal dimension $N_1+2N_2+6N_3-4$ and the residual gauge group is $\mathrm{H}_{(1,0)} = \mathrm{U}(1)\times\mathrm{SU}(2)$, i.e. allowing for dressings by degree 1 and 2 Casimirs. The resulting set of bare and dressed monopoles is

$$V_{(1,0)}^{\mathrm{dress},0} = (1,0) + (-1,0) + (0,1) + (0,-1) \;, \tag{19.7a}$$

$$V_{(1,0)}^{\mathrm{dress},2} = ((1,0) + (-1,0))\,(\phi_2)^2 + ((0,1) + (0,-1))\,(\phi_1)^2 \;, \tag{19.7b}$$

$$V_{(1,0)}^{\mathrm{dress},1} = ((1,0) - (-1,0))\,\phi_1 + ((0,1) - (0,-1))\,\phi_2 \;, \tag{19.7c}$$



(a) $N_3 \neq 0$ ***(b)*** $N_3 = 0$

Figure 19.1: *The various semi-groups for* USp(4) *depending on whether* $N_3 \neq 0$ *or* $N_3 = 0$. *For both cases the black circled points are the ray generators.*

$$V^{\text{dress},3}_{(1,0)} = ((1,0) - (-1,0))\,(\phi_1)^3 + ((0,1) - (0,-1))\,(\phi_2)^3\ . \tag{19.7d}$$

Thirdly, the bare monopole operators of GNO-charge $(1,1)$ has conformal dimension $2N_1 + 2N_2 + 8N_3 - 6$ and residual gauge group $\mathrm{H}_{(1,1)} = \mathrm{U}(1) \times \mathrm{SU}(2)$. The bare and dressed monopole operators can be written as

$$V^{\text{dress},0}_{(1,1)} = (1,1) + (1,-1) + (-1,1) + (-1,-1)\ , \tag{19.8a}$$
$$V^{\text{dress},2}_{(1,1)} = ((1,1) + (-1,-1))((\phi_1)^2 + (\phi_2)^2) + (1,-1)(\phi_2)^2 + (-1,1)(\phi_1)^2\ , \tag{19.8b}$$
$$V^{\text{dress},1}_{(1,1)} = (1,1)(\phi_1 + \phi_2) + (-1,-1)(-\phi_1 - \phi_2) + (1,-1)(-\phi_2) + (-1,1)(-\phi_1)\ , \tag{19.8c}$$
$$\begin{aligned} V^{\text{dress},3}_{(1,1)} = {}& (1,1)((\phi_1)^3 + (\phi_2)^3) + (-1,-1)(-(\phi_1)^3 - (\phi_2)^3) \\ &+ (1,-1)(-(\phi_2)^3) + (-1,1)(-(\phi_1)^3)\ . \end{aligned} \tag{19.8d}$$

The two magnetic weights $(1,0)$, $(1,1)$ lie at the boundary of the dominant Weyl chamber such that the dressing behaviour can be predicted by $P_{\mathrm{USp}(4)}(t, m_1, m_2)/P_{\mathrm{USp}(4)}(t,0,0) = 1{+}t{+}t^2{+}t^3$, following App. C. The above description of the bare and dressed monopole operators is therefore a valid choice of generating elements for the chiral ring.

Lastly, the bare monopole for $(2,1)$ has conformal dimension $3N_1 + 4N_2 + 12N_3 - 10$ and residual gauge group $\mathrm{H}_{(2,1)} = \mathrm{U}(1)^2$. Thus, the dressing proceeds by two independent degree 1 Casimir invariants.

$$\begin{aligned} V^{\text{dress},0}_{(2,1)} &= (2,1) + (2,-1) + (-2,1) + (1,2) + (1,-2) + (-1,2) + (-1,-2) + (-2,-1) \\ &\equiv (2,1) + (2,-1) + (-2,1) + (-2,-1) + \text{permutations}\ , \end{aligned} \tag{19.9a}$$
$$\begin{aligned} V^{\text{dress},2j-1,1}_{(2,1)} &= (2,1)(\phi_1)^{2j-1} + (2,-1)(\phi_1)^{2j-1} + (-2,1)(-\phi_1)^{2j-1} \\ &\quad + (-2,-1)(-\phi_1)^{2j-1} + \text{permutations} \qquad \text{for} \quad j = 1,2\ , \end{aligned} \tag{19.9b}$$
$$\begin{aligned} V^{\text{dress},2j-1,2}_{(2,1)} &= (2,1)(\phi_2)^{2j-1} + (2,-1)(-\phi_2)^{2j-1} + (-2,1)(\phi_2)^{2j-1} \\ &\quad + (-2,-1)(-\phi_2)^{2j-1} + \text{permutations} \qquad \text{for} \quad j = 1,2\ , \end{aligned} \tag{19.9c}$$
$$\begin{aligned} V^{\text{dress},2,1}_{(2,1)} &= (2,1)(\phi_1)^2 + (2,-1)(-(\phi_1)^2) + (-2,1)(-(\phi_1)^2) \\ &\quad + (-2,-1)(\phi_1)^2 + \text{permutations}\ , \end{aligned} \tag{19.9d}$$

$$V_{(2,1)}^{\text{dress},2,2} = (2,1)(\phi_1\phi_2) + (2,-1)(-\phi_1\phi_2) + (-2,1)(-\phi_1\phi_2) \\ + (-2,-1)(\phi_1\phi_2) + \text{permutations}\ , \tag{19.9e}$$

$$V_{(2,1)}^{\text{dress},4} = (2,1)(\phi_1^3\phi_2) + (2,-1)(-(\phi_1)^3\phi_2) + (-2,1)(-(\phi_1)^3\phi_2) \\ + (-2,-1)((\phi_1)^3\phi_2) + \text{permutations}\ . \tag{19.9f}$$

The number and the degrees of dressed monopole operators of charge $(2,1)$ are consistent with the quotient $P_{\mathrm{USp}(4)}(t, m_1 > m_2 > 0)/P_{\mathrm{USp}(4)}(t,0,0) = 1 + 2t + 2t^2 + 2t^3 + t^4$ of the dressing factors.

For *generic* values of N_1, N_2 and N_3 the Coulomb branch will be generated by the two Casimir invariants together with the bare and dressed monopole operators corresponding to the minimal generators of the Hilbert bases. However, we will encounter choices of the three parameters such that the set of monopole generators can be further reduced; for example, in the case of complete intersections.

19.4 Generic case

The computation for arbitrary N_1, N_2, and N_3 yields

$$\mathrm{HS}_{\mathrm{USp}(4)}(t, N_1, N_2, N_3) = \frac{R(t, N_1, N_2, N_3)}{P(t, N_1, N_2, N_3)}\ , \tag{19.10a}$$

with

$$P(t, N_1, N_2, N_3) = \left(1-t^2\right)\left(1-t^4\right)\left(1-t^{N_1+2N_2+6N_3-4}\right)\left(1-t^{2N_1+2N_2+8N_3-6}\right) \\ \times \left(1-t^{3N_1+4N_2+12N_3-10}\right)\ , \tag{19.10b}$$

$$\begin{aligned} R(t, N_1, N_2, N_3) = 1 &+ t^{N_1+2N_2+6N_3-3}(1+t+t^2) + t^{2N_1+2N_2+8N_3-5}(1+t+t^2) \\ &+ t^{3N_1+4N_2+12N_3-9}(2+2t+2t^2+t^3) \\ &- t^{3N_1+4N_2+14N_3-10}(1+2t+2t^2+2t^3) \\ &- t^{4N_1+6N_2+18N_3-13}(1+t+t^2) - t^{5N_1+6N_2+20N_3-15}(1+t+t^2) \\ &- t^{6N_1+8N_2+26N_3-16}\ . \end{aligned} \tag{19.10c}$$

The numerator (19.10c) is an anti-palindromic polynomial of degree $6N_1 + 8N_2 + 26N_3 - 16$; while the denominator is of degree $6N_1 + 8N_2 + 26N_3 - 14$. The difference in degrees is 2, which equals the quaternionic dimension of the moduli space. In addition, the pole of (19.10) at $t \to 1$ is of order 4, which matches the complex dimension of the moduli space. For that, one verifies explicitly $R(t = 1, N_1, N_2, N_3) = 0$, but $\frac{\mathrm{d}}{\mathrm{d}t}R(t, N_1, N_2, N_3)|_{t=1} \neq 0$.

Consequently, the above interpretation of bare and dressed monopoles from the Hilbert series (19.10) is correct for *generic* choices of N_1, N_2, and N_3. In particular, $N_3 \neq 0$ for this arguments to hold. Moreover, we will now exemplify the effects of the Casimir invariance in various special case of (19.10) explicitly. There are cases for which the inclusion of the Casimir invariance, i.e. dressed monopole operators, leads to a reduction of basis of monopole generators.

19.5 Category $N_3 = 0$

19.5.1 Representation $[1, 0]$

Hilbert series This choice is realised for $N_1 = N$, $N_2 = N_3 = 0$, and the Hilbert series simplifies drastically to a complete intersection

$$\mathrm{HS}_{\mathrm{USp}(4)}^{[1,0]}(t, N) = \frac{(1-t^{2N-4})(1-t^{2N-2})}{(1-t^2)(1-t^4)(1-t^{N-4})(1-t^{N-3})(1-t^{N-2})(1-t^{N-1})} \,, \tag{19.11}$$

which was first obtained in [191]. Due to the complete intersection property, the plethystic logarithm terminates and for $N > 4$ we obtain

$$\mathrm{PL}(\mathrm{HS}_{\mathrm{USp}(4)}^{[1,0]}) = t^2 + t^4 + t^{N-4}(1 + t + t^2 + t^3) - t^{2N-4} - t^{2N-2} \,. \tag{19.12}$$

Hilbert basis Naively, the Hilbert series (19.11) should be generated by the Hilbert basis (19.6) plus their dressings. However, due to the particular form of (19.2a) in the representation $[1, 0]$ and the Casimir invariance, the bare monopole operator of GNO-charge $(1, 1)$ can be generated by the dressings of $(1, 0)$. To see this, let us consider the Weyl-orbit $\mathcal{O}_{\mathcal{W}}(1, 0) = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$ and note the conformal dimensions align suitably, i.e. $\Delta(V_{(1,0)}^{\mathrm{dress},1}) = N - 3$, while $\Delta(V_{(1,1)}^{\mathrm{dress},0}) = 2N - 6$. Thus, we can symbolically write

$$V_{(1,1)}^{\mathrm{dress},0} = V_{(1,0)}^{\mathrm{dress},1} + V_{(0,1)}^{\mathrm{dress},1} \,. \tag{19.13}$$

The moduli space is then generated by the Casimir invariants and the bare and dressed monopole operators corresponding to $(1, 0)$, but this is to be understood as a rather *non-generic* situation.

19.5.2 Representation $[0, 1]$

This choice is realised for $N_2 = N$, and $N_1 = N_3 = 0$ and the Hilbert series simplifies to

$$\mathrm{HS}_{\mathrm{USp}(4)}^{[0,1]}(t, N) = \frac{1 + t^{2N-5} + t^{2N-4} + 2t^{2N-3} + t^{2N-2} + t^{2N-1} + t^{4N-6}}{(1-t^2)\,(1-t^4)\,(1-t^{2N-6})\,(1-t^{2N-4})} \,. \tag{19.14}$$

The Hilbert series (19.14) has a pole of order 4 at $t = 1$ as well as a palindromic polynomial as numerator. Moreover, the result (19.14) reflects the expected basis of monopole operators as given in the Hilbert basis (19.6).

19.5.3 Representation $[2, 0]$

This choice is realised for $N_1 = 2N$, $N_2 = N$, and $N_3 = 0$ and the Hilbert series reduces to

$$\mathrm{HS}_{\mathrm{USp}(4)}^{[2,0]}(t, N) = \frac{1 + t^{4N-3} + t^{4N-2} + t^{4N-1} + t^{6N-5} + t^{6N-4} + t^{6N-3} + t^{10N-6}}{(1-t^2)\,(1-t^4)\,(1-t^{4N-4})\,(1-t^{6N-6})} \,. \tag{19.15}$$

Also, the rational function (19.15) has a pole of order 4 for $t \to 1$ and a palindromic numerator. Evaluating the plethystic logarithm yields for all $N > 1$

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}_{\mathrm{USp}(4)}^{[2,0]}) = t^2 + t^4 + t^{4N-4}(1 + t + t^2 + t^3) \\ + t^{6N-6}(1 + t + t^2 + t^3) - t^{8N-6} + \mathcal{O}(t^{8N-5}) \,.\end{aligned} \tag{19.16}$$

This proves that bare monopole operators, corresponding to the the minimal generators of (19.6), together with their dressing generate all other monopole operators.

19.5.4 Representation $[0, 2]$

For $N_1 = 2N$, $N_2 = 3N$, and $N_3 = 0$ and the Hilbert series is given by

$$\mathrm{HS}^{[0,2]}_{\mathrm{USp}(4)}(t, N) = \frac{1 + t^{8N-3} + t^{8N-2} + t^{8N-1} + t^{10N-5} + t^{10N-4} + t^{10N-3} + t^{18N-6}}{(1-t^2)(1-t^4)(1-t^{8N-4})(1-t^{10N-6})} \ . \tag{19.17}$$

Evaluating the plethystic logarithm yields for all $N > 1$

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}^{[0,2]}_{\mathrm{USp}(4)}) = t^2 + t^4 + t^{8N-4}(1 + t + t^2 + t^3) \\ + t^{10N-6}(1 + t + t^2 + t^3) - t^{16N-6} + \mathcal{O}(t^{16N-5}) \ , \end{aligned} \tag{19.18a}$$

and for $N = 1$

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}^{[0,2]}_{\mathrm{USp}(4)}) = t^2 + t^4 + t^4(1 + t + t^2 + t^3) \\ + t^4(1 + t + t^2 + t^3) - 3t^{10} + \mathcal{O}(t^{11}) \ . \end{aligned} \tag{19.18b}$$

The inspection of the Hilbert series (19.17), together with the PL, proves that Hilbert basis (19.6), alongside all their dressings, are a sufficient set for all monopole operators.

19.6 Category $N_3 \neq 0$

19.6.1 Representation $[1, 1]$

This choice corresponds to $N_1 = 2N$, $N_2 = 0$, and $N_3 = N$ and we obtain the Hilbert series as

$$\mathrm{HS}^{[1,1]}_{\mathrm{USp}(4)}(t, N) = \frac{R(t, N)}{(1-t^2)(1-t^4)(1-t^{8N-4})(1-t^{12N-6})(1-t^{18N-10})} \ , \tag{19.19a}$$

$$\begin{aligned} R(t, N) = 1 + t^{8N-3}(1 + t + t^2) + t^{12N-5}(1 + t + t^2) \\ + t^{18N-9}(2 + 2t + 2t^2 + t^3) - t^{20N-10}(1 + 2t + 2t^2 + 2t^3) \\ - t^{26N-13}(1 + t + t^2) - t^{30N-15}(1 + t + t^2) - t^{38N-16} \ . \end{aligned} \tag{19.19b}$$

Considering the plethystic logarithm, we observe the following behaviour:

- For $N \geq 5$

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}^{[1,1]}_{\mathrm{USp}(4)}) = t^2 + t^4 + t^{8N-4}(1 + t + t^2 + t^3) + t^{12N-6}(1 + t + t^2 + t^3) \\ - t^{2(8N-4)+2}(1 + t + 2t^2 + t^3 + t^4) \\ + t^{18N-10}(1 + 2t + 2t^2 + 2t^3 + t^4) \\ - t^{20N-10}(1 + 2t + 3t^2 + 4t^3 + 3t^4 + 2t^5 + t^6) + \dots \end{aligned} \tag{19.20a}$$

- For $N = 4$

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}^{[1,1]}_{\mathrm{USp}(4)}) = t^2 + t^4 + t^{28}(1 + t + t^2 + t^3) + t^{42}(1 + t + t^2 + t^3) \\ - t^{58}(1 + t + 2t^2 + t^3 + t^4) \\ + t^{62}(1 + 2t + 2t^2 + 2t^3 + t^4) \end{aligned} \tag{19.20b}$$

$$- t^{70}(1+2t+3t^2+4t^3+3t^4+2t^5+t^6)+\dots$$

We see, employing the previous results for $N > 4$, that the bare monopole $(2,1)$ and the last relation at t^{62} coincide. Hence, the term $\sim t^{62}$ disappears from the PL.

- For $N = 3$

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[1,1]}_{\mathrm{USp}(4)}) = t^2 + t^4 + t^{20}(1+t+t^2+t^3) + t^{30}(1+t+t^2+t^3) \\ - t^{42}(1+t+2t^2+t^3+t^4) \\ + t^{44}(1+2t+2t^2+2t^3+t^4) \\ - t^{70}(1+2t+3t^2+4t^3+3t^4+2t^5+t^6)+\dots\end{aligned} \tag{19.20c}$$

We see, employing again the previous results for $N > 4$, that the some monopole contributions of $(2,1)$ and the some of the relations coincide, cf. the coloured terms. Hence, there are, presumably, cancellations between generators and relations. (See also App. C.)

- For $N = 2$

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[1,1]}_{\mathrm{USp}(4)}) &= t^2 + t^4 + t^{12}(1+t+t^2+t^3) + t^{18}(1+t+t^2+t^3) \\ &\quad - t^{26}(1+t+2t^2+t^3+t^4) \\ &\quad + t^{26}(1+2t+2t^2+2t^3+t^4) \\ &\quad - t^{30}(1+2t+3t^2+4t^3+3t^4+2t^5+t^6)+\dots\end{aligned} \tag{19.20d}$$

$$\begin{aligned}&= t^2 + t^4 + t^{12}(1+t+t^2+t^3) + t^{18}(1+t+t^2+t^3) \\ &\quad + t^{26}(0+t+0+t^3+0) \\ &\quad - t^{30}(1+2t+3t^2+4t^3+3t^4+2t^5+t^6)+\dots\end{aligned} \tag{19.20e}$$

- For $N = 1$

$$\mathrm{PL}(\mathrm{HS}^{[1,1]}_{\mathrm{USp}(4)}) = t^2 + 2t^4 + t^5 + 2t^6 + 2t^7 + 2t^8 + 3t^9 - t^{11} + \dots \tag{19.20f}$$

Summarising, the Hilbert series (19.19) and its plethystic logarithm display that the minimal generators of (19.5) are indeed the basis for the bare monopole operators, and the corresponding dressings generate the remaining operators.

19.6.2 Representation $[3,0]$

For the choice $N_1 = 5N$, $N_2 = 0$, and $N_3 = N$, the Hilbert series is given by

$$\mathrm{HS}^{[3,0]}_{\mathrm{USp}(4)}(t,N) = \frac{R(t,N)}{(1-t^2)(1-t^4)(1-t^{11N-4})(1-t^{18N-6})(1-t^{27N-10})}\,, \tag{19.21a}$$

$$\begin{aligned}R(t,N) = 1 &+ t^{11N-3}(1+t+t^2) + t^{18N-5}(1+t+t^2) \\ &+ t^{27N-9}(2+2t+2t^2+t^3) - t^{29N-10}(1+2t+2t^2+2t^3) \\ &- t^{38N-13}(1+t+t^2) - t^{45N-15}(1+t+t^2) - t^{56N-16}\,.\end{aligned} \tag{19.21b}$$

The inspection of the plethystic logarithm provides further insights:

- For $N \geq 3$

$$\mathrm{PL}(\mathrm{HS}^{[3,0]}_{\mathrm{USp}(4)}) = t^2 + t^4 + t^{11N-4}(1+t+t^2+t^3) + t^{18N-6}(1+t+t^2+t^3) \tag{19.22a}$$

$$
\begin{aligned}
&- t^{2(11N-4)+2}(1 + t + 2t^2 + t^3 + t^4) \\
&+ t^{27N-10}(1 + 2t + 2t^2 + 2t^3 + t^4) \\
&- t^{(11N-4)+(18N-6)}(1 + 2t + 3t^2 + 4t^3 + 3t^4 + 2t^5 + t^6) + \dots
\end{aligned}
$$

- For $N = 2$

$$
\begin{aligned}
\mathrm{PL}(\mathrm{HS}^{[3,0]}_{\mathrm{USp}(4)}) = t^2 + t^4 + t^{18}(1 + t + t^2 + t^3) + t^{30}(1 + t + t^2 + t^3) \\
- t^{38}(1 + t + 2t^2 + t^3 + t^4) \\
+ t^{44}(1 + 2t + 2t^2 + 2t^3 + t^4) \\
- t^{48}(1 + 2t + 3t^2 + 4t^3 + 3t^4 + 2t^5 + t^6) + \dots
\end{aligned} \tag{19.22b}
$$

  We see that, presumably, one generator and one relation cancel at t^{48}.

- For $N = 1$

$$
\mathrm{PL}(\mathrm{HS}^{[3,0]}_{\mathrm{USp}(4)}) = t^2 + t^4 + t^7(1 + t + t^2 + t^3) + t^{12}(1 + t + t^2 + t^3) - t^{16} - t^{20} + \dots \tag{19.22c}
$$

Again, we confirm that the minimal generators of the Hilbert basis (19.5) are the relevant generators (together with their dressings) for the moduli space.

20 Case: G_2

Here, we explore the Coulomb branch for the only exceptional simple Lie group of rank two.

20.1 Set-up

The group G_2 has irreducible representations labelled by two Dynkin labels and the dimension formula reads

$$\dim[a,b] = \frac{1}{120}(a+1)(b+1)(a+b+2)(a+2b+3)(a+3b+4)(2a+3b+5) \,. \qquad (20.1)$$

In the following, we study the representations given in Tab. 20.1. The three categories defined are due to the similar form of the conformal dimensions.

Dynkin label	$[1,0]$	$[0,1]$	$[2,0]$	$[1,1]$	$[0,2]$	$[3,0]$	$[4,0]$	$[2,1]$
Dim.	7	14	27	64	77	77	182	189
	category 1			category 2			category 3	

Table 20.1: *An overview of the* G_2*-representations considered in this chapter.*

The Weyl group of G_2 is D_6 and the GNO-dual group is another G_2. Any element in the Cartan subalgebra $\mathfrak{h} = \mathrm{span}(H_1, H_2)$ can be written as $H = n_1 H_1 + n_2 H_2$. Restriction to the principal Weyl chamber is realised via $n_1, n_2 \geq 0$.

