# Linear Logic and Linear Algebra

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(work in progress!)

# (Intuitionistic) Linear Logic

| | | | |
|---|---|---|---|
| $A, B$ | $::=$ | $0$ | *additive sum unit* |
| | $\mid$ | $1$ | *multiplicative product unit* |
| | $\mid$ | $\top$ | *additive product unit* |
| | $\mid$ | $\bot$ | *multiplicative sum unit* |
| | $\mid$ | $A \oplus B$ | *additive sum* |
| | $\mid$ | $A \& B$ | *additive product* |
| | $\mid$ | $A \otimes B$ | *multiplicative product* |
| | $\mid$ | $A \multimap B$ | *linear implication* |
| | $\mid$ | $!A$ | *exponential* |
| $\Gamma$ | $::=$ | $A$ | *contexts* |
| | $\mid$ | $\Gamma \otimes \Gamma$ | |
| | | $\Gamma \vdash A$ | *judgments* |

# Denotational (Categorical) Models

Basic idea:

- Interpret each type $A$ as some structure $[\![A]\!]$
- Interpret each judgement $\Gamma \vdash A$ as a morphism

$$[\![\Gamma \vdash A]\!] : [\![\Gamma]\!] \to [\![A]\!]$$

- Interpret inference rules compositionally

Interpretations should "respect" proof equivalences, *e.g.*:

$$\left[\!\!\left[ \frac{\overline{A \vdash A} \quad \overline{B \vdash B}}{A \otimes B \vdash A \otimes B} \right]\!\!\right] = [\![\overline{A \otimes B \vdash A \otimes B}]\!]$$

# Many Models of Linear Logic

(Fairly?) Simple:

- Sets and Relations

$$
\begin{aligned}
[\![0]\!] &= \emptyset \\
[\![1]\!] &= \{\bullet\} \\
[\![A \oplus B]\!] &= [\![A]\!] \uplus [\![B]\!] \\
&\dots \\
[\![A \vdash A]\!] &= \{(x, x) \mid x \in [\![A]\!]\} \\
[\![A \vdash A \oplus B]\!] &= \{(x, \mathrm{inl}\ x) \mid x \in [\![A]\!]\} \\
&\dots
\end{aligned}
$$

(Fairly?) Complex:

- Coherence Spaces, Proof Nets, Game Semantics

# Linear Logic and Linear Algebra

FinVect:

- Interpret a type as a *finite dimensional vector space* (over a *finite* field)
- Interpret a judgment as a *linear transformation* (*i.e.*, a matrix)

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Why?

- Next simplest reasonable model (after Set).
- I haven't seen this worked out in detail anywhere before.
- There are lots of interesting things that live in the category FinVect:
  - All of linear algebra: Matrix algebra, derivatives, eigenvectors, Fourier transforms, cryptography(?), *etc.*

# Linear Algebra

# Fields

A *field* $\mathbb{F} = (F, +, \cdot, 0, 1)$ is a structure such that:

- $F$ is a set containing distinct elements 0 and 1.
- *Addition*: $(F, +, 0)$ abelian group, identity 0
- *Multiplication*: $(F - \{0\}, \cdot, 1)$: abelian group, identity 1
- The *distributive law* holds:

$$\forall \alpha, \beta, \gamma \in F.\ \alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$$

- There are *no zero divisors*:

$$\forall \alpha, \beta \in F.\ \alpha \cdot \beta = 0 \implies \alpha = 0 \lor \beta = 0$$

# Vector Spaces

A vector space over $\mathbb{F}$ is just a set $V$ with addition and scalar multiplication:

$$\forall v, w \in V.\ (v + w) \in V$$

$$\forall \alpha \in \mathbb{F}.\ \forall v \in V.\ \alpha v \in V$$

Satisfying some laws:

- $(V, +, 0)$ form an abelian group
- $\alpha(v + w) = \alpha v + \alpha w$
- $(\alpha + \beta)v = \alpha v + \beta v$

# Functional Vector Spaces $\mathbb{F}^X$

Pick a *coordinate system* (*i.e.* a set X) and define $\mathbb{F}^X$, the "vector space with coordinates in X":

$$\mathbb{F}^X \triangleq \{v \mid v : X \to \mathbb{F}\}$$

- A vector is just a function that maps each coordinate to an element of $\mathbb{F}$
  - Example: In the plane, we might pick $X = \{\text{“x”}, \text{“y”}\}$
- Vector addition and scalar multiplication are defined *pointwise*
- The *dimension* of $\mathbb{F}^X$ is just the cardinality of $X$.

