The Crossed Menagerie (14 chapter version): an introduction to crossed gadgetry and cohomology in algebra and topology. (Notes, in part, prepared for the XVI Encuentro Rioplatense de Álgebra y Geometría Algebraica, in Buenos Aires, 12 - 15 December 2006, and for course MATH5312, Spring - Summer term 2007, University of Ottawa)[1]

Timothy Porter

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Introduction

These notes were originally intended to supplement lectures given at the Buenos Aires meeting in December 2006, were extended to give a lot more background for a course in cohomology at Ottawa (Summer term 2007) and then grew to include a lot more, including material on non-Abelian cohomology, stacks and gerbes, and related areas of TQFT and HQFT theory.

They introduce some of the family of crossed algebraic gadgetry that have their origins in combinatorial group theory in the 1930s and '40s, then were pushed much further by Henry Whitehead in the papers on Combinatorial Homotopy, and, in particular, in [277]. Since about 1970, more information and more examples have come to light, initially in the work of Ronnie Brown and Phil Higgins, (for which a useful central reference will be the forthcoming, [64]), in which crossed complexes were studied in depth. Explorations of crossed squares by Loday and Guin-Valery, [143, 186] and from about 1980 onwards indicated their relevance to many problems in algebra and algebraic geometry, as well as to algebraic topology have become clear. More recently in the guise of 2-groups, they have been appearing in parts of differential geometry, [16, 52] and have, via work of Breen and others, [48–51], been of central importance for *non-Abelian cohomology.* This connection between the crossed menagerie and non-Abelian cohomology is almost as old as the crossed gadgetry itself, dating back to Dedecker's work in the 1960s, [104]. Yet the basic message of what they are, why they work, how they relate to other structures, and how the crossed menagerie works, still need repeating, especially in that setting of non-Abelian cohomology in all its bewildering beauty.

The original notes have been augmented by lots of additional material, since the link with non-Abelian cohomology was worth pursuing in much more detail. These notes thus contain an introduction to the way 'crossed gadgetry' interacts with non-Abelian cohomology and areas such as topological and homotopical quantum field theory. This entails the inclusion of a fairly detailed introduction to torsors, gerbes, etc. This is based in part on Larry Breen's beautiful Minneapolis notes, [51].

If this is the first time you have met this sort of material, then some words of warning and welcome are in order.

There is much too much in these notes to digest in one go!

There is probably a lot more than you will need in your continuing research. For instance, the material on torsors, etc., is probably best taken at a later sitting and the chapter 'Beyond 2-types' is not directly used until a lot later, so can be glanced at. The notes are also designed with a 'dipper' in mind. I know that I do not always read a textbook or set of notes in a sequential manner, so after looking at crossed modules, you might need to look at sheaves of crossed modules, only to find you did need some ideas from the sections on crossed complexes as well. There are the, now usual, hyperlinks to help navigation, but also some comments are repeated, especially if they are 'warnings', say about terminology or notation, so do not think, 'he should have edited that out' if you find some such repeated point. That repetition is possibly, even probably, there for a purpose.

There are bits of bold text '**for the reader**'. Both the original audience in Buenos Aires and later the students of MATH 5312 in Ottawa had varied backgrounds, so some quite routine ideas had not been met by a subset of them. Different subsets for different ideas. The use of **bold** indicates where it might be a good idea to write down some more details for yourself, if you have not seen the idea before, or perhaps have not used it recently. It may indicate an **exercise**, although there are few formal exercises. Typically going through an easy example of some idea or through the 'routine' verification of some properties can help build the intuition as to the interpretation and meaning of a concept. ‘Meaning’ and ‘interpretation’ are highly dependent on the use the reader, *i.e.*, **you**, will make of an idea, so it is not for me to impose my meaning on you. I can however provide indications of intuition that has been helpful to me or to others, especially when at an entry point for a sequence of new ideas.

To start with I have concentrated on the group theoretic and geometric aspects of cohomology, since the non-Abelian theory is better developed there, but it is easy to attack other topics such as Lie algebra cohomology, once the basic ideas of the group case have been mastered and applications in differential geometry do need the torsors, etc. I have emphasised approaches using crossed modules (of groups). Analogues of these gadgets do exist in the other settings (Lie algebras, etc.), and most of the ideas go across without too much pain. If handling a non-group based problem (e.g. with monoids or categories), then the internal categorical aspect - crossed module as internal category in groups - would replace the direct method used here. Moreover the group based theory has the advantage of being central to both algebraic and geometric applications.

The aim of the notes is not to give an exhaustive treatment of cohomology and related concepts. That would be impossible. If, at the end of reading the relevant sections, the reader feels that they have some intuition on the *meaning* and *interpretation* of cohomology classes in their own area, and that they can more easily attack other aspects of cohomological and homotopical algebra by themselves, then the notes will have succeeded for them.

Although not ‘self contained’, I have tried to introduce topics such as sheaf theory as and when necessary, so as to give a natural development of the ideas. Some readers will already have been introduced to these ideas and they need not read those sections in detail. Such sections are, I think, clearly indicated. They do not give all the details of those areas, of course. For a start, those details are not needed for the purposes of the notes, but the summaries do try to sketch in enough ‘intuition’ to make it reasonable clear, I hope, what the notes are talking about!

I have not assumed much formal knowledge of category theory, at least to start with. The idea of category, a functor and a natural transformation will be used and not defined. Mention of *pullbacks*, *pushouts* and similar notions, if the reader has not met them before, should send them ‘hot foot’ to a category theory textbook or to web-sources such as Wikipedia. Only a passing acquaintance is needed to start with, as hopefully, as such ideas are used they will become clearer. Later on many more areas related to category theory will be introduced, but it will be assumed that by that point in the notes, the reader is more independently able to search out and read other sources for details when needed.

I have not tried to cover everything possible and would point the noses of researchers, who want to find out more, in the direction of the *nLab*. This is a ‘wiki’ devoted to, I quote:
collaborative work on Mathematics, Physics and Philosophy especially insofar as these subjects touch on n-categories and related ‘higher algebraic structures’.
The home page can be found at: http://ncatlab.org/nlab/show/HomePage, [221].

Acknowledgements

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The notes were continued and expanded for course MATH 5312 in the Spring of 2007 during a visit as a visiting professor to the Department of Mathematics and Statistics of the University of Ottawa. Thanks are due to Rick Blute, Pieter Hofstra, Phil Scott, Paul-Eugene Parent, Barry Jessup and Jonathan Scott for the warm welcome and the mathematical discussions on some of the material and the students of MATH 5312 for their interest and constructive comments. Financial assistance from the University of Ottawa and the NSERC is also gratefully acknowledged.

Some parts of the notes formed part of my session "Classifying spaces of categorical groups, and relations with non-Abelian cohomology" at the *Workshop on Categorical Groups, Barcelona*, (June 16th – 20th, 2008), and this resulted in some reworking of material, and additional material being added, as I learnt or relearnt various aspects of the theory of categorical groups that I had not seen recently, forgotten, never seen or whatever. Thanks are due to the organisers and the participants who together contributed to a delightful and rewarding meeting and thus, incidently, to the revision of the notes.

Again the first few chapters of the notes were used for a 'TQFT Club' minicourse in the CAMGSD, in the IST, Lisbon, (3 and 4 December, 2008), entitled *The 'Crossed' technology and its applications*. The feedback and questions that resulted again motivated certain additions and changes. Thanks are due to Roger Picken and João Faria Martins for the invitation, and, in addition, the participants, in particular Pedro Resende and Tim Van der Linden, for useful questions and comments, and the warm welcome. Since then I have benefitted from feedback, discussions and further ideas in visits to Lisbon, Louvain-la-Neuve, and more recently Leeds.

At Lyon, I used parts of the description of Schreier theory as a basis for a series of talks to the working group, *Invariants algébriques en informatique*, in the Spring of 2010, during a visit to the *Institut Camille Jordan, Lyon 1, (UMR 5208 du CNRS)*. Acknowledgement is due for the support from the CNRS and, of course, thanks for the warm welcome, and for the stimulating discussions on possible applications and analogues of these ideas in various areas, with Philippe Malbos and Yves Guiraud. More recently, they and others invited me to give a minicourse on the connections between '*Rewriting and Homotopy*' at a meeting (week 5 of LI2012) held at the C.I.R.M. at Luminy. (Thanks to the organisers and the staff at the C.I.R.M. for a very enjoyable visit.) The notes for this needed some additional material on syzygies and were a good occasion / excuse to prepare more material relating to the Abels and Holz methods of calculating syzygies, the links with work on algebraic K-theory and Haefliger's complexes of groups. This fits in well and hopefully will be useful for some reader. (I will put some of this on the nLab.)

Tim Porter, ex-Bangor, and other places including Ottawa, Paris, Granada, Lisbon, Lyon, Savoie, Leeds, Louvain-la-Neuve, ..., 2007 - 2018.

Contents

Chapter 1

Preliminaries

1.1 Groups and Groupoids, some basic ideas

Before launching into crossed modules, we need a word on groupoids.

1.1.1 Groupoids

By a groupoid, we mean a small category in which all morphisms are isomorphisms. (If you have not formally met categories then do not worry, the idea will come through without that specific formal knowledge, although a quick glance at Wikipedia for the definition of a category might be a good idea at some time soon. You do not need category theory as such at this stage.) These groupoids typically arise in three situations (i) symmetry objects of a fibered structure, (ii) equivalence relations, and (iii) group actions. It is worth noting that several of the initial applications of groups were thought of, by their discoverers, as being more naturally this type of groupoid structure.

For the first, assume we have a family of sets $\{X_a : a \in A\}$. Typically we have a function $f : X \to A$ and $X_a = f^{-1}a$ for $a \in A$. We form the symmetry groupoid of the family by taking the index set, A, as the set of objects of the groupoid, $\mathcal{G}$, and, if $a, a' \in A$, then $\mathcal{G}(a, a')$, the set of arrows in our symmetry groupoid from a to a', is the set $Bijections(X_a, X_{a'})$. This $\mathcal{G}$ will contain all the individual symmetry groups / permutation groups of the various X_a, but will also record comparison information between different X_as.

Of course, any group is a groupoid with one object and, conversely, if $\mathcal{G}$ is any groupoid, we have, for each object a of $\mathcal{G}$, a group $\mathcal{G}(a, a)$, of arrows that start and end at a. This is the 'automorphism group', $\mathsf{aut}_{\mathcal{G}}(a)$, of a within $\mathcal{G}$. It is also referred to as the vertex group of $\mathcal{G}$ at a, and denoted $\mathcal{G}(a)$. This later viewpoint and notation emphasise more the combinatorial, graph-like side of $\mathcal{G}$'s structure. Sometimes the notation $G[1]$ may be used for $\mathcal{G}$ as the process of regarding a group as a groupoid is a sort of 'suspension' or 'shift'. It is one aspect of 'categorification', cf. Baez and Dolan, [15].

That combinatorial side is strongly represented in the second situation, equivalence relations. Suppose that R is an equivalence relation on a set X. Going back to basics, R is a subset of $X \times X$ satisfying:

(a) if $a, b, c \in X$ and (a, b) and $(b, c) \in R$, then $(a, c) \in R$, *i.e.*, R is transitive;

(b) for all $a \in X$, $(a, a) \in R$, alternatively the diagonal $\Delta \subseteq R$, *i.e.*, R is reflexive;

(c) if $a, b \in X$ and $(a, b) \in R$, then $(b, a) \in R$, *i.e.*, R is symmetric.

Two comments might be made here. The first is 'everyone knows that!', the second 'that is not the usual order to put them in! Why the change?'

It is a well known, but often forgotten, fact that from R, you get a groupoid (which we will denote by $\mathcal{R}$). The objects of $\mathcal{R}$ are the elements of X and $\mathcal{R}(a, b)$ is a singleton if $(a, b) \in \mathcal{R}$ and is empty otherwise. (There is really no need to label the single element of $\mathcal{R}(a, b)$, when this is non empty, but it is sometimes convenient to call it (a, b) at the risk of over using the ordered pair notation.) Now transitivity of R gives us a composition function: for $a, b, c \in X$,

$$\circ : \mathcal{R}(a, b) \times \mathcal{R}(b, c) \to \mathcal{R}(a, c).$$

(Remember that a product of a set with the empty set is itself always empty, and that for any set, there is a unique function with domain $\emptyset$ and codomain the set, so checking that this composition works nicely is slightly more subtle than you might at first think. This *is* important when handling the analogues of equivalence relations in other categories., then you cannot just write $(a, b) \circ (b, c) = (a, c)$, or similar, as 'elements' may not be obvious things to handle.) Of course this composition *is* associative, but if you have not seen the verification, it is important to think about it, looking for subtle points, especially concerning the empty set and empty function and how to do the proof without 'elements'.

This composition makes $\mathcal{R}$ into a category, since (a) gives the existence of identities for each object. ($Id_a = (a, a)$ in 'elementary' notation.) Finally (c) shows that each (a, b) is invertible, so $\mathcal{R}$ is a groupoid. (You now see why that order was the natural one for the axioms. You cannot prove that (a, a) is an identity until you have a composition, and similarly until you have identities, inverses do not make sense.) We may call $\mathcal{R}$, the groupoid of the equivalence relation R.

This shows how to think of R as a groupoid, $\mathcal{R}$. The automorphism groups, $\mathcal{R}(a)$, are all singletons as sets, so are trivial groups[1]. Conversely any groupoid, $\mathcal{G}$, gives a diagram

$$\begin{tikzcd} Arr(\mathcal{G}) \arrow[r, shift left=2, "s"] \arrow[r, shift left=0.7, "t"'] & Ob(\mathcal{G}) \arrow[l, shift left=1.5, "i"] \end{tikzcd}$$

with $s =$ 'source', $t =$ 'target' and $i=$ 'identity' as it picks out the identity arrow on each object. It thus gives a function,

$$Arr(\mathcal{G}) \xrightarrow{(s,t)} Ob(\mathcal{G}) \times Ob(\mathcal{G}) \ .$$

The image of this function is an equivalence relation as is easily checked. We will call this equivalence relation, R, for the moment. If $\mathcal{G}$ is a groupoid such that each $\mathcal{G}(a)$ is a trivial group, then each $\mathcal{G}(a, b)$ has at most one element (check it), so (s, t) is a one-one function and it is then trivial to note that $\mathcal{G}$ is isomorphic to the groupoid of the equivalence relation, R.

Examples: (i) We denote by $\mathcal{I}$, the interval groupoid which has two objects, 0 and 1, and morphisms $\iota : 0 \to 1$ and its inverse, together with, of course, the identity arrows at each object. This is the groupoid that corresponds to the equivalence relation on $\{0, 1\}$ which makes 0 and 1 equivalent. It is used within the theory of groupoids as the analogue of the topological unit interval, $[0, 1]$.

[1]The groupoid is *simply connected*.

(ii) If X is any set, we get two 'extreme' forms of equivalence relation on X, and thus two 'extreme' examples of groupoids.

Definition: a) The *discrete groupoid on the set,* X, denoted $Disc(X)$, has X as its set of objects and, for each ordered pair, (x_1, x_2), of elements of X,

$$Disc(S)(x_1, x_2) = \begin{cases} \{id_{x_1}\} & \text{if } x_1 = x_2 \\ \emptyset & \text{if } x_1 \neq x_2 \end{cases}$$

b) The *codiscrete groupoid on a set,* X, denoted $Codisc(X)$, has X as its set of objects and, for each ordered pair, (x_1, x_2), of elements of X, $Codisc(X)(x_1, x_2)$ is a singleton set, the unique element of which is conveniently denoted (x_1, x_2).

The two extreme examples of equivalence relations on X are, of course, (i) the relation of *equality*, or *identity*, in which R *is* just Δ, the diagonal of $X \times X$, and $(a, b) \in R$ if, and only if, $x = y$, and (ii) the relation in which everything is related, so $R = X \times X$. The groupoid, $Disc(X)$, corresponds to the first, whilst $Codisc(X)$ corresponds to the second. The example, $\mathcal{I}$, given in (i), above, is the case $Codisc(\{0, 1\})$.

We have looked at these simple cases in some detail as in applications of the basic ideas, especially in algebraic geometry, arguments using elements are quite tricky to give and the initial intuition coming from these set-based examples can easily be forgotten.