The group G_2 has two Casimir invariants of degree 2 and 6. Therefore, the classical dressing function is [191]

$$P_{\mathrm{G}_2}(t, n_1, n_2) = \begin{cases} \frac{1}{(1-t^2)(1-t^6)} \,, & n_1 = n_2 = 0 \,, \\ \frac{1}{(1-t)(1-t^2)} \,, & n_1 > 0, n_2 = 0 \text{ or } n_1 = 0, n_2 > 0 \,, \\ \frac{1}{(1-t)^2} \,, & n_1, n_2 > 0 \,. \end{cases} \qquad (20.2)$$

20.2 Category 1

Hilbert basis The representations $[1,0]$, $[0,1]$, and $[2,0]$ schematically have conformal dimensions of the form

$$\Delta(n_1, n_2) = \sum_j A_j |a_j n_1 + b_j n_2| + B_1 |n_1| + B_2 |n_2| \qquad (20.3)$$

for $a_j, b_j \in \mathbb{N}$ and $A_j, B_1, B_2 \in \mathbb{Z}$. As a consequence, the usual fan within the Weyl chamber is simply one 2-dimensional rational polyhedral cone

$$C^{(2)} = \mathrm{Cone}((1,0),(0,1)) \,. \qquad (20.4)$$

The intersection with the weight lattice $\Lambda_w(\mathrm{G}_2)$ yields the relevant semi-group $S^{(2)}$, as depicted in Fig. 20.1. The Hilbert bases are trivially given by the ray generators

$$\mathcal{H}(S^{(2)}) = \Big\{(1,0),(0,1)\Big\} . \tag{20.5}$$

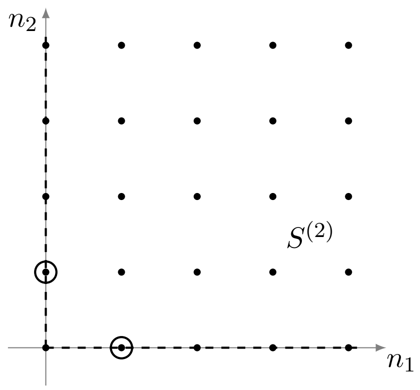


Figure 20.1: *The semi-group $S^{(2)}$ for the representations $[1,0]$, $[0,1]$, and $[2,0]$ obtained from the G_2 Weyl chamber (considered as rational cone) and its ray generators (black circled points).*

Dressings The two minimal generators lie at the boundary of the Weyl chamber and, therefore, have residual gauge group $\mathrm{H}_{(1,0)} = \mathrm{H}_{(0,1)} = \mathrm{U}(2)$. Recalling that G_2 has two Casimir invariants $\mathcal{C}_2$, $\mathcal{C}_6$ at degree 2 and 6, one can analyse the dressed monopole operators associated to $(1,0)$ and $(0,1)$.

The residual gauge group $\mathrm{U}(2) \subset \mathrm{G}_2$ has a degree one Casimir $C_1 \coloneqq \phi_1 + \phi_2$ and a degree two Casimir $C_2 \coloneqq \phi_1^2 + \phi_2^2$. Again, we employed the diagonal form of the adjoint-valued scalar Φ. Consequently, the bare monopole $V_{(0,1)}^{\mathrm{dress},0}$ exhibits five dressed monopoles $V_{(0,1)}^{\mathrm{dress},i}$ $(i = 1,\dots,5)$ of at degrees $\Delta(0,1)+1,\dots,\Delta(0,1)+5$. Since the highest degree Casimir invariant is of order 6 and the degree 2 Casimir invariant of G_2 differs from the pure sum of squares [217], one can build all dressings as follows:

$$C_1(0,1) \; , \quad C_2(0,1) \; , \quad C_1C_2(0,1) \; , \quad C_1^2C_2(0,1) \; , \quad (C_1C_2^2 + C_1^2C_2)(0,1) \; . \tag{20.6}$$

The very same arguments applies for the bare and dressed monopole generators associated to $(1,0)$. Thus, we expect six monopole operators: one bare $V_{(1,0)}^{\mathrm{dress},0}$ and five dressed $V_{(1,0)}^{\mathrm{dress},i}$ $(i = 1,\dots,5)$.

Comparing with App. C, we find that a magnetic weight at the boundary of the dominant Weyl chamber has dressings given by $P_{\mathrm{G}_2}(t,\{n_1 = 0 \text{ or } n_2 = 0\})/P_{\mathrm{G}_2}(t,0,0) = 1+t+t^2+t^3+t^4+t^5$, which is then consistent with the exposition above.

We will now exemplify the three different representations.

20.2.1 Representation $[1,0]$

The relevant computation has been presented in [191] and the conformal dimension reads

$$\begin{aligned}\Delta(n_1,n_2) =& N(|n_1+n_2| + |2n_1+n_2| + |n_1|) \\ &- (|n_1+n_2| + |2n_1+n_2| + |3n_1+n_2| + |3n_1+2n_2| + |n_1| + |n_2|) \; .\end{aligned} \tag{20.7}$$

Evaluating the Hilbert series for $N > 3$ yields

$$\mathrm{HS}_{\mathrm{G}_2}^{[1,0]}(t,N) = \frac{1+t^{2N-4}+t^{2N-3}+t^{2N-2}+t^{2N-1}+t^{4N-5}}{(1-t^2)\,(1-t^6)\,(1-t^{2N-6})\,(1-t^{2N-5})} \,. \tag{20.8}$$

We observe that the numerator of (20.8) is a palindromic polynomial of degree $4N-5$; while, the denominator has degree $4N-3$. Hence, the difference in degree between denominator and numerator is 2, which equals the quaternionic dimension of moduli space. In addition, the Hilbert series (20.8) has a pole of order 4 as $t \to 1$, which matches the complex dimension of the moduli space.

As discussed in [191], the plethystic logarithm has the following behaviour:

$$\mathrm{PL}(\mathrm{HS}_{\mathrm{G}_2}^{[1,0]}(t,N)) = t^2 + t^6 + t^{2N-6}(1+t+t^2+t^3+t^4+t^5) - t^{4N-8} + \dots \,. \tag{20.9}$$

Hilbert basis According to [191], the monopole corresponding to GNO-charge $(1,0)$, which has $\Delta(1,0) = 4N-10$, can be generate. Again, this is due to the specific form (20.7).

20.2.2 Representation $[0,1]$

Hilbert series For this representation, the conformal dimension is given as

$$\Delta(n_1,n_2) = (N-1)(|n_1+n_2| + |2n_1+n_2| + |3n_1+n_2| + |3n_1+2n_2| + |n_1| + |n_2|) \,, \tag{20.10}$$

and the computation of the Hilbert series for $N > 1$ yields

$$\mathrm{HS}_{\mathrm{G}_2}^{[0,1]}(t,N) = \frac{1+t^{6N-5}(1+t+t^2+t^3+t^4)+t^{10N-9}(1+t+t^2+t^3+t^4)+t^{16N-10}}{(1-t^2)\,(1-t^6)\,\left(1-t^{6(N-1)}\right)\left(1-t^{10(N-1)}\right)} . \tag{20.11}$$

The numerator of (20.11) is a palindromic polynomial of degree $16N-10$; while, the denominator is of degree $16N-8$. Hence, the difference in degree between denominator and numerator is 2, which matches the quaternionic dimension of moduli space. Moreover, the Hilbert series has a pole of order 4 as $t \to 1$, i.e. it equals complex dimension of the moduli space. Employing the knowledge of the Hilbert basis (20.5), the appearing objects in (20.11) can be interpreted as in Tab. 20.2.

object		$\Delta(n_1,n_2)$	$\mathrm{H}_{(n_1,n_2)}$
Casimir	$\mathcal{C}_2$	2	—
	$\mathcal{C}_6$	6	—
bare monopole	$V_{(0,1)}^{\mathrm{dress},0}$	$6(N-1)$	U(2)
dressings ($i=1,\dots,5$)	$V_{(0,1)}^{\mathrm{dress},i}$	$6(N-1)+i$	—
bare monopole	$V_{(1,0)}^{\mathrm{dress},0}$	$10(N-1)$	U(2)
dressings ($i=1,\dots,5$)	$V_{(1,0)}^{\mathrm{dress},i}$	$10(N-1)+i$	—

Table 20.2: *The chiral ring generators for a* G_2 *gauge theory and matter transforming in* $[0,1]$.

Plethystic logarithm For $N \geq 3$ the PL takes the form

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[0,1]}_{\mathrm{G}_2}(t,N)) = t^2 + t^6 + t^{6(N-1)}(1+t+t^2+t^3+t^4+t^5) \\ + t^{10(N-1)}(1+t+t^2+t^3+t^4+t^5) - t^{12N-10} + \dots \,,\end{aligned} \tag{20.12}$$

while for $N = 2$ the PL is

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[0,1]}_{\mathrm{G}_2}(t,2)) = t^2 + t^6 + t^6(1+t+t^2+t^3+t^4+t^5) \\ + t^{10}(1+t+t^2+t^3) - 2t^{16} + \dots \,.\end{aligned} \tag{20.13}$$

In other words, the 4th and 5th dressing of $(1,0)$ are absent, because they can be generated. (See also App. C for the degrees of the first relations.)

20.2.3 Representation $[2,0]$

Hilbert series For this representation, the conformal dimensions is given by

$$\begin{aligned}\Delta(n_1,n_2) =& N\Big(2\,|n_1+n_2| + 2\,|2n_1+n_2| + |3n_1+n_2| + |2n_1+2n_2| + |3n_1+2n_2| \\ &+ |4n_1+2n_2| + 2\,|n_1| + |2n_1| + |n_2|\Big) \\ &- \left(|n_1+n_2| + |2n_1+n_2| + |3n_1+n_2| + |3n_1+2n_2| + |n_1| + |n_2|\right) \,.\end{aligned} \tag{20.14}$$

The calculation for the Hilbert series is analogous to the previous cases and we obtain

$$\mathrm{HS}^{[2,0]}_{\mathrm{G}_2}(t,N) = \frac{1 + t^{12N-5}(1+t+t^2+t^3+t^4) + t^{22N-9}(1+t+t^2+t^3+t^4) + t^{34N-10}}{(1-t^2)\,(1-t^6)\,(1-t^{12N-6})\,(1-t^{22N-10})} \,. \tag{20.15}$$

One readily observes, the numerator of (20.15) is a palindromic polynomial of degree $34N-10$ and the denominator is of degree $34N-8$. Hence, the difference in degree between denominator and numerator is 2, which is precisely the quaternionic dimension of moduli space. Also, the Hilbert series has a pole of order 4 as $t \to 1$, which equals the complex dimension of the moduli space. Having in mind the minimal generators (20.5), the appearing objects in (20.15) can be summarised as in Tab. 20.3.

object		$\Delta(n_1,n_2)$	$\mathrm{H}_{(n_1,n_2)}$
Casimir	$\mathcal{C}_2$	2	—
	$\mathcal{C}_6$	6	—
bare monopole	$V^{\mathrm{dress},0}_{(0,1)}$	$12N-6$	$\mathrm{U}(2)$
dressings $(i=1,\dots,5)$	$V^{\mathrm{dress},i}_{(0,1)}$	$12N-6+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,0)}$	$22N-10$	$\mathrm{U}(2)$
dressings $(i=1,\dots,5)$	$V^{\mathrm{dress},i}_{(1,0)}$	$22N-10+i$	—

Table 20.3: *The chiral ring generators for a* G_2 *gauge theory and matter transforming in the representation* $[2,0]$.

Plethystic logarithm We complement the Hilbert series by its PL for all values of N.

- For $N \geq 3$ the PL takes the form

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[2,0]}_{\mathrm{G}_2}(t,N)) = t^2 + t^6 + t^{12N-6}(1+t+t^2+t^3+t^4+t^5) \\ + t^{22N-10}(1+t+t^2+t^3+t^4+t^5) - t^{12N-10} + \ldots\end{aligned} \quad (20.16)$$

- While for $N = 2$ the PL is

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[2,0]}_{\mathrm{G}_2}(t,2)) = t^2 + t^6 + t^{18}(1+t+t^2+t^3+t^4+t^5) \\ + t^{34}(1+t+t^2+t^3) - 2t^{40} + \ldots\end{aligned} \quad (20.17)$$

By the very same reasoning as before, $V^{\mathrm{dress},4}_{(1,0)}$ and $V^{\mathrm{dress},5}_{(1,0)}$ can be generated by monopoles associated to $(0,1)$.

- Moreover, for $N = 1$ the PL looks as follows

$$\mathrm{PL}(\mathrm{HS}^{[2,0]}_{\mathrm{G}_2}(t,1)) = t^2 + t^6 + t^6(1+t+t^2+t^3+t^4+t^5) + t^{12}(1+t) - t^{16} + \ldots \quad (20.18)$$

Looking at the conformal dimensions reveals that the missing dressed monopoles of GNO-charge $(1,0)$ can be generated.

20.3 Category 2

Hilbert basis The representations $[1,1]$, $[0,2]$, and $[3,0]$ schematically have conformal dimensions of the form

$$\Delta(n_1,n_2) = \sum_j A_j|a_j n_1 + b_j n_2| + B_1|n_1| + B_2|n_2| + C|n_1 - n_2| \quad (20.19)$$

for $a_j, b_j \in \mathbb{N}$ and $A_j, B_1, B_2, C \in \mathbb{Z}$. The novelty of this conformal dimension, compared to (20.3), is the difference $|n_1 - n_2|$, i.e. a hyperplane that intersects the Weyl chamber non-trivially. As a consequence, there is a fan generated by two 2-dimensional rational polyhedral cones

$$C^{(2)}_1 = \mathrm{Cone}((1,0),(1,1)) \qquad \text{and} \qquad C^{(2)}_2 = \mathrm{Cone}((1,1),(0,1)) \ . \quad (20.20)$$

The intersection with the weight lattice $\Lambda_w(\mathrm{G}_2)$ yields the relevant semi-groups S_p ($p = 1,2$), as depicted in Fig. 20.2. The Hilbert bases are again given by the ray generators

$$\mathcal{H}(S^{(2)}_1) = \Big\{(1,0),(1,1)\Big\} \qquad \text{and} \qquad \mathcal{H}(S^{(2)}_2) = \Big\{(1,1),(0,1)\Big\} \ . \quad (20.21)$$

Dressings The three minimal generators have different residual gauge groups, as two lie on the boundary and one in the interior of the Weyl chamber. The GNO-charges $(1,0)$ and $(0,1)$ are to be treated as in Sec. 20.2.

The novelty is the magnetic weight $(1,1)$ with $\mathrm{H}_{(1,1)} = \mathrm{U}(1)^2$. Thus, the dressing can be constructed with two independent U(1)-Casimir invariants, proportional to ϕ_1 and ϕ_2. We choose a basis of dressed monopoles

$$V^{\mathrm{dress},j,\alpha}_{(1,1)} = (1,1)(\phi_\alpha)^j \ , \qquad \text{for} \quad j = 1,\ldots 5 \ , \ \alpha = 1,2 \ , \quad (20.22a)$$

$$V^{\mathrm{dress},6}_{(1,1)} = (1,1)\left((\phi_1)^6 + (\phi_2)^6\right) \ . \quad (20.22b)$$

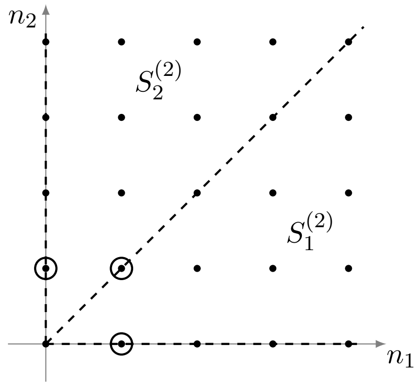


Figure 20.2: *The semi-groups* $S_p^{(2)}$ *(*$p = 1, 2$*) for the representations* $[1, 1]$*,* $[0, 2]$*, and* $[3, 0]$ *obtained from the* G_2 *Weyl chamber (considered as rational cone) and their ray generators (black circled points).*

The reason behind the large number of dressings of the bare monopole $(1, 1)$ lies in the delicate G_2 structure [217], i.e. the degree two Casimir $\mathcal{C}_2$ is not just the sum of the squares of ϕ_i and the next G_2-Casimir $\mathcal{C}_6$ is by four higher in degree and has a complicated structure as well.

The number and degrees of the dressed monopole operators associated to $(1, 1)$ can be confirmed by $P_{G_2}(t, n_1 > 0, n_2 > 0)/P_{G_2}(t, 0, 0) = 1 + 2t + 2t^2 + 2t^3 + 2t^4 + 2t^5 + t^6$, following App. C.

We will now exemplify the three different representations.

20.3.1 Representation $[1, 1]$

Hilbert series The conformal dimension of the 64-dimensional representation is given by

$$\begin{aligned}\Delta(n_1, n_2) = N\Big(& |n_1 - n_2| + 8\,|n_1 + n_2| + 8\,|2n_1 + n_2| + 2\,|3n_1 + n_2| + |4n_1 + n_2| \\ &+ |n_1 + 2n_2| + 2\,|3n_1 + 2n_2| + |5n_1 + 2n_2| + |4n_1 + 3n_2| + |5n_1 + 3n_2| \\ &+ 8\,|n_1| + 2\,|n_2|\Big) \\ &- \Big(|n_1 + n_2| + |2n_1 + n_2| + |3n_1 + n_2| + |3n_1 + 2n_2| + |n_1| + |n_2|\Big)\,. \end{aligned} \tag{20.23}$$

Computing the Hilbert series provides the following expression

$$\mathrm{HS}_{G_2}^{[1,1]}(t, N) = \frac{R(t, N)}{(1 - t^2)\,(1 - t^6)\,(1 - t^{36N-6})\,(1 - t^{64N-10})\,(1 - t^{98N-16})}\,, \tag{20.24a}$$

$$\begin{aligned} R(t, N) = 1 &+ t^{36N-5}(1 + t + t^2 + t^3 + t^4) + t^{64N-9}(1 + t + t^2 + t^3 + t^4) \\ &+ t^{98N-15}(2 + 2t + 2t^2 + 2t^3 + 2t^4 + t^5) \\ &- t^{100N-16}(1 + 2t + 2t^2 + 2t^3 + 2t^4 + 2t^5) \\ &- t^{134N-21}(1 + t + t^2 + t^3 + t^4) - t^{162N-25}(1 + t + t^2 + t^3 + t^4) - t^{198N-26}\,. \end{aligned} \tag{20.24b}$$

The numerator (20.24b) is a anti-palindromic polynomial of degree $198N - 26$; whereas the denominator is of degree $198N - 24$. Hence, the difference in degree between denominator and numerator is 2, which coincides with the quaternionic dimension of moduli space. The Hilbert series (20.24) has a pole of order 4 as $t \to 1$, which agrees with the complex dimension of the moduli space. (One can explicitly show that $R(t = 1, N) = 0$, but $\frac{\mathrm{d}}{\mathrm{d}t} R(t, N)|_{t=1} \neq 0$.) The appearing operators agree with the general setting outline above and we summarise them in

Tab. 20.4. The new monopole corresponds to GNO-charge $(1,1)$ and displays a different dressing

object		$\Delta(n_1,n_2)$	$\mathrm{H}_{(n_1,n_2)}$
Casimir	$\mathcal{C}_2$	2	—
	$\mathcal{C}_6$	6	—
bare monopole	$V_{(0,1)}^{\mathrm{dress},0}$	$134N-6$	$\mathrm{U}(2)$
dressings $(i=1,\dots,5)$	$V_{(0,1)}^{\mathrm{dress},i}$	$134N-6+i$	—
bare monopole	$V_{(1,0)}^{\mathrm{dress},0}$	$238N-10$	$\mathrm{U}(2)$
dressings $(i=1,\dots,5)$	$V_{(1,0)}^{\mathrm{dress},i}$	$238N-10+i$	—
bare monopole	$V_{(1,1)}^{\mathrm{dress},0}$	$364N-16$	$\mathrm{U}(1)\times\mathrm{U}(1)$
dressings $(i=1,\dots,5;\alpha=1,2)$	$V_{(1,1)}^{\mathrm{dress},i,\alpha}$	$364N-16+i$	—
dressing	$V_{(1,1)}^{\mathrm{dress},6}$	$364N-16+6$	—

Table 20.4: *The chiral ring generators for a* G_2 *gauge theory and matter content transforming in* $[1,1]$.

behaviour than $(1,0)$ and $(0,1)$. The reason behind lies in the residual gauge group being $\mathrm{U}(1)^2$.

Plethystic logarithm Although the bare monopole $V_{(1,1)}^{\mathrm{dress},0}$ is generically a necessary generator due to its origin as an ray generators of (20.21), not all dressings $V_{(1,1)}^{\mathrm{dress}}$ might be independent.

- For $N\geq 4$ the PL takes the form

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}_{\mathrm{G}_2}^{[1,1]}(t,N)) = t^2 &+ t^6 + t^{36N-6}(1+t+t^2+t^3+t^4+t^5) \\ &+ t^{64N-10}(1+t+t^2+t^3+t^4+t^5) \\ &- t^{2(36N-6)+2}(1+t+2t^2+2t^3+3t^4+2t^5+2t^6+t^7+t^8) \\ &+ t^{98N-16}(1+2t+2t^2+2t^3+2t^4+2t^5+t^6) - t^{100N-16}+\dots\end{aligned} \tag{20.25}$$

- For $N=3$ the PL is

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}_{\mathrm{G}_2}^{[1,1]}(t,N=3)) = t^2 &+ t^6 + t^{102}(1+t+t^2+t^3+t^4+t^5) \\ &+ t^{182}(1+t+t^2+t^3+t^4+t^5) \\ &- t^{206}(1+t+2t^2+2t^3+3t^4+2t^5+2t^6+t^7+t^8) \\ &+ t^{278}(1+2t+2t^2+2t^3+2t^4+2t^5) - 2t^{285}+\dots\end{aligned} \tag{20.26}$$

  Here, $\Delta(1,0)+\Delta(0,1)=284$ is precisely the conformal dimension of $V_{(1,1)}^{\mathrm{dress},6}$; i.e. it is generated and absent from the PL.

- For $N=2$ the PL is

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}_{\mathrm{G}_2}^{[1,1]}(t,N=2)) = t^2 &+ t^6 + t^{66}(1+t+t^2+t^3+t^4+t^5) \\ &+ t^{118}(1+t+t^2+t^3+t^4+t^5) \\ &- t^{134}(1+t+2t^2+2t^3+3t^4+2t^5+2t^6+t^7+t^8) \\ &+ t^{180}(1+2t+2t^2+2t^3+t^4) - 2t^{186}+\dots\end{aligned} \tag{20.27}$$

Here, $\Delta(1,0)+\Delta(0,1)=184$ is precisely the conformal dimension of $V_{(1,1)}^{\mathrm{dress},4,\alpha}$; i.e. only one of the dressings by the fourth power of a U(1)-Casimir is a generator. Consequently, the other one is absent from the PL.

- For $N=1$ the PL is

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}_{\mathrm{G}_2}^{[1,1]}(t,N=1)) = t^2 + t^6 + t^{30}(1+t+t^2+t^3+t^4+t^5) \\ + t^{54}(1+t+t^2+t^3+t^4+t^5) \\ - t^{62}(1+t+2t^2+2t^3+3t^4+2t^5+2t^6+t^7+t^8) \\ + t^{82}(1+2t+t^2) - t^{62} + \dots\end{aligned} \tag{20.28}$$

Here, $\Delta(1,0)+\Delta(0,1)=64$ is precisely the conformal dimension of $V_{(1,1)}^{\mathrm{dress},2,\alpha}$; i.e. only one of the dressings by the second power of a U(1)-Casimir is a generator. Consequently, the other one is absent from the PL.

20.3.2 Representation $[3,0]$

Hilbert series The conformal dimension in this representation is given by

$$\begin{aligned}\Delta(n_1,n_2) = N\Big(&|5n_1+3n_2| + |5n_1+2n_2| + |4n_1+3n_2| + |4n_1+n_2| + |n_1+2n_2| \\ &+ |n_1-n_2| + 10\big(|2n_1+n_2| + |n_1+n_2| + |n_1|\big) + 3\big(|3n_1+2n_2| \\ &+ |3n_1+n_2| + |n_2|\big)\Big) \\ &- \Big(|n_1+n_2| + |2n_1+n_2| + |3n_1+n_2| + |3n_1+2n_2| + |n_1| + |n_2|\Big)\,,\end{aligned} \tag{20.29}$$

such that we obtain for the Hilbert series

$$\mathrm{HS}_{\mathrm{G}_2}[3,0](t,N) = \frac{R(t,N)}{(1-t^2)\,(1-t^6)\,(1-t^{46N-6})\,(1-t^{82N-10})\,(1-t^{126N-16})}\,, \tag{20.30a}$$

$$\begin{aligned}R(t,N) = 1 &+ t^{46N-5}(1+t+t^2+t^3+t^4) + t^{82N-9}(1+t+t^2+t^3+t^4) \\ &+ t^{126N-15}(2+2t+2t^2+2t^3+2t^4+t^5) \\ &- t^{128N-16}(1+2t+2t^2+2t^3+2t^4+2t^5) \\ &- t^{172N-21}(1+t+t^2+t^3+t^4) - t^{208N-25}(1+t+t^2+t^3+t^4) - t^{254N-26}\,.\end{aligned} \tag{20.30b}$$

The numerator (20.30b) is a anti-palindromic polynomial of degree $254N-26$; while the denominator is of degree $254N-24$. Hence, the difference in degree between denominator and numerator is 2, which coincides with the quaternionic dimension of moduli space. The Hilbert series (20.30) has a pole of order 4 as $t\to 1$, which equals the complex dimension of the moduli space. (One can explicitly show that $R(t=1,N)=0$, but $\frac{\mathrm{d}}{\mathrm{d}t}R(t,N)|_{t=1}\neq 0$.) Interpreting the appearing operators leads to a list of chiral ring generators as presented in Tab. 20.5. The behaviour of the Hilbert series is absolutely identical to the case $[1,1]$, because the conformal dimension is structurally identical. Therefore, we do not provide further details.

