Oberwolfach Workshop 1803: The topology of arrangements and representation stability

A brief introduction to representation stability

Jenny Wilson • 15 January 2018

This is an expository talk given at the Oberwolfach workshop on *the topology of arrangements and representation stability*. The first half of the talk aims to give some background on the field of representation stability, and the second half a demonstration of a particular proof technique for proving representation stability in certain applications.

The talk comes with the disclaimer that my description of the history of representation stability is heavily biased by my own background: I was a graduate student of Benson Farb working in low-dimensional topology, and witnessed the field develop from that perspective. Other authors have come to the area from backgrounds in commutative algebra and algebraic geometry, from algebraic combinatorics, or from category theory and homological algebra – and they would likely give different motivation and a different origin story for the field.

With that disclaimer, here is one story about the advent of representation stability.

1 A brief and biased history of representation stability

Multiplicity stability (following Church–Farb [CF13])

This story starts in about 2010, when Church and Farb observed (both through computer experimentation and direct hands-on computation) patterns in the (co)homology of certain families of groups and spaces.

Notation 1. Let S_n denote the symmetric group on n letters. Recall that the group ring $\mathbb{Q}[S_n]$ is semisimple, that is, all rational S_n–representations decompose as a direct sum of irreducible representations. Recall moreover that there is a canonical bijection

$$\{\text{irreducible } S_n\text{–representations}\} \qquad \longleftrightarrow \qquad \{\text{partitions } \lambda \text{ of } n\}$$

We will use the notation V_λ to denote the irreducible representation associated to a Young diagram λ, eg, $V_{\yng(2,1)}$ is the irreducible S_3–representation associated to the partition $(2,1)$.

A motivating example: the pure braid groups

The prototypical family of groups studied by Church and Farb were the pure braid groups $\mathbf{PB}_n$. If $\mathbf{B}_n$ is Artin's braid group, then recall that the pure braid group is defined by the short exact sequence

$$1 \longrightarrow \mathbf{PB}_n \longrightarrow \mathbf{B}_n \longrightarrow S_n \longrightarrow 1,$$

that is, it is the subgroup of braids where each strand returns to its own starting position.

The group $\mathbf{B}_n$ acts on $\mathbf{PB}_n$ by conjugation, and the induced action of $\mathbf{B}_n$ on the cohomology groups $H^q(\mathbf{PB}_n;\mathbb{Q})$ factors through an action of the symmetric group

$$S_n \circlearrowright H^q(\mathbf{PB}_n;\mathbb{Q}).$$

Moreover, using the "forget the $(n+1)^{st}$ strand" projection maps $\mathbf{PB}_{n+1} \to \mathbf{PB}_n$ we can construct an S_n–equivariant sequence of representations

$$\cdots \longrightarrow H^q(\mathbf{PB}_n;\mathbb{Q}) \longrightarrow H^q(\mathbf{PB}_{n+1};\mathbb{Q}) \longrightarrow \cdots$$

We will illustrate the patterns that appear in these cohomology groups in the case $q = 1$.

Example 2 ($H^1(\mathbf{PB}_n;\mathbb{Q})$)**.** In cohomological degree $q = 1$, these S_n–representations decompose as follows.

$$\begin{array}{lcllll}
H^1(\mathbf{PB}_2;\mathbb{Q}) & \cong & V_{\yng(2)} & & & \\
H^1(\mathbf{PB}_3;\mathbb{Q}) & \cong & V_{\yng(3)} & \oplus \; V_{\yng(2,1)} & & \\
H^1(\mathbf{PB}_4;\mathbb{Q}) & \cong & V_{\yng(4)} & \oplus \; V_{\yng(3,1)} & \oplus \; V_{\yng(2,2)} & \\
H^1(\mathbf{PB}_5;\mathbb{Q}) & \cong & V_{\yng(5)} & \oplus \; V_{\yng(4,1)} & \oplus \; V_{\yng(3,2)} & \\
H^1(\mathbf{PB}_6;\mathbb{Q}) & \cong & V_{\yng(6)} & \oplus \; V_{\yng(5,1)} & \oplus \; V_{\yng(4,2)} & \\
H^1(\mathbf{PB}_7;\mathbb{Q}) & \cong & V_{\yng(7)} & \oplus \; V_{\yng(6,1)} & \oplus \; V_{\yng(5,2)} & \\
\cdots & & & & &
\end{array}$$

For all $n \geq 4$, this pattern continues: we can obtain one row of the decomposition from the previous by adding a single box to the top row of each of the Young diagrams.

Church and Farb called this "add a box to the top row" pattern *multiplicity stability*, and showed that it holds for all cohomology groups of $\mathbf{PB}_n$.

Proposition 3 (Church–Farb [CF13]). *For each $q \geq 0$, the sequence of S_n–representations $\{H^q(\mathbf{PB}_n; \mathbb{Q})\}_n$ is multiplicity stable, stabilizing for $n \geq 4q$.*

The decomposition of these cohomology groups into irreducible representations is not known explicitly except for some small values of q. Church and Farb's first proof of this result was combinatorial, using an orbit-stabilizer argument to realize the cohomology groups as a sum of certain induced representations, and then using the Littlewood-Richardson rules to analyze these induced representations.

Further examples

Church and Farb continued to dig, and they found variations on the same phenomenon in many other places. They wrote a paper [CF13] compiling some examples.

Theorem 4 (Church–Farb [CF13]). *There are mutliplicity stability patterns in the (co)homology of*

- *pure braid groups and certain generalized pure braid groups*
- *certain flag varieties*
- *certain Lie algebras and their homology*
- $\cdots$

At the time it was becoming clear that these patterns were prevalent across numerous mathematical areas, and perhaps that they were somehow connected to the Littlewood-Richardson rules. These reults raised the question,

Question 5. What underlying structure is driving these stability patterns?

In the case that the groups acting were the symmetric groups S_n, Church and Farb gave an answer to this question in joint work with Ellenberg: the sequences of S_n–representations are finitely generated FI–modules.

