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NUMBER THEORETIC BACKGROUND

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1. Weil Groups. If G is a topological group we shall let G^c denote the closure of the commutator subgroup of G, and $G^{ab} = G/G^c$ the maximal abelian Hausdorff quotient of G. Recall that if H is a closed subgroup of finite index in G there is a *transfer* homomorphism t: $G^{ab} \to H^{ab}$, defined as follows: if s: $H\backslash G \to G$ is any section, then for $g \in G$,

$$t(gG^c) = \prod_{x \in H\backslash G} h_{g,x} \pmod{H^c},$$

where $h_{g,x} \in H$ is defined by $s(x)g = h_{g,x}s\,(xg)$.

(1.1) *Definition of Weil group.* Let F be a local or global field and $\bar{F}$ a separable algebraic closure of F. Let $E, E', \cdots$ denote finite extensions of F in $\bar{F}$. For each such E, let $G_E = \mathrm{Gal}(\bar{F}/E)$. A *Weil group for* $\bar{F}/F$ is not really just a group but a triple $(W_F, \varphi, \{r_E\})$. The first two ingredients are a topological group W_F and a continuous homomorphism φ: $W_F \to G_F$ with dense image. Given W_F and φ, we put $W_E = \varphi^{-1}(G_E)$ for each finite extension E of F in $\bar{F}$. The continuity of φ just means that W_E is open in W_F for each E, and its having dense image means that φ induces a *bijection* of homogeneous spaces:

$$W_F/W_E \xrightarrow{\sim} G_F/G_E \approx \mathrm{Hom}_F(E, \bar{F})$$

for each E, and in particular, a group isomorphism $W_F/W_E \approx \mathrm{Gal}(E/F)$ when E/F is Galois. The last ingredient of a Weil group is, for each E, an isomorphism of topological groups r_E: $C_E \xrightarrow{\sim} W_E^{ab}$, where

$$C_E = \begin{cases} \text{The multiplicative group } E^* \text{ of } E \text{ in the local case,} \\ \text{the idele-class group } A_E^*/E^* \text{ in the global case.} \end{cases}$$

In order to constitute a Weil group these ingredients must satisfy four conditions:

(W_1) For each E, the composed map

$$C_E \xrightarrow[\sim]{r_E} W_E^{ab} \xrightarrow{\text{induced by } \varphi} G_E^{ab}$$

is the reciprocity law homomorphism of class field theory.

(W_2) Let $w \in W_F$ and $\sigma = \varphi(w) \in G_F$. For each E the following diagram is commutative:

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$$\begin{tikzcd} C_E \arrow[r, "r_E"] \arrow[d, "\text{induced by } \sigma"'] & W_E^{ab} \arrow[d, "\text{conjugation by } w"] \\ C_{E^\sigma} \arrow[r, "r_{E^\sigma}"] & W_{E^\sigma}^{ab} \end{tikzcd}$$

(W$_3$) For $E' \subset E$ the diagram

$$\begin{tikzcd} C_{E'} \arrow[r, "r_{E'}"] \arrow[d, "\text{induced by inclusion } E'\subset E"'] & W_{E'}^{ab} \arrow[d, "\text{transfer}"] \\ C_E \arrow[r, "r_E"] & W_E^{ab} \end{tikzcd}$$

is commutative.

(W$_4$) The natural map

$$W_F \longrightarrow \operatorname{proj}\lim_E \{W_{E/F}\}$$

is an isomorphism of topological groups, where

(1.1.1) $\quad W_{E/F}$ denotes W_F/W_E^c (*not* W_F/W_E),

and the projective limit is taken over all E, ordered by inclusion, as $E \to \bar{F}$.

This concludes our definition of Weil group. It is clear from the definition that if W_F is a Weil group for $\bar{F}/F$, then, for each finite extension E of F in $\bar{F}$, W_E (furnished with the restriction of φ and the isomorphisms $r_{E'}$ for $E' \supset E$) is a Weil group for $\bar{F}/E$.

If W_F is a Weil group, then for each $F \subset E' \subset E$ the diagram

(1.2.2)
$$\begin{tikzcd} C_E \arrow[r, "r_E"] \arrow[d, "\text{norm, } N_{E/E'}"'] & W_E^{ab} \arrow[d, "\text{induced by inclusion } W_E \subset W_{E'}"] \\ C_{E'} \arrow[r, "r_{E'}"] & W_{E'}^{ab} \end{tikzcd}$$

is commutative.

This follows from the fact that, when H is a normal subgroup of finite index in G, the composition

$$H^{ab} \xrightarrow{\text{induced by inclusion}} G^{ab} \xrightarrow{\text{transfer}} H^{ab}$$

is the map which takes an element $h \in H$ into the product of its conjugates by representatives of elements of G/H. (In the notation of the first paragraph above, $h_{g,x} = s(x)\, gs(x)^{-1}$, if $g \in H \subset G$.)

(1.2) *Cohomology; construction of Weil groups.* Let W_F be a Weil group for $\bar{F}/F$. Then for each Galois E/F the group $W_{E/F} = W_F/W_E^c$ is an extension of $W_F/W_E = \mathrm{Gal}(E/F)$ by $W_E/W_E^c \approx C_E$. Let $\alpha_{E/F} \in H^2(\mathrm{Gal}(E/F), C_E)$ denote the class of this group extension. For each $n \in \mathbf{Z}$, let

(1.2.3) $\quad \alpha_n(E/F)\colon H^n(\mathrm{Gal}(E/F), \mathbf{Z}) \longrightarrow H^{n+2}(\mathrm{Gal}(E/F), C_E)$

be the map given by cup product with $\alpha_{E/F}$. Since $C_{E'} \to C_E^{\mathrm{Gal}(E/E')}$ is a bijection for $F \subset E' \subset E$, the property (W_3) above implies, via an abstract cohomological theorem (combine the corollary of p. 184 of **[AT]** with Theorem 12, p. 154, of **[S1]**), that $\alpha_n(E/F)$ is an isomorphism for every n. Moreover, the canonical classes are interrelated by

$$(1.2.4) \qquad \mathrm{infl}\ \alpha_{E'/F} = [E:E']\alpha_{E/F} \quad \text{and} \quad \mathrm{res}\ \alpha_{E/F} = \alpha_{E/E'}$$

(for the first, use Theorem 6 on p. 188 of **[AT]**; the second is obvious). Thus, implicit in the existence of Weil groups is all the cohomology of class field theory.

For example, taking $n = -1$ in (1.2.3) we find $H^1(\mathrm{Gal}(E/F), C_E) = 0$. Taking $n = 0$, we find $H^2(\mathrm{Gal}(E/F), C_E)$ is cyclic of order $[E:F]$, generated by $\alpha_{E/F}$. Taking $n = -2$ we find an isomorphism $G_{E/F}^{\mathrm{ab}} \approx C_F/N_{E/F}C_E$ which, by (W_1), is that given by the reciprocity law. For E/F cyclic, this isomorphism determines $\alpha_{E/F}$, and it follows that $\alpha_{E/F}$ is the "canonical" or "fundamental" class of class field theory. The same is true for arbitrary E/F as one sees by taking a cyclic E_1/F of the same degree as E/F, and inflating $\alpha_{E/F}$ and $\alpha_{E_1/F}$ to EE_1/F, where they are equal by (1.2.4).

Conversely, if we are given classes $\alpha_{E/E'}$ satisfying (1.2.4) and such that the maps (1.2.3) are isomorphisms, then we can construct a Weil group W_F as the projective limit of group extensions $W_{E/F}$ made with these classes. This construction is abstracted and carried out in great detail in Chapter XIV of **[AT]**. The existence of such classes $\alpha_{E/E'}$ is proved in **[AT]** and **[CF]**.

Thus, a Weil group exists for every F; to what extent is it unique?

(1.3) *Unicity.* A Weil group for $\bar{F}/F$ is unique up to isomorphism. More precisely:

(1.3.1) PROPOSITION. *Let W_F and W'_F be two Weil groups for $\bar{F}/F$. There exists an isomorphism $\theta\colon W_F \xrightarrow{\sim} W'_F$ such that the diagrams*

$$\begin{tikzcd} W_F \arrow[dd, "\theta"'] \arrow[dr, "\varphi"] & \\ & G_F \\ W_{F'} \arrow[ur, "\varphi'"'] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & W_E^{\mathrm{ab}} \arrow[dd, "\text{induced by } \theta"] \\ C_E \arrow[ur, "r_E"] \arrow[dr, "r_{E'}"'] & \\ & (W'_E)^{\mathrm{ab}} \end{tikzcd}$$

are commutative.

For each finite Galois E/F, let $I(E)$ denote the set of isomorphisms f such that the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & C_E \arrow[r] \arrow[d, "\mathrm{id}"] & W_{E/F} \arrow[r] \arrow[d, "f"] & \mathrm{Gal}(E/F) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & C_E \arrow[r] & W'_{E/F} \arrow[r] & \mathrm{Gal}(E/F) \arrow[r] & 0 \end{tikzcd}$$

Since the two group extensions $W_{E/F}$ and $W'_{E/F}$ each have the same class, namely the canonical class $\alpha_{E/F}$, as their cohomology class, $I(E)$ is not empty. Since $H^1(\mathrm{Gal}(E/F), C_E) = 0$, an isomorphism $f \in I(E)$ is determined up to composing with an inner automorphism of $W_{E/F}$ by an element of $C_E \approx W_E^{ab}$. The center of $W_{E/F}$ is C_F, and C_E/C_F is compact. Hence $I(E)$, as principal homogeneous space for C_E/C_F, has a natural compact topology. For $E_1 \supset E$, the natural map $I(E_1) \to I(E)$ is continuous for this topology, since it is reflected in the norm map $N_{E_1/E}$, once we pick an element of $I(E_1)$. Hence the projective limit proj $\lim_E(I(E))$ is not empty. An element θ of this limit gives an isomorphism $W_F \xrightarrow{\sim} W_F'$ by (W_4), and has the required properties.

It turns out (cf. (1.5.2)) that θ is unique up to an inner automorphism of W_F by an element $w \in \mathrm{Ker}\ \varphi$, but we postpone the discussion of this question until after the next section.

(1.4) *Special cases.* We discuss now the special features of the four cases: F local nonarchimedean, F a global function field, F local archimedean, and F a global number field. In the first two of these, G_F is a completion of W_F; in the last two it is a quotient of W_F.

