A BESTIARY OF TOPOLOGICAL OBJECTS

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1. Introduction

These notes are intended to give graduate students in algebraic topology an overview of the range of examples that are understood. I have also added a long list of other examples which I have not had time to write anything about as yet. It takes quite a lot of theory to *justify* everything that I say about the examples I discuss, but I hope that it will not take too much theory to *understand* most of it.

These notes are new and not yet thoroughly debugged. Any comments, suggestions or corrections would be gratefully received.

1.1. **Background Reading.** In this section, we list some sources which provide the theory necessary to justify the examples which follow.

Maunder's book [27] is a pleasant introduction to general homology and homotopy theory. Dold's book [13] contains more information about homology, and Whitehead's book [44] is a very good reference for the more classical parts of homotopy theory. Simplicial methods are an indespensible tool in much of algebraic topology; the canonical reference is [28]. The book [19] also contains a good treatment of simplicial sets and CW complexes, together with a lot of useful background material from point-set topology. Another indespensable tool is the theory of spectral sequences, which is explained in a pleasant and approachable way in [31]. Lie groups crop up in topology in a number of ways; Adams' exposition in [1] is very elegant. His book [3] also provides an excellent survey of the theory of infinite loop spaces. For some purposes, one needs to approach homotopy theory from a more abstract point of view (to clarify the relationship between the homotopy theory of spaces and that of chain complexes, for example). One way to do this is to use Quillen's theory of closed model categories; there is a nice exposition in [16]. (There are a number of other useful survey articles in the same collection.)

For stable homotopy theory, the student should start by reading about the S-category in [21, Chapter 16] (actually, the whole book is highly recommended) and then read Adams' Chicago lecture notes [2, Part III]. Adams' construction of the smash product should be ignored, however, for reasons to be explained shortly. The first part of Margolis' book [26] is a very good treatment of the formal properties of the stable homotopy category, although it is almost entirely devoid of geometry. Kochman's new book [23] looks very promising, although I have not seen a copy yet. One of the most important themes in stable homotopy theory is the interection with the algebraic theory of formal groups via complex cobordism. This is explained in [2], and taken further in Ravenel's book [36]. The latter also contains valuable surveys of a number of areas of topology (as well as some large and elaborate calculations of stable homotopy groups). Another important (and related) theme is the exploitation of the Nilpotence Theorem of Hopkins, Devinatz and Smith. This is explained in another book by Ravenel [37], which does a remarkably good job of presenting a very deep result in a way which is comprehensible with a minimum of prerequisites.

There are now ways of setting up the foundations of stable homotopy theory which are more satisfactory in a number of ways than those known to Adams. In particular, Elmendorff, Kriz, Mandell and May have costructed a category whose homotopy category is the same as the one considered by Adams, but which has much better properties before passage to homotopy. This is explained in [17], which is unfortunately not light reading. The rather easier construction described in [30, Chapter XII] is just as good for most purposes. Incidentally, the book [30] also contains a wealth of other fascinating material.

2. Wedges of circles

Let W be a wedge of n circles, $W = S^1 \vee \ldots \vee S^1$. We shall show that W is the quotient of a contractible space $\widetilde{W}$ by a free action of the free group G on n generators. It will follow that $\pi_k W = 0$ unless $k = 1$, and that $\pi_1 W = G$. For this and related material, see [41, Chapter 2].

The inclusion $i_m \colon S^1 \to W$ of the m'th wedge summand in W can be thought of as an element $s_m \in \pi_1 W$ (for $m = 0, \ldots, l-1$). Recall the definition of the free group G generated by the elements s_m: an element g of G is a sequence $(t_1, \ldots, t_r)$ (possibly empty) in which each term t_j is either an s_m or an s_m^{-1} for some m, and s_m never occurs next to s_m^{-1}. Thus $()$ and (s_3, s_2^{-1}, s_1) are elements of G but (s_1, s_3^2, s_2) and (s_2, s_2^{-1}) are not. The group operation is defined by joining sequences together and discarding adjacent pairs of the form (s_m, s_m^{-1}) or (s_m^{-1}, s_m) until there are none left. There is an obvious homomorphism $G \to \pi_1 W$ sending a sequence $(t_1, \ldots, t_r)$ to the product $t_1 t_2 \ldots t_r$.

We next define $\widetilde{W}$ to be the (infinite) simplicial complex with vertices G and edges joining g to gs_m for each m. More explicitly, write $S = \{s_0, \ldots, s_{l-1}\}$ and

$$\widetilde{W} = (G \amalg (G \times S \times [0,1]))/\sim$$
$$(g, s, 0) \sim g \qquad\qquad (g, s, 1) \sim gs.$$

We take the element $() \in G$ as the basepoint of $\widetilde{W}$.

We now define a "truncation map" $k \colon \widetilde{W} \to \widetilde{W}$ as follows. The map $k \colon G \to G$ of vertices sends $(t_1, \ldots, t_r)$ to $(t_1, \ldots, t_{r-1})$ (and $0 = ()$ to itself). Clearly, if g and h are joined by an edge then the same is true of $k(g)$ and $k(h)$, so k extends to give a simplicial map $\widetilde{W} \to \widetilde{W}$. Moreover, it is easy to see that k is homotopic to the identity, so we have a family of maps $h_t \colon \widetilde{W} \to \widetilde{W}$ with $h_0 = 1$ and $h_1 = k$. We may also assume that $h_t(0) = 0$.

We now extend this and define $h_t(w)$ for all $0 \leq t \leq \infty$ by the formula $h_t(w) = h_s(k^m(w))$, where $t = m + s$ with $m \in \mathbb{N}$ and $0 \leq s \leq 1$, and $h_\infty(w) = 0$. One can check that this gives a continuous map $h \colon [0, \infty] \times \widetilde{W} \to \widetilde{W}$, where $[0, \infty]$ is topologised as the one-point compactification of $[0, \infty)$. Using a homeomorphism $[0,1] \simeq [0, \infty]$ (sending t to $t/(1-t)$, say), we conclude that $h_0 = 1$ is homotopic to $h_\infty = 0$, so that $\widetilde{W}$ is contractible as claimed, and thus $\pi_k \widetilde{W} = 0$ for all k.

We now define a map $p \colon \widetilde{W} \to W$. To do this, think of $\widetilde{W}$ as $(G \amalg G \times S \times [0,1])/\sim$ and W as $(0 \amalg S \times [0,1])/\sim$, where in the latter case $(s, 0) \sim (s, 1) \sim 0$ for all $s \in S$. In this picture, p sends G to 0 and $G \times S \times [0,1]$ to $S \times [0,1]$ by the projection. One can check that this is a bundle projection, and in fact it displays W as $\widetilde{W}/G$, where G acts on the left on $\widetilde{W}$ in an obvious way. It follows from the long exact sequence of the fibration $G \to \widetilde{W} \to W$ that $\pi_1 W = G$. (As yet, we have only proved that this is a bijection, not an isomorphism of groups, but that is also true). It also follows that $\pi_k W = 0$ for $k > 1$.

3. The configuration spaces $F_k\mathbb{C}$ and $B_k\mathbb{C}$

Define

$$\begin{aligned} F_k\mathbb{C} &= \{ \text{ injective maps } z\colon \{0,\dots,k-1\} \to \mathbb{C}\} \\ &= \{(z_0,\dots,z_{k-1}) \in \mathbb{C}^k \mid z_i \neq z_j \text{ for } i \neq j\} \end{aligned}$$

We take the usual inclusion $\{0,\dots,k-1\} \to \mathbb{C}$ as the basepoint in $F_k\mathbb{C}$.

Let Σ_k be the group of permutations of the set $\{0,\dots,k-1\}$. This acts on $F_k\mathbb{C}$ on the right by composition (i.e. $\sigma \in \Sigma_k$ sends $z \in F_k\mathbb{C}$ to $z \circ \sigma$). We define

$$B_k\mathbb{C} = F_k\mathbb{C}/\Sigma_k = \{ \text{ subsets of } \mathbb{C} \text{ of order } k\}.$$

Clearly $F_k\mathbb{C}$ is a noncompact open submanifold of $\mathbb{C}^k$ and thus has real dimension $2k$. The finite group Σ_k acts freely (i.e. $z \circ \sigma = z$ implies $\sigma = 1$), which implies that the orbit space $B_k\mathbb{C}$ is also a manifold, and that the projection $F_k\mathbb{C} \to B_k\mathbb{C}$ is a covering map.

Another nice description of $B_k\mathbb{C}$ is as follows. Let M be the set of monic polynomials of degree k over $\mathbb{C}$ in one variable t. This can be identified with $\mathbb{C}^k$ in an obvious way. Define a map $F_k\mathbb{C} \to M$ by sending $(z_0,\dots,z_{k-1})$ to the polynomial $\prod_i (t - z_i)$. This is clearly independent of the order of the z_i's, so it induces a map $j\colon B_k\mathbb{C} \to M$. One can check that this is a homeomorphism of $B_k\mathbb{C}$ with an open subset of M.

Now write $\mathbb{C}_l = \mathbb{C} \setminus \{0,\dots,l-1\}$ (with l as the basepoint). It is not hard to see that $\mathbb{C}_l$ is homotopy equivalent to a wedge of l circles, so that $\pi_k\mathbb{C}_l = 0$ for $k \neq 1$ and $\pi_1\mathbb{C}_l$ is a free group on l generators.

3.1. **Fibrations.** We next consider the map $q\colon F_{k+1}\mathbb{C} \to F_k\mathbb{C}$ defined by

$$q(z_0,\dots,z_k) = (z_0,\dots,z_{k-1}).$$

We claim that this is a fibre bundle projection with fibre $\mathbb{C}_k$, in other words that we can cover $F_k\mathbb{C}$ with open sets U and find homeomorphisms $f\colon q^{-1}U \simeq U \times \mathbb{C}_k$ such that $\pi f = q\colon q^{-1}U \to U$ (such an f is called a *local trivialisation*). First, note that this is at least plausible; the preimage of a point $z \in F_k\mathbb{C}$ under q is just the set of pairs (z,w) with $w \in \mathbb{C} \setminus \{z_0,\dots,z_{k-1}\}$, which certainly looks rather like $\mathbb{C}_k$.

To make this precise, we first let U be the set of k-tuples $(z_0,\dots,z_{k-1})$ such that $z_j = x_j + iy_j$ and $x_0 < x_1 < \dots < x_{k-1}$ (so $U \subset F_k\mathbb{C}$). Given a point $z \in U$, define a homeomorphism $\theta_z\colon \mathbb{C} \to \mathbb{C}$ as follows. The map θ_z will send vertical lines to vertical lines, it will send the point z_j to j, and it will send the vertical band $x_j \leq x \leq x_{j+1}$ to the band $j \leq x \leq j+1$. Explicitly, for $x \leq x_0$ we define $\theta_z(x+iy) = x+iy-z_0$, for $x \geq x_{k-1}$ we define $\theta_z(x+iy) = x+iy-z_{k-1}+k-1$, and for $x_j \leq x \leq x_{j+1}$ we define $t = (x-x_j)/(x_{j+1}-x_j)$ and

$$\theta_z(x+iy) = j + t + i(y - (1-t)y_j - ty_{j+1}).$$

Next, observe that $q^{-1}U$ is the set of pairs (z,w) where $z \in U$ and $w \in \mathbb{C}\setminus\{z_0,\dots,z_{k-1}\}$. Define $f\colon q^{-1}U \to U \times \mathbb{C}_k$ by $(z,w) \mapsto (z, \theta_z(w))$. This is a bijection, with inverse $(z,v) \mapsto (z, \theta_z^{-1}(v))$. With a bit more work, one can check that it is a homeomorphism, as required.

So far, we have only trivialised q over the open set U; we need to do the same over a covering family of open sets. For each permutation σ of $\{0,\dots,k-1\}$ and each complex number ζ with $|\zeta| = 1$, write $U(\sigma,\zeta) = \{z \in F_k\mathbb{C} \mid \zeta z \circ \sigma \in U\}$. Define $g\colon q^{-1}U(\sigma,\zeta) \to U(\zeta,\sigma) \times \mathbb{C}_k$ by

$$g(z,w) = (z, \theta_{\zeta z \circ \sigma}(\zeta w)).$$

One can check that this gives a trivialisation over $U(\zeta,\sigma)$, so we need only check that these sets cover $F_k\mathbb{C}$. For any $z \in F_k\mathbb{C}$, choose any angle ϕ that does not occur as $\arg(z_i - z_j)$ for any $i,j < k$, and set $\zeta = ie^{-i\phi}$. The real parts of the numbers $\zeta z_0,\dots,\zeta z_{k-1}$ are then all distinct, so there is a unique permutation σ such that $\Re(\zeta z_{\sigma(0)}) < \dots < \Re(\zeta z_{\sigma(k-1)})$. We thus have $z \in U(\zeta,\sigma)$, so the sets $U(\zeta,\sigma)$ cover $F_k\mathbb{C}$ as claimed.

This implies that the projection $q\colon F_{k+1}\mathbb{C} \to F_k\mathbb{C}$ is a fibration with fibre $\mathbb{C}_k$. The resulting long exact sequence shows (by induction on k) that $\pi_m F_k\mathbb{C} = 0$ for $m > 1$.

3.2. **Cohomology.** Recall that

$$[X, S^1] = [X, \mathbb{C}^\times] = H^1 X.$$

Thus, for each $i \neq j$ we have a class $a_{ij} \in H^1 F_k\mathbb{C}$ corresponding to the map $z \mapsto z_i - z_j$. Because the map $-1\colon \mathbb{C}^\times \to \mathbb{C}^\times$ is homotopic to the identity, we see that $a_{ij} = a_{ji}$.

To find some relations among these classes, consider a triple $i < j < k$. This gives a map $F_k\mathbb{C} \to F_3\mathbb{C}$ by $z \mapsto (z_i, z_j, z_k)$. Clearly a_{ij}, a_{jk} and a_{ki} are in the image of this map in cohomology. There is a homeomorphism

$$F_3\mathbb{C} = \mathbb{C} \times \mathbb{C}^\times \times \mathbb{C} \setminus \{0, 1\}$$

given by

$$(u, v, w) \mapsto (u, v - u, (w - u)/(v - u)).$$

It follows that $F_3\mathbb{C} \simeq S^1 \times (S^1 \vee S^1)$. Using this, one can check that

$$a_{ij}a_{jk} + a_{ki}a_{ij} + a_{jk}a_{ki} = 0.$$

In fact, we have

$$H^* F_n\mathbb{C} = E[a_{ij} \mid 0 \leq i < j < n]/(a_{ij}a_{jk} + a_{ki}a_{ij} + a_{jk}a_{ki} \mid i < j < k)$$

and this has a basis consisting of the monomials $a_{i_1j_1} \dots a_{i_rj_r}$ such that $i_1 < i_2 < \dots i_r$ and $i_k < j_k$ for all k. This is proved by making a similar statement for $H^* F_k\mathbb{C}_l$ and verifying it inductively using the Serre spectral sequences of the evident fibrations

$$F_{k-1,l+1} \to F_{k,l} \to F_{k-1,l}.$$

3.3. **The action of the symmetric group.** The first stage in computing $H^* B_n\mathbb{C}$ is to understand the action of Σ_n on $H^* F_n\mathbb{C}$. The character of this representation of Σ_n is the map

$$\xi\colon \sigma \mapsto \sum_k \operatorname{trace}(\sigma^*\colon H^k F_n\mathbb{C} \to H^k F_n\mathbb{C}).$$

Let $\kappa\colon F_n\mathbb{C} \to F_n\mathbb{C}$ be the complex conjugation map. This has $\kappa^* a_{ij} = -a_{ij}$ and thus $\kappa^* = (-1)^k$ on $H^k F_n\mathbb{C}$. It follows from the Lefschetz fixed point theorem that

$$\xi(\sigma) = \sum_k (-1)^k \operatorname{trace}(\kappa^*\sigma^*\colon H^k F_n\mathbb{C} \to H^k F_n\mathbb{C}) = \chi(\text{ fixed point set of } \sigma\kappa).$$

To understand the right hand side, suppose that $\sigma\kappa$ has a fixed point, say $z \circ \sigma = \overline{z}$ for some $z \in F_n\mathbb{C}$. We then have $z \circ \sigma^2 = z$ and thus $\sigma^2 = 1$, so that σ is a product of disjoint transpositions, say k of them. It is not hard to check that the fixed-point set of $\sigma\kappa$ is homeomorphic to a disjoint union of 2^k copies of $F_k(U) \times F_{n-2k}\mathbb{R}$, where U is the upper half plane, which is homeomorphic to $\mathbb{C}$. Using our earlier calculation of $H^* F_k\mathbb{C}$ we conclude that when $k > 1$, we have $\chi(F_k(U)) = 0$ and thus $\xi(\sigma) = 0$. From this we conclude that

$$\xi(\sigma) = \begin{cases} n! & \text{if } \sigma = 1 \\ 2(n-2)! & \text{if } \sigma \text{ is a transposition} \\ 0 & \text{otherwise.} \end{cases}$$

From this it is easy to check that $\xi = 2 \operatorname{ind}_{\Sigma_2}^{\Sigma_n}(1)$. It follows that there is an isomorphism of ungraded $\mathbb{Q}[\Sigma_n]$-modules

$$H^*(F_n\mathbb{C}; \mathbb{Q}) = \mathbb{Q}[\Sigma_n] \otimes_{\mathbb{Q}[\Sigma_2]} (\mathbb{Q} \oplus \mathbb{Q}).$$

More delicate analysis shows that $M_n = H^{n-1} F_n\mathbb{C}$ is a module over $\mathbb{Z}[\Sigma_n]$ that is free of rank one as a module over the subring $\mathbb{Z}[\Sigma_{n-1}]$. We can also describe the rest of $H^* F_n\mathbb{C}$ as a direct sum of modules of the form $\mathbb{Z}[\Sigma_n] \otimes_{\mathbb{Z}[G]} (M_{\mu_1} \otimes \dots \otimes M_{\mu_r})$.

Using this and the covering spectral sequence

$$H^*(\Sigma_n; H^* F_n\mathbb{C}) \Longrightarrow H^* B_n\mathbb{C}$$

one can show that $H^*(B_n; \mathbb{Q}) = E[a]$ and $H^*(B_p; \mathbb{F}_p) = E[a]$, where $a \in H^1(B_n\mathbb{C}; \mathbb{Z})$ maps to $\sum_{i<j} a_{ij}$ in $H^1(F_n; \mathbb{Z})$. The groups $H^*(B_n; \mathbb{F}_p)$ are in principle known for all n, but are complicated.

3.4. **References.** Most of the material here is originally due to Fred Cohen [11]. His later survey article [10] does many things more cleanly, and also gives further references. I first learnt the above calculation of the character of $H^*F_k\mathbb{C}$ as a Σ_k-module from Erich Ossa, although apparently a similar approach had been independently discovered by Atiyah and Jones some time previously. I have also written but not published an exposition with more details.

4. Projective spaces

Let V be a complex vector space of finite dimension n, equipped with a Hermitian inner product. The group $\mathbb{C}^\times$ acts by multiplication on $V\setminus\{0\}$, and the subgroup $S^1=\{z\in\mathbb{C}\mid |z|=1\}$ preserves the sphere $S(V)=\{v\in V\mid \|v\|=1\}$. The associated projective space $\mathbb{P}V$ is defined as

$$\mathbb{P}V=(V\setminus\{0\})/\mathbb{C}^\times=S(V)/S^1.$$

This is a compact complex manifold of complex dimension $n-1$. Clearly, the points of $\mathbb{P}V$ biject with one-dimensional subspaces of V via $[v]\mapsto\mathbb{C}v$. We can also identify PV with the space

$$P'V:=\{\alpha\in\operatorname{End}_{\mathbb{C}}(V)\mid \alpha^2=\alpha=\alpha^\dagger\ ,\ \operatorname{trace}(\alpha)=1\}:$$

an endomorphism $\alpha\in P'V$ corresponds to a line $L=\mathbb{C}v\in PV$ if $L=\alpha(V)$ or equivalently $\alpha(x)=\langle x,v\rangle v/\langle v,v\rangle$ for all x. We write $\mathbb{C}P^{n-1}$ for $\mathbb{P}(\mathbb{C}^n)$.

There is a "tautological" line bundle $L\to\mathbb{P}V$ with total space

$$E(L)=\{(u,[v])\in V\times\mathbb{P}V\mid u\in\mathbb{C}v\}.$$

If $\dim(V)\le\dim(W)$, then the space of linear embeddings $V\to W$ is connected, so we get a canonical homotopy class of maps $\mathbb{P}V\to\mathbb{P}W$.

If V is an infinite-dimensional vector space (topologised as the colimit of its finite-dimensional subspaces) then we also define $\mathbb{P}V=(V\setminus 0)/\mathbb{C}^\times$, and topologise it as the colimit of the $\mathbb{P}W$'s, where W runs over finite-dimensional subspaces. In particular, we will consider the infinite-dimensional projective space $\mathbb{C}P^\infty=\varinjlim_n\mathbb{C}P^n$. However, in the rest of this section we shall assume that V has finite dimension unless we explicitly state otherwise.

4.1. **(Generalised) Cohomology.** For any linear embedding $j\colon\mathbb{C}^2\to V$, we get an embedding $\mathbb{P}j\colon S^2=\mathbb{C}P^1\to\mathbb{P}V$. There is a unique element $x\in\widetilde{H}^2PV$ such that $\mathbb{P}j^*x$ is the usual generator of $\widetilde{H}^2S^2=\mathbb{Z}$, for every such j. Moreover,

$$H^*\mathbb{P}V=\mathbb{Z}[x]/x^{n+1}.$$

Dually, we have

$$H_*\mathbb{P}V=\mathbb{Z}\{\beta_k\mid 0\le k<n\}$$

with $\langle\beta_k,x^l\rangle=\delta_{k,l}$.