The third situation, that of group actions, is also a common one in algebra and algebraic geometry. Equivalence relations often come from group actions. If G is a group and X is a G-set with (left) G-action,

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ (g, x) & & g \cdot x \end{array} ,$$

(*i.e.*, a function $act(g, x) = g \cdot x$, which must satisfy the rules $1 \cdot x = x$ and for all $g_1, g_2 \in G$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$, a sort of associativity law), then we get a groupoid $\mathcal{A}ct_G(X)$, that will be called the *action groupoid* of the G-set, as follows:

- the objects of $\mathcal{A}ct_G(X)$ are the elements of X;
- if $a, b, \in X$,
$$\mathcal{A}ct_G(X)(a, b) \cong \{g \mid g \cdot a = b\}.$$

An important word of caution is in order here. Logical complications can occur here if $\mathcal{A}ct_G(X)(a, b)$ is set *equal* to $\{g \mid g \cdot a = b\}$, since then a g can occur in several different 'hom-sets'. A good way to avoid this is to take

$$\mathcal{A}ct_G(X)(a, b) = \{(g, a) \mid g \cdot a = b\}.$$

This is a non-trivial change. It basically uses a disjoint union, but although very simple, it is fundamental in its implications. We could also do it by taking $Arr_{\mathcal{G}}(X) = G \times X$ with source and target maps $s(g, x) = x$, $t(g, x) = g \cdot x$. (It is **useful**, if you have not seen this before, to see how the various parts of the definition of an action match with parts of the structural rules of a groupoid. This is important as it indicates how, much later on, we will relax those rules in various ways.)

We will sometimes use the notation, $G \curvearrowright X$, when discussing a left action of a group G on X.

In a groupoid, G, we say two objects, x and y are in the same *connected component* of G, if $G(x, y)$ is not empty. This gives an equivalence relation on the set of objects of G, as you **can easily check**. The equivalence classes re called the *connected components* of G and the set of connected components is usually denoted $\pi_0(G)$, by analogy with the usual notion for the set of connected components of a topological space.

We have not discussed morphisms of groupoids. These are straightforward to define and to work with. Together groupoids and the morphisms between them form a *category*, the *category of groupoids*, which will be denoted *Grpd*.

(As we introduced structures of various types, we will usually introduce a corresponding form of morphism and it will be rare that the resulting 'context' of objects and morphisms does not form a category. It is important to look up the definition of categories and functors, but *for the moment* you will not need to know very much 'category theory' to read the notes. It will suffice to get to grips with that as we go further and have good motivating examples for what is needed.)

Most of the concepts that we will be handling in what follows exist in many-object, groupoid versions as well as single-object, group based ones. For simplicity we will often, but not always, give concepts in the group based form, and will leave the other many-object form 'to the reader'. The conversion is usually not that difficult.

For more details on the theory of groupoids, two good sources are Ronnie Brown's book, [59] and Phil Higgins' monograph, now reprinted as [151].

1.2 A very brief introduction to cohomology

Partially as a case study, at least initially, we will be looking at various constructions that relate to group cohomology. Later we will explore a more general type of (non-Abelian) cohomology, including ideas about the non-Abelian cohomology of spaces, but that is for later. To start with we will look at a simple group theoretic problem that will be used for motivation at several places in what follows. Much of what is in books on group cohomology is the Abelian theory, whilst we will be looking more at the non-Abelian one. If you have not met cohomology at all, take a look at the Wikipedia entries for group cohomology. You may not understanding everything, but there are ideas there that will recur in what follows, and some terms that are described there or on linked entries, that will be needed later.

1.2.1 Extensions.

Given a group, G, an extension of G by a group K is a group E with an epimorphism $p : E \to G$ whose kernel is isomorphic to K (*i.e.*, a short exact sequence of groups

$$\mathcal{E} : 1 \to K \to E \xrightarrow{p} G \to 1.$$

As we asked that K be isomorphic to $Ker\, p$, we could have different groups E perhaps fitting into this, yet they would still be essentially the same extension. We say two extensions, $\mathcal{E}$ and $\mathcal{E}'$, are equivalent if there is an isomorphism between E and E' compatible with the other data. We can draw a diagram

$$\begin{tikzcd} \mathcal{E} \arrow[d] & 1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\cong"'] & G \arrow[r] \arrow[d, "="] & 1 \\ \mathcal{E}' & 1 \arrow[r] & K \arrow[r] & E' \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

A typical situation might be that you have an unknown group E' that you suspect is really E (*i.e.*, is isomorphic to E). You find a known normal subgroup K of E is isomorphic to one in E' and that the two quotient groups are isomorphic,

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "\cong"'] & E \arrow[r] \arrow[d, dashed, "?"'] & G \arrow[r] \arrow[d, "\cong"] & 1 \\ 1 \arrow[r] & K' \arrow[r] & E' \arrow[r] & G' \arrow[r] & 1 \end{tikzcd}$$

(But always remember, isomorphisms compare snap shots of the two structures and once chosen can make things more 'rigid' than perhaps they really 'naturally' are. For instance, we might have G a cyclic group of order 5 generated by an element a, and G' one generated by b. 'Naturally' we choose an isomorphism $\varphi : G \to G'$ to send a to b, but why? We could have sent a to any non-identity element of G' and need to be sure that this makes no difference. This is not just 'attention to detail'. It can be very important. It stresses the importance of $Aut(G)$, the group of automorphisms of G in this sort of situation.)

A simple case to illustrate that the extension problem is a valid one, is to consider $K = C_3 = \langle a \mid a^3\rangle$, $G = C_2 = \langle b \mid b^2\rangle$.

We could take $E = S_3$, the symmetric group on three symbols, or alternatively D_3 (also called D_6 to really confuse things, but being the symmetry group of the triangle). This has a presentation $\langle a, b \mid a^3, b^2, (ab)^2\rangle$. But what about $C_6 = \langle c \mid c^6\rangle$? This has a subgroup $\{1, c^2, c^4\}$ isomorphic to K and the quotient is isomorphic to G. Of course, S_3 is non-Abelian, whilst C_6 is. The presentation of C_6 needs adjusting to see just how similar the two situations are. This group also has a presentation $\langle a, b \mid a^3, b^2, aba^{-1}b\rangle$, since we can deduce $aba^{-1}b = 1$ from $[a, b] = 1$ and $b^2 = 1$ where in terms of the old generator c, $a = c^2$ and $b = c^3$. So there is a presentation of C_3 which just differs by a small 'twist' from that of S_3.

How could one be sure if S_3 and C_6 are the 'only' groups (up to isomorphism) that we could put in that central position? Can we classify all the extensions of G by K?

These extension problems were one of the impetuses for the development of a 'cohomological' approach to algebra, but they were not the only ones.

1.2.2 Invariants

Another group theoretic input is via group representation theory and the theory of invariants. If G is a group of $n \times n$ invertible matrices then one can use the simple but powerful tools of linear algebra to get good information on the elements of G and often one can tie this information in to some geometric context, say, by identifying elements of G as leaving invariant some polytope or pattern, so G acts as a subgroup of the group of the symmetries of that pattern or object.

If, therefore, we use the group $Gl(n, \mathbb{K})$ of such invertible matrices over some field $\mathbb{K}$, then we could map an arbitrary G into it and attempt to glean information on elements of G from the corresponding matrices. We thus consider a group homomorphism

$$\rho : G \to Gl(n, \mathbb{K}),$$

then look for nice properties of the $\rho(g)$. of course, ρ need not be a monomorphism and then we will loose information in the process, but in any case such a morphism will make G act (linearly) on the vector space $\mathbb{K}^n$. We could, more generally, replace $\mathbb{K}$ by a general commutative ring R, in particular we could use the ring of integers, $\mathbb{Z}$, and then replace $\mathbb{K}^n$ by a general *module*, M, over R. If $R = \mathbb{Z}$, then this is just an Abelian group. (If you have not formally met modules look up a definition. The theory feels very like that of vector spaces to start with at least, but as elements in R need not have inverses, care needs to be taken - you cannot cancel or divide in general, so $rx = ry$ does not imply $x = y$! Having looked up a definition, for most of the time you can think of modules as being vector spaces or Abelian groups and you will not be far wrong. We will shortly but briefly mention modules over a group algebra, $R[G]$, and that ring is not commutative, but again the complications that this does cause will not worry us at all.)

We can thus 'represent' G by mapping it into the automorphism group of M. This gives M the structure of a G-module. We look for invariants of the action of G on M - what are they? Suppose that G is some group of symmetries of some geometric figure or pattern, that we will call X, in $\mathbb{R}^n$, then for each $g \in G$, $gX = X$, since g acts by pushing the pattern around back onto itself. An invariant of G, considered as acting on M, or, to put it more neatly, of the G-module, M, is an element m in M such that $g.m = m$ for all $g \in G$. These form a submodule,

$$M^G = \{m \mid gm = m \text{ for all } g \in G\}.$$

Clearly, it will help in our understanding of the structure of G if we can calculate and analyse these modules of invariants. Now suppose we are looking at a submodule N of M, then N^G is a submodule of M^G and we can hope to start finding invariants, perhaps by looking at such submodules and the corresponding quotient modules, M/N. We have a short exact sequence

$$0 \to N \to M \to M/N \to 0,$$

but, although applying the (functorial) operation $(-)^G$ does yield

$$0 \to N^G \to M^G \to (M/N)^G,$$

the last map need not be onto so we may not get a short exact sequence and hence a nice simple way of finding invariants!

Example: Try $G = C_2 = \{1, a\}$, $M = \mathbb{Z}$, the Abelian group of integers, with G action, $a.n = -n$, and $N = 2\mathbb{Z}$, the subgroup of even integers, with the same G action. Now calculate the invariant modules M^G and N^G; they are both trivial, but $M/N \cong Z_2$, and ..., what is $(M/N)^G$ for this example?

The way of studying this in general is to try to to continue the exact sequence further to the right in some universal and natural way (via the theory of derived functors). This is what cohomology does. We can get a long exact sequence,

$$0 \to N^G \to M^G \to (M/N)^G \to H^1(G, N) \to H^1(G, M) \to H^1(G, M/N) \to H^2(G, N) \to \dots.$$

But what are these $H^k(G, M)$ and how does one get at them for calculation and interpretation? In fact, what is cohomology in general?

Its origins lie within Algebraic Topology as well as in Group Theory and that area provides some useful intuitions to get us started, before asking how to form group cohomology.

1.2.3 Homology and Cohomology of spaces.

Naively homology and cohomology give methods for measuring the holes in a space, holes of different dimensions yield generators in different (co)homology groups. The idea is easily seen for graphs and low dimensional simplicial complexes.

First we recall the definition of simplicial complex as we will need to be fairly precise about such objects and their role in relation to triangulations and related concepts.

Definition: A *simplicial complex*, K, is a set of objects, $V(K)$, called *vertices* and a set, $S(K)$, of finite non-empty subsets of $V(K)$, called *simplices*. All singletons are simplices and simplices satisfy the condition that if $\sigma \subset V(K)$ is a simplex and $\tau \subset \sigma$, $\tau \neq \emptyset$, then τ is also a simplex.

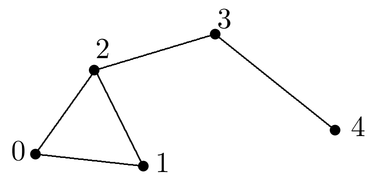
We say τ is a *face* of σ. If $\sigma \in S(K)$ has $p+1$ elements, it is said to be a *p-simplex*. The set of p-simplices of K is denoted by A. The dimension of K is the largest p such that K_p is non-empty.

We will sometimes use the notation, $\mathscr{P}(X)$, for the power set of a set X, *i.e.*, the set of subsets of X. Later when there are more demands on the notation, we will often use a substitute notation, writing V_K for $V(K)$, and S_K for $S(K)$, thus freeing up the notation used here for other uses. This should cause very little confusion, but does need noting.

Suppose, now, that $X = \{0, \ldots, p\}$, then there is a simple example of a simplicial complex, known as the *standard abstract p-simplex*, $\Delta[n]$, (or sometimes Δ^n),with vertex set, $V(\Delta[n]) = X$ and with $S(\Delta[n]) = \mathscr{P}(X) \setminus \{\emptyset\}$, in other words all non-empty subsets of X are to be simplices. (If you have not met simplicial complexes before this is a **good example to work with working out** what it looks like and 'feels like' for $n = 0, 1, 2$ and 3. It is too regular to be general, so we will, below, see another example which is perhaps a bit more typical.

Comment on notation: We have alternative notations, $\Delta[n]$ and Δ^n, for a good reason. The topic of simplicial complexes naturally leads on to that of simplicial sets both in these notes and historically. The notation $\Delta[n]$ can stand for both the simplicial complex version of the standard n-simplex or the analogous simplicial set. Most of the time this reuse is harmless, but once in a while, especially when comparing what happens in the two settings, it will be necessary to distinguish them in which case we will use $\Delta[n]$ for the simplicial set, with Δ^n for the simplicial complex. This latter notation will also serve for the geometric n-simplex. In any case, we will try to make it clear which context is the relevant one if this is not immediately clear from the surrounding discussion.

When thinking about simplicial complexes, it is important to have a picture in our minds of a triangulated space (probably a surface or similar, a wireframe as in computer graphics). The simplices are the triangles, tetrahedra, etc., and are determined by their sets of vertices. Not every set of vertices need be a simplex, but if a set of vertices does correspond to a simplex then all its non-empty subsets do as well, as they give the faces of that simplex. Here is an example:



Here $V(K) = \{0, 1, 2, 3, 4\}$ and $S(K)$ consists of $\{0, 1, 2\}$, $\{2, 3\}$, $\{3, 4\}$ and all the non-empty subsets of these. Note the triangle $\{0, 1, 2\}$ is intended to be solid, (but I did not work out how to do it on the Latex system I was using!)

Simplicial complexes are a natural combinatorial generalisation of (undirected) graphs. They not only have vertices and edges joining them, but also possible higher dimensional simplices relating paths in that low dimensional graph. It is often convenient to put a (total) order on the set $V(K)$ of vertices of a simplicial complex as this allows each simplex to be specified as a list $\sigma = \langle v_0, v_1, \ldots, v_n\rangle$ with $v_0 < v_1 < \ldots < v_n$, instead of as merely a set $\{v_0, v_1, \ldots, v_n\}$ of vertices. This, in turn, allows us to talk, unambiguously, of the k^{th} face of such a simplex, being the list with v_k omitted, so the zeroth face is $\langle v_1, \ldots, v_n\rangle$, the first is $\langle v_0, v_2, \ldots, v_n\rangle$ and so on.

Although strictly speaking different types of object, we tend to use the terms 'vertex' and '0-simplex' interchangeably and also use 'edge' as a synonym for '1-simplex'. We will usually write K_0 for $V(K)$ and may write K_1 for the set of edges of a graph, thought of as a 1-dimensional simplicial complex.

An abstract simplicial complex is a combinatorial gadget that models certain aspects of a spatial configuration. Sometimes it is useful, perhaps even necessary, to produce a topological space from that data in a simplicial complex.

Definition: To each simplicial complex K, one can associate a topological space called the *polyhedron* of K often also called or *geometric realisation* of K and denoted $|K|$.

This can be constructed by taking a copy $K(\sigma)$ of a standard topological p-simplex for each p-simplex of K and then 'gluing' them together according to the face relations encoded in K.

Definition: The *standard (topological) p-simplex* is usually taken to be the convex hull of the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{p+1}$ in $\mathbb{R}^{p+1}$, to represent each abstract p-simplex, $\sigma \in S(K)$, and then 'gluing' faces together, so whenever τ is a face of σ we identify $K(\tau)$ with the corresponding face of $K(\sigma)$. This space is usually denoted Δ^p.

There is a canonical way of constructing $|K|$ as follows: $|K|$ is the set of all functions from $V(K)$ to the closed interval $[0, 1]$ such that

- if $\alpha \in |K|$, the set
$$\{v \in V(K) \mid \alpha(v) \neq 0\}$$
is a simplex of K;

- for each $v \in V(K)$, $$\sum_{\alpha\in|K|} \alpha(v) = 1.$$

We can put a metric d on $|K|$ by

$$d(\alpha, \beta) = \Big(\sum_{v\in V(K)} (p_v(\alpha) - p_v(\beta))^2\Big)^{\frac{1}{2}}.$$

This however gives $|K|$ as a subspace of $\mathbb{R}^{\#(V(K))}$, and so is usually of much higher dimension then might seem geometrically significant in a given context. For instance, the above example would be represented as a subspace of $\mathbb{R}^5$, rather than $\mathbb{R}^2$, although that is the dimension of the picture we gave of it.

Given two simplicial complexes, K, L, then a function on the vertex sets, $f : V(K) \to V(L)$ is a *simplicial map* if it preserves simplices. (But that needs a bit of care to check out its exact meaning! ... **for you to do**. Look it up, or better try to see what the problem might be, try to resolve it yourself and then look it up!)

1.2.4 Betti numbers and Homology

One of the first sorts of invariant considered in what was to become Algebraic Topology was the family of Betti numbers. Given a simple shape, the most obvious piece of information to note would be the number of 'pieces' it is made up of, or more precisely, the number of *components*. The idea is very well known, at least for graphs, and as simplicial complexes are closely related to graphs, we will briefly look at this case first.

