

On Anomalies in Classical Dynamical Systems

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Received February 27, 2001; Revised April 25, 2001; Accepted May 5, 2001

Abstract

The definition of “classical anomaly” is introduced. It describes the situation in which a purely classical dynamical system which presents both a lagrangian and a hamiltonian formulation admits symmetries of the action for which the Noether conserved charges, endorsed with the Poisson bracket structure, close an algebra which is just the centrally extended version of the original symmetry algebra. The consistency conditions for this to occur are derived. Explicit examples are given based on simple two-dimensional models. Applications of the above scheme and lines of further investigations are suggested.

1 Introduction

Here I define as “*classically anomalous*” any classical dynamical system, described both in the lagrangian and in the hamiltonian formalism, whose symmetries of the action produce conserved Noether charges which, under the Poisson-bracket algebra, satisfy a centrally extended version of the original symmetry algebra.

In many and perhaps even most of the texts discussing quantum mechanics and quantum field theories it is commonly stated that anomalies are a purely quantum-mechanical effect. This statement reflects a widespread (but erroneous) belief in the scientific community, mostly shared by researchers who do not have a direct working experience with anomalies. While the specialists in the field are aware that some specific features, which can be reasonably named “anomalies”, can be present even in purely classical dynamical systems, it seems, however, that this correct interpretation passes largely unnoticed. One of the reasons is due to the fact that most of the results concerning anomalous effects in classical dynamical systems are scattered in the literature. Moreover, they appear in rather technical contexts and it seems that very little effort (if any) has been made in order to place them in a more general framework.

The aim of this paper is to furnish some clarification, emphasizing one single aspect of the appearance of “classical anomalies”. In the interpretation proposed here “classical anomalies”, as previously defined, lay and can be detected in the interplay between the lagrangian-versus-hamiltonian description of a dynamical system presenting a symmetry of the action. Very basic examples are explicitly constructed and analyzed. No new result will be discussed here, rather a re-interpretation of known results and techniques will be

given. Due to the mainly pedagogical character of the present note, a minimum level of mathematical sophistication has been purposely kept throughout the following discussion.

The present work is organized as follows. In the next section a drastically sketchy and far from complete resume of the history and importance of anomalies in quantum field theory will be made. The result obtained by Gervais and Neveu [1] in analyzing the Liouville theory will be mentioned. To my knowledge, they were the first authors who noticed an anomalous effect in a classical system. These authors indeed observed, according to the present definition, a “classical anomaly”.

In the following section a general argument is given suggesting the mechanism which gives rise to what has been here named of *classical anomalies*. The remaining sections are devoted to working out some specific examples with concrete models, by analyzing in each specific case whether the symmetries of the actions are preserved or not by the Poisson bracket structure. For simplicity reasons all the examples are worked out for two-dimensional field theories. In order of presentation, the following systems will be analyzed:

- i)* the free chiral fermion,
- ii)* the massless free boson,
- iii)* the Floreanini–Jackiw [2] chiral boson (FJ) model (introduced to complete the discussion of free and chiral models),
- iv)* finally, the Liouville theory will be revisited in light of the present interpretation.

In the conclusions some further remarks and comments will be made and possible future lines of development will be suggested.

2 A bit of history

In the last thirty years the investigation of anomalies in quantum field theories has been one of the major areas of research attracting the attention of theoretical physicists. The reason is clear: an objective relevance for physical applications, coupled with a beautiful mathematical structure.

Indeed, the first discovered anomaly (by Adler and Bell–Jackiw [3, 4], named ABJ after the authors) was the $U(1)$ chiral anomaly of gauge theories putting a consistency constraint on the existence of a quantized gauge theory. As an important application, the gauge group and the representation multiplets of the standard model for the electroweak interactions are carefully selected in order to guarantee an anomaly-free theory. Much in the same way, the critical dimensionality of a quantized string theory can be determined by requiring the cancellation of an anomaly [5] which can be associated to the Weyl invariance [6].

On the other hand, anomalies in physics are not always unwanted features to be eradicated. E.g. the trace anomalies associated to dilatation invariance lead to the Callan–Symanzik equations [7]. For a reference work concerning anomalies in the context of quantum current algebras and their related physical applications one can consult [8].

Since their original discovery, anomalies have been regarded as a feature of quantization. Some folklore was put on this aspect. In the light of the Feynman path-integral approach,

Fujikawa [9] developed a celebrated method which relies upon the fact that the functional measure is not always invariant under a symmetry of the classical action.