# Canonical Basis

Canonical basis for $\mathbb{F}^X$:

$$\{\delta_x \mid x \in X\}$$

- Here $\delta_x$ is the vector:

$$\delta_x[y] \;=\; \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

- Every vector in $\mathbb{F}^X$ can be written as a weighted sum of basis elements.

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3 \cdot \delta_x + 4 \cdot \delta_y$$

# Linear Maps

A linear transformation $f : \mathbb{F}^X \rightarrow \mathbb{F}^Y$ is a function such that:

$$f(\alpha\, v + \beta\, w) = \alpha\, f(v) + \beta\, f(w)$$

$f$ is completely characterized by its behavior on the set of basis vectors $\delta_x$.

$$f(\delta_x) = \sum_{y \in Y} M_f[y, x]\delta_y$$

Here: $M_f[y, x]$ is a (matrix) of scalars in $\mathbb{F}$

# Matrices

If $\mathbb{F}^X$ has $n$ coordinates and $\mathbb{F}^Y$ has $m$ coordinates, then any linear map $f : \mathbb{F}^X \to \mathbb{F}^Y$ can be represented as a matrix:

$$\begin{bmatrix} f[y_1, x_1] & f[y_1, x_2] & \cdots & f[y_1, x_n] \\ f[y_2, x_1] & f[y_2, x_2] & \cdots & f[y_2, x_n] \\ \vdots & \vdots & \ddots & \vdots \\ f[y_m, x_1] & f[y_m, x_2] & \cdots & f[y_m, x_n] \end{bmatrix}$$

For example, the 3x3 *identity* map:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \bullet & \cdot & \cdot \\ \cdot & \bullet & \cdot \\ \cdot & \cdot & \bullet \end{bmatrix}$$

# Linear Logic

# Multiplicative Unit: 1

Interpret 1 as a vector space:

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Interpret 1 as a vector space:

- Coordinates: $1^\dagger = \{\bullet\}$
- $[\![1]\!] = \mathbb{F}^{1^\dagger}$ $\qquad (= \{v \mid v : 1^\dagger \to \mathbb{F}\})$

Interpret the "1 introduction" inference rule as the 1x1 identity matrix:

$$[\![1 \vdash 1]\!] = [1]$$

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$$\frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1 \otimes \Gamma_2 \vdash A \otimes B}$$

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$$\frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1 \otimes \Gamma_2 \vdash A \otimes B} \qquad \frac{f : [\![\Gamma_1]\!] \to [\![A]\!] \quad g : [\![\Gamma_2]\!] \to [\![B]\!]}{f \otimes g : [\![\Gamma_1 \otimes \Gamma_2]\!] \to [\![A \otimes B]\!]}$$

$$(f \otimes g)[(a, b), (x, y)] = f[a, x] \cdot g[b, y]$$

# Multiplicative Product: Examples

$$f = \begin{bmatrix} \bullet & \cdot & \bullet \\ \cdot & \bullet & \cdot \\ \cdot & \bullet & \cdot \end{bmatrix} \qquad g = \begin{bmatrix} \bullet & \cdot \\ \bullet & \bullet \end{bmatrix}$$