(1.4.1) *F local nonarchimedean.* For each E, let k_E be the residue field of E and $q_E = \mathrm{Card}(k_E)$. Let $\bar{k} = \bigcup_E k_E$. We can take W_F to be the dense subgroup of G_F consisting of the elements $\sigma \in G_F$ which induce on $\bar{k}$ the map $x \to x^{q_F^n}$ for some $n \in \boldsymbol{Z}$. Thus W_F contains the *inertia group* I_F (the subgroup of G_F fixing $\bar{k}$), and $W_F/I_F \approx \boldsymbol{Z}$. The topology in W_F is that for which I_F gets the profinite topology induced from G_F, and is open in W_F. The map $\varphi\colon W_F \to G_F$ is the inclusion, and the maps $r_E\colon E^* \to W_E^{ab}$ *are* the reciprocity law homomorphisms. Concerning the *sign* of the reciprocity law, our convention will be that $r_E(a)$ acts as $x \mapsto x^{\|a\|_E}$ on $\bar{k}$, where $\|a\|_E$ is the normed absolute value of an element $a \in E^*$. (If π_E is a uniformizer in E, then $\|\pi_E\|_E = q_E^{-1}$; thus our convention is that uniformizers correspond to the inverse of the Frobenius automorphism, as in Deligne **[D3]**, *opposite* to the convention used in **[D1]**, **[AT]**, **[CF]**, and **[S1]**.)

(1.4.2) *F a global function field.* Here the picture is as in (1.4.1). Just change "residue field" to "constant field", "inertia group I_F" to "geometric Galois group $\mathrm{Gal}(\bar{F}/F\bar{k})$", and define the norm $\|a\|_E$ of an idele class $a \in C_E$ to be the product of the normed absolute values of the components of an idele representing the class.

(1.4.3) *F local archimedean.* If $F \approx \boldsymbol{C}$ we can take $W_F = F^*$, φ the trivial map, r_F the identity.

If $F \approx \boldsymbol{R}$, we can take $W_F = \bar{F}^* \cup j\bar{F}^*$ with the rules $j^2 = -1$ and $jcj^{-1} = \bar{c}$, where $c \mapsto \bar{c}$ is the nontrivial element of $\mathrm{Gal}(\bar{F}/F)$. The map φ takes $\bar{F}^*$ to 1 and $j\bar{F}^*$ to that nontrivial element. The map $r_{\bar{F}}$ is the identity, and r_F is characterized by

$$\begin{aligned} r_F(-1) &= jW_{\bar{F}}^c, \\ r_F(x) &= \sqrt{x}\ W_{\bar{F}}^c, \quad \text{for } x \in F,\ x > 0. \end{aligned}$$

($W_{\bar{F}}^c$ is the "unit circle" of elements $u \in \bar{F}$ with $\|u\| = N_{\bar{F}/F}\, u = 1$.)

(1.4.4) *F a global number field.* This is the only case in which there is, at present, no simple description of W_F, but merely the artificial construction by cocycles described in (1.2). This construction is due to Weil in **[W1]**, where he emphasizes the importance of the problem of finding a more natural construction, and proves the following facts. The map $\varphi\colon W_F \to G_F$ is surjective. Its kernel is the connected component of identity in W_F, isomorphic to the inverse limit, under the norm maps $N_{E/E'}$ of the connected components D_E of 1 in C_E. These norm maps $D_E \to D_{E'}$ are surjective, and $r_E(D_E) = (\mathrm{Ker}\,\varphi)W_E^c/W_E^c$ is the image of Ker φ in $W_{E/F}$. If E has r_1 real and r_2 complex places then D_E is isomorphic to the product of $\boldsymbol{R}$ with $r_1 + r_2 - 1$ solenoids and r_2 circles.

(1.4.5) Notice that in each of the four cases just discussed the subgroups of W_F which are of the form W_E for some finite extension E of F are just the open subgroups of finite index. Their intersection, Ker φ, is a divisible connected abelian group, trivial in the first two cases, isomorphic to $\boldsymbol{C}^*$ in the third, and enormous in the last case.

(1.4.6) In each case there is a homomorphism $w \mapsto \|w\|$ of W_F into the multiplicative group of strictly positive real numbers which reflects the norm or normed absolute value on C_F under the isomorphism $r_F : C_F \approx W_F^{\mathrm{ab}}$. By (1.1.2) and the rule $\|N_{E/F}a\|_F = \|a\|_E$, the restriction of this "norm" function $\|w\|$ from W_F to a subgroup W_E is the norm function for W_E, so we can write simply $\|w\|$ instead of $\|w\|_F$ without creating confusion. In each case the kernel W_F^1 of $w \mapsto \|w\|$ is *compact.* In the first two cases, the image of $w \mapsto \|w\|$ consists of the powers of q_F, and W_F is a semidirect product $\boldsymbol{Z} \ltimes W_F^1$. In the last two cases $w \mapsto \|w\|$ is surjective, and in fact, W_F is a direct product $\boldsymbol{R} \times W_F^1$.

Let us refer to the first two cases as the "$\boldsymbol{Z}$-cases" and the last two as the "$\boldsymbol{R}$-cases". In the $\boldsymbol{Z}$-cases, φ is injective, but not surjective; in the $\boldsymbol{R}$-cases, φ is surjective, but not injective.

(1.5) *Automorphisms of Weil groups.* Let W_F be a Weil group for $\bar{F}/F$. Let Aut$(\bar{F}, W_F)$ denote the set of pairs (σ, α), where $\sigma \in G_F$ is an automorphism of $\bar{F}/F$, and α is an automorphism of the group W_F such that the following diagrams are commutative, the second for all E:

$$\begin{tikzcd} W_F \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & G_F \arrow[d, "\mathrm{Inn}(\alpha)"] \\ W_F \arrow[r, "\varphi"] & G_F \end{tikzcd} \qquad \begin{tikzcd} C_E \arrow[r, "r_E"] \arrow[d, "\text{induced by } \sigma"'] & W_E^{\mathrm{ab}} \arrow[d, "\text{induced by } \alpha"] \\ C_{E^\sigma} \arrow[r, "r_{E^\sigma}"] & W_{E^\sigma}^{\mathrm{ab}} \end{tikzcd}$$

Here Inn(α) denotes the inner automorphism defined by α.

We shall call an automorphism of W_F *essentially inner* if it induces an inner automorphism on $W_{E/F}$ for each finite Galois E/F.

(1.5.1) Proposition. *In the $\boldsymbol{R}$-cases* Aut$(\bar{F}, W_F)$ *consists of the pairs* $(\varphi(w)$, Inn$(w))$, *for* $w \in W_F$.

In the $\boldsymbol{Z}$-cases, Aut$(\bar{F}, W_F)$ *consists of the pairs* (σ, α_σ), *for* $\sigma \in G_F$, *where* α_σ *denotes the restriction of* Inn(σ) *to* W_F, *viewed as a subgroup of* G_F *via* φ. *This automorphism* α_σ *of* W_F *is not an inner automorphism if* $\sigma \notin W_F$, *but it is essentially inner in the sense of the definition above.*

(1.5.2) Corollary. *The isomorphism* θ *in* (1.3.1) *is unique in the* $\boldsymbol{Z}$*-cases, and is* *unique to an inner automorphism of W_F by an element of the connected component $W_F^0 = \mathrm{Ker}\ \varphi$ in the **R**-cases.*

To prove (1.5.1) in the $\boldsymbol{R}$-cases, we note first that, since φ is surjective, we are reduced immediately to the case of the corollary: We must show that if $(1, \alpha) \in \mathrm{Aut}(\bar{F}, W_F)$, then $\alpha = \mathrm{Inn}(w)$ for some $w \in W_F^0$. Going back to the proof of (1.3.1) with $W_F' = W_F$ we find that the group of these α's is given by

$$\begin{aligned}
\operatorname{proj}\lim_E (C_E/C_F) &= \operatorname{proj}\lim_E (C_E^1/C_F^1) && \text{(1 for norm 1)}\\
&= \operatorname{proj}\lim_E C_E^1/\operatorname{proj}\lim_E C_F^1 && \text{(by compacity)}\\
&= W^{0,1}/(Z \cap W^{\circ,1}) && \text{(existence theorem; 0 for connected component)}\\
&= W^0/Z
\end{aligned}$$

as claimed, where Z is the center of W.

Suppose now we are in a $\boldsymbol{Z}$-case. Since φ is injective, i.e., $W_F \subset G_F$, it is clear that $\mathrm{Aut}(\bar{F}, W_F)$ consists only of the pairs (σ, α_σ). The center of G_F is 1, because $G_F/G_E \approx \mathrm{Gal}(E/F)$ acts faithfully on $C_E \subset G_E^{\mathrm{ab}}$ for each finite Galois E/F. Hence, since W_F is dense in G_F, α_σ is not an inner automorphism of W_F unless $\sigma \in W_F$. However, α_σ does induce an inner automorphism of $W_{E/F}$ for finite E/F. Since W_F is dense in G_F it suffices to prove this last statement for σ close to 1, say $\sigma \in G_E$. Then α_σ induces an isomorphism of the group extension $0 \to C_E \to W_{E/F} \to \mathrm{Gal}(E/F) \to 0$ which is identity on the extremities, and hence is an inner automorphism by an element of C_E, since $H^1(\mathrm{Gal}(E/F), C_E) = 0$.

(1.6) *The local-global relationship.* Suppose now F is global. Let v be a place of F and F_v the completion of F at v. Let $\bar{F}$ (resp. $\bar{F}_v$) be a separable algebraic closure of F (resp. F_v) and let W_F (resp. W_{F_v}) be a Weil group for $\bar{F}/F$ (resp. for $\bar{F}_v/F_v$).

(1.6.1) PROPOSITION. *Let $i_v : \bar{F} \to \bar{F}_v$ be an F-homomorphism. For each finite extension E of F in $\bar{F}$, let $E_v = i(E)F_v$ be the induced completion of E. There exists a continuous homomorphism $\theta_v \colon W_{F_v} \to W_F$ such that the following diagrams are commutative*

$$\begin{tikzcd}
W_{F_v} \arrow[r] \arrow[d, "\theta_v"'] & G_{F_v} \arrow[d, "\text{induced by } i_v"] \\
W_F \arrow[r] & G_F
\end{tikzcd}
\qquad
\begin{tikzcd}
E_v^* \arrow[r, "\sim"] \arrow[d, "n_v"'] & W_{E_v}^{\mathrm{ab}} \arrow[d, "\text{induced by } i_v"] \\
C_E \arrow[r, "\sim"] & W_E^{\mathrm{ab}}
\end{tikzcd}$$

where n_v maps $a \in E_v^*$ to the class of the idele whose v-component is a and whose other components are 1.

If F is a function field, then θ_v is unique. In the number field case, θ_v is unique up to composition with an inner automorphism of W_F defined by an element of the connected component $W_F^0 = \mathrm{Ker}\ \varphi$.

The proof of this is analogous to the proof of (1.3.1) and (1.5.1), using the standard relationship between global and local canonical classes, and the vanishing of $H^1(\mathrm{Gal}(E_v/F_v),\ C_E)$.

Combining (1.6.1) and (1.5.2) we obtain

COROLLARY. *The diagram*

$$\begin{array}{ccc} W_{F_v} & \longrightarrow & G_{F_v} \\ {\scriptstyle\theta_v}\downarrow & & \downarrow{\scriptstyle i_v} \\ W_F & \longrightarrow & G_F \end{array}$$

is unique up to isomorphism, and the automorphisms of it are the inner automorphisms of W, defined by elements $w \in W^0$, which induce an automorphism of W_{F_v} (i.e., for v nonarchimedean, w fixed by G_{F_v}; for v archimedean, w a product of an element of W^0 by an element of $(W^0)^{G_{F_v}}$).

