Intermediate Jacobians and Abel-Jacobi Maps

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Introduction

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The result is that questions about cycles can be translated into questions about complex tori . . .

Hodge Decomposition

Let X be a smooth projective complex variety of dimension n. The Hodge decomposition is a direct sum decomposition of the complex cohomology groups

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The subspaces $H^{p,q}(X)$ consist of classes $[\alpha]$ of differential forms that are representable by a closed form α of type (p, q) meaning that locally

$$\alpha = \sum_{\substack{I, J \subseteq \{1, \dots, n\} \\ |I| = p \, , \ |J| = q}} f_{I,J} \, dz_I \wedge d\overline{z}_J$$

for some choice of local holomorphic coordinates $z_1, \dots, z_n$.

Hodge Decomposition (Continued)

The subspaces $H^{p,q}(X)$ satisfy the Hodge symmetry

$$H^{p,q}(X) = \overline{H^{q,p}(X)}$$

where $\alpha \mapsto \overline{\alpha}$ is the natural action of complex conjugation on

$$H^k(X, \mathbb{C}) = H^k(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}\,.$$

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Since real forms are conjugate invariant, we may write $[\alpha] \in H^{2k}(X, \mathbb{R})$ in terms of the Hodge decomposition

$$\alpha = \alpha^{2k,0} + \cdots + \alpha^{k+1,k-1} + \alpha^{k,k} + \overline{\alpha^{k+1,k-1}} + \cdots + \overline{\alpha^{2k,0}}$$

and for $[\beta] \in H^{2k-1}(X, \mathbb{R})$ in odd degree

$$\beta = \beta^{2k+1,0} + \beta^{2k,1} + \cdots + \beta^{k+1,k} + \overline{\beta^{k+1,k}} + \cdots + \overline{\beta^{2k,1}} + \overline{\beta^{2k+1,0}}\,.$$

Hodge Decomposition (Continued)

The Hodge decomposition defines a filtration

$$0 \subseteq F^k H^k(X) \subseteq F^{k-1} H^k(X) \subseteq \cdots \subseteq F^0 H^k(X) = H^k(X, \mathbb{C})$$

where

$$F^r H^k(X) = \bigoplus_{p \geq r} H^{p,k-p}(X) \ , \ 0 \leq r \leq k \, .$$

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The cohomology groups of odd degree satisfy

$$H^{2k-1}(X, \mathbb{C}) = F^k H^{2k-1}(X) \oplus \overline{F^k H^{2k-1}(X)}$$

and the natural map

$$H^{2k-1}(X, \mathbb{R}) \longrightarrow H^{2k-1}(X, \mathbb{C}) / F^k H^{2k-1}(X)$$

is an isomorphism of real vector spaces.

Intermediate Jacobians

The rank of $H^{2k-1}(X,\mathbb{Z})$ is the dimension of $H^{2k-1}(X,\mathbb{R})$ therefore $H^{2k-1}(X,\mathbb{Z})$ defines a full lattice

$$L_k = \mathrm{im}\left(H^{2k-1}(X,\mathbb{Z}) \to H^{2k-1}(X,\mathbb{C})/F^kH^{2k-1}(X)\right)$$

in the complex vector space

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Definition

The kth intermediate Jacobian of X is the complex torus

$$J^k(X) = V_k/L_k = H^{2k-1}(X,\mathbb{C})/(F^kH^{2k-1}(X) \oplus H^{2k-1}(X,\mathbb{Z}))\,.$$

Intermediate Jacobians (Continued)

Poincaré duality asserts

$$H^{2n-2k+1}(X,\mathbb{C})^{\vee} = H^{2k-1}(X,\mathbb{C})$$

and so

$$F^{n-k+1}H^{2n-2k+1}(X)^{\vee} = H^{2k-1}(X,\mathbb{C})/F^{n-k+1}H^{2n-2k+1}(X)^{\perp}\,.$$

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We see that

$$F^{n-k+1}H^{2n-2k+1}(X)^\perp = F^k H^{2k-1}(X)$$

since a form of type $(p, 2n-2k+1-p)$ with $p \geq n-k+1$ wedged with a form of type $(p', 2k-1-p')$ with $p' \geq k$ is a form of type $(p'', 2n-p'')$ with $p'' \geq n+1$ which must be zero since X has complex dimension n.