FI–modules (following Church, Ellenberg, Farb, and Nagpal [CEF15, CEFN14])

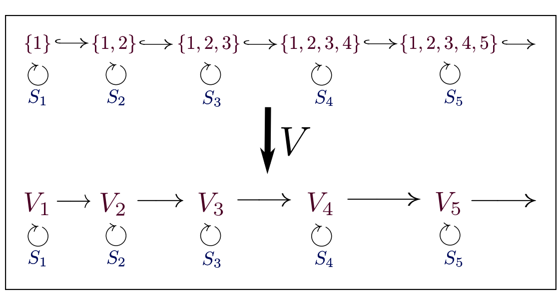
Definition 6. Church, Ellenberg, and Farb use the notation FI to denote the category of finite sets and injective maps. Up to equivalence, this is the category with one object for each integer $n \geq 0$ corresponding to the sets $[n] = \{1, 2, \ldots, n\}$, with endomorphisms $\text{End}([n]) = S_n$.

Given a commutative unital ring R, an FI–*module* (over R) is a functor

$$V : \mathsf{FI} \to R\text{–Mod}$$

to the category R–Mod of R–modules. We use the notation V_S to denote the image of V on a set S, or V_n when $S = \{1, 2, \ldots, n\}$.

The data of an FI–module is a sequence of S_n–representations with equivariant maps.



Examples and non-examples of FI–modules

A good first exercise is to verify the following.

Example 7 (Examples of FI–modules)**.** The following sequences of S_n–representations, along with the obvious inclusions, assemble to form rational FI–modules.

- $V_n = \mathbb{Q}$ (trivial S_n–representation)
- $V_n = \mathbb{Q}^n$ (canonical permutation representation)
- $V_n = \bigwedge^5 \mathbb{Q}^n$
- $V_n = \mathbb{Q}[x_1, \ldots, x_n]$ (diagonal action of S_n on monomials by permuting indices)

Example 8 (Non-examples of FI–modules)**.** The following equivariant sequences of S_n–representations do **not** have the structure of an FI–module.

- $V_n = \mathbb{Q}$ (alternating S_n–representation), with isomorphisms
- $V_n = \mathbb{Q}[S_n]$ (group ring), with maps induced by the inclusions $S_n \hookrightarrow S_{n+1}$ of groups[1]

For fixed R, the category of FI–modules over R comes with natural and well-behaved notions of maps of FI–modules, submodules, quotients, kernels, cokernels, direct sums, tensor products, ... In particular, it is an *abelian category*.

The pure braid group, continued

In our main example of the pure braid group, observe that for each q the sequence of cohomology groups $\{H^q(\mathbf{PB}_n; \mathbb{Q})\}_n$ forms an FI–module. And, for each q, this FI–module is finitely generated under the action of the category FI. To see this concretely when (for example) $q = 2$, we can use a 1969 result of Arnold that gives an explicit description of the algebra $H^*(\mathbf{PB}_n; \mathbb{Q})$ (and, in fact, the integral cohomology $H^*(\mathbf{PB}_n; \mathbb{Z})$).

Theorem 9 (Arnold [Arn69])**.** *The cohomology algebra $H^*(\mathbf{PB}_n; \mathbb{Q})$ can be described as a certain quotient of the exterior algebra on the symbols $\alpha_{i,j}$,*

$$H^*(\mathbf{PB}_n; \mathbb{Q}) \cong \bigwedge^* \langle \alpha_{i,j} \rangle / \sim \qquad \alpha_{i,j} = \alpha_{j,i}, \ i \neq j, \ i, j \in [n]$$

So for $q = 2$, the groups $H^2(\mathbf{PB}_n; \mathbb{Q})$ are spanned by monomials of the form $\alpha_{i,j} \wedge \alpha_{j,k}$ and $\alpha_{i,j} \wedge \alpha_{k,\ell}$. Thus, the FI–module $V_n = H^2(\mathbf{PB}_n; \mathbb{Q})$ is generated by the finite set of elements

$$\alpha_{1,2} \wedge \alpha_{2,3} \in V_3$$
$$\alpha_{1,2} \wedge \alpha_{3,4} \in V_4$$

in the sense that, if we hit these two elements with all our FI morphisms, we will recover a spanning set for V_n for all n. Since the generators appear in degrees $n = 3, 4$, we say that V is *finitely generated in degree* ≤ 4. And, Church–Ellenberg–Farb showed that this finite generation implies the multiplicity stability patterns we observed earlier.

The structure that drives multiplicity stability

Theorem 10 (Church–Ellenberg–Farb [CEF15])**.** *Let V be a finitely generated* FI*–module over $\mathbb{Q}$. Then the sequence $\{V_n\}$ of S_n–representations is multiplicity stable.*

The proof is harder than this, but (morally speaking) this theorem is a consequence of the following facts.

[1] Bonus exercise: Find an alternate action of FI that gives the sequence of rational group rings the structure of an FI–module. Note that this FI–module is necessarily not finitely generated.

1. We know that every finitely generated R–module is a quotient of a finite-rank free module. For the same reason, every finitely generated FI–module is a quotient of a finite sum of *representatble* FI–modules, functors of the form

$$[n] \longmapsto \mathbb{Q}\left[\mathrm{Hom}_{\mathsf{FI}}([m],[n])\right] \cong \mathbb{Q}\left[S_n/S_{n-m}\right] \qquad (m \text{ fixed}).$$

   These coset representations $\mathbb{Q}\left[S_n/S_{n-m}\right]$ are governed by the Littlewood–Richardson rules, and it is not difficult to check that for fixed m they are multiplicity stable as n grows.

2. FI–modules are *Noetherian* in the sense that submodules of finitely generated FI–modules are themselves finitely generated.

Remark 11. Given a finitely generated FI–module V,

- the *generation degree* of V constrains which irreducible S_n–representations can occur in V_n for $n \geq 0$,
- the *presentation degree* of V controls the stable range for its multiplicity stability. [2]

The FI–module perspective

This re-framing of the sequences studied by Church–Farb as FI–modules was a breakthrough for several reasons.

1. It gives a conceptual explanation and an easy-to-check criterion for multiplicity stability.

   For example, in the case of the pure braid group $\mathbf{PB}_n$, verifying that $\{H^q(\mathbf{PB}_n;\mathbb{Q})\}_n$ is finitely generated is much easier than the original direct combinatorial verification of multiplicty stability – and the pure braid group was the warm-up case.

2. The definition of a finitely generated FI–module makes sense for representations over the integers or other coefficients. It can be generalized to other groups or to maps with additional data. In general it makes sense even in situations where we multiplicity stability is not well-defined because, for example, the representations are not semisimple or the irreducible representations are not known.

Perhaps most importantly:

3. Benson and we his students learned what the algebraists had known all along: the power of working in an abelian category. In this framework we can draw on the tools of commutative or homological algebra to study our sequences of representations.