In the limit, we get

$$H^*\mathbb{C}P^\infty=\mathbb{Z}[\![x]\!]$$
$$H_*\mathbb{C}P^\infty=\mathbb{Z}\{\beta_k\mid k\ge 0\}.$$

If we use coefficients $\mathbb{F}_p$ (with $p>2$) then the action of the Steenrod algebra is

$$P^kx^l=\begin{pmatrix} l\\ k\end{pmatrix}x^{l+(p-1)k}\qquad\qquad \beta x^l=0.$$

We can also describe the complex K-theory of PV (compare [4]). We have an element $y=[L-1]\in\tilde{K}^0\mathbb{P}V$, in terms of which we have

$$K^0\mathbb{P}V=\mathbb{Z}[y]/y^n$$
$$K^*\mathbb{P}V=K^*[y]/y^n=\mathbb{Z}[\nu^{\pm1},y]/y^n\qquad \nu\in K^{-2}.$$

The action of the Adams operations is $\psi^k[L]=[L^k]$ and thus $\psi^k(y)=(1+y)^k-1$.

Let E be a multiplicative generalised cohomology theory. A *strict complex orientation* of E is a class $x\in\tilde{E}^2\mathbb{C}P^\infty$ such that the image of x in $\tilde{E}^2\mathbb{C}P^1=\tilde{E}^2S^2$ is the usual generator. We say that E is *complex-orientable* if it admits such an orientation; one can show that this is the case if E_* is torsion-free or concentrated in even degrees. We say that E is complex-oriented if we have chosen an orientation. In particular, H and K (with $x=y/\nu$) are complex-oriented. See [2, 36] for more information about such cohomology theories.

If E is complex-oriented, then we have

$$E^*\mathbb{P}V=E^*[x]/x^n$$
$$E^*\mathbb{C}P^\infty=E^*[\![x]\!]$$

$$E_*\mathbb{P}V = E_*\{\beta_k \mid 0 \leq k < n\}$$

just as with ordinary homology.

4.2. **Differential geometry.** Let W be a one-dimensional subspace of V, corresponding to a point of $\mathbb{P}V$. Let U_W be the set of lines that are not orthogonal to W, which is an open neighbourhood of W in $\mathbb{P}V$. There is a homeomorphism

$$\theta_W\colon \operatorname{Hom}(W, W^\perp) \simeq U_W$$

defined by

$$\theta_W(\alpha) = \operatorname{graph}(\alpha) \leq W \oplus W^\perp = V.$$

The maps θ_W are charts for a holomorphic atlas on $\mathbb{P}V$, making it a complex manifold.

This construction also shows that the tangent bundle of $\mathbb{P}V$ is given by

$$T_W\mathbb{P}V = \operatorname{Hom}(W, W^\perp) = \operatorname{Hom}(W, V - W) = V \otimes W^* - \mathbb{C}$$

(using the fact that $\operatorname{End}(W)$ is canonically isomorphic to $\mathbb{C}$). As V is a trivial bundle of rank n, this gives

$$T\mathbb{P}V = nL^{-1} - 1 \in K^0\mathbb{P}V.$$

Note also that this isomorphism gives a natural Riemannian metric on the tangent bundle.

There is a natural embedding of $\mathbb{P}V$ in a Euclidean space by the map

$$\phi\colon \mathbb{P}V = S(V)/S^1 \to \operatorname{End}(V)$$

$$\phi([v])(w) = \langle w, v\rangle v.$$

This gives a homeomorphism of $\mathbb{P}V$ with the set

$$\{\alpha \in \operatorname{End}(V) \mid \alpha^2 = \alpha = \alpha^* \text{ and } \operatorname{trace}(\alpha) = 1\}.$$

4.3. **Bundles, fibrations and Thom spaces.** The fibration $S^1 \to S(V) \to \mathbb{P}V$ can be easily identified with the evident fibration $S^1 \to S(L) \to \mathbb{P}V$, where $S(L)$ is the circle bundle associated to the line bundle L.

More generally, let L^k denote the k'th tensor power of L. There is thus a fibration

$$S^1 \to S(L^k) \to \mathbb{P}V.$$

This can be identified as follows. Let C_k be the cyclic subgroup of S^1 generated by $\exp(2\pi i/k)$. Let S^1 act on $S(V)/C_k$ by $z.[v] = [z^{1/k}v]$. We then have a fibration

$$S^1 \to S(V)/C_k \to S(V)/S^1 = \mathbb{P}V,$$

which is isomorphic to the fibration above.

Next, we claim that the Thom space $\mathbb{P}V^L$ is homeomorphic to $\mathbb{P}(\mathbb{C} \oplus V)$. Indeed, the Thom space is obtained from the disc bundle

$$D(L) = \{(u, [v]) \in V \times \mathbb{P}V \mid u \in \mathbb{C}v \text{ and } \|u\| \leq 1\}$$

by collapsing out $S(L)$. We define a map

$$D(L)/S(L) \to \mathbb{P}(\mathbb{C} \oplus V)$$

by

$$(u, [v]) \mapsto [\langle v, u\rangle, \sqrt{1 - \|u\|^2}v].$$

It is easy to check that this is a homeomorphism.

A more general result can be obtained in a slightly different way; we claim that

$$\mathbb{P}V^{L^{-1}\otimes U} = \mathbb{P}(U \oplus V)/\mathbb{P}U$$

or equivalently (using the conjugate-linear isomorphism $L \simeq L^{-1}$ given by the metric, and taking $V = \mathbb{C}^{n+1}$ and $U = \mathbb{C}^m$)

$$(\mathbb{C}P^n)^{mL} = \mathbb{C}P^{n+m}/\mathbb{C}P^{m-1}.$$

This space is often called a *stunted projective space* and written $\mathbb{C}P^n_m$.

4.4. **Cell structure.** The main result is that $\mathbb{C}P^n$ has a cell structure

$$\mathbb{C}P^n = S^2 \cup e^4 \cup \ldots \cup e^{2n}.$$

Indeed, we have seen above that $\mathbb{P}(\mathbb{C} \oplus V)$ is the cofibre of a map $S^{2n+1} = S(V) \to \mathbb{P}V$, so that $\mathbb{C}P^{n+1} = \mathbb{C}P^n \cup e^{2n+2}$ as required. Note that this also gives a cell structure

$$\mathbb{C}P^n_m = S^{2m} \cup e^{2m+2} \cup \ldots \cup e^{2m+2n}.$$

(However, $\mathbb{C}P^{n+k}_{m+k}$ is not usually homotopy equivalent to $\Sigma^{2k}\mathbb{C}P^n_m$, as one sees by examining the action of the Steenrod algebra on the mod p cohomology.)

The cell structure on $\mathbb{C}P^n$ can also be obtained by Morse theory (compare [32, Section I.4]). Define a map

$$f : \mathbb{C}P^n \to \mathbb{R} \qquad\qquad f[\underline{z}] = (\sum_{k=0}^{n} k z_k^2)/\|z\|^2.$$

One can show that this is a Morse function on $\mathbb{C}P^n$, in other words that all of its critical points are non-degenerate. The critical points are precisely the basis vectors $[e_k]$ (for $k = 0, \ldots, n$). The map $\theta(u) = [e_k + u]$ is a diffeomorphism of $e_k^{\perp}$ with a neighbourhood of $[e_k]$, and

$$f\theta(u) = f[e_k] + \sum_{i \neq k} (i - k)\|u_i\|^2$$

to second order in u. This shows that the (real) Hessian of f at $[e_k]$ has $2k$ negative eigenvalues (and none of the eigenvalues are zero, which is what nondegeneracy means). Thus, $[e_k]$ has index $2k$. Morse theory tells us that $\mathbb{C}P^n$ has a cell structure with one cell of dimension d for each critical point of index d, just as before.

4.5. **Product structure.** We can regard $\mathbb{C}^{n+1}$ as the space of polynomials of degree at most n, via the bijection $(z_0, \ldots, z_n) \mapsto \sum_i z_i t^i$. With this identification, multiplication of polynomials gives a bilinear map $\mathbb{C}^{n+1} \times \mathbb{C}^{m+1} \to \mathbb{C}^{n+m+1}$, which in turn induces a map $\mathbb{C}P^n \times \mathbb{C}P^m \to \mathbb{C}P^{n+m}$ (using the fact that a product of nonzero polynomials is nonzero).

In particular (using commutativity of $\mathbb{C}[t]$) we get a map

$$(\mathbb{C}P^1)^m/\Sigma_m \to \mathbb{C}P^m.$$

The fundamental theorem of algebra implies that this is a bijection (and hence a homeomorphism, as the source is compact and the target Hausdorff).

We also get a map $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$, which makes $\mathbb{C}P^\infty$ into an Abelian topological monoid. Up to homotopy, this also has inverses, given by the map $\nu : \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ sending $[\sum_i z_i t^i]$ to $[\sum_i \overline{z}_i t^i]$.

4.6. **Homotopy groups.** Using the fibration $S^1 \to S(V) \to \mathbb{P}V$ and the fact that $\pi_* S^1 = \mathbb{Z}$ concentrated in degree 1, we see that

$$\pi_k \mathbb{C}P^n = \begin{cases} \mathbb{Z} & \text{if } k = 2 \\ \pi_k S^{2n+1} & \text{if } k \neq 2 \end{cases}$$

In particular, we have $\pi_k \mathbb{C}P^n = 0$ for $k = 1$ or $2 < k < 2n+1$ and $\pi_{2n+1}\mathbb{C}P^n = \mathbb{Z}$. In the limiting case we get

$$\pi_k \mathbb{C}P^\infty = \begin{cases} \mathbb{Z} & \text{if } k = 2 \\ 0 & \text{otherwise} \end{cases}$$

4.7. **References.** The material in this section is all well-known, and can mostly be found in [44] or [21] (for example).

5. Grassmannians and flag varieties

We again let V be a complex vector space of finite dimension n. We let G_kV denote the set of subspaces $W \leq V$ such that $\dim(W) = k$ (so $PV = G_1V$). If we let $\mathcal{J}(\mathbb{C}^k, V)$ denote the space of inner-product preserving linear maps $j\colon \mathbb{C}^k \to V$, then the construction $j \mapsto j(\mathbb{C}^k)$ gives a bijection $\mathcal{J}(\mathbb{C}^k, V)/U(k) \to G_kV$, and we topologise G_kV so as to make this a homeomorphism. We can also identify PV with the space

$$G'_k(V) := \{\alpha \in \mathrm{End}_{\mathbb{C}}(V) \mid \alpha^2 = \alpha = \alpha^\dagger \text{ , } \mathrm{trace}(\alpha) = k\} :$$

an endomorphism $\alpha \in G'_kV$ corresponds to a subspace $W = j(\mathbb{C}^k) \in G_kV$ if $W = \alpha(V)$ or equivalently $\alpha = j^\dagger j$.

There is a "tautological" bundle $T \to G_kV$ with total space

$$E(T) = \{(u, W) \in V \times G_k(V) \mid u \in W\}.$$

If $\dim(V) \leq \dim(W)$, then the space of linear embeddings $V \to W$ is connected, so we get a canonical homotopy class of maps $G_kV \to G_kW$.

If V is an infinite-dimensional vector space (topologised as the colimit of its finite-dimensional subspaces) then we also define

$$G_kV = \{W < V \mid \dim(W) = k\} = \bigcup_{U<V \text{ , } \dim(U)<\infty} G_kU$$

In particular, we consider the spaces

$$BU(k) = G_k(\mathbb{C}^\infty) = \mathcal{J}(\mathbb{C}^k, \mathbb{C}^\infty)/U(k).$$

One checks that the space $\mathcal{J}(\mathbb{C}^k, \mathbb{C}^\infty)$ is contractible and that $U(k)$ acts freely on it, so $BU(k)$ is indeed a classifying space for principle $U(k)$-bundles, as indicated by the notation.

5.1. **Schubert cells.** Given $0 \leq k \leq n$, we let $D(n)$ denote the set of sequences

$$0 = i_0 \leq \ldots \leq i_n = k.$$

We define $d\colon G_k\mathbb{C}^n \to D(n)$ by **unfinished**.

Given $V \in G_k\mathbb{C}^n$, we define

5.2. **Infinite Grassmannians.**

5.3. **Fock spaces.** Let U be a complex universe. Given subuniverses $L, L' \leq U$ we write $L \sim L'$ iff $L/(L \cap L')$ and $L'/(L \cap L')$ are both finite-dimensional. This is easily seen to be an equivalence relation. We say that L is *standard* if both L and $L^\perp$ are infinite-dimensional, and $U = L \oplus L^\perp$. If L is standard and $L' \sim L$ then L' is also standard. A *polarisation* of U is an equivalence class of standard subuniverses. Let G be a polarisation.

Note that if $L, L' \in G$ then $L + L'$ and $L \cap L'$ also lie in G.

Definition 5.1. Given $M, N \in G$ with $M \leq N$, we put

$$G(M, N) = \{L \in G \mid M \leq L \leq N\}.$$

This is naturally identified with the Grassmannian of subspaces of N/M. The set G can thus be regarded as a filtered colimit of projective varieties.

Definition 5.2. Given $L, L' \in G$ we define $\dim(L, L') = \dim(L'/N) - \dim(L/N)$, for any $N \in G$ with $N \leq L \cap L'$. This is easily seen to be independent of N, and to satisfy $\dim(L, L) = 0$ and

$$\dim(L, L') + \dim(L', L'') = \dim(L, L'').$$

Definition 5.3. Given a vector space V, we write λ^kV for the k'th exterior power. We also write $\lambda^WV = \lambda^{\dim(W)}V$, for any finite-dimensional vector space W. Finally, we write $\det(V) = \lambda^VV$. We note that when $W \leq V$ there is an isomorphism $\det(V) = \det(W) \otimes \det(V/W)$, which is natural in the pair (V, W).

Definition 5.4. Given $L, L' \in G$ we define

$$\det(L,L') = \det(L/(L\cap L'))^* \otimes \det(L'/(L\cap L')) = \mathrm{Hom}(\det(L/(L\cap L')), \det(L'/(L\cap L')))$$

(which is a one-dimensional complex vector space).

Proposition 5.1. The set G can be made into a category, with $\det(L, L')$ as the morphisms from L to L'. Moreover, the composition map

$$\det(L', L'') \otimes \det(L, L') \to \det(L, L'')$$

is an isomorphism.

Proof. First, for any $N \leq L \cap L'$ we put

$$\det(L, L'; N) = \mathrm{Hom}(\det(L/N), \det(L'/N)).$$

If $M \leq N$ then we have canonical isomorphisms $\det(L/M) = \det(L/N) \otimes \det(M/N)$ and $\det(L'/M) = \det(L'/N) \otimes \det(M/N)$. As $\det(M/N)$ is invertible, these induce an isomorphism $\det(L, L'; N) \simeq \det(L, L'; M)$. These isomorphisms compose in the obvious way. Thus, we can replace $\det(L, L')$ by $\det(L, L'; N)$ for any convenient N. Now take $N \leq L \cap L' \cap L''$, and put $Q = L/N$, $Q' = L'/N$ and $Q'' = L''/N$. We have

$$\begin{aligned}
\det(L', L'') \otimes \det(L, L') &= \det(Q'') \otimes \det(Q')^* \otimes \det(Q') \otimes \det(Q)^* \\
&= \det(Q'') \otimes \det(Q)^* \\
&= \det(L, L'').
\end{aligned}$$

This identification is easily seen to be independent of N, and to be associative. □

Definition 5.5. For any $L \in G$, we define the Fock space $F_*(L) = F_*(U, L)$ as follows. For any N, M with $N \leq L \leq M$, we put

$$F_d(L; N, M) = \det(L/N)^* \otimes \lambda^{d+L/N}(M/N) = \mathrm{Hom}(\det(L/N), \lambda^{d+L/N}(M/N)).$$

Now suppose we have $N' \leq N \leq L \leq M \leq M'$. On the one hand, we have $\det(L/N') = \det(L/N) \otimes \det(N/N')$. On the other hand, the ring structure of $\lambda^*(M'/N')$ gives a map

$$\mu\colon \det(N/N') \otimes \lambda^{d+L/N}\left(\frac{M}{N'}\right) = \lambda^{N/N'}\left(\frac{N}{N'}\right) \otimes \lambda^{d+L/N}\left(\frac{M}{N'}\right) \to \lambda^{d+L/N'}\left(\frac{M'}{N'}\right).$$

Let I be the ideal in $\lambda^*(M/N')$ generated by $N/N' \leq \lambda^1(M/N')$. Then $\det(N/N')I = 0$ and $\lambda^*(M/N')/I = \lambda^*(M/N)$. Our map μ thus induces a map

$$\overline{\mu}\colon \det(N/N') \otimes \lambda^{d+L/N}\left(\frac{M}{N}\right) \to \lambda^{L/N'}\left(\frac{M'}{N'}\right),$$

and thus a map

$$\begin{aligned}
F_d(L; N, M) &= \mathrm{Hom}(\det(L/N), \lambda^{L/N}(M/N)) \\
&\simeq \mathrm{Hom}(\det(N/N') \otimes \det(L/N), \det(N/N') \otimes \lambda^{L/N}(M/N)) \\
&\simeq \mathrm{Hom}(\det(L/N'), \det(N/N') \otimes \lambda^{L/N}(M/N)) \\
&\xrightarrow{\overline{\mu}_*} \mathrm{Hom}(\det(L/N'), \lambda^{L/N'}(M'/N')) \\
&= F(L; N', M').
\end{aligned}$$

It is easy to see that these maps are injective, and that they compose together in the obvious way. We can thus define

$$F_*(L) = \varinjlim_{N,M} F_*(L; N, M).$$

Proposition 5.2. There are natural isomorphisms

$$F_*(L') = \det(L', L) \otimes \Sigma^{\dim(L',L)} F_*(L)$$

for all $L', L \in G$.

Proof. Put $m = \dim(L', L)$. It will suffice to give compatible isomorphisms $F_d(L'; N, M) \simeq \det(L', L) \otimes F_{d-e}(L; N, M)$ for all N, M with $N \leq L \cap L'$ and $M \geq L + L'$. Put $n = \dim(L/N)$, so $\dim(L'/N) = n - e$. We then have $\det(L', L) = \det(L'/N)^* \otimes \det(L/N)$, so

$$\begin{aligned} F_d(L'; N, M) &= \det(L'/N)^* \otimes \lambda^{d+n-e}(M/N) \\ &= \det(L, L') \otimes \det(L/N)^* \otimes \lambda^{d+n-e}(M/N) \\ &= \det(L, L') \otimes F_{d-e}(L; N, M) \end{aligned}$$

as required. □

Definition 5.6. Given $L \in G$, we put $G_0(L) = \{L' \in G \mid \dim(L, L') = 0\}$. Given N, M with $N \leq L \leq M$ we put

$$G_0(L; N, M) = G_0(L) \cap G(N, M) = \{L' \mid N \leq L' \leq M \text{ and } \dim(L, L') = 0\}.$$

We also let $D(L)$ denote the line bundle over $G_0(L)$ with fibre $\det(L', L)$ at L'

Proposition 5.3. There is a natural isomorphism

$$\Gamma(G_0(L; N, M); D(L)) = F_0(L; N, M)^*$$

(where $\Gamma(-, -)$ denotes the space of algebraic sections).

Proof. Put $d = \dim(L/N)$, so $\dim(L'/N) = d$ for $L' \in G_0(L)$. Let T be the bundle over $G(N, M)$ with fibre L'/N at L'. The restriction of $D(L)$ to $G_0(L; N, M)$ is $\det(L/N) \otimes \det(T)^*$, so

$$\Gamma(G_0(L; N, M); D(L)) = \det(L/N) \otimes \Gamma(G_0(L; N, M); \det(T)^*).$$

On the other hand, we have

$$F_0(L; N, M)^* = \det(L/N) \otimes \lambda^d(T)^*.$$

The claim now follows from Lemma 5.4 below. □

Lemma 5.4. Let V be a finite-dimensional complex vector space, and let T be the tautological bundle over $\mathrm{Grass}_k(V)$ (the Grassmannian variety of subspaces of V). Then $\Gamma(\mathrm{Grass}(V); \det(T)^*) = \lambda^k(V)^*$.

Proof. Suppose we have an element $\phi \in \lambda^k(V)^*$. For $W \in \mathrm{Grass}_k(V)$ we let $\sigma(\phi)_W$ denote the restriction of ϕ to $\det(W) = \lambda^k W \leq \lambda^k V$, so $\sigma(\phi)_W$ is an element of $\det(W)^*$, which is the fbre of the bundle $\det(T)^*$ at the point W. Thus, we can regard $\sigma(\phi)$ as a section of $\lambda^k(T)^*$, which is easily seen to be algebraic. Thus, we have a map

$$\sigma \colon \lambda^k(V)^* \to \Gamma(\mathrm{Grass}_k(V); \lambda^k(T)^*).$$

If $k = \dim(V)$ then $\mathrm{Grass}_k(V) = \{V\}$ and σ is obviously bijective. We therefore suppose that $k < \dim(V)$.

Now suppose we have $s \in \Gamma(\mathrm{Grass}_k(V); \lambda^k(T))^*$. Let X be the set of linearly independent lists $\underline{v} = (v_1, \ldots, v_k)$ in V^k. Given $\underline{v} \in X$, we define

$$\begin{aligned} W &= \mathrm{span}(\underline{v}) \in \mathrm{Grass}_k(V) \\ \tau(s)(\underline{v}) &= s_W(v_1 \wedge \ldots \wedge v_k). \end{aligned}$$

One checks that $V^k \setminus X$ has codimension $n - k + 1 \geq 2$ in V^k. As $\tau(s)$ is a rational function that is regular away from a closed subvariety of codimension at least two, it extends uniquely as a globally defined polynomial function. We also see from the definition that

$$\begin{aligned} \tau(s)(\lambda_1 v_1, \ldots, \lambda_k v_k) &= \left(\prod_i \lambda_i\right) \tau(s)(v_1, \ldots, v_k) \\ \tau(s)(v_{\pi(1)}, \ldots, v_{\pi(k)}) &= \mathrm{sgn}(\pi) \tau(s)(v_1, \ldots, v_k), \end{aligned}$$

showing that $\tau(s)$ is alternating and multilinear. It can thus be regarded as an element of $(\lambda^k V)^*$. It is easy to see that the maps σ and τ are mutually inverse isomorphisms. □

5.4. **Flag varieties.** For $I \subseteq \{0, \ldots, n\}$, put

$$\mathrm{Flag}_I(V) = \{\underline{W} \in \prod_{i \in I} G_i V \mid W_i < W_j \text{ whenever } i < j\}.$$

If $I = J \amalg \{i\}$ for some i, then $\mathrm{Flag}_I(V)$ is the projective space associated to a certain vector bundle over $\mathrm{Flag}_J(V)$.