20.3.3 Representation $[0,2]$

Hilbert series The conformal dimension reads as follows:

$$\Delta(n_1,n_2) = N\Big(|5n_1+3n_2| + |5n_1+2n_2| + |4n_1+3n_2| + |4n_1+n_2| + |n_1+2n_2|$$

object		$\Delta(n_1,n_2)$	$\mathrm{H}_{(n_1,n_2)}$
Casimir	$\mathcal{C}_2$	2	—
	$\mathcal{C}_6$	6	—
bare monopole	$V^{\mathrm{dress},0}_{(0,1)}$	$46N-6$	U(2)
dressings ($i=1,\dots,5$)	$V^{\mathrm{dress},i}_{(0,1)}$	$46N-6+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,0)}$	$82N-10$	U(2)
dressings ($i=1,\dots,5$)	$V^{\mathrm{dress},i}_{(1,0)}$	$82N-10+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,1)}$	$126N-16$	$\mathrm{U}(1)\times\mathrm{U}(1)$
dressings ($i=1,\dots,5;\alpha=1,2$)	$V^{\mathrm{dress},i,\alpha}_{(1,1)}$	$126N-16+i$	—
dressing	$V^{\mathrm{dress},6}_{(1,1)}$	$126N-16+6$	—

Table 20.5: *The chiral ring generators for a* G_2 *gauge theory and matter transforming in* $[3,0]$.

$$\begin{aligned}
&+|n_1-n_2|+10\big(|2n_1+n_2|+|n_1+n_2|+|n_1|\big) \\
&+5\big(|3n_1+2n_2|+|3n_1+n_2|+|n_2|\big)\Big) \\
&-\Big(|n_1+n_2|+|2n_1+n_2|+|3n_1+n_2|+|3n_1+2n_2|+|n_1|+|n_2|\Big)\,.
\end{aligned} \tag{20.31}$$

The computation of the Hilbert series results in

$$\mathrm{HS}^{[0,2]}_{\mathrm{G}_2}(t,N)=\frac{R(t,N)}{(1-t^2)\,(1-t^6)\,(1-t^{52N-6})\,(1-t^{90N-10})\,(1-t^{140N-16})}\,, \tag{20.32a}$$

$$\begin{aligned}
R(t,N)=1&+t^{52N-5}(1+t+t^2+t^3+t^4)+t^{90N-9}(1+t+t^2+t^3+t^4) \\
&+t^{140N-15}(2+2t+2t^2+2t^3+2t^4+t^5) \\
&-t^{142N-16}(1+2t+2t^2+2t^3+2t^4+2t^5) \\
&-t^{192N-21}(1+t+t^2+t^3+t^4)-t^{230N-25}(1+t+t^2+t^3+t^4)-t^{282N-26}\,.
\end{aligned} \tag{20.32b}$$

The numerator (20.32b) is a anti-palindromic polynomial of degree $282N-26$; while, the denominator is of degree $282N-24$. Hence, the difference in degree between denominator and numerator is 2, which agrees with the quaternionic dimension of moduli space. The Hilbert series (20.32) has a pole of order 4 as $t\to 1$, which equals complex dimension of the moduli space. (One can explicitly show that $R(t=1,N)=0$, but $\frac{\mathrm{d}}{\mathrm{d}t}R(t,N)|_{t=1}\neq 0$.) Tab. 20.6 summarises the appearing operators. The behaviour of the Hilbert series is identical to the cases $[1,1]$ and $[3,0]$, because the conformal dimension is structurally identical. Again, we do not provide further details.

20.4 Category 3

Hilbert basis Investigating the representations $[2,1]$ and $[4,0]$, one recognises the common structural form of the conformal dimensions

$$\Delta(n_1,n_2)=\sum_j A_j|a_jn_1+b_jn_2|+B_1|n_1|+B_2|n_2|+C|n_1-n_2|+D|2n_1-n_2| \tag{20.33}$$

object		$\Delta(n_1,n_2)$	$\mathrm{H}_{(n_1,n_2)}$
Casimir	$\mathcal{C}_2$	2	—
	$\mathcal{C}_6$	6	—
bare monopole	$V_{(0,1)}^{\mathrm{dress},0}$	$52N-6$	$\mathrm{U}(2)$
dressings $(i=1,\ldots,5)$	$V_{(0,1)}^{\mathrm{dress},i}$	$52N-6+i$	—
bare monopole	$V_{(1,0)}^{\mathrm{dress},0}$	$90N-10$	$\mathrm{U}(2)$
dressings $(i=1,\ldots,5)$	$V_{(1,0)}^{\mathrm{dress},i}$	$90N-10+i$	—
bare monopole	$V_{(1,1)}^{\mathrm{dress},0}$	$140N-16$	$\mathrm{U}(1)\times\mathrm{U}(1)$
dressings $(i=1,\ldots,5;\alpha=1,2)$	$V_{(1,1)}^{\mathrm{dress},i,\alpha}$	$140N-16+i$	—
dressing	$V_{(1,1)}^{\mathrm{dress},6}$	$140N-16+6$	—

Table 20.6: *The chiral ring generators for a* G_2 *gauge theory and matter transforming in* $[0,2]$.

for $a_j, b_j \in \mathbb{N}$ and $A_j, B_1, B_2, C, D \in \mathbb{Z}$. The novelty of this conformal dimension, compared to (20.3) and (20.19), is the difference $|2n_1 - n_2|$, i.e. a second hyperplane that intersects the Weyl chamber non-trivially. As a consequence, the Weyl chamber is decomposed into a fan generated by three rational polyhedral cones of dimension 2. These are

$$\begin{aligned} &C_1^{(2)} = \mathrm{Cone}((1,0),(1,1)) \ , \quad C_2^{(2)} = \mathrm{Cone}((1,1),(1,2)) \ , \\ &C_3^{(2)} = \mathrm{Cone}((1,2),(0,1)) \ . \end{aligned} \tag{20.34}$$

The intersection with the weight lattice $\Lambda_w(\mathrm{G}_2)$ yields the relevant semi-groups S_p (for $p = 1,2,3$), as depicted in Fig. 20.3. The Hilbert bases are again given by the ray generators

$$\begin{aligned} &\mathcal{H}(S_1^{(2)}) = \Big\{(1,0),(1,1)\Big\} \ , \quad \mathcal{H}(S_2^{(2)}) = \Big\{(1,1),(1,2)\Big\} \ , \\ &\mathcal{H}(S_3^{(2)}) = \Big\{(1,2),(0,1)\Big\} \ . \end{aligned} \tag{20.35}$$

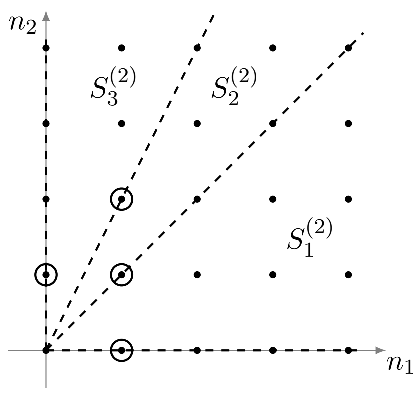


Figure 20.3: *The semi-groups* $S_p^{(2)}$ *(p=1,2,3) for the representations* $[2,1]$ *and* $[4,0]$ *obtained from the* G_2 *Weyl chamber (considered as rational cone) and their ray generators (black circled points).*

Dressings Compared to Sec. 20.2 and 20.3, the additional magnetic weight $(1,2)$ has the same dressing behaviour as $(1,1)$, because the residual gauge groups is $\mathrm{U}(1)^2$, too. Thus, the additional necessary monopole operators are the bare operator $V^{\mathrm{dress},0}_{(1,2)}$ and the dressed monopoles $V^{\mathrm{dress},i,\alpha}_{(1,2)}$ for $i = 1, \ldots, 5$, $\alpha = 1, 2$ as well as $V^{\mathrm{dress},6}_{(1,2)}$.

We will now exemplify the three different representations.

20.4.1 Representation $[4,0]$

Hilbert series The conformal dimension reads

$$\begin{aligned}\Delta(n_1,n_2) = N\Big(&3\,|n_1-n_2| + |2n_1-n_2| + 27\,|n_1+n_2| + 30\,|2n_1+n_2| + 7\,|3n_1+n_2| \\ &+ 3\,|4n_1+n_2| + |5n_1+n_2| + 3\,|n_1+2n_2| + 7\,|3n_1+2n_2| + 3\,|5n_1+2n_2| \\ &+ |2n_1+3n_2| + 3\,|4n_1+3n_2| + 3\,|5n_1+3n_2| + |7n_1+3n_2| + |5n_1+4n_2| \\ &+ |7n_1+4n_2| + 27\,|n_1| + 7\,|n_2|\Big) \\ &- \Big(|n_1+n_2| + |2n_1+n_2| + |3n_1+n_2| + |3n_1+2n_2| + |n_1| + |n_2|\Big)\,,\end{aligned} \tag{20.36}$$

from which we compute the Hilbert series to be

$$\mathrm{HS}^{[4,0]}_{\mathrm{G}_2}(t,N) = \frac{R(t,N)}{(1-t^2)\,(1-t^6)\,(1-t^{134N-6})\,(1-t^{238N-10})\,(1-t^{364N-16})\,(1-t^{496N-22})}, \tag{20.37a}$$

$$\begin{aligned}R(t,N) = 1 &+ t^{134N-5}(1+t+t^2+t^3+t^4) + t^{238N-9}(1+t+t^2+t^3+t^4) \\ &+ t^{364N-15}(2+2t+2t^2+2t^3+2t^4+t^5) \\ &- t^{372N-16}(1+2t+2t^2+2t^3+2t^4+2t^5) \\ &+ t^{496N-21}(2+2t+2t^2+2t^3+2t^4+t^5) \\ &- t^{498N-22}(1+3t+3t^2+3t^3+3t^4+3t^5+t^6) \\ &- t^{602N-25}(1+t+t^2+t^3+t^4) - t^{630N-27}(1+t+t^2+t^3+t^4) \\ &- t^{734N-32}(1+3t+3t^2+3t^3+3t^4+3t^5+t^6) \\ &+ t^{736N-32}(1+2t+2t^2+2t^3+2t^4+2t^5) \\ &- t^{860N-37}(2+2t+2t^2+2t^3+2t^4+t^5) \\ &+ t^{868N-38}(1+2t+2t^2+2t^3+2t^4+2t^5) \\ &+ t^{994N-43}(1+t+t^2+t^3+t^4) + t^{1098N-47}(1+t+t^2+t^3+t^4) + t^{1232N-48}\,.\end{aligned} \tag{20.37b}$$

The numerator (20.37b) is a palindromic polynomial of degree $1232N-48$; while, the denominator is of degree $1232N-46$. Hence, the difference in degree between denominator and numerator is 2, which equals the quaternionic dimension of moduli space. The Hilbert series (20.37) has a pole of order 4 as $t \to 1$, which coincides with the complex dimension of the moduli space. (One can explicitly show that $R(t=1,N) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}R(t,N)|_{t=1} = 0$, but $\frac{\mathrm{d}^2}{\mathrm{d}t^2}R(t,N)|_{t=1} \neq 0$.) The appearing operators can be summarised as in Tab. 20.7. The new monopole corresponds to GNO-charge $(1,2)$ and displays the same dressing behaviour as $(1,1)$. Contrary to the cases $[1,1]$, $[3,0]$, and $[0,2]$, the bare and dressed monopoles of GNO-charge $(1,1)$ are always independent generators as

$$\Delta(1,1) = 364N - 16 < 372N - 16 = 134N - 6 + 238N - 10 = \Delta(0,1) + \Delta(1,0) \tag{20.38}$$

object		$\Delta(n_1,n_2)$	$\mathrm{H}_{(n_1,n_2)}$
Casimir	$\mathcal{C}_2$	2	—
	$\mathcal{C}_6$	6	—
bare monopole	$V^{\mathrm{dress},0}_{(0,1)}$	$134N-6$	$\mathrm{U}(2)$
dressings $(i=1,\ldots,5)$	$V^{\mathrm{dress},i}_{(0,1)}$	$134N-6+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,0)}$	$238N-10$	$\mathrm{U}(2)$
dressings $(i=1,\ldots,5)$	$V^{\mathrm{dress},i}_{(1,0)}$	$238N-10+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,1)}$	$364N-16$	$\mathrm{U}(1)\times\mathrm{U}(1)$
dressings $(i=1,\ldots,5;\alpha=1,2)$	$V^{\mathrm{dress},i,\alpha}_{(1,1)}$	$364N-16+i$	—
dressing	$V^{\mathrm{dress},6}_{(1,1)}$	$364N-16+6$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,2)}$	$496N-22$	$\mathrm{U}(1)\times\mathrm{U}(1)$
dressings $(i=1,\ldots,5;\alpha=1,2)$	$V^{\mathrm{dress},i,\alpha}_{(1,2)}$	$496N-22+i$	—
dressing	$V^{\mathrm{dress},6}_{(1,2)}$	$496N-22+6$	—

Table 20.7: *The chiral ring generators for a* G_2 *gauge theory and hypermultiplets transforming in* $[4,0]$.

holds for all $N\geq 1$.

Plethystic logarithm By means of the minimal generators (20.35), the bare monopole $V^{\mathrm{dress},0}_{(1,2)}$ is a necessary generator. Nevertheless, not all dressings $V^{\mathrm{dress}}_{(1,2)}$ need to be independent. For $N\geq 1$ the PL takes the form

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[0,2]}_{\mathrm{G}_2}(t,N)) = t^2+t^6 &+ t^{134N-6}(1+t+t^2+t^3+t^4+t^5) \\ &+ t^{238N-10}(1+t+t^2+t^3+t^4+t^5) \\ &- t^{2(134N-6)+2}(1+t+2t^2+2t^3+3t^4+2t^5+2t^6+t^7+t^8) \\ &+ t^{364N-16}(1+2t+2t^2+2t^3+2t^4+2t^5+t^6) - t^{372N-16}\ldots\end{aligned} \tag{20.39}$$

Based purely in conformal dimension and GNO-charge, we can argue the following:

- For $N=3$, $\Delta(1,1)+\Delta(0,1)=1472$ is precisely the conformal dimension of $V^{\mathrm{dress},6}_{(1,2)}$, i.e. it is generated.
- For $N=2$, $\Delta(1,1)+\Delta(0,1)=974$ equals the conformal dimension of $V^{\mathrm{dress},4,\alpha}_{(1,2)}$, i.e. only one of the dressings by the fourth power of a U(1)-Casimir is a generator.
- For $N=1$, $\Delta(1,1)+\Delta(0,1)=476$ matches the conformal dimension of $V^{\mathrm{dress},2,\alpha}_{(1,2)}$, i.e. only one of the dressings by the second power of a U(1)-Casimir is a generator.

20.4.2 Representation $[2,1]$

Hilbert series The conformal dimension reads

$$\begin{aligned}\Delta(n_1,n_2) = N\Big(&3\,|n_1-n_2|+|2n_1-n_2|+24\,|n_1+n_2|+24\,|2n_1+n_2|+8\,|3n_1+n_2| \\ &+3\,|4n_1+n_2|+|5n_1+n_2|+3\,|n_1+2n_2|+8\,|3n_1+2n_2|+3\,|5n_1+2n_2|\end{aligned} \tag{20.40}$$

$$+ |2n_1 + 3n_2| + 3\,|4n_1 + 3n_2| + 3\,|5n_1 + 3n_2| + |7n_1 + 3n_2| + |5n_1 + 4n_2| + |7n_1 + 4n_2| + 24\,|n_1| + 8\,|n_2| \Big) - \Big(|n_1 + n_2| + |2n_1 + n_2| + |3n_1 + n_2| + |3n_1 + 2n_2| + |n_1| + |n_2| \Big) ,$$

from which we compute the Hilbert series to be

$$\mathrm{HS}_{\mathrm{G}_2}^{[2,1]}(t,N) = \frac{R(t,N)}{(1-t^2)\,(1-t^6)\,(1-t^{132N-6})\,(1-t^{232N-10})\,(1-t^{356N-16})\,(1-t^{486N-22})}, \tag{20.41a}$$

$$\begin{aligned} R(t,N) = 1 &+ t^{132N-5}(1+t+t^2+t^3+t^4) + t^{232N-9}(1+t+t^2+t^3+t^4) \\ &+ t^{356N-15}(2+2t+2t^2+2t^3+2t^4+t^5) \\ &- t^{364N-16}(1+2t+2t^2+2t^3+2t^4+2t^5) \\ &+ t^{486N-21}(2+2t+2t^2+2t^3+2t^4+t^5) \\ &- t^{488N-22}(1+3t+3t^2+3t^3+3t^4+3t^5+t^6) \\ &- t^{588N-25}(1+t+t^2+t^3+t^4) - t^{618N-27}(1+t+t^2+t^3+t^4) \\ &- t^{718N-32}(1+3t+3t^2+3t^3+3t^4+3t^5+t^6) \\ &+ t^{720N-32}(1+2t+2t^2+2t^3+2t^4+2t^5) \\ &- t^{842N-37}(2+2t+2t^2+2t^3+2t^4+t^5) \\ &+ t^{850N-38}(1+2t+2t^2+2t^3+2t^4+2t^5) \\ &+ t^{974N-43}(1+t+t^2+t^3+t^4) + t^{1074N-47}(1+t+t^2+t^3+t^4) + t^{1206N-48}. \end{aligned} \tag{20.41b}$$

The numerator (20.41b) is a palindromic polynomial of degree $1206N-48$; whereas, the denominator is of degree $1206N-46$. Hence, the difference in degree between denominator and numerator is 2, which agrees with the quaternionic dimension of moduli space. The Hilbert

object		$\Delta(n_1,n_2)$	$\mathrm{H}_{(n_1,n_2)}$
Casimir	$\mathcal{C}_2$	2	—
	$\mathcal{C}_6$	6	—
bare monopole	$V^{\mathrm{dress},0}_{(0,1)}$	$132N-6$	U(2)
dressings ($i=1,\ldots,5$)	$V^{\mathrm{dress},i}_{(0,1)}$	$132N-6+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,0)}$	$232N-10$	U(2)
dressings ($i=1,\ldots,5$)	$V^{\mathrm{dress},i}_{(1,0)}$	$232N-10+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,1)}$	$356N-16$	U(1) × U(1)
dressings ($i=1,\ldots,5;\alpha=1,2$)	$V^{\mathrm{dress},i,\alpha}_{(1,1)}$	$356N-16+i$	—
dressing	$V^{\mathrm{dress},6}_{(1,1)}$	$356N-16+6$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,2)}$	$486N-22$	U(1) × U(1)
dressings ($i=1,\ldots,5;\alpha=1,2$)	$V^{\mathrm{dress},i,\alpha}_{(1,2)}$	$486N-22+i$	—
dressing	$V^{\mathrm{dress},6}_{(1,2)}$	$486N-22+6$	—

Table 20.8: *The chiral ring generators for a* G_2 *gauge theory and matter transforming in* $[2,1]$.

series (20.41) has a pole of order 4 as $t \to 1$, which equals the complex dimension of the moduli space. (One can explicitly show that $R(t=1,N)=0$ and $\frac{\mathrm{d}}{\mathrm{d}t}R(t,N)|_{t=1}=0$, but $\frac{\mathrm{d}^2}{\mathrm{d}t^2}R(t,N)|_{t=1}\neq 0$.) The list of appearing operators is presented in Tab. 20.8. Due to the structure of the conformal dimension the behaviour of the $[2,1]$ representation is identical to that of $[4,0]$. Consequently, we do not discuss further details.

21 Case: SU(3)

The last rank two example we would like to cover is SU(3), for which the computation takes a detour over the corresponding U(3) theory, similar to [191]. The advantage is that we can simultaneously investigate the rank three example U(3) and demonstrate that the method of Hilbert bases for semi-groups works equally well in higher rank cases.

21.1 Set-up

In the following, we systematically study a number of SU(3) representation, where we understand a SU(3)-representation $[a, b]$ as an U(3)-representation with a fixed U(1)-charge.

Preliminaries for U(3) The GNO-dual group of U(3), which is again a U(3), has a weight lattice characterised by $m_1, m_2, m_3 \in \mathbb{Z}$ and the dominant Weyl chamber is given by the restriction $m_1 \geq m_2 \geq m_3$, cf. [191]. The classical dressing factors associated to the interior and boundaries of the dominant Weyl chamber are the following:

$$P_{\mathrm{U}(3)}(t^2, m_1, m_2, m_3) = \begin{cases} \frac{1}{(1-t^2)^3} , & m_1 > m_2 > m_3 , \\ \frac{1}{(1-t^2)^2(1-t^4)} , & (m_1 = m_2 > m_3) \vee (m_1 > m_2 = m_3) , \\ \frac{1}{(1-t^2)(1-t^4)(1-t^6)} , & m_1 = m_2 = m_3 . \end{cases} \tag{21.1}$$

Note that we already introduced the fugacity t^2 instead of t. Moreover, the GNO-dual U(3) has a non-trivial centre, i.e. $\mathcal{Z}(\mathrm{U}(3)) = \mathrm{U}(1)_J$; thus, the topological symmetry is a $\mathrm{U}(1)_J$ counted by $z^{m_1+m_2+m_3}$.