Intermediate Jacobians (Continued)

Poincaré duality further states that

$$H^{2k-1}(X, \mathbb{Z}) = H_{2n-2k+1}(X, \mathbb{Z})$$

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We abuse notation here as Voisin does when we do not distinguish between $H^{2k-1}(X, \mathbb{Z})$ and its image in $H^{2k-1}(X, \mathbb{C})$ and similarly for $H_{2n-2k-1}(X, \mathbb{Z})$ and its image in $H^{2n-2k-1}(X, \mathbb{C})^\vee$.

Cycle Classes

We can attach an integral cohomology class $[Z] \in H^{2k}(X, \mathbb{Z})$ to a cycle Z of codimension k in X and extending by $\mathbb{Z}$-linearity defines a homomorphism

$$\mathcal{Z}^k(X) \longrightarrow H^{2k}(X, \mathbb{Z}) : Z \mapsto [Z]$$

called the *cycle class map*.

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The image of $[Z]$ in $H^{2k}(X, \mathbb{R})$ is described by Poincaré duality

$$H^{2k}(X, \mathbb{R}) \cong H^{2n-2k}(X, \mathbb{R})^\vee .$$

The class $[Z]$ is the form of degree $2k$ such that

$$\langle [Z], \alpha \rangle_X = \int_{Z_{\mathrm{sm}}} \alpha \ , \ \text{for all } \alpha \in H^{2n-2k}(X, \mathbb{R}) \, ,$$

where Z_{sm} is the smooth locus of Z.

Constructing the Abel-Jacobi Map

The kernel of the cycle class map is the group of null-homologous cycles

$$\mathcal{Z}^k(X)_{\text{hom}} = \ker\left(\mathcal{Z}^k(X) \longrightarrow H^{2k}(X, \mathbb{Z})\right) .$$

In other words, if $Z \in \mathcal{Z}^k(X)_{\text{hom}}$ there exists some chain C_Z of dimension $2n - 2k + 1$ such that $\partial C_Z = Z$.

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Integrating over C_Z defines a functional on forms of degree $2n - 2k + 1$

$$\left(\omega \mapsto \int_{C_Z} \omega\right) \in A^{2n-2k+1}(X)^{\vee}$$

therefore we would like to attach to Z a cohomology class in degree $2k - 1$ modulo the ambiguity in choosing the chain C_Z.

Constructing the Abel-Jacobi Map (Continued)

Integrating over C_Z will only define a functional on cohomology if it is zero on exact forms. If $\omega = d\psi$ for some form ψ of degree $2n - 2k$, we have

$$\int_{C_Z} \omega = \int_Z \psi$$

which vanishes if ψ is of type $(p, q) \neq (n-k, n-k)$ since then $\psi|_Z = 0$ because Z has complex dimension $n - k$.

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Therefore, integrating over C_Z does not define a functional on the whole space $H^{2n-2k+1}(X, \mathbb{C})$ but we can choose a piece of the cohomology on which it does.

Constructing the Abel-Jacobi Map (Continued)

It can be shown that

$$F^{n-k+1}H^{2n-2k+1}(X,\mathbb{C}) = \frac{F^{n-k+1}A^{2n-2k+1}(X)\cap \ker d}{d\,F^{n-k+1}A^{2n-2k}(X)}$$

and we have shown that $\int_{C_Z} d\psi = 0$ for all $\psi \in F^{n-k+1}A^{2n-2k}(X)$ therefore

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It remains to consider the ambiguity in the choice of the chain C_Z bounding the cycle Z.

Constructing the Abel-Jacobi Map (Continued)

If C'_Z is another chain such that $\partial C'_Z = Z$ then $\partial(C_Z - C'_Z) = 0$ therefore $C_Z - C'_Z$ defines a class in $H_{2n-2k+1}(X, \mathbb{Z})$ and so

$$\left(\omega \mapsto \int_{C_Z} \omega\right) \in F^{n-k+1}H^{2n-2k+1}(X)^\vee / H_{2n-2k+1}(X, \mathbb{Z})$$

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Definition

The Abel-Jacobi Map of X of degree k is

$$\Phi_X^k : \mathcal{Z}^k(X)_{\text{hom}} \longrightarrow J^k(X) : Z \mapsto \left(\omega \mapsto \int_{C_Z} \omega\right).$$

Constructing the Abel-Jacobi Map (Continued)