Note also that the space $\mathrm{Flag}(V) = \mathrm{Flag}_{\{0,\ldots,n\}}(V)$ is a fibre bundle over $G_k(V)$, with fibre $\mathrm{Flag}(\mathbb{C}^k) \times \mathrm{Flag}(\mathbb{C}^{n-k})$.

5.5. **Presentations as homogeneous spaces.** The group $U(n)$ acts freely and transitively on the space $\mathcal{J}(\mathbb{C}^n, V)$. Given a set

$$I = \{i_1 < \cdots < i_r\} \subseteq \{0, \ldots, n\},$$

we put

$$U_I(n) = \{\alpha \in U(n) \mid \alpha(\mathbb{C}^{i_t}) = \mathbb{C}^{i_t} \text{ for } t = 1, \ldots, n\},$$

which is the stabiliser of the point

$$(\mathbb{C}^{i_1} < \ldots < \mathbb{C}^{i_t}) \in \mathrm{Flag}_I(V),$$

and is isomorphic to

$$U(i_1) \times U(i_2 - i_1) \times \cdots \times U(i_r - i_{r-1}) \times U(n - i_r).$$

We find that there is a canonical homeomorphism

$$\mathrm{Flag}_I(V) = \mathcal{J}(\mathbb{C}^n, V)/U_I(n).$$

If $V = \mathbb{C}^n$ we have $\mathcal{J}(\mathbb{C}^n, V) = U(n)$ and so $\mathrm{Flag}_I(\mathbb{C}^n) = U(n)/U_I(n)$. In particular, we have

$$G_k V = \mathrm{Flag}_{\{k\}}(V) = U(n)/(U(k) \times U(n-k))$$
$$PV = G_1 V = U(n)/(U(1) \times U(n-1)).$$

5.6. **(Generalised) Cohomology.** Let E be an even periodic cohomology theory

5.7. **Bott periodicity.**

5.8. **References.** The material in this section is all well-known, and can mostly be found in [44] or [21] (for example).

6. Milnor hypersurfaces

The Milnor hypersurface H_{mn} (where $m \leq n$) is the complex variety

$$H_{mn} = \{([z],[w]) \in \mathbb{C}P^m \times \mathbb{C}P^n \mid \sum_{i=0}^{m} z_i w_i = 0\}.$$

There are two evident line bundles over $\mathbb{C}P^m \times \mathbb{C}P^n$:

$$L_{([z],[w])} = \mathbb{C}v$$
$$M_{([z],[w])} = \mathbb{C}w.$$

We write $x = e(L)$ and $y = e(M)$ for the Euler classes.

Theorem 6.1. We have

$$H^* H_{mn} = \mathbb{Z}[x,y]/(x^{m+1}, y^n - y^{n-1}x + \ldots + (-x)^n).$$

We now outline a proof of this. It is easy to see that $x^{m+1} = 0$. Consider the bundle L^0 over $\mathbb{C}P^m \times \mathbb{C}P^n$ with fibres

$$L^0_{([z],[w])} = \{u \in \mathbb{C}^{n+1} \mid \sum_{i=0}^{m} z_i u_i = 0\}.$$

Over H_{mn}, the identity map gives an inclusion $M \to L^0$ and thus a nowhere-vanishing section of $\mathrm{Hom}(M, L^0) = \mathrm{Hom}(M, n+1-L^*) = (n+1)M^* - M^* \otimes L^*$, or dually a nowhere-vanishing section of $(n+1)M - M \otimes L$. It follows that the Euler class $e((n+1)M - M \otimes L)$ maps to zero in $H^* H_{mn}$. To compute this Euler class, we work with the evident analogous bundles over $\mathbb{C}P^m \times \mathbb{C}P^\infty$. Here

$$y^{n+1} = e((n+1)M) = e((n+1)M - M \otimes L)(x+y).$$

As $x + y$ is not a zero-divisor in $H^*(\mathbb{C}P^m \times \mathbb{C}P^\infty)$ and

$$(x+y)(y^n - y^{n-1}x + \ldots + (-x)^n) = y^{n+1} - x^{n+1} = y^{n+1},$$

we can conclude that $e((n+1)M - M \otimes L) = y^n - y^{n-1}x + \ldots + (-x)^n$. We thus get a map

$$A^* = \mathbb{Z}[x,y]/(x^{m+1}, y^n - y^{n-1}x + \ldots + (-x)^n) \to B^* = H^* H_{mn},$$

and the claim is that this is an isomorphism. To see this, consider the Serre spectral sequence for the evident fibration $\mathbb{C}P^{n-1} \to H_{mn} \to \mathbb{C}P^m$. This has the form

$$H^*(\mathbb{C}P^m; H^*\mathbb{C}P^{n-1}) = \mathbb{Z}[x,z]/(x^{m+1}, z^n) \Longrightarrow H^* H_{mn}.$$

The E_2 term is concentrated in even bidegrees, so all differentials must be zero, and $E_\infty = E_2$. Thus, the associated graded ring of B^* under the Serre filtration is $\mathbb{Z}[x,z]/(x^{m+1}, z^n)$.

We can now filter A^* by

$$F^s A^* = (\mathbb{Z}[x]/x^{m+1})\{1, y, \ldots, y^{\lfloor s/2 \rfloor}\}.$$

We find that the associated graded ring is just $\mathbb{Z}[x,y]/(x^{m+1}, y^n)$. The Serre filtration on $H^k E$ always stops with $F^k H^k E$, so trivially $y \in H^2 H_{mn}$ has Serre filtration at most two. We thus get a filtration-preserving map $f \colon A^* \to B^*$ and a resulting map of associated graded rings. It is easy to see that this sends y to z, and thus that it is an isomorphism. From this we can conclude that $f \colon A^* \to B^*$ is an isomorphism, as required.

7. Unitary groups

Let V be a complex vector space of finite dimension n, equipped with a Hermitian inner product. We define

$$\begin{aligned} U(V) &= \{ \text{ unitary automorphisms of } V \text{ } \} = \{\alpha \in \mathrm{End}(V) \mid \alpha\alpha^\dagger = 1\} \\ \mathfrak{u}(V) &= \{ \text{ antihermitian endomorphisms of } V \text{ } \} = \{\beta \in \mathrm{End}(V) \mid \beta + \beta^\dagger = 0\}. \end{aligned}$$

It is not hard to see that $U(V)$ is a subgroup and a closed submanifold of $\mathrm{Aut}(V)$, and thus a Lie group.

7.1. **Differential geometry.** The map $\beta \mapsto \alpha\exp(\beta) \simeq \alpha + \alpha\beta$ gives a diffeomorphism of a neighbourhood of 0 in $\mathfrak{u}(V)$ with a neighbourhood of α in $U(V)$, and thus an isomorphism $T_\alpha U(V) \simeq \mathfrak{u}(V)$. This shows that the tangent bundle of $U(V)$ is trivial (which is in fact true for any Lie group). Note also that

$$\dim U(V) = \dim \mathfrak{u}(V) = n^2.$$

We also write

$$\begin{aligned} SU(V) &= \{\alpha \in U(V) \mid \det(\alpha) = 1\} \\ \mathfrak{su}(V) &= \{\beta \in \mathfrak{u}(V) \mid \mathrm{trace}(\beta) = 0\} \end{aligned}$$

We again have $T_\alpha SU(V) = \mathfrak{su}(V)$ and

$$\dim SU(V) = \dim \mathfrak{su}(V) = n^2 - 1.$$

7.2. **Complex reflection maps.** Given a line $L \in \mathbb{P}V$ and $z \in S^1 < \mathbb{C}^\times$ we define

$$r(z, L) = z \oplus 1 \in \mathrm{End}(L \oplus L^\perp) = \mathrm{End}(V)$$

or equivalently

$$r(z, [v])(w) = w + (z - 1)\langle w, v\rangle v$$

(when $v \in S(V) = \{v \in V \mid \|v\| = 1\}$). It is easy to see that $r(z, L) \in U(V)$ and that $\det(r(z, L)) = z$. Given a chosen basepoint $L_0 \in \mathbb{P}V$ we define

$$\overline{r}(z, L) = r(z, L)r(z, L_0)^{-1} \in SU(V).$$

These constructions give continuous maps

$$\begin{aligned} r &\colon \Sigma(\mathbb{P}V_+) \to U(V) \\ \overline{r} &\colon \Sigma\mathbb{P}V \to SU(V) \end{aligned}$$

Moreover, the map $z \mapsto r(z, L_0)$ gives a splitting of the extension $SU(V) \rightarrowtail U(V) \xrightarrow{\det} S^1$ and thus a homeomorphism $U(V) = SU(V) \times S^1$ (which does not respect the group structure).

There is a commutative diagram as follows.

$$\begin{tikzcd} \Sigma\mathbb{C}P^{n-1}_+ \arrow[r, tail] \arrow[d, "r"'] & \Sigma\mathbb{C}P^n_+ \arrow[r, two heads] \arrow[d, "r"'] & S^{2n+1} \arrow[d, "="] \\ U(n) \arrow[r, tail] & U(n+1) \arrow[r, two heads, "\hat{e}_0"'] & S^{2n+1} \end{tikzcd}$$

Here $\hat{e}_0(\alpha) = \alpha(e_0)$, where e_0 is the first basis vector in $\mathbb{C}^{n+1}$. The top copy of S^{2n+1} should be thought of as $S^1 \wedge (\mathbb{C}^n \cup \infty)$ and the bottom one as $\{u \in \mathbb{C}^{n+1} \mid \|u\| = 1\}$. The right hand vertical map sends $z \wedge w$ to $e_0 + (z - 1)(e_0 + w)/(1 + \|w\|^2)$. Its inverse sends $(x + iy, u)$ to $(c + is) \wedge w$, where

$$\begin{aligned} w &= u/((x - 1) + iy) \\ c &= (1 + \|w\|^2)x - \|w\|^2 \\ s &= (1 + \|w\|^2)y. \end{aligned}$$

The top line of the diagram is a cofibration, and the bottom one is a fibration.

7.3. **Cell structure.** The top cell of $\mathbb{C}P^n$ gives a map $D^{2n+1} \to \Sigma\mathbb{C}P^n_+ \xrightarrow{r} U(n+1)$, which in turn gives a homeomorphism $D^{2n+1}/\partial D^{2n+1} \simeq S^{2n+1}$.

From this we get a map $D^{2n+1} \times U(n) \to U(n+1)$ sending $S^{2n} \times U(n)$ to $U(n)$. By induction, we get a cell structure on $U(n+1)$ with cells indexed by the evident basis for the exterior algebra $E[a_1, a_3, \dots, a_{2n+1}]$.

7.4. **(Generalised) cohomology.** For any complex-oriented multiplicative homology theory A (in particular, for $A = H$) we have

$$\begin{aligned} A_*U(V) &= E[\widetilde{A}_*\Sigma\mathbb{P}V_+] = E[a_1, \dots, a_{2n-1}] \\ A_*SU(V) &= E[\widetilde{A}_*\Sigma\mathbb{P}V] = E[a_3, \dots, a_{2n-1}] \\ A^*U(V) &= E[\widetilde{A}^*\Sigma\mathbb{P}V_+] = E[x_1, \dots, x_{2n-1}] \\ A^*SU(V) &= E[\widetilde{A}^*\Sigma\mathbb{P}V] = E[x_3, \dots, x_{2n-1}] \end{aligned}$$

All these rings are actually Hopf algebras, because $U(V)$ and $SU(V)$ are groups. All the generators are actually primitive, in other words the coproduct is $\psi(a_i) = a_i \otimes 1 + 1 \otimes a_i$ and similarly for x_i. Here $\widetilde{A}_*\Sigma\mathbb{P}V_+$ is embedded in $A_*U(V)$ by r_*, and $\widetilde{A}^*\Sigma\mathbb{P}V_+$ is embedded in $A^*U(V)$ by the inverse of isomorphism $\operatorname{Prim} A^*U(V) \rightarrowtail \widetilde{A}^*U(V) \xrightarrow{r^*} \widetilde{A}^*\Sigma\mathbb{P}V_+$.

The Steenrod operations in mod p cohomology (with $p > 2$) are given by

$$\begin{aligned} P^k x_{2l+1} &= \binom{l}{k} x_{2(l+(p-1)k)+1} \\ \beta x_{2l+1} &= 0. \end{aligned}$$

7.5. **The infinite unitary group.** Write $U = \varinjlim_n U(n)$. This has

$$\begin{aligned} A_*U &= E[\widetilde{A}_*\Sigma\mathbb{C}P^\infty_+] = E[a_{2k+1} \mid k \geq 0] \\ A^*U &= E[\widetilde{A}^*\Sigma\mathbb{C}P^\infty_+] = E[x_{2k+1} \mid k \geq 0] \\ \pi_*U &= (0, \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, \dots) \end{aligned}$$

This shows that $\pi_*U = \pi_{*+2}U = \pi_*\Omega^2 U$. In fact, we have a homotopy equivalence $U \simeq \Omega^2 U$; this is called Bott periodicity.

7.6. **The Miller splitting.**

7.7. **References.** Most of the above can be found in [39, Chapter IV]. The homology ring is also calculated in [44, Section VII.4]. For the theory of Lie groups in general, see [1].

8. Projective unitary groups

Lemma 8.1. *$PU(n)$ fits into diagrams as follows, in which each square is a homotopy-pullback:* The second diagram maps in an obvious way to the first.

Corollary 8.2. *$PU(n)$ is the homotopy fibre of the map $\mathbb{C}P^\infty \to BU(n)$ classifying nL^*, or equivalently the frame bundle $V_n(nL^*)$.*

This means we can approach $E^*PU(n)$ using the tower

$$PU(n) = V_n(nL^*) \to V_{n-1}(nL^*) \to \cdots \to V_1(nL^*) = \mathbb{C}P^{n-1} \to V_0(nL^*) = \mathbb{C}P^\infty.$$

One can also check that $V_k(nL^*) = PV_k(\mathbb{C}^n) = V_k(\mathbb{C}^n)/S^1$. This has an evident action of $PU(k)$, with orbit space $G_k(\mathbb{C}^n)$.

The pullback of nL^* to $V_k(nL^*)$ has a k-dimensional trivial summand, and we write W_k for its orthogonal complement; then $V_{k+1}(nL^*)$ is the sphere bundle in W_k.

If we have a complex orientation $x \in E^2\mathbb{C}P^\infty$ then the Chern polynomial of nL^* is $(t-x)^n$. In $E^*V_k(nL^*)$ this becomes divisible by t^k, and the Chern polynomial of W_k is $(t-x)^n/t^k$, so the Euler class is $\pm \left(\begin{array}{c} n \\ k \end{array} \right) x^{n-k}$. Now put

$$A(k) = A(E, n, k) = E^*V_k(nL^*).$$

We deduce that there are short exact sequences

$$A(k)/\left(\begin{array}{c} n \\ k \end{array} \right) x^{n-k} \to A(k+1) \to \Sigma^{2n-2k-1}\,\mathrm{ann}(\left(\begin{array}{c} n \\ k \end{array} \right) x^{n-k}, A(k)).$$

Note also that C_n acts on $SU(n)$ by translation, and $SU(n)$ is connected, so each element of C_n acts by a map homotopic to the identity. It follows that the projection $p\colon SU(n) \to PU(n)$ and the associated transfer $p^!\colon \Sigma^\infty PU(n)_+ \xrightarrow{\Sigma^\infty} SU(n)_+$ satisfy $p^!p = n.1_{SU(n)}$ and $pp^! = n.1_{PU(n)}$. Thus, if n is invertible in E^* then the map

$$p^*\colon E^*PU(n) \to E^*SU(n) = \lambda^*_{E^*}\widetilde{E}^*\Sigma\mathbb{C}P^{n-1}$$

is an isomorphism.

9. Lens spaces and BC_p

Fix an integer n and an odd prime p. Write $U = \mathbb{C}^n$ and $P = \mathbb{P}U = \mathbb{C}P^{n-1}$. Regard C_p as the subgroup $\{\zeta \mid \zeta^p = 1\}$ of $\mathbb{C}^\times$ generated by $e^{2\pi i/p} = \omega$. The lens space $B = B_n$ is the quotient $S(U)/C_p = S^{2n-1}/C_p$. The action of C_p on S^{2n-1} is free, so B is a manifold.

Let L be the tautological line bundle over P, with fibres $L_{[v]} = \mathbb{C}v$. We can also regard B as the sphere bundle of $L^{\otimes p}$ via the map $B \to S(L^{\otimes p})$ sending $[u]$ to $([u], u^{\otimes p})$.

9.1. **Cell structure.** We now describe a cell structure on $S(U)$ that is compatible with the action of C_p and thus induces a cell structure on B. This comes from [39, Section V.5] or [44, Section II.7]. Write

$$\begin{aligned} A &= \{z \in \mathbb{C} \mid |z| \leq 1 \text{ and } 0 \leq \arg(z) \leq 2\pi/p\} \\ e^{2k} &= \{(z_0, \ldots, z_k, 0, \ldots, 0) \mid z_k \in [0,1]\} \\ e^{2k+1} &= \{(z_0, \ldots, z_k, 0, \ldots, 0) \mid z_k \in A\} \end{aligned}$$

One can check that e^i is a closed i-cell and that the cells ζe^i with $\zeta \in C_p$ and $0 \leq i < 2n$ give a CW structure on B. The differential in the cellular chain complex is

$$\begin{aligned} \partial(e^{2k}) &= \sum_{\zeta \in C_p} \zeta e^{2k-1} \\ \partial(e^{2k+1}) &= \omega e^{2k} - e^{2k}. \end{aligned}$$

This gives a cell structure $B = e^0 \cup \ldots \cup e^{2n-1}$ with $\partial(e^{2k}) = pe^{2k-1}$ and $\partial(e^{2k+1}) = 0$. It follows that

$$H_*(B; \mathbb{F}_p) = \mathbb{F}_p\{e^0, \ldots, e^{2n-1}\}.$$

9.2. **Generalised cohomology.** As $B = S(L^{\otimes p})$, the cofibre of the evident projection $B \to P$ is the Thom space $P^{L^{\otimes p}}$. With this identification, the natural map from P to the cofibre becomes the zero-section $P \to P^{L^{\otimes p}}$. For any generalised cohomology theory A, this cofibration gives us a long exact sequence

$$\widetilde{A}^{*+1} P^{L^{\otimes p}} \leftarrow A^{*+1}P \leftarrow A^*B \leftarrow \widetilde{A}^* P^{L^{\otimes p}}.$$

If A^* is a complex-oriented multiplicative theory then $\widetilde{A}^* P^{L^{\otimes p}}$ is a free module over $A^*P = A^*[x]/x^n$ generated by the Thom class $u(L^{\otimes p})$. Under the zero-section $P \to P^{L^{\otimes p}}$, this Thom class pulls back to give the Euler class $e(L^{\otimes p}) = [p](e(L)) = [p](x)$.

In particular, if $A^* = H^*(-; \mathbb{F}_p)$ then $[p](x) = 0$. Putting this into our long exact sequence we find that the restriction maps $H^{2k}(P; \mathbb{F}_p) \to H^{2k}(B; \mathbb{F}_p)$ and the coboundary maps $H^{2k+1}(B; \mathbb{F}_p) \to H^{2k+2}(P; \mathbb{F}_p)$ are isomorphisms. From this, we conclude that

$$H^*(B; \mathbb{F}_p) = \mathbb{F}_p[x]/x^n \otimes E[a],$$

where $x \in H^2(B; \mathbb{F}_p)$ is the image of the usual generator of H^2P, and $a \in H^1(B; \mathbb{F}_p)$ is the unique class with $\delta(a) = x$, or equivalently the unique class whose Bockstein is $\beta(a) = x$.

The Steenrod action is

$$\begin{aligned} P^k x^l &= \binom{l}{k} x^{l+(p-1)k} \\ \beta x^l &= 0 \\ P^k a &= 0 \\ \beta a &= x. \end{aligned}$$

It follows that the Milnor Bockstein operations Q_k are given by

$$\begin{aligned} Q_k x &= 0 \\ Q_k a &= x^{p^k} \end{aligned}$$

We can also be more explicit in the case of complex K-theory. Write $y = L - 1 \in K^0 P$; it is not hard to check that

$$\begin{aligned} K^0 B &= \mathbb{Z}[y]/(y^n, (1+y)^p - 1) \\ K^1 B &= \mathbb{Z} \end{aligned}$$

9.3. **Homotopy groups.** Using the evident fibration $C_p \to S^{2n-1} \to B$, we find that

$$\begin{aligned} \pi_0 B &= 0 \\ \pi_1 B &= C_p \\ \pi_k B &= 0 && \text{for } 1 < k < 2n-1 \\ \pi_{2n-1} B &= \mathbb{Z}. \end{aligned}$$

9.4. **The infinite case.** Write $B_\infty = \varinjlim_n B_n = S^\infty / C_p$. As S^∞ is contractible, the long exact sequence for the fibration $C_p \to S^\infty \to B_\infty$ gives

$$\pi_k B_\infty = \begin{cases} C_p & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we see that B_∞ is a model for the classifying space BC_p. We have

$$\begin{aligned} H^*(BC_p; \mathbb{F}_p) &= \mathbb{F}_p[x] \otimes E[a] \\ H^*(BC_p; \mathbb{Z}) &= \mathbb{Z}[x]/px \\ K^0 BC_p &= \mathbb{Z}[y]/((1+y)^p - 1) \simeq \mathbb{Z}[C_p]. \end{aligned}$$

9.5. **The intermediate case.** It is also useful to consider the space $B'_n = B_n \cup e^{2n}$, so that $B_n \subset B'_n \subset B_{n+1}$. This has cohomology

$$H^*(B'; \mathbb{F}_p) = \mathbb{F}_p\{1, a, x, \dots, x^{2n}\} = \mathbb{F}_p[x] \otimes E[a]/(x^{2n+1}, ax^{2n}).$$

9.6. **Thom spectra.** We next consider the Thom spectrum B_n^{mL} for $m \in \mathbb{Z}$. When $m \geq 0$ this is a space, and it has a nice alternative description: there is a homeomorphism

$$B_n^{mL} = B_{n+m}/B_m = e^{2m} \cup \dots \cup e^{2(n+m)-1}.$$

To see this, observe that a point in the total space $E(mL)$ over B_n has the form $([u], v_1, \dots, v_m)$, with $u \in S^{2n-1}$ and $v_i \in \mathbb{C}u$ for each i. We define a map $\pi \colon \mathbb{C}^{n+m} \setminus 0 \to S^{2(n+m)-1}$ by $\pi(w) = w/\|w\|$, and a map $\theta \colon E(mL) \to B_{n+m} \setminus B_m$ by

$$\theta([u], v_1, \dots, v_m) = [\pi(u, \langle u, v_1 \rangle, \dots, \langle u, v_m \rangle)].$$

One can check that θ is a homeomorphism. By passing to the one-point compactifications, we obtain a homeomorphism $B_n^{mL} = B_{n+m}/B_m$ as claimed. This naturally identifies $H^*(B_n^{mL}; \mathbb{F}_p)$ with the subquotient $\mathbb{F}_p\{x^m, x^m a, \dots, x^{n+m-1}, x^{n+m-1}a\}$ of $H^*(BC_p; \mathbb{F}_p) = \mathbb{F}_p[x] \otimes E[a]$.