At this point, it became increasingly evident that there are close connections between the work of Church, Ellenberg, Farb, and Nagpal; the work of Djament, Pirashvili, Vespa, and others studying homological algebra and category theory; and work of Sam, Snowden, and others working in algebraic geometry and commutative algebra.

Current directions

Over the next 5 years, the field has taken several directions. Some goals of the field are:

- Exhibit representation stability phenomena in particular families of groups or spaces.

  Some applications that have been studied:

  - Congruence subgroups of linear groups. See (for example) work of Church, Ellenberg, Farb, Gan, Li, Miller, Nagpal, Patzt, Putman, Reinhold, Sam, and Wilson [Put15, CEFN14, CE17, PS17, GL17b, MPW17, CMNR17, MW17].

[2] Specifically, if we construct a resolution of V by "free" FI–modules (say, by FI♯–modules)

$$P^1 \longrightarrow P^0 \longrightarrow V \longrightarrow 0$$

then the generators of P^0 bound the *generation degree* of V and the generators of P^1 bound the *relation degree* of V. If P_0 is generated in degree $\leq g$ and P_1 is generated in degree $\leq r$, then V_n has *weight* $\leq g$, and the decomposition of V_n into irreducible representations stabilizes once $n \geq \max(g,r) + g$.

  - Complements of arrangements. See (for example) work of Berget, Bibby, Church, Casto, Ellenberg, Farb, Gadish, Rapp, and Wilson [CF13, CEF15, Wil14, Wil15, Bib16, Cas16, Ber16, Gad17b, Rap17].
  - Configuration spaces. See (for example) work of Arabia, Bahran, Church, Ellenberg, Farb, Hersh, Kupers, Lütgehetmann, Moseley, Nagpal, Palmer, Petersen, Proudfoot, Ramos, Reiner, Schiessl, Tosteson, Wilson, Wiltshire-Gordon, and Young [Chu12, CEF15, CEFN14, Pal13, KM15, EWG15, HR16, Ram16, Tos16, MW16, Ara16, Pet17, Ram17b, Lüt17, MPY17, CMNR17, Kra17, Bah18, Sch18].
  - Mapping class groups and moduli space. See (for example) work of Duque, Jiménez Rolland, and Tosteson [JR11, JR15, JRD15, Tos18].
  - Torelli groups. See (for example) work of Boldsen, Church, Day, Hauge Dollerup, Miller, Patzt, Putman, and Wilson [BD12, DP17, Pat16, CP15, MPW17].
  - Variations on the pure braid groups and related automorphisms groups, structures related to the combinatorics of graphs, and other examples. See (for example) work of Lee, Liu, Ramos, Saied, Szymik, Wilson, and White [Wil12, Lee13, Szy14, Sai15, Liu16, Ram17a, RW17]

- Develop analogous categories for actions by families of groups other than the symmetric group, or for sequences of symmetric group representations with additional structure.

  Some examples that have been studied are representations of wreath groups, classical Weyl groups, various linear groups, and products or decorated variants on FI. See (for example) work of Gan, Gadish, Miller, Patzt, Putman, Sam, Watterlond, Wilson, and Wu [Wil14, Wil15, SS14, PW16, GW16a, GW16b, Wat16, PS17, Gad17a, PSS17].

  A related generalization of FI–modules is via the theory of *twisted commutative algebras*. See for example work of Nagpal, Sam, and Snowden [SS12, SS16a, SS17a, SS17c, SS17d, NSS16b, NSS16a].

- Advance the algebraic theory of the category of FI–modules and its analogues.

  This may involve, for example, studying algebraic invariants of FI–modules.

  See (for example) work of Church, Ellenberg, Farb, Gan, Li, Miller, Nagpal, Patzt, Putman, Ramos, Reinhold, Sam, Snowden, Wilson, Wiltshire-Gordon, Xi, Yu [CF13, CEF15, CEFN14, WG14, GL14, WG15, GL15a, GL15b, GL15c, Li15, Ram15, Gan16, GL16, WG16, LR16, SS17b, CE17, LR17, GL17a, GLX17, Li17, LY17a, LY17b, Ram17c, Ram17d, NSS17, Pat17, CMNR17, MW17].

- Further develop the theory of *polynomial functors*.

  This area developed independently of the work of Church and Farb, and only later did the authors understand the connections between the work. See (for example) work of Collinet, Djament, Griffin, Pirashvili, Soulié, Vespa [DV10, Dja12, DV12, CDG12, DV13, HPV13, DPV16, Dja16, Sou17].

- Import tools from algebraic combinatorics or from the modular representation theory of the symmetric groups.

  See for example the work of Ashraf, Azam, Barter, Berceanu, Entova, Harman, Nagpal, Sam, and Snowden [AAB15, Har15, Nag15, SS16b, Har16, Har17, SS17e, BEAH17].

- Explore connections between representation stability results and objects in number theory.

  For instance, there is a relationship between stability results for the characters of finitely generated FI–modules, and point-counts on related varieties over finite fields.

  See (for example) the work of Casto, Chen, Church, Ellenberg, Farb, Fulman, Gadish, Howe, Hyde, Jiménez Rolland, Matei, Wilson, and Wolfson [CEF14, FW15, Ful16, Che16, How16a, How16b, Che17, JRW17a, FJRW17, JRW17b, Cas17a, Gad17c, Cas17b, Mat17, Hyd17].

What should "representation stability" mean?

The meaning of "representation stability" has evolved since the early work where it was a statement about patterns in the multiplicities of irreducible representations occuring in sequences of S_n–representations.

Arguably, the term should now refer to some sort of algebraic finiteness result for a module V over a suitable category – finiteness results like finite generation or relation degree, or the vanishing of some algebraic invariant like an associated functor homology group.

2 Quillen's methods in representation stability

For the second half of the talk I will change gears and describe a proof technique that has been useful in some representation stability applications.

This technique was first used in a representation stability context in work of Putman [Put15] to study congruence subgroups of $\mathrm{GL}_n(A)$, and has since been used by Church–Ellenberg [CE17], Putman–Sam [PS17], Miller–Patzt–Wilson [MPW17] and others to study families like congruence subgroups or Torelli groups.

For simplicity, I will describe the argument in the case of the pure braid group. I want to stress that this is not an efficient way to prove stability for the pure braid group, but merely intended to illustrate the argument – an argument which has the advantage that it adapts well to other categories and more complicated applications. The following theorem follows (for example) from work of Church–Ellenberg–Farb.