On the mathematical side, anomalies have been shown to satisfy consistency conditions induced by the (anomalous) Ward identities satisfied by the corresponding quantum field theories [10]. The (covariant) anomalies computed with the Fujikawa method do not, however, satisfy such conditions; nevertheless the relation between the two anomalies (consistent versus covariant) was made explicit in [11]. The reason for this discrepancy is that in the Fujikawa approach one regularizes the jacobian arising from a field transformation instead of the full partition function. A slightly modified version of the Fujikawa technique can be introduced, allowing the regularization of the full partition function. It turns out to be equivalent to the heat kernel technique.

The [10] consistency conditions for the anomalies allow one to determine their expressions via purely algebraic methods which make use of the so-called “transgression formula” [12]. An elegant reformulation of such consistency conditions is given in terms of the BRST-cohomology [13]. To summarize, the *form* of the possible anomalies is determined by the possible existence of non-trivial cocycles for a BRST-cohomology associated to the symmetry under consideration. On the other hand, the *coefficients* of such anomalies can be related to the index theorems for elliptic operators via heat kernel computations [14] (for this purpose a one-loop Euclidean version of the quantum field theory under investigation, regularized through zeta-functions [15], is required). A detailed account of the latter construction is given in [16].

Anomalous symmetries of quantized theories have received, therefore, a nice mathematical interpretation. As recalled, they have been regarded as genuine new features of quantization, not present in the underlining classical theories. This point of view is, by the way, still commonly shared by most of the researchers in the area and popularized in standard textbooks.

As far as I know, Gervais and Neveu were the first authors (in [1], pages 131/2) who observed in a classical theory (in a purely quantum context such features had already been observed in [17]) a new phenomenon that, with some good reasons, deserves to be named “classical anomaly” and for which the definition here introduced is applicable. The authors of [1] analyzed the Liouville theory appearing in the partition function of non-critical strings according to [6]. They showed that, even for the *classical* Liouville theory, the generators of (one chiral sector of) the conformal transformations satisfy, under classical Poisson brackets, the Virasoro algebra, i.e., the central extension of the Witt algebra. The remark appearing in [1] was not later developed (e.g, the Noether charges were not explicitly mentioned), since the main focus of their authors was on the quantum version of the Liouville theory (see also their related works [18]). Needless to say, most of the papers written by theoretical physicists on the Liouville field theory deal with the quantum version of the model, as a simple key-word inspection of the electronic bulletin boards reveals.

Subsequent works such as [19] and [20] on classical Liouville and Toda-field systems, were mostly concerned with the integrability properties of such models, like the presence of classical Sklyanin r -matrices in their Drinfeld–Sokolov exchange-algebras. No connection of such classical Poisson-brackets structures with the symmetries of the action, even if implicit, was explicitly stated.

3 General considerations on classical anomalies

The class of systems under consideration here consists of the classical dynamical systems which admit both a lagrangian and a hamiltonian description. It will be further assumed that the action \mathcal{S} admits an invariance under a group of symmetries which can be continuous (Lie), infinite-dimensional and/or super. The conserved Noether charges are associated to each generator of the symmetries of the action. When the hamiltonian dynamics is considered, the phase space of the theory possesses an algebraic structure given by the Poisson brackets. The existence of such a structure makes it possible to compute the Poisson bracket between any two given Noether charges. In the standard situation, the Poisson brackets among Noether charges realize a closed algebraic structure which is isomorphic to the original algebra of the symmetries of the action. It turns out, however, as it will be illustrated in the examples which follow, that this is not always the case. Indeed it can happen that the algebra of Noether charges under Poisson bracket structure close a centrally extended version of the original symmetry algebra. Mimicking the quantum case, the following definition can be proposed for a classical dynamical system. The system is said to possess an anomalously realized symmetry, or in short a “classical anomaly”, if the following condition is satisfied: the symmetry transformations of the action admit Noether generators whose Poisson brackets algebra is a centrally extended version of the algebra of symmetry transformations.

Therefore a classical anomaly is a very specific case of “non-equivariant map” (for a discussion in a finite-dimensional setting see [21]). Not all non-equivariant maps discussed in the literature are classical anomalies. For instance the one-dimensional free-particle conserved quantities p (the momentum) and $pt - mx$ generate a non-equivariant map (the Poisson bracket between p and $pt - mx$ is proportional to the mass m). However, despite being conserved, they do not generate a symmetry of the action and for that reason they are not Noether charges.

