Introduction to Stable Homotopy Theory

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We say that a phenomenon is "stable" if it can occur in any dimension, or in any sufficiently large dimension, and if it occurs in essentially the same way independent of dimension, provided, perhaps, that the dimension is sufficiently large.

- The Honorable Rev. John F. Adams

Introduction

The following theorem gives the canonical example of a stable phenomenon.

Theorem (Freudenthal, 1938)**.** *If X has dimension d and Y is $(n-1)$-connected, then $[X,Y] \to [\Sigma X, \Sigma Y]$ is an isomorphism when $d \leq 2n-2$.*

Corollary. *When X and Y are finite, the sequence $[X,Y] \to [\Sigma X, \Sigma Y] \to \cdots$ stabilizes.*

Corollary. *We can define the stable homotopy groups of spheres $\pi_n^S = \mathrm{colim}_k \pi_{n+k}(S^n)$.*

Duality

Suppose we have $X \hookrightarrow S^n$. Then Alexander duality gives us that $\tilde{H}^r(X) \simeq \tilde{H}_{n-r-1}(X^c)$, where X^c denotes the complement. Of course we can have different embeddings $X \hookrightarrow S^n$, but it turns out that the complement of X is nevertheless *stably* determined by X.

Definition 1. If X and Y are finite complexes, we say that X and Y are *stably homotopy equivalent* and write $X \sim Y$ if there is some m such that $\Sigma^m X \simeq \Sigma^m Y$. We define the *stable mapping space* by $\{X,Y\} = \mathrm{colim}[\Sigma^k X, \Sigma^k Y]$.

Theorem (Spanier-Whitehead, 1953)**.** *If $X, Y \subseteq S^n$ are stably homotopy equivalent, then so are X^c and Y^c.*

Proof. Since we have $X \subseteq S^n$ we have $\Sigma X \subseteq S^{n+1}$. So without loss of generality, we can say that $f : X \to Y$ is a homotopy equivalence. Then we embed $M_f \subseteq S^n * S^n$ (the *join*). Now we have $X \hookrightarrow M_f \hookleftarrow Y$, and this gives us the homotopy equivalences $S^{2n+1} \backslash X \leftarrow S^{2n+1} \backslash M_f \to S^{2n+1} \backslash Y$. □

The Hopf Invariant Problem

We can often turn geometric problems into problems in stable homotopy theory, which gives us a rich set of tools to attack our problems. For example, the *Hopf invariant problem* asks: For which n do we have $S^{2n-1} \to S^n$ with Hopf invariant 1? This ends up requiring that in the the mapping cone $X = S^n \cup e^{2n}$, we have that $Sq^n : H^n X \to H^{2n} X$ is nontrivial. The Adem relations then immediately give us that we need $n = 2^t$. In fact, we can make this construction more generally; since the Steenrod squares are stable, then geometric questions about homotopy groups turn into stable ones. More broadly, many of these questions can be rephrased as questions about whether certain classes in the E_2 page of the Adams spectral sequence are permanent.

Cobordism

We say that two k-manifolds M and N are *cobordant* if there is some compact $(k+1)$-manifold W such that $\partial W = M \coprod N$; in this case we write $M \equiv N$. We denote the equivalence classes $\mathcal{N}_k$; these form a group under disjoint union, and the graded group $\mathcal{N}_*$ becomes a ring under Cartesian product.