Theorem 12 (eg, Church–Ellenberg–Farb [CEF15])**.** *For each homological degree q, the sequence $\{H_q(\mathbf{PB}_n;\mathbb{Q})\}_n$ of S_n–representations is representation stable: it is finitely generated as an FI–module with generators in degree $\leq q$ and no relations.*

Proof. Straight-forward from Arnold's computation [Arn69]. □

The following method would allow us to prove a finite presentation result for $\{H_q(\mathbf{PB}_n;\mathbb{Q})\}$ even if we did not have an explicit computation of the cohomology groups. The proof can be used to give an explicit stable range, but for simplicity here I will not keep track of the bounds. We will prove:

Theorem 13. *For each homological degree q, the sequence of S_n–representations $\{H_q(\mathbf{PB}_n;\mathbb{Q})\}$ forms a finitely presented FI–module (with bounds on generation and relation degree that can be made explicit).*

Remark 14. The same proof should also hold when we replace $\mathbf{PB}_n$ with the pure mapping class group of a manifold with boundary and n marked points, or if we replace $\mathbf{PB}_n$ with the surface braid group of a surface with nonempty boundary. These results are due to Jiménez Rolland [JR15] or (eg) Church–Ellenberg–Farb [CEF15].

A key tool to the proof are the following functor homology groups.

Functor homology

Definition 15. Given an FI–module V, define an associated augmented chain complex of FI–modules by

$$\begin{aligned}\widetilde{C}_{-1}(V)_n &= V_n \\ \widetilde{C}_p(V)_n &= \bigoplus_{f:[p+1]\hookrightarrow[n]} V_{[n]\setminus \mathrm{im}(f)} \\ &\cong \mathrm{Ind}_{S_{n-(p+1)}}^{S_n} V_{n-(p+1)}\end{aligned}$$

with differential

$$\begin{aligned}d &: \widetilde{C}_p(V) \to \widetilde{C}_{p-1}(V)_n \\ d &= \sum_{i=1}^{p+1} (-1)^{i+1} d_i\end{aligned}$$

where

$$d_i : \bigoplus_{f:[p+1]\hookrightarrow[n]} V_{[n]\setminus \mathrm{im}(f)} \longrightarrow \bigoplus_{\overline{f}=f|_{[p+1]\setminus\{i\}}} V_{[n]\setminus \mathrm{im}(\overline{f})}$$

is defined by forgetting the element i from the domain of the injective map f, and using the maps

$$V_{[n]\setminus \mathrm{im}(f)} \longrightarrow V_{[n]\setminus \mathrm{im}(\overline{f})}$$

induced by the inclusion of sets

$$\Big([n]\setminus \mathrm{im}(f)\Big) \hookrightarrow \Big([n]\setminus \mathrm{im}(\overline{f})\Big) = \Big([n]\setminus \mathrm{im}(f)\Big) \cup \{f(i)\}.$$

The utility of this chain complex comes from the significance of its homology groups in degrees -1 and 0.[3] Consider the tail of the complex

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \widetilde{C}_0(V)_n & \longrightarrow & \widetilde{C}_{-1}(V)_n & \longrightarrow & 0. \\ & & \| & & \| & & \\ & & \mathrm{Ind}_{S_{n-1}}^{S_n} V_{n-1} & & V_{n-1} & & \end{array}$$

Hence,

$$\begin{aligned} & \widetilde{H}_{-1}(V)_n = 0 \quad \text{for } n > N \\ \iff \quad & \mathrm{Ind}_{S_{n-1}}^{S_n} V_{n-1} \longrightarrow V_{n-1} \quad \text{surjects for } n > N \\ \iff \quad & V \text{ is generated in degree} \leq N \end{aligned}$$

and thus the vanishing of $\widetilde{H}_{-1}(V)_n$ controls the generation degree of V. Similarly, it turns out that the vanishing of $\widetilde{H}_{-1}(V)_n$ and $\widetilde{H}_0(V)_n$ control the relation degree of V.

Fact 16. $\widetilde{H}_{-1}(V)_n = \widetilde{H}_0(V)_n = 0 \quad$ for $n >> 0 \quad \iff \quad V$ has small presentation degree

We note that it is possible to bound the degree of generators and relators explicitly in terms of the support of $\widetilde{H}_{-1}(V)_n$ and $\widetilde{H}_0(V)_n$ in n. Hence, to prove Theorem 13, it suffices to do the following.

Goal 17. Fix q, and let V be the FI–module $\{H_q(\mathbf{PB}_n; \mathbb{Q})\}_n$. Show that $\widetilde{H}_{-1}(V)_n$ and $\widetilde{H}_0(V)_n$ vanish for $n >> q$.

We will accomplish this goal as follows. From the short exact sequences

$$1 \longrightarrow \mathbf{PB}_n \longrightarrow \mathbf{B}_n \longrightarrow S_n \longrightarrow 1$$

we can construct a double complex for each n. The two associated spectral sequences both converge to the same limit. To prove that this limit vanishes in a range, we study the E^1 pages of the first spectral sequence. The statement that the groups $E^1_{p,q}$ vanish for small $p+q$ reduces to showing that a certain simplicial complex associated to the braid group is highly connected – and this result was proven by Hatcher–Wahl as part of a special case of a homological stability proof for mapping class groups.

Proposition 18 (Hatcher–Wahl [HW10])**.** *These spectral sequences converge to zero for $n >> p + q$.*

The second spectral sequence has E^2 page

$$E^2_{p,q}(n) = \widetilde{H}_p\Big(H_q(\mathbf{PB}_\bullet; \mathbb{Z})\Big)_n$$

In other words, the q^{th} row of this E^2 page is the functor homology groups of the FI–module $H_q(\mathbf{PB}_n; \mathbb{Z})$.