The contributions of $N_{(a,b)}$ hypermultiplets transforming in $[a, b]$ to the conformal dimension are as follows:

$$\Delta_{\mathrm{h-plet}}^{[1,0]} = \frac{N_{(1,0)}}{2} \sum_i |m_i| , \tag{21.2a}$$

$$\Delta_{\mathrm{h-plet}}^{[2,0]} = \frac{3N_{(2,0)}}{2} \sum_i |m_i| , \tag{21.2b}$$

$$\Delta_{\mathrm{h-plet}}^{[1,1]} = N_{(1,1)} \sum_{i<j} |m_i - m_j| , \tag{21.2c}$$

$$\Delta_{\mathrm{h-plet}}^{[3,0]} = \frac{3N_{(3,0)}}{2} \sum_i |m_i| + N_{[3,0]} \sum_{i<j} |m_i - m_j| , \tag{21.2d}$$

$$\Delta_{\mathrm{h-plet}}^{[2,2]} = 3N_{(2,2)} \sum_i |m_i| + 4N_{(2,2)} \sum_{i<j} |m_i - m_j| , \tag{21.2e}$$

$$\Delta_{\mathrm{h-plet}}^{[2,1]} = 4N_{(2,1)} \sum_i |m_i| + \frac{N_{(2,1)}}{2} \sum_{i<j} \left(|2m_i - m_j| + |m_i - 2m_j|\right) , \tag{21.2f}$$

where $i, j = 1, 2, 3$. In addition, the contribution of the vector multiplet reads as

$$\Delta_{\mathrm{v-plet}} = - \sum_{i<j} |m_i - m_j| . \tag{21.3}$$

Consequently, one can study a pretty wild matter content if one considers the conformal dimension to be of the form

$$\begin{aligned}\Delta(m_1,m_2,m_3) = \frac{N_F}{2}\sum_i |m_i| + (N_A-1)\sum_{i<j}|m_i-m_j| \\ + \frac{N_R}{2}\sum_{i<j}\left(|2m_i-m_j|+|m_i-2m_j|\right) \ ,\end{aligned} \tag{21.4}$$

and the relation to the various representations (21.2) is established via

$$N_F = N_{(1,0)} + 3N_{(2,0)} + 3N_{(3,0)} + 6N_{(2,2)} + 4N_{(2,1)} \ , \tag{21.5a}$$

$$N_A = N_{(1,1)} + N_{(3,0)} + 4N_{(2,2)} \ , \tag{21.5b}$$

$$N_R = N_{(2,1)} \ . \tag{21.5c}$$

Preliminaries for SU(3) As noted in [191], the reduction from U(3) to SU(3) (with the same matter content) is realised by averaging over $\mathrm{U}(1)_J$, for the purpose of setting $m_1+m_2+m_3=0$, and multiplying by $(1-t^2)$, such that $\mathrm{tr}(\Phi)=0$ holds for the adjoint scalar Φ. In other words

$$\mathrm{HS}^{[a,b]}_{\mathrm{SU}(3)}(t^2) = (1-t^2)\oint_{|z|=1}\frac{\mathrm{d}z}{2\pi\mathrm{i}z}\mathrm{HS}^{[a,b]}_{\mathrm{U}(3)}(t^2,z) \ . \tag{21.6}$$

As a consequence, the conformal dimension for SU(3) itself is obtained from (21.4) via

$$\Delta(m_1,m_2) \coloneqq \Delta(m_1,m_2,m_3)\big|_{m_3=-m_1-m_2} \ . \tag{21.7}$$

The Weyl chamber is now characterised by $m_1 \geq \max\{m_2,-2m_2\}$. Multiplying (21.1) by $(1-t^2)$ and employing $m_3=-m_1-m_2$ results in the classical dressing factors for SU(3)

$$P_{\mathrm{SU}(3)}(t^2,m_1,m_2) = \begin{cases} \frac{1}{(1-t^2)^2} \ , & m_1 > \max\{m_2,-2m_2\} \ , \\ \frac{1}{(1-t^2)(1-t^4)} \ , & (m_1=m_2)\vee(m_1=-2m_2) \ , \\ \frac{1}{(1-t^4)(1-t^6)} \ , & m_1=m_2=0 \ . \end{cases} \tag{21.8}$$

21.2 Hilbert basis

21.2.1 Fan and cones for U(3)

Following the ideas outline previously, $\Lambda_w(\widehat{\mathrm{U}(3)})/\mathcal{W}_{\mathrm{U}(3)}$ can be described as a collection of semi-groups that originate from a fan. Since this is our first 3-dimensional example, we provide a detail description on how to obtain the fan. Consider the absolute values $|am_1+bm_2+cm_2|$ in (21.7) as *Hesse normal form* for the hyperplanes

$$\vec{n}\cdot\vec{m} \equiv \begin{pmatrix} a\\ b\\ c\end{pmatrix}\cdot\begin{pmatrix} m_1\\ m_2\\ m_3\end{pmatrix} = 0 \tag{21.9}$$

which pass through the origin. Take all normal vectors $\vec{n}_j$, define the matrices $M_{i,j}=(\vec{n}_i,\vec{n}_j)^T$ (for $i<j$) and compute the null spaces (or kernel) $K_{i,j} \coloneqq \ker(M_{i,j})$. Linear algebra tell us that

$\dim(K_{i,j}) \geq 1$, but by the specific form[31] of Δ we have the stronger condition $\mathrm{rk}(M_{i,j}) = 2$ for all $i < j$; thus, we always have $\dim(K_{i,j}) = 1$. Next, we select a basis vector $e_{i,j}$ of $K_{i,j}$ and check if $e_{i,j}$ or $-e_{i,j}$ intersect the Weyl-chamber. If it does, then it is going to be an edge for the fan and, more importantly, will turn out to be a ray generator (provided one defines $e_{i,j}$ via the intersection with the corresponding weight lattice). Now, one has to define all 3-dimensional cones, merge them into a fan, and, lastly, compute the Hilbert bases. The program *Sage* is a convenient tool for such tasks.

As two examples, we consider the conformal dimension (21.7) for $N_R = 0$ and $N_R \neq 0$ and preform the entire procedure. That is: firstly, compute the edges of the fan; secondly, define the all 3-dimensional cones; and, thirdly, compute the Hilbert bases.

Case $N_R = 0$ In this circumstance, we deduce the following edges

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \begin{pmatrix}0\\0\\-1\end{pmatrix}, \quad \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \quad \begin{pmatrix}-1\\-1\\-1\end{pmatrix}. \tag{21.10}$$

All these vectors are on the boundaries of the Weyl chamber. The set of 3-dimensional cones that generate the corresponding fan is given by

$$C_1^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}, \qquad C_2^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}, \tag{21.11a}$$

$$C_3^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}, C_4^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}. \tag{21.11b}$$

A computation shows that all four cones are strictly convex, smooth, and simplicial. The Hilbert bases for the resulting semi-groups consist solely of the ray generators

$$\mathcal{H}(S_1^{(3)}) = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}, \qquad \mathcal{H}(S_2^{(3)}) = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}, \tag{21.12a}$$

$$\mathcal{H}(S_3^{(3)}) = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}, \mathcal{H}(S_4^{(3)}) = \left\{\begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}. \tag{21.12b}$$

From the above, we expect 6 bare monopole operators plus their dressings for a generic theory with $N_R = 0$. Since all ray generators lie at the boundary of the Weyl chamber, the residual gauge groups are $\mathrm{U}(3)$ for $\pm(1,1,1)$ and $\mathrm{U}(2) \times \mathrm{U}(1)$ for the other four GNO-charges.

Case $N_R \neq 0$ Here, we compute the following edges:

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \begin{pmatrix}2\\1\\0\end{pmatrix}, \quad \begin{pmatrix}2\\1\\1\end{pmatrix}, \quad \begin{pmatrix}2\\2\\1\end{pmatrix}, \quad \begin{pmatrix}4\\2\\1\end{pmatrix}, \tag{21.13a}$$

[31] Δ is homogeneous and all hyperplanes pass through the origin; hence, no two hyperplanes can be parallel. This implies that no two normal vectors can be multiples of each other.

$$\begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\-2\end{pmatrix}, \begin{pmatrix}-1\\-1\\-2\end{pmatrix}, \begin{pmatrix}-1\\-2\\-2\end{pmatrix}, \begin{pmatrix}-1\\-2\\-4\end{pmatrix}. \tag{21.13b}$$

Now, we need to proceed and define all 3-dimensional cones that constitute the fan and, in turn, will lead to the semi-groups that we wish to study. We obtain the following 16 cones:

$$C_1^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}4\\2\\1\end{pmatrix}\right\}, \qquad C_2^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}4\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}\right\}, \tag{21.14a}$$

$$C_3^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}\right\}, \qquad C_4^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}4\\2\\1\end{pmatrix}\right\}, \tag{21.14b}$$

$$C_5^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}4\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}\right\}, \qquad C_6^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}, \tag{21.14c}$$

$$C_7^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}\right\}, \qquad C_8^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}\right\}, \tag{21.14d}$$

$$C_9^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\-2\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\}, \quad C_{10}^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}0\\-1\\-2\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\}, \tag{21.14e}$$

$$C_{11}^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\-2\end{pmatrix}, \begin{pmatrix}-1\\-2\\-4\end{pmatrix}\right\}, C_{12}^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}-1\\-2\\-4\end{pmatrix}, \begin{pmatrix}-1\\-1\\-2\end{pmatrix}\right\}, \tag{21.14f}$$

$$C_{13}^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}0\\-1\\-2\end{pmatrix}, \begin{pmatrix}-1\\-2\\-4\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\end{pmatrix}\right\}, C_{14}^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\-2\\-4\end{pmatrix}, \begin{pmatrix}-1\\-2\\-2\end{pmatrix}\right\}, \tag{21.14g}$$

$$C_{15}^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}-1\\-2\\-4\end{pmatrix}, \begin{pmatrix}-1\\-2\\-2\end{pmatrix}, \begin{pmatrix}-1\\-1\\-2\end{pmatrix}\right\}, C_{16}^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}-1\\-2\\-2\end{pmatrix}, \begin{pmatrix}-1\\-1\\-2\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right\}. \tag{21.14h}$$

All of the rational polyhedral cones are strictly convex and simplicial, but only the cones C_p for $p = 1, 2, 3, 6, \ldots, 13, 16$ are smooth. Now, we compute the Hilbert bases for semi-groups $S_p^{(3)}$ for $p = 1, 2, \ldots, 16$ and obtain

$$\mathcal{H}(S_1^{(3)}) = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}4\\2\\1\end{pmatrix}\right\}, \qquad \mathcal{H}(S_2^{(3)}) = \left\{\begin{pmatrix}4\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}\right\}, \tag{21.15a}$$

$$\mathcal{H}(S_3^{(3)}) = \left\{\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}\right\}, \qquad \mathcal{H}(S_4^{(3)}) = \left\{\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}4\\2\\1\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}\right\}, \tag{21.15b}$$

$$\mathcal{H}(S_5^{(3)}) = \left\{\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}4\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}\right\}, \quad \mathcal{H}(S_6^{(3)}) = \left\{\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}, \tag{21.15c}$$

$$\mathcal{H}(S_7^{(3)})=\left\{\begin{pmatrix}0\\0\\-1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right\}, \qquad \mathcal{H}(S_8^{(3)})=\left\{\begin{pmatrix}0\\0\\-1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right\}, \tag{21.15d}$$

$$\mathcal{H}(S_9^{(3)})=\left\{\begin{pmatrix}0\\0\\-1\end{pmatrix},\begin{pmatrix}0\\-1\\-2\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}, \qquad \mathcal{H}(S_{10}^{(3)})=\left\{\begin{pmatrix}0\\-1\\-2\end{pmatrix},\begin{pmatrix}0\\-1\\-1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}, \tag{21.15e}$$

$$\mathcal{H}(S_{11}^{(3)})=\left\{\begin{pmatrix}0\\0\\-1\end{pmatrix},\begin{pmatrix}0\\-1\\-2\end{pmatrix},\begin{pmatrix}-1\\-2\\-4\end{pmatrix}\right\}, \quad \mathcal{H}(S_{12}^{(3)})=\left\{\begin{pmatrix}0\\0\\-1\end{pmatrix},\begin{pmatrix}-1\\-2\\-4\end{pmatrix},\begin{pmatrix}-1\\-1\\-2\end{pmatrix}\right\}, \tag{21.15f}$$

$$\mathcal{H}(S_{13}^{(3)})=\left\{\begin{pmatrix}0\\-1\\-2\end{pmatrix},\begin{pmatrix}-1\\-2\\-4\end{pmatrix},\begin{pmatrix}0\\-1\\-1\end{pmatrix}\right\}, \mathcal{H}(S_{14}^{(3)})=\left\{\begin{pmatrix}0\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\-2\\-4\end{pmatrix},\begin{pmatrix}-1\\-2\\-2\end{pmatrix},\begin{pmatrix}-1\\-2\\-3\end{pmatrix}\right\}, \tag{21.15g}$$

$$\mathcal{H}(S_{15}^{(3)})=\left\{\begin{pmatrix}-1\\-2\\-4\end{pmatrix},\begin{pmatrix}-1\\-2\\-2\end{pmatrix},\begin{pmatrix}-1\\-1\\-2\end{pmatrix},\begin{pmatrix}-1\\-2\\-3\end{pmatrix}\right\}, \mathcal{H}(S_{16}^{(3)})=\left\{\begin{pmatrix}-1\\-2\\-2\end{pmatrix},\begin{pmatrix}-1\\-1\\-2\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right\}. \tag{21.15h}$$

We observe that there are four semi-groups S_p for $p = 4, 5, 14, 15$ for which the Hilbert bases exceeds the set of ray generators by an additional element. Consequently, we expect 16 bare monopoles plus their dressings for a generic theory with $N_R \neq 0$. However, the dressings exhibit a much richer structure compared to $N_R = 0$, because some minimal generators lie in the interior of the Weyl chamber. The residual gauge groups are U(3) for $\pm(1, 1, 1)$; $\mathrm{U}(2) \times \mathrm{U}(1)$ for $(1, 0, 0)$, $(0, 0, -1)$, $(1, 1, 0)$, $(0, -1, -1)$, $(2, 1, 1)$, $(-1, -1, -2)$, $(2, 2, 1)$, and $(-1, -2, -2)$; and $\mathrm{U}(1)^3$ for $(2, 1, 0)$, $(0, -1, -2)$,$(4, 2, 1)$, $(-1, -2, -4)$, $(3, 2, 1)$, and $(-1, -2, -3)$.

21.2.2 Fan and cones for SU(3)

The conformal dimension (21.7) divides the Weyl chamber of the GNO-dual into two different fans, depending on $N_R = 0$ or $N_R \neq 0$.

Case $N_R = 0$ For this situation, which is depicted in Fig. 21.1a, there are the following three rays $\sim |m_1|, |m_1 - m_2|, |m_1 + 2m_2|$ present that intersect the Weyl chamber non-trivially. The corresponding fan is generated by two 2-dimensional cones

$$C_1^{(2)} = \mathrm{Cone}((2, -1), (1, 0)) \qquad \text{and} \qquad C_2^{(2)} = \mathrm{Cone}((1, 0), (1, 1)) \; . \tag{21.16}$$

The Hilbert bases for the semi-groups, obtained by intersecting the cones with the weight lattice, are solely given by the ray generators, i.e.

$$\mathcal{H}(S_1^{(2)}) = \Big\{(2, -1), (1, 0)\Big\} \qquad \text{and} \qquad \mathcal{H}(S_2^{(2)}) = \Big\{(1, 0), (1, 1)\Big\} \; . \tag{21.17}$$

As a consequence, we expect three bare monopole operators (plus dressings) for a generic $N_R = 0$ theory. The residual gauge group is $\mathrm{SU}(2) \times \mathrm{U}(1)$ for $(2, -1)$ and $(1, 1)$, because these GNO-charges are at the boundary of the Weyl-chamber. In contrast, $(1, 0)$ has residual gauge group $\mathrm{U}(1)^2$ as it lies in the interior of the dominant Weyl chamber.

Case $N_R \neq 0$ For this circumstance, there are two additional rays $\sim |m_1 - 2m_2|, |m_1 + 3m_2|$ present, compared to $N_R = 0$, that intersect the Weyl chamber non-trivially. We refer to

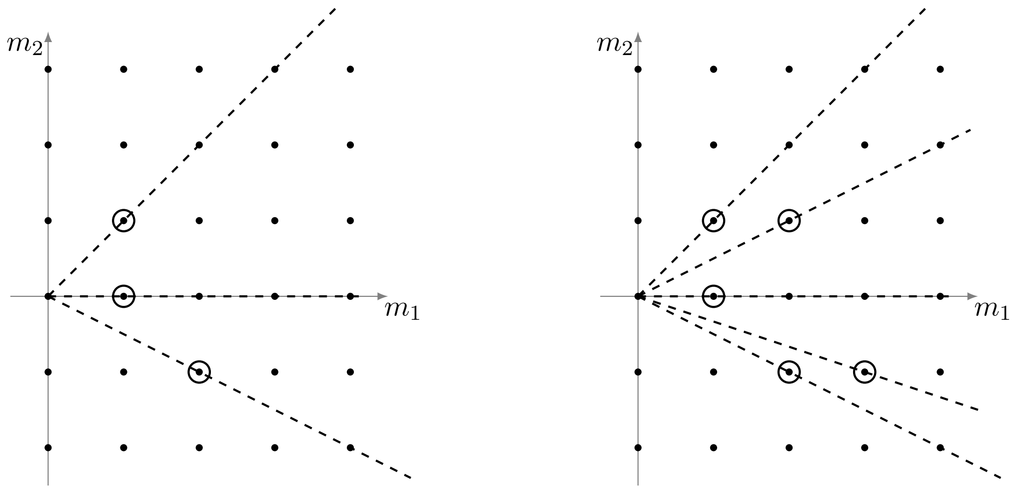
Fig. 21.1b. The corresponding fan is now generated by four 2-dimensional cones

$$C_{1-}^{(2)} = \mathrm{Cone}((2,-1),(3,-1)) \ , \qquad C_{1+}^{(2)} = \mathrm{Cone}((3,-1),(1,0)) \ , \tag{21.18a}$$
$$C_{2-}^{(2)} = \mathrm{Cone}((1,0),(2,1)) \ , \qquad C_{2+}^{(2)} = \mathrm{Cone}((2,1),(1,1)) \ . \tag{21.18b}$$

The Hilbert bases for the resulting semi-groups are given by the ray generators, i.e.

$$\mathcal{H}(S_{1-}^{(2)}) = \Big\{(2,-1),(3,-1)\Big\} \ , \qquad \mathcal{H}(S_{1+}^{(2)}) = \Big\{(3,-1),(1,0)\Big\} \ , \tag{21.19a}$$
$$\mathcal{H}(S_{2-}^{(2)}) = \Big\{(1,0),(2,1)\Big\} \ , \qquad \mathcal{H}(S_{2+}^{(2)}) = \Big\{(2,1),(1,1)\Big\} \ . \tag{21.19b}$$

Judging from the Hilbert bases, there are five bare monopole operators present in the generic case. The residual gauge group for $(1,0)$, $(3,-1)$, and $(2,1)$ is $\mathrm{U}(1)^2$, as they lie in the interior. For $(1,1)$ and $(2,-1)$ the residual gauge group is $\mathrm{SU}(2) \times \mathrm{U}(1)$, because these points lie at the boundary of the Weyl chamber.



(a) $N_R = 0$ ***(b)*** $N_R \neq 0$

Figure 21.1: *The semi-groups for* SU(3) *and the corresponding ray generators (black circled points).*

21.3 Casimir invariance

Now, we need to discuss the dressed monopole operators associated to each element of the Hilbert basis. For U(3) we will heavily rely on the results of App. C; while we can provide a more detailed description for the dressings in the SU(3) case.

21.3.1 Dressings for U(3)

Following the description of dressed monopole operators as in [191], we diagonalise the adjoint-valued scalar Φ along the moduli space, i.e.

$$\mathrm{diag}\,\Phi = (\phi_1, \phi_2, \phi_3) \ . \tag{21.20}$$

Moreover, the Casimir invariants of U(3) can then be written as $\mathcal{C}_j = \mathrm{tr}(\Phi^j) = \sum_{l=1}^{3} (\phi_l)^j$ for $j = 1, 2, 3$. We will now elaborate on the possible dressed monopole operators by means of the insights gained in Sec. 15.3 and App. C.

To start with, for a monopole with GNO-charge such that $\mathrm{H}_{(m_1,m_2,m_3)} = \mathrm{U}(3)$ the dressings are described by

$$\frac{P_{\mathrm{U}(3)}(t,m_1,m_1,m_1)}{P_{\mathrm{U}(3)}(t,0)} - 1 = 0 \ , \tag{21.21}$$

i.e. there are no dressings, because the Casimir invariants of the centraliser $\mathrm{H}_{(m_1,m_2,m_3)}$ are identical to those of G, since the groups coincide. Prominent examples are the (bare) monopoles of GNO-charge $\pm(1,1,1)$.

Next, a monopole of GNO-charge such that $\mathrm{H}_{(m_1,m_2,m_3)} = \mathrm{U}(1)\times\mathrm{U}(2)$ exhibit dressings governed by

$$\frac{P_{\mathrm{U}(3)}(t,m_1,m_2,m_3)}{P_{\mathrm{U}(3)}(t,0)} - 1 = \frac{(1-t^2)(1-t^4)(1-t^6)}{(1-t^2)^2(1-t^4)} - 1 = t^2 + t^4 \ , \tag{21.22}$$

implying there to be exactly one dressing by a degree 2 Casimir and one dressing by a degree 4 Casimir. The two degree 2 Casimir invariants of $\mathrm{H}_{(m_1,m_2,m_3)}$, one by U(1) and one by U(2), are not both independent because there is the overall Casimir $\mathcal{C}_1$ of U(3). Therefore, only one of them leads to an independent dressed monopole generator. The second dressing is then due to the second Casimir of U(2). For example, the monopole of GNO-charge $(1,1,0)$, $(0,-1,-1)$, $(2,1,1)$, $(-1,-2,-2)$, $(2,2,1)$, and $(-1,-2,-2)$ exhibit these two dressings options.

Lastly, if the residual gauge group is $\mathrm{H}_{(m_1,m_2,m_3)} = \mathrm{U}(1)^3$ then the dressings are determined via

$$\frac{P_{\mathrm{U}(3)}(t,m_1,m_2,m_3)}{P_{\mathrm{U}(3)}(t,0)} - 1 = \frac{(1-t^2)(1-t^4)(1-t^6)}{(1-t^2)^3} - 1 = 2t^2 + 2t^4 + t^6 \ . \tag{21.23}$$

Consequently, there are generically five dressings for each such bare monopole operator. Examples for this instance are $(2,1,0)$, $(0,-1,-2)$, $(3,2,1)$, $(-1,-2,-3)$, $(4,2,1)$, $(-1,-2,-4)$.

21.3.2 Dressings for SU(3)

To determine the dressings, we take the adjoint scalar Φ and diagonalise it, keeping in mind that it now belongs to SU(3), that is

$$\mathrm{diag}\Phi = (\phi_1, \phi_2, -(\phi_1+\phi_2)) \ . \tag{21.24}$$

Recalling that each ϕ_i has dimension one, we can write down the dressings (in the dominant Weyl chamber): $(1,0)$ can be dressed by two independent U(1)-Casimir invariants, i.e. directly by ϕ_1 and ϕ_2

$$V_{(1,0)}^{\mathrm{dress},(0,0)} \equiv (1,0) \longrightarrow \begin{cases} V_{(1,0)}^{\mathrm{dress},(1,0)} \equiv \phi_1 \ (1,0) \ , \\ V_{(1,0)}^{\mathrm{dress},(0,1)} \equiv \phi_2 \ (1,0) \ , \end{cases} \tag{21.25}$$

such that the dressings have conformal dimension $\Delta(1,0)+1$. Next, out of the three degree 2 combinations of ϕ_i, only two of them are independent and we choose them to be

$$V_{(1,0)}^{\mathrm{dress},(0,0)} \equiv (1,0) \longrightarrow \begin{cases} V_{(1,0)}^{\mathrm{dress},(2,0)} \equiv \phi_1^2 \ (1,0) \ , \\ V_{(1,0)}^{\mathrm{dress},(0,2)} \equiv \phi_2^2 \ (1,0) \ , \end{cases} \tag{21.26}$$

and these second order dressings have conformal dimension $\Delta(1,0)+2$. Finally, one last dressing is possible

$$V_{(1,0)}^{\mathrm{dress},(0,0)} \equiv (1,0) \longrightarrow V_{(1,0)}^{\mathrm{dress},(3,0)+(0,3)} \equiv (\phi_1^3+\phi_2^3) \ (1,0) \ , \tag{21.27}$$

having dimension $\Delta(1,0)+3$. Alternatively, we utilise App. C and compute the number and degrees of the dressed monopole operators of magnetic charge $(1,0)$ via the quotient of dressing factors $P_{\mathrm{SU}(3)}(t^2,1,0)/P_{\mathrm{SU}(3)}(t^2,0,0) = 1+2t^2+2t^4+t^6$.