$f \otimes g$ $\qquad$ $g \otimes f$

# Multiplicative Product: Examples

$$f = \begin{bmatrix} \bullet & \cdot & \bullet \\ \cdot & \bullet & \cdot \\ \cdot & \bullet & \cdot \end{bmatrix} \qquad g = \begin{bmatrix} \bullet & \cdot \\ \bullet & \bullet \end{bmatrix}$$

$$f \otimes g = \begin{bmatrix} \bullet & \cdot & \cdot & \cdot & \bullet & \cdot \\ \bullet & \bullet & \cdot & \cdot & \bullet & \bullet \\ \cdot & \cdot & \bullet & \cdot & \cdot & \cdot \\ \cdot & \cdot & \bullet & \bullet & \cdot & \cdot \\ \cdot & \cdot & \bullet & \cdot & \cdot & \cdot \\ \cdot & \cdot & \bullet & \bullet & \cdot & \cdot \end{bmatrix} \qquad g \otimes f = \begin{bmatrix} \bullet & \cdot & \bullet & \cdot & \cdot & \cdot \\ \cdot & \bullet & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bullet & \cdot & \cdot & \cdot & \cdot \\ \bullet & \cdot & \bullet & \bullet & \cdot & \bullet \\ \cdot & \bullet & \cdot & \cdot & \bullet & \cdot \\ \cdot & \bullet & \cdot & \cdot & \bullet & \cdot \end{bmatrix}$$

# Multiplicative Product: Structural Rules

Contexts:

$$\Gamma ::= A \mid \Gamma \otimes \Gamma$$

Structural Rule:

$$\frac{\Gamma_1 \vdash A \quad \Gamma_1 \equiv \Gamma_2}{\Gamma_2 \vdash A}$$

$\Gamma_1 \equiv \Gamma_2$

- reflexivity, symmetry, transitivity
- associativity: $(\Gamma_1 \otimes \Gamma_2) \otimes \Gamma_3 \equiv \Gamma_1 \otimes (\Gamma_2 \otimes \Gamma_3)$
- unit law: $\Gamma \equiv \Gamma \otimes 1$
- commutativity: $\Gamma_1 \otimes \Gamma_2 \equiv \Gamma_2 \otimes \Gamma_1$
- $[\![\Gamma_1 \equiv \Gamma_2]\!]$ is an isomorphism

# Function Composition

## Function Composition

Given $f : \mathbb{F}^X \to \mathbb{F}^Z$ and $g : \mathbb{F}^Z \to \mathbb{F}^Y$, define

$$(f;g)[y,x] = \sum_{z \in Z} g[y,z] \cdot f[z,x]$$

(a.k.a. matrix multiplication)

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Note: We *sum* over all elements of $Z$, so this is *not necessarily defined* if $Z$ is infinite!

- Option 1: Allow infinite vectors but only those with "finite support" (zero almost everywhere)
  $\implies$ Ehrhard's Finiteness spaces
- Option 2: Work with only finite matrices.
  $\implies$ How to ensure that $!A$ remains finite?

# Identity and Cut

Identity:

$$\frac{}{A \vdash A}$$

$$id_A[y,x] \;=\; \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

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Identity:

$$\overline{A \vdash A}$$

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Cut:

$$\frac{\Gamma_1 \vdash A \quad A \otimes \Gamma_2 \vdash B}{\Gamma_1 \otimes \Gamma_2 \vdash B}$$

$$\frac{f : [\![\Gamma_1]\!] \to [\![A]\!] \quad g : [\![A \otimes \Gamma_2]\!] \to [\![B]\!]}{(f \otimes id_{\Gamma_2}); g : [\![\Gamma_1 \otimes \Gamma_2]\!] \to [\![A \otimes B]\!]}$$

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Interpret $\oplus$ introduction:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}$$

$$\mathrm{inl}\ _{A,B}[y, x] \ = \ \begin{cases} 1 & \text{if } y = \mathrm{inl}\ x \\ 0 & \text{otherwise} \end{cases}$$