For convenience we will assume the vertices $V = V(\Gamma)$ of a given finite graph, Γ, are ordered, so for each edge e of Γ, we can assign a source $s(e)$ and a target $t(e)$ amongst the vertices. (This gives us a 'directed graph' or 'quiver'[2]). Two vertices v and w are said to be in the same component of Γ if there is a sequence of edges, $e_1, \ldots, e_k$, of Γ joining them[3]. There are, of course, several ways of thinking about this, for instance, define a relation $\sim$ on V by : for each e, $s(e) \sim t(e)$. Extend $\sim$ to an equivalence relation on V in the standard way, then $v \sim w$ if and only if they are in the same component. The zeroth Betti number, $\beta_0(\Gamma)$, is the number of components of Γ.

The first Betti number, $\beta_1(\Gamma)$, somewhat similarly, counts the number of cycles of Γ. We have ordered the vertices of Γ, so have effectively also directed its edges. If e is an edge, going from u to v, (so $u < v$ in the order on Γ_0), we write e also for the path going just along e and $-e$ for that going backwards along it, then extend our notation so $s(-e) = t(e) = v$, etc. Adding in these 'negative edges' corresponds to the formation of the symmetric closure of $\sim$. For the transitive closure we need to concatenate these simple one-edge paths: if e' is an edge or a 'negative edge' from v to w, we write $e + e'$ for the path going along e then e'. Playing algebraically with s and t and making them respect addition, we get a 'pseudo-calculation' for their difference $\partial = t - s$:

$$\partial(e + e') = t(e + e') - s(e + e') = t(e) + t(e') - s(e) - s(e') = t(e') - s(e) = u - w,$$

since $t(e) = v = s(e')$. In other words, defined in a suitable way, we would get that ∂, equal to 'target minus source', applies nicely to paths as well as edges, so that, for instance, two vertices

[2]We will return to discuss *directed graphs* aka *quivers*, in section 2.2.3.

[3]In fact here, the ordering we have assumed on the vertices complicates the exposition a little, but it is useful later on so will stick with it here.

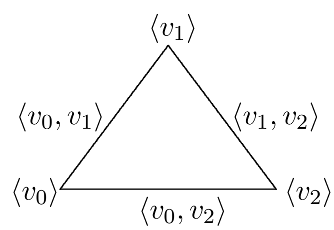
would be related in the transitive closure of $\sim$ if there was a 'formal sum' of edges that mapped down to their 'difference'. We say 'formal sum' as this is just what it is. We will need 'negative vertices' as well as 'negative edges'.

We set this up more formally as follows: Let
$C_0(\Gamma)$ = the set of formal sums, $\sum_{v\in\Gamma_0} a_v v$ with $a_v \in \mathbb{Z}$, the additive group of integers, (an alternative form is to take $a_v \in \mathbb{R}$.;
$C_1(\Gamma)$ = the set of formal sums, $\sum_{e\in\Gamma_1} b_e e$ with $b_e \in \mathbb{Z}$,
where Γ_1 denotes the set of edges of Γ, and $\partial : C_1(\Gamma) \to C_0(\Gamma)$ defined by extending additively the mapping given on the edges by $\partial = t - s$.

The task of determining components is thus reduced to calculating when integer vectors differ by the image of one in $C_1(\Gamma)$. The Betti number $\beta_0(\Gamma)$ is just the *rank* of the quotient $C_0(\Gamma)/Im(\partial)$, that is, the number of free generators of this commutative group. This would be exactly the dimension of this 'vector space' if we had allowed real coefficients in our formal sums not just integer ones.

Having reformulated components and $\sim$ in an algebraic way, we immediately get a pay-off in our determination of cycles. A cycle is a path which starts and ends at the same vertex; a path is being modelled by an element in $C_1(\Gamma)$, so a cycle is an element x in $C_1(\gamma)$ satisfying $\partial(x) = 0$. With this we have $\beta_1(\Gamma) = rank(Ker(\partial))$, a similar formulation to that for β_0. The similarity is even more striking if we replace the graph Γ by a simplicial complex K. We can then define in general and in any dimension p, $C_p(K)$ to be the commutative group of all formal sums $\sum_{\sigma\in K_p} a_\sigma \sigma$.

We next need to get an analogue of the $\partial = t - s$ formula. We want this to correspond to the boundary of the objects to which it is applied. For instance, if σ was the triangle / 2-simplex, $\langle v_0, v_1, v_2\rangle$, we would want $\partial\sigma$ to be $\langle v_1, v_2\rangle + \langle v_0, v_1\rangle - \langle v_0, v_2\rangle$, since going (clockwise) around the triangle, that cycle will be traced out:



If we write, in general, $d_i\sigma$ for the i^{th} face of a p-simplex $\sigma = \langle v_0, \ldots, v_p\rangle$, then in this 2-dimensional example $\partial\sigma = d_0\sigma - d_1\sigma + d_2\sigma$, changing the order for later convenience. This is the sum of the faces with weighting $(-1)^i$ given to $d_i\sigma$. This is consistent with $\partial = t - s$ in the lower dimension as $t = d_0$ and $s = d_1$. We can thus suggest that

$$\partial = \partial_p : C_p(K) \to C_{p-1}(K)$$

be defined on p-simplices by

$$\partial_p\sigma = \sum_{i=0}^{p} (-1)^i d_i\sigma,$$

and then extended additively to all of $C_p(K)$.

As an example of what this does, look at a square K, with vertices v_0, v_1, v_2, v_3, edges $\langle v_i, v_{i+1}\rangle$ for $i = 0, 1, 2$ and $\langle v_0, v_2\rangle$, and 2-simplices $\sigma_1 = \langle v_0, v_1, v_2\rangle$ and $\sigma_2 = \langle v_0, v_2, v_3\rangle$. As the square has these two 2-simplices, we can think of it as being represented by $\sigma_1 + \sigma_2$ in $C_2(K)$, then $\partial(\sigma_1 + \sigma_2) = \langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_3\rangle - \langle v_0, v_3\rangle$, as the two occurrences of the diagonal $\langle v_0, v_2\rangle$ cancel out as they have opposite sign, and this is the path around the actual boundary of the square.

It is important to note that the boundary of a boundary is always trivial, that is, the composite mapping

$$C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \xrightarrow{\partial_{p-1}} C_{p-2}(K)$$

is the mapping sending everything to $0 \in C_{p-1}(K)$.

The idea of the higher Betti numbers, $\beta_p(K)$, is that they measure the number of p-dimensional 'holes' in K. Imagine we has a tunnel-shaped hole through a space K, then we would have a cycle around the hole at one end of the tunnel and another around the hole at the other end. If we merely count cycles then we will get at least two such coming from this hole, but these cycles are linked as there is the cylindrical hole itself and that gives a 2 dimensional element with boundary the difference of the two cycles. In general, a *p-cycle* will be an element x of $C_p(K)$ with trivial boundary, *i.e.*, such that $\partial x = 0$, and we say that two p-cycles x and x' are *homologous* if there is an element y in $C_{p+1}(K)$ such that $\partial y = x - x'$. The 'holes' correspond to classes of homologous cycles as in our tunnel.

The number of 'independent' cycle classes in the various dimensions give the corresponding Betti number. Using some algebra, this is easier to define rigorously, but, at the same time, the geometric insights from the vaguer description are important to try to retain. (They are not always put in a central enough position in textbooks!) This algebraic approach identifies $\beta_p(K)$ as the (torsion free) rank of a certain commutative group formed as follows: the p^{th} homology group of K is defined to be the quotient:

$$H_p(K) = \frac{Ker(\partial_p : C_p(K) \to C_{p-1}(K))}{Im(\partial_p : C_{p+1}(K) \to C_p(K))},$$

and then $\beta_p(K) = rank(H_p(K))$.

Thus far we have from K built a sequence of modules, $C(K)_n$, generated by the n-simplices of K and with homomorphisms $\partial_p : C_p(K) \to C_{p-1}(K)$ satisfying $\partial_{p-1}\partial_p = 0$.. (We abstract this structure calling it a *chain complex*. We will look at in more detail at several places later in these notes.)

Exercises: Try to investigate this homology in some very simple situations perhaps including some of the following:
(a) $V(K) = \{0, 1, 2, 3\}$, $S(K) = \mathscr{P}(V(K)) \setminus \{\emptyset, \{0, 1, 2, 3\}\}$. This is an empty tetrahedron so one expects one 3-dimensional hole., *i.e.*, $\beta_3(K) = 1$ but the others are zero.
(b) $\Delta[2]$ is the (full) triangle and $\partial\Delta[2]$ its boundary, so is an empty triangle. Find the homology of $\partial\Delta[2] \times \partial\Delta[2]$, which is a triangulated torus.
(c) Find the homology of $\Delta[1] \times \partial\Delta[2]$, which is a cylinder.

Note, it is up to you to find the meaning of product in this context. Remember the discussion of the square, above, which is, of course $\Delta[1] \times \Delta[1]$.

Often cohomology is more use than homology. Starting with K and a module M work out $C^n(K, M) = Hom(C(K)_n, M)$. Now the boundary maps increase (upper) degree by one. The cohomology is $H^n(K, M) = Ker\, \partial^n / Im\, \partial^{n-1}$. Again this measures 'holes' detectable by M! What does that mean? The cohomology groups are better structured than the homology ones, but how are these invariants be interpreted?

A simplicial map, $f : K \to L$, will induce a map on cohomology groups. Try it! We can equally well do this for chain or 'cochain complexes'. There is a notion of chain map between chain complexes, say, $\varphi : C \to D$ and such a map will induce maps on both homology ad cohomology. Of special interest is when the induced maps are isomorphisms. The chain map is then called a *quasi-isomorphism.*

1.2.5 Interpretation

The question of interpretation is a very crucial one but, rather than answering it now, we will return to the cohomology of groups. The terminology may seem a bit strange. Here we have been talking about measuring holes in a space, so how does that relate to groups. The idea is that one builds a space from a group in such a way as the properties of the space reflect those of the group in some sense. The simplest case of this is an Eilenberg - Mac Lane space, $K(G, 1)$. The defining property of such a space is that its fundamental group is G whilst all other homotopy groups are trivial. Eilenberg and Mac Lane showed that however such a space was constructed its cohomology could be got just from G itself and that cohomology was related with the extension problem and the invariant module problem. Their method was to build a chain complex that would copy the structure of the chain complex on the $K(G, 1)$. This chain complex, *the bar resolution*, was very important because although in the group case there was an alternative route via the topological space $K(G, 1)$, for many other types of algebraic system (Lie algebras, associative algebras, commutative algebras, etc.), the analogous basic construction could be used, and in those contexts no space was available. Thus from G, we want to construct a nice chain complex directly. The construction is reasonably simple. It gives a natural way of getting a chain complex, but it does not exploit any particular features of the group so if the group is infinite, the modules will be infinitely generated, which will occupy us later, as we use insights from combinatorial group theory to construct smaller models for equivalent resolutions, and better still look at 'crossed' versions.

For the moment we just need the definition (adapted from the account given in Wikipedia):

1.2.6 The bar resolution

The input data is a group G and a module M with a left G-action (*i.e.*, a left G-module).

For $n \geq 0$, we let $C^n(G, M)$ be the group of all *functions* from the n-fold product G^n to M:

$$C^n(G, M) = \{\varphi : G^n \to M\}$$

This is an Abelian group; its elements are called the *n-cochains*. We further define group homomorphisms

$$\partial^n : C^n(G, M) \to C^{n+1}(G, M)$$

by

$$\begin{aligned} \partial^n(\varphi)(g_0, \ldots, g_n) \quad = \quad & g_0 \cdot \varphi(g_1, \ldots, g_n) \\ & + \sum_{i=0}^{n-1} (-1)^{i+1} \varphi(g_0, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) \\ & + (-1)^{n+1} \varphi(g_0, \ldots, g_{n-1}) \end{aligned}$$

These are known as the *coboundary homomorphisms*. The crucial thing to check here is $\partial^{n+1} \circ \partial^n = 0$, thus we have a chain complex and we can 'compute' its cohomology. For $n \geq 0$, define the group of n-cocycles as:

$$Z^n(G, M) = Ker\,\partial^n$$

and the group of n-coboundaries as

$$\begin{cases} B^0(G, M) = 0 \\ B^n(G, M) = Im(\partial^{n-1}) \qquad n \geq 1 \end{cases}$$

and

$$H^n(G, M) = Z^n(G, M)/B^n(G, M).$$

Thinking about this topologically, it is as if we had constructed a sort of space / simplicial complex, K, out of G by taking $K_n = G^n$. We will see this idea many times later on. This cochain complex is often called the *bar resolution*. It exists in a normalised and a unnormalised form. This is the unnormalised one. It can also be constructed via a chain complex, sometimes denoted βG, so that this $C(G, M)$ is formed by taking $Hom(\beta G, M)$, in a suitable sense.

There are lots of properties that are easy to check here. Some will be suggested as exercises for you to do. For others, you can refer to some of the standard textbooks that deal with introductions to group cohomology, for instance, K. Brown's [56].

One further point is that this cohomology used a module, and so encodes 'commutative' or Abelian information. We will be also looking at the non-Abelian case.

Before we leave this introduction to cohomology, it should be mentioned that in the topological case, if we do not have a simplicial complex to start with, we either use the singular complex (see next section) which is a simplicial set and not a simplicial complex, but the theory extends easily enough, or we use open covers of the space to build a system of simplicial complexes approximating to the space. We will see this later as Čech cohomology. This is most powerful when the module M of coefficients is allowed to vary over the various points of the space. For this we will need the notion of sheaf, which will be discussed in some detail later.

1.3 Simplicial things in a category

1.3.1 Simplicial Sets

Simplicial objects are extremely useful. Simplicial sets extend ideas of simplicial complexes in a neat way. They combine a reasonably simple combinatorial definition with subtle algebraic properties. Their original construction was motivated in algebraic topology by the singular complex of a space.

If X is a topological space, $Sing(X)$ denotes the collection of sets and mappings defined by

$$Sing(X)_n = Top(\Delta^n, X), \qquad\qquad n \in \mathbb{N},$$

where Δ^n is the usual topological n-simplex given, for example, by

$$\{\underline{x} \in \mathbb{R}^{n+1} \mid \sum x_i = 1; \;\text{ all } x_i \geq 0\}.$$

There are inclusion maps, $\delta_i : \Delta^{n-1} \to \Delta^n$, and 'squashing' maps, $\sigma_i : \Delta^{n+1} \to \Delta^n$, and these induce the face maps,

$$d_i : Sing(X)_n \to Sing(X)_{n-1}, \qquad 0 \leq i \leq n,$$

and degeneracy maps,

$$s_i : Sing(X)_n \to Sing(X)_{n+1}, \qquad 0 \leq i \leq n.$$

These satisfy the simplicial identities,

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \qquad \text{if } i < j, \\ d_i s_j &= \begin{cases} s_{j-1} d_i & \text{if } \quad i < j, \\ id & \text{if } i = j \quad \text{or } j+1, \\ s_j d_{i-1} & \text{if } i > j+1, \end{cases} \\ s_i s_j &= s_j s_{i-1} \qquad \text{if } \quad i > j. \end{aligned}$$

This structure is abstracted to give a family of sets, $\{K_n : n \geq 0\}$, face maps, $d_i : K_n \to K_{n-1}$ and degeneracy maps, $s_i : K_n \to K_{n+1}$, satisfying these simplicial identities. The result is a *simplicial set*.

Remark: Using the singular complex, we can proceed much as in our earlier discussion to define *singular homology groups* for a space. Starting from $Sing(X)$, take a free Abelian group in each dimension then take the alternating sum of the faces to get a boundary map and thus a chain complex, $C(X)$, then take the homology of that. (We do not give details as this is very readily available in standard texts on algebraic topology.)

If $\mathcal{C}$ is any category, a simplicial object in $\mathcal{C}$ is given by a family of objects of $\mathcal{C}$, $\{K_n : n \geq 0\}$, and *morphisms* d_i and s_i as above. If $\mathbf{\Delta}$ denotes the category of finite ordinal sets, $[n] = \{0 < 1 < \ldots < n\}$ and order preserving functions between them, then a simplicial object in $\mathcal{C}$ is simply a functor, $K : \mathbf{\Delta}^{op} \to \mathcal{C}$, so the obvious definition of a simplicial map will be a natural transformation of functors, $f : K \to L$. This translates as a family of morphisms, $f_n : K_n \to L_n$, compatible in the obvious way with the d_i and s_i.

Notation and terminology: Of particular importance amongst the maps of $\mathbf{\Delta}$ are two families of very simple maps, the *coface morphisms* and the *codegeneracy morphisms*. The coface maps, δ_i or, using a more precise notation, $\delta_i^n : [n-1] \to [n]$ for $0 \leq i \leq n$ are injective maps, and δ_i^n misses out i from its image. For instance, for $n = 2$, δ_1^s sends 0 to 0 and 1 to 2, so picks out the face opposite the object 1 in the above picture.