The two monopoles of GNO-charge $(1,1)$ and $(2,-1)$ have residual gauge group $\mathrm{SU}(2)\times\mathrm{U}(1)$, i.e. the monopoles can be dressed by a degree one Casmir invariant of the U(1) and by a degree two Casimir invariant of the SU(2). These increase the dimensions by one and two, respectively. Consequently, we obtain

$$V^{\mathrm{dress},0}_{(1,1)} \equiv (1,1) \longrightarrow \begin{cases} V^{\mathrm{dress},\mathrm{U}(1)}_{(1,1)} \equiv (\phi_1+\phi_2)\ (1,1)\ , \\ V^{\mathrm{dress},\mathrm{SU}(2)}_{(1,1)} \equiv (\phi_1^2+\phi_2^2)\ (1,1)\ , \end{cases} \tag{21.28}$$

and similarly

$$V^{\mathrm{dress},0}_{(2,-1)} \equiv (2,-1) \longrightarrow \begin{cases} V^{\mathrm{dress},\mathrm{U}(1)}_{(2,-1)} \equiv (\phi_1+\phi_2)\ (2,-1)\ , \\ V^{\mathrm{dress},\mathrm{SU}(2)}_{(2,-1)} \equiv (\phi_1^2+\phi_2^2)\ (2,-1)\ . \end{cases} \tag{21.29}$$

Since the magnetic weights $(1,1)$, $(2,-1)$ lie at the boundary of the dominant Weyl chamber, we can derive the dressing behaviour via $P_{\mathrm{SU}(3)}(t^2,(1,1) \text{ or } (2,-1))/P_{\mathrm{SU}(3)}(t^2,0,0) = 1+t^2+t^4$ and obtain agreement with our choice of generators.

The remaining monopoles of GNO-charge $(2,1)$ and $(3,-1)$ can be treated analogously to $(1,0)$ and we obtain

$$V^{\mathrm{dress},(0,0)}_{(2,1)} \equiv (2,1) \longrightarrow \begin{cases} V^{\mathrm{dress},(1,0)}_{(2,1)} \equiv \phi_1\ (2,1)\ , \\ V^{\mathrm{dress},(0,1)}_{(2,1)} \equiv \phi_2\ (2,1)\ , \\ V^{\mathrm{dress},(2,0)}_{(2,1)} \equiv \phi_1^2\ (2,1)\ , \\ V^{\mathrm{dress},(0,2)}_{(2,1)} \equiv \phi_2^2\ (2,1)\ , \\ V^{\mathrm{dress},(3,0)+(0,3)}_{(2,1)} \equiv (\phi_1^3+\phi_2^3)\ (2,1)\ , \end{cases} \tag{21.30}$$

$$V^{\mathrm{dress},(0,0)}_{(3,-1)} \equiv (3,-1) \longrightarrow \begin{cases} V^{\mathrm{dress},(1,0)}_{(3,-1)} \equiv \phi_1\ (3,-1)\ , \\ V^{\mathrm{dress},(0,1)}_{(3,-1)} \equiv \phi_2\ (3,-1)\ , \\ V^{\mathrm{dress},(2,0)}_{(3,-1)} \equiv \phi_1^2\ (3,-1)\ , \\ V^{\mathrm{dress},(0,2)}_{(3,-1)} \equiv \phi_2^2\ (3,-1)\ , \\ V^{\mathrm{dress},(3,0)+(0,3)}_{(3,-1)} \equiv (\phi_1^3+\phi_2^3)\ (3,-1)\ . \end{cases} \tag{21.31}$$

There can be circumstances in which not all dressings for the minimal generators determined by the Hilbert bases (21.19) are truly independent. However, this will only occur for special configurations of (N_F, N_A, F_R) and, therefore, is considered as *non-generic* case.

21.4 Category $N_R = 0$

At first, we restrict to a matter content consisting of hypermultiplets transforming in the fundamental or adjoint representation of SU(3). From (21.5), we know that this choice includes several other representations as well.

21.4.1 N_F hypermultiplets in $[1,0]$ and N_A hypermultiplets in $[1,1]$

Intermediate step at U(3) The conformal dimension (21.4) reduces for $N_R = 0$ to the following:

$$\Delta(m_1,m_2,m_3) = \frac{N_F}{2}\sum_i |m_i| + (N_A-1)\sum_{i<j}|m_i-m_j| \,. \tag{21.32}$$

The Hilbert series is then readily computed

$$\mathrm{HS}_{\mathrm{U}(3)}^{[1,0]+[1,1]}(N_F,N_A,t,z) = \frac{R(N_F,N_A,t,z)}{P(N_F,N_A,t,z)} \,, \tag{21.33a}$$

$$\begin{aligned} P(N_F,N_A,t,z) = \prod_{j=1}^{3}&\left(1-t^{2j}\right)\left(1-\tfrac{1}{z}t^{4N_A+N_F-4}\right)\left(1-zt^{4N_A+N_F-4}\right) \\ &\times\left(1-\tfrac{1}{z^2}t^{4N_A+2N_F-4}\right)\left(1-z^2t^{4N_A+2N_F-4}\right)\left(1-\tfrac{1}{z^3}t^{3N_F}\right)\left(1-z^3t^{3N_F}\right) \end{aligned} \tag{21.33b}$$

$$\begin{aligned} R(N_F,N_A,t,z) = 1 &+ t^{8N_A+2N_F-2} - t^{8N_A+4N_F-8}(1+2t^2+2t^4) \\ &+ 2t^{8N_A+6N_F-8}(1-t^6) + t^{8N_A+8N_F-6}(2+2t^2+t^4) - t^{8N_A+10N_F-8} \\ &+ t^{16N_A+6N_F-10} - t^{16N_A+12N_F-10} - t^{6N_F} \\ + \left(z+\frac{1}{z}\right)&\Big(t^{4N_A+N_F-2}(1+t^2) + t^{4N_A+7N_F-4} - t^{4N_A+5N_F-4}(1+t^2+t^4) \\ &\quad - t^{8N_A+3N_F-6}(1+t^2) + t^{8N_A+9N_F-6}(1+t^2) - t^{12N_A+5N_F-6} \\ &\quad + t^{12N_A+7N_F-10}(1+t^2+t^4) - t^{12N_A+11N_F-10}(1+t^2)\Big) \\ + \left(z^2+\frac{1}{z^2}\right)&\Big(t^{4N_A+2N_F-2} + t^{4N_A+2N_F} - t^{4N_A+4N_F-4}(1+t^2+t^4) \\ &\quad + t^{4N_A+8N_F-4} - t^{12N_A+4N_F-6} \\ &\quad + t^{12N_A+8N_F-10}(1+t^2+t^4) - t^{12N_A+10N_F-10}(1+t^2)\Big) \\ + \left(z^3+\frac{1}{z^3}\right)&\Big(t^{8N_A+3N_F-2} - t^{8N_A+5N_F-6}(1+t^2+t^4) \\ &\quad + t^{8N_A+7N_F-8}(1+t^2+t^4) - t^{8N_A+9N_F-8}\Big) \,. \end{aligned} \tag{21.33c}$$

One can verify that $R(N_F,N_A,t=1,z) = 0$ and $\frac{\mathrm{d}^n}{\mathrm{d}t^n}R(N_F,N_A,t,z)|_{t=1,z=1} = 0$ for $n = 1,2$. Thus, the Hilbert series (21.33) has a pole of order 6, which matches the dimension of the moduli space. Moreover, one can compute the degree of the numerator (21.33c) to be $12N_F+16N_A-10$ and the degree of the denominator (21.33b) to be $12N_F+16N_A-4$, such that their difference equals the dimension of the moduli space. The interpretation follows the results (21.12) obtained from the Hilbert bases and we summarise the minimal generators in Tab. 21.1.

Reduction to SU(3) Following the prescription (21.6), we derive the following Hilbert series:

$$\mathrm{HS}_{\mathrm{SU}(3)}^{[1,0]+[1,1]}(N_F,N_A,t) = \frac{R(N_F,N_A,t)}{(1-t^4)(1-t^6)(1-t^{8N_A+2N_F-8})(1-t^{12N_A+4N_F-12})} \,, \tag{21.34a}$$

$$R(N_F,N_A,t) = 1 + t^{8N_A+2N_F-6}(2+2t^2+t^4) \tag{21.34b}$$

(m_1, m_2, m_3)		$2\Delta(m_1, m_2, m_3)$	$\mathrm{H}_{(m_1,m_2,m_3)}$
$(1,0,0)$	$(0,0,-1)$	$N_F + 4N_A - 4$	$\mathrm{U}(1) \times \mathrm{U}(2)$
$(1,1,0)$	$(0,-1,-1)$	$2N_F + 4N_A - 4$	$\mathrm{U}(1) \times \mathrm{U}(2)$
$(1,1,1)$	$(-1,-1,-1)$	$3N_F$	$\mathrm{U}(3)$

Table 21.1: *The monopole generators for a* U(3) *gauge theory with* $N_R = 0$ *that together with the Casimir invariants generate the chiral ring.*

$$+ t^{12N_A+4N_F-12}(1 + 2t^2 + 2t^4) + t^{20N_A+6N_F-14} .$$

An inspection yields that the numerator (21.34b) is a palindromic polynomial of degree $20N_A + 6N_F - 14$; while the degree of the denominator is $20N_A + 6N_F - 10$. Thus, the difference in the degrees is 4, which equals the complex dimension of the moduli space. In addition, the Hilbert series (21.34) has a pole of order four at $t \to 1$, which agrees with the dimension of Coulomb branch as well.

The minimal generators of (21.17) are given by $V^{\mathrm{dress},(0,0)}_{(1,0)}$ with $2\Delta(1,0) = 8N_A + 2N_F - 8$, and $V^{\mathrm{dress},0}_{(1,1)}$ and $V^{\mathrm{dress},0}_{(2,-1)}$ with $2\Delta(2,-1) = 2\Delta(1,1) = 12N_A + 4N_F - 12$. The dressed monopole operators are as described in Sec. 21.3.2.

21.4.2 N hypermultiplets in $[1,0]$

Considering N hypermultiplets in the fundamental representation is on extreme case of (21.4), as $N_A = 0 = N_R$. We recall the results of [191] and discuss them in the context of Hilbert bases for semi-groups.

Intermediate step at U(3) The Hilbert series has been computed to read

$$\mathrm{HS}^{[1,0]}_{\mathrm{U}(3)}(N, t, z) = \prod_{j=1}^{3} \frac{1 - t^{2N+2-2j}}{(1 - t^{2j})(1 - z t^{N+2-2j})(1 - \frac{t^{N+2-2j}}{z})} . \tag{21.35}$$

Notably, it is a complete intersection in which the (bare and dressed) monopole operators of GNO-charge $(1,0,0)$ and $(0,0,-1)$ generate all other monopole operators. The expected minimal generators $(1,1,0)$, $(0,-1,-1)$, $(1,1,1)$, and $(-1,-1,-1)$ are now generated because

$$V^{\mathrm{dress},0}_{(1,1,0)} = V^{\mathrm{dress},1}_{(1,0,0)} + V^{\mathrm{dress},1}_{(0,1,0)} , \tag{21.36a}$$

$$V^{\mathrm{dress},0}_{(1,1,0)} = V^{\mathrm{dress},2}_{(1,0,0)} + V^{\mathrm{dress},2}_{(0,1,0)} + V^{\mathrm{dress},2}_{(0,0,1)} . \tag{21.36b}$$

Reduction to SU(3) The reduction leads to

$$\mathrm{HS}^{[1,0]}_{\mathrm{SU}(3)}(N, t) = \frac{1 + t^{2N-6} + 2t^{2N-4} + t^{2N-2} + t^{4N-8}}{(1 - t^4)(1 - t^6)(1 - t^{2N-6})(1 - t^{2N-8})} . \tag{21.37}$$

Although the form of the Hilbert series (21.37) is suggestive: it has a pole of order 4 for $t \to 1$ and the numerator is palindromic, there is one drawback: no monopole operator of conformal dimension $(2N - 6)$ exists. Therefore, we provide a equivalent rational function to emphasis the minimal generators:

$$\mathrm{HS}^{[1,0]}_{\mathrm{SU}(3)}(N, t) = \frac{1 + t^{2N-6}(2 + 2t^2 + t^4) + t^{4N-12}(1 + 2t^2 + 2t^4) + t^{6N-14}}{(1 - t^4)(1 - t^6)(1 - t^{2N-8})(1 - t^{4N-12})} . \tag{21.38}$$

The equivalent form (21.38) still has a pole of order 4 and a palindromic numerator. Moreover, the monopole generators are clearly visible, as we know the set of minimal generators (21.17), and can be summarise for completeness: $2\Delta(1,0) = 2N-8$ and $2\Delta(1,1) = 2\Delta(2,-1) = 4N-12$.

21.4.3 N hypermultiplets in $[1,1]$

Investigating N hypermultiplets in the adjoint representation is another extreme case of (21.4) as $N_F = 0 = N_R$. The conformal dimension in this circumstance reduces to

$$\Delta(m_1, m_2, m_3) = (N-1) \sum_{i<j} |m_i - m_j| \,, \tag{21.39}$$

and we notice that there is the shift symmetry $m_i \to m_i + a$ present. Due to this, the naive calculation of the U(3) Hilbert series is divergent, which we understand as follows: Define overall U(1)-charge $M \coloneqq m_1 + m_2 + m_3$, then the Hilbert series becomes

$$\begin{aligned} \mathrm{HS}^{(1,1)}_{\mathrm{U}(3)} = \sum_{M\in\mathbb{Z}} \sum_{\substack{m_1,m_2 \\ m_1 \geq \max(m_2, M-2m_2)}} & t^{2(N-1)(3m_1+3m_2-2M+|m_1-m_2|)} \, z^M \\ & \times P_{\mathrm{U}(3)}(t, m_1, m_2, m_3) \,. \end{aligned} \tag{21.40}$$

Since we want to use the U(3)-calculation as an intermediate step to derive the SU(3)-case, the only meaningful choice to fix the shift-symmetry is $m_1 + m_2 + m_3 = 0$. But then

$$\mathrm{HS}^{(1,1)}_{\mathrm{U}(3),\mathrm{fixed}} = \sum_{\substack{m_1,m_2 \\ m_1 \geq \max(m_2, -2m_2)}} t^{2(N-1)(3m_1+3m_2+|m_1-m_2|)} \, P_{\mathrm{U}(3)}(t, m_1, m_2, -m_1-m_2) \tag{21.41}$$

and the transition to SU(3) is simply

$$\begin{aligned} \mathrm{HS}^{(1,1)}_{\mathrm{SU}(3)} &= (1-t^2) \int_{|z|=1} \frac{\mathrm{d}z}{2\pi z} \sum_{\substack{m_1,m_2 \\ m_1 \geq \max(m_2, -2m_2)}} t^{2(N-1)(3m_1+3m_2+|m_1-m_2|)} \\ &\qquad \times P_{\mathrm{U}(3)}(t, m_1, m_2, -m_1-m_2) \\ &= \sum_{\substack{m_1,m_2 \\ m_1 \geq \max(m_2, -2m_2)}} t^{2(N-1)(3m_1+3m_2+|m_1-m_2|)} \, P_{\mathrm{SU}(3)}(t, m_1, m_2) \,. \end{aligned} \tag{21.42}$$

The computation then yields

$$\mathrm{HS}^{(1,1)}_{\mathrm{SU}(3)} = \frac{1 + t^{8N-6}(2+2t^2+t^4) + t^{12N-12}(1+2t^2+2t^4) + t^{20N-14}}{(1-t^4)(1-t^6)(1-t^{8N-8})(1-t^{12N-12})} \,. \tag{21.43}$$

We see that numerator of (21.43) is a palindromic polynomial of degree $20N-14$; while the degree of the denominator is $20N-10$. Hence, the difference in the degrees is 4, which coincides with the complex dimension of the moduli space. The same holds for the order of the pole of (21.43) at $t \to 1$.

The interpretation of the appearing monopole operators, and their dressings, is completely analogous to (21.34) and reproduces the picture concluded from the Hilbert bases (21.12). To be specific, $2\Delta(1,0) = 8N-8$ and $2\Delta(1,1) = 2\Delta(2,-1) = 12N-12$.

21.4.4 N hypermultiplets in $[3,0]$

Intermediate step at U(3) The conformal dimension reads

$$\Delta(m_1,m_2,m_3) = \frac{3}{2}N\sum_i |m_i| + (N-1)\sum_{i<j}|m_i-m_j| \, . \tag{21.44}$$

We then obtain for $N>2$ the following Hilbert series:

$$\mathrm{HS}_{\mathrm{U}(3)}^{[3,0]}(t,z) = \frac{R(N,t,z)}{P(N,t,z)} \, , \tag{21.45a}$$

$$\begin{aligned} P(N,t,z) = \prod_{j=1}^{3}\left(1-t^{2j}\right)\left(1-\tfrac{1}{z}t^{7N-4}\right)\left(1-zt^{7N-4}\right)\left(1-\tfrac{1}{z^2}t^{10N-4}\right)\left(1-z^2t^{10N-4}\right) \\ \times\left(1-\tfrac{1}{z^3}t^{9N}\right)\left(1-z^3t^{9N}\right) \, , \end{aligned} \tag{21.45b}$$

$$\begin{aligned} R(N,t,z) = 1 &+ t^{14N-2} - t^{18N} - t^{20N-8} - 2t^{20N-6} - 2t^{20N-4} + 2t^{26N-8} - 2t^{26N-2} \\ &+ 2t^{32N-6} + 2t^{32N-4} + t^{32N-2} + t^{34N-10} - t^{38N-8} - t^{52N-10} \\ &+ (z+\tfrac{1}{z})\Big(t^{7N-2} + t^{7N} - t^{17N-6} - t^{17N-4} - t^{19N-4} - t^{19N-2} - t^{19N} + t^{25N-4} \\ &\quad - t^{27N-6} + t^{33N-10} + t^{33N-8} + t^{33N-6} + t^{35N-6} + t^{35N-4} - t^{45N-10} - t^{45N-8}\Big) \\ &+ (z^2+\tfrac{1}{z^2})\Big(t^{10N-2} + t^{10N} - t^{16N-4} - t^{16N-2} - t^{16N} - t^{24N-6} + t^{28N-4} + t^{36N-10} \\ &\quad + t^{36N-8} + t^{36N-6} - t^{42N-10} - t^{42N-8}\Big) \\ &+ (z^3+\tfrac{1}{z^3})\Big(t^{17N-2} - t^{23N-6} - t^{23N-4} - t^{23N-2} + t^{29N-8} + t^{29N-6} \\ &\quad + t^{29N-4} - t^{35N-8}\Big) \, . \end{aligned} \tag{21.45c}$$

The Hilbert series (21.45) has a pole of order 6 as $t\to 1$, because $R(N,t=1,z)=0$ and $\frac{\mathrm{d}^n}{\mathrm{d}t^n}R(N,t,z)|_{t=1}=0$ for $n=1,2$. Therefore, the moduli space is 6-dimensional. Also, the degree of (21.45c) is $52N-10$, while the degree of (21.45b) us $52N-4$; thus, the difference in degrees equals the dimension of the moduli space.

As this example is merely a special case of (21.33), we just summarise the minimal generators in Tab. 21.2.

(m_1,m_2,m_3)		$2\Delta(m_1,m_2,m_3)$	$\mathrm{H}_{(m_1,m_2,m_3)}$
$(1,0,0)$	$(0,0,-1)$	$7N-4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(1,1,0)$	$(0,-1,-1)$	$10N-4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(1,1,1)$	$(-1,-1,-1)$	$9N$	$\mathrm{U}(3)$

Table 21.2: *The monopole generators for a* U(3) *gauge theory with matter transforming in* $[3,0]$ *that together with the Casimir invariants generate the chiral ring.*

Reduction to SU(3) Following the recipe (21.6), we obtain for the Hilbert series the following rational function:

$$\mathrm{HS}_{\mathrm{SU}(3)}^{[3,0]}(t) = \frac{1+t^{14N-6}(2+2t^2+t^4)+t^{24N-12}(1+2t^2+2t^4)+t^{38N-14}}{(1-t^4)\,(1-t^6)\,(1-t^{14N-8})\,(1-t^{24N-12})} \tag{21.46}$$

It is apparent that the numerator of (21.46) is a palindromic polynomial of degree $38N - 14$; while the degree of the denominator is $38N - 10$; hence, the difference in the degrees is 4, which equals the complex dimension of the moduli space.

The structure of (21.46) is merely a special case of (21.34), and the conformal dimensions of the minimal generators are $2\Delta(1,0) = 14N - 8$ and $2\Delta(1,1) = 2\Delta(2,-1) = 24N - 12$.

21.5 Category $N_R \neq 0$

In this section, we allow an non-vanishing number of hypermultiplets transforming in the representation $[2,1]$. As outlined in Sec. 21.2, the fan becomes a more sophisticated and we expect several new monopole operators to appear as minimal generators.

21.5.1 N_F hypermultiplets in $[2,1]$, N_A hypermultiplets in $[1,1]$, and N_R hypermultiplets in $[2,1]$

Intermediate step at U(3) The conformal dimension reads

$$2\Delta(m_1,m_2,m_3) = (4N_R + N_A)\sum_{i=1}^{3}|m_i| + N_R\sum_{i<j}\left(|2m_i - m_j| + |m_i - 2m_j|\right) + 2(N_A - 1)\sum_{i<j}|m_i - m_j| \ . \tag{21.47}$$

The Hilbert series reads

$$\mathrm{HS}_{\mathrm{U}(3)}^{[1,0]+[1,1]+[2,1]}(t,z) = \frac{R(N_F,N_A,N_R,t,z)}{P(N_F,N_A,N_R,t,z)} \ , \tag{21.48a}$$

$$\begin{aligned} P(N_F,N_A,N_R,t,z) = \prod_{j=1}^{3}\left(1-t^{2j}\right)&\left(1-\frac{t^{N_F+4N_A+10N_R-4}}{z}\right)\left(1-zt^{N_F+4N_A+10N_R-4}\right) \\ &\times\left(1-\frac{t^{2N_F+4N_A+16N_R-4}}{z^2}\right)\left(1-z^2t^{2N_F+4N_A+16N_R-4}\right) \\ &\times\left(1-\frac{t^{3N_F+18N_R}}{z^3}\right)\left(1-z^3t^{3N_F+18N_R}\right) \\ &\times\left(1-\frac{t^{3N_F+8N_A+24N_R-8}}{z^3}\right)\left(1-z^3t^{3N_F+8N_A+24N_R-8}\right) \\ &\times\left(1-\frac{t^{4N_F+4N_A+24N_R-4}}{z^4}\right)\left(1-z^4t^{4N_F+4N_A+24N_R-4}\right) \\ &\times\left(1-\frac{t^{5N_F+4N_A+30N_R-4}}{z^5}\right)\left(1-z^5t^{5N_F+4N_A+30N_R-4}\right) \\ &\times\left(1-\frac{t^{7N_F+12N_A+46N_R-12}}{z^7}\right)\left(1-z^7t^{7N_F+12N_A+46N_R-12}\right) , \end{aligned} \tag{21.48b}$$

and the numerator $R(N_F,N_A,N_R,t,z)$ is too long to be displayed, because it contains 28650 monomials. One can verified explicitly that $R(N_F,N_A,N_R,t=1,z)=0$ and, moreover, $\frac{\mathrm{d}^n}{\mathrm{d}t^n}R(N_F,N_A,N_R,t,z)|_{t=1,z=1} = 0$ for all $n = 1,2\ldots,10$. Therefore, the Hilbert series (21.48) has a pole of order 6 at $t=1$, which equals the dimension of the moduli space. In addition, $R(N_F,N_A,N_R,t,z)$ is a polynomial of degree $50N_F + 72N_A + 336N_R - 66$, while the denominator (21.48b) is of degree $50N_F + 72N_A + 336N_R - 60$. The difference in degrees reflects the dimension of the moduli space as well.