2. Representations. Let G be a topological group. By a *representation* of G we shall mean, in this section, a continuous homomorphism $\rho: G \to \mathrm{GL}(V)$ where V is a finite-dimensional complex vector space. By a *quasi-character* of G we mean a continuous homomorphism $\chi: G \to C^*$. If (ρ, V) is any representation of G, then $\det \rho$ is a quasi-character which we may sometimes denote also by $\det V$. The map $V \mapsto \det V$ sets up a bijection between the isomorphism classes of representations V of dimension 1 and quasi-characters. Of course we can identify quasi-characters of G with quasi-characters of G^{ab}.

We let $M(G)$ denote the set of isomorphism classes of representations of G, and $R(G)$ the group of virtual representations. A function λ on $M(G)$ with values in an abelian group X can be "extended" to a homomorphism $R(G) \to X$ if and only if it is *additive*, i.e., satisfies $\lambda(V) = \lambda(V')\,\lambda(V'')$ whenever $0 \to V' \to V \to V'' \to 0$ is an exact sequence of representations of G.

(2.1) Let F be a local or global field, $\bar{F}$ an algebraic closure of F, and W_F a Weil group for $\bar{F}/F$. Let (ρ, V) be a representation of W_F. Since $W_F = \mathrm{proj\,lim}\,\{W_{E/F}\}$ and $\mathrm{GL}(V)$ has no nontrivial small subgroups, ρ must factor through $W_{E/F}$, for some finite Galois extension E of F in $\bar{F}$. It follows that if α is an essentially inner automorphism of W_F in the sense of (1.5), then $V^\alpha \approx V$. Thus essentially inner automorphisms act as identity on $M(W_F)$ and $R(W_F)$. By (1.5.1) we can therefore safely think of $M(W_F)$ as *a set depending only on* F, not on a particular choice of $\bar{F}$ or of Weil group W_F for $\bar{F}/F$, and the same for $R(W_F)$. In this sense, if v is a place of a global F, the "restriction" map $M(W_F) \to M(W_{F_v})$ induced by the map θ_v of Proposition (1.6.1) *depends only on* v, not on a particular choice of the maps i_v and θ_v in that proposition, and the same for $R(W_F) \to R(W_{F_v})$. We shall indicate this map by $\rho \mapsto \rho_v$ or $V \mapsto V_v$. (The independence from θ_v results from (1.6.1), and the independence from i_v, from (1.5.1).)

If E/F is any finite separable extension, we have *canonical* maps

$$R(W_F) \underset{\substack{\text{induction} \\ \mathrm{ind}_{E/F}}}{\overset{\substack{\mathrm{res}_{E/F} \\ \text{restriction}}}{\rightleftarrows}} R(W_E)$$

satisfying the usual Frobenius reciprocity, for we can identify W_E with a closed subgroup of finite index in W_F.

(2.2) *Quasi-characters and representations of Galois type*. Using the isomorphism $C_F \approx W_F^{ab}$ we can identify quasi-characters of C_F with quasi-characters of W_F^{ab}. For example, we will denote by ω_s, for $s \in \boldsymbol{C}$, the quasi-character of W_F associated with the quasi-character $c \mapsto \|c\|_F^s$, where $\|c\|_F$ is the norm of $c \in C_F$. Thus $\omega_s(w) = \|w\|^s$ in the notation of (1.4).

On the other hand, since φ: $W_F \to G_F$ has dense image, we can identify the set $M(G_F)$ of isomorphism classes of representations of G_F with a subset of $M(W_F)$. We will call the representations in this subset "of Galois type". Thus, by (1.4.5), a representation ρ of W_F is of Galois type if and only if $\rho(W_F)$ is finite.

With these identifications, a character χ of G_F is identified with the character χ of C_F to which χ corresponds by the reciprocity law homomorphism.

(2.2.1) In the $\boldsymbol{Z}$-cases, i.e., if F is a global function field, or a nonarchimedean local field, then every *irreducible* representation ρ of W_F is of the form $\rho = \sigma \otimes \omega_s$, where σ is of Galois type. This is a general fact about irreducible representations of a group which is an extension of $\boldsymbol{Z}$ by a profinite group; some twist of ρ by a quasi-character trivial on the profinite subgroup has a finite image; see [D3, §4.10].

(2.2.2) If F is an archimedean local field, the quasi-characters of W_F, i.e., of $F^* \approx W_F^{ab}$, are of the form $\chi = z^{-N}\omega_s$, where z: $F \to \boldsymbol{C}$ is an embedding and N an integer $\geqq 0$, restricted to be 0 or 1 if F is real. If F is complex, these are the only irreducible representations of $F^* = W_F$. If F is real, W_F has an abelian subgroup $W_{\bar{F}} = \bar{F}^*$ of index 2, and the irreducible representations of W_F which are not quasi-characters are of the form $\rho = \mathrm{Ind}_{\bar{F}/F}(z^{-N}\omega_s)$ with $N > 0$. (For $N = 0$ this induced representation is reducible:

$$\mathrm{Ind}_{\bar{F}/F}\omega_s = \omega_s \oplus x^{-1}\omega_{s+1} \tag{2.2.2.1}$$

where x: $F \to \boldsymbol{C}$ is the embedding of F in $\boldsymbol{C}$.)

(2.2.3) Suppose F is a global number field. A *primitive* (i.e., not induced from a proper subgroup) *irreducible* representation ρ of W_F is of the form $\rho = \sigma \otimes \chi$ where σ is of Galois type and χ a quasi-character.

Choose a finite Galois extension E of F big enough so that ρ factors through $W_{E/F} = W_F/W_E^c$. Since ρ is primitive and irreducible, $\rho(W_E^{ab})$ must be in the center of GL(V), because W_E^{ab} is an abelian normal subgroup of $W_{E/F}$. In other words, the composed map $W_F \xrightarrow{\rho} \mathrm{GL}(V) \to \mathrm{PGL}(V)$ kills W_E and therefore gives a projective representation of Gal(E/F). This projective representation of Gal(E/F) can be lifted to a linear representation σ_0: $G_F \to \mathrm{GL}(V)$ (see [S3, Corollary of Theorem 4]). Let $\sigma = \sigma_0 \cdot \varphi$. The two compositions

$$W_F \underset{\sigma}{\overset{\rho}{\rightrightarrows}} \mathrm{GL}(V) \longrightarrow \mathrm{PGL}(V)$$

are equal; hence $\rho = \sigma \otimes \chi$ for some quasi-character χ.

(2.2.4) Note that, in all cases, global and local, the *primitive irreducible representations of W_F are twists of Galois representations by quasi-characters.*

(2.2.5) In the local nonarchimedean case one can say much more about the structure of primitive irreducible representations (see [K]). A first result of this sort is

(2.2.5.1) PROPOSITION. *Let F be local nonarchimedean and let V be a primitive irreducible representation of W_F. Then the restriction of V to the wild ramification group P is irreducible.*

This result is proved in [K] and [B]. The proof depends on the supersolvability of G_F/P.

(2.2.5.2) COROLLARY. *The dimension of V is a power of the residue characteristic p.*

Indeed, P is a pro-p-group.

(2.2.5.3) COROLLARY. *If U is an irreducible (not necessarily primitive) representation of W_F of degree prime to p, then U is monomial.*

Because U is induced from a primitive irreducible V whose dimension is prime to p and a p-power, hence 1.

(2.3) *Inductive functions of representations.* Let F be a local or global field. For a representation V of W_F, let $[V] \in R(W_F)$ denote the virtual representation determined by V. Let $R^0(W_F)$ denote the group of virtual representations of degree 0 of W_F, i.e., those of the form $[V]-[V']$, with $\dim V = \dim V'$.

(2.3.1) PROPOSITION. *The group $R(W_F)$ is generated by the elements of the form* $\mathrm{Ind}_{E/F}[\chi]$ *for E/F finite and χ a quasi-character of W_E. Similarly, $R^0(W_F)$ is generated by the elements of the form* $\mathrm{Ind}_{E/F}([\chi] - [\chi'])$.

It suffices to prove the second statement, because $R(W_F) = R^0(W_F) + \boldsymbol{Z}\cdot[1]$. Let $R^0_*(W_F)$ denote the subgroup of $R^0(W_F)$ generated by the elements $\mathrm{Ind}_{E/F}([\chi] - [\chi'])$. By the degree 0 variant of Brauer's theorem [D3, Proposition 1.5] we have $R^0(G_F) \subset R^0_*(W_F)$. The formula $\mathrm{Ind}(\rho \otimes \mathrm{Res}\,\chi) = (\mathrm{Ind}\,\rho)\otimes\chi$ shows that $R^0_* \cdot \chi \subset R^0_*$ for each quasi-character χ of W_F.

To prove the proposition we must show for each irreducible representation ρ of W_F that $[\rho]-(\dim\rho)[1] \in R^0_*(W_F)$. For each ρ there is a finite extension E of F and a *primitive* irreducible representation ρ_E of W_E such that $\rho = \mathrm{Ind}_{E/F}\,\rho_E$. Then $[\rho]-(\dim\rho)\cdot[1]$ is the sum of $\mathrm{Ind}_{E/F}([\rho_E]-(\dim\rho_E)[1_E])$ and $(\dim\rho_E)\,(\mathrm{Ind}_{E/F}[1_E]-[E{:}F]\,[1_F])$. The latter is of Galois type, so by the transitivity of induction we are reduced to the case in which ρ is primitive and irreducible. But then $\rho = \sigma\otimes\chi$ with $\sigma \in R(G_F)$ and χ a quasi-character (2.2.4). If $n = \dim\rho = \dim\sigma$

$$[\rho] - n[1] = ([\sigma] - n[1])\,[\chi] + n([\chi] - [1])$$

and this is in $R^0_*(W_F)$ by the remarks above, since $[\sigma] - n\,[1] \in R^0(G_F)$.

(2.3.2) DEFINITION. Let F be a local or global field. Let λ be a function which assigns to each finite separable extension E/F and each $V \in M(W_E)$ an element $\lambda(V)$ in an abelian group X. We say λ is *additive over* F is for each E and each exact sequence $0 \to V' \to V \to V'' \to 0$ of representations of W_E we have $\lambda(V) = \lambda(V')\lambda(V'')$. When that is so we can define λ on virtual representations so that λ: $R(E) \to X$ is a homomorphism for each E. We say λ is *inductive over* F if it is additive over F and the diagram

$$\begin{tikzcd} R(W_E) \arrow[dd, "\mathrm{Ind}_{E/E'}"'] \arrow[dr, "\lambda"] & \\ & X \\ R(W_{E'}) \arrow[ur, "\lambda"'] & \end{tikzcd}$$

is commutative for finite separable extensions $E/E'/F$. We say λ is *inductive in degree* 0 *over* F if the same is true with R replaced by R^0.