For $0 \leq i \leq n$, the i^{th} codegeneracy morphism, $\sigma_i^n : [n+1] \to [n]$, is the surjective order preserving map that 'doubles up' the image of i, so, for instance, for $n = 1$, $\sigma_1^2(0) = 0$, whilst $\sigma_1^2(1) = \sigma_1^2(2) = 1$. Usually for both coface and codegeneracy morphisms, we will omit the dimensional superfix if there is little danger of confusion.

These morphisms generate $\mathbf{\Delta}$ with some simple relations between them given by 'cosimplicial identities'.

We denote the category of simplicial objects in $\mathcal{C}$ by $Simp(\mathcal{C})$ or $Simp.\mathcal{C}$, but will shorten $Simp(Sets)$ to $\mathcal{S}$.

The category, $\mathcal{S}$, models all homotopy types of spaces. It is a presheaf category, so is a topos and has a lot of nice structure including products, and mapping space objects $\underline{\mathcal{S}}(K, L)$, where

$$\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L).$$

Here $\Delta[n] = \mathbf{\Delta}(-, [n])$, the standard simplicial n-simplex. This has a special n-simplex, namely the element ι_n in $\Delta[n]_n$ determined by the identity map.

The Yoneda lemma, from category theory, gives us an isomorphism $\mathcal{S}(\Delta[n], K) \cong K_n$, and so, for any n-simplex, x, gives us a simplicial map $\ulcorner x \urcorner : \Delta[n] \to K$, which is sometimes called the *name*, or *representing map* of x. From $\ulcorner x \urcorner$, you get x back by evaluating on $\ulcorner x \urcorner$ on ι_n.

Examples of simplicial sets.

First let us have a trivial example, ..., trivial but often very useful.

Definition: Given a set, X, the discrete simplicial set, $K(X, 0)$, is defined to have $K(X, 0)_n = X$ for all n and to have all face and degeneracy maps given by the identity function on X. A simplicial set, K, is said to be *discrete* if it is isomorphic to one of form $K(X, 0)$ for some set X. (An easy extension gives the notion of discrete simplicial object in a category.)

With more substance, we have the following examples:

(i) If $\mathcal{A}$ is a small category or a groupoid, we can form a simplicial set, $Ner(\mathcal{A})$, defined by $Ner(\mathcal{A})_n = Cat([n], \mathcal{A})$, with the obvious face and degeneracy maps induced by composition with the analogues of the δ_i and σ_i. The simplicial set, $Ner(\mathcal{A})$, is called the *nerve of the category* $\mathcal{A}$. An n-simplex in $Ner(\mathcal{A})$ is a sequence of n composable arrows in $\mathcal{A}$.

This is easier to understand in pictures:

$Ner(A)_0$ is the set of objects;

$Ner(A)_1$ is the set of arrows or morphisms;

$Ner(A)_2$ is the set of composable pairs of morphisms, so $\sigma \in Ner(\mathcal{A})_2$ will be of form $\sigma = (a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2)$. Visualising this as a triangle shows the faces more clearly:

$$\begin{tikzcd} & a_1 \arrow[dr, "\alpha_2"] & \\ a_0 \arrow[ur, "\alpha_1"] \arrow[rr, "\alpha_1\alpha_2"'] & & a_2 \end{tikzcd}$$

The case $Ner(A)_n$ for $n = 3$, etc. are left to you. This *is* worth doing if you have not seen it before.

Note that in these contexts, we will sometimes use composition in the 'left-to-right' order, but in general categorical settings will use gf being first do f then g. To stick exclusively to one or the other is usually awkward, so we use both as appropriate. This sometimes means we have to take extra care over the conventions that we are using at a particular time.

If we have a group, G, consider it as the one object groupoid $G[1]$ as before, then $Ner(G[1])$ is really the simplicial set corresponding to our construction of the bar resolution of G. It is called the *nerve of* G, and is a *classifying space* for G, an aspect that we will explore later in some detail.

If we have a *discrete category* $\mathcal{A}$, *i.e.*, $\mathcal{A}$ has no non-identity morphisms between objects, then $\mathcal{A}$ is really just a set, and $Ner(\mathcal{A})$ is a discrete simplicial set.

(ii) Suppose we have a simplicial complex K, then it almost is a simplicial set. There are some problems, but they are easily resolved. If we, a bit naïvely, set K_n to be the set of n-simplices of K, then how are we to define the face maps, and if K has no simplices in dimensions greater than n say, K_{n+1} will be empty so degeneracies cause problems as you cannot map from a non-empty set to an empty one!

That was too naïve, so we pick a partial order on the vertices of K such that any simplex is totally ordered, (for instance, a total order on $V(K)$ does the job, but may not be convenient sometimes and so may be 'overkill'). Now, reset K_n to be the set of all ordered strings, $\sigma = \langle x_0, \ldots, x_n\rangle$ of vertices, for which the underlying (unordered) set is a simplex of K. The degeneracies now can be handled simply. For example, if $\sigma = \langle x_0, x_1\rangle$ is a 1-simplex in this simplicial set, then $s_0\sigma = \langle x_0, x_0, x_1\rangle$, whilst $s_1\sigma = \langle x_0, x_1, x_1\rangle$. (The details are left to you to complete. Note we did not specify how to define the face maps, so you need to do that as well and to verify that it all fits together neatly.) This construction gives a reasonably small simplicial set model of the simplicial complex, but is not a functorial construction.

There is another way of going from simplicial complexes to simplicial sets that we will explore in detail later on, in section 4.4.3. For the moment, we will merely give the 'formula'. If K is a simplicial complex, this time we take K_n^{simp} to be the set of $(n+1)$-tuples of elements from the set, $V(K)$, of vertices of K such that the underlying set (*i.e.*, removing any repetitions) is a simplex of K. This construction is functorial, so it converts maps of simplicial complexes to maps of simplicial sets as well as converting the objects. This is an advantage over the first method we mentioned, but at the price of a *much* larger set of simplices in all dimensions.

If you want to learn more about basic simplicial set theory, the old paper of Curtis, [97] and Peter May's monograph, [198], are very readable. There is a fairly well behaved notion of homotopy in $\mathcal{S}$, and simplicial homotopy theory is the subject of many good books. A chatty introduction to it can be found in Kamps and Porter, [171], which, of course, is highly recommended!

The homotopy theory of simplicial sets yields a notion of weak equivalence. (This is similar to 'quasi-isomorphism' in the homotopy theory of chain complexes.) There are homotopy groups and $f : K \to L$ is a weak equivalence if f induces isomorphisms on all homotopy groups. We will not need the detailed definition yet.

We next look at some simplicial algebraic gadgets, especially simplicial groups and simplicially enriched groupoids. We will concentrate on the first but must mention the second for completeness.

1.3.2 Simplicial Objects in Categories other than *Sets*

If $\mathcal{A}$ is any category, we can form $Simp.\mathcal{A} = \mathcal{A}^{\mathbf{\Delta}^{op}}$. (Sometimes we will use a variant notation: $Simp(\mathcal{A})$, as occasionally the first notation may be ambiguous.)

These categories often have a good notion of homotopy as briefly mentioned above; see also the discussion of simplicially enriched categories in [171]. Of particular use are:

(i) *Simp.Ab*, the category of simplicial Abelian groups. This is equivalent to the category of chain complexes by the Dold-Kan theorem, which we will mention in more detail later.

(ii) *Simp.Grps*, the category of simplicial groups. This 'models' all connected homotopy types, by Kan, [173] (cf., Curtis, [97]). There are adjoint functors, $G : \mathcal{S}_{conn} \to Simp.Grps$, $\overline{W} : Simp.Grps \to \mathcal{S}_{conn}$, with the two natural maps $G\overline{W} \to Id$ and $Id \to \overline{W}G$ being weak equivalences.

Results on simplicial groups by Carrasco, [76], generalise the Dold-Kan theorem to the non-Abelian case, (cf., Carrasco and Cegarra, [77]).

(iii) '*Simp.Grpd*': in 1984 Dwyer and Kan, [113], (and also Joyal and Tierney, and Duskin and van Osdol, cf., Nan Tie, [218, 219]) noted how to generalise the $(G, \overline{W})$ adjoint pair to handle all simplicial sets, not just the connected ones. (Beware there are several important printing errors in the paper [113].) For this they used a special type of simplicial groupoid. Although the term used in [113] was exactly that, 'simplicial groupoid', this is really a misnomer and may give the wrong impression, as not all simplicial objects in the category of groupoids are used. A probably better term would be 'simplicially enriched groupoid', although 'simplicial groupoid with discrete objects' is also used. We will denote this category by $\mathcal{S}-Grpd$.

This category 'models' all homotopy types using a mix of algebra and combinatorial structure.

We will later describe both G and $\overline{W}$ in some detail, and will use simplicially enriched groupoids and simplicially enriched categories as well.

(iv) *Nerves of internal categories:* Suppose that $\mathcal{D}$ is a category with finite limits and C is an internal category in $\mathcal{D}$. What does that mean? In our earlier discussion on groupoids, we had the diagram that looked a bit like

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & C_0 \arrow[l, shift left=2, "i"] \end{tikzcd} .$$

We complete this one stage to build in the set of composable pairs $C_2 = C_1 \times_{C_0} C_1$ and the multiplication/ composition map, which we denote here by m.

$$\begin{tikzcd} C_2 \arrow[r, shift left=3, "p_1"] \arrow[r, "m"] \arrow[r, shift right=3, "p_2"'] & C_1 \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & C_0 \arrow[l, shift left=2, "i"] \end{tikzcd} .$$

We did this previously within the category of sets, but could do it equally well in $\mathcal{D}$. We should also mention an object C_3 given by a 'triple pullback', which is useful when discussing the associativity of composition. This will give us the analogue of a small category, but in which the object of objects and the object of arrows are both themselves objects of $\mathcal{D}$ and the source target and composition maps are all morphisms in that category.

If one interprets this for $\mathcal{D} = Sets$, it becomes clear that this diagram that we seem to be building is part of the diagram specifying the nerve of the small category, C, with C_0 the set of objects, C_1 that of morphisms, C_2 that of composable pairs and so on. (We have not specified the two degeneracies from C_1 to C_2 in the diagram, but this is merely because we left the details of the rules governing identities out of our earlier discussion.) This builds a simplicial object in $\mathcal{D}$ as follows: take an n-fold pullback to get

$$C_n = \underbrace{C_1 \times_{C_0} C_1 \times_{C_0} C_1 \times_{C_0} \ldots \times_{C_0} C_1}_{n},$$

define face and degeneracies by the same sort of rules as in the set based nerve, that is, in dimension n, d_0 and d_n each leave out an end, whilst the d_i use the composition in the category to get a composite of two adjacent 'arrows', and the degeneracies are 'insertion of identities'. (Working out how to do these morphisms in terms of diagrams is quite fun!) We thus get a simplicial object in

$\mathcal{D}$ called the *nerve of the internal category, C*. We will use this in several situations later in a key way. In particular, we will use the case $\mathcal{D} = Grps$.

Later on, we will use internal functors and natural transformations as well. For the moment, the description of these structures is **left to you**. Notationally, we will write $Cat(\mathcal{D})$ for the category of internal categories in $\mathcal{D}$. As you might expect, the above nerve construction is a functor from $Cat(\mathcal{D})$ to $Simp(\mathcal{D})$. (If you know about such things, you might also expect that $Cat(\mathcal{D})$ can be thought of as a 2-category, ..., you would be right, but we will leave that until much later on.)

(v) *Bisimplicial and multisimplicial objects:* A useful category in which we can take simplicial objects is $\mathcal{S}$ itself, and the same is true for other categories of form $Simp(\mathcal{A})$. For simplicity we will start by looking at simplicial objects in $\mathcal{S}$.

As a simplicial object in a category $\mathcal{A}$ is just a functor from $\mathbf{\Delta}^{op}$ to $\mathcal{A}$, a simplicial object in $\mathcal{S}$ is such a functor taking values that themselves are functors from $\mathbf{\Delta}^{op}$ to $Sets$. Another way to look at these is a 'functor of two variables' using a categorical version of the way that a function of two variables, $f : X \times Y \to Z$, can be thought of as a function $\tilde{f} : X \to Z^Y$ from X to the set of functions from Y to Z. Of course, $f(x, y) = \tilde{f}(x)(y)$ and similarly for the functors. We thus have a description of a simplicial object in $\mathcal{S}$ as corresponding to a functor $X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to Sets$.

Definition: A *bisimplicial set* is a functor $X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to Sets$. . A *morphism of bisimplicial sets*, $f : X \to Y$ is a natural transformation between the corresponding functors. More generally a *bisimplicial object in a category* $\mathcal{A}$ is a functor $X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathcal{A}$, similarly for the corresponding morphisms. The corresponding categories will denoted $Bi\mathcal{S} := BiSimp(Sets)$ and in general $BiSimp(\mathcal{A})$.

A simplicial set can be specified by giving sets X_n and face and degeneracy 'operators' between them satisfying the simplicial idenities. A bisimplicial set is similarly specified by a bi-indexed family of sets $X_{p,q}$ and two families of simplicial operators. We may use the terms 'horizontal' and 'vertical' for these two families as that is how the corresponding diagrams are often drawn. For instance, the bottom part of a bisimplicial set will look a bit like the following:

$$
\begin{tikzcd}
 & \vdots \arrow[d, shift right=2, "d_2^v"'] \arrow[d] \arrow[d, shift left=2, "d_0^v"] & \vdots \arrow[d, shift right=2, "d_2^v"'] \arrow[d] \arrow[d, shift left=2, "d_0^v"] \\
\cdots \arrow[r, shift left=2, "d_0^h"] \arrow[r] \arrow[r, shift right=2, "d_2^h"'] & X_{1,1} \arrow[r, shift left, "d_0^h"] \arrow[r, shift right, "d_1^h"'] \arrow[d, shift right, "d_1^v"'] \arrow[d, shift left, "d_0^v"] & X_{0,1} \arrow[d, shift right, "d_1^v"'] \arrow[d, shift left, "d_0^v"] \\
\cdots \arrow[r, shift left=2, "d_0^h"] \arrow[r] \arrow[r, shift right=2, "d_2^h"'] & X_{1,0} \arrow[r, shift left, "d_0^h"] \arrow[r, shift right, "d_1^h"'] & X_{0,0}
\end{tikzcd}
$$

(As usual in such diagrams, there is not really room to show the degeneracy maps and so these are omitted from the picture.) In addition to the simplicial identities holding in each direction, each horizontal face or degeneracy has to be a simplicial map between the vertical simplicial sets. Practically this means that the diagram must commute.

We will later meet bisimplicial groups, and also briefly multisimplicial objects in which the number of variables is not limited to two. For instance, the nerve of a simplicial group is most naturally viewed as a bisimplicial set, and similarly the nerve of a bisimplicial group is a trisimplicial set, that is a functor from $\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$ to $Sets$. There are ways of passing between such things as we will see later.

(vi) *Cosimplicial things:* At certain points in the development of cohomology and related areas we will have need to talk of cosimplicial sets.

Definition: A *cosimplicial set* is a functor, $K : \mathbf{\Delta} \to Sets$, and a morphism of such is a natural transformation between the corresponding functors. The category of such will be denoted $CoSimp(Sets)$, and similarly for the obvious generalisations to other settings, namely *cosimplicial objects in a category* $\mathcal{A}$, being functors $K : \mathbf{\Delta} \to \mathcal{A}$ with corresponding morphisms forming a category $CoSimp(\mathcal{A})$.

This looks at one and the same time very similar and very different to simplicial objects. Certainly analysis of, say, simplicial groups is much easier than that of cosimplicial groups, but, as any functor, $K : \mathbf{\Delta} \to \mathcal{A}$, gives uniquely a functor, $K^{op} : \mathbf{\Delta}^{op} \to \mathcal{A}^{op}$, a cosimplicial object is also a simplicial object in the opposite category. The problem, thus, is that often the opposite category of a well known category, such as that of groups, is a lot less nice. Even the dual of $Sets$ is not that 'well behaved'.

Conjugation: There is an 'inversion' operation on each finite ordinal in $\mathbf{\Delta}$, which reverses the order on the ordinal, that is, it sends $\{0 < 1 < \ldots < n\}$ to $\{0 > 1 > \ldots > n\}$. Of course the resulting object is isomorphic to the original, but this isomorphism is not compatible with the face or degeneracy maps. This operation induces an operation on simplicial objects, that we will call *conjugation*.

Definition: Given a simplicial object, X in a category $\mathcal{A}$, the *conjugate simplicial object*, $ConjX$, is defined by

$$(ConjX)_n = X_n,$$

$$d_i : (ConjX)_n \to (ConjX)_{n-1} = d_{n-i} : X_n \to X_{n-1}$$

for each $0 \leq i \leq n$, and, similarly,

$$s_i : (ConjX)_n \to (ConjX)_{n+1} = s_{n-i} : X_n \to X_{n+1}.$$

Clearly X and $ConjX$ are closely related. For instance, they have isomorphic geometric realisation, isomorphic homotopy groups, ..., but the actual comparisons are quite difficult to give because there are, in general, very few simplicial morphisms from X to $ConjX$. The conjugate of a simplicial set is also sometimes called its *opposite*. This works well with regard to the construction of the nerve of a category.