Following the analysis of the Hilbert bases (21.19), we identify the bare monopole operators and provide their conformal dimensions in Tab. 21.3. The result (21.48) has been tested against

(m_1, m_2, m_3)		$2\Delta(m_1, m_2, m_3)$	$\mathrm{H}_{(m_1,m_2,m_3)}$
$(1,0,0)$	$(0,0,-1)$	$N_F + 4N_A + 10N_R - 4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(1,1,0)$	$(0,-1,-1)$	$2N_F + 4N_A + 16N_R - 4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(1,1,1)$	$(-1,-1,-1)$	$3N_F + 18N_R$	$\mathrm{U}(3)$
$(2,1,0)$	$(0,-1,-2)$	$3N_F + 8N_A + 24N_R - 8$	$\mathrm{U}(1)^3$
$(2,1,1)$	$(-1,-1,-2)$	$4N_F + 4N_A + 24N_R - 4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(2,2,1)$	$(-1,-2,-2)$	$5N_F + 4N_A + 30N_R - 4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(3,2,1)$	$(-1,-2,-3)$	$6N_F + 8N_A + 38N_R - 8$	$\mathrm{U}(1)^3$
$(4,2,1)$	$(-1,-2,-4)$	$7N_F + 12N_A + 46N_R - 12$	$\mathrm{U}(1)^3$

Table 21.3: *The monopole generators for a* U(3) *gauge theory with a mixture of matter transforming in* $[1,0]$*,* $[1,1]$*, and* $[2,1]$*.*

the independent calculations of the cases: N hypermultiplets in $[1,0]$; N_F hypermultiplets in $[1,0]$ together with N_A hypermultiplets in $[1,1]$; and N hypermultiplets in $[2,1]$. All the calculations agree.

Reduction to SU(3) The Hilbert series for the SU(3) theory reads

$$\mathrm{HS}_{\mathrm{SU}(3)}^{[1,0]+[1,1]+[2,1]}(N_F, N_A, N_R, t) = \frac{R(N_F, N_A, N_R, t)}{P(N_F, N_A, N_R, t)}\,, \tag{21.49a}$$

$$\begin{aligned} P(N_F, N_A, N_R, t) &= \left(1-t^4\right)\left(1-t^6\right)\left(1-t^{2N_F+8N_A+20N_R-8}\right) \\ &\quad \times \left(1-t^{4N_F+12N_A+36N_R-12}\right)\left(1-t^{6N_F+20N_A+54N_R-20}\right)\,, \end{aligned} \tag{21.49b}$$

$$\begin{aligned} R(N_F, N_A, N_R, t) &= 1 + t^{2N_F+8N_A+20N_R-6}(2+2t^2+t^4) \\ &\quad + t^{4N_F+12N_A+36N_R-12}(1+2t^2+2t^4) \\ &\quad + t^{6N_F+20N_A+54N_R-20}(1+4t^2+4t^4+2t^6) \\ &\quad - t^{6N_F+20N_A+56N_R-20}(2+4t^2+4t^4+t^6) \\ &\quad - t^{8N_F+28N_A+74N_R-26}(2+2t^2+t^4) \\ &\quad - t^{10N_F+32N_A+90N_R-32}(1+2t^2+2t^4) - t^{12N_F+40N_A+110N_R-34}\,. \end{aligned} \tag{21.49c}$$

Again, the numerator (21.49c) is an anti-palindromic polynomial of degree $12N_F + 40N_A + 110N_R - 34$; while the denominator (21.49b) is of degree $12N_F + 40N_A + 110N_R - 30$, such that the difference is again 4.

The minimal generators from (21.19) are now realised with the following conformal dimensions: $2\Delta(1,0) = 2N_F + 8N_A + 20N_R - 8$, $2\Delta(1,1) = 2\Delta(2,-1) = 4N_F + 12N_A + 36N_R - 12$, and $2\Delta(2,1) = 2\Delta(3,-1) = 6N_F + 20N_A + 54N_R - 20$. Moreover, the appearing dressed monopoles are as described in Sec. 21.3.2.

Remark The SU(3) result (21.49) has been tested against the independent calculations of the cases: N hypermultiplets in $[1,0]$; N hypermultiplets in $[1,1]$; N_F hypermultiplets in $[1,0]$ together with N_A hypermultiplets in $[1,1]$; and N hypermultiplets in $[2,1]$. All the calculations agree.

Dressings of $(2,1)$ **and** $(3,-1)$ From the generic analysis (21.19) the bare monopoles of GNO-charges $(3,-1)$ and $(2,1)$ are necessary generators. However, not all of their dressings need to be independent generators, see for instance the degrees of the first relations in App. C.

- $N_R = 0$: $(2,1)$ and $(3,-1)$ are generated by $(1,0)$, $(1,1)$, and $(2,-1)$, which is the generic result of (21.17).
- $N_R = 1$: Here, $(2,1)$ and $(3,-1)$ are independent, but not all of their dressings, as we see

$$(2,1) = (1,1) + (1,0) \qquad \text{and} \qquad \Delta(2,1) + 1 = \Delta(1,1) + \Delta(1,0)\ . \tag{21.50}$$

  Hence, *only one* of the degree one dressings $V_{(2,1)}^{\mathrm{dress},(1,0)}$, $V_{(2,1)}^{\mathrm{dress},(0,1)}$ is independent, while the other can be generated. (Same holds for $(3,-1)$.)

- $N_R = 2$: Here, $(2,1)$ and $(3,-1)$ are independent, but not all of their dressings, as we see

$$(2,1) = (1,1) + (1,0) \qquad \text{and} \qquad \Delta(2,1) + 2 = \Delta(1,1) + \Delta(1,0)\ . \tag{21.51}$$

  Hence, *only one* of the degree two dressings $V_{(2,1)}^{\mathrm{dress},(2,0)}$, $V_{(2,1)}^{\mathrm{dress},(0,2)}$ is independent, while the other can be generated. However, *both* degree one dressings $V_{(2,1)}^{\mathrm{dress},(1,0)}$, $V_{(2,1)}^{\mathrm{dress},(0,1)}$ are independent. (Same holds for $(3,-1)$.)

- $N_R = 3$: Here, $(2,1)$ and $(3,-1)$ are independent, but still not all of their dressings, as we see

$$(2,1) = (1,1) + (1,0) \qquad \text{and} \qquad \Delta(2,1) + 3 = \Delta(1,1) + \Delta(1,0)\ . \tag{21.52}$$

  Hence, the degree three dressing $V_{(2,1)}^{\mathrm{dress},(3,0)+(0,3)}$ is not independent. However, *both* degree one dressings $V_{(2,1)}^{\mathrm{dress},(1,0)}$, $V_{(2,1)}^{\mathrm{dress},(0,1)}$ and *both* degree two dressings $V_{(2,1)}^{\mathrm{dress},(2,0)}$, $V_{(2,1)}^{\mathrm{dress},(0,2)}$ are independent. (Same holds for $(3,-1)$.)

- $N_R \geq 4$: The bare and the all dressed monopoles corresponding to $(2,1)$ and $(3,-1)$ are independent.

21.5.2 N hypermultiplets in $[2,1]$

Intermediate step at U(3) The conformal dimension reads

$$2\Delta(m_1,m_2,m_3) = 4N\sum_{i=1}^{3}|m_i| + N\sum_{i<j}\left(|2m_i - m_j| + |m_i - 2m_j|\right) - 2\sum_{i<j}|m_i - m_j|\ . \tag{21.53}$$

From the calculations we obtain the Hilbert series

$$\mathrm{HS}_{\mathrm{U}(3)}^{[2,1]}(N,t,z) = \frac{R(N,t,z)}{P(N,t,z)}\ , \tag{21.54a}$$

$$\begin{aligned} P(N,t,z) = \prod_{j=1}^{3}\left(1-t^{2j}\right)\left(1-\frac{t^{10N-4}}{z}\right)\left(1-zt^{10N-4}\right)\left(1-\frac{t^{16N-4}}{z^2}\right)\left(1-z^2t^{16N-4}\right) \\ \times\left(1-\frac{t^{18N}}{z^3}\right)\left(1-z^3t^{18N}\right)\left(1-\frac{t^{24N-8}}{z^3}\right)\left(1-z^3t^{24N-8}\right) \end{aligned} \tag{21.54b}$$

$$\times \left(1 - \frac{t^{24N-4}}{z^4}\right)\left(1 - z^4 t^{24N-4}\right)\left(1 - \frac{t^{30N-4}}{z^5}\right)\left(1 - z^5 t^{30N-4}\right)$$
$$\times \left(1 - \frac{t^{46N-12}}{z^7}\right)\left(1 - z^7 t^{46N-12}\right) ,$$

and the numerator $R(N,t,z)$ is with 13492 monomials too long to be displayed. Nevertheless, we checked explicitly that $R(N,t=1,z) = 0$ and $\frac{\mathrm{d}^n}{\mathrm{d}t^n} R(N,t,z)|_{t=1,z=1} = 0$ for all $n = 1, 2 \ldots, 10$. Therefore, the Hilbert series (21.54) has a pole of order 6 at $t = 1$, which equals the dimension of the moduli space. In addition, the degree of $R(N,t,z)$ is $296N - 62$, while the denominator (21.54b) is of degree $296N - 56$; therefore, the difference in degrees is again equal to the dimension of the moduli space.

The Hilbert series (21.54) appears as special case of (21.48) and as such the appearing monopole operators are the same. For completeness, we provide in Tab. 21.4 the conformal dimensions of all minimal (bare) generators (21.15). The GNO-charge $(3,2,1)$ is not apparent in

(m_1,m_2,m_3)		$2\Delta(m_1,m_2,m_3)$	$\mathrm{H}_{(m_1,m_2,m_3)}$
$(1,0,0)$	$(0,0,-1)$	$10N-4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(1,1,0)$	$(0,-1,-1)$	$16N-4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(1,1,1)$	$(-1,-1,-1)$	$18N$	$\mathrm{U}(3)$
$(2,1,0)$	$(0,-1,-2)$	$24N-8$	$\mathrm{U}(1)^3$
$(2,1,1)$	$(-1,-1,-2)$	$24N-4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(2,2,1)$	$(-1,-2,-2)$	$30N-4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(3,2,1)$	$(-1,-2,-3)$	$38N-8$	$\mathrm{U}(1)^3$
$(4,2,1)$	$(-1,-2,-4)$	$46N-12$	$\mathrm{U}(1)^3$

Table 21.4: *The monopole generators for a* U(3) *gauge theory with matter transforming in* $[2,1]$ *that generate the chiral ring (together with the Casimir invariants).*

the Hilbert series, but we know it to be present due to the analysis of the Hilbert bases (21.15).

Reduction to SU(3) After reduction (21.6) of (21.54) to SU(3) we obtain the following Hilbert series:

$$HS^{(2,1)}_{\mathrm{SU}(3)} = \frac{R(N,t)}{(1-t^4)(1-t^6)(1-t^{20N-8})(1-t^{36N-12})(1-t^{54N-20})} , \tag{21.55a}$$

$$R(N,t) = 1 + t^{20N-6}(2+2t^2+t^4) + t^{36N-12}(1+2t^2+2t^4) \tag{21.55b}$$
$$+ t^{54N-20}(1+4t^2+4t^4+2t^6) - t^{56N-20}(2+4t^2+4t^4+t^6)$$
$$- t^{74N-26}(2+2t^2+t^4) - t^{90N-32}(1+2t^2+2t^4) - t^{110N-34} .$$

The numerator of (21.55b) is an anti-palindromic polynomial of degree $110N - 34$; while the numerator is of degree $110N - 30$. Consequently, the difference in degree reflects the complex dimension of the moduli space.

The Hilbert series (21.55) is merely a special case of (21.49) and, thus, the appearing (bare and dressed) monopole operators are the same. For completeness we provide their conformal dimensions: $2\Delta(1,0) = 20N-8$, $2\Delta(1,1) = 2\Delta(2,-1) = 36N-12$, and $2\Delta(2,1) = 2\Delta(3,-1) = 54N - 20$.

22 Conclusions and outlook

In this last part we introduced a geometric concept to identify and compute the set of bare and dressed monopole operators that are sufficient to describe the entire chiral ring $\mathbb{C}[\mathcal{M}_C]$ of any 3-dimensional $\mathcal{N}=4$ gauge theory. The methods can be summarised as follows:

1. The matter content together with the positive roots of the gauge group G define the conformal dimension, which in turn defines an arrangement of hyperplanes that divides the dominant Weyl chamber of $\widehat{\mathrm{G}}$ into a fan.
2. The intersection of the fan with the weight lattice of the GNO-dual group leads to a collection of affine semi-groups. All semi-groups are finitely generated and the unique, finite generating set is called Hilbert basis.
3. The knowledge of the minimal generators, together with their properties $\mathrm{SU}(2)_R$-spin, residual gauge group H_m, and topological charges $J(m)$, is sufficient to explicitly sum and determine the Hilbert series as rational function.

Most importantly, the entire procedure works for any rank of the gauge group, as indicated in Ch. 21 for U(3). For the majority of this third part we, however, have chosen to provide a comprehensive collection of rank two examples.

Utilising the fan and the Hilbert bases for each semi-group also allows to deduce the dressing behaviour of monopole operators. The number of dressed operators is determined by a ratio of orders of Weyl groups, while the degrees are determined by the ratio of the dressing factors associated to the GNO-charge m divided by the dressing factor of the trivial monopole $m=0$.

With the advent of Hilbert bases for affine semi-groups, there are a various points we hope to address in the future. The subject of combinatorial commutative algebra, see for instance [195], provides a variety of plausible links to the monopole formula:

- To each affine semi-group S one can associate a *semi-group ring* $\mathbb{K}[S]$ over a field $\mathbb{K}$, which is isomorphic to a polynomial ring in as many variables as elements in the Hilbert basis of S module a so-called *lattice ideal*. The lattice ideal captures the relations among the minimal generators, which is a manifestation of the instance that the Hilbert basis is a generating set, but not necessarily a linearly independent set.
- The Hilbert series for a semi-group ring is in spirit cunningly similar to the monopole formula, i.e. one counts monomials associated to each point in S. Hence, we suspect that the contribution of each semi-group to the monopole formula can be understood as Hilbert series of a semi-group ring. Moreover, these semi-group rings are naturally multi-graded and it is reasonable to expect that the corresponding Hilbert series allows to include the counting of quantum numbers with respect to global symmetries.
- Building on semi-group rings, the form of the Hilbert series is dictated by the contribution of the Hilbert basis in the denominator and a polynomial with integer coefficients in the numerator, which is known as K-polynomial. The K-polynomial in turn is determined by the *free resolution* of the lattice ideal. Hence, we suspect that the free resolution provides all syzygies for the Hilbert basis of a semi-group, which might yield some insight in the relations on the Coulomb branch itself.

In addition, the monopole formula is a delicate mixture of two phenomena: (i) the classical dressing factors $P_{\mathrm{G}}(t,m)$, which we argued to be Poincaré series, and (ii) the Hilbert series for affine semi-groups. The dressing factors are sensitive to the bulk and boundary structure of the Weyl chamber of the GNO dual group; whereas, the Hilbert series for semi-groups are sensitive to the fan induced by the conformal dimension, i.e. by the matter content. Therefore, we hope to describe the monopole formula as a twisted product of the Poincaré series and the Hilbert series, which might improved geometric understanding of the Coulomb branch.

C Appendix: Plethystic logarithm

In this appendix we summarise the main properties of the plethystic logarithm. Starting with the definition, for a mulit-valued function $f(t_1, \ldots, t_m)$ with $f(0, \ldots, 0) = 1$, one defines

$$\mathrm{PL}[f] \coloneqq \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log\left(f(t_1^k, \ldots, t_m^k)\right) \,, \tag{C.1}$$

where $\mu(k)$ denotes the Möbius function [199]. Some basic properties include

$$\mathrm{PL}[f \cdot g] = \mathrm{PL}[f] + \mathrm{PL}[g] \qquad \text{and} \qquad \mathrm{PL}\left[\frac{1}{\prod_n (1-t^n)^{a_n}}\right] = \sum_n a_n \, t^n \,. \tag{C.2}$$

The plethystic logarithm is accompanied by the plethystic exponential which is defined by

$$\mathrm{PE}[f] \coloneqq \exp\left(\sum_{k=1}^{\infty} \frac{f(t_1^k, \ldots, t_m^k) - f(0, \ldots, 0)}{k}\right) . \tag{C.3}$$

It follows that the PE and PL are the inverse of one another. The plethystic exponential provides a means of symmetrisation of its argument. Given, for example, a freely generated, graded, commutative algebra $\boldsymbol{A}$ with algebraically independent generators $\{t_r\}$, then the PE of the function $\sum_r t_r$ equals

$$\mathrm{PE}\left[\sum_r t_r\right] = \prod_r \mathrm{PE}[t_r] = \frac{1}{\prod_r (1-t_r)} = 1 + \sum_r t_r + \sum_{r \leq s} t_r \, t_s + \ldots \,. \tag{C.4}$$

Thus, the PE provides the Hilbert series of $\boldsymbol{A}$ as it counts all possible symmetric polynomials in the generators. The PL, on the other hand, does the opposite and allows to study to what extend a given function is the symmetrisation of another. Hence, we employ the plethystic logarithm to identify generators and relations. For further details on the so-called *plethystic program* we refer to [199–202].

Now, we wish to compute the plethystic logarithm. Given a Hilbert series as rational function, i.e. of the form (15.28) or (15.35), the denominator can be taken care of by means of (C.2), while the numerator is a polynomial with integer coefficients. In order to obtain an approximation of the PL, we employ the following two equivalent transformations for the numerator:

$$\begin{aligned}
\mathrm{PL}\left[1 + at^n + \mathcal{O}(t^{n+1})\right] &= \mathrm{PL}\left[\frac{(1-t^n)^a \;\; (1 + at^n + \mathcal{O}(t^{n+1}))}{(1-t^n)^a}\right] \\
&= at^n + \mathrm{PL}\left[1 + \mathcal{O}(t^{n+1})\right] \,, \qquad \text{(C.5a)} \\
\mathrm{PL}\left[1 - at^n + \mathcal{O}(t^{n+1})\right] &= \mathrm{PL}\left[\frac{(1-t^n)^a \; (1+t^n)^a \;\; (1 - at^n + \mathcal{O}(t^{n+1}))}{(1-t^{2n})^a}\right] \\
&= -at^n + at^{2n} + \mathrm{PL}\left[1 + \mathcal{O}(t^{n+1})\right] \,. \qquad \text{(C.5b)}
\end{aligned}$$

Now, we derive an approximation of the PL for a generic rank two gauge group in terms of t^{Δ}.

More precisely, consider the Hilbert basis $\{X_i\}$ then we provide an approximation of the PL up to second order, i.e.

$$\mathrm{PL} = \text{Casimir inv.} + \left\{ t^{\Delta(X_i)}\text{-terms} \right\} + \left\{ t^{\Delta(X_i)+\Delta(X_j)}\text{-terms} \right\} + \mathcal{O}\left(t^{\Delta(X_i)+\Delta(X_j)+\Delta(X_k)} \right) \tag{C.6}$$

Considering (15.28), the numerator is denoted by $R(t)$, while the denominator $Q(t)$ is given by

$$Q(t) = \prod_{i=1}^{2} (1 - t^{d_i}) \prod_{p=0}^{L} \left(1 - t^{\Delta(x_p)} \right) \,, \tag{C.7}$$

with d_i the degrees of the Casimir invariants. Then expand the numerator as follows:

$$\begin{aligned} R(t) = 1 &+ \sum_{q=0}^{L} \left(\frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)} - 1 \right) t^{\Delta(x_q)} + \sum_{q=0}^{L} \sum_{s \in \mathrm{Int}(\mathcal{P}(C_q^{(2)}))} \frac{P_{\mathrm{G}}(t,s)}{P_{\mathrm{G}}(t,0)} t^{\Delta(s)} \\ &- \sum_{\substack{q,p=0 \\ q \neq p}}^{L} \left(\frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)} - \frac{1}{2} \right) t^{\Delta(x_p)+\Delta(x_q)} + \sum_{q=1}^{L} \frac{P_{\mathrm{G}}(t,C_q^{(2)})}{P_{\mathrm{G}}(t,0)} t^{\Delta(x_{q-1})+\Delta(x_q)} \\ &- \sum_{q=1}^{L} \sum_{s \in \mathrm{Int}(\mathcal{P}(C_q^{(2)}))} \sum_{\substack{r=0 \\ r \neq q-1,q}}^{L} \frac{P_{\mathrm{G}}(t,s)}{P_{\mathrm{G}}(t,0)} t^{\Delta(s)+\Delta(x_r)} \,. \end{aligned} \tag{C.8}$$

Note that the appearing factor $\frac{1}{2}$ avoids double counting when changing summation $\sum_{q<p}$ to $\sum_{q \neq p}$. Still, the numerator is a polynomial with integer coefficients. The PL then reads

$$\mathrm{PL}\left[\mathrm{HS}_{\mathrm{G}}(t)\right] = \sum_{i=1}^{2} t^{d_i} + \sum_{p=0}^{L} t^{\Delta(x_p)} + \mathrm{PL}\left[R(t)\right] \,. \tag{C.9}$$

By step (C.5a) we factor out the order $t^{\Delta(x_q)}$ and $t^{\Delta(s)}$ terms. However, this introduces further terms at order $t^{\Delta(x_q)+\Delta(s)}$ and so forth, which are given by

$$- \left(\sum_{q=0}^{L} \left(\frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)} - 1 \right) t^{\Delta(x_q)} + \sum_{q=1}^{L} \sum_{s \in \mathrm{Int}(\mathcal{P}(C_q^{(2)}))} \frac{P_{\mathrm{G}}(t,s)}{P_{\mathrm{G}}(t,0)} t^{\Delta(s)} \right)^2 \,. \tag{C.10}$$

Subsequently factoring the terms of this order by means of (C.5b), one derives at the following expressing of the PL

$$\begin{aligned} \mathrm{PL}\left[\mathrm{HS}_{\mathrm{G}}(t)\right] = \sum_{i=1}^{2} t^{d_i} &+ \sum_{q=0}^{L} \frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)} \, t^{\Delta(x_q)} + \sum_{q=1}^{L} \sum_{s \in \mathrm{Int}(\mathcal{P}(C_p^{(2)}))} \frac{P_{\mathrm{G}}(t,s)}{P_{\mathrm{G}}(t,0)} \, t^{\Delta(s)} \\ &- \sum_{\substack{q,p=0 \\ q \neq p}}^{L} \left(\frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)} - \frac{1}{2} \right) t^{\Delta(x_p)+\Delta(x_q)} + \sum_{q=1}^{L} \frac{P_{\mathrm{G}}(t,C_q^{(2)})}{P_{\mathrm{G}}(t,0)} t^{\Delta(x_{q-1})+\Delta(x_q)} \\ &- \sum_{q=1}^{L} \sum_{s \in \mathrm{Int}(\mathcal{P}(C_q^{(2)}))} \sum_{\substack{r=0 \\ r \neq q-1,q}}^{L} \frac{P_{\mathrm{G}}(t,s)}{P_{\mathrm{G}}(t,0)} t^{\Delta(s)+\Delta x_r} \end{aligned} \tag{C.11}$$

$$
\begin{aligned}
&- \sum_{q,p=0}^{L} \left(\frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)} - 1 \right) \left(\frac{P_{\mathrm{G}}(t,x_p)}{P_{\mathrm{G}}(t,0)} - 1 \right) t^{\Delta(x_q)+\Delta(x_p)} \\
&- 2 \sum_{p=0}^{L} \sum_{q=1}^{L} \left(\frac{P_{\mathrm{G}}(t,x_p)}{P_{\mathrm{G}}(t,0)} - 1 \right) \sum_{s\in \mathrm{Int}(\mathcal{P}(C_q^{(2)}))} \frac{P_{\mathrm{G}}(t,s)}{P_{\mathrm{G}}(t,0)} \, t^{\Delta(x_p)+\Delta(s)} \\
&- \sum_{q,p=1}^{L} \sum_{s\in \mathrm{Int}(\mathcal{P}(C_p^{(2)}))} \sum_{s'\in \mathrm{Int}(\mathcal{P}(C_q^{(2)}))} \frac{P_{\mathrm{G}}(t,s)}{P_{\mathrm{G}}(t,0)} \frac{P_{\mathrm{G}}(t,s')}{P_{\mathrm{G}}(t,0)} t^{\Delta(s)+\Delta(s')} \\
&+ \mathrm{PL} \left[1 + \mathcal{O}\left(t^{\Delta(X_i)+\Delta(X_j)+\Delta(X_j)} \right) \right] \, .
\end{aligned}
$$

Strictly speaking, the truncation (C.11) is only meaningful if

$$
\begin{aligned}
&\max\{\Delta(X)\} + \max\{d_i | i=1,2\} < \min\{\Delta(X)+\Delta(Y)\} = 2\cdot \min\{\Delta(X)\} \\
&\quad \text{for} \quad X,Y = x_q \text{ or } s \, , s \in \mathrm{Int}(\mathcal{P}(C_p^{(2)})), q=0,1,\ldots,l
\end{aligned}
\tag{C.12}
$$

holds. Only in this case do the positive contributions, i.e. the generators, of the first line in (C.11) not mix with the negative contributions, i.e. first syzygies or relations, of the remaining lines. Moreover, the condition (C.12) ensures that the remained $\mathcal{O}\left(t^{\Delta(X_i)+\Delta(X_j)+\Delta(X_k)}\right)$ does not spoil the truncation.