(2.3.3) REMARK. By (2.3.1) a λ which is inductive over F, or even only inductive in degree 0, is uniquely determined by its value on quasi-characters χ of W_E (i.e., of C_E), for all finite separable E/F. In [D3, §1.9] there is a discussion, for finite groups, of the relations a function λ of characters of subgroups must satisfy in order that it extend to an inductive function of representations.

(2.3.4) EXAMPLE. Let $a \in C_F$. Put

$$\lambda(V) = (\det V)(r_E(a)) \quad \text{for } V \in M(W_E).$$

Then λ is inductive in degree 0 over F. This follows from property (W_3) of Weil groups and the rule.

$$\det(\mathrm{Ind}\, V) = (\det V) \cdot \text{transfer}, \quad \text{for } V \text{ virtual of degree } 0$$

(cf. [D3, §1]).

(2.3.5) EXAMPLE. Suppose v is a place of a global field F, and λ is an inductive function over F_v. If we put for each finite separable E/F and each $V \in M(W_E)$

$$\lambda_v(V) = \prod_{w \text{ place of } E;\, w|v} \lambda(V_w)$$

we obtain an inductive function λ_v over F. If λ is only inductive in degree 0, then λ_v is inductive in degree 0.

Indeed, by a standard formula for the result of inducing from a subgroup and restricting to a different subgroup we have

$$(\mathrm{Ind}_{E/F}\, V)_v = \bigoplus_{w|v} \mathrm{Ind}_{E_w/F_v} V_w,$$

because if w_0 is one place of E over v, then the map $\sigma \mapsto \sigma w_0$ puts the set of double cosets $W_E\backslash W_F/W_{F_v}$ in bijection with the set of all such places, and for each σ we can identify $W_{E_{\sigma w_0}}$ with $(\sigma\, W_{F_v}\, \sigma^{-1}) \cap W_E$.

3. ***L*-series, functional equations, local constants.** The L-functions considered in this section are those associated by Weil [W1] to representations of Weil groups. They include as special cases the "abelian" L-series of Hecke, made with "Grössencharakteren" (= quasi-character of C_F), and the "nonabelian" L-functions of Artin, made with representations of Galois groups. Our discussion follows quite closely that of [D3, §§3, 4, 5] which we are just copying in many places.

(3.1) *Local abelian L-functions.* Let F be a local field.

For a quasi-character χ of F^* one defines $L(\chi) \in C^* \cup \{\infty\}$ as follows.

(3.1.1) $F \approx \boldsymbol{R}$. For x the embedding of F in C and $N = 0$ or 1,

$$L(x^{-N}\omega_s) = \Gamma_{\boldsymbol{R}}(s) \stackrel{\text{defn}}{=} \pi^{-s/2}\,\Gamma(s/2).$$

(3.2.1) $F \approx \boldsymbol{C}$. For z an embedding of F in $\boldsymbol{C}$ and $N \geqq 0$,

$$L(z^{-N}\omega_s) = \Gamma_{\boldsymbol{C}}(s) \stackrel{\text{defn}}{=} 2(2\pi)^{-s}\Gamma(s).$$

(3.1.3) F nonarchimedean. For π a uniformizer in F,

$$L(\chi) = \begin{cases} (1 - \chi(\pi))^{-1}, & \text{if } \chi \text{ is unramified,} \\ 1, & \text{if } \chi \text{ is ramified.} \end{cases}$$

In every case, L is a meromorphic function of χ, i.e., $L(\chi\omega_s)$ is meromorphic in s, and L has no zeros.

(3.2) *Local abelian ε-functions.* We will denote by dx a Haar measure on F, by d^*x a Haar measure on F^* (e.g., $d^*x = \|x\|^{-1}\,dx$) and by ψ a nontrivial additive character of F.

Given ψ and dx, one has a "Fourier transform"

$$\hat{f}(y) = \int f(x)\,\psi\,(xy)\,dx.$$

The local functional equation

$$\frac{\int \hat{f}(x)\,\omega_1\chi^{-1}(x)\,d^*x}{L(\omega_1\chi^{-1})} = \varepsilon(\chi, \psi, dx)\,\frac{\int f(x)\chi(x)\,d^*x}{L(\chi)} \tag{3.2.1}$$

defines a number $\varepsilon(\chi, \psi, dx) \in \boldsymbol{C}^*$ which is independent of f, for f's such that the two sides make sense. If f is continuous such that $f(x)$ and $\hat{f}(x)$ are $O(e^{-\|x\|})$ as $\|x\| \to \infty$, then the two sides make sense naively for χ such that $\chi(x) = \|x\|^{\sigma}$ with $0 < \sigma < 1$, and each side is a meromorphic function of χ. One takes the same multiplicative Haar measure d^*x on each side. The dependence of ε on ψ and dx comes from the dependence of the Fourier transform $\hat{f}(x)$ on ψ and dx. One finds

$$\varepsilon(\chi, \psi, r\,dx) = r\varepsilon(\chi, \psi, dx), \qquad \text{for } r > 0, \tag{3.2.2}$$

$$\varepsilon(\chi, \psi(ax), dx) = \chi(a)\|a\|^{-1}\,\varepsilon(\chi, \psi, dx) \quad \text{for } a \in F^*. \tag{3.2.3}$$

Easy computations carried out in **[T1]** and **[W2]** show that the function ε is given by (3.2.2), (3.2.3) and the following explicit formulas:

(3.2.4) $F \simeq \boldsymbol{R}$. Let x be the embedding of F in $\boldsymbol{C}$ and $N = 0$ or 1. For $\psi = \exp(2\pi i x)$ and dx the usual measure, $\varepsilon(x^{-N}\omega_s, \psi, dx) = i^N$.

(3.2.5) $F \approx \boldsymbol{C}$. Let z be an embedding of F in $\boldsymbol{C}$ and $N \geqq 0$. For $\psi = \exp(2\pi i\ \mathrm{Tr}_{\boldsymbol{C}/\boldsymbol{R}} z)$ and $dx = i\,dz \wedge d\bar{z}$ ($= 2\,da\,db$ for $z = a + bi$), $\varepsilon(z^{-N}\omega_s, \psi, dx) = i^N$.

(3.2.6) *F nonarchimedean.* Let $\mathcal{O}$ be the ring of integers in F. Put

$n(\psi) =$ the largest integer n such that $\psi(\pi^{-n}\mathcal{O}) = 1$,

$a(\chi) =$ the (exponent of the) conductor of χ ($= 0$ if χ is unramified, the smallest integer m such that χ is trivial on units $\equiv 1 \pmod{\pi^m}$ if π is ramified),

$c =$ an element of F^* of valuation $n(\psi) + a(\chi)$. If χ is unramified,

$$\varepsilon(\chi, \psi, dx) = \frac{\chi(c)}{\|c\|}\int_{\mathcal{O}} dx. \tag{3.2.6.1}$$

(In particular, $\varepsilon(\chi, \psi, dx) = 1$, if $\int_{\mathcal{O}} dx = 1$, and $n(\psi) = 0$ when χ is unramified.)

For χ ramified,

$$\begin{aligned}(3.2.6.2)\qquad \varepsilon(\chi, \psi, dx) &= \int_{F^*} \chi^{-1}(x)\,\psi(x)\,dx \overset{\text{defn}}{=} \sum_{n\in\mathbf{Z}} \int_{\pi^n\mathcal{O}^*} \chi^{-1}(x)\,\psi(x)\,dx \\ &= \int_{c^{-1}\mathcal{O}^*} \chi^{-1}(x)\,\psi(x)\,dx.\end{aligned}$$

From these formulas one deduces, for χ arbitrary and ω unramified

$$(3.2.6.3)\qquad \varepsilon(\chi\omega, \psi, dx,) = \varepsilon(\chi, \psi, dx)\,\omega(\pi^{n(\psi)+a(\chi)}).$$

(3.3) *Local nonabelian L-functions.* We owe to Artin the discovery that there is an inductive (2.3.2) function L of representations of Weil groups of local fields such that $L(V) = L(\chi)$ when V is a representation of degree 1 corresponding to the quasi-character χ. The explicit description of L is as follows:

(3.3.1) *F archimedean.* Since L is additive, we can define it by giving its value on irreducible V. For F complex, $W_F = F^*$ is abelian, and the only irreducible V's are the quasi-characters χ, for which L has already been defined. For F real, the only irreducible V's which are not of dimension 1 are those of the form $V = \mathrm{Ind}_{\bar{F}/F}\,\chi$, where χ is a quasi-character of $\bar{F}^* = W_{\bar{F}}$ which is not invariant under "complex conjugation". For such V we put $L(V) = L(\chi)$, as we are forced to do in order that L be inductive.

(3.3.2) *F nonarchimedean.* Let I be the inertia subgroup of W_F. Let Φ be an "inverse Frobenius", i.e., an element of W_F such that $\|\Phi\| = \|\pi\|_F$. This condition determines Φ uniquely mod I and we put $L(V) = \det(1 - \Phi|V^I)^{-1}$, where V^I is the subspace of elements in V fixed by I.

A proof that the "nonabelian" function L defined as above is inductive can be found in [**D3**, Proposition 3.8] (as well as in [**A**]). In the archimedean case one uses the relation $\Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1)$. Technically, in order that L have values in a *group*, we should view L as a function which associates with V the meromorphic function $s \mapsto L(V\omega_s)$, and take the X in Definition (2.3.2) to be the multiplicative group of nonzero meromorphic functions of s.

(3.4) *The local "nonabelian" ε-function, $\varepsilon(V, \psi, dx)$.* For this there is at present only an existence theorem (see below), no explicit formula.[1] This lack is not surprising if we recall that the formulas defining ε in (3.2) make essential use of the interpretation of χ as a quasi-character of F^*; if we think of χ as a quasi-character of W_F we have no way to define $\varepsilon(\chi, \psi, dx)$ without using the reciprocity law isomorphism $F^* \approx W_F^{\mathrm{ab}}$. In fact it was his idea about "nonabelian reciprocity laws" relating representations of degree n of W_F to irreducible representations π of $\mathrm{GL}(n, F)$, and the possibility of defining $\varepsilon(\pi, \psi, dx)$ for the latter, which led Langlands to conjecture and prove a version of the following big

(3.4.1) THEOREM. *There is a unique function ε which associates with each choice of a local field F, a nontrivial additive character ψ of F, an additive Haar measure dx on F and a representation V of W_F a number $\varepsilon(V, \psi, dx) \in \mathbf{C}^*$ such that $\varepsilon(V, \psi, dx) = \varepsilon(\chi, \psi, dx)$ if V is a representation of degree 1 corresponding to a quasi-character χ, and such that if F is a local field and we choose for each finite separable extension E*

[1] Except for Deligne's expression in terms of Stiefel-Whitney classes for *orthogonal* representations [**D5**, Proposition 5.2].

of F an additive Haar measure μ_E on E, then the function which associates with each such E and each $V \in M(W_E)$ the number $\varepsilon(V, \psi \cdot \mathrm{Tr}_{E/F}, \mu_E)$ is inductive in degree 0 *over F in the sense of* (2.3.2).