Let $d_k = \dim_{\mathbb{C}} F^{n-k+1} H^{2n-2k+1}(X)$ and choose a basis

$$\omega_1, \ldots, \omega_{d_k} \in F^{n-k+1} H^{2n-2k+1}(X) .$$

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Choose a basis of the homology group

$$\gamma_1, \ldots, \gamma_{2d_k} \in H_{2n-2k+1}(X, \mathbb{Z}) ,$$

let $\Omega_i = \left(\int_{\gamma_i} \omega_1, \ldots, \int_{\gamma_i} \omega_{d_k} \right) \in \mathbb{C}^{d_k}$ and let

$$\Lambda = \operatorname{span}_{\mathbb{Z}} \{\Omega_1, \ldots, \Omega_{2d_k}\} \subset \mathbb{C}^{d_k} .$$

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The Abel-Jacobi map of degree k is then

$$\Phi_X^k(Z) = \left(\int_{C_Z} \omega_1, \ldots, \int_{C_Z} \omega_{d_k}\right) \in \mathbb{C}^{d_k}/\Lambda\,.$$

Example: The Albanese Variety ($k = n$)

In the case $k = n$, a cycle $Z \in \mathcal{Z}^n(X)$ is a point and the complex torus $J^n(X) = \mathrm{Alb}_X$ is called the Albanese variety of X.

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We may fix a point $x_0 \in X$ and consider the null-homologous 0-cycles $(x) - (x_0)$ as x varies in X. The Abel-Jacobi Map on these cycles then defines a holomorphic map

$$\mathrm{alb}_X : X \longrightarrow \mathrm{Alb}_X : x \mapsto \Phi_X^n((x) - (x_0))$$

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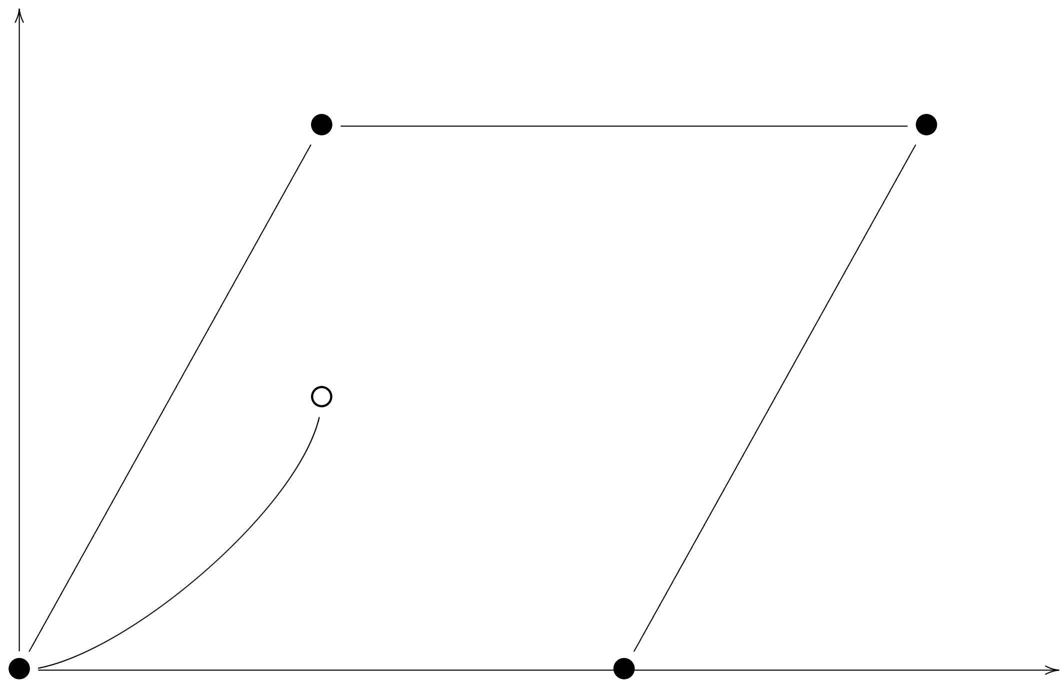
Theorem

If $f : X \longrightarrow T$ is a holomorphic map from X to a complex torus T such that $f(x_0) = 0$, then there exists a unique morphism of complex tori $g : Alb_X \longrightarrow T$ such that $g \circ alb_X = f$.