On the other hand, infinite-dimensional non-equivariant moment maps were constructed in [22]. In those papers the only explicit application concerned the dynamical systems of KdV type (classical integrable hierarchies). Such systems, in contrast with the examples discussed here, admits a hamiltonian description, but not a lagrangian formulation. Even if conserved quantities can be constructed, they can not be interpreted as Noether charges.

The possibility for a classical anomaly to occur is based on very simple and nice mathematical consistency conditions, implemented by the Jacobi-identity property of the given symmetry algebra. Let us illustrate this point by considering some generic (but not the most general) scheme. Let us suppose that the (bosonic) generators δ_a 's of a symmetry invariance of the action satisfy a linear algebra whose structure constants satisfy the Jacobi identity, i.e.

$$[\delta_a, \delta_b] = f_{ab}{}^c \delta_c,$$

while

$$[\delta_a, [\delta_b, \delta_c]] + [\delta_b, [\delta_c, \delta_a]] + [\delta_c, [\delta_a, \delta_b]] = 0.$$

The associated Noether charges Q_a 's are further assumed to be the generators of the algebra, i.e., applied on a given field ϕ they produce

$$\delta_a \phi = \{Q_a, \phi\}, \tag{3.1}$$

where the brackets obviously denote the Poisson-brackets.

The condition

$$[\delta_a, \delta_b]\phi = f_{ab}{}^c \delta_c \phi, \quad (3.2)$$

puts restriction on the possible Poisson brackets algebra satisfied by the Noether charges. It is certainly true that

$$\{Q_a, Q_b\} = f_{ab}{}^c Q_c,$$

(which corresponds to the standard case) is consistent with both (3.1) and (3.2). However, in a generic case, it is not at all a necessary condition since more general solutions can be found. Indeed, the presence of a central extension, expressed through the relation

$$\{Q_a, Q_b\} = f_{ab}{}^c Q_c + k * \Delta_{ab},$$

(where k is a central charge and the function Δ_{ab} is antisymmetric in the exchange of a and b), is allowed.

Indeed, since the relation

$$\{Q_a, \{Q_b, \phi\}\} - \{Q_b, \{Q_a, \phi\}\} = \{\{Q_a, Q_b\}, \phi\} \quad (3.3)$$

holds due to the Jacobi property of the Poisson bracket structure (which is assumed to be satisfied), no contradiction can be found with (3.2); the right hand side of (3.3) in fact is given by

$$\{f_{ab}{}^c Q_c + k * \Delta_{ab}, \phi\} = \{f_{ab}{}^c Q_c, \phi\} = f_{ab}{}^c \delta_c \phi,$$

due to the fact that k is a central term and has vanishing Poisson brackets with any field.

This observation on one hand puts restrictions on the possible symmetries for which a classical anomaly can be detected; the symmetries in question, on a purely algebraic ground, must admit a central extension. This is not the case, e.g., for the Lie groups of symmetry based on finite simple Lie algebras. On the other hand one is warned that, whenever a symmetry *does* admit an algebraically consistent central extension, it should be carefully checked, for any specific dynamical model which concretely realizes it, whether it is satisfied exactly or anomalously. This remark already holds at the classical level, not just for purely quantum theories.

Some further points deserve to be mentioned. The first one concerns the fact that the quantization procedure (which, for the cases we are here considering, can be understood as an explicit realization of an abstract Poisson brackets algebra as an algebra of commutators between operators acting on a given Hilbert space) can induce anomalous terms for theories which, in their classical version, are not anomalous in the sense previously specified. It therefore turns out that the occurrence of classical anomalies is a phenomenon which is “more difficult to observe” than the corresponding appearance of quantum anomalies since it occurs more seldom.

A second point concerns the fact that the algebra of Poisson brackets, as an abstract algebra, is assumed to satisfy the Leibniz property. This is no longer the case for its concrete realization given by the algebra of commutators. The Noether charges are in

general non-linearly constructed with the original fields ϕ_i (which collectively denote the basic fields and their conjugate momenta) of a given theory. For such a reason it is only true in the classical case that, whenever an anomalous central charge in an infinitesimal linear algebra of symmetries is detected, it can be normalized at will by a simultaneous rescaling of all the fields ϕ_i involved ($\phi_i \mapsto \alpha \cdot \phi_i$) and of the Poisson brackets normalization ($\{.,.\} \mapsto \frac{1}{\alpha}\{.,.\}$), for an arbitrary real constant α . In the classical case any central charge different from zero can therefore be consistently set equal to 1. However in the quantum case a specific value of the central charge is fixed by the type of representation of the symmetry algebra associated with the given model and is a genuine physical parameter (the role of the Virasoro central charge in labeling the conformal minimal models is an example). The above argument is not, however, (at least directly) applicable to non-linear symmetries, such as those leading to the classical counterparts of the Fateev–Zamolodchikov W -algebras. Classical non-linear symmetries fall outside the scope of the present paper and deserve to be analyzed separately.