In fact, for all $m \in \mathbb{Z}$ there is a similar identification of $H^*(B_n^{mL}; \mathbb{F}_p)$ as a subquotient of $\mathbb{F}_p[x, x^{-1}] \otimes E[a]$. There is a unique action of the Steenrod algebra on $\mathbb{F}_p[x, x^{-1}] \otimes E[a]$ compatible with this, given by

$$P^k x^l = \binom{l + p^m}{k} x^{l + (p-1)k} \qquad \text{for } m \gg 0.$$

The spectra B_n^{mL} also have an interesting periodicity property:

Proposition 9.1. $B_n^{(m+k)L} \simeq \Sigma^{2k} B_n^{mL}$ whenever p^{n-1} divides k.

Proof. It is enough to show that $p^{n-1}L$ is stably isomorphic to the trivial bundle $p^{n-1}\mathbb{C}$ over B_n, or equivalently that $p^{n-1}y = 0$ in $K^0 B_n = \mathbb{Z}[y]/(y^n, (1+y)^p - 1)$. In this ring we have $py + p(p-1)/2y^2 + \dots + y^p = 0$ so $py \in (y^2)$ so $p^{n-1}y \in (y^n) = 0$ as required. □

Note that we really only needed $p^{n-1}(L-1) = 0$ as stable spherical fibrations rather than as vector bundles, or in other words that $p^{n-1}(L-1) = 0$ in Adams' group $J(B_n)$ rather than in $K(B_n)$. See [21] for more about the groups $J(X)$.

9.7. **Lin's theorem.** One can assemble the spectra B_∞^{mL} into a tower

$$B_\infty \leftarrow B_\infty^{-L} \leftarrow B_\infty^{-2L} \leftarrow \dots.$$

By applying the functor $H^*(-;\mathbb{F}_p)$, we get a sequence of inclusions

$$R \rightarrowtail R.x^{-1} \rightarrowtail R.x^{-2} \rightarrowtail R.x^{-3} \rightarrowtail \dots,$$

where $R = \mathbb{F}_p[x] \otimes E[a]$ and thus every term is a subgroup of $\widehat{R} = \mathbb{F}_p[x^{\pm 1}] \otimes E[a]$. If we let $\widehat{B}$ be the homotopy inverse limit of the tower, we might thus expect that $H^*(\widehat{B};\mathbb{F}_p) = \widehat{R}$. However, $\widehat{R}$ contains many elements that are in the image of the Steenrod operation P^k for an unbounded set of k's, and this cannot happen in the cohomology of a spectrum, so we cannot have $H^*(\widehat{B};\mathbb{F}_p) = \widehat{R}$. Instead, we have the celebrated theorem of Lin, which says that $\widehat{B} = (S^{-1})^\wedge_p$, and thus $H^*(\widehat{B};\mathbb{F}_p) = \mathbb{F}_p\{ax^{-1}\}$. For a modern proof in its proper context, see [30, Corollary XX.6.2]. (The notational conventions in force there are that $G = (\mathbb{Z}/p)^r$ and that all spectra are implicitly completed at p.)

9.8. **Stable splitting.** Another interesting phenomenon is the stable splitting of $B_\infty = BC_p$. Note that B_∞ is homotopy equivalent to $(\mathbb{C}[t]\setminus 0)/C_p$, so we take this as our model. For any $k \geq 0$ we define $\theta_k \colon B_\infty \to B_\infty$ by $\theta_k[f] = [f^k]$. If $k = l \pmod p$ then the map $\phi_t[f] = [tf^k + (1-t)f^l]$ is well-defined, so $\theta_k \simeq \theta_l$. It is also clear that $\theta_k\theta_l = \theta_{kl}$. Thus, the maps θ_k give an action of $\mathrm{Aut}(C_p) = \mathbb{F}_p^\times$ on B_∞.

One can check that the spectrum $X = S_p^0 \vee \Sigma^\infty B_\infty$ is p-complete, so we get an action of the p-adic group ring $\mathbb{Z}_p[\mathbb{F}_p^\times]$ on X. For each $k \in \mathbb{F}_p^\times$ there is a unique "Teichmüller representative" $\hat{k} = \lim_{i\to\infty} k^{p^i} \in \mathbb{Z}_p^\times$ such that $\hat{k} = k \pmod p$ and $\hat{k}^{p-1} = 1$. The elements

$$e_i = \frac{1}{p-1} \sum_{k \in \mathbb{F}_p^\times} \hat{k}^{-i}[k] \in \mathbb{Z}_p[\mathbb{F}_p^\times]$$

are orthogonal idempotents with $\sum_{i=0}^{p-2} e_i = 1$, so we have a stable splitting $X = \bigvee_{i=0}^{p-2} e_i X$. Moreover, we have

$$H^*(e_i X;\mathbb{F}_p) = \mathbb{F}_p\{x^j a^\epsilon \mid j + \epsilon = i \pmod{p-1}\}.$$

In particular, we have

$$H^*(e_0 X;\mathbb{F}_p) = \mathbb{F}_p[x^{p-1}] \otimes E[x^{p-2}a],$$

and in fact there is a homotopy equivalence $e_0 X = (\Sigma^\infty B\Sigma_{p+})^\wedge_p$.

It is a very common phenomenon for classifying spaces of finite groups to split stably as a wedge of simpler pieces. See [7] for a survey.

10. Fermat hypersurfaces

Fix an odd integer n and an odd prime p. Write $U = \mathbb{C}^{n+1}$ and $P = \mathbb{P}U = \mathbb{C}P^n$. Let $\phi\colon U \to \mathbb{C}$ be the map $\phi(z) = \sum_{i=0}^n z_i^p$. We shall study the Fermat hypersurface

$$V = \{[u] \in P \mid \phi(u) = 0\}.$$

Proposition 10.1. *V is a smooth complex manifold.*

Proof. Suppose that $u \neq 0$ and $\phi(u) = 0$. Note that the map $u^\perp \to P$ sending v to $[u+v]$ is an open embedding. It is thus enough to check that $\{v \in u^\perp \mid \phi(u+v) = 0\}$ looks like a manifold near $0 \in u^\perp$, and thus enough to show that $\phi(u+\epsilon v) \neq 0$ to first order in ϵ, for some $v \in u^\perp$. Let v be the vector $(\overline{u}_0^{p-1}, \dots, \overline{u}_n^{p-1})$, so that

$$\langle u, v\rangle = \sum_i u_i^p = \phi(u) = 0$$

and thus $v \in u^\perp$. One can check directly that $\phi(u+\epsilon v) = p\epsilon\|v\|^2$ to first order in ϵ, as required. $\square$

Proposition 10.2. *The normal bundle to V in P is L^{-p}.*

proof?

10.1. Morse theory.

Consider the function $f\colon P \to \mathbb{R}$ defined by

$$f[z] = |\sum_i z_i^p| / \sum_i |z_i|^p.$$

It is clear that $0 \leq f \leq 1$, and that $f(a) = 0$ if and only if $a \in V$.

We next consider the space

$$E = \{z \in S^{2n+1} \mid z_i^p \in [0,1] \text{ for } 0 \leq i \leq n\}.$$

This has a free action of C_p, and we write $B = E/C_p$. It is not hard to see that the evident map $B \to S^{sn+1}/S^1 = P$ is injective, so we think of B as a subspace of P. Explicitly, it is the subspace of all points $[z_0 : \ldots : z_n]$ such that all the numbers z_i^p have the same argument. By the triangle inequality, we thus have

$$B = \{[z] \in P \mid f[z] = 1\}.$$

We next claim that f is continuously differentiable on $P\setminus V$, with no critical points in $P\setminus(V \cup B)$. To see this, note that for any $u, v \in \mathbb{C}$ with $u \neq 0$ we have $|u + \epsilon v| = |u| + \mathrm{Re}(\epsilon u\overline{v}/|u|)$ to first order in ϵ. Thus, when $f[z] \neq 0$ we have

$$f[z+\epsilon w] = \frac{|\sum z_i^p| + p\epsilon |\sum z_i^p|^{-1}\,\mathrm{Re}(\sum \overline{z}_i^p \sum z_j^{p-1} w_j)}{\sum |z_i|^p + p\epsilon\,\mathrm{Re}(\sum |z_i|^{p-2}\overline{z}_i w_i)}$$

(to first order). To simplify this, write

$$\begin{aligned}
\overline{z}^{p-1} &= (\overline{z}_0^{p-1}, \dots, \overline{z}_n^{o-1}) \\
z|z|^{p-2} &= (z_0|z_0|^{p-2}, \dots, z_n|z_n|^{p-2}) \\
\alpha &= \sum |z_i|^p \\
\beta &= \sum z_i^p.
\end{aligned}$$

We find that

$$f[z+\epsilon w] = f[z] + p\epsilon\,\mathrm{Re}\left(\frac{\overline{\beta}}{\alpha|\beta|}\langle w, \overline{z}^{p-1}\rangle - \frac{|\beta|}{\alpha^2}\langle w, z|z|^{p-2}\rangle\right).$$

This shows that f is continuously differentiable. If $[z]$ is a critical point then the real part of the expression in brackets must vanish for all w. As we can replace w by iw, we see that the expression in brackets must itself vanish for all w. It follows that $\overline{z}^{p-1}$ and $z|z|^{p-2}$ must be linearly dependent, say $\lambda^p \overline{z}_i^{p-1} = z_i|z_i|^{p-2}|$ for some λ and all i. It is easy to see that $|\lambda| = 1$, and multiplying by $\overline{z}_i$ gives $(\lambda\overline{z}_i)^p = |z_i|^p \geq 0$. This means that $z \in B$, as required.

We now apply the ideas of Morse theory. Write $M = f^{-1}\{\frac{1}{2}\} \subset P$. By standard results on the solution of differential equations on manifolds, there is a continuously differentiable function $\phi\colon (0,1) \times M \to P$ with $\phi(\frac{1}{2}, a) = a$ and

$$\frac{d}{dt}\phi(t,a) = \left(\frac{\nabla f}{\|\nabla f\|^2}\right)(\phi(t,a)).$$

It follows that $f(\phi(t,a)) = t$, and that the map $\phi_t\colon a \mapsto \phi(t,a)$ gives a homeomorphism $M \simeq f^{-1}\{t\}$.

For small $t > 0$, one expects that $f^{-1}\{t\}$ should be homeomorphic to the sphere bundle of the normal bundle to $V = f^{-1}\{0\}$ in P. This can be made precise, and in fact we obtain a homeomorphism between the total space of the normal bundle and $P \setminus B$. After recalling that the normal bundle is L^{-p} and adding a point at infinity, we obtain a homeomorphism

$$V^{L^{-p}} = P/B.$$

Note that V is homotopy equivalent to the total space of L^{-p}, which is obtained from P by removing B. It is easy to see that B has codimension n in P, and so transversality arguments show that the map $\pi_k V \to \pi_k P$ is an isomorphism for $k < n-1$ and an epimorphism for $k = n-1$.

10.2. **Cohomology.** We need to consider the following three maps:

$$\begin{aligned} i\colon \mathbb{C}P^{(n-1)/2} &\to V & i[w] &= [w : -w] \\ j\colon V &\to P & &\text{(inclusion)} \\ q\colon V &\to \mathbb{C}P^{n-1} & q[z_0 : \ldots : z_n] &= [z_0 : \ldots : z_{n-1}]. \end{aligned}$$

The definition of q is legitimate because whenever $(z_0, \ldots, z_n) \neq 0$ and $\sum_i z_i^p = 0$ we must have $(z_0, \ldots, z_{n-1}) \neq 0$. Write x for the usual generator of $H^2\mathbb{C}P^d$, for any $d > 0$. It is easy to see that qi and ji are homotopic to the usual inclusions, so that $i^*q^*x = i^*j^*x = x$. We also write x for $q^*x \in H^2V$.

Next, observe that i is a smooth embedding of real codimension $2n-2-(n-3) = n+1$, so we have a Gysin map $i_!\colon H^*\mathbb{C}P^{(n-3)/2} \to H^{*+n+1}V$, using which we define $y = i_!(1) \in H^{n+1}V$. More explicitly, we can collapse out a tubular neighbourhood of $\mathbb{C}P^{(n-3)/2}$ in V to get a map from V to the Thom space of an $(n+1)$-dimensional complex bundle, and we define y to be the pullback of the Thom class.

Note that $i_!$ is a map of modules over H^*V, so we have $x^{(n-3)/2}y = i_!(x^{(n-3)/2})$. On the other hand, $x^{(n-3)/2}$ is the fundamental class in $H^*\mathbb{C}P^{(n-3)/2}$, so it is equal to $k_!(1)$, where k is the inclusion of a point in $\mathbb{C}P^{(n-3)/2}$. It follows that $x^{(n-3)/2}y = i_!k_!(1)$ is the fundamental class in H^*V.

Using this, Poincaré duality, and the fact that i is $(n-1)$-connected, we conclude that

$$H^*V = \mathbb{Z}\{1, x, \ldots, x^{(n-3)/2}, y, xy, \ldots, x^{(n-3)/2}y\} \oplus H^{n-1}V.$$

Using the fact that q has degree p, we see that x^{n-1} is p times the fundamental class $x^{(n-3)/2}y$, and thus that $x^{(n+1)/2} = py$.

10.3. **The middle dimension.** We now analyse the middle-dimensional cohomology $H^{n-1}V$. For this, we need to understand the cohomology of B.

Proposition 10.3. There is a stable splitting $B = S^n/C_p \vee Z$, where Z is a finite wedge of copies of S^n.

Proof. Recall that $B = E/C_p$, where

$$E = \{z \in S^{2n+1} \mid z_i^p \in [0,1] \text{ for } 0 \leq i \leq n\}.$$

We can identify E with the $(n+1)$-fold join $C_p * \ldots * C_p$, so the usual formula $\widetilde{H}^m(X * Y) = (\widetilde{H}^*X \otimes \widetilde{H}^*Y)^{m-1}$ shows that $\widetilde{H}^*E = \widetilde{H}^0(C_p)^{\otimes(n+1)}$, concentrated in degree n.

Note that both E and S^n are n-dimensional, $(n-1)$-connected CW complexes with a free action of C_p. The usual proof of the essential uniqueness of BC_p gives canonical $(n-1)$-connected maps $B \xrightarrow{f} BC_p \xleftarrow{g} S^n/C_p$, and non-canonical maps $B \xrightarrow{r} S^n/C_p \xrightarrow{s} B$ compatible with f and g.

As $g^*\colon H^*BC_p \to H^*S^n/C_p$ is surjective, we conclude that $(rs)^*\colon H^*S^n/C_p \to H^*S^n/C_p$ is the identity and thus that rs is a homotopy equivalence. After replacing s by $s(rs)^{-1}$ if necessary, we may assume that $rs = 1$. It follows that there is a stable splitting $B = S^n/C_p \vee Z$ for some Z. Because f, g, r and s are $(n-1)$-connected and B is n-dimensional, we see that the homology of Z with any coefficients is concentrated in dimension n and thus that Z is a wedge of copies of S^n. □

Proposition 10.4. The number of spheres in Z is $N = ((p-1)^{n+1} - 1)/p$.

Proof. In this proof, all cohomology has coefficients $\mathbb{F}_p$. It will be enough to show that the rank of H^nB is $N+1$.

We will need a little structure theory of modules over the ring $A = \mathbb{F}_p[C_p]$. Write $u = [e^{2\pi i/p}] - [1]$. One can check that $A = \mathbb{F}_p[u]/u^p$ and that $\sum_{\zeta\in C_p}[\zeta] = u^{p-1}$. It follows that the reduced cohomology of the discrete space C_p is isomorphic to $I = A/u^{p-1}$ as an A-module, and thus that the reduced cohomology of E is $\widetilde{H}^nE = I^{\otimes(n+1)}$. By applying Schanuel's lemma to the short exact sequences $\mathbb{F}_p \xrightarrow{u^{p-1}} A \to I$ and $I\otimes I \to A\otimes I \to I$ we find that

$$I\otimes I \oplus A \simeq A\otimes I \oplus \mathbb{F}_p = (p-1)A \oplus \mathbb{F}_p.$$

By the Krull-Schmidt theorem, we can cancel to get

$$I\otimes I \simeq (p-2)A \oplus \mathbb{F}_p.$$

(In fact, it is not too hard to just write down an isomorphism here.) It follows easily that $H^nE = I^{\otimes(n+1)}$ is the direct sum of $\mathbb{F}_p$ with a free module F over A. By counting dimensions, the rank of F must be $N = ((p-1)^{n+1} - 1)/p$.

Now consider the covering spectral sequence

$$H^s(C_p; H^tE) \Longrightarrow H^{s+t}B.$$

The line $t = 0$ is just $H^*BC_p = \mathbb{F}_p[x]\otimes E[a]$. The line $t = n$ is a free module of rank one over H^*BC_p on one generator $b \in E_2^{0n}$, together with the group F^{C_p} (which is also located in E_2^{0n} and is annihilated by a and x). All other lines are zero, so the only possible differential is d_{n+1}. The image of F^{C_p} must be annihilated by x, which acts regularly on H^*BC_p, so we must have $d_{n+1}(F^{C_p}) = 0$. Because B has dimension n, the element $x^{(n+1)/2}$ must be hit, so $d_{n+1}(b)$ must be a unit multiple of $x^{(n+1)/2}$. From this it is easy to see that the E_∞ page consists of $\mathbb{F}_p\{1, a, \dots, x^{(n-1)/2}a\}$ on the bottom line, together with F^{C_p} in E_∞^{0n}. It follows that H^nB has dimension $N+1$, as required. □

Corollary 10.5. The group $H^{n-1}V$ is free Abelian of rank $N+2$.

Proof. Our earlier study of the cohomology of V showed that it is concentrated in even degrees, and it is easy to see that this is valid with any field coefficients. This implies that H^*V is a free Abelian group concentrated in even degrees, so we need only check that $H^{n-1}(V;\mathbb{F}_p)$ has rank $N+2$. For the rest of this proof, all cohomology has coefficients $\mathbb{F}_p$.

The cofibration $B \to P \to V^{L^{-p}}$ together with the Thom isomorphism $\widetilde{H}^*V^{L^{-p}} \simeq H^{*-2}V$ gives an exact sequence

$$0 = H^nP \to H^nB \to H^{n-1}V \to H^{n+1}P \to H^{n+1}B = 0.$$

Here the first term is zero because n is odd, and the last is zero because B has dimension n. It follows easily that $H^{n-1}V$ has rank $N+2$. □

10.4. **References.** Most of the material here comes from [24]. The fact that the inclusion $V \to P$ is $(n-1)$-connected is a special case of the Lefschetz hyperplane theorem in complex algebraic geometry. For a Morse-theoretic proof, see [32, Section I.7].

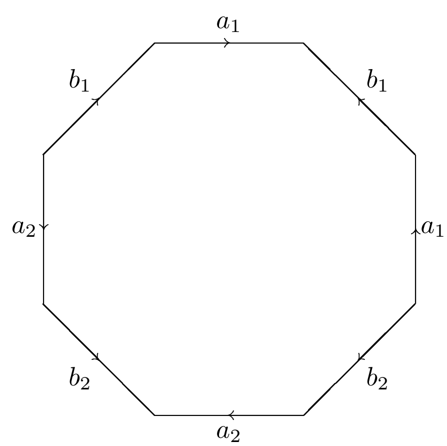
11. Two-dimensional manifolds

In this section we consider compact connected closed oriented smooth manifolds of dimension two (which we shall refer to as surfaces). For more discussion, see [41, Chapter I]. It is well-known that surfaces are classified by their genus, which is a nonnegative integer. A surface of genus zero is homeomorphic to S^2, and a surface of genus one is homeomorphic to the torus $S^1 \times S^1$.

The simplest model of a surface of genus $g > 0$ is given by identifying sides in a polygon. Let $P = P_g$ be a $4g$-gon in $\mathbb{R}^2$. We give each edge a label and a direction. The labels are

$$a_1, b_1, a_1, b_1, a_2, b_2, a_2, b_2, \dots, a_g, b_g, a_g, b_g$$

The first edge with each label is oriented anticlockwise, and the second one is oriented clockwise. We illustrate the scheme in the case $g = 2$.



Let $M = M_g$ be the quotient space of P where edges with the same label are identified together (and thus all the vertices of P are identified to a single point, which we take as the basepoint). It is trivial to check that each point of P other than the basepoint has a neighbourhood homeomorphic to a disc, and only a little harder to check that this holds for the basepoint as well. This means that M is a topological manifold, and one can easily give it a smooth structure.

Note that the image of each edge a_i or b_i in M is a closed loop passing through the basepoint, so we can regard it as an element of $\pi_1 M$. One can show that

$$\pi_k M = \begin{cases} \langle a_1, \dots, a_g, b_1, \dots b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This shows that the universal cover of M is contractible; in fact, it is homeomorphic to $\mathbb{R}^2$. Moreover, we have

$$H^k M = \begin{cases} k = 0 & \mathbb{Z} \\ k = 1 & \mathbb{Z}\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\} \\ k = 2 & \mathbb{Z}\gamma \end{cases}$$

The multiplicative structure is as follows: we have $\alpha_i \beta_i = \gamma$ for all i, and all other products of α's and β's vanish.