The Simplest Case (k=n=1)

The chain $C_{(x)-(x_0)}$ varying in a curve X of genus 1 as x varies in X:



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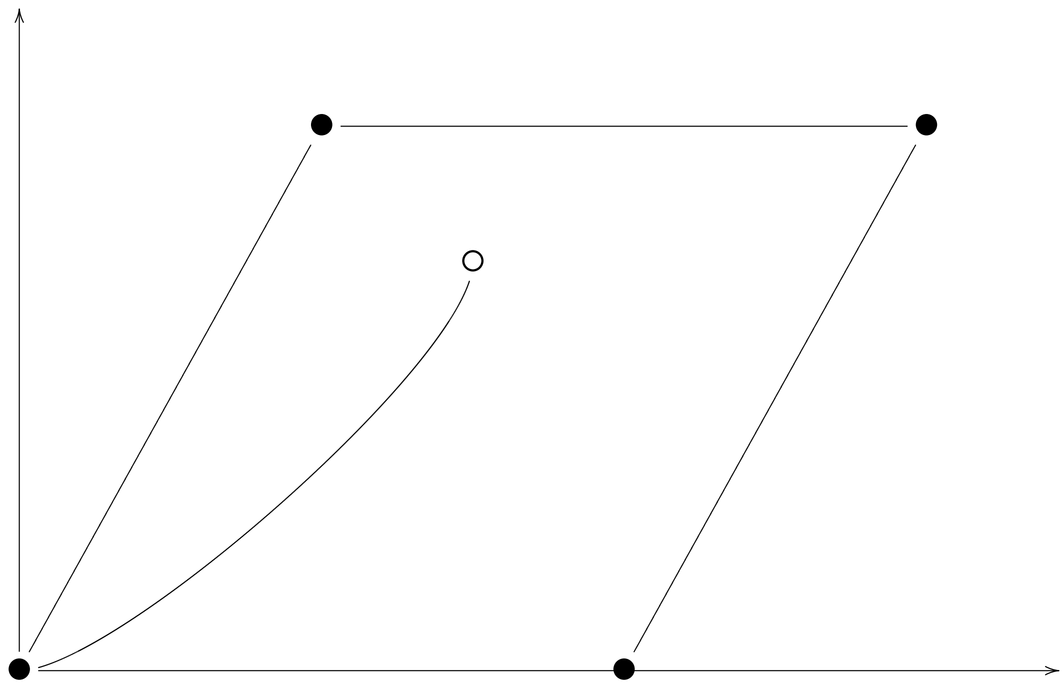

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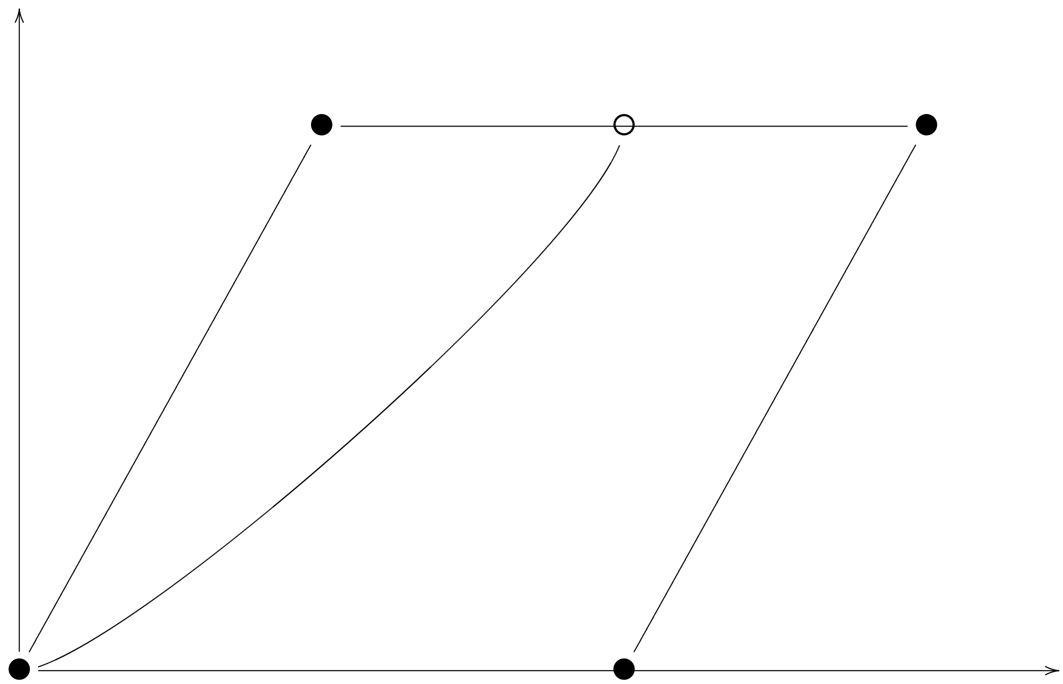