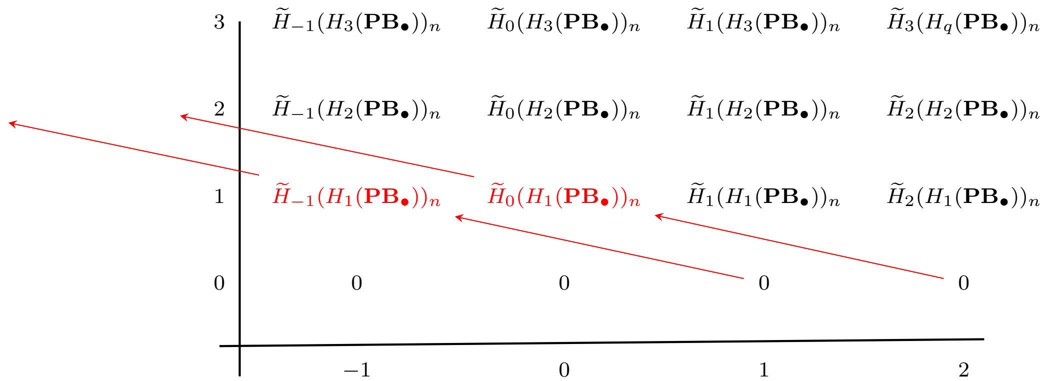
3	$\widetilde{H}_{-1}(H_3(\mathbf{PB}_\bullet))_n$	$\widetilde{H}_0(H_3(\mathbf{PB}_\bullet))_n$	$\widetilde{H}_1(H_3(\mathbf{PB}_\bullet))_n$	$\widetilde{H}_3(H_q(\mathbf{PB}_\bullet))_n$
2	$\widetilde{H}_{-1}(H_2(\mathbf{PB}_\bullet))_n$	$\widetilde{H}_0(H_2(\mathbf{PB}_\bullet))_n$	$\widetilde{H}_1(H_2(\mathbf{PB}_\bullet))_n$	$\widetilde{H}_2(H_2(\mathbf{PB}_\bullet))_n$
1	$\widetilde{H}_{-1}(H_1(\mathbf{PB}_\bullet))_n$	$\widetilde{H}_0(H_1(\mathbf{PB}_\bullet))_n$	$\widetilde{H}_1(H_1(\mathbf{PB}_\bullet))_n$	$\widetilde{H}_2(H_1(\mathbf{PB}_\bullet))_n$
0	$\widetilde{H}_{-1}(H_0(\mathbf{PB}_\bullet))_n$	$\widetilde{H}_0(H_0(\mathbf{PB}_\bullet))_n$	$\widetilde{H}_1(H_0(\mathbf{PB}_\bullet))_n$	$\widetilde{H}_2(H_0(\mathbf{PB}_\bullet))_n$
	-1	0	1	2

[3] Confusingly, the homology groups $\widetilde{H}_{-1}(V)$ corresponds to the functor FI–Mod $\to$ FI–Mod denoted by $H_0(V)$ by Church–Ellenberg–Farb [CEF15]. The indexing convention used here is natural if we view our chain complex as arising from a semi-simplicial object. Church–Ellenberg [CE17] and others study the derived functors of the functor $H_0(V)$, which are closely related but not the same as the homology groups $\widetilde{H}_p(V)$ defined here.

Since $H_0(\mathbf{PB}_n) = \mathbb{Z}$ for all n, the functor homology groups $\widetilde{H}_p(H_0(\mathbf{PB}_n))$ all vanish for $n >> p$; this follows from a classical result of Farmer [Far79] proving that a certain semi-simplicial set called the *complex of injective words* is highly connected.

Now, observe that once n is large enough, the first four terms in the bottom row $q = 0$ vanish, and there is no opportunity for nonzero differentials to or from the groups

$$E^2_{-1,1} = \widetilde{H}_{-1}\Big(H_1(\mathbf{PB}_\bullet)\Big)_n \qquad \text{or} \qquad E^2_{0,1} = \widetilde{H}_0\Big(H_1(\mathbf{PB}_\bullet)\Big)_n$$



Since (when n is large enough) the spectral sequence converges to zero at these two points $E^*_{-1,1}$ and $E^*_{0,1}$, we conclude that the groups $\widetilde{H}_{-1}(H_1(\mathbf{PB}_\bullet))_n$ and $\widetilde{H}_0(H_1(\mathbf{PB}_\bullet))_n$ must vanish for large n.

This proves the desired result for the case $q = 1$. Now, to propogate the argument, we can invoke the following fact.

Fact 19 (Variation on Putman [Put15] or Church–Ellenberg [CE17]; see (eg) Patzt [Pat17])**.** Let V be an FI–module. If $\widetilde{H}_{-1}(V)_n = \widetilde{H}_0(V)_n = 0$ for $n >> 0$, then $\widetilde{H}_p(V)_n = 0$ for $n >> p$.

Since we've deduced that $\widetilde{H}_{-1}(H_1(\mathbf{PB}_\bullet))_n$ and $\widetilde{H}_0(H_1(\mathbf{PB}_\bullet))_n$ vanish for n large, this fact implies that the second row $q = 1$ of the E^2 page must also vanish in a range[4]. We can therefore repeat the argument to conclude the vanishing of the groups $\widetilde{H}_{-1}\Big(H_2(\mathbf{PB}_\bullet)\Big)_n$ and $\widetilde{H}_0\Big(H_2(\mathbf{PB}_\bullet)\Big)_n$ vanish for n sufficiently large.

By induction on the row q, we can show that $\widetilde{H}_{-1}\Big(H_q(\mathbf{PB}_\bullet)\Big)_n$ and $\widetilde{H}_0\Big(H_q(\mathbf{PB}_\bullet)\Big)_n$ vanish for $n >> q$, which accomplishes Goal 17 and proves the theorem.

References

[AAB15] Samia Ashraf, Haniya Azam, and Barbu Berceanu, *Representation stability of power sets and square free polynomials*, Canadian Journal of Mathematics – Journal Canadien de Mathematiques **67** (2015), no. 5, 1024–1045.

[Ara16] Alberto Arabia, *Espaces de configuration généralisés. Espaces topologiques i-acycliques. Suites spectrales "basiques"*, arXiv preprint arXiv:1609.00522 (2016).

[Arn69] V. I. Arnol'd, *The cohomology ring of the colored braid group*, Mathematical Notes **5** (1969), no. 2, 138–140.

[Bah18] Cihan Bahran, *An improvement in the linear stable ranges for ordered configuration spaces*, arXiv preprint arXiv:1801.03302 (2018).

[BD12] Søren K Boldsen and Mia Hauge Dollerup, *Towards representation stability for the second homology of the Torelli group*, Geometry & Topology **16** (2012), no. 3, 1725–1765.

[BEAH17] Daniel Barter, Inna Entova-Aizenbud, and Thorsten Heidersdorf, *Deligne categories and representations of the infinite symmetric group*, arXiv preprint arXiv:1706.03645 (2017).