The unicity of such an ε is clear, by (2.3.1); the problem is *existence*. The experience of Dwork and Langlands indicates that the local proof of existence, based on showing that the $\varepsilon(\chi, \psi, dx)$ satisfy the necessary relations, is too involved to publish completely. Deligne found a relatively short proof (see [D3, §4]; possibly also [T2]). It has two main ingredients, one global, one local: (1) the existence of a global $\varepsilon(V)$ coming from the global functional equation for $L(V)$ (cf. (3.5) below), and (2) the fact that if F is local nonarchimedean and α a wildly ramified quasi-character of F^*, there is an element $y = y(\alpha, \psi)$ in F^* such that for all quasi-characters χ of F^* with $a(\chi) \leqq \frac{1}{2}a(\alpha)$, we have $\varepsilon(\chi\alpha, \psi, dx) = \chi^{-1}(y)\varepsilon(\alpha, \psi, dx)$, a rather harmless function of χ.

Granting the existence of $\varepsilon(V, \psi, dx)$ the following properties of it are easy consequences of the corresponding properties of $\varepsilon(\chi, \psi, dx)$, via inductivity in degree 0 and (2.3.1).

(3.4.2) ε is additive in V, so makes sense for V virtual.

(3.4.3) $\varepsilon(V, \psi, rdx) = r^{\dim V}\varepsilon(V, \psi, dx)$, for $r > 0$. In particular, for V virtual of degree 0, $\varepsilon(V, \psi, dx) = \varepsilon(V, \psi)$ is independent of dx.

(3.4.4) $\varepsilon(V, \psi a, dx) = (\det V)(a)\, \|a\|^{-\dim V}\varepsilon(V, \psi, dx)$, for $a \in F^*$ (cf. (2.3.4)).

(3.4.5) $\varepsilon(V\omega_s, \psi, dx) = \varepsilon(V, \psi, dx) f(V)^{-s}\,\delta(\psi)^{-s\dim V}$, where:

$\delta(\psi) = q_F^{n(\psi)}$ in the nonarchimedean case and is characterized in the archimedean case by the fact that $\delta(\psi a) = \|a\|^{-1}\delta(\psi)$, and $\delta(\psi) = 1$ for ψ as in (3.2.4) and (3.2.5).

$f(V) = 1$ in the archimedean case, and $= q_F^{a(V)}$, the absolute norm of the Artin conductor of V in the nonarchimedean case. This f can be characterized as the unique function inductive in degree 0 such that $f(\chi) = q_F^{a(\chi)}$ for quasi-characters χ. For the well-known explicit formula for $a(V)$ in terms of higher ramification groups, see [S1] or [D3, (4.5)].

(3.4.6) Suppose F nonarchimedean, W unramified. Then

$$\varepsilon(V \otimes W, \psi, dx) = \varepsilon(V, \psi, dx)^{\dim W} \cdot \det W(\pi^{a(V)+\dim V n(\psi)}).$$

(3.4.7) Let V^* denote the dual of V and dx' the Haar measure dual to dx relative to ψ. Then

$$\varepsilon(V, \psi, dx)\,\varepsilon(V^*\omega_1, \psi(-x), dx') = 1.$$

In particular

$$|\varepsilon(V, \psi, dx)|^2 = f(V)(\delta(\psi)dx/dx')^{\dim V}, \quad \text{if } V^* = \bar{V},$$

i.e., if V is unitary.

(3.4.8) If E/F is a finite separable extension, V_E a virtual representation of degree 0 of W_E and V_F the induced representation of W_F, then $\varepsilon(V_F, \psi) = \varepsilon(V_E, \psi \circ \mathrm{Tr})$.

(3.5) *Global L-functions, functional equations.* Let F be a global field, ψ a nontrivial additive character of A_F/F, and dx the Haar measure on A_F such that $\int_{A_F/F} dx = 1$ (Tamagawa measure). Call ψ_v the local component of ψ at a place v, and let $dx = \prod_v dx_v$ be any factorization of dx into a product of local measures such that the ring of integers in F_v gets measure 1 for almost all v.

Let V be a representation of "the" global Weil group W_F, and put

$$L(V, s) = \prod_v L(V_v \omega_s), \tag{3.5.1}$$

$$\varepsilon(V, s) = \prod_v \varepsilon(V_v \omega_s, \psi_v, dx_v). \tag{3.5.2}$$

(3.5.3) THEOREM. *The product* (3.5.1) *converges for s in some right half-plane and defines a function $L(V, s)$ which is meromorphic in the whole s-plane and satisfies the functional equation*

$$L(V, s) = \varepsilon(V, s)L(V^*, 1 - s) \tag{3.5.4}$$

where V^* is the dual of V.

For V a quasi-character χ this result was proved by Hecke. In the modern version of his proof (**[T1]**, **[W2]**) one shows by Poisson summation that for suitable functions f on A

$$\int_{A^*} f(x)\omega_{1-s}\chi^{-1}(x)\, dx^* = \int_{A^*} f(x)\omega_s \chi(x)\, dx^*,$$

the integrals being defined for all s by analytic continuation. Taking $f = \prod f_v$ and using the local functional equation (3.2.1) (with χ replaced by $\chi\omega_s$) one finds that (3.5.4) holds in the "abelian" case, $V = \chi$.

At this point, even without having a theory of the local nonabelian $\varepsilon(V_v, \psi_v, dx_v)$'s, one gets, via (2.3.1), (2.3.5), and the inductivity of the local L's, that $L(V, s)$ is meromorphic in the whole plane for each V, being defined by the product (3.5.1) in a right half-plane, and that $L(V, s)$ is inductive as a function of V. It follows that

$$\varepsilon'(V, s) \overset{\text{defn}}{=} \frac{L(V, s)}{L(V^*, 1 - s)}$$

is inductive in V and satisfies $\varepsilon'(\chi, s) = \prod_v \varepsilon(\chi_v\omega_s, \psi_v, dx_v)$ for quasi-characters χ of A^*/F^*. It is this fact about the local $\varepsilon(\chi_v, \psi_v, dx_v)$'s—that their product over all v for a global χ has an inductive extension to all global V—that Deligne uses in his "global" proof of the existence of local nonabelian ε's. Once their existence is proved, we have $\varepsilon'(V, s) = \varepsilon(V, s)$ by the unicity of inductive functions since $\varepsilon(V, s)$, defined by the product (3.5.2), is inductive in degree 0 by (2.3.5).

(3.5.5) Hecke's global function $L(\chi, s)$ is entire if χ is not of the form ω_s. Artin conjectured (in the Galois case) that $L(V, s)$ is entire for any V which has no constituent of the form ω_s. Weil proved Artin's conjecture for function fields. Recently Langlands, using ideas of Saito and Shintani, made a first breakthrough in the number field case, treating certain V's of dimension 2 by base change, using the trace formula. (See *The solution of a base change problem for* GL(2) (*following Langlands, Saito, Shintani*), these PROCEEDINGS, part 2, pp. 115–133.) These methods work for all V's of dimension 2 for which the image of W_F in PGL(V) is the tetrahedral group. They also work for some octahedral cases, but a new idea will be needed to apply them in the nonsolvable icosahedral case. However, J. Buhler **[B]**, with the aid of the Harvard Science Center PDP11 and the main result of **[DS]**, has proved the Artin conjecture for one particular icosahedral V of conductor 800, by checking the existence of the corresponding modular form of weight 1 and level 800.

Although the Riemann hypothesis concerning the zeros of $L(\chi, s)$ has been proved by Weil in the function field case, there seems to be no breakthrough in sight in the number field case. The conjunction of the Artin conjecture for all V and the Riemann hypothesis for all χ is equivalent to the positivity of a certain distribution on W_F (cf. [W3]).

(3.6) *Comparison of different conventions for local constants.* The modern references for the material we have been discussing are Deligne [D3] and Langlands [L], and we have here followed the conventions of [D3]. Happily, the definition of L-functions, both local and global, in [D3] coincides with that in [L]. But Deligne's local constants $\varepsilon(V, \psi, dx)$, which we will designate in this section by ε_D instead of just ε, differ somewhat from Langlands' $\varepsilon(V, \psi)$ which we will denote by ε_L here. The relationship is

$$\varepsilon_L(V, \psi) = \varepsilon_D(V\omega_{1/2}, \psi, dx_\psi), \tag{3.6.1}$$

where dx_ψ is the additive measure which is self-dual with respect to ψ. The other way around we have

$$\varepsilon_D(V, \psi, dx) = (dx/dx_\psi)^{\dim V}\, \varepsilon_L(V\omega_{-1/2}, \psi). \tag{3.6.2}$$

In the nonarchimedean case the constant dx/dx_ψ is given explicitly by $q^{-n(\psi)/2}\int_{\mathcal{O}} dx$. Also, in that case if V corresponds to a quasi-character χ of F^* we have

$$\varepsilon_L(\chi, \psi) = \chi(c)\,\frac{\int_{\mathcal{O}^*}\chi^{-1}(u)\psi(u/c)\,du}{\left|\int_{\mathcal{O}^*}\chi^{-1}(u)\psi(u/c)\,du\right|}, \tag{3.6.3}$$

where c is an element of F^* of valuation $a(\chi) + n(\psi)$ as in (3.2.6).

Langlands puts

$$\varepsilon_L(s, V, \psi) \overset{\text{defn}}{=} \varepsilon_L(V\omega_{s-(1/2)}, \psi) = \varepsilon_D(V\omega_s, \psi, dx_\psi). \tag{3.6.4}$$

Then the "constant" $\varepsilon(V, s)$ in the global functional equation (3.5.4) is given by $\varepsilon(V, s) = \prod_v \varepsilon_L(s, V_v, \psi_v)$ for any nontrivial character ψ of A/F, because if dx_v is self-dual on F_v with respect to ψ_v for each place v, then $dx = \prod_v dx_v$ is self-dual on A with respect to ψ, and is therefore the Tamagawa measure on A.