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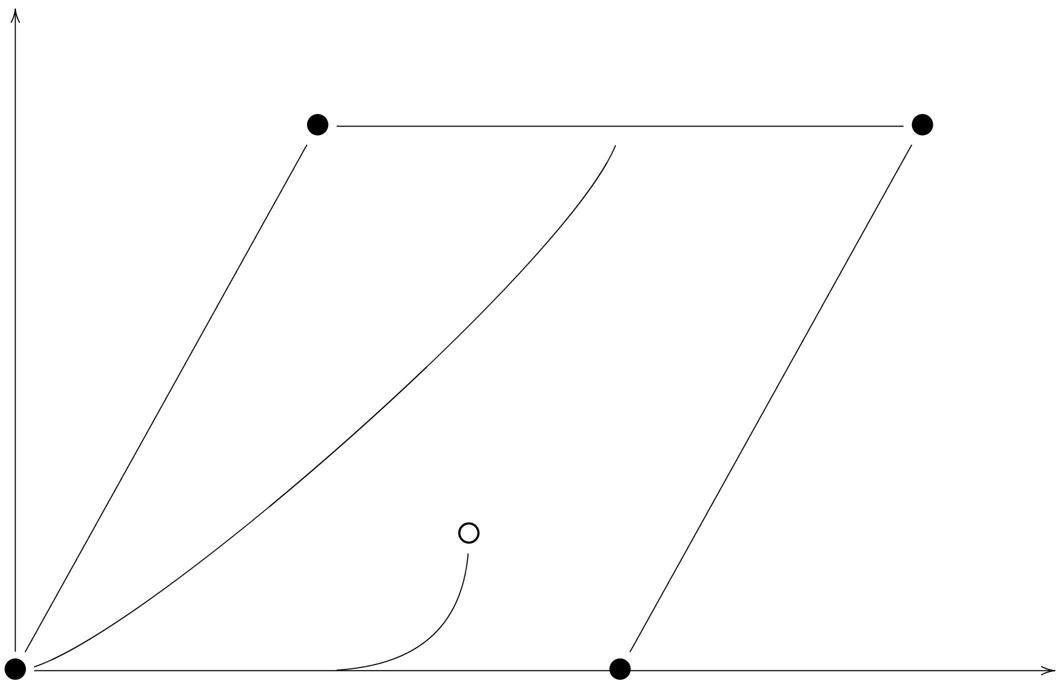

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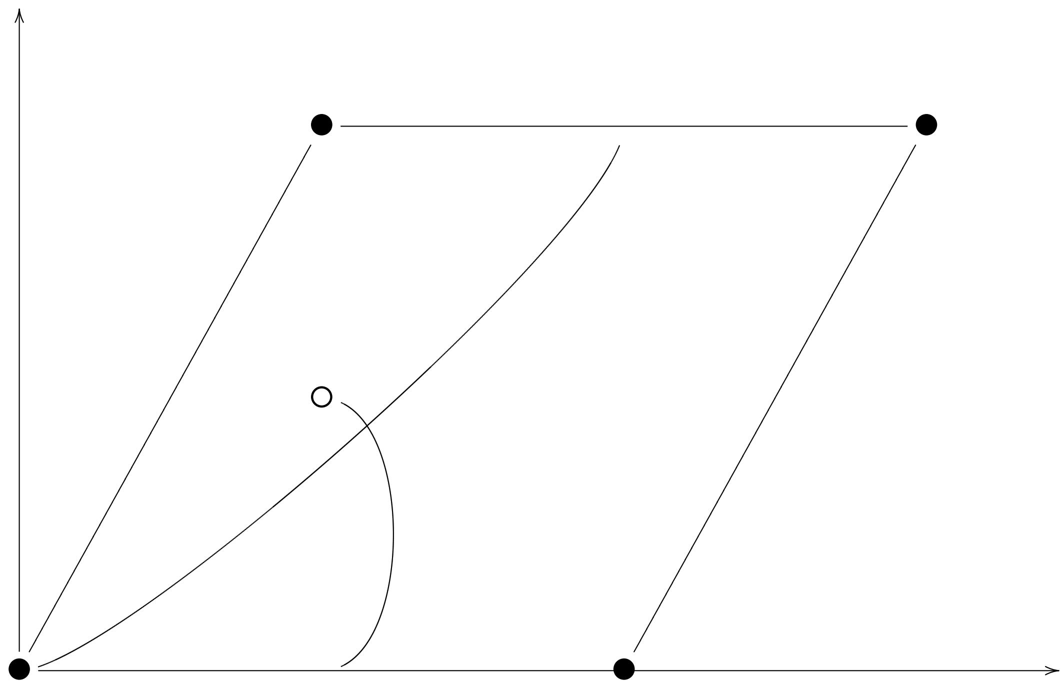