It should be noticed that the presence of a classical anomaly in the construction of [23] is the underlining reason which allows overcoming a no-go theorem and realizing a partial breaking of an $N = 4$ extended supersymmetry.

It is worth mentioning that in a different context, the appearance of centrally extended algebras has been studied in [24] (and references therein). This analysis however, developed for lagrangian dynamical systems, is not directly related with the present results.

Furthermore, let me remark that the presence of a centrally extended algebra of classical symmetries is not always a sign of the presence of an anomaly (at least not in the sense specified here). In [25] it was shown that a classical two-dimensional complex bosonic field, coupled to an external constant electromagnetic field, admits a symmetry corresponding to the central extension of the two-dimensional Poincaré algebra. This model is not anomalous, within the definition here proposed, because, due to the presence of the constant external field, the symmetry algebra of the classical action itself is centrally extended and not given by the ordinary $2D$ Poincaré algebra.

The term “classical anomaly” has been employed in [26] as well, in a different context however and to denote a different phenomenon than the one here discussed.

Finally, in the present work no effort is made to derive the hamiltonian dynamics associated to a given lagrangian. It is simply assumed to exist, based on the results furnished in the literature. This is especially true for the chiral boson model in Section 6, whose hamiltonian analysis is somewhat delicate, but has nevertheless been performed in [27], see also [28].

4 The free chiral fermion

The first example that will be discussed here concerns the theory of the free chiral fermion. It is described by the Grassmann field $\psi(x, t)$, where x is a one-dimensional space coordinate and t the time. The dynamics is specified by the action

$$\mathcal{S} = \int dxdt \cdot \psi \partial_- \psi,$$

where

$$z_{\pm} = x \pm t \quad \text{and} \quad \partial_{\pm} = \frac{1}{2}(\partial_x \pm \partial_t).$$

For our purposes we will assume the space-coordinate x to be compactified on a circle S^1 of radius R and ψ to satisfy periodic boundary conditions.

The equation of motion is given by

$$\partial_- \psi = 0.$$

The action \mathcal{S} , besides being off-shell invariant under the transformation specified by the infinitesimal function $\epsilon(z_+)$

$$\delta(\psi) = \epsilon(z_+) \partial_x \psi + \frac{1}{4} \epsilon'(z_+) \psi,$$

(the prime in the r.h.s. denotes the derivative) admits a global fermionic symmetry given by

$$\delta_{\kappa}(\psi) = \kappa,$$

where κ is a global fermionic parameter.

It is convenient to expand $\epsilon(z_+)$ as a Laurent series according to

$$\epsilon(z_+) = - \sum_n \epsilon_n (z_+)^{n+1}.$$

The two above symmetries can be expressed through

$$\delta(\psi) = \sum_n \epsilon_n \cdot l_n \psi, \quad \delta_{\kappa}(\psi) = \kappa \cdot g \psi,$$

where the operators l_n and g are respectively given by

$$l_n = -(z_+)^{n+1} \partial_x - \frac{1}{2} (n+1) (z_+)^n, \quad g = \oint dx \cdot \frac{\delta}{\delta \psi(x)}.$$

While the commutators among the l_n 's operators realize the Witt algebra

$$[l_n, l_m] = (n-m) l_{n+m},$$

the anticommutator of g with itself satisfies

$$[g, g]_+ = 0,$$

so that g is nilpotent.

The conserved Noether charges associated to the above symmetries are given by

$$L_n = - \oint dx (z_+)^{n+1} \psi \partial_x \psi, \quad G = 2 \oint dx \psi(x, t).$$

In the hamiltonian description the equation of motion is expressed through

$$\dot{\psi} = \{H, \psi\}_t,$$

where H is the hamiltonian

$$H = - \oint dx \cdot (\psi \partial_x \psi),$$

while the equal-time Poisson brackets $\{.,.\}_t$ are introduced through

$$\{\psi(x), \psi(y)\}_t = \frac{1}{2} \delta(x - y).$$

We are now in the position to compute the Poisson brackets among the Noether conserved charges, which are the generators of the symmetries, according to

$$l_n \psi = \{L_n, \psi\}_t, \quad g\psi = \{G, \psi\}_t.$$

While the Noether generators L_n associated with the Witt generators reproduce, under the Poisson brackets structure, the Witt algebra, i.e.

$$\{L_n, L_m\} = (n - m)L_{n+m},$$

it is no longer true that the fermionic conserved charge G satisfies the same nilpotency condition as g . Indeed we have that the Poisson bracket of G with itself produces a central element, given by

$$\{G, G\} = 4\pi R.$$

It follows that the generator of the fermionic symmetry is now no longer nilpotent. Even in this trivial free model the presence of a symmetry which presents a classical anomaly can be detected.