# Additive Sums

Booleans (over $\mathbb{F}_2$):

$$\mathbb{B} = 1 \oplus 1$$

$$\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \quad \begin{bmatrix} \bullet \\ \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot \\ \bullet \end{bmatrix} \quad \begin{bmatrix} \bullet \\ \bullet \end{bmatrix}$$

$$\mathrm{inl}_{\mathbb{B},\mathbb{B}} : [\![\mathbb{B}]\!] \to [\![\mathbb{B}]\!] \oplus [\![\mathbb{B}]\!] \qquad \mathrm{inr}_{\mathbb{B},\mathbb{B}} : [\![\mathbb{B}]\!] \to [\![\mathbb{B}]\!] \oplus [\![\mathbb{B}]\!]$$

$$\begin{bmatrix} \bullet & \cdot \\ \cdot & \bullet \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \qquad \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \bullet & \cdot \\ \cdot & \bullet \end{bmatrix}$$

# Exponential Types

# Linear Logic: Exponentials

### Dereliction

$$\frac{\Gamma \otimes A \vdash B}{\Gamma \otimes !A \vdash B}$$

### Weakening

$$\frac{\Gamma \otimes 1 \vdash B}{\Gamma \otimes !A \vdash B}$$

### Contraction

$$\frac{\Gamma \otimes (!A \otimes !A) \vdash B}{\Gamma \otimes !A \vdash B}$$

### Introduction

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}$$

# ! is a Comonad

- ! is a functor:
  - On types: for vector space $[\![A]\!]$, need a vector space $![\![A]\!]$
  - On functions: For $f : [\![A]\!] \to [\![B]\!]$, need $!f : ![\![A]\!] \to ![\![B]\!]$

$$\mathrm{coreturn}_A : ![\![A]\!] \to [\![A]\!]$$

$$\mathrm{comultiply}_A : ![\![A]\!] \to !![\![A]\!]$$

- Satisfying the comonad laws.
- Plus some other operations: $m : !A \otimes !B \to !(A \otimes B)$

# Defining !

For objects: interpret $!A$ as a vector space:

- Coordinates: $(!A)^\dagger = [\![A]\!]$
- $[\![!A]\!] = \mathbb{F}^{(!A)^\dagger}$
- The canonical basis for $[\![!A]\!]$ is $\{\delta_v \mid v \in [\![A]\!]\}$.

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For functions: suppose $f : A \to B$ then:

$$(!f)(\delta_v) = \delta_{f(v)}$$

# Finite Fields

A field $\mathbb{F}$ is finite if $|F|$ is finite.

Some beautiful theorems:

- Every finite field $\mathbb{F}_q$ with $q$ elements has $q = p^k$, where $p$ is a prime.
- For every element $\alpha \in \mathbb{F}_q$ we have:
  - $\underbrace{\alpha + \alpha + \ldots + \alpha}_{\text{p times}} = 0$
  - $\alpha^q = \alpha$

# Comonadic structure

- $\mathrm{coreturn}_A : !\llbracket A \rrbracket \to \llbracket A \rrbracket$

$$\mathrm{coreturn}_A(\delta_v) = v$$

- $\mathrm{comultiply}_A : !\llbracket A \rrbracket \to !!\llbracket A \rrbracket$

$$\mathrm{comultiply}_A(\delta_v) = \delta_{\delta_v}$$

# Back to the Comonad: Coreturn

Example: $\mathrm{coreturn}_{\mathbb{B}} : [\![!\mathbb{B}]\!] \to \mathbb{B}$ over $\mathbb{F}_2$

$$\begin{bmatrix} \cdot & \bullet & \cdot & \bullet \\ \cdot & \cdot & \bullet & \bullet \end{bmatrix}$$

More generally: The $n^{th}$ column of the matrix is just $n$ written in base $q$

# Dimensionality

$$\begin{aligned}
\dim[0] &= 0 \\
\dim[\top] &= 0 \\
\dim[1] &= 1 \\
\dim[\bot] &= 1 \\
\dim[A \oplus B] &= \dim[A] + \dim[B] \\
\dim[A \mathbin{\&} B] &= \dim[A] + \dim[B] \\
\dim[A \otimes B] &= \dim[A] \times \dim[B] \\
\dim[A \multimap B] &= \dim[A] \times \dim[B] \\
\dim[!A] &= ??
\end{aligned}$$

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\dim[A \mathbin{\&} B] &= \dim[A] + \dim[B] \\
\dim[A \otimes B] &= \dim[A] \times \dim[B] \\
\dim[A \multimap B] &= \dim[A] \times \dim[B] \\
\dim[!A] &= q^{\dim[A]}
\end{aligned}
$$

# Observations

# Basic Properties

- This model is sound with respect to (simply-typed) lambda calculus.
- One way to gain completeness is to move to an *algebraic lambda calculus*.