The behavior of ε_L under twisting by an unramified quasi-character is given by

$$\varepsilon_L(V\omega_s, \psi) = \varepsilon_L(V, \psi) f(V)^{-s}\, \delta(\psi)^{-s \dim V} \tag{3.6.5}$$

as in (3.4.5), but its dependence on ψ is according to

$$\varepsilon_L(V, \psi_a) = (\det V)(a)\varepsilon(V, \psi), \tag{3.6.6}$$

instead of as in (3.4.4). If V^* is the contragredient of V, then

$$\varepsilon_L(V, \psi)\varepsilon_L(V^*, \psi^{-1}) = 1. \tag{3.6.7}$$

Hence, by (3.6.6)

$$\varepsilon_L(V, \psi)\, \varepsilon_L(V^*, \psi) = (\det V)(-1) \tag{3.6.8}$$

and on the other hand,

$$|\varepsilon_L(V, \psi)| = 1, \quad \text{if } V \text{ is unitary.} \tag{3.6.9}$$

The ε_L-system has the advantage that it avoids carrying along the measure dx, but it has the following disadvantage: in the nonarchimedean case, if σ is a discontinuous automorphism of C, then V^σ is again a representation of W_F, and ψ^σ an additive character, but $\varepsilon_L(V^\sigma, \psi^\sigma)$ is not in general equal to $\varepsilon_L(V, \psi)^\sigma$ (nor is $\varepsilon_L(0, V^\sigma, \psi^\sigma) = \varepsilon_L(0, V, \psi)^\sigma$). The trouble is that the absolute value (in (3.6.3)) may not be preserved by σ, and/or that the self-dual measure dx_ψ in (3.6.1) may involve $\sqrt{p}$, and hence may not be preserved by σ. If one does wish to eliminate the measure dx, it is probably preferable to define, say,

$$\varepsilon_1(V, \psi) = \varepsilon_D(V, \psi, dx_1), \tag{3.6.10}$$

where dx_1 is the measure for which $\mathcal{O}$ gets measure 1 in the nonarchimedean case, and is the measure described in (3.2.4) and (3.2.5) in the archimedean case. This convention has the minor disadvantage that the $\varepsilon(V)$ in the global functional equation is not equal to the product of the local $\varepsilon_1(V_v, \psi_v)$'s, but is, rather, a^{-1} times that product, where a is the square root of the discriminant for a number field, and is q^{g-1} for a function field of genus g with q elements in its constant field. But the $\varepsilon_1(V, \psi)$ has the advantage that in the nonarchimedean case we do have $\varepsilon_1(V^\sigma, \psi^\sigma) = \varepsilon_1(V, \psi)^\sigma$ for all automorphisms σ of C. This is clear, by unicity (2.3.3) and the formula

$$\varepsilon_1(\chi, \psi) = \chi(c)q^{n(\psi)} \sum_{u \in \mathcal{O}^* \bmod \pi^{a(\chi)}} \chi(u)\left(\frac{u}{c}\right) \tag{3.6.11}$$

which follows from (3.2.6.1) and (3.2.6.2) where the notation is explained. Thus in the nonarchimedean case we can define, for V and ψ over any field E of characteristic 0 (an open subgroup of I acting trivially on V, and ψ trivial on some $\pi^n\mathcal{O}$), an $\varepsilon_1(V, \psi) \in E^*$, in a unique way such that $\varepsilon_1(V^\alpha, \psi^\alpha) = \varepsilon_1(V, \psi)^\alpha$ for any homomorphism $\alpha\colon E \to E'$ and such that ε_1 is the old ε_1, given by (3.6.10), when $E = C$. So defined, $\varepsilon_1(V_E, \psi \cdot \mathrm{Tr}_{E/F})$ is inductive in degree 0 (2.3.2) for every field of scalars E of characteristic 0, and $\varepsilon_1(V, \psi)$ will be given by (3.6.11) if V corresponds to a quasi-character $\chi\colon F^* \to E^*$.

In writing these notes I was tempted to shorten things a bit by using only $\varepsilon_1(V, \psi)$ instead of $\varepsilon_D(V, \psi, dx)$, but decided against it because (1) the ε_D-system avoids all choices and is the most general and flexible—any other system, like ε_L or ε_1 can be immediately described as a special case of ε_D; (2) the dependence of ε on dx shows "why" ε is inductive only in degree 0, and (3) in case our local field F is nonarchimedean, the ε_D-system, like the ε_1, works over any field E of characteristic 0, as soon as one defines the notion of Haar measure on F with values in E (cf. [D3, (6.1)]).

4. The Weil-Deligne group, λ-adic representations, L-functions of motives. The representations considered in §3 are just the beginning of the story. Those of Galois type are effective motives of degree 0—which Deligne calls *Artin motives* in his article [D6, §6] in these PROCEEDINGS—with coefficients in C. We cannot discuss the notion of motive here (cf., e.g., [D1] and [D6] for this) but we do want to discuss the way in which L-functions and ε-functions are attached to motives of any degree. Only very special motives of degree $\neq 0$ correspond to the representations of W_F considered in §3, namely, those *of type* A_0, i.e., those which, after a finite extension E/F, correspond to direct sums of Hecke characters of type A_0 over E. (A candidate for a "motivic Galois group" for these is constructed by Langlands in these

PROCEEDINGS [L3].) The simplest motives not of this type are those given by elliptic curves with no complex multiplication; their L-functions are the "Hasse-Weil zeta-functions" which are not expressible in terms of Hecke's L-functions.

The procedure for attaching L-functions to motives in the form given it by Deligne [D3], [D6] can be outlined schematically as follows:

F is a global field.

v is a place of F.

E is a field of finite degree over $\boldsymbol{Q}$.

λ runs through the finite places of E whose residue characteristic is prime to char(F).

σ is an embedding of E in $\boldsymbol{C}$.

$$\begin{tikzcd}[column sep=large]
\begin{array}{l}\text{Motive } M \text{ over } F\\ \text{with cx. multn. by } E\end{array} \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\begin{array}{l}\text{gd. fld.}\\ \text{extension}\end{array}"'] & \begin{array}{l}\text{System } (H_\lambda(M)) \text{ of } \lambda\text{-adic}\\ \text{representations of } G_F\end{array} \arrow[d, Rightarrow, "\text{restriction}"] \\
\begin{array}{l}\text{Motive } M_v \text{ over } F_v\\ \text{with cx. multn. by } E\end{array} \arrow[r, Rightarrow, "\begin{array}{l}l\text{-adic}\\ \text{coho.}\end{array}"'] \arrow[d, Rightarrow, "\begin{array}{l}\text{Hodge theory}\\ \text{cf.(4.4)}\end{array}"', "v \text{ archimedean}"] & \begin{array}{l}\text{System } (H_\lambda(M_v)) \text{ of } \lambda\text{-adic}\\ \text{representations of } G_{F_v}\end{array} \arrow[d, Rightarrow, "\begin{array}{l}v \text{ nonarchimedean;}\\ \text{cf. (4.2)}\end{array}"] \\
\begin{array}{l}\text{Hodge structure } \mathrm{Hdg}(M_v)\\ \text{over } F_v \text{ with cx. multn. by } E\end{array} \arrow[d, Rightarrow, "\text{(cf. (4.4))}"'] & \begin{array}{l}\text{Equiv. class } V(M_{v,\sigma}) \text{ of}\\ \text{repns. of the Weil-Deligne}\\ \text{group } W'_{F_v} \text{ over } \boldsymbol{C}\text{, invariant}\\ \text{under } \mathrm{Aut}(\boldsymbol{C}/E)\end{array} \arrow[dd, Rightarrow, "\text{cf. (4.1)}"] \\
\begin{array}{l}\text{Equiv. class } V(M_{v,\sigma}) \text{ of repns.}\\ \text{over } \boldsymbol{C} \text{ of the Weil group } W_{F_v}\end{array} \arrow[d, Rightarrow, "\text{cf. (3.3), (3.4)}"'] & \\
\begin{array}{l}L\text{- (i.e. } \Gamma\text{-) and } \varepsilon\text{-factors}\\ \text{at the archimedean place } v\end{array} & \begin{array}{l}L\text{- and } \varepsilon\text{-factors at the}\\ \text{nonarchimedean place } v\end{array}
\end{tikzcd}$$

In the next sections we discuss some of the steps and concepts indicated in the above chart. We begin with the Weil-Deligne group. This is a group scheme over $\boldsymbol{Q}$, but what counts, its points in and representations over fields of characteristic 0, can be described naively with no reference to schemes.

(4.1) *The Weil-Deligne group and its representations.* Let F be a nonarchimedean local field and let $\bar{F}$, $G_F = \mathrm{Gal}(\bar{F}/F)$, W_F (Weil group), and I (inertia group) have their usual meaning. For $w \in W_F$, let $\|w\|$ denote the power of q to which w raises elements of the residue field, as in (1.4.6). Thus we have $\|w\| = 1$ for $w \in I$, and $\|\Phi\| = q^{-1}$ for a geometric Frobenius element Φ. We view W_F as a group scheme over $\boldsymbol{Q}$ as follows: for each open normal subgroup J of I, we view W_F/J as a "discrete" scheme, and we put $W_F = \text{proj lim } (W_F/J)$, the limit taken over all J. In other words, we have

$$W_F = \coprod_{n \in \boldsymbol{Z}} \Phi^n I = \coprod_{n \in \boldsymbol{Z}} \operatorname{spec} A_n,$$

where A_n is the ring of locally constant Q-valued functions on $\Phi^n I$.

(4.1.1) DEFINITION [D3, (8.3.6)]. The *Weil-Deligne group* W_F' is the group scheme over Q which is the semidirect product of W_F by G_a, on which W_F acts by the rule $wxw^{-1} = \|w\|x$.

Let E be a field of characteristic 0. The group $W_F'(E)$ of points of W_F' with coordinates in E is just $E \times W_F$ with the law of composition $(a_1, w_1)(a_2, w_2) = (a_1 + \|w_1\| a_2, w_1w_2)$ for $a_1, a_2 \in E$ and $w_1, w_2 \in W_F$.

Let V be a finite-dimensional vector space over E. A homomorphism of group schemes over E

$$\rho' \colon W_F' \times_Q E \longrightarrow \mathrm{GL}(V)$$

determines, and is determined by, a pair (ρ, N) as in (4.1.2) below, such that, on points, $\rho'((a, w)) = \exp(aN) \cdot \rho(w)$. That is the explanation for the following definition:

(4.1.2) DEFINITION [D3, (8.4.1)]. Let E be a field of characteristic 0. A *representation of W_F' over E* is a pair $\rho' = (\rho, N)$ consisting of:

(a) A finite-dimensional vector space V over E and a homomorphism $\rho \colon W_F \to \mathrm{GL}(V)$ whose kernel contains an open subgroup of I, i.e., which is continuous for the discrete topology in $\mathrm{GL}(V)$.

(b) A nilpotent endomorphism N of V, such that $\rho(w)N\rho(w)^{-1} = \|w\|N$, for $w \in W$.

(4.1.3) *Φ-semisimplicity*. Let $\rho' = (\rho, N)$ be a representation of W_F' over E. Define $v : W_F \to Z$ by $\|w\| = q^{-v(w)}$. There is a unique unipotent automorphism u of V such that u commutes with N and with $\rho(W_F)$ and such that $\exp(aN)\rho(w)u^{-v(w)}$ is a semisimple automorphism of V for all $a \in E$ and all $w \in W_F—I$ [D3, (8.5)]. Then $\rho'_{ss} = (\rho u^{-v}, N)$ is called the *Φ-semisimplification* of ρ', and ρ' is called *Φ-semisimple* if and only if $\rho' = \rho'_{ss}$, i.e., $u = 1$, i.e., the Frobeniuses act semisimply. For this it is necessary and sufficient that the representation ρ of W_F be semisimple in the ordinary sense, because $\rho(\Phi)$ generates a subgroup of finite index in $\rho(W_F)$, and in characteristic 0 a representation of a group is semisimple if and only if its restriction to a subgroup of finite index is semisimple. In his article in these PROCEEDINGS, Borel discusses *admissible* morphisms $W_F' \to {}^L G$; when $G = \mathrm{GL}_n$, these are just our Φ-semisimple (ρ, N)'s.