Let W_{2g} be a wedge of $2g$ circles, so that $pi_1 W_{2g}$ is a free group on $2g$ generators, which we can call $a_1, \dots, a_g, b_1, \dots, b_g$. Write

$$c = [a_1, b_1] \dots [a_g, b_g] \in \pi_1 W_{2g} = [S^1, W_{2g}].$$

It is not hard to see that there is a cofibration

$$S^1 \xrightarrow{c} W_{2g} \to M_g \to S^2.$$

Note that any smooth Riemann surface is a surface of the type we are considering, so it will be homeomorphic to M_g for some g. Every Riemann surface of genus g is holomorphically isomorphic to a projective complex variety of the form

$$V = \{[x : y : z] \in \mathbb{C}P^2 \mid y^2 z^g = \prod_{i=0}^{g+1} (x - \lambda_i z)\},$$

for suitable complex numbers $\lambda_0, \ldots, \lambda_{g+1}$ (with no two the same).

One can also tesselate the open unit disc D with $4g$-gons whose sides are geodisics in the hyperbolic metric, and let G be the group of holomorphic automorphisms of this tesselation. This can be done in such a way that each polygon in the tesselation is a fundamental domain and D/G is holomorphically isomorphic to V.

12. Three-dimensional manifolds

In this section we consider compact connected closed oriented smooth manifolds of dimension three, which we refer to simply as 3-manifolds. Most of the material here comes from John Hempel's book [20].

Let M be such a thing, and write $\pi = \pi_1 M$, which is a finitely-presented group. Using Poincaré duality and the Hurewicz theorem, we have

$$\begin{aligned} H_0 M &= H^0 M = \mathbb{Z} \\ H_1 M &= H^2 M = \pi/[\pi,\pi] = H_1\pi \\ H_2 M &= H^1 M = \mathrm{Hom}(\pi,\mathbb{Z}) = H^1\pi \\ H_3 M &= H^3 M = \mathbb{Z}. \end{aligned}$$

Note that $H_2 M$ is a finitely generated free Abelian group.

12.1. Finite fundamental groups.

If $\pi = 0$ then the above shows that $H_1 = H_2 = 0$ and $\pi_3 = H_3 M = \mathbb{Z}$ (by the Hurewicz theorem). It follows that the generator of $\pi_3 M$ is a homotopy equivalence $S^3 \to M$. In fact, in this situation M is always homeomorphic to S^3. This was originally conjectured by Poincaré and eventually resolved as a consequence of Perelman's celebrated work on Thurston's geometrization conjecture.

Now suppose instead that π is merely finite. Then the universal cover $\widetilde{M}$ is a simply connected compact closed 3-manifold, and is therefore homeomorphisc to S^3 as above. It is interesting to ask which groups π can occur like this. If a finite group π admits a map $\pi \to SO(4)$ such that the resulting action on $S^3 \subset \mathbb{R}^4$ is free, then we can take $M = S^3/\pi$. Let P be the set of groups which admit such a map. The identification of P is a problem in classical representation theory, which is completely understood; we will not give the details here.

It is not the case that every 3-manifold with finite fundamental group arises as a quotient of S^3 by a linear action of a finite group, as discussed above. Nonetheless, it is conjectured that P is the same as the set of finite fundamental groups of 3-manifolds. To see how one might approach this, let M be a 3-manifold with finite fundamental group π and consider the covering spectral sequence

$$H^*(\pi; H^*\widetilde{M}) \Longrightarrow H^* M.$$

As M is oriented, the action of π on $H^3\widetilde{M} = \mathbb{Z}$ is trivial. The spectral sequence therefore looks like this:

$$\begin{tikzcd}
H^0\pi \arrow[ddrrrr, "d_4"] & H^1\pi & H^2\pi & H^3\pi & H^4\pi & H^5\pi \\
 & & & & & \\
 & & & & & \\
H^0\pi & H^1\pi & H^2\pi & H^3\pi & H^4\pi & H^5\pi
\end{tikzcd}$$

The only possible differential is d_4. The target ring H^*M is concentrated in degrees less than or equal to 3 (because M is a 3-manifold); it follows that $d_4 \colon H^k\pi \to H^{k+4}\pi$ must be an isomorphism for all $k > 0$. Let u be the generator of $E_2^{0,3} = H^0\pi = \mathbb{Z}$, and write $v = d_4(u) \in H^4\pi$. Using the multiplicative properties of the spectral sequence, we see that multiplication by v is an isomorphism $H^k\pi \to H^{k+4}\pi$ for $k > 0$. The existence of such an element v is a strong constraint on the structure of the finite group π. For example, one can show quite easily that for each odd prime p the Sylow p-subgroups of π are cyclic.

12.2. **Connected sums.** Given two 3-manifolds M and N, we can form their connected sum $M\#N$ by cutting small open 3-balls out of M and N to get two manifolds with boundary S^2, and gluing them together along the boundary in a way that respects the orientations. It turns out that the result is well-defined up to diffeomorphism. It is easy to see that $S^3\#M = M$.

We say that a 3-manifold M is prime if it cannot be written as a connected sum of two manifolds not diffeomorphic to S^3. It is natural to try to decompose an arbitrary 3-manifold M as a connected sum $M = M_1\#\ldots\#M_n$, where each M_i is prime. It turns out that this is always possible, and that the decomposition is essentially unique. Moreover, we have

$$\begin{aligned} H_1M &= H_1M_1 \oplus \ldots \oplus H_1M_n \\ H_2M &= H_2M_1 \oplus \ldots \oplus H_2M_n \\ \pi_1M &= \pi_1M_1 * \ldots * \pi_1M_n. \end{aligned}$$

We next attempt a crude classification of prime 3-manifolds. It is easy to see that if π_1M is finite and the Poincaré conjecture holds then M is prime. If $\pi_1M = \mathbb{Z}$ then it can be shown that $M = S^1 \times S^2$, which is prime.

Now suppose that M is prime and that $\pi = \pi_1M$ is infinite and not cyclic. We claim that $\widetilde{M}$ is contractible. Indeed, $\widetilde{M}$ is a simply-connected non-compact 3-manifold, so $H_1\widetilde{M} = H_3\widetilde{M} = 0$, so the only possible reduced homology is $H_2\widetilde{M} = \pi_2\widetilde{M} = \pi_2M$. If $\pi_2M \neq 0$ then a result called the Sphere Theorem tells us that there is an *embedding* $S^2 \to M$ which represents a nontrivial element of π_2M. It is at least plausible that we could cut M along this embedded sphere and thus write M as a connected sum. The example of $S^1 \times S^2$ shows that this is a little too naive, but it turns out that that is essentially the only counterexample, and we have excluded it by our conditions on π_1M. Thus, we must have $\pi_2M = 0$ and thus $\widetilde{H}_*\widetilde{M} = 0$ and thus $\widetilde{M}$ is contractible as claimed. This implies that M is homotopy equivalent to the classifying space $B\pi$.

We now return to the case of a general 3-manifold M, decomposed as $M_1\#\ldots\#M_n$ say, with each M_i prime. It is no longer the case that the higher homotopy groups of M vanish, but we can still calculate π_2M. For simplicity, we assume that each group $\pi_1(M_i)$ is infinite and not cyclic. Note that π_2M has a natural action of $\pi = \pi_1M$, so it is a module over the group ring $\mathbb{Z}[\pi]$. For each i we can choose a based embedding $\sigma_i \colon S^2 \to M$ which separates M_i from the other M_j's, and we also write σ_i for the corresponding element of π_2M. It turns out that

$$\pi_2M = \mathbb{Z}[\pi]\{\sigma_1, \ldots, \sigma_n\}/(\sigma_1 + \ldots + \sigma_n).$$

For a proof, see [42].

12.3. **Swarup's homotopy classification.** Let $\mathcal{M}$ be the category of 3-manifolds with a given basepoint. The morphisms from M to M' are homotopy classes of pointed maps which have degree one (in other words, we require that $f_*[M] = [M'] \in H_3M'$).

Let $\mathcal{G}$ be the category of pairs (π, u), where π is a group and $u \in H_3B\pi$. The morphisms from (π, u) to (π', u') are homomorphisms $f \colon \pi \to \pi'$ such that $(Bf)_*u = u'$.

Given a manifold $M \in \mathcal{M}$, there is an obvious map $q \colon M \to B\pi_1M$, and we can define $\zeta_M = q_*[M] \in H_3B\pi_1M$. We thus get an object $FM = (\pi_1M, \zeta_M)$ of $\mathcal{G}$, and it is easy to see that this gives a functor $F \colon \mathcal{M} \to \mathcal{G}$. Swarup proved that this functor is full [43]. It follows that FM is isomorphic to FM' if and only if M is homotopy equivalent to M', by an orientation preserving equivalence. In other words, FM is a complete invariant of the homotopy type of M.

12.4. **Heegaard splittings.** An important way of constructing 3-manifolds is via Heegaard splittings. Let F be an oriented surface of genus $g > 0$, and T the usual solid with $\partial T = F$. Write j for the inclusion $F \to T$, and let $f \colon F \to F$ be an orientation-preserving diffeomorphism. We can then form a pushout diagram

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[d, "jf"'] & T \arrow[d] \\ T \arrow[r] & M. \end{tikzcd}$$

It is not hard to check that M is a 3-manifold. Such a decomposition of a given 3-manifold M is called a Heegaard splitting of genus g. It can be shown that any 3-manifold admits such a splitting for some g.

Note that

$$\begin{aligned} \pi_1 F &= \langle a_1, \dots, a_g, b_1, \dots b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle \\ \pi_1 T &= \langle b_1, \dots, b_g \rangle \qquad\qquad (\text{and } j_* a_i = 0) \\ \pi_1 M &= \langle b_1, \dots, b_g \mid (jf^{-1})_*(a_i) = 1 \rangle. \end{aligned}$$

12.5. **Surgery on knots.** Another important construction of 3-manifolds is by surgery on knots and links. We explain a version which creates homology 3-spheres, in other words manifolds M with $H_*M = (\mathbb{Z}, 0, 0, \mathbb{Z}) = H_*S^3$. (It is equivalent to say that $\pi_1 M$ is perfect.)

Let K be a knot in S^3 (that is, a smoothly embedded copy of S^1). Let N be a tubular neighbourhood of K. Then ∂N is an oriented S^1-bundle over $K \simeq S^1$, so that $\partial N \simeq S^1 \times K$. Write $L = \overline{S^3 \setminus N}$, which is homotopy equivalent to $S^3 \setminus K$. By Alexander duality we have $\widetilde{H}_k L = \widetilde{H}^{2-k} K$, so $\widetilde{H}_* L = \mathbb{Z}$ concentrated in degree one. Moreover, the Mayer-Vietoris sequence

$$0 = H_2 S^3 \to H_1 \partial N \to H_1 N \oplus H_1 L \to H_1 S^3 = 0$$

shows that $H_1 \partial N = H_1 K \oplus H_1 L = \mathbb{Z}^2$. There is thus a basis $H_1 \partial N = \mathbb{Z}\{\lambda, \mu\}$ such that $\lambda \mapsto 0$ in $H_1 L$ and $\mu \mapsto 0$ in $H_1 K$. Here λ is parallel to K, and μ is a fibre of $\partial N \to K$, so it winds once around K.

For any integer n, there is an automorphism f of $\partial N \simeq S^1 \times S^1$ that fixes λ and sends μ to $\mu + n\lambda$. We can form a pushout diagram as follows:

$$\begin{tikzcd} \partial N \arrow[r, "f"] \arrow[d, tail] & \partial N \arrow[r, tail, "j"] & L \arrow[d] \\ N \arrow[rr] & & M. \end{tikzcd}$$

In other words, M is obtained from S^3 by removing the solid torus N, twisting it n times around K and gluing it back in. As $j_* f_* = j_*$, the Mayer-Vietoris sequence for $M = N \cup_{\partial N} L$ shows that $H_* M = H_* S^3$ as claimed.

Any 3-manifold may be obtained from S^3 by a related procedure in which K is allowed to be a link and more general twisting is permitted.

12.6. **Hyperbolic 3-manifolds. Write this**

12.7. **Other constructions.** Other important classes of 3-manifolds include fibre bundles $F \to M \to S^1$, with F a surface (so $M = F \times [0,1]/((x,0) \sim (fx,1))$ for some automorphism f of F). Similarly, one has bundles $S^1 \to M \to F$, which are classified by $H^2 F \simeq \mathbb{Z}$. More generally, one can start with such a bundle and perform a certain kind of surgery on a finite set of fibres to obtain a so-called Seifert-fibred manifold.

13. Simply-connected four-dimensional manifolds

In this section we consider compact connected closed oriented topological manifolds of dimension four, which we refer to simply as 4-manifolds. The following result gives a complete classification of simply-connected 4-manifolds up to oriented homotopy equivalence. It is essentially the same as [22, Theorem 2.1].

Theorem 13.1. Let M be a simply-connected 4-manifold. Then

$$H_0M = H_4M = H^0M = H^4M = \mathbb{Z}$$
$$H_1M = H_3M = H^1M = H^3M = 0.$$

Moreover, H^2M is a finitely-generated free Abelian group. There is a natural unimodular symmetric bilinear form b on H^2M. The set of oriented homotopy types of 4-manifolds bijects with the set of isomorphism classes of finitely-generated free Abelian groups equipped with a unimodular symmetric bilinear form.

We now sketch the proof of this fact. Let M be a simply-connected 4-manifold. By Poincaré duality and the Hurewicz theorem we have

$$H_0M = H^4M = \mathbb{Z}$$
$$H_1M = H^3M = 0$$
$$H_2M = H^2M.$$

We next consider the universal coefficient sequence

$$\mathrm{Ext}(H_{k-1}M, \mathbb{Z}) \rightarrowtail H^kM \twoheadrightarrow \mathrm{Hom}(H_kM, \mathbb{Z}).$$

Using this we find that

$$H^0M = \mathbb{Z}$$
$$H^1M = 0$$
$$H^2M = \mathrm{Hom}(H_2M, \mathbb{Z})$$

It follows that H^2M is a finitely-generated torsion-free Abelian group, and thus isomorphic to $\mathbb{Z}^d$ for some d. We have already seen that $H_2M = H^2M$, so $H_2M = \mathbb{Z}^d$ also.

The intersection form b is defined by $b(u, v) = \langle uv, [M]\rangle$. This is easily seen to be a symmetric bilinear form, and the Poincaré duality isomorphism $H^2M \simeq H_2M$ tells us that b is unimodular. Clearly, if $f\colon M \to N$ is an orientation-preserving homotopy equivalence of simply-connected 4-manifolds then f induces an isomorphism $(H^2M, b_M) \simeq (H^2N, b_N)$. We thus get a map from the set of oriented homotopy types of simply-connected 4-manifolds to the set of isomorphism types of free Abelian groups with a unimodular form.

Now let M' be the result of removing a small open ball from M. We have a cofibration

$$S^3 \xrightarrow{v} M' \to M \to S^4,$$

in which the last map has degree one. It follows that $\widetilde{H}_*M'$ is concentrated in degree 2, where it is isomorphic to $\pi_2M = H_2M = \mathbb{Z}^d$ say. Choose maps $u_1, \ldots, u_d\colon S^2 \to M'$ giving a basis for this group. It is easy to see that the resulting map $u\colon X = S^2 \vee \ldots \vee S^2 \to M'$ is a homotopy equivalence. Using this, we conclude that

$$\pi_3M' = \mathbb{Z}\{[u_i, u_j] \mid i < j\} \oplus \mathbb{Z}\{u_i \circ \eta\}.$$

Here $[u_i, u_j]$ denotes the Whitehead product, and $\eta\colon S^3 \to S^2$ is the Hopf map. In particular, we have

$$v = \sum_{i<j} a_{ij}[u_i, u_j] + \sum_i a_{ii}u_i \circ \eta$$

for some integers a_{ij}. One can show that the matrix (a_{ij}) is the same as the matrix of the bilinear form b written in terms of the basis dual to $\{u_i\}$. It follows that b determines the homotopy type of M.

Conversely, suppose that we start with a unimodular form b on $\mathbb{Z}^d$. We can then let u_i denote the inclusion of the i'th summand in $X = S^2 \vee \ldots \vee S^2$ and define a map $v = \sum_{i<j} b_{ij}[u_i, u_j] + \sum_i b_{ii} u_i \circ \eta \colon S^3 \to X$. We let M be the cofibre of v. It is easy to check that M is a finite CW complex which looks homologically like a simply-connected 4-manifold with intersection form b. It can be shown that there is a topological manifold homotopy equivalent to M. This essentially completes the proof of the theorem.

It turns out that there are far fewer smooth 4-manifolds than topological 4-manifolds. Work of Seiberg and Witten (which simplifies an earlier proof due to Donaldson) shows that the symmetric bilinear forms arising from smoothable 4-manifolds are all diagonalisable over $\mathbb{Z}$, which rarely the case for an arbitrary form. This is the end of a long story, which starts by considering how quantum electrodynamics would work if spacetime were diffeomorphic to a given 4-manifold M. For Donaldson's theory, see [15]. I don't know an appropriate reference for the Seiberg-Witten theory.

14. Moore spectra

Let p be an odd prime. Let $\mathcal{M}$ be the category of p-local spectra X such that $\pi_k X = 0$ for $k < 0$ and $H_k X = 0$ for $k > 0$. Such spectra are called *Moore spectra*. Note that if X is a Moore spectrum then the Hurewicz theorem gives $\pi_0 X = H_0 X$ and $H_k X = 0$ for $k \neq 0$. Let $\mathcal{A}$ be the category of p-local Abelian groups, so that $\pi_0 X \in \mathcal{A}$.

Theorem 14.1. The functor $\pi_0 \colon \mathcal{M} \to \mathcal{A}$ is an equivalence of categories. If $X, Y \in \mathcal{M}$ then

$$\begin{aligned}[X, Y] &= \mathrm{Hom}(\pi_0 X, \pi_0 Y)\\ [X, \Sigma Y] &= \mathrm{Ext}(\pi_0 X, \pi_0 Y)\\ [X, \Sigma^k Y] &= 0 \qquad \text{for } k > 1.\end{aligned}$$

To prove this, let A be a p-local Abelian group. Choose a set $\{a_j \mid j \in J\}$ of generators of A, and thus an epimorphism $F = \bigoplus_J \mathbb{Z}_{(p)} \twoheadrightarrow A$, with kernel R say. As R is a subgroup of a free module over $\mathbb{Z}_{(p)}$, it is free. (This is well-known if F is finitely-generated. Essentially the same proof works in general, except that one has to choose a well-ordering of J and use transfinite recursion.) We may therefore choose an isomorphism $R = \bigoplus_{i \in I} \mathbb{Z}_{(p)}$ and thus a short exact sequence

$$\bigoplus_I \mathbb{Z}_{(p)} \rightarrowtail \bigoplus_J \mathbb{Z}_{(p)} \twoheadrightarrow A.$$

Define $SF = \bigvee_J S^0_{(p)}$ and $SR = \bigvee_I S^0_{(p)}$, so we have given isomorphisms $\pi_0 SF = F$ and $\pi_0 SR = R$. For any p-local spectrum Y, the universal property of the wedge gives

$$[SR, Y] = \prod_I [S^0_{(p)}, Y] = \mathrm{Hom}(R, \pi_0 Y).$$

In particular, we have a map $SR \to SF$ whose effect on π_0 is the given map $R \rightarrowtail F$. Write SA for the cofibre of this map. It is not hard to see that $SA \in \mathcal{M}$ and that there is a canonical isomorphism $\pi_0 SA = A$. This shows that the functor $\pi_0 \colon \mathcal{M} \to \mathcal{A}$ is essentially surjective.

From the defining cofibration and the fact that $\pi_1 S = \mathbb{Z}/2$, we find easily that $\pi_1 SA = A \otimes \mathbb{Z}/2$, which is zero because A is p-local and p is odd.

Consider again an arbitrary p-local spectrum Y. As above, we have $[SR, Y] = \mathrm{Hom}(R, \pi_0 Y)$ and similarly $[SF, Y] = \mathrm{Hom}(F, Y)$ and $[\Sigma SR, Y] = \mathrm{Hom}(R, \pi_1 Y)$ and $[\Sigma SF, Y] = \mathrm{Hom}(F, \pi_1 Y)$. Using this and the cofibration

$$SR \to SF \to SA \to \Sigma SR \to \Sigma SF,$$

we obtain a short exact sequence

$$\mathrm{Ext}(A, \pi_1 Y) \rightarrowtail [SA, Y] \twoheadrightarrow \mathrm{Hom}(A, \pi_0 Y).$$

In particular, if $X \in \mathcal{M}$ then there exists a map $S(\pi_0 X) \to X$ inducing the identity on $\pi_0 = H_0$, and thus on all homology groups (because the other ones are zero). The map is thus an equivalence. It follows that all spectra in $\mathcal{M}$ have the form SA for some $A \in \mathcal{A}$.

Given that $\pi_1 SB = 0$, the above short exact sequence shows that $[SA, SB] = \mathrm{Hom}(A, B)$, and thus that $[X, Y] = \mathrm{Hom}(\pi_0 X, \pi_0 Y)$ for all $X, Y \in \mathcal{M}$. This shows that the functor $\pi_0 \colon \mathcal{M} \to \mathcal{A}$ is also full and faithful, and thus an equivalence of categories. The other claims are now easy to check.

14.1. **References.** For this and much other material, the books [26] and [2] are good references.

15. Eilenberg-MacLane spaces

Consider an integer $n > 1$ and write

$$\mathcal{C}_n = \{ \text{ pointed CW complexes } X \text{ such that } \pi_m X = 0 \text{ for all } m \neq n \ \}.$$

We make this into a category, with homotopy classes of maps as morphisms. We also consider the category $\mathcal{A}$ of Abelian groups.