Example: In some contexts, a situation naturally leads to a variant form of the nerve functor being used. Suppose that $\mathcal{A}$ is a category. Our usual notation for an n-simplex in $Ner(\mathcal{A})$ would be something like $(a_0 \overset{\alpha_1}{\to} a_1 \to \ldots \overset{\alpha_n}{\to} a_n)$, but sometimes the order of the terms is reversed as it may be more natural, in certain situations, to use $(a'_n \overset{\alpha'_n}{\to} a'_{n-1} \to \overset{\alpha'_1}{\to} a'_0)$. This might typically arise if one has a right action of some group instead of the left actions that we will tend to meet more often. It also occurs sometimes in the way that terms of the Bousfield-Kan form of the homotopy colimit construction are presented, (see the comment on page 688). The link between the two forms is $a'_i = a_{n-i}$ and $\alpha'_i = \alpha_{n-i+1}$. The face operators delete or compose in the conjugate way. Of course, the nerve based on this notational form is the conjugate of the one we have defined earlier. We will refer to it as the *conjugate nerve* of the category.

We should also note that, if $\mathcal{A}$ is a (small) category, and $\mathcal{A}^{op}$ its opposite then $Ner(\mathcal{A}^{op})$ and $ConjNer(\mathcal{A})$ are isomorphic simplicial sets. Because of this we may occasionally write $Ner(\mathcal{A})^{op}$ as a synonym for $ConjNer(\mathcal{A})$.

1.3.3 The Moore complex and the homotopy groups of a simplicial group

Given a simplicial group G, the *Moore complex*, (NG, ∂), *of* G is the chain complex defined by

$$NG_n = \bigcap_{i=1}^{n} Ker\, d_i^n$$

with $\partial_n : NG_n \to NG_{n-1}$ induced from d_0^n by restriction. (Note there is no assumption that the NG_n are Abelian.)

The n^{th} *homotopy group*, π_n(G), of G is the n^{th} homology of the Moore complex of G, *i.e.*,

$$\begin{aligned} \pi_n(G) &\cong H_n(NG, \partial), \\ &= \big(\textstyle\bigcap_{i=0}^{n} Ker\, d_i^n\big)/d_0^{n+1}\big(\bigcap_{i=1}^{n+1} Ker\, d_i^{n+1}\big). \end{aligned}$$

(You should check that $\partial NG_{n+1} \lhd NG_n$.)

The interpretation of NG and $\pi_n(G)$ is as follows:

for $n = 1$, $g \in NG_1$,

$$1\bullet \xrightarrow{g} \bullet\partial g$$

and $g \in NG_2$ looks like

$$\begin{tikzcd} & \bullet \arrow[dr, "\partial g"] \arrow[d, phantom, "g"] & \\ \bullet \arrow[ur, "1"] \arrow[rr, "1"'] & & \bullet \end{tikzcd}$$

and so on.

We note that $g \in NG_2$ is in $Ker\,\partial$ if it looks like

$$\begin{tikzcd} & \bullet \arrow[dr, "1"] \arrow[d, phantom, "g"] & \\ \bullet \arrow[ur, "1"] \arrow[rr, "1"'] & & \bullet \end{tikzcd}$$

whilst it will give the trivial element of $\pi_2(G)$ if there is a 3-simplex x with g on its third face and all other faces are the identity element of the corresponding dimension.

This simple interpretation of the elements of NG and $\pi_n(G)$ will 'pay off' later by aiding interpretation of some of the elements in other situations. The homotopy groups we have introduced above have been defined purely algebraically as homology of a related complex. Any simplicial group gives us a base pointed simplicial set simply by forgetting the group structure and taking the identity element as the base point. Any pointed simplicial set gives homotopy groups in two different ways. There is an intrinsic way that is described in detail in, for instance, May's book, [198], but they can also be defined via a geometric realisation, which produces a space from the simplicial set. These two ways always give the same answer, and in the case that we are looking at of an underlying simplicial set of a simplicial group, this group coincides with that defined via the Moore complex. (This is easily found in the literature if you want to check up on it, so we will not repeat it here.)

n-equivalences and homotopy n-types Let $n \geq 0$. A morphism, $f : G \to H$, of simplicial group(oid)s is an *n-equivalence* if the induced homomorphisms, $\pi_k(f) : \pi_k(G) \to \pi_k(H)$ are isomorphisms for all $k < n$.

Inverting the n-equivalences in $Simp.Grps$ gives a category $Ho_n(Simp.Grps)$ and two simplicial groups have the same *n-type* if they are isomorphic in $Ho_n(Simp.Grps)$.

Remark and warning: For a space or simplicial set K, $\pi_k(K) \cong \pi_{k-1}(\mathcal{G}(K))$, so these simplicial group n-types correspond to restrictions on $\pi_k(K)$ for $k \leq n$ in the spatial context.

To consider the application of this to homotopical and homological algebra, we will also need the following:

Definitions: (i) A simplicial group, G, is *augmented* by specifying a constant simplicial group $K(G_{-1}, 0)$ and a surjective group homomorphism, $f = d_0^0 : G_0 \to G_{-1}$ with $fd_0^1 = fd_1^1 : G_1 \to G_{-1}$. An *augmentation* of the simplicial group G is then a map

$$G \longrightarrow K(G_{-1}, 0),$$

where $K(G_{-1}, 0)$ is the constant simplicial group with value G_{-1}.

(ii) An augmented simplicial group, (G, f), is *acyclic* if the corresponding complex is acyclic, *i.e.*, $H_n(NG) \cong 1$ for $n > 0$ and $H_0(NG) \cong G_{-1}$.

Remarks: (i) The above notions are just particular instances of the general notion of an *augmented simplicial object* in a category, and the corresponding idea of *acyclic* such things in settings where the definition makes sense.

(ii) When considering augmented simplicial objects, we sometimes use the notation d_0 or d_0^0 for the augmentation map as then the condition $fd_0^1 = fd_1^1$ becomes $d_0 d_0 = d_0 d_1$, which is a natural extension of the simplicial identities.

1.3.4 Kan complexes and Kan fibrations

Within the category of simplicial sets, there is an important subcategory determined by those objects that satisfy the Kan condition, that is the *Kan complexes*.

As before we set $\Delta[n] = \mathbf{\Delta}(-, [n]) \in \mathcal{S}$, then, for each i, $0 \leq i \leq n$, we can form, within $\Delta[n]$, a subsimplicial set, $\Lambda^i[n]$, called the *(n, i)-horn* or *(n, i)-box*, by discarding the top dimensional n-simplex (given by the identity map on $[n]$) and its i^{th} face. We must also discard all the degeneracies of those simplices.

By an *(n, i)-horn* or *box* in a simplicial set K, we mean a simplicial map $f : \Lambda^i[n] \to K$. Such a simplicial map corresponds geometrically and intuitively to a family of n simplices of dimension

$(n-1)$, fitting together to form a 'funnel' or 'empty horn' shaped subcomplex within K. The family is thus a sequence, $(k_0, \ldots, k_{i-1}, -, k_{i+1}, \ldots, k_n)$, with each $k_\ell \in K_{n-1}$, satisfying $d_\ell k_j = d_{j-1}k_\ell$, for $\ell < j$, whenever both k_ℓ and k_j are in the sequence. The idea is that a Kan fibration of simplicial sets is a map in which the horns in the domain can be 'filled' if their images in the codomain can be. More formally:

Definition: A map, $p : E \to B$, is a *Kan fibration* if, for any n, i as above, given any (n, i)-horn in E, specified by a map $f_1 : \Lambda^i[n] \to E$, together with an n-simplex, $f_0 : \Delta[n] \to B$, such that

$$\begin{tikzcd}
\Lambda^i[n] \arrow[r, "f_1"] \arrow[d, "inc"'] & E \arrow[d, "p"] \\
\Delta[n] \arrow[r, "f_0"'] & B
\end{tikzcd}$$

commutes, then there is an $f : \Delta[n] \to E$ such that $pf = f_0$ and $f.inc = f_1$, *i.e.*, f lifts f_0 and extends f_1.

We also say that p satisfies the *Kan lifting condition* if this is true.

Definition: A simplicial set, K, is a *Kan complex* if the unique map $K \to \Delta[0]$ is a Kan fibration. This is equivalent to saying that every horn in K has a filler, *i.e.*, any $f_1 : \Lambda^i[n] \to Y$ extends to an $f : \Delta[n] \to Y$.

Singular complexes, $Sing(X)$, and the simplicial mapping spaces, $\underline{Top}(X, Y)$, are always Kan complexes.

Lemma 1 *The nerve of a category, $\mathcal{C}$, is a Kan complex if and only if the category is a groupoid.* ■

The proof is left to **the reader**.

This is very important as the filler structure involves compositions and inverses, so encodes the *algebraic* structure of $\mathcal{C}$. Later we will use this many times, sometimes explicitly, but often it will be giving structure behind the scenes, for instance, internally within some other category.

There is a property of Kan fibrations, that is *very useful*, namely that *the pullback of a Kan fibration along a simplicial map is again a Kan fibration.* More precisely:

Proposition 1 *Let $p : E \to B$ be a Kan fibration, and let $f : X \to B$ be a simplicial map, and form the pullback of p along f, written $f^*(p) : E_f \to X$. This map is a Kan fibration.*

Proof: (Just to help you think about $f^*(p) : E_f \to X$ more concretely, first note that $f^*(p) : E_f \to X$ is only really defined up to isomorphism as it is given by a universal property in the usual way, but we can find a particular 'model' of that isomorphism class of potential things as follows. Look at the simplicial set $X \times_B E$, where

$$(X \times_B E)_n = \{(x, e) \mid x \in X_n, e \in E_n, f(x) = p(e)\}$$

and where face and degeneracy maps are defined componentwise, so $d_i(x,e) = (d_i(x), d_i(e))$, etc. The map, $f^*(p)$ is then represented by the first projection. We will not use this model explicitly. It is just there to help you if need be. Make sure you have looked up the universal property of pullbacks as we will need it.)

We have a pullback square:

$$\begin{tikzcd} E_f \arrow[r, "f'"] \arrow[d, "f^*(p)"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B. \end{tikzcd}$$

Now assume we are given a diagram

$$\begin{tikzcd} \Lambda^i[n] \arrow[r, "f_1"] \arrow[d, "inc"'] & E_f \arrow[d, "f^*(p)"] \\ \Delta[n] \arrow[r, "f_0"'] & X \end{tikzcd}$$

and we seek a lift of f_0 to E_f. Composing f_0 and f on the base, and f_1 and f' up top, and using the Kan fibration property of p, we get a lift, g, of ff_0 to E. (**Draw the diagram.**) Using the maps f_0 and g, you check that $ff_0 = pg$, and the universal property of the original pullback square gives you a map, h, say, to E_f. It now just remains to check that this is a lift of f_0, and an extension of f_1, and checking that is **left to you**. ■

This result is often stated by saying that the class of Kan fibrations is *pullback stable.*

1.3.5 Simplicial groups are Kan

If G is a simplicial group, then its *underlying simplicial set* is a Kan complex. Moreover, given a box in G, there is an algorithm for filling it using products of degeneracy elements. A form of this algorithm is given below. More generally if $f : G \to H$ is an epimorphism of simplicial groups, then the underlying map of simplicial sets is a Kan fibration.

The following description of the algorithm is adapted from May's monograph, [198], page 67.

Proposition 2 *Let G be a simplicial group, then every box has a filler.*

Proof: Let $(y_0, \ldots, y_{k-1}, -, y_{k+1}, \ldots, y_n)$ give a horn in G_{n-1}, so the y_is are $(n-1)$ simplices that fit together as if they were all but one, the k^{th} one, of the faces of an n-simplex. There are three cases:

(i) $k = 0$: Let $w_n = s_{n-1}y_n$ and then $w_i = w_{i+1}(s_{i-1}d_iw_{i+1})^{-1}s_{i-1}y_i$ for $i = n, \ldots, 1$, then w_1 satisfies $d_iw_1 = y_i$, $i \neq 0$;

(ii) $0 < k < n$: Let $w_0 = s_0y_0$ and $w_i = w_{i-1}(s_id_iw_{i-1})^{-1}s_iy_i$ for $i = 0, \ldots, k-1$, then take $w_n = w_{k-1}(s_{n-1}d_nw_{k-1})^{-1}s_{n-1}y_n$, and finally a downwards induction given by $w_i = w_{i+1}(s_{i-1}d_iw_{i+1})^{-1}s_{i-1}y_i$, for $i = n, \ldots, k+1$, then w_{k+1} gives $d_iw_{k+1} = y_i$ for $i \neq k$;

(iii) the third case, $k = n$ uses $w_0 = s_0y_0$ and $w_i = w_{i-1}(s_id_iw_{i-1})^{-1}s_iy_i$ for $i = 0, \ldots, n-1$, then w_{n-1} satisfies $d_iw_{n-1} = y_i$, $i \neq n$. ■

Some discussion of how you can think of this algorithm can be found in [171].

(You could see if you can adapt the idea of this proof to prove the result mentioned immediately before the statement, namely: if $f : G \to H$ is an epimorphism of simplicial groups, then the underlying map of simplicial sets is a Kan fibration. What about the converse?)

Later on we will meet the simplicial mapping space, $\underline{\mathcal{S}}(K, L)$, of simplicial maps from K to L. It is defined by $\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L)$, with the obvious induced maps. It is easy to see that if L is a Kan complex, then so is $\underline{\mathcal{S}}(K, L)$, for any K. (**Try to prove it**, but then look at May, [198], to compare your attempt with his proof.) This result has a useful generalisation that we will state as a lemma, but again will leave **you to give or find a proof**.

Lemma 2 *If $p : L \to M$ is a Kan fibration, and K is an arbitrary simplicial set, then the induced map, $\underline{\mathcal{S}}(K, p) : \underline{\mathcal{S}}(K, L) \to \underline{\mathcal{S}}(K, M)$, is also one.* ■

(To give you a hint consider what a horn in $\underline{\mathcal{S}}(K, L)$ looks like, and likewise what an n-simplex in $\underline{\mathcal{S}}(K, M)$ is. Why should you be able to put the information together to build an n-simplex in $\underline{\mathcal{S}}(K, L)$? Look at low dimensional examples to build up some geometric intuition about what is going on. That is important even if you later look up a proof as not every proof that you will find gives the intuitive idea behind.)

1.3.6 T-complexes

There is quite a difference between the Kan complex structure of the nerve of a groupoid, G, and that of a singular complex. In the first, if we are given a (n, i)-horn, then there is *exactly one* n-simplex in $Ner(G)$, since the (n, i)-horn has a chain of n-composable arrows of G in it (at least unless $(n, i) = (2, 0)$ or $(2, 2)$, which cases are **left to you**) and that chain gives the required n-simplex. In other words, there is a 'canonical' filler for any horn. In $Sing(X)$, there will usually be many fillers. (Think about why this is true.)

One attempt to handle 'canonical fillers' interacts with a notion that we will encounter later on, namely that of crossed complexes, for which see section 3.1. The resulting notion of a simplicial T-complex is one sort of 'Kan complex with canonical fillers' and various of the intuitions and arguments that this introduces will recur frequently in the following chapters. It assumes there is always a unique special filler. There may be other non-special ones, but that is not controlled in the process, as we will see. Simplicial T-complexes were introduced by Dakin, [98]:

Definition: A *simplicial T-complex* consists of a pair (K, T), where K is a simplicial set and $T = (T_n)_{n\geq 1}$ is a graded subset of K with $T_n \subseteq K_n$. Elements of T are called *thin*. The thin structure satisfies the following axioms:

$T.1$ Every degenerate element is thin.

$T.2$ Every box in K has a unique thin filler.

$T.3$ A thin filler of a thin box also has its last face thin.

Example: The nerve of a groupoid has a T-complex structure in which each simplex of dimension greater than or equal to 2 is thin. Our earlier comments give the proof. Conversely, if (K, T) is a T-complex with $T_n = K_n$ for all $n \geq 2$, then K is the nerve of a groupoid with set of objects

K_0 and set of arrows, K_1. (It is **left to you** to see how to compose arrows, to prove that it is an associative composition, and that there are identities at all objects.)

A box or horn is, of course, as in section 1.3.4, a collection of n-simplices that fits together like the collection of all but one faces of an $(n+1)$-simplex. The collection of such n-boxes with given face missing can be formulated in terms of a pullback and hence axioms $T2$ and $T3$ can be encoded in a form suitable for adapting to other contexts. Similar ideas are used by Duskin, [107], and Nan-Tie, [218, 219], and also by Street, [253], and Verity, [266–269], and we will have occasion to refer back to both of these later. We will need to adapt those ideas initially to T-complexes within the setting of groups (group T-complexes as below) but later we may need them in various other settings. Group T-complexes were briefly considered by Ashley, [12, 13], but their main theory has been clarified and extended by Carrasco, [76], and Cegarra and Carrasco, [77], using ideas that will be discussed briefly later.