(4.1.4) EXAMPLE. $\mathrm{Sp}(n)$ is the following representation (ρ, N) of W_F' over Q.

$$\begin{aligned} V &= Q^n = Qe_0 + Qe_1 + \cdots + Qe_{n-1}, \\ \rho(w)e_i &= \omega_i(w)e_i \qquad (= \|w\|^i e_i), \\ Ne_i &= e_{i+1} \qquad (0 \leq i < n-1),\ Ne_{n-1} = 0. \end{aligned}$$

(4.1.5) Given any (ρ, N), Ker N is stable under W_F. Hence (ρ, N) is irreducible $\Leftrightarrow$ $N = 0$ and ρ is irreducible. It is not hard to show that the Φ-semisimple indecomposable representations of W_F' are those of the form $\rho' \otimes \mathrm{Sp}(n)$ with ρ' irreducible. (The $\otimes$ is defined by $(\rho, N) \otimes (\rho_1, N_1) = (\rho \otimes \rho_1, N \otimes 1 + 1 \otimes N_1)$.)

(4.1.6) Let (ρ, N, V) be a representation of W_F' over E. We put $V_N^I = (\mathrm{Ker}\, N)^I$ and define a local L-factor, a conductor, and a local constant by

$$Z(V, t) = \det(1 - \Phi t \mid V_N^I)^{-1}, \quad \text{and} \quad L(V, s) = Z(V, q^{-s}), \text{ when } E \subset C;$$

$$a(V) = a(\rho) + \dim V^I - \dim V_N^I,$$

$$\varepsilon(V) = \varepsilon(\rho)\det(-\Phi \mid V^I/V_N^I),$$

and

$$\varepsilon(V, t) = \varepsilon(V)t^{a(V)}.$$

Here, for ε, the usual ψ and dx are understood, but omitted from the notation.

These quantities do not change if we replace V by its Φ-semisimplification; but note that they are *not* additive as functions of V, because V_N is not. If $N = 0$, they are the same as before.

One of the main reasons for introducing the Weil-Deligne group is the fantastic generalization of local class field theory embodied in:

(4.1.7) *Conjecture.* Let F be a nonarchimedean local field and n an integer $\geqq 1$. There is a (in fact more than one) natural bijection between isomorphism classes of Φ-semisimple representations of W_F' of degree n, and of irreducible admissible representations of $\mathrm{GL}(n, F)$.

For $n = 1$ this is local class field theory. For $n = 2$, it is discussed at length in [**D2**, (3.2)]. In this conjecture, for any n, the *irreducible* representations of W_F' (which are just irreducible representations of W_F) should correspond to the *cuspidal* representations of $\mathrm{GL}(n, F)$. I understand that Bernštein and Želevinsky have shown that the way in which arbitrary admissible representations of $\mathrm{GL}(n, F)$ are built out of cuspidal ones follows the same pattern as the way in which arbitrary Φ-semisimple representations of W_F' are built up out of irreducible ones. Thus the main problem is now the correspondence between irreducibles and cuspidals.

A more general conjecture, involving an arbitrary reductive group G rather than just $\mathrm{GL}(n)$, relates admissible representations of $G(F)$ to homomorphisms of W_F' into the "Langlands dual" of G (see Borel's article in these PROCEEDINGS). This more general conjecture is the nonarchimedean local case of "Langlands' philosophy".

(4.2) *λ-adic representations.* Now suppose l is a prime different from the residue characteristic p of F and let $t_l\colon I_F \to \boldsymbol{Q}_l$ be a nonzero homomorphism. (Such a t_l exists and is unique up to a constant multiple, because the wild ramification group P is a pro-p-group, and the quotient I/P is isomorphic to the product $\prod_{l' \neq p} \boldsymbol{Z}_{l'}$.) We have

$$t_l(w\sigma w^{-1}) = \|w\| \, t_l(\sigma), \quad \text{for } \sigma \in I, \ w \in W,$$

because conjugation by w induces raising to the $\|w\|$ power in I/P. Let Φ be an inverse Frobenius element (4.1.8). Suppose E_λ is a finite extension of $\boldsymbol{Q}_l$. A λ-adic representation of W_F is a finite-dimensional vector space V_λ over E_λ and a homomorphism of topological groups $\rho_\lambda\colon W_F \to \mathrm{GL}_{E_\lambda}(V_\lambda)$ where $\mathrm{GL}_{E_\lambda}(V_\lambda)$ has the λ-adic topology (i.e., the topology given by the valuation).

(4.2.1) THEOREM (DELIGNE [**D3**, §8]). *The relationship $V_\lambda = V$ and*

$$\rho_\lambda(\Phi^n\sigma) = \rho(\Phi^n\sigma)\exp(t_l(\sigma)\, N), \qquad \sigma \in I, \ n \in \boldsymbol{Z},$$

sets up a bijection between the set of λ-adic representations $(\rho_\lambda, V_\lambda)$ of W_F and the set of representations (ρ, N, V) of W_F' over E_λ. The corresponding bijection between isomorphism classes of each is independent of the choice of t_l and Φ.

To show that every ρ_λ is of this form one uses

(4.2.2) COROLLARY (GROTHENDIECK). *Let $(\rho_\lambda, V_\lambda)$ be a λ-adic representation of W_F. There exists a nilpotent endomorphism N of V_λ such that $\rho_\lambda(\sigma) = \exp(t_l(\sigma)N)$ for σ in an open subgroup of I.*

A proof of the corollary can be found in the appendix of [ST]. Here is a sketch. Since I is compact, $\rho_\lambda(I)$ stabilizes a "lattice" L in V_λ. Replacing F by a finite extension we can assume that $\rho_\lambda(I)$ fixes L (mod l^2). Then $\rho_\lambda(I)$ is a pro-l-group, so is a homomorphic image of $t_l(I)$, since Ker t_l is prime to l. Choose $c \in \boldsymbol{Q}_l$ such that $ct_l(I) = \boldsymbol{Z}_l$. Then there is an $\alpha \in \mathrm{GL}(V_\lambda)$ fixing L (mod l^2) such that

$$\rho_\lambda(\sigma) = \alpha^{ct_l(\sigma)} = \exp(t_l(\sigma)N)$$

for all $\sigma \in I$, where $N = c \log \alpha$. Conjugating by $\rho_\lambda(\Phi)$ we find $\rho_\lambda(\Phi)N\rho_\lambda(\Phi)^{-1} = q^{-1}N$. Thus the set of eigenvalues of N is stable under multiplication by q^{-1}. Since q is not a root of unity in characteristic 0, it follows that the only eigenvalue of N is zero, i.e., N is nilpotent.

(4.2.3) COROLLARY. *If V_λ is a semisimple λ-adic representation of W_F then some open subgroup of I acts trivially on V_λ, so V_λ can be viewed as an "ordinary" representation of W_F.*

For any V_λ the kernel of N is stable under W_F because $\rho_\lambda(w)N_{\rho_\lambda}(w)^{-1} = \|w\|N$. So if V_λ is irreducible, then $N=0$, and the statement follows from (4.2.2). A semisimple V_λ is a direct sum of irreducible subrepresentations.

(4.2.4) In view of (4.2.3), $\varepsilon(V_\lambda)$ and $a(V_\lambda)$ have meaning if V_λ is semisimple. For arbitrary V_λ, if $(\rho_\lambda, V_\lambda)$ and (ρ, N, V) correspond as in (4.2.1), we define the L- and ε-factors associated to V_λ to be those associated to V. These can be expressed directly in terms of V_λ as follows:

$$Z(V, t) = \det(1 - \Phi \mid V_\lambda^I) = Z(V_\lambda, t),$$

$$a(V) = a(V_\lambda^{\mathrm{ss}}) + \dim(V_\lambda^{\mathrm{ss}})^I - \dim V_\lambda = a(V_\lambda),$$

$$\varepsilon(V) = \varepsilon(V_\lambda^{\mathrm{ss}}) \frac{\det(-\Phi \mid (V_\lambda^{\mathrm{ss}})^I)}{\det(-\Phi \mid V_\lambda^I)} = \varepsilon(V_\lambda),$$

and

$$\varepsilon(V, t) = \varepsilon(V_\lambda)t^{a(V_\lambda)} = \varepsilon(V_\lambda, t),$$

where V_λ^{ss} is the semisimplification of V_λ in the ordinary sense. One can define a "Φ-semisimplification" of V_λ, analogous to that of V (4.1.3). The quantities on the right do not change if we replace V_λ by its Φ-semisimplification, but they are not additive in V_λ, because V_λ^I is not.

(4.2.4) *Motives.* Suppose now E is a finite extension of $\boldsymbol{Q}$. Let M be a motive with complex multiplication by E, defined over our nonarchimedean local field F ([D1], [D6]). Let n be the rank of M. Attached to M will be l-adic representations $H_l(M)$, vector spaces of dimension n over $\boldsymbol{Q}_l$ on which G_F acts continuously, one for each $l \neq \operatorname{char}(F)$. The field E will act on these, and for each l we get a decomposition $H_l(M) = \bigoplus_{\lambda | l} H_\lambda(M)$, where for each place λ of E above l, we put $H_\lambda(M) = E_\lambda \otimes_{E \otimes \boldsymbol{Q}_l} H_l(M)$, a vector space of dimension m over E, where m is the rank of M over E, given by $n = m[E: \boldsymbol{Q}]$.

For each $l \neq p$ and each λ above l let $H_\lambda'(M)$ be the representation of W_F' over E_λ corresponding to $H_\lambda(M)$ by (4.2.1). If our motive M lives up to expectations, the system of λ-adic representations $H_\lambda(M)$ will be *compatible over* E in the sense that the system $H_\lambda'(M)$ is compatible over E in the following naive sense: for any two finite places λ, μ of E not over p and every commutative diagram

$$\begin{tikzcd} & E_\lambda \arrow[dr] & \\ E \arrow[ur] \arrow[dr] & & C \\ & E_\mu \arrow[ur] & \end{tikzcd}$$

the m-dimensional representations of W_F' over C, $H_\lambda'(M) \otimes_{E_\lambda} C$ and $H_\mu'(M) \otimes_{E_\mu} C$, are isomorphic (or at the very least, have isomorphic Φ-semisimplifications). If so, then the isomorphism class of (the Φ-semisimplifications of) these representations depends only on the embedding $E \subset C$ in the diagram above.

We denote this isomorphism class by $V(M_\sigma)$, where σ denotes the embedding of E in C and $M_\sigma = M \otimes_{E,\sigma} C$ is the motive of rank m with coefficients in C deduced from the original M, the action of E on it, and the embedding σ of E in C, cf. [D6, 2.1]. Associated to $V(M_\sigma)$ as explained in (4.1.6) are the local quantities a, L, and ε which we shall denote by $L(M_\sigma, s)$, etc.