The following theorem is proved in [44, Section V.7] (for example):

Theorem 15.1. The functor $\pi_n \colon \mathcal{C}_n \to \mathcal{A}$ is an equivalence of categories. If X is a CW complex and $Y \in \mathcal{C}_n$ then there is a natural isomorphism $[X, Y] = H^n(X; \pi_n Y)$. □

It follows that for any Abelian group A there is an essentially unique CW complex $K(A, n)$ equipped with an isomorphism $\pi_n K(A, n) = A$. This is called the Eilenberg-MacLane space of type (A, n). We also have

$$[X, K(A, n)] = H^n(X; A)$$
$$[K(A, n), K(B, n)] = \mathrm{Hom}(A, B).$$

There are a number of ways of constructing $K(A, n)$.

(a) Let G be a topological group which is a CW complex. Then Milgram's classifying-space construction [25, Chapter 1] gives a CW complex BG which is a covariant functor of G, together with a natural map $G \to \Omega BG$ which is a homotopy equivalence. Moreover, there is a natural homeomorphism $B(G \times H) = BG \times BH$.

If A is an Abelian topological group then the addition map $\sigma \colon A \times A \to A$ is a group homomorphism so we can apply B to get a map of spaces $BA \times BA = B(A \times A) \xrightarrow{B\sigma} BA$. It is easy to see that this makes BA into an Abelian topological group with $\Omega BA = A$.

Thus, given a discrete Abelian group A, we can define $K(A, n) = B^n A$; it is straightforward to check that $\pi_* K(A, n) = A$ concentrated in degree n, as required.

(b) For any pointed space X, we define the k'th symmetric power $\mathrm{SP}^k X = X^k/\Sigma_k$. The map $X^k \to X^{k+1}$ sending $(x_1, \ldots, x_k)$ to $(x_1, \ldots, x_k, 0)$ induces an inclusion $\mathrm{SP}^k X \to \mathrm{SP}^{k+1} X$, so we can define $\mathrm{SP}^\infty X = \varinjlim_k \mathrm{SP}^k X$. This is also the free Abelian topological monoid generated by X modulo one relation, which sets the basepoint equal to the identity. In other words, for any Abelian topological monoid A we have a bijection between continuous monoid maps $\mathrm{SP}^\infty X \to A$ and pointed maps $X \to A$.

The Dold-Thom theorem [14] says that if X is a connected CW complex then there is a homotopy equivalence

$$\mathrm{SP}^\infty X = \prod_{n>0} K(H_n X, n).$$

In particular, we have

$$K(\mathbb{Z}, n) = \mathrm{SP}^\infty(S^n).$$

We outline a proof of the Dold-Thom theorem. There is a well-known homeomorphism $\mathrm{SP}^n \mathbb{C} = \mathbb{C}^n$, which sends $[z_1, \ldots, z_n]$ to the list of coefficients of the polynomial $\prod_i (t - z_i)$. This restricts to give a homotopy equivalence

$$\mathrm{SP}^n S^1 \simeq \mathrm{SP}^n \mathbb{C}^\times \simeq \mathbb{C}^\times \times \mathbb{C}^{n-1} \simeq S^1.$$

In the limit we get $\mathrm{SP}^\infty S^1 = S^1 = K(\mathbb{Z}, 1)$ (which is a special case of the theorem). One can also show that the functor SP^∞ converts cofibrations to quasifibrations, and thus that the functor $X \mapsto \pi_* \mathrm{SP}^\infty X$ is a homology theory. The rest is general nonsense.

(c) There is an essentially unique space $S^n\mathbb{Q}$ equipped with a map $S^n \to S^n\mathbb{Q}$ that induces an isomorphism $\pi_*(S^n) \otimes \mathbb{Q} \simeq \pi_*(S^n\mathbb{Q})$. If n is odd then a theorem of Serre tells us that $\mathbb{Q} \otimes \pi_* S^n = \mathbb{Q}$, concentrated in degree n. Thus $S^n\mathbb{Q} = K(\mathbb{Q}, n)$ when n is odd.

We recall the construction of $S^n\mathbb{Q}$: it is the telescope of the sequence

$$S^n \xrightarrow{2} S^n \xrightarrow{3} S^n \xrightarrow{4} S^n \to \ldots.$$

Here we write $k\colon S^n \to S^n$ for any map of degree k. The telescope of any sequence $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots$ is the space $\coprod_k [k, k+1] \times X_k / \sim$, where $(k+1, x)$ is identified with $(k+1, f_k x)$ for all $x \in X_k$. One can show that $H_* \operatorname{Tel}(X_k) = \varinjlim_k H_* X_k$, and that $\pi_* \operatorname{Tel}(X_k) = \varinjlim_k \pi_*(X_k)$.

There is a well-developed and satisfying theory of spaces whose homotopy groups are all rational vector spaces. The original reference is [35], and there are some more recent survey articles in [18].

(d) By methods similar to those of Section 14, one can construct a simply-connected space $S^n A$ with $H_* S^n A = A$, concentrated in degree n. It follows that $\pi_k S^n A = 0$ for $k < n$, and that $\pi_n S^n A = A$. However, $S^n A$ is not an Eilenberg-MacLane space, because the homotopy groups $\pi_k S^n A$ for $k > n$ will not vanish in general. Nonetheless, these higher homotopy groups can be killed off by attaching extra cells of dimension greater than n, and after doing this we are left with an Eilenberg-MacLane space. This sort of procedure is explained in [44, Section V.2].

16. Wilson spaces

Let p be an odd prime, and $k > 0$ an integer. In this section we discuss the properties of certain space Y_k which was first studied by Steve Wilson [45, 46]. See also [8, Sections 22-23].

The space Y_k is a $(k-1)$-connected p-local H-space, such that for all m the groups H_mY_k and π_mY_k are finitely generated free modules over $\mathbb{Z}_{(p)}$. Moreover, the first nontrivial group is $\pi_kY_k = H_kY_k = \mathbb{Z}_{(p)}$.

The space Y_k is also atomic: if $f\colon Y_k \to Y_k$ is such that $f_*\colon H_k(Y_k;\mathbb{F}_p) \to H_k(Y_k;\mathbb{F}_p)$ is nonzero, then f is a homotopy equivalence. It follows that Y_k is indecomposable: it cannot be written as $V \times W$ with V and W both not contractible.

Wilson proved that the above facts characterise Y_k uniquely: if Z is an indecomposable $(k-1)$-connected p-local H-space of finite type such that π_*Z and H_*Z are free over $\mathbb{Z}_{(p)}$ and $\pi_kZ = \mathbb{Z}_{(p)}$, then Z is (noncanonically) homotopy equivalent to Y_k.

One construction of the spaces Y_k is that $Y_k = \Omega^\infty\Sigma^k BP\langle n\rangle$, where n is characterised by $|v_n| < (p-1)k \leq |v_{n+1}|$, where $|v_m| = 2(p^m-1)$.

It follows that the space Ω^kY_k can be made into a ring up to homotopy, and we have

$$\pi_{*+k}Y_k = \pi_*\Omega^kY_k = \mathbb{Z}_{(p)}[v_i \mid 0 < |v_i| < (p-1)k].$$

16.1. **Bipolynomial Hopf algebras.** The Hopf algebras H^*Y_{2k} and H_*Y_{2k} (with coefficients $\mathbb{Z}_{(p)}$) are both polynomial rings; in other words, H_*Y_k is a bipolynomial Hopf algebra. The structure theory of such Hopf algebras is known [38]; we will give a brief outline.

There is a unique Hopf algebra structure on $W = \mathbb{Z}_{(p)}[x_k \mid k \geq 0]$ such that the elements $w_k = \sum_{i+j} p^j x_j^{p^i}$ are primitive. Indeed, we have $\mathbb{Q}\otimes W = \mathbb{Q}[w_k \mid k \geq 0]$, so it is clear that there is a unique coproduct ψ on $\mathbb{Q}\otimes W$ such that the elements w_k are primitive. We need only check that the elements $\psi(x_k) \in \mathbb{Q}\otimes W\otimes W$ actually lie in $W\otimes W$. This can be done by induction on k. We call W with this coproduct the Witt Hopf algebra. It can be shown that W is self-dual, and thus bipolynomial.

Given any integer $i > 0$, we can make W into a connected graded Hopf algebra by putting x_k in degree $2p^ki$. We call this graded algebra $W(i)$. It can be shown that any connected graded bipolynomial Hopf algebra of finite type is noncanonically isomorphic to a tensor product $\bigotimes_{i>0} W(i)^{\otimes d_i}$ for certain (uniquely determined) integers $d_i \geq 0$.

In the case of H_*Y_k, the integers d_i are implicitly given by

$$\sum_i d_it^i = f(t) - f(t^p)$$

where

$$f(t) = t^k\prod_j(1-t^{p^j-1})^{-1},$$

where the product runs over all $j > 0$ for which $p^j - 1 < (p-1)k$.

16.2. **Torsion-free spaces.** Let X be a connected p-local CW complex of finite type, and suppose that π_*X and H_*X are free over $\mathbb{Z}_{(p)}$ (or equivalently, torsion-free). Suppose also that $X\mathbb{Q}$ is a product of Eilenberg-MacLane spaces (which follows from the other assumptions if X is an H-space). Then it can be shown that $X \simeq \prod_{k>0} Y_k^{d_k}$ for certain uniquely determined integers $d_k \geq 0$.

Using this, we find that the connected components of the infinite loop spaces in the spectra KU, MU, BP, $BP\langle n\rangle$, and $E(n)$ are all p-locally products of Wilson spaces.

17. The building for $GL_n\mathbb{F}_p$

Let P be a finite partially ordered set. Recall that a subset $C \subseteq P$ is called a chain iff it is totally ordered, in other words for all $u, v \in C$ we have either $u \leq v$ or $v \leq u$. Given a map $x\colon P \to [0,1]$, we write $\operatorname{supp}(x) = \{u \in P \mid x(u) > 0\}$. Write

$$|P| = \{x\colon P \to [0,1] \mid \operatorname{supp}(u) \text{ is a chain }\}.$$

This is a simplicial complex with vertex set P. It has one k-simplex for each chain of length $k+1$. For more details, see [6, Chapter 6].

Let V be a vector space over $\mathbb{F}_p$ of finite dimension n, and write

$$\operatorname{Sub}(V) = \{ \text{ proper nonzero subspaces of } V\}.$$

This is a partially ordered set under inclusion, so we can form the simplicial complex $|\operatorname{Sub}(V)|$. This is called the Bruhat-Tits building for the group $\operatorname{Aut}(V) \simeq GL_n\mathbb{F}_p$.

The following theorem is due to Quillen, and is also proved as [6, Theorem 6.8.5].

Theorem 17.1. $|\operatorname{Sub}(V)|$ is homotopy equivalent to a wedge of $p^{n(n-1)/2}$ copies of the sphere S^{n-2}.

Note that $\operatorname{Aut}(V)$ acts on $\operatorname{Sub}(V)$ and thus on $M = H_{n-2}(|\operatorname{Sub}(V)|;\mathbb{Z}_{(p)})$. Thus M is a module over the group ring $R = \mathbb{Z}_{(p)}[\operatorname{Aut}(V)]$, called the Steinberg module. It crops up in a number of places in topology: see [30, Section XX.6] for a connection with the Segal conjecture, and [33] for an application to the stable homotopy theory of finite complexes.

Before discussing the properties of M, we recall some facts about $GL_n\mathbb{F}_p$. Let U be the subgroup of matrices with ones on the diagonal and zeros below it (so $|U| = p^{n(n-1)/2}$). Equivalently, $g \in U$ if and only if $g(\mathbb{F}_p^k) = \mathbb{F}_p^k$ and g acts as the identity on $\mathbb{F}_p^k/\mathbb{F}_p^{k-1}$, for $1 \leq k \leq n$.

It is well-known that

$$|GL_n\mathbb{F}_p| = (p^n-1)(p^n-p)\dots(p^n-p^{n-1}) = p^{n(n-1)/2}(p^n-1)(p^{n-1}-1)\dots(p-1).$$

(Consider the number of possible choices for $g(e_k)$ after $g(e_1),\dots,g(e_{k-1})$ have been chosen.)

Theorem 17.2. (a) M is free of rank one as a module over $\mathbb{Z}_{(p)}[U]$.

(b) M is an indecomposable projective module over $\mathbb{Z}_{(p)}[GL_n\mathbb{F}_p]$; in fact, there is an idempotent $e \in \mathbb{Z}_{(p)}[GL_n\mathbb{F}_p]$ such that $M \simeq \mathbb{Z}_{(p)}[GL_n\mathbb{F}_p]e$.

(c) $\overline{\mathbb{F}}_p \otimes M$ is a simple module over $\overline{\mathbb{F}}_p[GL_n\mathbb{F}_p]$.

Proof of Theorem 17.1. The claim is trivial when $n = 2$, so suppose that $n > 2$. Choose a line $L < V$, and let $\mathcal{H}$ be the set of hyperplanes $H < V$ such that $V = L \oplus H$. If $H_0 \in \mathcal{H}$ then any $H \in \mathcal{H}$ is the graph of a unique map $H_0 \to L$, and thus $|\mathcal{H}| = |\operatorname{Hom}(H_0, L)| = p^{n-1}$.

Next, observe that a map $f\colon P \to Q$ of finite posets induces a map $|f|\colon |P| \to |Q|$. Explicitly, we have

$$|f|(x)(v) = \sum_{f(u)=v} x(u).$$

If $f_0, f_1\colon P \to Q$ are two such maps and $f_0(u) \leq f_1(u)$ for all $u \in P$ then there is an obvious way to construct a map of posets $\{0,1\} \times P \to Q$, and one can check that $|\{0,1\} \times P| = [0,1] \times |P|$, so $|f_0|$ and $|f_1|$ are homotopic.

Now take $P = \operatorname{Sub}(V) \setminus \mathcal{H}$. Define $f, g, h\colon P \to P$ by

$$\begin{aligned} f(W) &= W \\ g(W) &= W + L \\ h(W) &= L. \end{aligned}$$

We then have $f \leq g \geq h$, so the identity map $|f|$ is homotopic to the constant map $|h|$, so $|P|$ is contractible. We thus have a homotopy equivalence $|\operatorname{Sub}(V)| \simeq |\operatorname{Sub}(V)|/|P|$.

For each $H \in \mathcal{H}$, write $X_H = \{x \in |\operatorname{Sub}(V)| : x(H) > 0\}$. These spaces are disjoint, and their union is $|\operatorname{Sub}(V)| \setminus |P|$. Moreover, one can see that $X_H = (0,1] \times |\operatorname{Sub}(H)|/\sim$, where all points of the form $(1, x)$ are identified together. Using this, we see that $|\operatorname{Sub}(V)|/|P| = \bigvee_{H\in\mathcal{H}} \widetilde{\Sigma}|\operatorname{Sub}(H)|$.

Here $\widetilde{\Sigma}Y$ denotes the unreduced suspension $[0,1]\times Y/\sim$, where all the points $(0,y)$ are identified to one point, and all the points $(1,y)$ are identified to a different point. By induction, we know that $|\operatorname{Sub}(H)|$ is a wedge of $p^{(n-1)(n-2)/2}$ copies of the sphere S^{n-3}. As $\widetilde{\Sigma}S^{n-3}=S^{n-2}$ and $|\mathcal{H}|p^{(n-1)(n-2)/2}=p^{n(n-1)/2}$, we see that $|\operatorname{Sub}(V)|$ is a wedge of $p^{n(n-1)/2}$ copies of S^{n-2}, as claimed. □

We now take $V=\mathbb{F}_p^n$, and write V_i for the evident copy of $\mathbb{F}_p^i$ in V, so we have a flag $0=V_0<\ldots<V_n=V$. Let $\mathcal{C}$ be the chains $W_1<\ldots<W_{n-1}$ such that $V=W_i\oplus V_{n-i}$ for all i. By examining carefully the induction in the above proof, we see that $|\operatorname{Sub}(V)|$ is homotopy equivalent to the space obtained by collapsing out all simplices not contained in $\mathcal{C}$. Moreover, $\mathcal{C}$ and its complement are invariant under the subgroup U. It follows that M is U-equivariantly isomorphic to $\mathbb{Z}_{(p)}\{\mathcal{C}\}$. On the other hand, one can check that U permutes $\mathcal{C}$ freely and transitively, so that $\mathbb{Z}_{(p)}\{\mathcal{C}\}\simeq\mathbb{Z}_{(p)}[U]$. This proves the first part of Theorem 17.2. For the second part, we simply record the formula

$$e=\frac{1}{|GL_n\mathbb{F}_p:U|}\sum_{\sigma,g}\epsilon(\sigma)[\sigma g]\in\mathbb{Z}_{(p)}[GL_n\mathbb{F}_p].$$

Here σ runs over $\Sigma_n<GL_n\mathbb{F}_p$, and g runs over the group $B=\{g\in GL_n\mathbb{F}_p\mid g(V_k)=V_k \text{ for all } k\}$. We shall not check here that this has the properties claimed.

18. Looped spheres

Let $n > 0$ be an integer, and consider the loop space ΩS^{2n+1}.

18.1. **Homology and cohomology.** The adjunction $[\Sigma X, Y] = [X, \Omega Y]$ shows that $\pi_k \Omega S^{2n+1} = \pi_{k+1} S^{2n+1}$. In particular, the identity element of $\pi_{2n+1} S^{2n+1}$ gives a map $\eta\colon S^{2n} \to \Omega S^{2n+1}$; explicitly, we have $\eta(x)(t) = t \wedge x$. The Hurewicz image of η is a class $x \in H_{2n} \Omega S^{2n+1}$.

Proposition 18.1. x is a primitive element, and $H_* \Omega S^{2n+1} = \mathbb{Z}[x]$ (as an algebra under the Pontrjagin product).

Proof. Write u for the usual generator of $H^{2n+1} S^{2n+1}$, so that $H^* S^{2n+1} = E[u]$. The fibration $\Omega S^{2n+1} \to PS^{2n+1} \to S^{2n+1}$ gives a Serre spectral sequence

$$E[u] \otimes H^* \Omega S^{2n+1} \Longrightarrow \mathbb{Z}.$$

The E_2 term is concentrated in columns 0 and $2n+1$, so the only possible differential is

$$d_{2n+1}\colon E_2^{0,m} \to E_2^{n+1,m-2n}.$$

As $E_{2n+2} = E_\infty = H^* PS^{2n+1} = \mathbb{Z}$, this differential must be an isomorphism for $m > 0$. By induction on degree, we see that $H^* \Omega S^{2n+1} = \mathbb{Z}\{x'_k \mid k \geq 0\}$ with $|x'_k| = 2nk$ and $x'_0 = 1$ and $d_{2n+1}(x'_k) = x'_{k-1} u$ for $k > 0$.

As d_{n+1} is a derivation, we see by induction on $k + l$ that $x'_k x'_k = (k, l) x'_{k+l}$, and thus that $H^* \Omega S^{2n+1}$ is the divided-power algebra $D[x'_1]$. Moreover, x'_1 is primitive for dimensional reasons, so that

$$\psi(x'_k) = \psi(x'_1)^k / k! = \sum_{k=i+j} (x'_1)^i / i! \otimes (x'_1)^j / j! = \sum_{k=i+j} x'_i \otimes x'_j.$$

We next consider the dual Hopf algebra $H_* \Omega S^{2n+1}$. By the Hurewicz theorem, we know that $H_{2n} \Omega S^{2n+1}$ is freely generated by x, so we must have $\langle x, x'_1 \rangle = \pm 1$. It follows easily that $\langle x^k, x'_l \rangle = \delta_{kl}$, and thus that $H_* \Omega S^{2n+1} = \mathbb{Z}[x]$. □

We could also have proved the above theorem using the Eilenberg-Moore spectral sequence [31]

$$\mathrm{Tor}_{s,t}^{H^* S^{2n+1}}(\mathbb{Z}, \mathbb{Z}) \Longrightarrow H^{t-s} \Omega S^{2n+1}.$$

To compute the Tor groups, we consider the differential graded algebra $C_* = D[x'_1] \otimes E[u]$ with differential

$$d((x'_1)^{[k]}) = (x'_1)^{[k-1]} u$$
$$d((x'_1)^{[k]} u) = 0.$$

(Note that this is a derivation.) There is an evident augmentation $\epsilon\colon C_* \to \mathbb{Z}$. Clearly C_* is a resolution of $\mathbb{Z}$ by free $E[u]$-modules, so $\mathrm{Tor}_{**}^{E[u]}(\mathbb{Z}, \mathbb{Z})$ is just $H_*(C_* \otimes_{E[u]} \mathbb{Z}) = D[x'_1]$, with $(x'_1)^{[k]}$ in bidegree $(k, (2n+1)k)$. As the E^2 page is so sparse, there is no room for differentials. As before, we conclude that $H^* \Omega S^{2n+1} = D[x'_1]$.

18.2. **The James model.** Let JS^{2n} be the free monoid generated by the set S^{2n}, modulo the relation that identifies the basepoint of S^{2n} with the identity element of the monoid. There is an evident map $(S^{2n})^k \to JS^{2n}$, and we write $J_k S^{2n}$ for its image. We topologise $J_k S^{2n}$ as a quotient of $(S^{2n})^k$, and JS^{2n} as the colimit of the spaces $J_k S^{2n}$. This makes JS^{2n} into the free topological monoid generated by S^{2n} modulo the basepoint, so that for any topological monoid M there is a natural bijection between maps $JS^{2n} \to M$ of monoids, and pointed maps $S^{2n} \to M$ of spaces.

We would like to apply this with $M = \Omega S^{2n+1}$, but unfortunately this is only a monoid up to homotopy. Th fix this, we consider instead the measured loop space

$$\Omega^* S^{2n+1} = \{(t, \omega) \mid t \geq 0, \omega\colon [0, t] \to S^{2n+1}, \omega(0) = \omega(1) = *\}.$$

See [44, Section III.2] for more detailed discussion of this space. It can be given a topology in a natural way. It is a topological monoid with product $(s,\alpha).(t,\beta)=(s+t,\gamma)$, where

$$\gamma(u)=\begin{cases}\alpha(u) & \text{if } 0\le u\le s\\ \beta(u-s) & \text{if } s\le u\le s+t.\end{cases}$$

One can also show that Ω^*S^{2n+1} is homotopy equivalent to ΩS^{2n+1}. The unit map $\eta\colon S^{2n}\to \Omega S^{2n+1}\simeq\Omega^*S^{2n+1}$ thus extends to give a map $JS^{2n}\to\Omega^*S^{2n+1}\simeq\Omega S^{2n+1}$ of H-spaces.