1.3.7 Group T-complexes

Definition: A *group T-complex* is "a T-complex, (G,T), in which G is a simplicial group and T is a graded subgroup of G", (Ashley, [13]).

Ashley proved a series of results that gave a neat alternative formulation of this concept. We note the following observations:

Lemma 3 *Let $D = (D_n)_{n\geq 1}$ be the graded subgroup of G generated by the images of the degeneracy maps, $s_i : G_n \to G_{n+1}$, for all i and n, then any box in G has a standard filler in D.*

Proof: In fact, the algorithmic formulae used when proving that any simplicial group is a Kan complex (cf., Proposition 2) give a filler defined as a product of degenerate copies of the faces of the box. ■

Proposition 3 *If (G,T) is a group T-complex then $T = D$.*

Proof: To see this, we note that axiom $T1$ implies that $D \subseteq T$. Conversely if $t \in T_n$, then it fills the box made up of $(\_, d_1t, \ldots, d_nt)$. This, in turn, has a filler, d, in D, but, as this filler is also thin, it must be that $t = d$, since thin fillers are uniquely determined ($T2$). ■

This is neat since it says there is essentially at most one group T-complex structure on any given simplicial group. The next results says when such a structure does exist.

Theorem 1 *(Ashley, [13]) If G is a simplicial group, then (G,D) is a group T-complex if and only if $NG \cap D$ is the trivial graded subgroup.*

Proof: One way around, this is nearly trivial. If (G,D) is a group T-complex and $x \in NG_n$, then x fills a box $(\_, 1, \ldots, 1)$, so if $x \in NG_n \cap D_n$, x must itself be the thin filler, however 1 is also a thin filler for this box, so $x = 1$ as required.

Conversely if $NG \cap D = \{1\}$, then we must check $T2$ and $T3$, $T1$ being trivial. As any box has a standard filler in D, we only have to check uniqueness, but if x and y are in D_n, and both fill the same box (with the k^{th} face missing) then $z = xy^{-1}$ fills a box with 1s on all faces (and the k^{th} face missing).

If $k = 0$, then as $z \in NG_n \cap D_n$, we have $z = 1$ and x and y are equal. If $k > 0$, assume that if $\ell < k$ and $z \in D_n \cap \bigcap_{i \neq \ell} Ker\, d_i$ then $z = 1$, (i.e, that we have uniqueness up to at least the $(k-1)^{st}$ case). Consider $w = zs_{k-1}d_kz^{-1}$. This is still in D_n and $d_iw = 1$ unless $i = k-1$, hence by assumption $w = 1$. Of course, this implies that $z = s_{k-1}d_kz$, but then $d_{k-1}z = d_kz$. We know that $d_{k-1}z = 1$, so $d_kz = 1$ and $z = 1$, *i.e.*, $x = y$ and we have uniqueness at the next stage.

To verify $T3$, assume that $x \in D_{n+1}$ and each $d_ix \in D_n$ for $i \neq k$, then we can assume that $k = 0$, since otherwise we can skew the situation around as before to get that to be true, verify it in that case and 'skew' it back again later. Suppose therefore that $d_ix \in D_n$ for all $0 < i < n$. As x must be the degenerate filler given by the standard method, we can calculate x as follows: let $w_n = s_{n-1}d_nx$, $w_i = w_{i+1}(s_{i-1}d_iw_{i+1})^{-1}s_{i-1}y_i$ for $i = 1$, then $x = w_1$. We can therefore check that $d_0x \in D_n$ as required. ■

Remark: Ashley, [13], in fact assumes a seemingly stronger conclusion, namely that $D_n \cap \bigcup_{\ell=0}^{n}(\bigcap_{i\neq\ell} Ker\, d_i) = 1$. The reduction to the single case is noted by Carrasco, [76].

A group T-complex is, thus, a simplicial group in which the Moore complex contains no non-trivial product of degenerate elements.

It is often useful to have a 'dimensionwise' terminology in the following sense. We could say that a group T-complex satisfies the *thin filler condition* or simply, the *T-condition*, in all dimensions. That suggests that we extract that condition 'dimensionwise' as follows:

Definition: A simplicial group G satisfies the *thin filler condition in dimension* n if $NG_n \cap D_n$ is trivial. We may abbreviate that to *T-condition in dimension n*.

This terminology lends itself well to such variants as 'G satisfies the *thin filler condition in dimensions greater that* k' meaning that $NG_n \cap D_n$ is trivial for all $n > k$, and so on.

It is left as an exercise to prove that any simplicial Abelian group is a group T-complex. (At this stage, this is moderately challenging, and it may help to take a brief look at the later section on Conduché's decomposition and the Dold-Kan theorem.)

Chapter 2

Crossed modules - definitions, examples and applications

We will give these for groups, although there are analogues for many other algebraic settings.

2.1 Crossed modules

Definition: A *crossed module*, (C, G, δ), consists of groups C and G with a left action of G on C, written $(g, c) \to {}^g c$ for $g \in G$, $c \in C$, and a group homomorphism $\delta : C \to G$ satisfying the following conditions:
CM1) for all $c \in C$ and $g \in G$,

$$\delta({}^g c) = g\delta(c)g^{-1},$$

CM2) for all $c_1, c_2 \in C$,

$${}^{\delta(c_2)}c_1 = c_2 c_1 c_2^{-1}.$$

(CM2 is called the *Peiffer identity*.)

If (C, G, δ) and (C', G', δ') are crossed modules, a *morphism*, $(\mu, \eta) : (C, G, \delta) \to (C', G', \delta')$, *of crossed modules* consists of group homomorphisms $\mu : C \to C'$ and $\eta : G \to G'$ such that

(i) $\delta'\mu = \eta\delta$ and (ii) $\mu({}^g c) = {}^{\eta(g)}\mu(c)$ for all $c \in C$, $g \in G$.

Crossed modules and their morphisms form a category, of course. It will usually be denoted $CMod$.

There is, for a fixed group G, a subcategory $CMod_G$ of $CMod$, which has, as objects, those crossed modules with G as the "base", *i.e.*, all (C, G, δ) for this fixed G, and having as morphisms from (C, G, δ) to (C', G, δ') just those (μ, η) in $CMod$ in which $\eta : G \to G$ is the identity homomorphism on G.

Several well known situations give rise to crossed modules. The verification will be left to you.

2.1.1 Algebraic examples of crossed modules

(i) Let H be a normal subgroup of a group G with $i : H \to G$ the inclusion, then we will say (H, G, i) is a *normal subgroup pair*. In this case, of course, G acts on the left of H by conjugation and the inclusion homomorphism i makes (H, G, i) into a crossed module, an 'inclusion crossed modules'. Conversely it is an easy exercise to prove

Lemma 4 *If (C, G, ∂) is a crossed module, ∂C is a normal subgroup of G.* ∎

(ii) Suppose G is a group and M is a left G-module; let $0 : M \to G$ be the trivial map sending everything in M to the identity element of G, then $(M, G, 0)$ is a crossed module.

Again conversely:

Lemma 5 *If (C, G, ∂) is a crossed module, $K = Ker\,\partial$ is central in C and inherits a natural G-module structure from the G-action on C. Moreover, $N = \partial C$ acts trivially on K, so K has a natural G/N-module structure.* ∎

Again the proof is left as an exercise.

As these two examples suggest, general crossed modules lie between the two extremes of normal subgroups and modules, in some sense, just as groupoids lay between equivalence relations and G-sets. Their structure bears a certain resemblance to both - they are "external" normal subgroups, but also are "twisted" modules.

(iii) Let G be a group, then, as usual, let $Aut(G)$, denote the group of automorphisms of G. Conjugation gives a homomorphism

$$\iota : G \to Aut(G).$$

Of course, $Aut(G)$ acts on G in the obvious way and ι is a crossed module. We will need this later so will give it its own name, the *automorphism crossed module of the group, G* and its own notation: $\mathsf{Aut}(G)$.

More generally if L is some type of algebra then $U(L) \to Aut(L)$ will be a crossed module, where $U(L)$ denotes the units of L and the morphism send a unit to the automorphism given by conjugation by it.

This class of example has a very nice property with respect to general crossed modules. For a general crossed module, (C, P, ∂), we have an action of P on C, hence a morphism, $\alpha : P \to Aut(C)$, so that $\alpha(p)(c) = {}^pc$. There is clearly a square

$$\begin{tikzcd} C \arrow[r, "="] \arrow[d, "\partial"'] & C \arrow[d, "\iota"] \\ P \arrow[r, "\alpha"'] & Aut(C) \end{tikzcd}$$

and we can ask if this gives a morphism of crossed modules. 'Clearly' it should. The requirements are that the square commutes and that the actions are compatible in the obvious sense, (recall page 41). To see that the square commutes, we just note that, given $c \in C$, ∂c acts on an $x \in C$, by conjugation by c: ${}^{\partial c}x = c.x.c^{-1} = \iota(c)(x)$, whilst to check that the actions match correctly remember that $\alpha(p)(c) = {}^px$ by definition, so we do have a morphism of crossed modules as expected.

(iv) We suppose given a morphism
$$\theta : M \to N$$
of left G-modules and form the semi-direct product $N \rtimes G$. This group we make act on M via the projection from $N \rtimes G$ to G.

We define a morphism
$$\partial : M \to N \rtimes G$$
by $\partial(m) = (\theta(m), 1)$, where 1 denotes the identity element of G, then $(M, N \rtimes G, \partial)$ is a crossed module. In particular, if A and B are Abelian groups, and B is considered to act trivially on A, then any homomorphism, $A \to B$ is a crossed module.

(v) Suppose that we have a crossed module, $\mathsf{C} = (C, G, \delta)$, and a group homomorphism $\varphi : H \to G$, then we can form the 'pullback group' $H \times_G C = \{(h, c) \mid \varphi(h) = \delta c\}$, which is a subgroup of the product $H \times C$. There is a group homomorphism, $\delta' : H \times_G C \to H$, namely the restriction of the first projection morphism of the product, (so $\delta'(h, c) = h$). **You are left** to construct an action of H on this group, $H \times_G C$ such that $\varphi^*(\mathsf{C}) := (H \times_G C, H, \delta')$ is a crossed module, and also such that the pair of maps φ and the second projection $H \times_G C \to C$ give a morphism of crossed modules.

Definition: The crossed module, $\varphi^*(\mathsf{C})$, thus defined, is called the *pullback crossed module of* C *along* φ

(vi) As a last algebraic example for the moment, let
$$1 \to K \xrightarrow{a} E \xrightarrow{b} G \to 1$$
be an extension of groups with K a central subgroup of E, *i.e.*, a *central extension* of G by K. For each $g \in G$, pick an element $s(g) \in b^{-1}(g) \subseteq E$. Define an action of G on E by: if $x \in E$, $g \in G$, then
$${}^g x = s(g) x s(g)^{-1}.$$
This is well defined, since if $s(g)$, $s'(g)$ are two choices, $s(g) = ks'(g)$ for some $k \in K$, and K is central. (This also shows that this *is* an action.) The structure, (E, G, b), is a crossed module.

A particular important case is: for R a ring, let $E(R)$ be the group of elementary matrices of R, $E(R) \subseteq G\ell(R)$ and $St(R)$, the corresponding Steinberg group with $b : St(R) \to E(R)$, the natural morphism, (see later, page 111, or [205], for the definition). This, then, gives a central extension
$$1 \to K_2(R) \to St(R) \to E(R) \to 1$$
and thus a crossed module. In fact, more generally,
$$b : St(R) \to G\ell(R)$$
is a crossed module. The group, $G\ell(R)/Im(b)$, is $K_1(R)$, the first algebraic K-group of the ring.

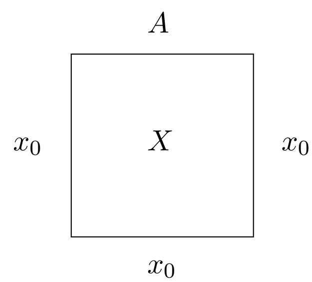
2.1.2 Topological Examples

In topology there are several examples that deserve looking at in detail as they do relate to aspects of the above algebraic cases. They require slightly more topological knowledge than has been assumed so far.

(vii) Let X be a pointed space, with $x_0 \in X$ as its base point, and A a subspace with $x_0 \in A$. Recall that the second relative homotopy group, $\pi_2(X, A, x_0)$, consists of relative homotopy classes of continuous maps

$$f : (I^2, \partial I^2, J) \to (X, A, x_0)$$

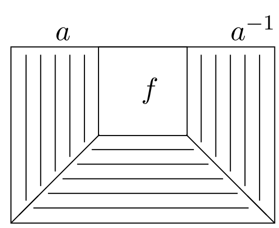
where ∂I^2 is the boundary of I^2, the square, $[0,1] \times [0,1]$, and $J = \{0,1\} \times [0,1] \cup [0,1] \times \{0\}$. Schematically f maps the square as:



so the top of the boundary goes to A, the rest to x_0 and the whole thing to X. The relative homotopies considered then deform the maps in such a way as to preserve such structure, so intermediate mappings also send J to x_0, etc. Restriction of such an f to the top of the boundary clearly gives a homomorphism

$$\partial : \pi_2(X, A, x_0) \to \pi_1(A, x_0)$$

to the fundamental group of A, based at x_0. There is also an action of $\pi_1(A, x_0)$ on $\pi_2(X, A, x_0)$ given by rescaling the 'square' given by



where f is partially 'enveloped' in a region on which the mapping is behaving like a.

Of course, this gives a crossed module

$$\pi_2(X, A, x_0) \to \pi_1(A, x_0).$$

A direct proof is quite easy to give. One can be found in Hilton's book, [153] or in Brown-Higgins-Sivera, [64]. Alternatively one can use the argument in the next example.

(viii) Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration sequence of pointed spaces. Thus p is a fibration, $F = p^{-1}(b_0)$, where b_0 is the basepoint of B. The fibre F is pointed at f_0, say, and f_0 is taken as the basepoint of E as well.

There is an induced map on fundamental groups

$$\pi_1(F) \xrightarrow{\pi_1(i)} \pi_1(E)$$

and if a is a loop in E based at f_0, and b a loop in F based at f_0, then the composite path corresponding to aba^{-1} is homotopic to one wholly within F. To see this, note that $p(aba^{-1})$ is null homotopic. Pick a homotopy in B between it and the constant map, then lift that homotopy back up to E to one starting at aba^{-1}. This homotopy is the required one and its other end gives a well defined element ${}^a b \in \pi_1(F)$ (abusing notation by confusing paths and their homotopy classes). With this action $(\pi_1(F), \pi(E), \pi_1(i))$ is a crossed module. This will not be proved here, but is not that difficult. Links with previous examples are strong.

If we are in the context of the above example, consider the inclusion map, f of a subspace A into a space X (both pointed at $x_0 \in A \subset X$). Form the corresponding fibration,

$$i^f : M^f \to X,$$

by forming the pullback

$$\begin{tikzcd}
M^f \arrow[r, "\pi^f"] \arrow[d, "j^f"'] & X^I \arrow[d, "e_0"] \\
A \arrow[r, "f"'] & X
\end{tikzcd}$$

so M^f consists of pairs, (a, λ), where $a \in A$ and λ is a path from $f(a)$ to some point $\lambda(1)$. Set $i^f = e_1\pi^f$, so $i^f(a, \lambda) = \lambda(1)$. It is standard that i^f is a fibration and its fibre is the subspace $F_h(f) = \{(a, \lambda) \mid \lambda(1) = x_0\}$, often called the *homotopy fibre* of f. The base point of $F_h(f)$ is taken to be the constant path at x_0, (x_0, c_{x_0}).

If we note that

$$\pi_1(F_h(f)) \cong \pi_2(X, A, x_0)$$

$$\pi_1(M^f) \cong \pi_1(A, x_0)$$

(even down to the descriptions of the actions, etc.), the link with the previous example becomes clear, and thus furnishes another proof of the statement there.

(ix) The link between fibrations and crossed modules can also be seen in the category of simplicial groups. A morphism $f : G \to H$ of simplicial groups is a fibration if and only if each f_n is an epimorphism. This means that a fibration is determined by the fibre over the identity which is, of course, the kernel of f. The $(G, \overline{W})$-links between simplicial groups and simplicial sets mean that the analogue of π_1 is π_0. Thus the fibration f corresponds to

$$Ker\, f \xrightarrow{\triangleleft} G$$

and each level of this is a crossed module by our earlier observations. Taking π_0, it is easy to check that

$$\pi_0(Ker\, f) \to \pi_0(G)$$

is a crossed module. In fact any crossed module is isomorphic to one of this form. (Proof left to the reader.)