From the examples of Ch. 16-21, we see that (C.12) is at most satisfied for scenarios with just a few generators, but not for elaborate cases. Nevertheless, there are some observations we summarise as follows:

- The bare and dressed monopole operators associated to the GNO-charge m are described by $\frac{P_{\mathrm{G}}(t,m)}{P_{\mathrm{G}}(t,0)} t^{\Delta(m)}$. In particular, we emphasis that the quotient of dressing factors provides information on the number and degrees of the dressed monopole operators.
- The previous observation provides an upper bound on the number of dressed monopole operators associated to a magnetic weight m. In detail, the value of $\frac{P_{\mathrm{G}}(t,m)}{P_{\mathrm{G}}(t,0)}$ at $t=1$ equals the number of bare and dressed monopole operators associated to m. Let $\{d_i\}$ and $\{b_i\}$, for $i=1,\ldots,\mathrm{rk}(\mathrm{G})$ denote the degree of the Casimir invariants for G and H_m, respectively. Then

$$
\begin{aligned}
\begin{matrix} \#\text{ dressed monopoles} \\ +1 \text{ bare monopole} \end{matrix} &= \lim_{t\to 1} \frac{P_{\mathrm{G}}(t,m)}{P_{\mathrm{G}}(t,0)} = \lim_{t\to 1} \frac{\prod_{i=1}^{\mathrm{rk}(\mathrm{G})} \left(1-t^{d_i}\right)}{\prod_{j=1}^{\mathrm{rk}(\mathrm{G})} \left(1-t^{b_j}\right)} \\
&= \frac{\prod_{i=1}^{\mathrm{rk}(\mathrm{G})} d_i}{\prod_{j=1}^{\mathrm{rk}(\mathrm{G})} b_j} = \frac{|\mathcal{W}_{\mathrm{G}}|}{|\mathcal{W}_{\mathrm{H}_m}|} \, ,
\end{aligned}
\tag{C.13}
$$

  where the last equality holds because the order of the Weyl group equals the product of the degrees of the Casimir invariants. Since $\mathcal{W}_{\mathrm{H}_m} \subset \mathcal{W}_{\mathrm{G}}$ is a subgroup of the finite group $\mathcal{W}_{\mathrm{G}}$, Lagrange's theorem implies that $\frac{|\mathcal{W}_{\mathrm{G}}|}{|\mathcal{W}_{\mathrm{H}_m}|} \in \mathbb{N}$ holds.

  The situation becomes obvious whenever m belongs to the interior of the Weyl chamber, because $\mathrm{H}_m = \mathrm{T}$ and thus

$$
\left. \begin{matrix} \#\text{ dressed monopoles} \\ +1 \text{ bare monopole} \end{matrix} \right|_{\substack{\text{interior of} \\ \text{Weyl chamber}}} = |\mathcal{W}_{\mathrm{G}}| \qquad \text{and} \qquad \frac{P_{\mathrm{G}}(t,m)}{P_{\mathrm{G}}(t,0)} = \prod_{i=1}^{\mathrm{rk}(\mathrm{G})} \sum_{l_i=0}^{d_i-1} t^{l_i} \, . \tag{C.14}
$$

- The significance of the PL is limited, as, for instance, a positive contribution $\sim t^{\Delta(X_1)}$ can coincide with a negative contribution $\sim t^{\Delta(X_2)+\Delta(X_3)}$, but this does not necessarily imply that the object of degree $\Delta(X_1)$ can be generated by others. The situation becomes clearer if there exists an additional global symmetry $Z(\widehat{\mathrm{G}})$ on the moduli space. The truncated PL for (15.35) is obtained from (C.11) by the replacement

$$t^{\Delta(X)} \mapsto \vec{z}^{\vec{J}(X)}\ t^{\Delta(X)}\ . \tag{C.15}$$

Then the *syzygy* $\vec{z}^{\vec{J}(X_2+X_3)} t^{\Delta(X_2)+\Delta(X_3)}$ can cancel the *generator* $\vec{z}^{\vec{J}(X_1)} t^{\Delta(X_1)}$ only if the symmetry charges agree $\vec{z}^{\vec{J}(X_1)} = \vec{z}^{\vec{J}(X_2+X_3)}$, in addition to the $\mathrm{SU}(2)_R$ iso-spin.

Lastly, we illustrate the truncation with the two simplest examples:

Example: one simplicial cone For the Hilbert series (15.32) we obtain

$$\begin{aligned}\mathrm{PL} = \sum_{i=1}^{2} t^{d_i} + \frac{P_1(t)}{P_0(t)}\left(t^{\Delta(x_0)} + t^{\Delta(x_1)}\right) - \left(2\frac{P_1(t)}{P_0(t)} - 1 - \frac{P_2(t)}{P_0(t)}\right) t^{\Delta(x_0)+\Delta(x_1)} \\ - \left(\frac{P_1(t)}{P_0(t)}\right)^2 \left(t^{2\Delta(x_0)} + t^{2\Delta(x_1)} + 2t^{\Delta(x_0)+\Delta(x_1)}\right) + \dots\ .\end{aligned} \tag{C.16}$$

Example: one non-simplicial cone In contrast, for the Hilbert series (15.33) we arrive at

$$\begin{aligned}\mathrm{PL} = \sum_{i=1}^{2} t^{d_i} + \frac{P_1(t)}{P_0(t)}\left(t^{\Delta(x_0)} + t^{\Delta(x_1)}\right) + \sum_{s\in\mathrm{Int}\mathcal{P}} \frac{P_2(t)}{P_0(t)} t^{\Delta(s)} \\ - \left(2\frac{P_1(t)}{P_0(t)} - 1 - \frac{P_2(t)}{P_0(t)}\right) t^{\Delta(x_0)+\Delta(x_1)} \\ - \left(\frac{P_1(t)}{P_0(t)}\right)^2 \left(t^{2\Delta(x_0)} + t^{2\Delta(x_1)} + 2t^{\Delta(x_0)+\Delta(x_1)}\right) \\ - 2\left(\frac{P_1(t)}{P_0(t)} - 1\right)\frac{P_2(t)}{P_0(t)} \sum_{s\in\mathrm{Int}\mathcal{P}} \left(t^{\Delta(s)+\Delta(x_0)} + t^{\Delta(s)+\Delta(x_1)}\right) \\ - \sum_{s\in\mathrm{Int}\mathcal{P}} \sum_{s'\in\mathrm{Int}\mathcal{P}} \left(\frac{P_2(t)}{P_0(t)}\right)^2 t^{\Delta(s)+\Delta(s')} + \dots\ .\end{aligned} \tag{C.17}$$

Bibliography

[1] ATLAS collaboration, G. Aad et al., *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, *Phys. Lett. B* **716** (2012) 1–29, [1207.7214].

[2] CMS collaboration, S. Chatrchyan et al., *Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC*, *Phys. Lett. B* **716** (2012) 30–61, [1207.7235].

[3] Virgo, LIGO Scientific collaboration, B. P. Abbott et al., *Observation of Gravitational Waves from a Binary Black Hole Merger*, *Phys. Rev. Lett.* **116** (2016) 061102, [1602.03837].

[4] D. Rickles, *A Brief History of String Theory - From Dual Models to M-Theory.* The Frontiers Collection. Springer Berlin Heidelberg, 2014.

[5] P. Ramond, *Dual Theory for Free Fermions*, *Phys. Rev. D* **3** (1971) 2415–2418.

[6] J. Scherk and J. H. Schwarz, *Dual Models for Nonhadrons*, *Nucl. Phys. B* **81** (1974) 118–144.

[7] M. B. Green and J. H. Schwarz, *Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory*, *Phys. Lett. B* **149** (1984) 117–122.

[8] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *The Heterotic String*, *Phys. Rev. Lett.* **54** (1985) 502–505.

[9] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *Heterotic String Theory. 1. The Free Heterotic String*, *Nucl. Phys. B* **256** (1985) 253.

[10] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *Heterotic String Theory. 2. The Interacting Heterotic String*, *Nucl. Phys. B* **267** (1986) 75.

[11] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys. B* **443** (1995) 85–126, [hep-th/9503124].

[12] J. Polchinski, *Dirichlet Branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724–4727, [hep-th/9510017].

[13] S. H. Katz, A. Klemm and C. Vafa, *Geometric engineering of quantum field theories*, *Nucl. Phys. B* **497** (1997) 173–195, [hep-th/9609239].

[14] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics*, *Nucl. Phys. B* **492** (1997) 152–190, [hep-th/9611230].

[15] M. R. Douglas and G. W. Moore, *D-branes, quivers, and ALE instantons*, hep-th/9603167.

[16] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113–1133, [hep-th/9711200].

[17] L. J. Dixon, *Some World Sheet Properties of Superstring Compactificaions, on Orbifolds and Otherwise*, in *Summer Workshop in High-energy Physics and Cosmology Trieste, Italy, June 29-August 7, 1987*, 1987.

[18] W. Lerche, C. Vafa and N. P. Warner, *Chiral Rings in N=2 Superconformal Theories*, *Nucl. Phys. B* **324** (1989) 427.

[19] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, *Strings on Orbifolds*, *Nucl. Phys. B* **261** (1985) 678–686.

[20] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, *Strings on Orbifolds. 2.*, *Nucl. Phys. B* **274** (1986) 285–314.

[21] D. Tong, *TASI lectures on solitons: Instantons, monopoles, vortices and kinks*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Many Dimensions of String Theory (TASI 2005) Boulder, Colorado, June 5-July 1, 2005*, 2005. `hep-th/0509216`.

[22] A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Yu. S. Tyupkin, *Pseudoparticle Solutions of the Yang-Mills Equations*, *Phys. Lett. B* **59** (1975) 85–87.

[23] G. 't Hooft, *Computation of the Quantum Effects Due to a Four-Dimensional Pseudoparticle*, *Phys. Rev. D* **14** (1976) 3432–3450.

[24] S. K. Donaldson, *Self-dual connections and the topology of smooth* 4*-manifolds*, *Bull. Amer. Math. Soc. (N.S.)* **8** (1983) 81–83.

[25] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Yu. I. Manin, *Construction of Instantons*, *Phys. Lett. A* **65** (1978) 185–187.

[26] E. Witten, *Small instantons in string theory*, *Nucl. Phys. B* **460** (1996) 541–559, [`hep-th/9511030`].

[27] M. R. Douglas, *Branes within branes*, in *Strings, branes and dualities. Proceedings, NATO Advanced Study Institute, Cargese, France, May 26-June 14, 1997*, 1995. `hep-th/9512077`.

[28] S. Benvenuti, A. Hanany and N. Mekareeya, *The Hilbert Series of the One Instanton Moduli Space*, *JHEP* **06** (2010) 100, [`1005.3026`].

[29] A. Hanany, N. Mekareeya and S. S. Razamat, *Hilbert Series for Moduli Spaces of Two Instantons*, *JHEP* **01** (2013) 070, [`1205.4741`].

[30] A. Dey, A. Hanany, N. Mekareeya, D. Rodríguez-Gómez and R.-K. Seong, *Hilbert Series for Moduli Spaces of Instantons on* $\mathbb{C}^2/\mathbb{Z}_n$, *JHEP* **01** (2014) 182, [`1309.0812`].

[31] S. Cremonesi, G. Ferlito, A. Hanany and N. Mekareeya, *Coulomb Branch and The Moduli Space of Instantons*, *JHEP* **12** (2014) 103, [`1408.6835`].

[32] N. Mekareeya, *The moduli space of instantons on an ALE space from 3d* $\mathcal{N} = 4$ *field theories*, *JHEP* **12** (2015) 174, [`1508.06813`].

[33] A. Hanany and R. Kalveks, *Construction and Deconstruction of Single Instanton Hilbert Series*, *JHEP* **12** (2015) 118, [`1509.01294`].

[34] E. Corrigan, C. Devchand, D. B. Fairlie and J. Nuyts, *First Order Equations for Gauge Fields in Spaces of Dimension Greater Than Four*, *Nucl. Phys. B* **214** (1983) 452–464.

[35] S. K. Donaldson, *Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, *Proc. London Math. Soc. (3)* **50** (1985) 1–26.

[36] S. K. Donaldson, *Infinite determinants, stable bundles and curvature*, *Duke Math. J.* **54** (1987) 231–247.

[37] K. Uhlenbeck and S. T. Yau, *On the existence of hermitian-Yang-Mills connections in stable vector bundles*, *Comm. Pure Appl. Math.* **39** (1986) S257–S293.

[38] W. Nahm, *A simple formalism for the bps monopole*, *Phys. Lett. B* **90** (1980) 413 – 414.

[39] N. J. Hitchin, *On the construction of monopoles*, *Comm. Math. Phys.* **89** (1983) 145–190.

[40] S. K. Donaldson, *Nahm's equations and the classification of monopoles*, *Comm. Math. Phys.* **96** (1984) 387–407.

[41] J. Hurtubise, *The classification of monopoles for the classical groups*, *Comm. Math. Phys.* **120** (1989) 613–641.

[42] G. 't Hooft, *Magnetic Monopoles in Unified Gauge Theories*, *Nucl. Phys. B* **79** (1974) 276–284.

[43] A. M. Polyakov, *Particle Spectrum in the Quantum Field Theory*, *JETP Lett.* **20** (1974) 194–195.

[44] P. A. M. Dirac, *Quantized Singularities in the Electromagnetic Field*, *Proc. Roy. Soc. Lond. A* **133** (1931) 60–72.

[45] D.-E. Diaconescu, *D-branes, monopoles and Nahm equations*, *Nucl. Phys. B* **503** (1997) 220–238, [`hep-th/9608163`].

[46] D. Tsimpis, *Nahm equations and boundary conditions*, *Phys. Lett. B* **433** (1998) 287–290, [`hep-th/9804081`].

[47] V. Borokhov, A. Kapustin and X.-k. Wu, *Topological disorder operators in three-dimensional conformal field theory*, *JHEP* **11** (2002) 049, [`hep-th/0206054`].

[48] V. Borokhov, A. Kapustin and X.-k. Wu, *Monopole operators and mirror symmetry in three-dimensions*, *JHEP* **12** (2002) 044, [`hep-th/0207074`].

[49] V. L. Ginzburg and L. D. Landau, *On the Theory of superconductivity*, *Zh. Eksp. Teor. Fiz.* **20** (1950) 1064–1082.

[50] A. M. Jaffe and C. H. Taubes, *Vortices and Monopoles. Structure of Static Gauge Theories.* Birkhäuser, 1980.

[51] O. García-Prada, *Invariant connections and vortices*, *Comm. Math. Phys.* **156** (1993) 527–546.

[52] O. García-Prada, *Dimensional reduction of stable bundles, vortices and stable pairs*, *Int. J. Math.* **5** (1994) 1–52.

[53] L. Álvarez-Cónsul and O. García-Prada, *Dimensional reduction, SL(2,C) equivariant bundles and stable holomorphic chains*, *Int. J. Math.* **12** (2001) 159–201, [`math/0112159`].

[54] L. Álvarez-Cónsul and O. García-Prada, *Dimensional reduction and quiver bundles*, *J. Reine Angew. Math.* **556** (2003) 1–46, [`math/0112160`].

[55] L. Álvarez-Cónsul and O. García-Prada, *Hitchin-Kobayashi correspondence, quivers, and vortices*, *Comm. Math. Phys.* **238** (2003) 1–33.

[56] E. Witten, *Monopoles and four manifolds*, *Math. Res. Lett.* **1** (1994) 769–796, [hep-th/9411102].

[57] R. Rajaraman, *Solitons and instantons. An introduction to solitons and instantons in quantum field theory.* North-Holland, Amsterdam, 1982.

[58] N. Manton and P. Sutcliffe, *Topological solitons.* Cambridge University Press, Cambridge, 2004.

[59] E. J. Weinberg, *Classical solutions in quantum field theory.* Cambridge University Press, Cambridge, 2012.

[60] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge University Press, Cambridge, 1998.

[61] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond.* Cambridge University Press, Cambridge, 1998.

[62] K. Becker, M. Becker and J. H. Schwarz, *String theory and M-theory: A modern introduction.* Cambridge University Press, Cambridge, 2007.

[63] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, *Vacuum Configurations for Superstrings*, *Nucl. Phys. B* **258** (1985) 46–74.

[64] S. Ivanov, *Heterotic supersymmetry, anomaly cancellation and equations of motion*, *Phys. Lett. B* **685** (2010) 190–196, [0908.2927].

[65] L. B. Anderson, *Heterotic and M-theory Compactifications for String Phenomenology.* PhD thesis, Oxford University, 2008. 0808.3621.

[66] M. Grana, *Flux compactifications in string theory: A Comprehensive review*, *Phys. Rept.* **423** (2006) 91–158, [hep-th/0509003].

[67] R. Blumenhagen, B. Kors, D. Lüst and S. Stieberger, *Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes*, *Phys. Rept.* **445** (2007) 1–193, [hep-th/0610327].

[68] C. Hull, *Superstring compactifications with torsion and space-time supersymmetry*, in *1st Torino Meeting on Superunification and Extra Dimensions* (P. Fre and R. D'Auria, eds.). World Scientific Publishing Co Pte Ltd., 1986.

[69] C. Hull, *Anomalies, ambiguities and superstrings*, *Phys. Lett. B* **167** (1986) 51–55.

[70] C. Hull, *Compactifications of the heterotic superstring*, *Phys. Lett. B* **178** (1986) 357–364.

[71] A. Strominger, *Superstrings with torsion*, *Nucl. Phys. B* **274** (1986) 253–284.

[72] J. Louis and A. Micu, *Heterotic string theory with background fluxes*, *Nucl. Phys. B* **626** (2002) 26–52, [hep-th/0110187].

[73] J. Louis and A. Micu, *Heterotic-type IIA duality with fluxes*, *JHEP* **03** (2007) 026, [hep-th/0608171].

[74] G. Lopes Cardoso, G. Curio, G. Dall'Agata and D. Lüst, *BPS action and superpotential for heterotic string compactifications with fluxes*, *JHEP* **10** (2003) 004, [hep-th/0306088].

[75] A. R. Frey and M. Lippert, *AdS strings with torsion: Non-complex heterotic compactifications*, *Phys. Rev. D* **72** (2005) 126001, [hep-th/0507202].

[76] P. Manousselis, N. Prezas and G. Zoupanos, *Supersymmetric compactifications of heterotic strings with fluxes and condensates*, *Nucl. Phys. B* **739** (2006) 85–105, [hep-th/0511122].

[77] K. Becker, M. Becker, K. Dasgupta and P. S. Green, *Compactifications of heterotic theory on nonKähler complex manifolds. 1.*, *JHEP* **04** (2003) 007, [hep-th/0301161].

[78] K. Becker, M. Becker, P. S. Green, K. Dasgupta and E. Sharpe, *Compactifications of heterotic strings on nonKähler complex manifolds. 2.*, *Nucl. Phys. B* **678** (2004) 19–100, [hep-th/0310058].

[79] M. Fernandez, S. Ivanov, L. Ugarte and R. Villacampa, *Non-Kähler heterotic string compactifications with non-zero fluxes and constant dilaton*, *Comm. Math. Phys.* **288** (2009) 677–697, [0804.1648].

[80] G. Papadopoulos, *Heterotic supersymmetric backgrounds with compact holonomy revisited*, *Class. Quant. Grav.* **27** (2010) 125008, [0909.2870].

[81] J. Held, D. Lüst, F. Marchesano and L. Martucci, *DWSB in heterotic flux compactifications*, *JHEP* **06** (2010) 090, [1004.0867].

[82] O. Lechtenfeld, C. Nölle and A. D. Popov, *Heterotic compactifications on nearly Kähler manifolds*, *JHEP* **09** (2010) 074, [1007.0236].

[83] M. Klaput, A. Lukas and C. Matti, *Bundles over nearly-Kähler homogeneous spaces in heterotic string theory*, *JHEP* **09** (2011) 100, [1107.3573].

[84] D. Harland and C. Nölle, *Instantons and Killing spinors*, *JHEP* **03** (2012) 082, [1109.3552].

[85] K.-P. Gemmer, A. S. Haupt, O. Lechtenfeld, C. Nölle and A. D. Popov, *Heterotic string plus five-brane systems with asymptotic AdS3*, *Adv. Theor. Math. Phys.* **17** (2013) 771–827, [1202.5046].

[86] R. Harvey and H. B. Lawson Jr., *Calibrated geometries*, *Acta Math.* **148** (1982) 47–157.

[87] A. Chatzistavrakidis, O. Lechtenfeld and A. D. Popov, *Nearly Kähler heterotic compactifications with fermion condensates*, *JHEP* **04** (2012) 114, [1202.1278].

[88] K.-P. Gemmer and O. Lechtenfeld, *Heterotic G_2-manifold compactifications with fluxes and fermionic condensates*, *JHEP* **11** (2013) 182, [1308.1955].

[89] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Sasaki-Einstein metrics on $S^2 \times S^3$*, *Adv. Theor. Math. Phys.* **8** (2004) 711–734, [hep-th/0403002].

[90] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, *A New infinite class of Sasaki-Einstein manifolds*, *Adv. Theor. Math. Phys.* **8** (2006) 987–1000, [hep-th/0403038].

[91] M. Cvetic, H. Lu, D. N. Page and C. Pope, *New Einstein-Sasaki spaces in five and higher dimensions*, *Phys. Rev. Lett.* **95** (2005) 071101, [hep-th/0504225].

[92] M. Cvetic, H. Lu, D. N. Page and C. Pope, *New Einstein-Sasaki and Einstein spaces from Kerr-de Sitter*, *JHEP* **07** (2009) 082, [hep-th/0505223].