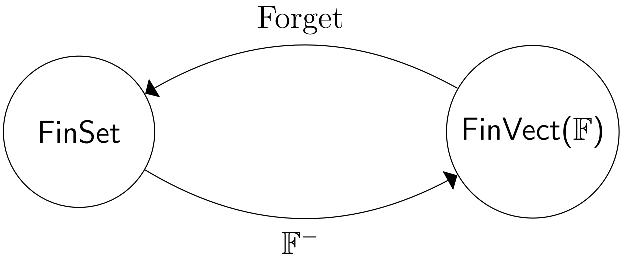
$$
\begin{array}{rcl}
M, N, P & ::= & x \mid \lambda x.M \mid MN \mid \pi_l(M) \mid \pi_r(M) \mid \langle\, M, N \,\rangle \mid \\
 & & \mathtt{tt} \mid \mathtt{ff} \mid \mathtt{if}\; M \;\mathtt{then}\; N \;\mathtt{else}\; P \mid \\
 & & 0 \mid M + N \mid \alpha \cdot M \\
A, B & ::= & \mathtt{Bool} \mid A \to B \mid A \times B.
\end{array}
$$

# Added Typing Rules

$$\frac{}{\Delta \vdash 0 : A} \qquad \frac{\Delta \vdash M : A \quad \Delta \vdash N : A}{\Delta \vdash M + N : A} \qquad \frac{\Delta \vdash M : A}{\Delta \vdash \alpha \cdot M : A}$$

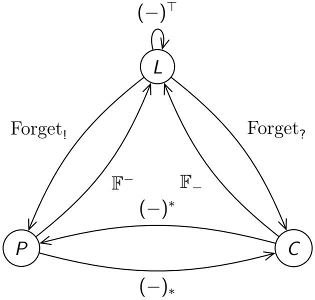
# Linear–Nonlinear Adjunction

Benton-style Linear-Nonlinear Decomposition:



# Classical Linear Logic

- The Linear/Nonlinear approach generalizes to full classical linear logic.
- Duality in FinVect($\mathbb{F}$) is given by *transposition*.



# Porting ideas from Linear Algebra to Lambda Calculus

- Example: eigenvalues of a square matrix. In $\mathbb{F}_2$, given a lambda calculus function $f : A \to B$ it is possible to construct $\hat{f} : A \,\&\, B \to A \,\&\, B$ (a square matrix) such that:

$$v \in \mathrm{eigvalues}(\hat{\mathrm{f}}) \implies f(\mathrm{fst}\ v) = \mathrm{snd}\ v$$

# Conclusions

The category of finite dimensional vector spaces over finite fields is a model of linear logic.

- Very pretty mathematics!
- Connects lambda calculus and linear algebra
- Interpretation of $(!A)$ in $\mathsf{FinVect}(\mathbb{F})$ is interesting.
- What are the implications of picking a particular $\mathbb{F}_q$?
- Applications?

# Multinomials

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Some beautiful theorems:

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Consequence:
When working with multinomials whose variables range over elements of $\mathbb{F}$, we have $\mathsf{x}^q = \mathsf{x}$.
For example, in $\mathbb{F}_2$:

$$(\mathsf{x} + \mathbf{1})^2 \quad = \quad \mathsf{x}^2 + 2\mathsf{x} + \mathbf{1} \quad = \quad \mathsf{x}^2 + 1 \quad = \quad \mathsf{x} + 1$$

# Another Endo-Functor: $M : A \to !A$

Morally, we have:

$$!A \approx 1 \,\&\, A \,\&\, A^2 \,\&\, A^3 \,\&\, \ldots$$

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Analogy: In Set $[\![!A]\!]$ is the set of all finite multisets whose elements are drawn from $[\![A]\!]$.