If $\mathsf{M} = (C, G, \partial)$ is a crossed module, then we sometimes write $\pi_0(\mathsf{M}) := G/\partial C$, $\pi_1(\mathsf{M}) := Ker\,\partial$, and then have a 4-term exact sequence:

$$0 \to \pi_1(\mathsf{M}) \to C \xrightarrow{\partial} G \to \pi_0(\mathsf{M}) \to 1.$$

In topological situations when M provides a model for (part of) the homotopy type of a space X or a pair (X, A), then typically $\pi_1(\mathsf{M}) \cong \pi_2(X)$, $\pi_0(\mathsf{M}) \cong \pi_1(X)$.

Mac Lane and Whitehead, [195], showed that crossed modules give algebraic models for all homotopy 2-types of connected spaces. We will visit this result in more detail later, but loosely a 2-equivalence between spaces is a continuous map that induces isomorphisms on π_1 and π_2, the first two homotopy groups. Two spaces have the same 2-type if there is a zig-zag of 2-equivalences joining them.

2.1.3 Restriction along a homomorphism φ/ 'Change of base'

Given a crossed module, (C, H, ∂), over H and a homomorphism $\varphi : G \to H$, we can form the pullback:

$$\begin{tikzcd} D \arrow[r, "\psi"] \arrow[d, "\partial'"'] & C \arrow[d, "\partial"] \\ G \arrow[r, "\varphi"'] & H \end{tikzcd}$$

in $Grps$. Clearly the universal property of pullbacks gives a good universal property for this, namely that any morphism $(\varphi', \varphi) : (C', G, \delta) \to (C, H, \partial)$ factors uniquely through (ψ, φ) and a morphism in $CMod_G$ from (C', G, δ) to (D, G, ∂'). Of course this statement depends on verification that (D, G, ∂') is a crossed module and that the resulting maps are morphisms of crossed modules, but this is routine, and will be **left as an exercise**. (You may need to recall that D can be realised, up to isomorphism, as $G \times_H C = \{(g, c) \mid \varphi(g) = \partial c\}$. It is for you to see what the action is.)

This construction also behaves nicely on morphisms of crossed modules over H and yields a functor,

$$\varphi^* : CMod_H \to CMod_G,$$

which will be called *restriction along* φ.

We next turn to the use of crossed modules in combinatorial group theory. This will involve us in giving some background on presentations, and Cayley graphs, which will provide some useful examples for later use as well as, hopefully, providing some motivation / intuition for the constructions involved.

2.2 Group presentations, identities and 2-syzygies

Before turning towards the uses of crossed modules in the theory of group presentations, we must set up some of the basics of group presentations for later use. We first sketch the links with identities among relations for a presentation, but then will look at Cayley graphs and some of the ideas there, before returning to crossed modules in earnest.

2.2.1 Presentations and Identities

(The source for some of this material is the article by Brown and Huebschmann, [65]).

We consider a presentation, $\mathcal{P} = (X : R)$, of a group G. The elements of X are called *generators* and those of R *relators*. We then have a short exact sequence,

$$1 \to N \to F \to G \to 1,$$

where $F = F(X)$, the free group on the set X, R is a subset of F and $N = N(R)$ is the normal closure in F of the set R.

Sometimes we may use an alternative notation $\langle X : R\rangle$, for a presentation. The type of 'braces' used in a context usually has no significance. It is sometimes useful to refer to a group together with a presentation of it as a *presented group*.

A standard if somewhat trivial example is given by the *standard presentation* of a group, G. We take $X = \{x_g \mid g \in G, g \neq 1\}$, to be a set in bijective correspondence with the underlying set of G. (You can take X equal to that set if you like, but sometimes it is better to have a distinct set, for instance, it make for an easier notation for the description of certain morphisms.) The set of relations will be $R = \{x_g.x_h = x_{gh} \mid g, h \in G\}$, so as a *set* is just a copy of $G \times G$ as the relations are indexed by pairs of elements of G.

The group F acts on N by conjugation: ${}^uc = ucu^{-1}$, $c \in N, u \in F$ and the elements of N are words in the conjugates of the elements of R:

$$c = {}^{u_1}(r_1^{\varepsilon_1})^{u_2}(r_2^{\varepsilon_2}) \ldots {}^{u_n}(r_n^{\varepsilon_n})$$

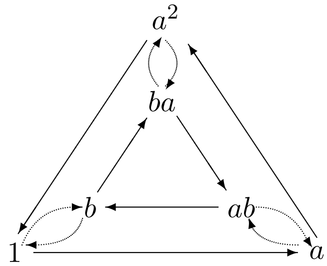
where each ε_i is $+1$ or -1. One also says such elements are *consequences* of R. Heuristically an *identity among the relations* of $\mathcal{P}$ is such an element c which equals 1. The problem of what this means is analogous to that of working with a relation in R. For example, in the presentation $(a : a^3)$ of C_3, the cyclic group of order 3, if a is thought of as being an element of C_3, then $a^3 = 1$, so why is this different from the situation with the 'presentation', $(a : 1)$? Surely if the two things, a^3 and 1 are equal, then substituting one for the other should not give a different answer, yet, clearly, these are giving different groups. To get around that difficulty the free group on the generators $F(X)$ is introduced and, of course, in $F(\{a\})$, a^3 is not 1. A similar device, namely *free crossed modules* on the presentation will be introduced in a moment to handle the identities. Before that consider some examples which indicate that identities exist even in some quite common-or-garden cases.

Example 1: Suppose $r \in R$, but it is a power of some element $s \in F$, i.e. $r = s^m$. Of course, $rs = sr$ and

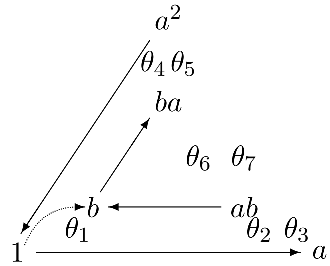
$${}^srr^{-1} = 1$$

so ${}^sr.r^{-1}$ is an identity. In fact, there will be a unique $z \in F$ with $r = z^q$, q maximal with this property. This z is called the *root of* r and if $q > 1$, r is called a *proper power*.

Example 2: Consider one of the standard presentations of S_3, $(a, b : a^3, b^2, (ab)^2)$. Write $r = a^3$, $s = b^2$, $t = (ab)^2$. Here the presentation leads to F, free of rank 2, but $N(R) \subset F$, so it must be free as well, by the Nielsen-Schreier theorem. Its rank will be 7, given by the Neilsen index formula or, geometrically, it will be the fundamental group of the Cayley quiver or Cayley graph of the presentation[1]. This group is free on generators corresponding to edges outside a maximal tree as in the following diagram:



The Cayley graph or Cayley quiver of S_3



and a maximal tree in it.

The set of normal generators of $N(R)$ has 3 elements; $N(R)$ is free on 7 elements (corresponding to the edges not in the tree), but is specified as consisting of products of conjugates of r, s and t, and there are infinitely many of these. Clearly there must be some slight redundancy, *i.e.*, there must be some identities among the relations!

A path around the outer triangle corresponds to the relation r; each other region corresponds to a conjugate of one of r, s or t. (It may help in what follows to think of the graph being embedded on a 2-sphere, so 'outer' and 'outside' mean 'round the back face'.) Consider a loop around a region. Pick a path to a start vertex of the loop, starting at 1. For instance the path that leaves 1 and goes along a, b and then goes around aaa before returning by $b^{-1}a^{-1}$ gives $abrb^{-1}a^{-1}$. Now the path around the outside can be written as a product of paths around the inner parts of the graph, e.g. $(abab)b^{-1}a^{-1}b^{-1}(bb)(b^{-1}a^{-1}b^{-1}a^{-1})\ldots$ and so on, thus r can be written in a non-trivial way as a product of conjugates of r, s and t. (An explicit identity constructed like this is given in [65].)

Example 3: In a presentation of the free Abelian group on 3 generators, one would expect the commutators, $[x, y]$, $[x, z]$ and $[y, z]$. The well-known identity, usually called the Jacobi identity, expands out to give an identity among these relations (again see [65], p.154, or Loday, [187].)

2.2.2 Directed graphs, quivers and their reflexive 'cousins'

To finish up with the basics on group presentations, we will recall, fairly formally, the construction of the Cayley graph, or *Cayley quiver*, of a presentation that we have used above, as it will be convenient to have it available later for reference purposes. We need this in several equivalent forms as, being a basic construction, it tends to crop up (or creep in) in different contexts in which its appearance is slightly 'variable'. We will also discuss results on the construction of free categories and free groupoids on a quiver.

To start we had better be a bit more formal about some ideas on directed graphs, also known as quivers, so we will start by recalling the notion of 'directed graph' or'quiver'.

A graph in the combinatorial sense of the word has vertices or nodes, and edges going between (some of) them, but there is often no 'direction' to the edges. For instance, a simplicial complex,

[1]For convenience, we will discuss the Cayley quiver of a presentation separately in section 2.2.3.

K, in which there are no simplices of dimension 2 (or higher therefore) is essentially just a graph in this combinatorial sense.

If there is a direction (so the edge, e, 'goes from vertex v to vertex w' say), the terminology '*directed graph*' is sometimes used. As a directed edge is usually drawn as an arrow, a directed graph is pictured as a collection of arrows, and hence the term '*quiver*' is used as a more graphic way of referring to such. Both terms are used in the literature, so we will use both in these notes, but they are synonyms. More formally we give:

Definition: A *quiver*, Q, consists of a set, VQ, of vertices and a collection, EQ, of edges, together with a function,

$$(s,t) : EQ \to VQ \times VQ,$$

that assigns a source vertex, $s(e)$, and a target vertex, $t(e)$, to each edge $e \in EQ$. We write $Q = (EQ, VQ, s, t)$, or sometimes just $Q = (EQ, VQ)$.

Other terminology is sometimes used including 'initial' and 'final vertices' of an edge, or sometimes 'tail' and 'head' of an arrow[2]. We note that several edges may share a source and target.

We often write the quiver Q as a diagram,

$$EQ \underset{t}{\overset{s}{\rightrightarrows}} VQ.$$

In this form it is easy to generalise the notion of quiver (in the category of sets) to an *internal quiver* or *internal directed graph* in an arbitrary category.

There is a category, *Quiv*, of quivers with a fairly obvious notion of morphism between them[3]. (We will also write *DGrph* for this category following the literature where appropriate to make reading beyond the brief summary notes here slightly easier than otherwise.)

Remark: There is another form of quiver that we will be needing later on and so will introduce it here in preparation. A *reflexive quiver*, or *reflexive directed graph*, has the additional structure that at each vertex there is a distinguished 'loop', so if v is a vertex in such a quiver, then there is an edge $i(v)$ with $si(v) = ti(v) = v$. This gives a map, $i : VQ \to EQ$, satisfying some obvious equations. We think of, and talk of $i(v)$ as being the identity loop at the vertex v, although that is there is no composition around to make sense of that term.

This notion of reflexive quiver leads to a very natural idea of morphism of reflexive quivers and this has the advantage over the non-reflexive version that a morphism can 'kill off' an edge. For instance, let Q be the quiver with one edge, e, going between two vertices, 0 and 1, and Q', be a quiver with one vertex, v, and no edges. There are no morphisms from Q to Q', as there are no maps from a non-empty set to an empty one. The category of reflexive quivers will be denoted $Quiv_0$, or $DGrph_0$, depending on the context.

There is a functor from $Quiv_0$ to $Quiv$ that forgets the extra piece of information.

We can make any quiver into a reflexive one by replacing EQ by $EQ \sqcup VQ$, adjusting s and t accordingly, and finally setting $i : VQ \to EQ \cup VQ$ to be the inclusion into the disjoint union. We will write Q^{refl} for this reflexive variant of Q, but now going back to our two quivers, Q and

[2]... so always check on notation since $t(e)$ may mean the tail of a directed edge, *i.e.*, its source, in some sources in the literature.

[3]... but see below.

Q', there is a morphism from Q^{refl} to $(Q')^{refl}$ as a morphism can 'kill' e, sending it to $i(v)$. **The reader is left to explore the relationship between this construction and the forgetful functor mentioned just above**[4]**.**

Examples of quivers: (i) Taking V to be a singleton gives us that any set, E, gives a graph with one vertex. For instance, let $E = \{a, b\}$ and $V = \{*\}$. There is no choice for the maps, $s, t, : E \to V$, as both must send all edges to $*$. We can picture this as

$$a \circlearrowright * \circlearrowleft b\ ,$$

a bouquet of two loops. In general, if X is a set, we write $X[1]$ for the quiver having $E = X$, $V = \{*\}$, and s and t the evident source and target maps. This follows and extends the convention that a group, G, gives a groupoid, $G[1]$, with a single object, and which is thought of as a 'shifted group'; see page 15. It may also be convenient to write $Bouq(X)$ for this same object for instance if the other notion risks potential confusion in a discussion.

If the set, X, is 'pointed', *i.e.*, has a 'basepoint' or 'distinguished point', then $X[1]$ is more naturally considered as a reflexive quiver.

(ii) If K is a simplicial set, then its 1-skeleton is the simplicial subset determined by the vertices and the 1-simplices, *i.e.*, the edges of K. This gives us a (reflexive) quiver. It is reflexive as each vertex x in K_0 gives an edge $s_0(x)$, (but remember that, in the theory of simplicial sets, s is nearly always used for the degeneracy operations, whilst if $e \in K_1$ is thought of as an edge in the corresponding quiver, $s(e) = d_1(e)$, and $t(e) = d_0(e)$, as d_i is the face opposite the i^{th} vertex[5]). This 1-skeleton as a simplicial set still has higher dimensional simplices but those will all be degenerate and all the information is given by the vertices and edges, so in dimensions 0 and 1.

(iii) Given any (small) category, $\mathcal{C}$, we have a (reflexive) directed graph, $U(\mathcal{C})$ *underlying* $\mathcal{C}$. In fact, from one point of view, a category can be specified as being a (reflexive) directed graph together with a partial composition operation, and the above functor U just forgets the composition (and thus, if forgetting 'all the way down to *Quiv*', also the 'identity' label on the special loops). There are thus two things going on here. If we think of the category as being the quiver,

$$\begin{tikzcd} Arr(\mathcal{C}) \arrow[r, shift left=2, "s"] \arrow[r, shift left=0.5, "t"'] & Ob(\mathcal{C}) \arrow[l, shift left=1.5, "i"] \end{tikzcd}$$

plus the composition, then the corresponding quiver is just, of course,

$$\begin{tikzcd} Arr(\mathcal{C}) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Ob(\mathcal{C}) \end{tikzcd}\ .$$

Of course, this *is* a functor, $U : Cat \to Quiv_0$, or to *Quiv*, depending how much you 'forget', and those functors have left adjoints giving rise to the free category on a quiver. We will briefly sketch this construction, leaving the reader to search out more detailed treatments[6] if they have not met the ideas before. We will start with a reflexive quiver / directed graph[7], Q. A string of *non-identity* edges, $\sigma = e_1 \ldots e_n$, in Q, is said to be *composable* if, for $i = 1, 2, \ldots, n-1$, $t(e_i) = s(e_{i+1})$,

[4]For instance, checking it is functorial, which is straightforward, then looking for if they form an adjoint pair.

[5]This notational convention may sometimes lead to a little bit of confusion, but as the two subject areas interact in useful and interesting ways the reader just has, as always, to take care with the notation being used in any source.

[6] for instance, in Phil Higgins monograph, [151], page 25 of the main text.

[7]..., as otherwise, the first part of the construction for a quiver, Q, that is not reflexive, would be to make it reflexive, replacing it by Q^{refl}, as above!

so the edges fit together to give a 'directed path' in Q. Such a string has *source*, $s(\sigma) := s(e_1)$ and target, $t(\sigma) := t(e_n)$, thus extending both the terminology and the notation for source and target to composable strings. In addition, we write the 'empty string at vertex v' as if it was the (invalid) string $1_v := i(v)$. Composition is by concatenation of strings and the identities as they are empty strings do act like identities. There has to be some checking that this does give one a category, but that is routine and well known. (The only tricky problem that occurs is deciding how to handle the empty string at a vertex. Various means, such as replacing an edge e by a notation, such as, (v, e, w) with $v = s(e)$ and $w = t(e)$, so a path is of form $(v_0, e_1, v_1, e_2, \ldots, e_n, v_n)$, and then defining $i(v) = (v)$, allowing there to be no edge, and so on, can be used; see page 180 for more discussion of this in a slightly more general context.)

If we restrict to bouquets, then for $Q = X[1]$, the free category on Q is $Mon(X)[1]$, the small one object category corresponding to the free monoid on X.

Restricting to groupoids, and thus to *Grpd*, the underlying quiver functor again has a left adjoint, namely the *free groupoid* functor. The free groupoid, $FreeGrpd(Q)$, on Q can also be obtained from the free category, $FreeCat(Q)$, on Q by using the construction that inverts all arrows. In other words, using a hopefully fairly obvious notion,

$$FreeGrpd(Q) \cong FreeCat(Q)(Arr(FreeCat(Q))^{-1}).$$

Again we **leave the reader to check up on the details of these constructions in the literature**, but will give some pointers below.