[93] H. Lu, C. Pope and J. F. Vazquez-Poritz, *A New construction of Einstein-Sasaki metrics in $D \geq 7$*, *Phys. Rev. D* **75** (2007) 026005, [hep-th/0512306].

[94] M. Fernández, S. Ivanov, V. Munoz and L. Ugarte, *Nearly hypo structures and compact nearly Kähler 6-manifolds with conical singularities*, *J. London Math. Soc.* **78** (2008) 580–604, [math/0602160].

[95] K.-P. Gemmer, O. Lechtenfeld, C. Nölle and A. D. Popov, *Yang-Mills instantons on cones and sine-cones over nearly Kahler manifolds*, *JHEP* **09** (2011) 103, [1108.3951].

[96] B. S. Acharya, J. Figueroa-O'Farrill, C. Hull and B. J. Spence, *Branes at conical singularities and holography*, *Adv. Theor. Math. Phys.* **2** (1999) 1249–1286, [hep-th/9808014].

[97] P. Koerber, D. Lüst and D. Tsimpis, *Type IIA AdS(4) compactifications on cosets, interpolations and domain walls*, *JHEP* **07** (2008) 017, [0804.0614].

[98] C. Nölle, *Instantons, Five-Branes and Fractional Strings*, 1207.7268.

[99] M. Mamone Capria and S. M. Salamon, *Yang-Mills fields on quaternionic spaces*, *Nonlinearity* **1** (1988) 517–530.

[100] R. R. Carrión, *A generalization of the notion of instanton*, *Differential Geom. Appl.* **8** (1998) 1–20.

[101] L. Baulieu, H. Kanno and I. M. Singer, *Special quantum field theories in eight and other dimensions*, *Comm. Math. Phys.* **194** (1998) 149–175.

[102] M. Blau and G. Thompson, *Euclidean SYM theories by time reduction and special holonomy manifolds*, *Phys. Lett. B* **415** (1997) 242–252.

[103] B. Acharya, J. Figueroa-O'Farrill, B. Spence and M. O'Loughlin, *Euclidean D-branes and higher-dimensional gauge theory*, *Nucl. Phys. B* **514** (1998) 583–602.

[104] S. Donaldson and R. Thomas, *Gauge theory in higher dimensions*, in *The Geometric Universe* (S. A. Huggett, L. J. Mason, K. P. Tod, S. T. Tsou and N. M. J. Woodhouse, eds.), pp. 31–47. Oxford University Press, Oxford, 1996.

[105] S. Donaldson and E. Segal, *Gauge theory in higher dimensions, II*, in *Surveys in Differential Geometry* (N. C. Leung and S.-T. Yau, eds.), vol. 16. International Press of Boston, Inc., 2009. 0902.3239.

[106] G. Tian, *Gauge theory and calibrated geometry, I*, *Ann. Math.* **151** (2000) 193–268, [math/0010015].

[107] D. Fairlie and J. Nuyts, *Spherically symmetric solutions of gauge theories in eight-dimensions*, *J. Phys. A* **17** (1984) 2867–2872.

[108] S. Fubini and H. Nicolai, *The octonionic instanton*, *Phys. Lett. B* **155** (1985) 369–372.

[109] T. Ivanova and A. Popov, *Self-dual Yang-Mills fields in $d = 7, 8$, octonions and Ward equations*, *Lett. Math. Phys.* **24** (1992) 85–92.

[110] T. Ivanova and A. Popov, *(Anti)self-dual gauge fields in dimension $d \geq 4$*, *Theor. Math. Phys.* **94** (1993) 225–242.

[111] M. Günaydin and H. Nicolai, *Seven-dimensional octonionic Yang-Mills instanton and its extension to an heterotic string soliton*, *Phys. Lett. B* **351** (1995) 169–172, [hep-th/9502009].

[112] D. Baraglia and P. Hekmati, *Moduli Spaces of Contact Instantons*, *Adv. Math.* **294** (2016) 562–595, [1401.5140].

[113] T. A. Ivanova and A. D. Popov, *Instantons on special holonomy manifolds*, *Phys. Rev. D* **85** (2012) 105012, [1203.2657].

[114] F. P. Correia, *Hermitian Yang-Mills instantons on Calabi-Yau cones*, *JHEP* **12** (2009) 004, [0910.1096].

[115] F. P. Correia, *Hermitian Yang-Mills instantons on resolutions of Calabi-Yau cones*, *JHEP* **02** (2011) 054, [1009.0526].

[116] S. Groot Nibbelink, M. Trapletti and M. Walter, *Resolutions of C^n/Z_n Orbifolds, their U(1) Bundles, and Applications to String Model Building*, *JHEP* **03** (2007) 035, [hep-th/0701227].

[117] S. Bunk, T. A. Ivanova, O. Lechtenfeld, A. D. Popov and M. Sperling, *Instantons on sine-cones over Sasakian manifolds*, *Phys. Rev. D* **90** (2014) 065028, [1407.2948].

[118] S. Bunk, O. Lechtenfeld, A. D. Popov and M. Sperling, *Instantons on conical half-flat 6-manifolds*, *JHEP* **01** (2015) 030, [1409.0030].

[119] M. Sperling, *Instantons on Calabi–Yau cones*, *Nucl. Phys. B* **901** (2015) 354–381, [1505.01755].

[120] C. P. Boyer and K. Galicki, *Sasakian geometry.* Oxford University Press, Oxford, 2008.

[121] P. B. Kronheimer, *A Hyper-Kählerian structure on coadjoint orbits of a semisimple complex group*, *J. London Math. Soc.* **42** (1990) 193–208.

[122] P. B. Kronheimer, *Instantons and the geometry of the nilpotent variety*, *J. Diff. Geom.* **32** (1990) 473–490.

[123] D. Conti and S. Salamon, *Generalized Killing spinors in dimension* 5, *Trans. Amer. Math. Soc.* **359** (2007) 5319–5343, [math/0508375].

[124] S. Chiossi and S. Salamon, *The intrinsic torsion of $SU(3)$ and G_2 structures*, in *Differential Geometry, Valencia 2001* (O. Gil-Medrano and V. Miquel, eds.), pp. 115–133. World Sci. Publishing, 2002. math/0202282.

[125] D. Harland, T. A. Ivanova, O. Lechtenfeld and A. D. Popov, *Yang-Mills flows on nearly Kähler manifolds and G_2-instantons*, *Comm. Math. Phys.* **300** (2010) 185–204, [0909.2730].

[126] T. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, *Asian J. Math.* **6** (2002) 303–336, [math/0102142].

[127] J.-M. Bismut, *A local index theorem for non-Kähler manifolds*, *Math. Ann.* **284** (1989) 681–699.

[128] D. Harland and A. D. Popov, *Yang-Mills fields in flux compactifications on homogeneous manifolds with SU(4)-structure*, *JHEP* **02** (2012) 107, [1005.2837].

[129] S. Bunk, *A method of deforming G-structures*, *J. Geom. Phys.* **96** (2015) 72–80, [1410.5849].

[130] C. P. Boyer and K. Galicki, *Sasakian geometry, holonomy, and supersymmetry*, math.DG/0703231.

[131] G. Lopes Cardoso, G. Curio, G. Dall'Agata, D. Lust, P. Manousselis and G. Zoupanos, *NonKahler string backgrounds and their five torsion classes*, *Nucl. Phys. B* **652** (2003) 5–34, [hep-th/0211118].

[132] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. I.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996.

[133] C. Nölle, *Heterotic Supergravity on manifolds with Killing spinors.* PhD thesis, Leibniz Universität Hannover, 2012.

[134] M. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, *Phil. Trans. Roy. Soc. Lond. A* **308** (1982) 523–615.

[135] A. Deser, O. Lechtenfeld and A. D. Popov, *Sigma-model limit of Yang–Mills instantons in higher dimensions*, *Nucl. Phys. B* **894** (2015) 361–373, [1412.4258].

[136] R. P. Thomas, *Notes on GIT and symplectic reduction for bundles and varieties*, in *Surveys in differential geometry. Vol. X*, vol. 10 of *Surv. Differ. Geom.*, pp. 221–273. Int. Press, Somerville, MA, 2006. DOI.

[137] S. Donaldson, *Boundary value problems for Yang-Mills fields*, *J. Geom. Phys.* **8** (1992) 89–122.

[138] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. 2 of *Wiley Classics Library.* John Wiley & Sons, Inc., New York, 1996.

[139] D. Kapetanakis and G. Zoupanos, *Coset space dimensional reduction of gauge theories*, *Phys. Rept.* **219** (1992) 4–76.

[140] I. Bauer, T. A. Ivanova, O. Lechtenfeld and F. Lubbe, *Yang-Mills instantons and dyons on homogeneous G_2-manifolds*, *JHEP* **10** (2010) 044, [1006.2388].

[141] A. D. Popov and R. J. Szabo, *Double quiver gauge theory and nearly Kähler flux compactifications*, *JHEP* **02** (2012) 033, [1009.3208].

[142] O. Lechtenfeld, *Instantons and Chern-Simons flows in 6, 7 and 8 dimensions*, *Phys. Part. Nucl.* **43** (2012) 569–576, [1201.6390].

[143] M. Wolf, *Contact Manifolds, Contact Instantons, and Twistor Geometry*, *JHEP* **07** (2012) 074, [1203.3423].

[144] F. Xu, *Geometry of $SU(3)$ manifolds.* PhD thesis, Duke University, 2008.

[145] N. Hitchin, *Hyper-Kähler Manifolds*, *Asterisque* **206** (1992) 137.

[146] A. A. Kirillov, *Lectures on the orbit method*, vol. 64 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2004, 10.1090/gsm/064.

[147] O. Lechtenfeld, A. D. Popov and R. J. Szabo, *Sasakian quiver gauge theories and instantons on Calabi-Yau cones*, 1412.4409.

[148] J. C. Geipel, O. Lechtenfeld, A. D. Popov and R. J. Szabo, *Sasakian quiver gauge theories and instantons on the conifold*, *Nucl. Phys. B* **907** (2016) 445–475, [1601.05719].

[149] O. Lechtenfeld, A. D. Popov, M. Sperling and R. J. Szabo, *Sasakian quiver gauge theories and instantons on cones over lens 5-spaces*, *Nucl. Phys. B* **899** (2015) 848–903, [1506.02786].

[150] A. D. Popov and R. J. Szabo, *Quiver gauge theory of nonAbelian vortices and noncommutative instantons in higher dimensions*, *J. Math. Phys.* **47** (2006) 012306, [hep-th/0504025].

[151] I. Biswas, *Holomorphic Hermitian vector bundles over the Riemann sphere*, *Bull. Sci. Math.* **132** (2008) 246–256.

[152] B. P. Dolan and R. J. Szabo, *Dimensional Reduction, Monopoles and Dynamical Symmetry Breaking*, *JHEP* **03** (2009) 059, [0901.2491].

[153] R. J. Szabo and O. Valdivia, *Covariant Quiver Gauge Theories*, *JHEP* **06** (2014) 144, [1404.4319].

[154] O. Lechtenfeld, A. D. Popov and R. J. Szabo, *Rank two quiver gauge theory, graded connections and noncommutative vortices*, *JHEP* **09** (2006) 054, [hep-th/0603232].

[155] O. Lechtenfeld, A. D. Popov and R. J. Szabo, SU(3)*-equivariant quiver gauge theories and nonabelian vortices*, *JHEP* **08** (2008) 093, [0806.2791].

[156] B. P. Dolan and R. J. Szabo, *Dimensional reduction and vacuum structure of quiver gauge theory*, *JHEP* **08** (2009) 038, [0905.4899].

[157] F. Klein, *Vorlesungen über das Ikosaeder: und die Auflösung der Gleichungen vom fünften Grade*. Teubner, 1884.

[158] P. Du Val, *On isolated singularities of surfaces which do not affect the conditions of adjunction (part i.)*, *Math. Proc. Cambridge* **30** (1934) 453–459.

[159] P. Du Val, *On isolated singularities of surfaces which do not affect the conditions of adjunction (part ii.)*, *Math. Proc. Cambridge* **30** (1934) 460–465.

[160] P. Du Val, *On isolated singularities of surfaces which do not affect the conditions of adjunction (part iii.)*, *Math. Proc. Cambridge* **30** (1934) 483–491.

[161] J. McKay, *Graphs, singularities, and finite groups*, *Proc. Symp. Pure Math* **37** (1980) 1866.

[162] P. B. Kronheimer, *Instantons gravitationnels et singularités de Klein*, *C. R. Acad. Sci. Paris Sér. I Math.* **303** (1986) 53–55.

[163] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, *J. Diff. Geom.* **29** (1989) 665–683.

[164] C. V. Johnson and R. C. Myers, *Aspects of type IIB theory on ALE spaces*, *Phys. Rev. D* **55** (1997) 6382–6393, [hep-th/9610140].

[165] H. F. Blichfeldt, *Blichfeldt's finite collineation groups*, *Bull. Amer. Math. Soc.* **24** (1918) 484–487.

[166] A. V. Sardo-Infirri, *Partial resolutions of orbifold singularities via moduli spaces of HYM-type bundles*, alg-geom/9610004.

[167] A. V. Sardo-Infirri, *Resolutions of orbifold singularities and flows on the McKay quiver*, alg-geom/9610005.

[168] M. R. Douglas, B. R. Greene and D. R. Morrison, *Orbifold resolution by D-branes*, *Nucl. Phys. B* **506** (1997) 84–106, [hep-th/9704151].

[169] M. R. Douglas, B. Fiol and C. Romelsberger, *The spectrum of BPS branes on a noncompact Calabi-Yau*, *JHEP* **09** (2005) 057, [hep-th/0003263].

[170] L. C. Biedenharn, *Invariant operators of the Casimir type*, *Phys. Lett.* **3** (1962) 69–70.

[171] G. E. Baird and L. C. Biedenharn, *On the representations of semisimple Lie groups. 2*, *J. Math. Phys.* **4** (1963) 1449–1466.

[172] N. Mukunda and L. K. Pandit, *Tensor methods and a unified representation theory of* SU(3), *J. Math. Phys.* **6** (1965) 746–765.

[173] M. Cirafici, A. Sinkovics and R. J. Szabo, *Instantons, quivers and noncommutative Donaldson-Thomas theory*, *Nucl. Phys. B* **853** (2011) 508–605, [1012.2725].

[174] M. Cirafici, A. Sinkovics and R. J. Szabo, *Instanton counting and wall-crossing for orbifold quivers*, *Ann. Henri Poincaré* **14** (2013) 1001–1041, [1108.3922].

[175] M. Cirafici and R. J. Szabo, *Curve counting, instantons and McKay correspondences*, *J. Geom. Phys.* **72** (2013) 54–109, [1209.1486].

[176] M. K. Murray and M. A. Singer, *Gerbes, Clifford modules and the index theorem*, *Ann. Global Anal. Geom.* **26** (2004) 355–367, [math/0302096].

[177] K. A. Intriligator and N. Seiberg, *Mirror symmetry in three-dimensional gauge theories*, *Phys. Lett. B* **387** (1996) 513–519, [hep-th/9607207].

[178] D. Gaiotto and E. Witten, *S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory*, *Adv. Theor. Math. Phys.* **13** (2009) 721–896, [0807.3720].

[179] D. Bashkirov and A. Kapustin, *Supersymmetry enhancement by monopole operators*, *JHEP* **05** (2011) 015, [1007.4861].

[180] G. Chalmers and A. Hanany, *Three-dimensional gauge theories and monopoles*, *Nucl. Phys. B* **489** (1997) 223–244, [hep-th/9608105].

[181] M. Bullimore, T. Dimofte and D. Gaiotto, *The Coulomb Branch of 3d $\mathcal{N} = 4$ Theories*, 1503.04817.

[182] N. Seiberg, *IR dynamics on branes and space-time geometry*, *Phys. Lett. B* **384** (1996) 81–85, [hep-th/9606017].

[183] N. Seiberg and E. Witten, *Gauge dynamics and compactification to three-dimensions*, in *The mathematical beauty of physics: A memorial volume for Claude Itzykson. Proceedings, Conference, Saclay, France, June 5-7, 1996*, 1996. hep-th/9607163.

[184] N. Dorey, V. V. Khoze, M. P. Mattis, D. Tong and S. Vandoren, *Instantons, three-dimensional gauge theory, and the Atiyah-Hitchin manifold*, *Nucl. Phys. B* **502** (1997) 59–93, [hep-th/9703228].

[185] N. Dorey, D. Tong and S. Vandoren, *Instanton effects in three-dimensional supersymmetric gauge theories with matter*, *JHEP* **04** (1998) 005, [hep-th/9803065].

[186] D. Tong, *Three-dimensional gauge theories and ADE monopoles*, *Phys. Lett. B* **448** (1999) 33–36, [hep-th/9803148].

[187] G. W. Gibbons and N. S. Manton, *The Moduli space metric for well separated BPS monopoles*, *Phys. Lett. B* **356** (1995) 32–38, [hep-th/9506052].

[188] H. Nakajima, *Towards a mathematical definition of Coulomb branches of* 3*-dimensional* $\mathcal{N} = 4$ *gauge theories, I*, 1503.03676.

[189] H. Nakajima, *Questions on provisional Coulomb branches of* 3*-dimensional* $\mathcal{N} = 4$ *gauge theories*, 1510.03908.

[190] A. Braverman, M. Finkelberg and H. Nakajima, *Towards a mathematical definition of Coulomb branches of* 3*-dimensional* $\mathcal{N} = 4$ *gauge theories, II*, 1601.03586.

[191] S. Cremonesi, A. Hanany and A. Zaffaroni, *Monopole operators and Hilbert series of Coulomb branches of* 3*d* $\mathcal{N} = 4$ *gauge theories*, *JHEP* **01** (2014) 005, [1309.2657].

[192] G. 't Hooft, *On the Phase Transition Towards Permanent Quark Confinement*, *Nucl. Phys. B* **138** (1978) 1.

[193] F. Englert and P. Windey, *Quantization Condition for 't Hooft Monopoles in Compact Simple Lie Groups*, *Phys. Rev. D* **14** (1976) 2728.

[194] P. Goddard, J. Nuyts and D. I. Olive, *Gauge Theories and Magnetic Charge*, *Nucl. Phys. B* **125** (1977) 1.

[195] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, vol. 227 of *Graduate Texts in Mathematics*. Springer New York, 2005.

[196] R. P. Stanley, *Hilbert functions of graded algebras*, *Advances in Math.* **28** (1978) 57–83.

[197] J. Gray, A. Hanany, Y.-H. He, V. Jejjala and N. Mekareeya, *SQCD: A Geometric Apercu*, *JHEP* **05** (2008) 099, [0803.4257].

[198] D. Forcella, A. Hanany, Y.-H. He and A. Zaffaroni, *The Master Space of N=1 Gauge Theories*, *JHEP* **08** (2008) 012, [0801.1585].

[199] S. Benvenuti, B. Feng, A. Hanany and Y.-H. He, *Counting BPS Operators in Gauge Theories: Quivers, Syzygies and Plethystics*, *JHEP* **11** (2007) 050, [hep-th/0608050].

[200] A. Hanany and C. Romelsberger, *Counting BPS operators in the chiral ring of N=2 supersymmetric gauge theories or N=2 braine surgery*, *Adv. Theor. Math. Phys.* **11** (2007) 1091–1112, [hep-th/0611346].

[201] A. Hanany, *Counting BPS operators in the chiral ring: The plethystic story*, *AIP Conf. Proc.* **939** (2007) 165–175.

[202] B. Feng, A. Hanany and Y.-H. He, *Counting gauge invariants: The Plethystic program*, *JHEP* **03** (2007) 090, [hep-th/0701063].

[203] A. Hanany and N. Mekareeya, *Complete Intersection Moduli Spaces in N=4 Gauge Theories in Three Dimensions*, *JHEP* **01** (2012) 079, [1110.6203].

[204] M. K. Benna, I. R. Klebanov and T. Klose, *Charges of Monopole Operators in Chern-Simons Yang-Mills Theory*, *JHEP* **01** (2010) 110, [0906.3008].

[205] A. Hanany and M. Sperling, *Coulomb branches for rank 2 gauge groups in 3d N=4 gauge theories*, 1605.00010.

[206] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, vol. 9 of *Graduate Texts in Mathematics*. Springer New York, 1972.

[207] D. Cox, J. Little and H. Schenck, *Toric Varieties*, vol. 124 of *Graduate Studies in Mathematics*. American Mathematical Soc., 2011.

[208] A. Kapustin, *Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality*, *Phys. Rev. D* **74** (2006) 025005, [hep-th/0501015].

[209] G. M. Ziegler, *Lectures on Polytopes*, vol. 152 of *Graduate Texts in Mathematics*. Springer New York, 1995.

[210] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, vol. 8 of *University Lecture Series*. American Mathematical Society, 1996.

[211] V. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, vol. 102 of *Graduate Texts in Mathematics*. Springer New York, 1984.

[212] J. E. Humphreys, *Reflection groups and Coxeter groups*, vol. 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1990.

[213] C. Chevalley, *Invariants of finite groups generated by reflections*, *Amer. J. Math.* **77** (1955) 778–782.

[214] T. Molien, *Über die Invarianten der linearen Substitutionsgruppen.*, *Berl. Ber.* **1897** (1897) 1152–1156.

[215] Harish-Chandra, *On some applications of the universal enveloping algebra of a semisimple Lie algebra*, *Trans. Amer. Math. Soc.* **70** (1951) 28–96.

[216] S. Cremonesi, A. Hanany, N. Mekareeya and A. Zaffaroni, T_ρ^σ *(G) theories and their Hilbert series*, *JHEP* **01** (2015) 150, [1410.1548].

[217] S. Okubo, *Casimir invariants and vector operators in simple and classical lie algebras*, *J. Math. Phys.* **18** (1977) 2382–2394.

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List of Publications

(i) A. Hanany and **M. Sperling**, *Coulomb branches for rank 2 gauge groups in 3d N=4 gauge theories*, to appear in JHEP, `hep-th/1605.00010`.

(ii) O. Lechtenfeld, A. D. Popov, **M. Sperling** and R. J. Szabo, *Sasakian quiver gauge theories and instantons on cones over lens 5-spaces*, Nucl. Phys. B **899**, 848-903 (2015), `hep-th/1506.02786`.

(iii) **M. Sperling**, *Instantons on Calabi-Yau cones*, Nucl. Phys. B **901**, 354-381 (2015), `hep-th/1505.01755`.

(iv) S. Bunk, O. Lechtenfeld, A. D. Popov and **M. Sperling**, *Instantons on conical half-flat 6-manifolds*, JHEP **01**, 030 (2015), `hep-th/1409.0030`.

(v) S. Bunk, T. A. Ivanova, O. Lechtenfeld, A. D. Popov and **M. Sperling**, *Instantons on sine-cones over Sasakian manifolds*, Phys. Rev. D **90**, 065028 (2014), `hep-th/1407.2948`.

(vi) C. Gnendiger, A. Signer, **M. Sperling**, D. Stöckinger and A. Voigt, *Two-loop results on the renormalization of vacuum expectation values and infrared divergences in the FDH*, C14-04-27 Proceedings **LL2014**, 076 (2014), `hep-ph/1407.3082`.

(vii) **M. Sperling**, D. Stöckinger and A. Voigt, *Renormalization of vacuum expectation values in spontaneously broken gauge theories: Two-loop results*, JHEP **01**, 068 (2014), `hep-ph/1310.7629`.

(viii) **M. Sperling**, D. Stöckinger and A. Voigt, *Renormalization of vacuum expectation values in spontaneously broken gauge theories*, JHEP **07**, 132 (2013), `hep-ph/1305.1548`

Curriculum Vitae

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