Theorem (Thom, 1954)**.** *There is a sequence of spaces $MO(n)$ and maps $\Sigma MO(n) \to MO(n+1)$ such that $\mathcal{N}_k \xrightarrow{\simeq} \mathrm{colim}_n \pi_{n+k}(MO(n)) =: \pi_k(MO)$. It works out that $\mathcal{N}_* = \pi_*(MO) = \mathbb{Z}/2[x_i : i \neq 2^s - 1] = \mathbb{Z}/2[x_2, x_4, x_5, x_6, x_8, \ldots]$.*

The construction is very pretty. Suppose we have an embedding $M^k \hookrightarrow \mathbb{R}^{n+k} \hookrightarrow \mathbb{R}^{n+k} \cup \infty = S^{n+k}$. Let N be a tubular neighborhood of M. Then we have the *Thom collapse map* $S^{n+k} \to N/\partial N \cong T(\nu)$, where ν is the normal bundle of the embedding $M^k \hookrightarrow \mathbb{R}^{n+k}$. But not that we can consider this as $\nu : M \to BO(n)$, and so we get a map on Thom spaces $T(\nu) \to T(\gamma_n) =: MO(n)$. This ends up being well-defined since different embeddings of M are isotopic in sufficiently large Euclidean spaces, and we obtain from M an element of $\pi_*(MO)$.

Spectra

Of course, things would be much easier if we could treat the $MO(n)$ as a single object. So, we define a "spectrum" (which is in quotes because we'll soon see fancier, more high-tech definitions).

Definition 2. A *spectrum* is a sequence of spaces $\{E_n\}$ together with maps $\Sigma E_n \to E_{n+1}$.

We won't say exactly how maps are constructed here, but the point is that built into the morphisms of this category is the idea that they don't need to be defined until some arbitrarily high suspension.

When spectra?

Let K be a (reduced) (generalized) cohomology theory. We have the following beautiful theorem.

Theorem (Brown, 1962)**.** *There is a spectrum $\{E_n\}$ such that $K^n(X) = [X, E_n]$.*

In this setup, the coboundary map in the long exact sequence and excision give the diagram

$$\begin{array}{ccc} K^n(X) & \xrightarrow[\simeq]{\delta} & K^{n+1}(CX, X) \\ & \searrow^{\simeq} & \downarrow \simeq \\ & & K^{n+1}(\Sigma X). \end{array}$$

Representability gives us that these are $[X, E_n]$, $[\Sigma X, E_{n+1}]$, and $[X, \Omega E_{n+1}]$; this tells us that the adjoint map $E_n \to \Omega E_{n+1}$ must be a weak equivalence.

So to a cohomology theory we can associated a spectrum. Conversely, to a spectrum E we can define the cohomology theory via $E^k(X) := \mathrm{colim}_n[\Sigma^{n+k} X, E_n]$. (If E came from a cohomology theory, then $[\Sigma^{n-k} X, E_n] \simeq [X, \Omega^{n-k} E_n] \simeq [X, E_k]$. So this is good.)

We want a category that allows us to study both of these phenomena at once: homotopy theory of spaces and cohomology theories.

<u>What should the stable homotopy category SH look like?</u>

1. We should have CW-spectra. More explicitly, we should be able to attach an n-cell by taking the cofiber of a map $CS^n \to X$. This is important so that we can carry over CW approximation from the classical setting, so that instead of having to invert weak equivalences we can just look at CW replacements. Then weak homotopy equivalences are necessarily homotopy equivalences.
2. We want our category to be additive: it should be that $X \vee Y \simeq X \times Y$.
3. If we take a finite complex X and look at its suspension spectrum $\Sigma^\infty X$, our definitions should yield that $\pi_k \Sigma^\infty X = \mathrm{colim}_n \pi_{n+k} \Sigma^n X$. This gives us a faithful embedding of spaces into spectra.
4. The suspension functor $\Sigma : SH \to SH$ should be an equivalence. (Since this is adjoint to looping, this will give that taking loops is also an equivalence.)
5. We want Brown representability internal to the category of spectra.
6. Cohomology theories often come with more structure than just graded groups, e.g. cup products. We'd like this structure to be witnessed by a map of spectra. For example, $E^n X \otimes E^m X \to E^{m+n} X$ should come from a map of spectra $EE \to E$.

The good news is that we can do this! The bad news is that it's quite difficult to get a sufficiently nice construction that one can work with, and in fact there's a theorem to the effect that no category of spectra can have a smash product with all the properties we'd like.