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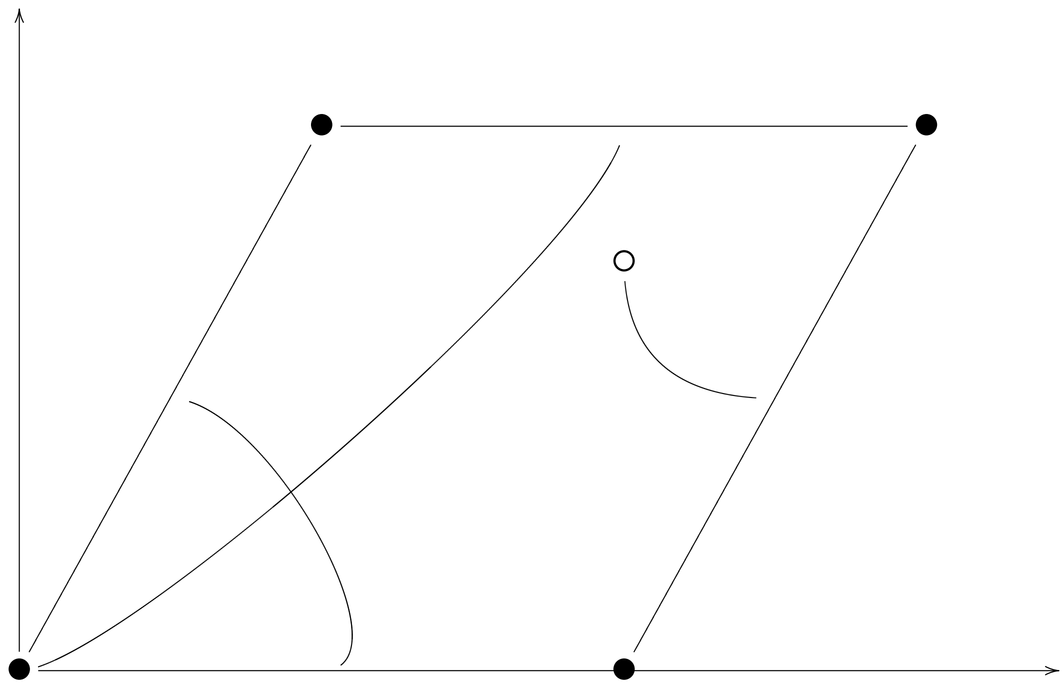