In the quantum case, due to the double contractions in the Wick expansion, the quantum analogs of the L_n generators satisfy the centrally extended version of the Witt algebra, i.e. the Virasoro algebra, with central charge $c = \frac{1}{2}$. This is in accordance with the statement that the “quantization” is a more effective way to produce anomalies than the plain introduction of a classical Poisson bracket structure. Still, as the fermionic symmetry shows, in many cases the introduction of a classical Poisson bracket structure is sufficient to induce anomalies at the level of the Noether charges.

5 The free massless boson in $2D$

The next example that we would like to discuss concerns the 2-dimensional free massless boson model, described by the following action

$$\mathcal{S} = -2 \int dx dt \cdot \partial_- \phi \partial_+ \phi.$$

The field $\phi(x, t)$ satisfies the free equation of motion

$$\square \phi \equiv 4\partial_- \partial_+ \phi = 0.$$

This system admits an (anomalous-free) two-dimensional conformal invariance which corresponds to the direct sum of two copies of the Witt algebra ($Witt \oplus Witt$). Actually the symmetry algebra of the system is richer. Indeed, the following transformations are symmetries of the action

$$\delta_+ \phi = \epsilon(z_+) \partial_+ \phi + \mu(z_+), \quad \delta_- \phi = \bar{\epsilon}(z_-) \partial_- \phi + \bar{\mu}(z_-),$$

for arbitrary infinitesimal functions $\epsilon(z_+)$, $\bar{\epsilon}(z_-)$, $\mu(z_+)$, $\bar{\mu}(z_-)$. Such a set of transformations is anomalous in the sense discussed here. This point can be easily understood when we consider a specific case of $\mu(z_+)$ ($\bar{\mu}(z_-)$) given by

$$\mu(z_+) = \lambda_+ \partial_+ \epsilon(z_+), \quad \bar{\mu}(z_-) = \lambda_- \partial_- \bar{\epsilon}(z_-).$$

for arbitrary fixed values of the parameters λ_{\pm} .

In full analogy with the previous case, after Laurent series expansion for $\epsilon(z_+)$, $\bar{\epsilon}(z_-)$,

$$\epsilon(z_+) = - \sum_n \epsilon_n(z_+)^{n+1}, \quad \bar{\epsilon}(z_-) = - \sum_n \bar{\epsilon}_n(z_-)^{n+1},$$

we obtain two mutually commuting set of λ_+ and λ_- -dependent symmetry generators, each set generating a copy of the Witt algebra. They are given by

$$l_n(\lambda_+) = -(z_+)^{n+1} \partial_+ - \lambda_+(n+1)(z_+)^n \cdot \int \frac{\delta}{\delta \phi(x, t)},$$

$$\bar{l}_n(\lambda_-) = -(z_-)^{n+1} \partial_- - \lambda_-(n+1)(z_-)^n \cdot \int \frac{\delta}{\delta \phi(x, t)}.$$

For any given couple of values λ_{\pm} , we obtain the closure of the $Witt \oplus Witt$ algebra

$$[l_n(\lambda_+), l_m(\lambda_+)] = (n-m)l_{n+m}(\lambda_+),$$

$$[\bar{l}_n(\lambda_-), \bar{l}_m(\lambda_-)] = (n-m)\bar{l}_{n+m}(\lambda_-),$$

$$[l_n(\lambda_+), \bar{l}_m(\lambda_-)] = 0.$$

The free massless boson model admits a hamiltonian formulation, with the hamiltonian given by

$$H = \frac{1}{2} \int dx (\pi^2 + (\partial_x \phi)^2).$$

The equations of motions, expressed through

$$\frac{d}{dt} f = \{H, f\} + \frac{\partial}{\partial t} f$$

imply

$$\dot{\phi} = \pi, \quad \dot{\pi} = (\partial_x)^2 \phi.$$

The equal-time Poisson brackets are obviously given by

$$\{\pi(x), \phi(y)\} = \delta(x-y)$$

and vanishing otherwise.

