Monads and Interaction
Lecture 3

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Comonads

Comonads = Co-monads

- Comonads on $\mathcal{C}$ are monads on $\mathcal{C}^{\mathrm{op}}$.

- You don't like this, what?

- Ok, they are comonoids
  in the (strict) monoidal category $([\mathcal{C}, \mathcal{C}], \mathrm{Id}_{\mathcal{C}}, \cdot)$.

- What else can there be to say?

Comonads

- A *comonad* on a category $\mathcal{C}$ is given by a
    - a functor $D : \mathcal{C} \to \mathcal{C}$,
    - a natural transformation $\varepsilon : D \xrightarrow{.} \mathrm{Id}_{\mathcal{C}}$ (the *counit*),
    - a natural transformation $\delta : D \xrightarrow{.} D \cdot D$ (the *comultiplication*)

  such that

$$
\begin{array}{ccc}
DY & \xrightarrow{\delta_Y} & D(DY) \\
{\scriptstyle \delta_Y}\downarrow & & \downarrow{\scriptstyle D\varepsilon_Y} \\
D(DY) & \xrightarrow[\varepsilon_{DY}]{} & DY
\end{array}
\qquad
\begin{array}{ccc}
DY & \xrightarrow{\delta_Y} & D(DY) \\
{\scriptstyle \delta_{DY}}\downarrow & & \downarrow{\scriptstyle D\delta_Y} \\
D(DY) & \xrightarrow[\delta_{DY}]{} & D(D(DY))
\end{array}
$$

CoKleisli triples

- A *coKleisli triple* (*comonad in extension form, no-iteration form*) is given by
  - an object mapping $D : |\mathcal{C}| \to |\mathcal{C}|$,
  - a family of maps $\varepsilon_Y : DY \to Y$
    indexed by $X \in |\mathcal{C}|$,
  - a family of maps $(-)^{\dagger}_{X,Y} : \mathcal{C}(DX, Y) \to \mathcal{C}(DX, DY)$
    indexed by $X, Y \in |C|$ (the *coKleisli coextension* operation)

  such that
  - $\varepsilon_y \circ k^{\dagger} = k$ for $k : DX \to Y$,
  - $\varepsilon_Y^{\star} = \mathrm{id}_{DY}$,
  - $(\ell \circ k^{\dagger})^{\dagger} = \ell^{\dagger} \circ k^{\dagger} : DX \to DZ$ for $k : DX \to Y$, $\ell : DY \to Z$

- Functoriality of D, naturality of ε, $(-)^{\dagger}$ are not required, but follow.

- Comonads = coKleisli triples, for the same $D : |\mathcal{C}| \to |C|$, ε.

FP intuition

- D – a "notion" of environment (environments of some flavor with whose help computations can run)

- An environment $\approx$ a process that is able to serve computations and is at any moment in some state.

- Y – states

- DY – environments with states of type Y

- $D(DY)$ – environments with, as states, environments with states of type Y

- $\varepsilon_Y : DY \rightarrow Y$ – extract from an environment its initial state (useful for "halting" the environment at the start moment)

- $\delta_Y : DY \rightarrow D(DY)$ – "blow up" an environment into one that has as its state at any given moment the current remainder of the environment
  (useful for "pausing" the environment whenever, with the intent to "resume")

CoKleisli category of a comonad

- A comonad D on a category $\mathcal{C}$ induces a category **CoKl**(D) called the *coKleisli category* of D defined by
    - an object is an object of $\mathcal{C}$,
    - a map of from X to Y is a map of $\mathcal{C}$ from DX to Y,
- $^D\mathrm{id}_Y = DY \xrightarrow{\varepsilon_Y} Y$,
- $\ell\ ^D \circ k = DX \xrightarrow{\delta_X} DDX \xrightarrow{Dk} DY \xrightarrow{\ell} Z$

    $$\underbrace{\phantom{DX \xrightarrow{\delta_X} DDX \xrightarrow{Dk} DY}}_{k^\dagger}$$

    for $k : X\ ^D \to Y$, $\ell : Y\ ^D \to Z$

- From $\mathcal{C}$ there is an identity-on-objects functor J to **CoKl**(T), defined on maps by
    - $Jf = DX \xrightarrow{\varepsilon_X} X \xrightarrow{f} Y$ for $f : X \to Y$
- If ε is epi, then J is faithful.