[4] Since the groups $V_n = H_q(\mathbf{PB}_n)$ actually form FI♯–modules, we have a stronger result that does not require Fact 19: the groups $\widetilde{H}_p(V)$ vanish in a range determined by $\widetilde{H}_{-1}(V)$. Specifically,

$$\text{if} \quad \widetilde{H}_{-1}(V)_n = 0 \quad \text{for } n > N \qquad \text{then} \qquad \widetilde{H}_p(V)_n = 0 \quad \text{for } n > p + 1 + N.$$

To illustrate the general Quillen argument, however, we will pretend that we are not aware of this extra structure on the FI–module $H_q(\mathbf{PB}_\bullet)$.

[Ber16] Andrew Berget, *Internal zonotopal algebras and the monomial reflection groups*, arXiv preprint arXiv:1611.06446 (2016).

[Bib16] Christin Bibby, *Representation stability for the cohomology of arrangements associated to root systems*, arXiv preprint arXiv:1603.08131, to appear in the Journal of Algebraic Combinatorics (2016).

[Cas16] Kevin Casto, FI_G*–modules, orbit configuration spaces, and complex reflection groups*, arXiv preprint arXiv:1608.06317 (2016).

[Cas17a] ______, FI_G*-modules and arithmetic statistics*, arXiv preprint arXiv:1703.07295 (2017).

[Cas17b] ______, *Representation stability and arithmetic statistics of spaces of 0-cycles*, arXiv preprint arXiv:1710.06850 (2017).

[CDG12] Gaël Collinet, Aurélien Djament, and James T Griffin, *Stabilité homologique pour les groupes dautomorphismes des produits libres*, International Mathematics Research Notices **2013** (2012), no. 19, 4451–4476.

[CE17] Thomas Church and Jordan Ellenberg, *Homology of* FI*–modules*, Geometry & Topology **21** (2017), no. 4, 2373–2418.

[CEF14] Thomas Church, Jordan Ellenberg, and Benson Farb, *Representation stability in cohomology and asymptotics for families of varieties over finite fields*, Contemporary Mathematics **620** (2014), 1–54.

[CEF15] Thomas Church, Jordan S Ellenberg, and Benson Farb, FI*–modules and stability for representations of symmetric groups*, Duke Mathematical Journal **164** (2015), no. 9, 1833–1910.

[CEFN14] Thomas Church, Jordan S Ellenberg, Benson Farb, and Rohit Nagpal, FI*–modules over Noetherian rings*, Geometry & Topology **18** (2014), no. 5, 2951–2984.

[CF13] Thomas Church and Benson Farb, *Representation theory and homological stability*, Advances in Mathematics **245** (2013), 250–314.

[Che16] Weiyan Chen, *Twisted cohomology of configuration spaces and spaces of maximal tori via point-counting*, arXiv preprint arXiv:1603.03931 (2016).

[Che17] ______, *Homology of braid groups, the Burau representation, and points on superelliptic curves over finite fields*, Israel Journal of Mathematics (2017), 1–24.

[Chu12] Thomas Church, *Homological stability for configuration spaces of manifolds*, Inventiones mathematicae **188** (2012), no. 2, 465–504.

[CMNR17] Thomas Church, Jeremy Miller, Rohit Nagpal, and Jens Reinhold, *Linear and quadratic ranges in representation stability*, arXiv preprint arXiv:1706.03845 (2017).

[CP15] Thomas Church and Andrew Putman, *Generating the Johnson filtration*, Geom. Topol. **19** (2015), no. 4, 2217–2255. MR 3375526

[Dja12] Aurélien Djament, *Sur l'homologie des groupes unitaires à coefficients polynomiaux*, Journal of K-theory **10** (2012), no. 1, 87–139.

[Dja16] ______, *Des propriétés de finitude des foncteurs polynomiaux*, Fundamenta Mathematicae **233** (2016), 197–256.

[DP17] Matthew Day and Andrew Putman, *On the second homology group of the Torelli subgroup of* $\mathrm{Aut}(fn)$, Geometry & Topology **21** (2017), no. 5, 2851–2896.

[DPV16] Aurélien Djament, Teimuraz Pirashvili, and Christine Vespa, *Cohomologie des foncteurs polynomiaux sur les groupes libres*, Documenta Mathematica **21** (2016), 205–222.

[DV10] Aurélien Djament and Christine Vespa, *Sur l'homologie des groupes orthogonaux et symplectiques à coefficients tordus*, Ann. Sci. Éc. Norm. Supér.(4) **43** (2010), no. 3, 395–459.

[DV12] ______, *Sur l'homologie des groupes d'automorphismes des groupes libres à coefficients polynomiaux*, arXiv preprint arXiv:1210.4030 (2012).

[DV13] ______, *Foncteurs faiblement polynomiaux*, International Mathematics Research Notices (2013).

[EWG15] Jordan S Ellenberg and John D Wiltshire-Gordon, *Algebraic structures on cohomology of configuration spaces of manifolds with flows*, arXiv preprint arXiv:1508.02430 (2015).

[Far79] Frank D. Farmer, *Cellular homology for posets*, Math. Japon. **23** (1978/79), no. 6, 607–613. MR 529895 (82k:18013)

[FJRW17] Jason Fulman, Rita Jiménez Rolland, and Jennifer C. H. Wilson, *Generating functions and statistics on spaces of maximal tori in classical Lie groups*, New York J. Math **23** (2017), 165–191.

[Ful16] Jason Fulman, *A generating function approach to counting theorems for square-free polynomials and maximal tori*, Annals of Combinatorics **20** (2016), no. 3, 587–599.

[FW15] Benson Farb and Jesse Wolfson, *Étale homological stability and arithmetic statistics*, arXiv preprint arXiv:1512.00415 (2015).

[Gad17a] Nir Gadish, *Categories of* FI *type: a unified approach to generalizing representation stability and character polynomials*, Journal of Algebra **480** (2017), 450–486.

[Gad17b] ______, *Representation stability for families of linear subspace arrangements*, Adv. Math. **322** (2017), 341–377. MR 3720801

[Gad17c] ______, *A trace formula for the distribution of rational G-orbits in ramified covers, adapted to representation stability*, New York J. Math. **23** (2017), 987–1011. MR 3690238

[Gan16] Wee Liang Gan, *A long exact sequence for homology of* FI*–modules*, New York J. Math **22** (2016), 1487–1502.

[GL14] Wee Liang Gan and Liping Li, *Koszulity of directed categories in representation stability theory*, arXiv preprint arXiv:1411.5308 (2014).