As a straightforward computation shows, the conserved Noether charges $L_n(\lambda_+)$, $\bar{L}_n(\lambda_-)$, associated to the symmetry generators $l_n(\lambda_+)$, $\bar{l}_n(\lambda_-)$ respectively, are recovered from the Laurent expansions

$$L_n(\lambda_+) = \int dx(z_+)^{n+1} \cdot T, \quad \bar{L}_n(\lambda_-) = \int dx(z_-)^{n+1} \cdot \bar{T},$$

where T, \bar{T} are given by

$$T = \frac{1}{4}(\pi^2 + (\partial_x \phi)^2 + 2\pi \partial_x \phi - 4\lambda_+ \partial_x^2 \phi - 4\lambda_+ \partial_x \pi),$$

$$\bar{T} = -\frac{1}{4}(\pi^2 + (\partial_x \phi)^2 - 2\pi \partial_x \phi - 4\lambda_- \partial_x^2 \phi + 4\lambda_- \partial_x \pi).$$

The conservation law for L_n, \bar{L}_n is a consequence of the (anti-)chiral equations satisfied by $T (\bar{T})$ respectively, i.e.

$$\partial_- T = 0, \quad \partial_+ \bar{T} = 0.$$

L_n, \bar{L}_n are the generators of the l_n, \bar{l}_n transformations since

$$l_n \phi = \{L_n, \phi\}, \quad \bar{l}_n \phi = \{\bar{L}_n, \phi\}. \tag{5.1}$$

L_n, \bar{L}_n generate the direct sum of two copies of the Virasoro algebra, $Vir \oplus Vir$, as can be directly read from the equal-time Poisson brackets between $T(x), \bar{T}(x)$, namely

$$\{T(x), T(y)\} = -2\lambda_+^2 \partial_y^3 \delta(x - y) + 2T(y) \partial_y \delta(x - y) + \partial_y T(y) \cdot \delta(x - y),$$

$$\{T(x), \bar{T}(y)\} = 0,$$

$$\{\bar{T}(x), \bar{T}(y)\} = 2\lambda_-^2 \partial_y^3 \delta(x - y) + 2\bar{T}(y) \partial_y \delta(x - y) + \partial_y \bar{T}(y) \cdot \delta(x - y).$$

For given values of $\lambda_{\pm} \neq 0$, central terms are produced which are proportional to λ_{\pm}^2 . The corresponding transformations can therefore be regarded as anomalous.

The two-dimensional conformal symmetry itself however is not anomalous in this free case, since for the choice $\lambda_{\pm} = 0$, the symmetry is preserved at the Poisson bracket level.

It should be stressed the fact that the freedom in choosing inhomogeneous transformations acting on ϕ , for $\lambda_{\pm} \neq 0$, can be held as responsible for the preservation (i.e. not anomalous realization) of the two-dimensional conformal invariance even in the quantum case. The choice $\lambda_{\pm} \neq 0$ corresponds to the introduction of the Feigin–Fuchs term in the Coulomb gas formalism¹.

¹Let me recall that in the quantum OPE language, given a chiral propagator $\phi(z)\phi(w) \sim -\log(z - w)$, a stress-energy tensor $T(z)$ satisfying a Virasoro algebra can be introduced through $T(z) = -\frac{1}{4} : \partial\phi\partial\phi : + i\alpha\partial^2\phi$. The linear term in $\phi(z)$ is inserted in order to allow modifying the value of the central charge c of the Virasoro algebra, given by $c = 1 - 24\alpha^2$. This construction can be repeated in the classical case too.

6 The Floreanini–Jackiw chiral boson model

For completeness, let us discuss the last chiral and free model, namely the Floreanini–Jackiw chiral boson model [2] introduced through the lagrangian

$$\mathcal{L} = \partial_t \phi \partial_x \phi - (\partial_x \phi)^2,$$

which leads to the equation of motion

$$\partial_x \partial_- \phi = 0.$$

Despite the fact that it is not manifestly Lorentz-invariant, it can nevertheless be shown to be Poincaré invariant in 2 dimensions. This model defines the dynamics of a chiral boson. The treatment is much in the same lines as the free boson model with a notable exception. Since we are in presence of chiral dynamics the invariance of the model is given by a single (chiral) copy of the Witt algebra and its central extension. A class of λ -dependent infinitesimal symmetries of the above action is given by

$$\delta_\lambda \phi = \epsilon(z_+) \partial_x \phi + \lambda \partial_x \epsilon(z_+). \quad (6.1)$$