CoKleisli adjunction

- In the opposite direction of $J : \mathcal{C} \to \mathbf{CoKl}(D)$, there is a functor $L : \mathbf{CoKl}(D) \to \mathcal{C}$ defined by
  - $LY = DY$,
  - $Lk = DX \xrightarrow{k^\dagger} DY$ for $k : X^D \to Y$.
- L is left adjoint to J.

$$
\begin{array}{cc}
\mathbf{CoKl}(D) & \\
L \left(\;\; \dashv \;\; \right) J & \dfrac{X^D \to \overbrace{Y}^{JY}}{\underbrace{DX}_{LX} \to Y} \\
\mathcal{C} &
\end{array}
$$

- We have $L \cdot J = D$. Indeed,
  - $L(JX) = TX$,
  - if $f : X \to Y$, then $L(Jf) = (f \circ \varepsilon_X)^\dagger = Df$.
- And the counit of the adjunction is ε.

Comonad coalgebras

- A *coalgebra* of a comonad (D, ε, δ) is an object Y with a map $\chi : Y \to DY$ such that

$$
\begin{array}{ccc}
Y & & \\
\chi \downarrow & \big\Vert & \\
DY & \xrightarrow{\varepsilon_Y} & Y
\end{array}
\qquad
\begin{array}{ccc}
Y & \xrightarrow{\chi} & DY \\
\chi \downarrow & & \downarrow D\chi \\
DY & \xrightarrow{\delta_Y} & D(DY)
\end{array}
$$

- A *map* between two coalgebras (X, ξ) and (Y, χ) is a map h such that

$$
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\xi \downarrow & & \downarrow \chi \\
DX & \xrightarrow{Th} & DY
\end{array}
$$

- The coalgebras of the comonad and maps between them form a category **Coalg**(D), called the *coEilenberg-Moore category*, with an obvious forgetful functor $U : \textbf{Coalg}(T) \to \mathcal{C}$.

FP intuition

- A coalgebra $(Y, \xi : Y \to DY)$ – a coeffect producer (should we call it a cohandler?)

- Y – a fixed type of states
- $\xi : Y \to DY$ –
  a (consistent) assignment of an environment to every initial state

  (think, e.g., that you have a state machine over state set Y that you can start in any state and obtain a behavior)

CoEilenberg-Moore adjunction

- In the opposite direction of $U : \mathbf{Coalg}(D) \to \mathcal{C}$ there is a functor $R : \mathcal{C} \to \mathbf{Coalg}(D)$ defined by
  - $RY = (DY, \delta_Y)$,
  - $Rf = Df : (DX, \delta_X) \to (DY, \delta_Y)$ for $f : X \to Y$.
- R is right adjoint to U.

$$
\mathbf{Coalg}(D) \qquad\qquad \cfrac{(X, \xi) \to \overbrace{(DY, \delta_Y)}^{RY}}{\underbrace{X}_{U(X,\xi)} \to Y}
$$

$$
U \left(\ \dashv \ \right) R
$$

$$
\mathcal{C}
$$

- This is says that (DY, δ_X) is a coalgebra of the comonad D, moreover, it is the cofree one.
- $U \cdot R = D$. Indeed,
  - $U(RY) = U(DY, \delta_Y) = DY$,
  - if $f : X \to Y$, then $U(Rf) = U(Df) = Df$.
- The counit of the adjunction is ε.

Comonads from adjunctions

- Any adjunction gives rise to a comonad.
- Given an adjunction

$$
\begin{array}{c}
\mathcal{D} \\
L \left(\;\dashv\; \right) R \\
\mathcal{C}
\end{array}
\qquad
\frac{LX \to Y}{X \to RY}
$$

  the endofunctor $D = L \cdot R$ on $\mathcal{C}$ carries a comonad structure with ε the counit of the adjunction.

- Adjunctions so related to a comonad are called its resolutions.
- Resolutions of a comonad form a category.
- The coKleisli and coE-M adjunctions are resolutions. The coKleisli resolution is the initial, the coE-M resolution the final object of this category.