[GL15a] ______, *Coinduction functor in representation stability theory*, Journal of the London Mathematical Society **92** (2015), no. 3, 689–711.

[GL15b] ______, *Noetherian property of infinite EI categories*, New York J. Math **21** (2015), 369–382.

[GL15c] ______, *A remark on* FI*–module homology*, arXiv preprint arXiv:1505.01777 (2015).

[GL16] ———, *An inductive machinery for representations of categories with shift functors*, arXiv preprint arXiv:1610.09081 (2016).

[GL17a] ———, *Asymptotic behaviors of representations of graded categories with inductive functors*, arXiv preprint arXiv:1705.00882 (2017).

[GL17b] ———, *Linear stable range for homology of congruence subgroups via* FI*–modules*, arXiv preprint arXiv:1712.00890 (2017).

[GLX17] Wee Liang Gan, Liping Li, and Changchang Xi, *An application of Nakayama functor in representation stability theory*, arXiv preprint arXiv:1710.05493 (2017).

[GW16a] Wee Liang Gan and John Watterlond, *A representation stability theorem for* VI*–modules*, Algebras and Representation Theory (2016), 1–14.

[GW16b] ———, *Stable decompositions of certain representations of the finite general linear groups*, arXiv preprint arXiv:1605.08434 (2016).

[Har15] Nate Harman, *Stability and periodicity in the modular representation theory of symmetric groups*, arXiv preprint arXiv:1509.06414 (2015).

[Har16] ———, *Counting SET-free sets*, arXiv preprint arXiv:1604.07811, to appear in the Annals of Combinatorics (2016).

[Har17] ———, *Virtual Specht stability for* FI*–modules in positive characteristic*, Journal of Algebra (2017).

[How16a] Sean Howe, *Motivic random variables and representation stability I: Configuration spaces*, arXiv preprint arXiv:1610.05723 (2016).

[How16b] ———, *Motivic random variables and representation stability II: Hypersurface sections*, arXiv preprint arXiv:1610.05720 (2016).

[HPV13] Manfred Hartl, Teimuraz Pirashvili, and Christine Vespa, *Polynomial functors from algebras over a set-operad and nonlinear Mackey functors*, International Mathematics Research Notices **2015** (2013), no. 6, 1461–1554.

[HR16] Patricia Hersh and Victor Reiner, *Representation stability for cohomology of configuration spaces in* $\mathbb{R}^d$, International Mathematics Research Notices **2017** (2016), no. 5, 1433–1486.

[HW10] Allen Hatcher and Nathalie Wahl, *Stabilization for mapping class groups of 3-manifolds*, Duke Math. J. **155** (2010), no. 2, 205–269. MR 2736166 (2012c:57001)

[Hyd17] Trevor Hyde, *Factorization statistics and the twisted Grothendieck–Lefschetz formula*, arXiv preprint arXiv:1712.00071 (2017).

[JR11] Rita Jiménez Rolland, *Representation stability for the cohomology of the moduli space* $\mathcal{M}_g^n$, Algebraic & Geometric Topology **11** (2011), no. 5, 3011–3041.

[JR15] R Jiménez Rolland, *On the cohomology of pure mapping class groups as* FI*–modules*, Journal of Homotopy and Related Structures **10** (2015), no. 3, 401–424.

[JRD15] R Jiménez Rolland and J Maya Duque, *Representation stability for the pure cactus group*, Preprint (2015).

[JRW17a] Rita Jiménez Rolland and Jennifer C. H. Wilson, *Convergence criteria for* FI_W*–algebras and polynomial statistics on maximal tori in type B/C*, Advances in Mathematics **310** (2017), 121–158.

[JRW17b] ———, *Stability for hyperplane complements of type B/C and statistics on squarefree polynomials over finite fields*, arXiv preprint arXiv:1703.06349 (2017).

[KM15] Alexander Kupers and Jeremy Miller, *Representation stability for homotopy groups of configuration spaces*, Journal für die reine und angewandte Mathematik (Crelles Journal) (2015).

[Kra17] Manuel Krannich, *Homological stability of topological moduli spaces*, arXiv preprint arXiv:1710.08484 (2017).

[Lee13] Peter Lee, *On the action of the symmetric group on the cohomology of groups related to (virtual) braids*, arXiv preprint arXiv:1304.4645 (2013).

[Li15] Liping Li, *Upper bounds of homological invariants of* FI_G*–modules*, arXiv preprint arXiv:1512.05879 (2015).

[Li17] ———, *Homological degrees of representations of categories with shift functors*, Transactions of the American Mathematical Society (2017).

[Liu16] Ye Liu, *On chromatic functors and stable partitions of graphs*, arXiv preprint arXiv:1603.08310 (2016).

[LR16] Liping Li and Eric Ramos, *Depth and the local cohomology of* FI_G*–modules*, arXiv preprint arXiv:1602.04405 (2016).

[LR17] ———, *Local cohomology and the multi-graded regularity of* FI^m*–modules*, arXiv preprint arXiv:1711.07964 (2017).

[Lüt17] Daniel Lütgehetmann, *Representation stability for configuration spaces of graphs*, arXiv preprint arXiv:1701.03490 (2017).

[LY17a] Liping Li and Nina Yu, *Filtrations and homological degrees of* FI*–modules*, Journal of Algebra **472** (2017), 369–398.

[LY17b] ———, FI^m*–modules over noetherian rings*, arXiv preprint arXiv:1705.00876 (2017).

[Mat17] Vlad Matei, *A geometric perspective on landau's problem over function fields*, arXiv preprint arXiv:1705.10727 (2017).

[MPW17] Jeremy Miller, Peter Patzt, and Jennifer C. H. Wilson, *Central stability for the homology of congruence subgroups and the second homology of torelli groups*, arXiv preprint arXiv:1704.04449 (2017).

[MPY17] Daniel Moseley, Nicholas Proudfoot, and Ben Young, *The Orlik–Terao algebra and the cohomology of configuration space*, Experimental Mathematics **26** (2017), no. 3, 373–380.

[MW16] Jeremy Miller and Jennifer C. H. Wilson, *Higher order representation stability and ordered configuration spaces of manifolds*, arXiv preprint arXiv:1611.01920 (2016).

[MW17] Jeremy Miller and Jennifer CH Wilson, *Quantitative representation stability over linear groups*, arXiv preprint arXiv:1709.03638 (2017).

[Nag15] Rohit Nagpal, *Fi-modules and the cohomology of modular sn-representations*, arXiv preprint arXiv:1505.04294 (2015).