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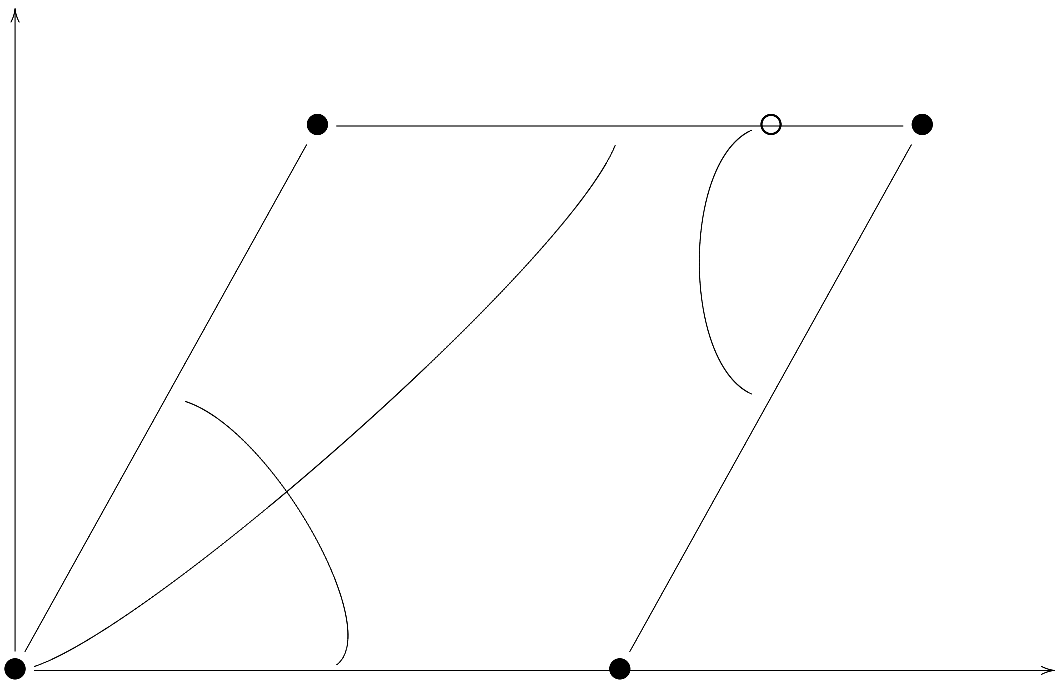

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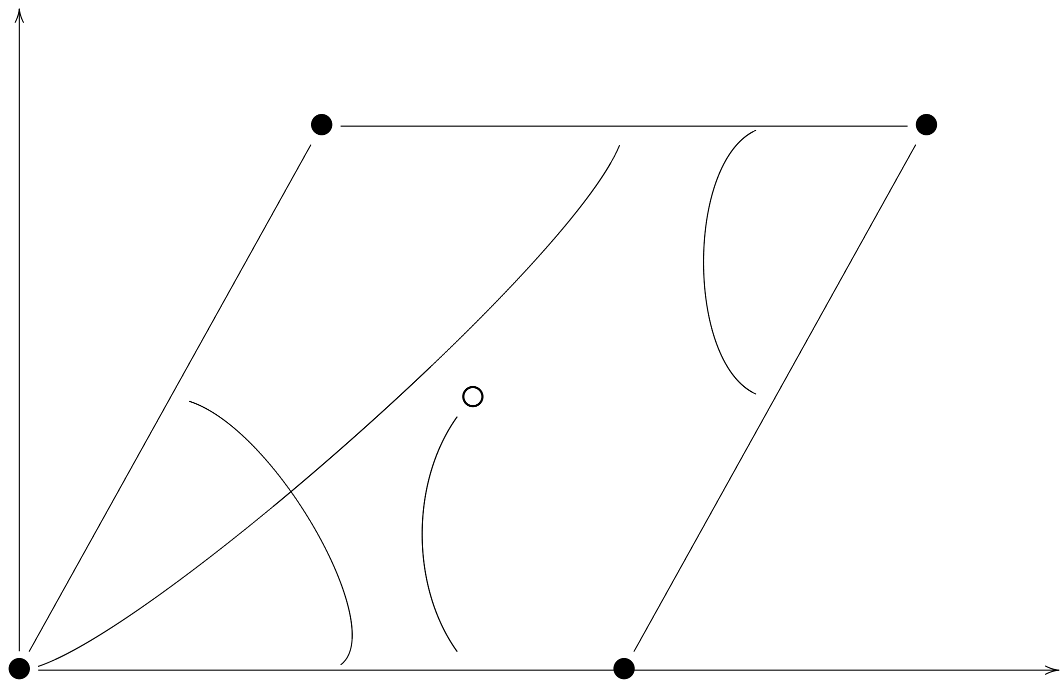