The corresponding Noether conserved charges are given by the following expressions

$$L_n(\lambda) = \frac{1}{2} \int dx (z_+)^{n+1} ((\partial_x \phi)^2 + \lambda \partial_x^2 \phi). \quad (6.2)$$

The hamiltonian of the system is

$$H = \frac{1}{2} \int dx (\partial_x \phi)^2,$$

while the Poisson-brackets structure system is non-local

$$\{\phi(x), \phi(y)\} = \partial_y^{-1} \delta(x - y).$$

Despite its non-locality however, since in (6.2) only the derivatives of the field ϕ enter, the algebra satisfied by the $L_n(\lambda)$ Noether charges is a local algebra which, as in the previous example, corresponds to the Virasoro algebra with central extension proportional to λ^2 . For $\lambda \neq 0$ we are in the presence of an anomaly induced by the Poisson bracket structure. The Noether charges L_n are, as in the previous example, the generators of the transformations in (6.1).

7 The Liouville theory revisited

The last model that we are going to discuss is the Liouville theory, revisited in view of the considerations which motivated the present paper.

The action of the Liouville model can be written as

$$\mathcal{S} = - \int dx dt \cdot (2\partial_- \phi \partial_+ \phi + e^{2\phi}).$$

The equation of motion is

$$2\partial_-\partial_+\phi = e^{2\phi}.$$

In the hamiltonian description the hamiltonian is given by

$$H = \oint dx \cdot \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\phi)^2 + e^{2\phi} \right),$$

while the Poisson-bracket structure between π , ϕ is the same as in the free case

$$\{\pi(x), \phi(y)\} = \delta(x - y).$$

We obtain

$$\dot{\phi} = \pi, \quad \dot{\pi} = \partial_x^2\phi - 2e^{2\phi}.$$

The theory is conformally invariant, with transformations given, as in the free-case, by the infinitesimal transformations

$$\delta_+\phi = \epsilon(z_+)\partial_x\phi + \lambda_+(\partial_x\epsilon(z_+))\phi, \quad \delta_-\phi = \bar{\epsilon}(z_-)\partial_x\phi + \lambda_-(\partial_x\bar{\epsilon}(z_-))\phi.$$

However, due to the presence of the potential term, the action is no longer off-shell invariant for arbitrary values of λ_{\pm} . The invariance is indeed satisfied only for

$$\lambda_+ = \lambda_- = \frac{1}{2}.$$

There is no longer a whole class of λ_{\pm} -dependent symmetry transformations, but just a given, point-like in the λ_{\pm} parametric space, set of symmetry transformations. In this particular case, the analysis plainly follows the one conducted for the free massless boson. The conserved Noether charges L_n, \bar{L}_n can be introduced through the Laurent-expansion of

$$\int dx\epsilon(z_+)T(x), \quad \int dx\bar{\epsilon}(z_-)\bar{T}(x).$$

They are conserved provided that

$$\partial_-T = \partial_+\bar{T} = 0.$$

T, \bar{T} can be unambiguously fixed to be given by

$$T = \frac{1}{4} \left(\pi^2 + (\partial_x\phi)^2 + 2\pi\partial_x\phi + e^{2\phi} - 2\partial_x^2\phi - 2\partial_x\pi \right),$$

$$\bar{T} = -\frac{1}{4} \left(\pi^2 + (\partial_x\phi)^2 - 2\pi\partial_x\phi + e^{2\phi} - 2\partial_x^2\phi + 2\partial_x\pi \right).$$

The two sets of Noether charges, L_n, \bar{L}_n , generate the symmetry transformation of the field ϕ according to (5.1).

Their Poisson-bracket algebra however is anomalous and coincides with $Vir \oplus Vir$, with fixed values of the two central charges c_{\pm} given by $c_{\pm} = \mp 6$, for the given normalization of the field ϕ and of the action,

$$\begin{aligned} \{T(x), T(y)\} &= -\frac{1}{2} \partial_y^3 \delta(x-y) + 2T(y) \partial_y \delta(x-y) + \partial_y T(y) \cdot \delta(x-y), \\ \{T(x), \bar{T}(y)\} &= 0, \\ \{\bar{T}(x), \bar{T}(y)\} &= \frac{1}{2} \partial_y^3 \delta(x-y) + 2\bar{T}(y) \partial_y \delta(x-y) + \partial_y \bar{T}(y) \cdot \delta(x-y). \end{aligned}$$