[NSS16a] Rohit Nagpal, Steven V Sam, and Andrew Snowden, *Noetherianity of some degree two twisted commutative algebras*, Selecta Mathematica **22** (2016), no. 2, 913–937.

[NSS16b] ______, *Noetherianity of some degree two twisted skew-commutative algebras*, arXiv preprint arXiv:1610.01078 (2016).

[NSS17] ______, *Regularity of* FI*–modules and local cohomology*, arXiv preprint arXiv:1703.06832 (2017).

[Pal13] Martin Palmer, *Twisted homological stability for configuration spaces*, arXiv preprint arXiv:1308.4397 (2013).

[Pat16] Peter Patzt, *Representation stability for filtrations of torelli groups*, arXiv preprint arXiv:1608.06507 (2016).

[Pat17] ______, *Central stability homology*, arXiv preprint arXiv:1704.04128 (2017).

[Pet17] Dan Petersen, *A spectral sequence for stratified spaces and configuration spaces of points*, Geometry & Topology **21** (2017), no. 4, 2527–2555.

[PS17] Andrew Putman and Steven V Sam, *Representation stability and finite linear groups*, Duke Mathematical Journal **166** (2017), no. 13, 2521–2598.

[PSS17] Andrew Putman, Steven V Sam, and Andrew Snowden, *Stability in the homology of unipotent groups*, arXiv preprint arXiv:1711.11080 (2017).

[Put15] Andrew Putman, *Stability in the homology of congruence subgroups*, Inventiones mathematicae **202** (2015), no. 3, 987–1027.

[PW16] Peter Patzt and Xiaolei Wu, *Stability results for houghton groups*, Algebraic & Geometric Topology **16** (2016), no. 4, 2365–2377.

[Ram15] Eric Ramos, *Homological invariants of* FI*-modules and* FI_G*-modules*, arXiv preprint arXiv:1511.03964 (2015).

[Ram16] ______, *Stability phenomena in the homology of tree braid groups*, arXiv preprint arXiv:1609.05611 (2016).

[Ram17a] ______, *Asymptotic behaviors in the homology of symmetric group and finite general linear group quandles*, arXiv preprint arXiv:1706.02809 (2017).

[Ram17b] ______, *Configuration spaces of graphs with certain permitted collisions*, arXiv preprint arXiv:1703.05535 (2017).

[Ram17c] ______, *Generalized representation stability and* FI_d*–modules*, Proceedings of the American Mathematical Society **145** (2017), no. 11, 4647–4660.

[Ram17d] ______, *On the degree-wise coherence of* FI_G*–modules.*, New York Journal of Mathematics **23** (2017).

[Rap17] Artur Rapp, *Representation stability on the cohomology of complements of subspace arrangements*, arXiv preprint arXiv:1711.08924 (2017).

[RW17] Eric Ramos and Graham White, *Families of nested graphs with compatible symmetric-group actions*, arXiv preprint arXiv:1711.07456 (2017).

[Sai15] Amin Saied, *On the* FI*–module structure of* $h^i(\gamma_{n,s})$, arXiv preprint arXiv:1506.05861 (2015).

[Sch18] Christoph Schiessl, *Integral cohomology of configuration spaces of the sphere*, arXiv preprint arXiv:1801.04273 (2018).

[Sou17] Arthur Soulié, *The Long–Moody construction and polynomial functors*, arXiv preprint arXiv:1702.08279 (2017).

[SS12] Steven V Sam and Andrew Snowden, *Introduction to twisted commutative algebras*, arXiv preprint arXiv:1209.5122 (2012).

[SS14] ______, *Representations of categories of G–maps*, Journal für die reine und angewandte Mathematik (Crelles Journal) (2014).

[SS16a] Steven Sam and Andrew Snowden, *Gl-equivariant modules over polynomial rings in infinitely many variables*, Transactions of the American Mathematical Society **368** (2016), no. 2, 1097–1158.

[SS16b] Steven V Sam and Andrew Snowden, *Proof of Stembridges conjecture on stability of Kronecker coefficients*, Journal of Algebraic Combinatorics **43** (2016), no. 1, 1–10.

[SS17a] ______, *Gl-equivariant modules over polynomial rings in infinitely many variables, ii*, arXiv preprint arXiv:1703.04516 (2017).

[SS17b] Steven V. Sam and Andrew Snowden, *Gröbner methods for representations of combinatorial categories*, J. Amer. Math. Soc. **30** (2017), no. 1, 159–203. MR 3556290

[SS17c] Steven V Sam and Andrew Snowden, *Hilbert series for twisted commutative algebras*, arXiv preprint arXiv:1705.10718 (2017).

[SS17d] ______, *Regularity bounds for twisted commutative algebras*, arXiv preprint arXiv:1704.01630 (2017).

[SS17e] ______, *Some generalizations of Schur functors*, arXiv preprint arXiv:1708.06410 (2017).

[Szy14] Markus Szymik, *Twisted homological stability for extensions and automorphism groups of free nilpotent groups*, Journal of K-theory **14** (2014), no. 1, 185–201.

[Tos16] Philip Tosteson, *Lattice spectral sequences and cohomology of configuration spaces*, arXiv preprint arXiv:1612.06034 (2016).

[Tos18] ______, *Stability in the homology of Deligne–Mumford compactifications*, arXiv preprint arXiv:1801.03894 (2018).

[Wat16] John Patrick Watterlond, *Representation stability of consistent sequences of* $\mathrm{GL}(n, \mathbb{F}_q)$*–modules*, Ph.D. thesis, 2016, Thesis (Ph.D.)–University of California, Riverside, p. 85. MR 3653633

[WG14] John D Wiltshire-Gordon, *Uniformly presented vector spaces*, arXiv preprint arXiv:1406.0786 (2014).

[WG15] ______, *Categories of dimension zero*, arXiv preprint arXiv:1508.04107 (2015).

[WG16] ______, *On computing the eventual behavior of a finitely presented* FI*–module.*

[Wil12] Jennifer C. H. Wilson, *Representation stability for the cohomology of the pure string motion groups*, Algebraic & Geometric Topology **12** (2012), no. 2, 909–931.

[Wil14] ______, FI_W*–modules and stability criteria for representations of classical Weyl groups*, Journal of Algebra **420** (2014), 269–332.

[Wil15] ______, FI_W*–modules and constraints on classical Weyl group characters*, Mathematische Zeitschrift **281** (2015), no. 1-2, 1–42.