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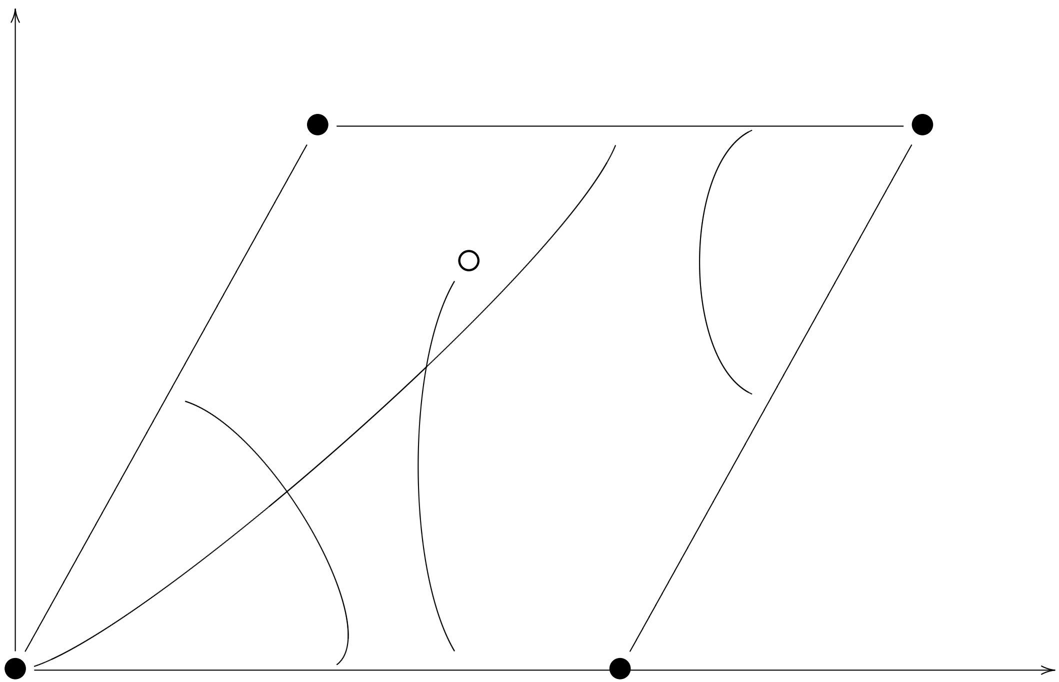

Holomorphic Families of Cycles

The Albanese map is the simplest example of the Abel-Jacobi map applied to a holomorphic family of cycles. More generally, we can consider a holomorphic family $(Z_y)_{y\in Y}$ of cycles of codimension k in X parametrized by a connected complex manifold Y.

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Theorem (Griffiths, 1968)

Let X be a complex Kähler manifold, let Y be a connected complex manifold and let $Z \subset Y \times X$ be a cycle of codimension k and flat over Y. The fibers Z_y are all homologous in X and, given a reference point $y_0 \in Y$, the map

$$\phi : Y \longrightarrow J^k(X) : y \mapsto \Phi_X^k(Z_y - Z_{y_0})$$

is holomorphic.

Rational Equivalence

Theorem

If $Z \in \mathcal{Z}^k(X)$ is rationally equivalent to 0 then $\Phi_X^k(Z) = 0$. In particular, the Abel-Jacobi map factors through the (null-homologous) Chow group

$$\Phi_X^k : CH^k(X)_{hom} \longrightarrow J^k(X)\,.$$

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The idea is that if Z_1 and Z_2 are rationally equivalent then these cycles lie in a family of cycles parametrized by $\mathbb{P}^1$ and the Abel-Jacobi map then defines a holomorphic map $\Phi_X^k : \mathbb{P}^1 \longrightarrow J^k(X)$. But such a holomorphic map must be constant so $\Phi_X^k(Z_1) = \Phi_X^k(Z_2)$.

We now have a tool with which to study the Chow groups $\mathrm{CH}^k(X)_{\mathrm{hom}}$ in terms of the simpler objects $J^k(X)$.

Applications: The Work of Schoen

The recurring fantasy of Bloch conjectures the relationship

$$\operatorname{rank} \mathrm{CH}^k(X_F)_{\mathrm{hom}} = \operatorname{ord}_{s=k} L(H^{2k-1}(X_F), s)$$

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The rough idea of (a part of) Schoen's work is to consider a specific moduli space $\widetilde{W}$ of elliptic curves and to construct a large class of cycles over an imaginary quadratic field F and of codimension 2 in $\widetilde{W}$ using of the theory of complex multiplication.

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The rough idea of (a part of) Schoen's work is to consider a specific moduli space $\widetilde{W}$ of elliptic curves and to construct a large class of cycles over an imaginary quadratic field F and of codimension 2 in $\widetilde{W}$ using of the theory of complex multiplication. Since proving rational equivalence amongst these cycles is much too difficult, the idea is to provide modest numerical evidence that the rank of the subgroup these cycles generate, via the Abel-Jacobi map, in the intermediate Jacobian $J^2(\widetilde{W})$ is 1 when the corresponding L-function is "known" to have a simple zero at s=2.

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