Coreader and cowriter comonads

- The *coreader* comonad for an object S (of states) is
  - $DY = S \times Y$
  - $\varepsilon_Y : S \times Y \to Y$
    $\varepsilon\,(\_, y) = y$
  - $\delta_Y : S \times Y \to S \times (S \times Y)$
    $\delta\,(s, y) = (s, (s, y))$

- The *cowriter* comonad for a monoid $(P, \mathrm{o}, \oplus)$ (of updates) is
  - $DY = P \Rightarrow Y$
  - $\varepsilon_Y : P \Rightarrow Y \to Y$
    $\varepsilon\,e = e\,\mathrm{o}$
  - $\delta_Y : P \Rightarrow Y \to P \Rightarrow P \Rightarrow Y$
    $\delta\,e = \lambda p.\,\lambda p'.\,e\,(p \oplus p')$

Costate comonads

- The *costate comonad* (originally called *array comonad*) for an object S (of states) is
  - $DY = S \times (S \Rightarrow Y)$
  - $\varepsilon_Y : S \times (S \Rightarrow Y) \to Y$
    $\varepsilon(s, f) = f\, s$
  - $\delta_Y : S \times (S \Rightarrow Y) \to S \times (S \Rightarrow S \times (S \Rightarrow Y))$
    $\delta(s, f) = (s, \lambda s'.\, (s', f))$
- A variation, the *external costate comonad* for an object S of states, is
  - $DY = \mathcal{C}(S, Y) \bullet S$
- The costate monads for S arise from the adjunction

$$
-\times S \left(\begin{array}{c} \mathcal{C} \\ \dashv \\ \mathcal{C} \end{array}\right) S \Rightarrow - \qquad \frac{X \times S \to Y}{X \to S \Rightarrow Y} \qquad\qquad -\bullet S \left(\begin{array}{c} \mathcal{C} \\ \dashv \\ \mathbf{Set} \end{array}\right) \mathcal{C}(S, -) \qquad \frac{I \bullet S \to Y}{I \to \mathcal{C}(S, Y)}
$$

Coalgebras of costate comonads

- Coalgebras of the costate comonad for S are (by definition) objects Y with a map $\chi : Y \to S \times (S \Rightarrow Y)$ subject to two equations.

- They are in a bijection very well-behaved lenses between Y and S.

  Those are (by definition) pairs of maps $coget : Y \to S$, $coput : Y \times S \to Y$ such that
    - $y = coput\,(y, coget\,y))$,
    - $coget\,(coput\,(y, s)) = s$,
    - $coput\,(coput\,(y, s), s') = coput\,(y, s')$.

Cofree functor-coalgebras comonads

- The following comonad $G^\dagger$ delivers (carriers of) cofree algebras of a functor G:

  - $DY = \nu W.\, Y \times GW$ $\qquad\qquad\qquad\qquad\qquad DY \cong^\nu Y \times G(DY)$
    (G-branching trees with Y-labelled branching nodes)

  - $\varepsilon\, t = \mathsf{fst}\,(\mathsf{out}\, t)$
    (extracts the root label of a tree)

  - $\delta\, t = \mathsf{out}^{-1}\,(t,\, G\, \delta\,(\mathsf{snd}\,\mathsf{out}\, t)$
    (relabels each node of a tree with the subtree rooted by that node)

- For $GY = Y$, we get $DY = \nu W.\, Y \times W \cong \mathsf{Str}\, Y$
  (streams and suffixes).

- For $GY = 1 + Y$, we get $DY = \nu W.\, Y \times (1 + W) \cong \mathsf{NEColist}\, Y$
  (nonempty colists and suffixes).

Coalgebras of cofree functor-algebras comonads

- The category **Coalg**$(G^\dagger)$ of coalgebras of the comonad $G^\dagger$ is isomorphic to the category **coalg**(G) of coalgebras of the functor G. The isomorphism is identity on carriers.

- E.g., the a coalgebra of the stream comonad D is an object Y with a map $\chi : Y \to \mathsf{Str}\, Y$ subject to two equations

- It is described by map $\psi : Y \to Y$ subject to no equations.

- χ is defined from ψ by $\mathsf{shd}\,(\chi\, y) = y$, $\mathsf{stl}\,(\chi\, y) = \chi\,(\psi\, y)$

- ψ is constructed from χ by $\psi\, y = \mathsf{shd}\,(\mathsf{stl}\,(\chi\, y))$.

- The comonad $G^\dagger$ is the free comonad on the functor G.