I Factorization algebras in perturbative quantum field theory By Kevin Costello (Northwestern) May 2009

I.1 Deformation quatization (work in progress with Owen Guilliam).

Classical mechanics: described by a Poisson algebra $(A, \{\cdot, \cdot\})$. In deformation quantization, want to replace A by an associative product $A[[\hbar]]$, such that if $f, g \in A$, $\{f, g\} = \lim_{\hbar \to 0} \frac{1}{\hbar}[f, g]$.

Want an analogy in QFT:

1) Describe calssical and quantum algebraic structures.
2) Show that we can get classical structure from classical field theory.
3) Stat a "quantization" theorem.

Let M be a manifold. A *factorization algebra* on M can be described as follows. Let $B(M)$ be the space of balls in M and $B_n(M)$ the collection of n disjoint unions of balls $B_1, ..., B_n \in B(M)$ ambedded into B_{n+1} (like in the definition of the little cubes operad).

A factorization algebra is a vector bundle V on $B(m)$, together with maps

$$B(M) \xleftarrow{p} B_n(M) \xrightarrow{q} B(M)^n, \quad q^{*(V^{\otimes n})} \to p^* V$$

satisfying some evident compatibility: $V(B_1) \otimes V(B_2) \to V(B_3)$. The two maps

$$V^{\otimes \text{three balls}} \to V^{\otimes \text{two intermediate balls}} \to V^{\text{outer ball}}$$

and $V^{\otimes \text{three balls}} \to V^{\text{outer ball}}$ commute.

$\rightsquigarrow$ Close relation of E_n-algebras for $\dim(M) = n$. Say a top. factorization algebra in M is this structure when V is a locally constant sheaf and the maps are morphisms of locally constant sheafs.

Theorem I.1. *An E_n-algebra yields a top. factorization algebra on any framed manifold M with $n = \dim(M)$.*

A factorization algebra in $\mathbb{C}$, where everything is holomorphic an invariant under $\text{Aff}(\mathbb{C})$:

$$V(\{z \in \mathbb{C} : |z| < 1\}) =: W \quad \text{a vector space}$$

and the factorization algebra gives a map $m_z : W \otimes W \to W$, depending holomorphically on z in an annulus. Thus $m_z \sim \sum_{k \in \mathbb{Z}} \varphi_k z^k$, where $\varphi_k \in$ some completion of $\text{Hom}(W, W) \rightsquigarrow$ reminiscent of the operator product expansion in vertex algebras.

Claim: Factorization algebras on M encode structure one expects from a quantum field theory on M. This is motivated by the 2-dim. holomorphic setting, which fits with the known picture. In one dimension, this reduces to an associative algebra, the algebra of observables of quantum mechanics. In any dimension, one can construct a factorization algebra using perturbative quantum field theory.

Factorization algebras are symmetric monoidal algebras. A classical fact. algebra is a commutative algebra in this category. Suppose we have a classical field theory. For instance, take M compact riemannian, fields to be $C^\infty(M, \mathbb{R})$ and the action

$$S(\varphi) = \int_M \varphi \Delta \varphi + \varphi^3$$

Let EL be the sheaf of solutions to the Euler-Lagrange equations $2\Delta\varphi + 3\varphi^2 = 0$. If $B \subseteq M$, let $\mathcal{O}(EL(B))$ be functions on $EL(B)$. This is a commutative fact. algebra $EL(B_3) \to EL(B_1) \times EL(B_2)$ and applying $\mathcal{O}$ to this gives a fact. algebra.

Claim: The E_∞-algebra "wants" to become E_0, i.e. just a factorization algebra.

Poisson degree	???
1	E_0
0	E_1
-1	E_2
-2	E_3

Definition I.2. The BV_0-operad is the operad over $\mathbb{R}[[\hbar]]$, generated by a commutative product $*$ and a Poisson bracket of degree 1, s.th. $d* = \hbar\{\cdot,\cdot\}$. ∎

Invert $\hbar$: $BV_0 \simeq E_0$. If $\hbar = 0$, then $BV_0/\hbar$ is the operad of commutative algebras with bracket of degree 1.

Let X be a manifold, $f \in \mathcal{O}(X)$. Then $h(r,t(f))$, the derived critical scheme has a $\{\cdot,\cdot\}$ of degree 1 and "wants to become" E_0. $\mathcal{O}(h(r,t(f)))$ is a dga

$$\Lambda^2 TX \xrightarrow{\iota_v(df)} TX \xrightarrow{\iota_v(df)} \mathcal{O}(X)$$

The Schouten bracket gives $\mathcal{O}(h(r,t(f)))$ a Poisson bracket of degree 1. Apply this to the Euler-Lagrange situation and thus look at a "derived space of solutions". This is $\mathrm{Crit}(S)$ and should acquire the Poisson bracket.

Example I.3. In free field theory, the derived critical locus of $\int \varphi\Delta\varphi$ is the 2-term complex

$$C^\infty(M) \xrightarrow{\Delta} C^\infty(M)$$

If we had an action with a cubic term, then differentiation acquires a non-linear term. ∎

If $B \subseteq M$, the $\mathcal{O}(\hbar EL(B))$ looks like

$$\prod \mathrm{Hom}(\mathrm{Sym}^n(C^\infty(B) \otimes \Lambda^k C^\infty(B)), \mathbb{R})$$

with a differential coming from the action S. This has a bracket of degree 1, so it "wants to become" E_0, thus a factorization algebra. If X has a measure, i.e. a trivialization of the top-degree sheaf of differential forms, then we get a BV operator $\Delta : \Lambda^\bullet TX \to \Lambda^\bullet TX$ such that $d + \hbar\Delta$ gives an algebra over BV_0.

Theorem I.4. *The commutative factorization algebra, associated to the Euler-Lagrange equation of a classical field theory can be quantized into a factorization algebra in many interesting situations, e.g.*

- *scalar field theories*
- *Yang-Mills theories on* $\mathbb{R}^4$.

Remark I.5. This is false over $\overline{\mathbb{Q}}$! ∎