- So the *coordinates* of the vector space corresponding to $!A$ should (morally) be finite multisets drawn from $A$.
- Example: Write $\mathbb{B}^\dagger = \{\mathrm{inl}\ \bullet, \mathrm{inr}\ \bullet\}$ as $\{0, 1\}$

$$(!\mathbb{B})^\dagger = \{\emptyset, \{0\}, \{1\}, \{0,0\}, \{0,1\}, \{1,1\}, \{0,0,0\}, \dots\}$$

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Problem: This isn't finite! (But we persevere anyway...)

# Vectors With Multisets as Coords

$$(!\mathbb{B})^\dagger = \{\emptyset, \{0\}, \{1\}, \{0,0\}, \{0,1\}, \{1,1\}, \{0,0,0\}, \ldots\}$$

One more observation: What would a vector with coordinates as above look like?

$$\begin{array}{rcc} v & = & \alpha_\emptyset \cdot \delta_\emptyset \\ & + & \alpha_{\{0\}} \cdot \delta_{\{0\}} \\ & + & \alpha_{\{1\}} \cdot \delta_{\{1\}} \\ & + & \alpha_{\{0,0\}} \cdot \delta_{\{0,0\}} \\ & + & \alpha_{\{0,1\}} \cdot \delta_{\{0,1\}} \\ & + & \alpha_{\{1,1\}} \cdot \delta_{\{1,1\}} \\ & + & \alpha_{\{0,0,0\}} \cdot \delta_{\{0,0,0\}} \\ & \vdots & . \end{array}$$

# Multinomials

Suppose we knew that we would only ever need multisets with at most two of each element?

$$(!\mathbb{B})^\dagger = \{ \quad \emptyset, \{0\}, \{1\}, \{0,0\}, \{0,1\}, \\ \{1,1\}, \{1,1,0\}, \{1,0,0\}, \{1,1,0,0\} \quad \}$$

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Suppose we knew that we would only ever need multisets with at most two of each element?

$$(!\mathbb{B})^\dagger = \{ \quad \emptyset, \{0\}, \{1\}, \{0,0\}, \{0,1\}, \{1,1\}, \{1,1,0\}, \{1,0,0\}, \{1,1,0,0\} \quad \}$$

$$\begin{array}{rcc} v & = & \alpha_\emptyset \cdot \delta_\emptyset \\ & + & \alpha_{\{0\}} \cdot \delta_{\{0\}} \\ & + & \alpha_{\{1\}} \cdot \delta_{\{1\}} \\ & + & \alpha_{\{0,0\}} \cdot \delta_{\{0,0\}} \\ & + & \alpha_{\{0,1\}} \cdot \delta_{\{0,1\}} \\ & + & \alpha_{\{1,1\}} \cdot \delta_{\{1,1\}} \\ & + & \alpha_{\{1,1,0\}} \cdot \delta_{\{1,1,0\}} \\ & + & \alpha_{\{1,0,0\}} \cdot \delta_{\{1,0,0\}} \\ & + & \alpha_{\{1,1,0,0\}} \cdot \delta_{\{1,1,0,0\}} \end{array} \quad \Rightarrow \quad \begin{array}{rcl} v & = & \alpha_{00} \cdot x_0^0 x_1^0 \\ & + & \alpha_{10} \cdot x_0^1 x_1^0 \\ & + & \alpha_{01} \cdot x_0^0 x_1^1 \\ & + & \alpha_{20} \cdot x_0^2 x_1^0 \\ & + & \alpha_{11} \cdot x_0^1 x_1^1 \\ & + & \alpha_{02} \cdot x_0^0 x_1^2 \\ & + & \alpha_{21} \cdot x_0^2 x_1^1 \\ & + & \alpha_{12} \cdot x_0^1 x_1^2 \\ & + & \alpha_{22} \cdot x_0^2 x_1^2 \end{array}$$