(4.3) *Reduction.* Let r be an integer $\geqq 0$ and X a projective nonsingular variety over F. In this paragraph we shall restrict our attention to the special motive $M = H^r(X)$ given by the r-dimensional cohomology of X, and we shall ignore any complex multiplication. For the moment F can be any field. Put $\bar{X} = X \times_F \bar{F}$, the scheme obtained by extending scalars from F to $\bar{F}$. For each prime $l \neq \operatorname{char}(F)$ the l-adic étale cohomology group $H^r(\bar{X}_{\text{et}}, \boldsymbol{Q}_l)$ is defined, and gives an l-adic representation of $G_F = \operatorname{Gal}(\bar{F}/F)$ (by functoriality, G_F acting on $\bar{X}$ through $\bar{F}$). In the notation of the previous paragraph we have now $E = \boldsymbol{Q}$, $\lambda = l$, $H_l(M) = H^r(\bar{X}_{\text{et}}, \boldsymbol{Q}_l)$. I do not know to what extent the compatibility of the $H_l(M)$'s is known (assuming now again that F is local nonarchimedean), but the compatibility at least of their Φ-semisimplifications is known in one very important case—that of

(4.3.1) *Good reduction.* Let $\mathcal{O}$ be the ring of integers in F, and $k = \mathcal{O}/\pi\mathcal{O}$ the residue field. The scheme X is said to have *good reduction* if there exists a scheme $\mathfrak{X}$ projective and smooth over $\mathcal{O}$ such that $X = \mathfrak{X} \times_{\mathcal{O}} F$. Choosing such an $\mathfrak{X}$, one calls $\mathfrak{X} \times_{\mathcal{O}} k$ the *reduction* of X. Let us denote this reduction by X_0. Putting $\bar{X}_0 = X_0 \times_k \bar{k}$, where $\bar{k}$ is the residue field of $\bar{F}$, the base-change theorem gives a canonical isomorphism

$$(*) \qquad H_l(M) = H^r(\bar{X}, \boldsymbol{Q}_l) \approx H^r(\bar{X}_0, \boldsymbol{Q}_l)$$

compatible with the action of the Galois groups. Hence $H_l(M)$ is *unramified*, i.e., fixed by I, and the structure of $H_l(M)$ as representation of W_F is given by the action of Φ. Let $\varphi: X_0 \to X_0$ be the Frobenius *morphism*, and $\sigma: \bar{k} \to \bar{k}$ the Frobenius automorphism. The composition $\varphi \times \sigma$ acts on $\bar{X}_0 = X_0 \times \bar{k}$ by fixing points and by mapping $f \mapsto f^q$ in the structure sheaf. This map induces (a morphism canonically isomorphic to) the identity on the site $(\bar{X}_0)_{\text{et}}$, so the action of the Frobenius morphism φ on $H^r(\bar{X}_0, \boldsymbol{Q}_l)$ is the same as that of σ^{-1}, which is the one corresponding to our Φ under the isomorphism (∗). That is why Deligne calls Φ the *geometric Frobenius.*

Deligne [D4] has proved Weil's conjecture, that the characteristic polynomial of φ acting on $H^r(\bar{X}_0, \boldsymbol{Q}_l)$ has coefficients in $\boldsymbol{Z}$, is independent of l, and that its complex roots have absolute value $q^{r/2}$. From the independence of l it follows in this case of good reduction that the Φ-semisimplifications of the $H_l(M)$'s form a compatible system; and the $H_l(M)$'s are known to be Φ-semisimple for $r = 1$.

It is natural to say that a motive M over F has *good reduction*, or is *unramified* if and only if $H_l(M) = H_l(M)^I$, i.e., if $V(M) = V(M)^I_N$. In case $M = H_1(A)$, A an abelian variety, this is equivalent to A having good reduction (criterion of Néron-Ogg-Shafaryevitch in [ST]).

Similarly we say M has *potential good reduction* $\Leftrightarrow N = 0$, and M has *semistable* reduction if $V(M) = V(M)^I$. Clearly this latter can always be achieved by a finite extension of the ground field.

(4.4) *F archimedean.* Let now M, E, n, m be as in (4.2.4), but take F to be archimedean, instead of nonarchimedean. Let $z: F \to \boldsymbol{C}$ be the embedding of F in $\boldsymbol{C}$ if F is real, or one of the two isomorphisms of F on $\boldsymbol{C}$ if F is complex. Such a z gives us a motive M_z over $\boldsymbol{C}$ and M_z has a "Betti realization" $H_B(M_z)$ which is an n-dimensional vector space over $\boldsymbol{Q}$ whose complexification $H_B(M_z) \otimes \boldsymbol{C} = \bigoplus H^{pq}(M_z)$ is doubly graded in such a way that the map $1 \otimes c$ (c = complex conjugation) takes H^{pq} to H^{qp}. (For example, if $M = H^r(X)$ as in (4.3), then $H_B(M_z) = H^r(X_z^{\text{an}}, \boldsymbol{Q})$, where X_z^{an} is the complex analytic variety underlying the scheme $X \times_{F,z} \boldsymbol{C}$, and the complexification of this space, $H^r(X_z^{\text{an}}, \boldsymbol{C})$, is doubly graded by Hodge theory.)

Let $\bar{z} = c \circ z: F \to \boldsymbol{C}$ be the map conjugate to z. By transport of structure, there is an isomorphism $\tau: H_B(M_z) \to H_B(M_{\bar{z}})$ such that $\tau \otimes c$ preserves the bigrading on the complexifications; hence $\tau \otimes 1$ carries $H^{pq}(M_z)$ onto $H^{qp}(M_{\bar{z}})$. The field E of complex multiplications acts on $H_B(M_z)$ preserving the bigradation on the complexification, and τ is an E-homomorphism. Let $\sigma: E \to \boldsymbol{C}$. Putting $V_z(M_\sigma) = H_B(M_z) \otimes_{E,\sigma} \boldsymbol{C}$ we obtain a bigraded complex vector space of dimension m and a linear isomorphism $\tau \otimes 1: V_z(M_\sigma) \to V_{\bar{z}}(M_\sigma)$ taking V_z^{pq} to $V_{\bar{z}}^{qp}$.

There is a natural action of the Weil group W_F on these spaces as follows:

F complex. $z: F \approx \boldsymbol{C}$ an isomorphism, $W_F = F^*$, and W_F acts on V_z^{pq} by scalar multiplication via the character $z^{-p}(\bar{z})^{-q}$. Clearly, $\tau \otimes 1$ is W_F-equivariant, so the two representations $V_z(M_\sigma)$ and $V_{\bar{z}}(M_\sigma)$ are isomorphic. We let $V(M_\sigma)$ denote their isomorphism class.

F real. $z = \bar{z}: F \to \boldsymbol{C}$ is *the* embedding, and $W_F = \boldsymbol{C}^* \cup j\boldsymbol{C}^*$. This time $M_z = M_{\bar{z}}$, so we have only one space, $V_z(M_\sigma) = V_{\bar{z}}(M_\sigma)$, and $\tau \otimes 1$ is an automorphism of it. The action of W_F on it is as follows:

$u \in \boldsymbol{C}^*$ acts as multiplication by $u^{-p}(\bar{u})^{-q}$ on V_z^{pq}.

j acts as $i^{p+q}(\tau \otimes 1)$ on V_z^{pq}.

Again, let $V(M_\sigma)$ denote the equivalence class of this representation.

Notice that the representations obtained from motives via Hodge theory are very special, in that the p and q are integers.

Finally, define $L(M_\sigma, s)$ and $\varepsilon(M_\sigma, s)$ to be the L- and ε-factors associated to the representation $V(M_\sigma)$ as in §3. For a table making these explicit see [**D6**, 5.3].

(4.5) *F global.* Let F be a global field and M a motive with complex multiplication by E, defined over F. For each place v of F, let M_v denote the restriction of M to F_v. Let $\sigma: E \to C$. The product $L(M_\sigma, s) = \prod_v L(M_{v,\sigma}, s)$ converges in a right half-plane. It is conjectured that it is meromorphic in the whole s-plane and satisfies the functional equation

$$L(M_\sigma, s) = \varepsilon(M_\sigma, s)\, L(M_\sigma^*, 1 - s)$$

with $M^* = \mathrm{Hom}(M, Q)$ and $\varepsilon(M_\sigma, s) = \prod_v \varepsilon(M_{v,\sigma}, s)$.

In the function field case this conjecture has been proved by Deligne. Let q be the number of elements in the constant field k of F. Grothendieck proved that for any given λ-adic representation V of G_F which is unramified at all but a finite number of places v, the corresponding L-function $L(V, s) = \prod_v L(V_v, s)$ is a rational function of q^{-s} (even a polynomial if $V^{\bar{G}} = 0$ and $V_{\bar{G}} = 0$, where $\bar{G}$ is the geometric Galois group, i.e., the kernel of the map of G_F to G_k), and satisfies a functional equation of the form $L(V, s) = \varepsilon(V, s)L(V^*, 1 - s)$ with an ε which is a monomial in q^{-s} of degree $\sum_v [k(v){:}k]\; a(V_v)$. Later, Deligne showed that Grothendieck's $\varepsilon(V, s)$ is equal to the product of the local $\varepsilon(V_v, s)$'s if $V = V_{\lambda_0}$ is a member of a family $(V_\lambda)_{\lambda \in \mathscr{L}}$ of λ-adic representations of G_F for some infinite set of places $\mathscr{L}$ of a number field E, and the family is compatible in the following weak sense: for each λ, $\mu \in \mathscr{L}$ there is a finite set S of places of F such that for $v \notin S$, the representations V_λ and V_μ are unramified at v and the characteristic polynomals of Φ_v acting on V_λ and V_μ have coefficients in E and are equal. Deligne's method is to prove that Grothendieck's ε is congruent to the product of the local ε's modulo λ for all $\lambda \in \mathscr{L}$ and is therefore equal to that product. By (4.3) any λ-adic representation coming from l-adic cohomology, i.e., from a motive, is a member of a system which is weakly compatible in the above sense.

When $\dim(V) = 2$, then by Jacquet-Langlands (resp. Weil), Springer Lecture Notes 114 (resp. 189), these results show that $L(V, s)$ comes from an automorphic representation of (resp. modular form on) $\mathrm{GL}_2(A_F)$. On the other hand, Drinfeld has recently shown that automorphic representations of GL_2 give rise to systems of l-adic representations occurring as constituents in tensor products of those coming from 1-dimensional l-adic cohomology, hence from motives. Thus for GL_2 over function fields, the equivalence between motives, compatible systems of l-adic representations, and automorphic representations is pretty well established.

In this connection it should be mentioned that Zarhin [**Z**] has proved the isogeny theorem over function fields: if two abelian varieties A and B over a global function field F give isomorphic l-adic representations, then they are isogenous; more precisely,

$$Q_l \otimes \mathrm{Hom}_F(A, B) = \mathrm{Hom}_{G_F}(V_l(A), V_l(B)).$$

Over number fields our knowledge is not nearly so advanced. For Artin motives of rank 2, Langlands has made a beginning with the theory of base change (see the remarks (3.5.5)). For elliptic curves M over Q, it is not even known whether

$L(M, s)$ has a meromorphic continuation throughout the s-plane, or whether the isogeny theorem is true. For a more detailed account of our ignorance, as well as of a few things which are known, see [S4].

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