(I recall that, by definition, the central charge is normalized to be the coefficient in front of the inhomogeneous term δ''' normalized by a factor 12). The conformal invariance of the $2D$ Liouville theory is classically anomalous, satisfying the definition proposed here. This is in contrast with the free massless boson model, where the symmetry can be restored both at the classical and quantum level, as well as the free chiral fermion theory. In that case the chiral (i.e. Witt) invariance is classically preserved, while it is violated at the quantum level for a fixed value of the central charge ($c = 1/2$). The Liouville theory, on the other hand, admits a non-vanishing classical central charge. Its normalization is meaningless in the classical case, since it can always be reabsorbed through a simultaneous rescaling of the fields and of the Poisson brackets, as previously mentioned. Nevertheless, in the quantum theory, the effect of such “freedom of rescaling” of the underlining classical theory can be seen in the arbitrariness of the Liouville quantum central charge, which is not fixed by the theory, apart the restriction coming from unitarity requirement. This is in sharp contrast to the free chiral theory, where such freedom is not allowed.

8 Conclusions

In the present work I have stressed the fact that features which correspond to an anomalous realization of a symmetry can be present even in purely classical dynamical systems.

I introduced the definition of “classical anomaly” to describe the situation of a classical system whose conserved Noether charges, which are associated to a symmetry of the action, admit a Poisson brackets algebra, induced by the hamiltonian dynamics, which is only isomorphic to a centrally extended version of the original symmetry algebra. A classical anomaly is therefore a specific example of a non-equivariant moment map realized by Noether charges.

The underlining mathematical reason which makes possible the realization of such a case has been illustrated in Section 3. It has been shown there that eventual central extensions of the original symmetry realized by the Poisson brackets algebra of the Noether charges satisfy the compatibility constraint required by the simultaneous validity of (3.1) and (3.2). Later, a list of simple models which show the concrete application of this mechanism and the appearance of anomalies have been given.

It is certainly true, as discussed in the text, that quantization is still the “preferred” mechanism to produce anomalies (a good example is given by the chiral fermion theory of Section 4, which is quantum anomalous under 1-dimensional diffeomorphisms, but satisfies the ordinary Witt algebra for what concerns classical Poisson brackets). It turns out, however, that in general and in many cases of interest, it is not necessary to perform

the quantization of a dynamical system in order to induce anomalies. In some cases the introduction of a classical Poisson brackets structure is sufficient for the purpose. The anomalous nilpotent fermionic symmetry of the free chiral fermion of Section 4 is perhaps the simplest example, as well as the anomalous conformal symmetry of the classical Liouville equation analyzed in Section 7.

Moreover, any symmetry which algebraically admits a central extension, is potentially anomalous. The investigation of the Poisson brackets algebra of its Noether charges realized on specific models can lead to non-trivial results even for classical dynamics.

Specific differences with respect to the “quantum anomalies” have been pointed out throughout the text. At least for the case of symmetries associated to linear algebras (in the present analysis no effort was put in including infinitesimal symmetries of non-linear W -type), the Leibniz rule observed by classical Poisson brackets allow, through the simultaneous rescaling of the fields and the Poisson brackets, to freely normalize the value of the central charge, which can be conveniently chosen.

The examples chosen and the techniques employed in the present work are elementary. The main motivation of this paper is to illustrate, in the simplest possible contexts, the mathematical framework, deep and simple at the same time, behind the appearance of anomalies in classical dynamical systems.

The techniques which are usually encountered in the literature and which appear in disguised form in the analysis of the examples here illustrated (e.g., the introduction of the Feigin–Fuchs term in the Coulomb gas approach to “shift” the value of the quantum central charge), often appear to a layman reader as just a set of ad hoc prescriptions to perform technical computations. While it is certainly true that they are technically very helpful, the deep symmetry principles which make them possible are somehow hidden. To place them in the proper context of lagrangian and hamiltonian dynamics is the main issue of the present paper.

The analysis here conducted suggests many possible lines of development. On a purely mathematical ground one can ask which kind of centrally extended algebras can find a dynamical interpretation as (anomalous) symmetry for some given dynamical system.

D. Leites [29] has recently proposed a very specific problem on the extended superization of the Liouville equation which could be investigated in the light of the present considerations.

On the other hand, the interplay between lagrangian and hamiltonian methods seems quite fruitful. It seems likely that, by employing superspace techniques, the embedding of certain classes of hamiltonian solitonic equations in some superized system, which also admits a (super)lagrangian description, could be given. This subject is currently under investigation. Nice and neat results concerning the symmetry algebra of these systems should be derivable. Needless to say, the presence of central charges in the Virasoro subalgebra is mandatory for any integrable system which contains KdV as its consistent reduction.

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