# Multinomials

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$$\begin{array}{lcc}
v & = & \alpha_\emptyset \cdot \delta_\emptyset \\
 & + & \alpha_{\{0\}} \cdot \delta_{\{0\}} \\
 & + & \alpha_{\{1\}} \cdot \delta_{\{1\}} \\
 & + & \alpha_{\{0,0\}} \cdot \delta_{\{0,0\}} \\
 & + & \alpha_{\{0,1\}} \cdot \delta_{\{0,1\}} \\
 & + & \alpha_{\{1,1\}} \cdot \delta_{\{1,1\}} \\
 & + & \alpha_{\{1,1,0\}} \cdot \delta_{\{1,1,0\}} \\
 & + & \alpha_{\{1,0,0\}} \cdot \delta_{\{1,0,0\}} \\
 & + & \alpha_{\{1,1,0,0\}} \cdot \delta_{\{1,1,0,0\}}
\end{array}
\Rightarrow
\begin{array}{lcl}
v & = & \alpha_{00} \cdot \mathbf{x}_0^0 \mathbf{x}_1^0 \\
 & + & \alpha_{10} \cdot \mathbf{x}_0^1 \mathbf{x}_1^0 \\
 & + & \alpha_{01} \cdot \mathbf{x}_0^0 \mathbf{x}_1^1 \\
 & + & \alpha_{20} \cdot \mathbf{x}_0^2 \mathbf{x}_1^0 \\
 & + & \alpha_{11} \cdot \mathbf{x}_0^1 \mathbf{x}_1^1 \\
 & + & \alpha_{02} \cdot \mathbf{x}_0^0 \mathbf{x}_1^2 \\
 & + & \alpha_{21} \cdot \mathbf{x}_0^2 \mathbf{x}_1^1 \\
 & + & \alpha_{12} \cdot \mathbf{x}_0^1 \mathbf{x}_1^2 \\
 & + & \alpha_{22} \cdot \mathbf{x}_0^2 \mathbf{x}_1^2
\end{array}
\Rightarrow
\begin{array}{lcl}
v & = & \alpha_{00} \cdot \mathbf{1} \\
 & + & \alpha_{10} \cdot \mathbf{x}_0 \\
 & + & \alpha_{01} \cdot \mathbf{x}_1 \\
 & + & \alpha_{20} \cdot \mathbf{x}_0^2 \\
 & + & \alpha_{11} \cdot \mathbf{x}_0 \mathbf{x}_1 \\
 & + & \alpha_{02} \cdot \mathbf{x}_1^2 \\
 & + & \alpha_{21} \cdot \mathbf{x}_0^2 \mathbf{x}_1 \\
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# Multinomials