It is not hard to see that $JS^{2n}=S^{2n}\cup e^{4n}\cup\dots$ and thus that $H_*JS^{2n}=\mathbb{Z}[x]=H_*\Omega S^{2n+1}$, and thus that our map $JS^{2n}\to\Omega S^{2n+1}$ is a weak equivalence. In fact, it can be shown that ΩS^{2n+1} has the homotopy type of a CW complex, and thus that our map is an actual homotopy equivalence. All this is discussed in more detail in [44, Section VII.2].

18.3. **The James-Hopf maps.** We can use the James model to construct maps out of ΩS^{2n+1} by combinatorial means. The most important example is the James-Hopf map $j_m\colon \Omega S^{2n+1}\to\Omega S^{2nm+1}$. Consider an element $x_1\dots x_r\in JS^{2n}$, so $x_i\in S^{2n}$ (we allow $x_i=*$). For any set $S=\{i_1<\dots<i_m\}\subseteq\{1,\dots,r\}$ of order m, we write $x_S=x_{i_1}\wedge\dots\wedge x_{i_m}\in S^{2nm}$. We order the collection of all such subsets lexicographically, to obtain a list $S_1,\dots,S_t$ say. Write

$$j_m(x_1\dots x_r)=x_{S_1}\dots x_{S_t}\in J_tS^{2nm}.$$

One can check that this is well-defined (it does not change if we drop some x_i's with $x_i=*$) and that it gives a continuous map $JS^{2n}\to JS^{2nm}$. It is easy to see that j_m sends $J_{m-1}S^{2n}$ to the basepoint, and that the induced map $S^{2nm}=J_mS^{2n}/J_{m-1}S^{2n}\to JS^{2nm}$ is the usual inclusion. It follows that $j_m^*x_1'=x_m'\in H^{2nm}JS^{2n}$, and thus that

$$j_m^*x_k'=\frac{(mk)!}{k!(m!)^k}x_{mk}'\in H^{2nmk}JS^{2n}.$$

The James-Hopf maps are discussed in [5, Section II.2]; this book also contains much other material of the same kind.

18.4. **Fibrations.** We can use the James-Hopf maps to construct some interesting p-local fibrations. Let p be an odd prime, and let $f\colon\Omega S^{2n+1}\to\Omega S^{2np^k+1}$ be a map which induces a injection in mod p cohomology. The formulae of the previous section show that this applies when $f=j_{p^k}$, or more generally when $f=j_{p^{k_1}}\circ\dots\circ j_{p^{k_r}}$ with $k_1+\dots+k_r=k$.

Proposition 18.2. There is a p-local fibration $J_{p^k-1}S^{2n}\to\Omega S^{2n+1}\xrightarrow{f}\Omega S^{2np^k+1}$.

Proof. Let F be the homotopy fibre of f. Because $J_{p^k-1}S^{2n}=S^{2n}\cup\dots\cup e^{2n(p^k-1)}$ is $(2np^k-1)$-connected, we see that f is null on $J_{p^k-1}S^{2n}$. Thus, the inclusion $J_{p^k-1}S^{2n}\to\Omega S^{2n+1}$ factors (uniquely) through a map $g\colon J_{p^k-1}\to F$. We claim that g is a p-local equivalence.

To see this, consider the Eilenberg-Moore spectral sequence (with mod p coefficients)

$$\mathrm{Tor}_{st}^{H^*\Omega S^{2np^k+1}}(\mathbb{F}_p,H^*\Omega S^{2n+1})\Longrightarrow H^{t-s}F.$$

Note that $H^*\Omega S^{2np^k+1}=D[x_1']$, with $|x_1'|=2np^k$. It is well-known that such a divided power algebra over $\mathbb{F}_p$ is isomorphic to a tensor product of truncated polynomial algebras $T[y]=\mathbb{F}_p[y]/y^p$. Specifically, we have

$$H^*\Omega S^{2np^k+1}=\bigotimes_{i\ge0}T[x_{p^i}'] \qquad \text{with } |x_{p^i}'|=2np^{k+i}$$

and similarly

$$H^*\Omega S^{2n+1}=\bigotimes_{i\ge0}T[y_{p^i}'] \qquad \text{with } |y_{p^i}'|=2np^{i}.$$

As f^* is injective, we see that $f^*x_{p^i}'$ is a unit multiple of $y_{p^{i+k}}'$. It follows that

$$H^*\Omega S^{2n+1}=H^*\Omega S^{2np^k+1}\otimes\bigotimes_{i=0}^{k-1}T[y_{p^i}']=(H^*\Omega S^{2np^k+1})\{y_j'\mid 0\le j<p^k\}.$$

As this is a free module over $H^*\Omega S^{2np^k+1}$, we see that the E^2 page of the Eilenberg-Moore spectral sequence is concentrated on the line $s = 0$, where we just have $\mathbb{F}_p\{y'_j \mid 0 \leq j < p^k\}$, which is isomorphic to $H^*J_{p^k-1}S^{2n}$. It follows easily that our map $J_{p^k-1}S^{2n} \to F$ is an equivalence in mod p cohomology, as required. □

Related methods give a p-local fibration

$$S^{2n-1} \to \Omega J_{p-1}S^{2n} \to \Omega S^{2np-1}.$$

I think that this is explained in [34]. These fibrations, together with the ones in the proposition, give a maze of interlocking exact sequences relating the homotopy groups $\pi_k S^m_{(p)}$ and $\pi_k J_{p-1} S^m_{(p)}$, which can be assembled into the so-called EHP spectral sequence. This is an effective tool for calculating $\pi_k S^m_{(p)}$. For more discussion, see [36].

19. Doubly looped spheres

We now consider the double loop space $\Omega^2 S^{2n+1}$. Let p be an odd prime. All our (co)homology groups will have mod p coefficients.

19.1. Homology.

We will outline a proof of the following theorem:

Theorem 19.1. There are elements $x_k \in H_{2np^k-1}\Omega^2 S^{2n+1}$ (for $k \geq 0$) and $y_k \in H_{2np^k-2}\Omega^2 S^{2n+1}$ (for $k > 0$) such that
$$H_*\Omega^2 S^{2n+1} = \mathbb{F}_p[y_k \mid k > 0] \otimes E[x_k \mid k \geq 0].$$

We remark that much stronger results are known; in fact, Fred Cohen has described $H_*\Omega^k\Sigma^k X$ as a functor of H_*X (see [11]). Our discussion follows Cohen's, but is much simpler because we restrict attention to the case $n = 2$ and $X = S^{2n-1}$.

There is an evident unit map $\eta\colon S^{2n-1} \to \Omega^2 S^{2n+1}$, which carries a homology class $x_0 \in H_{2n-1}\Omega^2 S^{2n+1}$.

We next need to define a certain natural (nonadditive) operation $\xi\colon H_{2k-1}\Omega^2 X \to H_{2kp-1}\Omega^2 X$, which will help us to produce all the algebra generators for $H_*\Omega^2 S^{2n+1}$.

Consider a list $z_1, \dots, z_p$ of distinct points in $\mathbb{C}$, and write $\rho = \min_{i\neq j} |z_i - z_j|/3$. Consider also a list $\alpha_1, \dots, \alpha_p$ of pointed maps $S^2 \to X$, for some space X. We define a new map $\alpha\colon \mathbb{C} \cup \{\infty\} = S^2 \to X$ as follows. Let B_i be the closed disc of radius ρ about z_i. Define $\alpha(z) = 0$ for all z outside $\coprod_i B_i$, and let the restriction of α to B_i be the evident composite
$$B_i \twoheadrightarrow B_i/\partial B_i \simeq S^2 \xrightarrow{\alpha_i} X.$$
One can check that this gives a continuous map
$$\alpha = \theta(z_1, \dots, z_p; \alpha_1, \dots, \alpha_p)\colon S^2 \to X,$$
and moreover that this construction gives a continuous map
$$\theta\colon F_p\mathbb{C} \times_{\Sigma_p} (\Omega^2 X)^p \to \Omega^2 X.$$
This construction is probably due to Boardman, but has been elaborated and extended by many people. See [3] for a pleasant survey of these ideas.

We next need to construct a certain singular chain $e \in C_{p-1}F_p\mathbb{C}$. Consider the point $x = (0, 1, \dots, p-1) \in F_p\mathbb{R}$, and let D be the convex hull of its orbit under the action of Σ_p. This is a cell of dimension $(p-1)$, contained in the hyperplane $\sum_i y_i = p(p-1)/2$. Of course, D is not contained in $F_p\mathbb{R}$. However, it turns out that one can choose a map $f\colon D \to \mathbb{R}^p$ with $f(\partial D) = 0$ such that the map $g(x) = x + if(x)$ gives an embedding $g\colon D \rightarrowtail F_p\mathbb{C}$. Moreover, if $\overline{D}$ is obtained from D by identifying points in ∂D that lie in the same orbit under Σ_p, then the natural map $\overline{D} \to B_p\mathbb{C}$ is a homotopy equivalence. All this can be done completely explicitly, using a certain combinatorial triangulation of D. In particular, one can show that the degree of $\sigma \in \Sigma_p$ acting on $\partial D = S^{p-2}$ is the signature $\epsilon(\sigma)$.

We let e denote the image of D under the map g mentioned above. Consider a cycle $a \in Z_{2k-1}\Omega^2 X$. One can consider $e \otimes_{\Sigma_p} a^{\otimes p}$ as a chain on $F_p\mathbb{C} \times_{\Sigma_p} (\Omega^2 X)^p$. The action of Σ_p on ∂D is free, so we have $\partial e = \sum_{\sigma\in\Sigma_p} \epsilon(\sigma)\sigma.e'$ for some chain $e' \in C_{p-2}F_p\mathbb{C}$. It follows that
$$\partial(e \otimes_{\Sigma_p} a^{\otimes p}) = \sum_\sigma \epsilon(\sigma) e \otimes_{\Sigma_p} \sigma.a^{\otimes p} = pe \otimes_{\Sigma_p} a^{\otimes p} = 0 \pmod{p}.$$
Here we have used the fact that $\sigma.a^{\otimes p} = \epsilon(\sigma)a^{\otimes p}$, because $|a|$ is odd.

This shows that $e \otimes_{\Sigma_p} a^{\otimes p}$ is a cycle, defining a homology class in dimension $(p-1) + p(2k-1) = 2kp - 1$. We can thus define
$$\xi(a) = \theta_*[e \otimes_{\Sigma_p} a^{\otimes p}] \in H_{2kp-1}\Omega^2 X.$$
One can check that this only depends on the homology class of a, so we get a well-defined operation
$$\xi\colon H_{2k-1}\Omega^2 X \to H_{2kp-1}\Omega^2 X.$$

We will need to know how this interacts with the homology suspension map $\sigma_*\colon \widetilde{H}_{k-1}\Omega Y = \widetilde{H}_k\Sigma\Omega Y \to \widetilde{H}_k Y$, which is induced by the evaluation map $\Sigma\Omega Y \to Y$. A special case of the Kudo transgression theorem says that for any $a \in H_{2k-1}\Omega^2 X$ we have

$$\sigma_*\xi(a) = \sigma_*(a)^p \in H_{2kp}\Omega X.$$

Recall that x_0 was defined as the obvious generator of $H_{2n-1}\Omega^2 S^{2n+1}$. For $k > 0$ we define

$$\begin{aligned} x_k &= \xi^k x_0 \in H_{2np^k-1}\Omega^2 S^{2n+1} \\ y_k &= \beta\xi^k x_0 \in H_{2np^k-2}\Omega^2 S^{2n+1}. \end{aligned}$$

Note that $\sigma_*(x_0)$ is the usual generator z of $H_*\Omega S^{2n+1} = \mathbb{F}_p[z]$, so the Kudo theorem tells us that $\sigma_*(x_k) = z^{p^k}$. The usual relation between σ_* and differential in the Serre spectral sequence for the fibration $\Omega^2 S^{2n+1} \to P\Omega S^{2n+1} \to \Omega S^{2n+1}$ shows that z^{p^k} survives to the $2np^k$'th page and that $d_{2np^k}(z^{p^k}) = x_k$. Related arguments show that $d_{2np^k(p-1)}(z^{p^k(p-1)}x_k = y_k$. We can thus set up a model spectral sequence $\widehat{E}^r_{st}$ with

$$\begin{aligned} \widehat{E}^{2np^k}_{**} &= \mathbb{F}_p[z^{p^k}] \otimes E[x_j \mid j \geq k] \otimes \mathbb{F}_p[y_j \mid j > k] \\ d_{2np^k}(z^{p^k}) &= x_k \\ \widehat{E}^{2np^k(p-1)}_{**} &= \mathbb{F}_p[z^{p^{k+1}} \otimes E[x_j \mid j > k] \otimes \mathbb{F}_p[y_j \mid j > k]\{1, z^{p^k(p-1)}x_k\} \\ d_{2np^k(p-1)}(z^{p^k(p-1)}x_k) &= y_k. \end{aligned}$$

If E^r_{st} denotes the Serre spectral sequence, then we get a map $\widehat{E}^*_{**} \to E^*_{**}$ of first quadrant spectral sequences. This is an isomorphism on the bottom line at E^2, and also on the whole E^∞ page. A standard comparison result now tells us that it is an isomorphism on the whole E^2 page, and we conclude that

$$H_*\Omega^2 S^{2n+1} = \mathbb{F}_p[y_k \mid k > 0] \otimes E[x_k \mid k \geq 0]$$

as claimed.

19.2. **The Snaith splitting.** In this section, we outline a proof that $\Omega^2 S^{2n+1}$ splits stably as a wedge of finite spectra. Our proof follows [12]. We start with the following combinatorial approximation to $\Omega^2 S^{2n+1}$. Let $C_k S^{2n-1}$ denote the set of (discontinuous) maps $x\colon \mathbb{C} \to S^{2n-1}$ such that there are at most k points $z \in \mathbb{C}$ for which $x(z)$ is not the basepoint. There is an evident surjective map $F_k\mathbb{C} \times_{\Sigma_k} (S^{2n-1})^k \to C_k S^{2n-1}$: it sends a point $[z_1, \dots, z_k; a_1, \dots a_k]$ of $F_k\mathbb{C}$ to the map $x\colon \mathbb{C} \to S^{2n}$ with $x(z_i) = a_i$ and $x(z) = 0$ for all points other than the z_i's. We topologise $C_k S^{2n-1}$ as a quotient of $F_k\mathbb{C} \times_{\Sigma_k} (S^{2n-1})^k$. We also write $CS^{2n-1} = \bigcup_k C_k S^{2n-1}$, and we topologise this as the direct limit.

The map $\theta_k\colon F_k\mathbb{C} \times_{\Sigma_k} (\Omega^2 X)^k \to \Omega^2 X$ that we discussed earlier, together with the unit map $\eta\colon S^{2n-1} \to \Omega^2 S^{2n+1}$, gives a map $\overline{\theta}_k\colon C_k S^{2n-1} \to \Omega^2 S^{2n+1}$, and in the limit a map $\overline{\theta}\colon CS^{2n-1} \to \Omega^2 S^{2n+1}$. It can be shown that this is a homotopy equivalence [29].

We next write D_k for the quotient $C_k S^{2n-1}/C_{k-1}S^{2n-1}$. One can see directly that

$$D_k = (F_k\mathbb{C})_+ \wedge_{\Sigma_k} S^{(2n-1)k}.$$

This is also the Thom space $(B_k\mathbb{C})^{(2n-1)V}$, where V is the vector bundle $F_k\mathbb{C} \times_{\Sigma_k} \mathbb{R}^k$ over $B_k\mathbb{C}$. Moreover, the bundle $2V = \mathbb{C} \otimes V = F_k\mathbb{C} \times_{\Sigma_k} \mathbb{C}^k$ is trivial. To see this, let W be the space of complex polynomials of degree at most k, and note that the map $B_k\mathbb{C} \times W \to F_k\mathbb{C} \times_{\Sigma_k} \mathbb{C}^k$ sending $([z_1, \dots, z_p], f)$ to $[z_1, \dots, z_p; f(z_0), \dots, f(z_p)]$ is an isomorphism (by Lagrange interpolation, for example). We conclude that

$$D_k = \Sigma^{2(n-1)k}(B_k\mathbb{C})^V.$$

We next show that the filtration of CS^{2n-1} by the spaces $C_k S^{2n-1}$ is stably split. We first generalise the definition of CS^{2n-1}: for any space U and any pointed space X we let $C(U, X)$ denote the set of discontinuous maps $x\colon U \to X$ such that $x(u) = 0$ for all but finitely many points $u \in U$. Thus $CS^{2n-1} = C(\mathbb{C}, S^{2n-1})$. The same procedure that gave our map $CS^{2n-1} \to \Omega^2 S^{2n+1}$ gives maps $C(\mathbb{R}^n, X) \to \Omega^n\Sigma^n X$. After some minor adjustments we can make these compatible as n varies, and thus get a map $C(\mathbb{R}^\infty, X) \to QX = \varinjlim_n \Omega^n\Sigma^n X$ in the limit. (The cleanest way to make these adjustments is to use the constructions in [40].) It can be shown that this is a homotopy equivalence when X is a connected CW complex.

We now define a map $j_k \colon CS^{2n-1} \to C(\mathbb{R}^\infty, D_k)$ as follows. We take the polynomial ring $\mathbb{C}[t]$ as our model for $\mathbb{R}^\infty$, and $\mathbb{R}^{2n-1} \cup \{\infty\}$ as our model for S^{2n-1}. There is a natural identification

$$D_k = \{(A, z) \mid A \subset \mathbb{C},\ |A| = k,\ z \colon A \to \mathbb{R}^{2n-1}\} \cup \{\infty\}.$$

Suppose that $x \in CS^{2n-1}$, and let B be the finite set of points $z \in \mathbb{C}$ such that $x(z) \neq \infty$. For each subset $A \subseteq B$ with $|A| = k$, we define $g_A(t) = \prod_{z\in A}(t - z)$, so that $g_A \in \mathbb{C}[t]$. We then define a map $y \colon \mathbb{C}[t] \to D_k$ by setting $y(g_A) = (A, x|_A)$ for each subset A as above, and $y(f) = \infty$ for all other polynomials f. Thus y is a point of $C(\mathbb{R}^\infty, D_k)$, and we define $j_k(x) = y$. One can check that this gives a continuous map $j_k \colon CS^{2n-1} \to C(\mathbb{R}^\infty, D_k)$.

If we identify $C(\mathbb{R}^\infty, D_k)$ with QD_k and take adjoints, we get a map of spectra

$$j'_k \colon \Sigma^\infty CS^{2n-1} \to \Sigma^\infty D_k.$$

It is easy to see that $j_k C_{k-1} S^{2n-1} = 0$ and that the induced map

$$j'_k \colon \Sigma^\infty D_k = \Sigma^\infty C_k S^{2n-1}/C_{k-1}S^{2n-1} \to \Sigma^\infty D_k$$

is just the identity. Using this, it is not hard to conclude that

$$\Sigma^\infty \Omega^2 S^{2n+1}_+ \simeq \bigvee_{k\geq 0} \Sigma^\infty D_k.$$

(We need to observe here that the connectivity of $D_k = \Sigma^{2(n-1)k}(B_k\mathbb{C})^V$ tends to ∞ as k does, so that the wedge of the spectra $\Sigma^\infty D_k$ is the same as their product.)

19.3. **More about homology.** One can show that the loop sum map $\Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1} \to \Omega^2 S^{2n+1}$ splits stably as a wedge of maps $D_k \wedge D_l \to D_{k+l}$, and that the ξ operation sends $\widetilde{H}_* D_k$ to $\widetilde{H}_* D_{pk}$. This is enough to tell us how the Snaith splitting splits the homology ring $H_*\Omega^2 S^{2n+1} = \mathbb{F}_p[y_k \mid k > 0] \otimes E[x_k \mid k \geq 0]$. Explicitly, we can define a weight function on monomials by $\|x_i\| = \|y_i\| = p^i$, and $\|uv\| = \|u\| + \|v\|$. It follows easily from the above that $\widetilde{H}_* D_k$ is the span of those monomials u such that $\|u\| = k$. Note that this is zero unless k is 0 or 1 mod p, and that $\widetilde{H}_* D_{pk+1} = x_0 \widetilde{H}_* D_{pk}$. In fact, there is a p-local equivalence $D_{pk+1} = \Sigma^{2n-1} D_{pk}$ and D_j is p-locally stably contractible unless $j \in \{0, 1\} \pmod p$. The lowest degree generator on $\widetilde{H}_* D_{pk}$ is $y_1^k \in H_{2(np-1)k} D_{pk}$.

We can also determine the (co)action of the (dual) Steenrod algebra on $H_*\Omega^2 S^3$. There is a general formula for $P^k_* \xi(a)$ (for $a \in H_{2k-1}\Omega^2 X$), which involves the so-called Browder bracket operation $\lambda(-,-)$. In the case $X = S^{2n+1}$ one can show that λ vanishes. Given this, the formula reduces to

$$P^k_* \xi(a) = \begin{cases} \xi P^{k/p}_* a & \text{if } p \text{ divides } k \\ 0 & \text{otherwise.} \end{cases}$$

If we write $x(s) = \sum_{i\geq 0} x_i s^{p^i}$ and $y(s) = \sum_{i>0} y_i s^{p^i}$ then we find that

$$\begin{aligned} (\chi P(t))_* x(s) &= x(s) \\ (\chi P(t))_* y(s) &= y(s) + t y(s)^p \\ \alpha x(s) &= x(s) - (\chi\tau)(y(s)) \\ \alpha y(s) &= (\chi\zeta)(y(s)) = \zeta^{-1}(y(s)). \end{aligned}$$

One can also show that

$$\widetilde{H}^* D_{pk} = \mathcal{A}/(\chi(\beta^\epsilon P^j) \mid pj + \epsilon \geq k) u_{pk},$$

as modules over the Steenrod algebra $\mathcal{A}$. Here $u_{pk} \in H^{2(np-1)k} D_{pk}$ is dual to y_1^k. If p does not divide k then we have $\widetilde{H}^* D_{pk} = \mathcal{A}/(\chi(\beta^\epsilon P^j) \mid j > \lfloor k/p \rfloor)$, which means that D_{pk} is a so-called Brown-Gitler spectrum. A great deal is known about these [9].