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$$\begin{array}{rcl} v & = & \alpha_{\emptyset}\cdot\delta_{\emptyset} \\ & + & \alpha_{\{0\}}\cdot\delta_{\{0\}} \\ & + & \alpha_{\{1\}}\cdot\delta_{\{1\}} \\ & + & \alpha_{\{0,0\}}\cdot\delta_{\{0,0\}} \\ & + & \alpha_{\{0,1\}}\cdot\delta_{\{0,1\}} \\ & + & \alpha_{\{1,1\}}\cdot\delta_{\{1,1\}} \\ & + & \alpha_{\{1,1,0\}}\cdot\delta_{\{1,1,0\}} \\ & + & \alpha_{\{1,0,0\}}\cdot\delta_{\{1,0,0\}} \\ & + & \alpha_{\{1,1,0,0\}}\cdot\delta_{\{1,1,0,0\}} \end{array} \Rightarrow \begin{array}{rcl} v & = & \alpha_{00}\cdot\mathbf{x}_0^0\mathbf{x}_1^0 \\ & + & \alpha_{10}\cdot\mathbf{x}_0^1\mathbf{x}_1^0 \\ & + & \alpha_{01}\cdot\mathbf{x}_0^0\mathbf{x}_1^1 \\ & + & \alpha_{20}\cdot\mathbf{x}_0^2\mathbf{x}_1^0 \\ & + & \alpha_{11}\cdot\mathbf{x}_0^1\mathbf{x}_1^1 \\ & + & \alpha_{02}\cdot\mathbf{x}_0^0\mathbf{x}_1^2 \\ & + & \alpha_{21}\cdot\mathbf{x}_0^2\mathbf{x}_1^1 \\ & + & \alpha_{12}\cdot\mathbf{x}_0^1\mathbf{x}_1^2 \\ & + & \alpha_{22}\cdot\mathbf{x}_0^2\mathbf{x}_1^2 \end{array} \Rightarrow \begin{array}{rcl} v & = & \alpha_{00}\cdot\mathbf{1} \\ & + & \alpha_{10}\cdot\mathbf{x}_0 \\ & + & \alpha_{01}\cdot\mathbf{x}_1 \\ & + & \alpha_{20}\cdot\mathbf{x}_0^2 \\ & + & \alpha_{11}\cdot\mathbf{x}_0\mathbf{x}_1 \\ & + & \alpha_{02}\cdot\mathbf{x}_1^2 \\ & + & \alpha_{21}\cdot\mathbf{x}_0^2\mathbf{x}_1 \\ & + & \alpha_{12}\cdot\mathbf{x}_0\mathbf{x}_1^2 \\ & + & \alpha_{22}\cdot\mathbf{x}_0^2\mathbf{x}_1^2 \end{array}$$

Upshot: A vector whose coordinates are multisets over $A$ can be thought of as a *multinomial* with one variable for each element of $A$.

# Definition of $M$

- A multiset $\{0, 0, 1\}$ corresponds to a *term* $\mathbf{x}_0^2\mathbf{x}_1$ of the multinomial.
- The set of these terms form a basis.
  $f : [A] \to [B]$ acts on each $\mathbf{x}_a$ by:

$$\mathbf{x}_a \overset{f}{\longmapsto} \sum_{b \in B} f[b, a] \cdot \mathbf{y}_b$$

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- A multiset $\{0, 0, 1\}$ corresponds to a *term* $\mathsf{x}_0^2\mathsf{x}_1$ of the multinomial.
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$$\mathsf{x}_a \xmapsto{f} \sum_{b \in B} f[b, a] \cdot \mathbf{y}_b$$

So $M(f)$ acts on a term like $\mathsf{x}_0^2\mathsf{x}_1$ by:

$$\mathsf{x}_0^2\mathsf{x}_1 \xmapsto{!f} (\sum_{b \in B} f[b, 0] \cdot \mathbf{y}_b) \times (\sum_{b \in B} f[b, 0] \cdot \mathbf{y}_b) \times (\sum_{b \in B} f[b, 1] \cdot \mathbf{y}_b)$$

This is multinomial multiplication, modulo $\mathbf{y}^q = \mathbf{y}$.

# Example in $\mathbb{F}_2$

Let $f : [\![1 \oplus 1 \oplus 1]\!] \to [\![1 \oplus 1 \oplus 1]\!]$ be:

$$\begin{bmatrix} \bullet & \cdot & \bullet \\ \cdot & \bullet & \cdot \\ \cdot & \bullet & \cdot \end{bmatrix}$$

Then $M(f) : ![\![1 \oplus 1 \oplus 1]\!] \to ![\![1 \oplus 1 \oplus 1]\!]$ is:

$$\begin{bmatrix} \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \color{green}{\bullet} & \cdot & \cdot & \color{green}{\bullet} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \color{green}{\bullet} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \bullet & \cdot \\ \cdot & \cdot & \color{green}{\bullet} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \bullet & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

## Theorem (Functoriality of M)

*For any* $f : [\![A]\!] \to [\![B]\!]$ *and* $g : [\![B]\!] \to [\![C]\!]$*:*

$$M(f;g) = M(f);M(g) \quad : ![\![A]\!] \to ![\![C]\!]$$