The element $y_1 \in H_{2(np-1)}D_p$ is carried by a stable map $S^{2(np-1)} \to D_p$, which we also call y_1. Multiplication by y_1 gives maps

$$D_0 \to D_p \to D_{2p} \xrightarrow{\cdots} .$$

It turns out that the telescope of this sequence is just the Eilenberg-MacLane spectrum $H\mathbb{F}_p$.

Now take $n = 1$, so we consider $\Omega^2 S^3$. The evident map $S^3 \to K(\mathbb{Z}, 3)$ gives a map $f\colon \Omega^2 S^3 \to \Omega^2 K(\mathbb{Z}, 3) = S^1$ of H-spaces, whose composite with the unit map $\eta\colon S^1 \to \Omega^2 S^3$ is the identity. Using this, we see that the fibre W of f is an H-space, and that $\Omega^2 S^3$ is homotopy equivalent to $S^1 \times W$ (but not as H-spaces). It can be shown that there is a p-local sphere bundle over W whose Thom spectrum is the Eilenberg-MacLane spectrum $H\mathbb{Z}_{(p)}$.

19.4. **References.** A very important application of the above theory is to the proof of the nilpotence theorem of Hopkins, Devinatz and Smith. See [37] for a discussion of this.

20. Homology of Eilenberg-MacLane spaces

Let p be an odd prime. In this section, we discuss the mod p homology of the Eilenberg-MacLane spaces $K(\mathbb{F}_p, n)$. The key to an efficient computation of these groups is to fit them together into an elaborate algebraic object called a Hopf ring and compute them all simultaneously. This is discussed in more detail in [47].

20.1. **The Hopf ring $H_*\underline{H}_*$.** We will write H for the Eilenberg-MacLane spectrum with $\pi_* H = \mathbb{F}_p$. The k'th space in this spectrum is $\underline{H}_k = K(\mathbb{F}_p, k)$. We will need to consider the following maps:

$$\begin{aligned}
\sigma\colon \underline{H}_k \times \underline{H}_k &\to \underline{H}_k \\
\mu\colon \underline{H}_k \times \underline{H}_l &\to \underline{H}_{k+l} \\
\delta\colon \underline{H}_k &\to \underline{H}_k \times \underline{H}_k \\
\zeta\colon \underline{H}_k &\to 1
\end{aligned}$$

The map σ induces the addition map in the group $H^k X = [X, \underline{H}_k]$. The map μ induces the multiplication map

$$H^k X \times H^l X = [X, \underline{H}_k \times \underline{H}_l] \to [X, \underline{H}_{k+l}] = H^{k+l} X.$$

The maps δ and ζ are the diagonal map and the constant map.

These maps induce

(a) A product
$$\sigma_*\colon H_i\underline{H}_k \otimes H_j\underline{H}_k \to H_{i+j}\underline{H}_k,$$
written $\sigma_*(a \otimes b) = a * b$ or just ab.

(b) Another product
$$\mu_*\colon H_i\underline{H}_k \otimes H_j\underline{H}_l \to H_{i+j}\underline{H}_{k+l},$$
written $\mu_*(a \otimes b) = a \circ b$.

(c) A coproduct
$$\psi = \delta_*\colon H_i\underline{H}_l \to \bigoplus_{i=j+k} H_j\underline{H}_l \otimes H_k\underline{H}_l.$$

(d) An augmentation
$$\epsilon = \zeta_*\colon H_k\underline{H}_l \to \mathbb{F}_p.$$

Note that $H_*\underline{H}_0 = \mathbb{F}_p[\mathbb{F}_p] = \mathbb{F}_p\{[i] \mid i \in \mathbb{F}_p\}$. It is easy to see that

$$\begin{aligned}
[i] * [j] &= [i + j] \\
[i] \circ [j] &= [ij] \\
\psi[i] &= [i] \otimes [i] \\
\epsilon[i] &= 1.
\end{aligned}$$

Next, we consider $H_*\underline{H}_1 = H_* BC_p$. It is well-known that $H^* BC_p = E[x] \otimes \mathbb{F}_p[y]$, where $|x| = 1$, $|y| = 2$, and both x and y are primitive. It follows that $H_*\underline{H}_1 = E[e] \otimes D[a]$. The second factor here is a divided-power algebra, spanned by elements $a^{[k]}$ (to be thought of as $a^k/k!$) with

$$\begin{aligned}
a^{[k]} a^{[l]} &= \binom{k+l}{k} a^{[k+l]} \\
\psi(a^{[k]}) &= \sum_{k=i+j} a^{[i]} \otimes a^{[j]}.
\end{aligned}$$

If we write $a_i = a^{[p^i]}$ then $D[a]$ is isomorphic as an algebra to $\bigotimes_{i \geq 0} T[a_i]$, where $T[a]$ means the truncated polynomial algebra $\mathbb{F}_p[a]/a^p$.

Similarly, we find that

$$H_* K(\mathbb{Z}, 2) = H_* \mathbb{C}P^\infty = D[b] = \bigotimes_{i \geq 0} T[b_i],$$

where $b_i = b^{[p^i]} \in H_{2p^i}\mathbb{C}P^\infty$. We also write b_i for the image of b_i in $H_*K(\mathbb{F}_p, 2) = H_*\underline{H}_2$.

We can combine the elements e, a_i and b_j using our two products $*$ and $\circ$ to get many more elements of $H_*\underline{H}_*$; in fact, it turns out that we get all of $H_*\underline{H}_*$ this way. To keep this process under control, we need to understand some of the properties of these products. The relevant properties can be summarised by saying that they make $H_*\underline{H}_*$ into a graded ring in the category of graded coalgebras (otherwise known as a Hopf ring). More explicitly: $*$ and $\circ$ are both associative products, with units [0] and [1] respectively. The $*$-product is graded-commutative, but we have

$$a \circ b = (-1)^{ij}[(-1)^{kl}] \circ b \circ a$$

if $a \in H_i\underline{H}_k$ and $b \in H_j\underline{H}_l$. Both products are compatible with ψ and ϵ, in the sense that

$$\begin{aligned}\psi(a \circ b) &= \sum (-1)^{|b'||a''|} a' \circ b \otimes a'' \circ b'' \\ \epsilon(a \circ b) &= \epsilon(a)\epsilon(b),\end{aligned}$$

and similarly for $a * b$. The first equation is written in the usual Hopf algebra notation, with $\psi(a) = \sum a' \otimes a''$ and $\psi(b) = \sum b' \otimes b''$. Finally, we have a distributivity formula

$$a \circ (b * c) = \sum (-1)^{|b||a''|}(a' \circ b) * (a'' \circ c).$$

(Note that the "obvious" formula $a \circ (b * c) = (a \circ b) * (a \circ c)$ is both dimensionally inconsistent and nonlinear in a. Instead of the "obvious" diagonal map $a \mapsto a \otimes a$, we need to use the coproduct map ψ, which is induced by the diagonal map of spaces.)

For any sequence of integers $\nu = (\nu_0, \nu_1, \dots)$ with $0 \leq \nu_i < p$ and $\nu_i = 0$ for $i \gg 0$, we define

$$b^{\circ\nu} = b_0^{\circ\nu_0} \circ b_1^{\circ\nu_1} \circ \dots.$$

It is convenient to set $b^{\circ\nu} = [1] - [0]$ if $\nu = 0$. This ensures that $(b^{\circ\nu})^{*p} = 0$ and $\epsilon(b^{\circ\nu}) = 0$ for all ν. Similarly, given a sequence $\mu = (\mu_0, \mu_1, \dots)$ with $\mu_i \in \{0, 1\}$ and $\mu_i = 0$ for $i \gg 0$, we define

$$a^{\circ\mu} = a_0^{\circ\mu_0} \circ a_1^{\circ\mu_1} \circ \dots.$$

The main result is that

$$\bigotimes_k H_*\underline{H}_k = \bigotimes_{\mu,\nu} T[a^{\circ\mu} \circ b^{\circ\nu}] \otimes E[e \circ a^{\circ\mu} \circ b^{\circ\nu}].$$

The subring $H_*\underline{H}_k$ consists of those factors $T[a^{\circ\mu} \circ b^{\circ\nu}]$ for which $\sum \mu_i + 2\sum \nu_i = k$, together with those factors $E[e \circ a^{\circ\mu} \circ b^{\circ\nu}]$ for which $\sum \mu_i + 2\sum \nu_i = k - 1$.

The proof uses the Rothenberg-Steenrod spectral sequence (also called the bar spectral sequence)

$$\mathrm{Tor}_{**}^{H_*\underline{H}_k}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow H_*\underline{H}_{k+1}.$$

The E_2 term is easy to compute (given that $H_*\underline{H}_k$ is as claimed) using the formulae

$$\begin{aligned}\mathrm{Tor}^{A\otimes B}(\mathbb{F}_p, \mathbb{F}_p) &= \mathrm{Tor}^A(\mathbb{F}_p, \mathbb{F}_p) \otimes \mathrm{Tor}^B(\mathbb{F}_p, \mathbb{F}_p) \\ \mathrm{Tor}^{E[e]}(\mathbb{F}_p, \mathbb{F}_p) &= D[b] = \bigotimes_{k\geq 0} T[b^{[k]}] \qquad \text{with } b \in \mathrm{Tor}_{1,2|e|} \\ \mathrm{Tor}^{T[a]}(\mathbb{F}_p, \mathbb{F}_p) &= E[c] \otimes D[d] \qquad \text{with } c \in \mathrm{Tor}_{1,|a|} \text{ and } d \in \mathrm{Tor}_{2,p|a|}\,.\end{aligned}$$

With more work, one can show that the spectral sequence always collapses, and deduce that $H_*\underline{H}_{k+1}$ is as claimed.

20.2. **Stable homology.** We next consider the homology of the spectrum H, in other words the ring

$$H_*H = \varinjlim_k \widetilde{H}_{*+k}\underline{H}_k.$$

This is called the dual Steenrod algebra. The colimit implicitly uses a map

$$\sigma_* \colon \widetilde{H}_{k+l}\underline{H}_k \to \widetilde{H}_{k+l+1}\underline{H}_{k+1}.$$

This is just the homology suspension map, which is the same as the map $x \mapsto e \circ x$. This kills $*$-decomposable elements, which implies that H_kH is spanned by the images of the elements $a^{\circ\mu} \circ b^{\circ\nu}$

and $e \circ a^{\circ\mu} \circ b^{\circ\nu}$. It turns out that we do not need the latter elements, because $e \circ e = b_0$. Let $\zeta_k \in H_{2p^k-2}H$ and $\tau_k \in H_{2p^k-1}H$ be the images of $b_k \in \widetilde{H}_{2p^k}\underline{H}_2$ and $a_k \in \widetilde{H}_{2p^k}\underline{H}_1$ respectively. It turns out that $\zeta_0 = 1$ and that

$$H_*H = \mathbb{F}_p[\zeta_k \mid k > 0] \otimes E[\tau_k \mid k \geq 0].$$

This ring has a natural Hopf algebra structure, arising from the map

$$H_*H = \pi_*(H \wedge H) \xrightarrow{\pi_*(1\wedge\eta\wedge 1)} \pi_*(H \wedge H \wedge H) = H_*H \otimes H_*H.$$

For more discussion of this, see [36, Section 2.2] or [2, Part III, Section 12]. It turns out that the coproduct is

$$\psi(\zeta_k) = \sum_{k=i+j} \zeta_i^{p^j} \otimes \zeta_j$$
$$\psi(\tau_k) = \tau_k \otimes 1 + \sum_{k=i+j} \zeta_i^{p^j} \otimes \tau_j.$$

20.3. **Automorphisms of the additive group.** The Hopf algebra $\mathbb{F}_p[\zeta_k \mid k > 0] < H_*H$ can be described in the following more conceptual way. For any $\mathbb{F}_p$-algebra A, write

$$\widehat{G}_a(A) = \{ \text{ nilpotent elements of } A\},$$

considered as a group under addition. Define

$$\mathrm{Aut}(\widehat{G}_a)(A) = \{ \text{ isomorphisms } f\colon \widehat{G}_a(B) \to \widehat{G}_a(B), \text{ natural for } A\text{-algebras } B \}.$$

One can show that any $f \in \mathrm{Aut}(\widehat{G}_a)(A)$ has the form $f(b) = \sum_i a_i b^{p^i}$ for uniquely determined coefficients $a_i \in A$, with $a_0 \in A^\times$. We write $\mathrm{Aut}_1(\widehat{G}_a)(A)$ for the subgroup where $a_0 = 1$.

Given a map $\theta\colon \mathbb{F}_p[\zeta_k \mid k > 0] \to A$, we obtain an element $f\colon b \mapsto \sum_{i\geq 0} \theta(\zeta_i) b^{p^i}$ of $\mathrm{Aut}_1(\widehat{G}_a)(A)$. This clearly gives a bijection

$$\mathrm{Rings}(\mathbb{F}_p[\zeta_k \mid k > 0], A) = \mathrm{Aut}_1(\widehat{G}_a)(A).$$

The coproduct on $\mathbb{F}_p[\zeta_k \mid k > 0]$ makes the left hand side into a group, the right hand side is a group under composition, and the coproduct is the unique one for which these two group structures coincide.

If Z is a space and A is an $\mathbb{F}_p$-algebra, we can define $X_Z(A) = \mathrm{Rings}(H^*Z, A)$. Because of the above relationship of $\mathbb{F}_p[\zeta_k]$ with H_*H and $\mathrm{Aut}_1(\widehat{G}_a)$, it turns out that there is a natural action of the group $\mathrm{Aut}_1(\widehat{G}_a)(A)$ on the set $X_Z(A)$. It turns out that for many popular spaces Z, there is a simple and conceptual description of $X_Z(A)$ as a set with action of $\mathrm{Aut}_1(\widehat{G}_a)(A)$.

20.4. **Cohomology.** We can dualise the above calculation to describe H^*H and $H^*\underline{H}_*$. Firstly, we let $P(\alpha) \in H^*H$ be dual to $\zeta^\alpha = \prod_{i>0} \zeta_i^{\alpha_i}$ with respect to the obvious monomial basis, and let Q_i be dual to τ_i. It follows that $\pm Q^\epsilon P(\alpha)$ is dual to $\tau^\epsilon \zeta^\alpha$, and that these elements form a basis for the Hopf algebra H^*H (which is called the Steenrod algebra). The coproduct is given by

$$\psi P(\alpha) = \sum_{\alpha=\beta+\gamma} P(\beta) \otimes P(\gamma)$$
$$\psi Q_i = Q_i \otimes 1 + 1 \otimes Q_i.$$

There is an explicit but elaborate formula for the product $P(\alpha)P(\beta)$. See [26] for a discussion of this, and an extensive structure theory of modules over H^*H.

Let ι_m be the tautological generator of $H^m\underline{H}_m$. It can be shown (I think) that $H^*\underline{H}_m$ is the free graded-commutative algebra generated by the elements $Q^\epsilon P(\alpha)\iota_m$ for which $\sum_{k>0}(\epsilon_k + 2\alpha_k) < m$ (note that ϵ_0 does not contribute). This element has degree $m + \sum_{k\geq 0}(2p^k - 1)\epsilon_k + \sum_{k>0}(2p^k - 2)\alpha_k$, and it generates an exterior (resp. polynomial) algebra if the degree is odd (resp. even). The algebra $H^*\underline{H}_m$ is the tensor product of all these exterior and polynomial algebras.

An alternative basis for H^*H is given by the so-called admissible monomials [39]. Write $\beta = Q_0$ and $P^k = P(k, 0, 0, \ldots)$. These elements satisfy the Adem relations:

$$P^a P^b = \sum_{t=0}^{\lfloor a/p \rfloor} (-1)^{a+t} \left(\begin{array}{c} (p-1)(b-t)-1 \\ a-pt \end{array} \right) P^{a+b-t} P^t$$

$$\begin{aligned} P^a \beta P^b = & \sum_{t=0}^{\lfloor a/p \rfloor} (-1)^{a+t} \left(\begin{array}{c} (p-1)(b-t) \\ a-pt \end{array} \right) \beta P^{a+b-t} P^t + \\ & \sum_{t=0}^{\lfloor (a-1)/p \rfloor} (-1)^{a+t-1} \left(\begin{array}{c} (p-1)(b-t)-1 \\ a-pt-1 \end{array} \right) P^{a+b-t} \beta P^t. \end{aligned}$$

A sequence $I = (\epsilon_0, s_1, \epsilon_1, \ldots, s_k, \epsilon_k, 0, 0, \ldots)$ is said to be admissible if $\epsilon_i \in \{0, 1\}$ and $s_i \geq ps_{i+1} + \epsilon_i$ for all i. We write

$$P^I = \beta^{\epsilon_0} P^{s_1} \beta^{\epsilon_1} P^{s_2} \ldots P^{s_k} \beta^{\epsilon_k} \in H^*H.$$

It can be shown that these elements form a basis for H^*H, as I runs over the set of all admissible sequences. Moreover, we have

$$\langle P^I, \zeta_k \rangle = \begin{cases} 1 & \text{if } P^I = P^{p^{k-1}} P^{p^{k-2}} \ldots P^1 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle P^I, \tau_k \rangle = \begin{cases} 1 & \text{if } P^I = P^{p^{k-1}} P^{p^{k-2}} \ldots P^1 \beta \\ 0 & \text{otherwise} \end{cases}$$

20.5. **References.** Find a more precise reference for E_*E being a Hopf algebroid. Add a reference for free unstable modules.

21. Things left to do

(1) Spaces of long knots
(2) The Kummer surface.
(3) Brieskorn manifolds.
(4) The dodecahedral manifold.
(5) Knot complements, the Massey product in the complement of the Borromean rings.
(6) Grassmannians, Stiefel manifolds, (bounded) flag varieties.
(7) Buhstaber-Ray manifolds representing generators for MU_* and MU_*MU
(8) Matrices of bounded rank, intersection loci.
(9) Projective unitary groups.
(10) Loop groups, algebraic loops, central extensions, representations.
(11) Toric varieties.
(12) Moduli of stable marked curves of genus zero.
(13) Manifolds of isospectral tridiagonal matrices. (See papers by Gaifullin, Tomei, M. Davis ('Some aspherical manifolds'). Fix a monic polynomial $p(t)$ of degree $n+1$ with n distinct real roots. Let M be the space of $(n+1)\times(n+1)$ real symmetric matrices A such that $A_{ij}=0$ when $|i-j|>1$ and $\det(tI-A)=p(t)$. Then M is a $K(\pi,1)$ and a compact oriented n-manifold. It can be written as a union of 2^n permutahedra.)
(14) Springer varieties
(15) $\mathbb{Z}\times BU$.
(16) Projective unitary groups
(17) Spaces related to degeneracy loci
(18) The connective covers $BU\langle m\rangle$.
(19) General things about BG; some stable splittings.
(20) Stable retracts of BV, $U(\mathbb{C}[V])/V$, (Mitchell's complexes with cohomology free over $A(n)$) and $(\mathbb{C}P^n)^m$ (Jeff Smith's examples for the smash product theorem).
(21) Stable retracts of U and BU.
(22) The spectra $M(k)$, $L(k)$, $D(k)$, $Sp^{p^k}S^0$ occuring in Nick Kuhn's work on the Whitehead conjecture.
(23) Bruhat-Tits building for $PGL_n\mathbb{Q}_p$.
(24) The contractible spaces S^∞, $F_n\mathbb{R}^\infty$, and the quotient space $B_n\mathbb{R}^\infty=B\Sigma_n$.
(25) The space $\mathcal{L}(\mathcal{U},\mathcal{V})$ of linear isometries, and the associated operad.
(26) Spaces of embeddings of manifolds. I seem to remember that Goodwillie and/or Waldhausen have some stuff about this. Contractibility of the space of embeddings in $\mathbb{R}^\infty$.
(27) Some generalities about topology of manifolds: Atiyah duality, Morse theory, connected sums, handlebodies, elementary ideas about surgery.
(28) Brown-Gitler spectra.
(29) The Brown-Comenetz spectrum I.
(30) $\mathrm{Map}(BV,X)$.
(31) Generalised Moore spectra $V(n)$, S/I.
(32) Spectra realising small $\mathcal{A}$-modules.
(33) KU, kU.
(34) MP, $MP\wedge MP$.
(35) $BTop$, BPL, SF, $MTop$ and all that.
(36) Surgery spectra, Waldhausen K-theory
(37) BP, $P(n)$, $B(n)$, $E(n)$, $K(n)$ etc.
(38) Tate spectra P_GH, P_GMU, P_GS.
(39) $X(n)$, $T(n)$, and the nilpotence theorem.
(40) Spectra associated with the Adams conjecture and etale homotopy theory.
(41) Spaces occuring in unstable v_n-periodic homotopy theory and Brayton Gray's theory of cospectra.
(42) $BP\langle n\rangle$ and stuff about the projective dimension of BP_*X.
(43) Elliptic spectra, tmf, TMF and so on.

(44) $K(kU)$ and related objects.
(45) Spectra occuring in the disproof of the telescope conjecture.
(46) LM^{-T}, string topology, the cactus operad.
(47) Floer pro-spectra.
(48) Seiberg-Witten and Yang-Mills moduli spaces.
(49) Teichmüller space, and compactifications.
(50) Spaces of representations of discrete groups in $PSL_2(\mathbb{C})$.
(51) Calabi-Yau manifolds.
(52) Examples from rational homotopy theory.
(53) Groups of homotopy spheres.
(54) Fuchsian groups.
(55) Something about Goodwillie calculus.
(56) Something about Π-algebras.
(57) $\mathbb{C}P^{\infty}_{-1}$, Madsen-Weiss
(58) $\mathrm{Aut}(\mathrm{Free}_n)$, outer space, associated K-theory spectra.
(59) p-compact groups. $U(n)$ as a homotopy colimit of p-toral groups, Sullivan's classifying spaces for S^{2p-3}, p-compact groups with Weyl group $\Sigma_n \wr C_p$
(60) Anick spaces, Gray's delooping of the fibre of the double suspension.
(61) Examples from the theory of LS-category.
(62) Simultaneous conjugacy spaces $(U(n)^{\mathrm{ad}})^k/U(n)$.
(63) $\mathrm{Hom}(\mathbb{Z}^n, U(m))$
(64) The space of maximal tori in $U(n)$.

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