The free groupoid on Q is also its *fundamental groupoid*, $\Pi(Q)$. This, of course, has VQ as its set of objects and all *reduced* paths between vertices as its arrows. As we are handling the groupoid case, every directed edge, e, in Q will give not only an arrow $e : s(e) \to t(e)$, but that arrow's inverse $e^{-1} : t(e) \to s(e)$. A (composable) string or path is then a string, $e_1^{\pm 1} \ldots e_n^{\pm 1}$, almost as before, but with a letter of the string now being either an edge or an inverse of an edge. The path is reduced if no edge in the string is adjacent to its inverse. If in a path, we find a ee^{-1} or $e^{-1}e$, we can reduce the length of the path by 2 by deleting that two element substring. In particular, the path ee^{-1} reduces to the empty path / identity, $1_{s(e)}$ at the source of e, whilst $e^{-1}e$ reduces to $1_{t(e)}$.

We list some results below without proof.

a) If $Q = X[1]$ for a set, X, then $\Pi(Q) \cong F(X)[1]$, where $F(X)$ is the free group on the set X. More generally, if Q is connected and v is a vertex of Q, then the vertex group $\Pi(Q)(v)$ is a free group.

b) If Q is a tree then $\Pi(Q)$ is trivial as any path between two vertices, v and w, can be reduced to a unique form depending on the single direct path from v to w.

c) If Q is a finite quiver having n_v vertices and n_e edges, then a maximal tree in Q has $n_v - 1$ edges. It then follows that $\Pi(Q)(v)$ is free on $n_e - n_v + 1$ elements (corresponding to the edges not in a chosen maximal tree in Q). This is almost the *Niesen index formula.*

2.2.3 Cayley quiver of a 'generated group'

Given a group, G, and some set of generators, X, of G, we will sometimes call the resulting pair, (G, X), a *generated group.* It may be necessary, sometimes, to augment the notation (G, X) by a function $\varphi : X \to U(G)$, from X to the underlying set of G, a function that interprets the elements of X as actual elements of G. This would give (G, X, φ). Most of the time this is being a bit pedantic, but, for instance, when working with transformations of a presentation, it can be crucial as a transformation may lead to two copies of a particular generator. We will return to this shortly.

How, then, is one to find a *complete* set of relations for G relative to these generators. 'Complete' means that no more are needed to get G. Later on, in section 4.1.1, we will continue this by asking for complete sets of identities among relations, and then on to higher *syzygies*.

Returning to finding enough relations, 'trial and error' might work with a bit of luck, but is haphazard. In this a useful aid is to try to 'visualise' the information that we have. (This will be illustrated here for finite groups, G, but some of the ideas will work for any discrete group, and the definition we will give *is* independent of any finiteness assumption.)

We assume that G is known and X is known to be a set of generators[8]. We construct a quiver having a set, $V = V(Q) = \{v_g \mid g \in G\}$, of vertices corresponding to the elements of G (and usually thought of as *being* the elements of G), and, for each vertex v_g, and generator, $x \in X$, we draw an edge, labelled by x, with source v_g and target, v_{gx}. (We note that this edge has a definite direction *from* v_g *to* v_{gx}, so this is a quiver.)

Definition: The above quiver is usually called the *Cayley quiver* of G, but really we should add 'with respect to the given set, X, of generators'. We will say, more briefly, that it is the Cayley quiver, $Cay(G, X)$ of the generated group, (G, X).

If we are given a presentation $\mathcal{P} = (X : R)$ of G, then we will say this is the *Cayley quiver* of the presentation or of the presented group $(G, \mathcal{P})$, even though the relations do not enter into the actual definition of the quiver.

The terms 'Cayley graph', 'Cayley diagram', etc. are also used interchangeably for this, and hopefully, if we use a different term later on, no confusion will arise! In fact, it is not quite exact to say that a Cayley graph is a quiver, as clearly it is more than that. Each edge in a Cayley quiver, $Cay(G, X)$, is labelled[9] by an element of the generating set, X, so it is a labelled quiver. Similar ideas of labelling combinatorial objects with algebraic data will be encountered several times in what follows.

For a fairly elementary introduction to Cayley quivers, see section 6.3. of [133], starting on p. 118.

Examples: The Cayley quiver of S_3 with respect to the set of generators consisting of $a = (123)$ and $b = (12)$ is shown here in section 2.2.1, (page 48), and for an even simpler example, $C_5 = (x : x^5)$ yields a pentagon.

Drawing a Cayley quiver rarely is as easy as it looks when you see nice neat examples done in textbooks! If you manage to draw one as a planar graph then relations can easily be spotted, as in our example os S_3 above. If the quiver is non-planar, then a neat embedding on a surface then may enable the further analysis of relations; see Coxeter and Moser, [94], for some discussion of this, although the theory has been developed much further since that book was first written. This subject has links with the action of a group on a graph, as clearly G acts on the Cayley quiver relative to any set of generators. This then relates to the action of groups on graphs giving graphs of groups as in the Bass-Serre theory, [246], and on simplicial complexes, and thus to the idea of a

[8]This avoids us having to handle non-connected graphs which would be the result if X did not generate G.
[9]Some people say 'coloured'.

complex of groups as we will meet later on in section 4.5.2. (We leave that aspect here, although it would be great fun to develop it further, and perhaps we will return to it later!)

Another aspect of drawing a Cayley quiver is that it may, in fact, be very difficult or even impossible actually to draw a Cayley quiver in some cases. The difficulty occurs when you have v_g and an x in X. You want to draw the edge, labelled x from v_g to the vertex v_{gx}, but how are you to determine which of the elements of G is this new element gx. For an example in which it is not that hard to see what is going on, we have $S_3 = \{1, a, a^2, b, ab, ba\}$, according to the list of elements that we used on page 48. When building that graph, we have to know how to rewrite ga and gb in terms of the listed elements. As we have that a stands for (123) and b for (12), we can do this quite easily, and if we did have a presentation of the group concerned, we could use that, e.g., to discover what the target is of the edge whose source is v_{ab}, and which corresponds to the generator a. In fact as $abab = 1$, and $b^2 = 1$, we see that $v_{aba} = v_b$, and that is clear from the picture we had! This seems frustrating, but it just says that trying to find a complete set of relations for a given set of relations can be hard. All this is further complicated by its nearness to the 'Word Problem', that is, in general deciding whether two elements in the free group on the generators represent the same element of G. In its full generality it is undecidable. If normal forms for the elements of G can be specified, then things are much better.

Despite these difficulties Cayley quivers have considerable importance and also there are important related constructions in other contexts.

We have given the construction of the Cayley graph for a 'generated group', (G, X), and have noted how there is a sort of interplay between knowing the relations to work out the quiver, and using the quiver to 'see' the relations. Of course, there is always one presentation where all the relations are known from the start[10], and that is the standard presentation of G given by $X = \{x_g \mid g \in G, g \neq 1\}$, the set of element of G, and $R = \{x_g.x_h = x_{gh} \mid g, h \in G\}$, so X 'is' the set of elements in G, and R is similarly $G \times G$.

For this, the Cayley graph of the group G, relative to $X = G$ has vertices labelled by the elements of G, and, if v_g and v_h are two such, there is always a unique $k \in X$ such that $v_h = v_{gk}$ simply because $k = g^{-1}h$ is a generator! We thus have that Q is the underlying quiver of $Codisc(G)$, the codiscrete groupoid on the set of elements of G. In fact, this is such that the evident morphism

$$Codisc(G) \to G[1]$$

is a covering groupoid.

This is an important special case of a type of (covering) map of (labelled) quivers that has a role to play in our later discussion of crossed modules and their applications to group presentations and which we will explore in more generality. We start with our example of the Cayley graph for S_3 and the generators a and b. This will be the domain of our map. The codomain will be $Bouq(\{a, b\})$, the bouquet on $X = \{a, b\}$. The morphism,

$$p : Cay(S_3, X) \to Bouq(X),$$

sends, as it must, all the vertices of the Cayley quiver to the single vertex of the bouquet, whilst the corresponding map on edges is determined by requiring that it preserves the labelling. If we look at each vertex of $Cay(S_3, X)$, it has two incoming edges and two outgoing ones, with labels

[10] as they correspond to the multiplication

as at the single vertex in $Bouq(\{a,b\})$, thus locally the above map looks like an isomorphism. It is a covering map in a fairly obvious graph theoretic sense[11]. This map induces a map of groupoids:

$$p_* : \Pi(Cay(S_3, X)) \to \Pi(Bouq(X)) \cong F(a,b)[1],$$

which is a covering map of groupoids[12]. We note that the fibre of this covering, $p^{-1}(*)$, is a discrete groupoid on the elements of S_3, *i.e.*, essentially just that set of elements. There is an action of $F(a,b)$ on that discrete groupoid since, as you would expect, laths in the base lift to the top quiver. Of course, that action is generated, algebraically, by the obvious action of the two generators, a and b, which can be visualised as an automorphism of the fibre (but note this is by multiplication, so is not a group automorphism; the abstract fibre identifies with the underlying set of S_3 only when we pick one element to correspond to the identity element). If we look at the action of any relation, then it will be trivial on the fibre, so the action is actually one of S_3 on its underlying set, by multiplication on the right. If we look at the subgroup of $F(a,b)$ corresponding to the image of a vertex group of $\Pi(Cay(S_3, X))$ in $\Pi(Bouq(X))$, it will be a free subgroup (and one does not need to quote the Nielsen - Schreier theorem[13] from the theory of free groups, to see this, as that theorem's proof can be framed in the setting of covering graphs, and covering groupoids, and this is then just a very special instance of that). The Nielsen index formula[14] gives that that subgroup has rank 7 as we indicated back in section 2.2.1.

Of course, this example yields a general construction. If we form $Cay(G, X)$ for a generated group, (G, X), and then $Bouq(X) = X[1]$, then mapping all the vertices of $Cay(G, X)$ to the single vertex of $X[1]$ and the edges according to their labels gives a covering of quivers

$$p : Cay(G, X) \to X[1].$$

We can also view this on the level of groupoids, or look at the fundamental groups of the two quivers, etc., but we will **leave this to the reader to explore** as we need to get back to crossed modules, and identities among relations at a more formal level.

2.2.4 Free crossed modules and identities

The idea that an identity is an equation in conjugates of relations leads one to consider formal conjugates of symbols that label relations. Abstracting this a bit, suppose G is a group and $f : Y \to G$, a function 'labelling' the elements of some subset of G. To form a conjugate, you need a thing being conjugated and an element 'doing' the conjugating, so form pairs (p, y), $p \in G$, $y \in Y$, to be thought of as ${}^p y$, the *formal conjugate* of y by p. Consequences are words in conjugates of relations; *formal consequences* are elements of $F(G \times Y)$. There is a function extending f from $G \times Y$ to G given by

$$\bar{f}(p, y) = pf(y)p^{-1},$$

[11] If one realises the quivers as topological space, for instance by thinking of them as simplicial sets and taking their geometric realisation, the the realisation of this covering map of quivers *is* a covering map of spaces in the classical sense.

[12] For the moment, this is in a sense we have not seen yet, but we will meet it in some detail in section 4.5.14. The idea is similar both to that of graphs, as here, and to topological covering spaces. For the moment we argue more by analogy, sketching some consequences of the theory of coverings.

[13] Every subgroup H of a (discrete) free group G is itself a free group.

[14] Continuing from the previous footnote: if G is free on k generators and H has index n in G, then H is free on $nk - n + 1$ generators; see Phil Higgins' monograph, [151], for a groupoid proof, or Gilbert and Porter, [133], for a discussion. The proof is discussed in many other sources.

converting a formal conjugate to an actual one and this extends further to a group homomorphism

$$\varphi : F(G \times Y) \to G$$

defined to be $\bar{f}$ on the generators. The group. G, acts on the left on $G \times Y$ by multiplication: $p.(p', y) = (pp', y)$. This extends to a group action of G on $F(G \times Y)$. For this action, φ is G-equivariant if G is given its usual G-group structure by conjugations / inner automorphisms. Naively identities are the elements in the kernel of this, but there are some elements in that kernel that are there regardless of the form of function f. In particular, suppose that $g_1, g_2 \in G$ and $y_1, y_2 \in Y$ and look at

$$(g_1, y_1)(g_2, y_2)(g_1, y_1)^{-1}((g_1 f(y_1) g_1^{-1}) g_2, y_2)^{-1}.$$

Such an element is always annihilated by φ. The normal subgroup generated by such elements is called the Peiffer subgroup. We divide out by it to obtain a quotient group. This is the construction of the free crossed module on the function f. If f is, as in our initial motivation, the inclusion of a set of relators into the free group on the generators we call the result the *free crossed module on the presentation* $\mathcal{P}$ and denote it by $C(\mathcal{P})$.

We can now formally define the module of identities of a presentation, $\mathcal{P} = (X : R)$. We form the free crossed module on $R \to F(X)$, which we will denote by $\partial : C(\mathcal{P}) \to F(X)$. The *module of identities* of $\mathcal{P}$ is $Ker\,\partial$. By construction, the group presented by $\mathcal{P}$ is $G \cong F(X)/Im\,\partial$, where $Im\,\partial$ is just the normal closure of the set, R, of relations and we know that $Ker\,\partial$ is a G-module. We will usually denote the module of identities by $\pi_{\mathcal{P}}$.

We can get to $C(\mathcal{P})$ in another way. Construct a space from the combinatorial information in $\mathcal{P}$ as follows. Take a bunch of circles labelled by the elements of X; call it $K(\mathcal{P})_1$, it is the 1-skeleton of the space we want. We have $\pi_1(K(\mathcal{P})_1 \cong F(X)$. Each relator $r \in R$ is a word in X so gives us a loop in $K(\mathcal{P})_1$, following around the circles labelled by the various generators making up r. This loop gives a map $S^1 \xrightarrow{f_r} K(\mathcal{P})_1$. For each such r, we use f_r to glue a 2-dimensional disc e^2_r to $K(\mathcal{P})_1$ yielding the space $K(\mathcal{P})$. This space is sometimes known as the *presentation complex* of $\mathcal{P}$, whilst its universal covering complex is the *Cayley complex* of the presentation. The crossed module, $C(\mathcal{P})$, is isomorphic to $\pi_2(K(\mathcal{P}), K(\mathcal{P})_1) \xrightarrow{\partial} \pi_1(K(\mathcal{P})_1$.

The main immediate problem here is how to calculate $\pi_{\mathcal{P}}$, or equivalently $\pi_2(K(\mathcal{P}))$. One approach is via an associated chain complex. This can be viewed as the chains on the universal cover of $K(\mathcal{P})$, but can also be defined purely algebraically, for which see Brown-Huebschmann, [65], or Loday, [187]. That algebraic - homological approach leads to 'homological syzygies'.

2.3 Cohomology, crossed extensions and algebraic 2-types

2.3.1 Cohomology and extensions, continued

Suppose we have any group extension

$$\mathcal{E}: \quad 1 \to K \to E \xrightarrow{p} G \to 1,$$

with K Abelian, but not necessarily central. We can look at various possibilities.

If we can split p, by a homomorphism $s : G \to E$, with $ps = Id_G$, then, of course, $E \cong K \rtimes G$ by the isomorphisms,

$$e \longrightarrow (esp(e)^{-1}, p(e)),$$

$$ks(g) \longleftarrow (k, g),$$

which are compatible with the projections etc., so there is an equivalence of extensions

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\cong"] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & K \arrow[r] & K \rtimes G \arrow[r] & G \arrow[r] & 1. \end{tikzcd}$$

Our convention for multiplication in $K \rtimes G$ will be

$$(k, g)(k', g') = (k^g k', gg').$$

But what if p does not split. We can build a (small) category of extensions $\mathcal{E}xt(G, K)$ with objects such as $\mathcal{E}$ above and in which a morphism from $\mathcal{E}$ to $\mathcal{E}'$ is a diagram

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\alpha"] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & K \arrow[r] & E' \arrow[r] & G \arrow[r] & 1. \end{tikzcd}$$

By the 5-lemma, α will be an isomorphism, so $\mathcal{E}xt(G, K)$ is a groupoid.

In $\mathcal{E}$, the epimorphism p is usually not splittable, but as a function between sets, it is onto so we can pick an element in each $p^{-1}(g)$ to get a transversal (or set of coset representatives), $s : G \to E$. We get a comparison pairing / obstruction map or 'factor set' :

$$f : G \times G \to E$$

$$f(g_1, g_2) = s(g_1)s(g_2)s(g_1 g_2)^{-1},$$

which will be trivial, (*i.e.*, $f(g_1, g_2) = 1$ for all $g_1, g_2 \in G$) exactly if s splits p, *i.e.*, if s is a homomorphism. This construction assumes that we know the multiplication in E, otherwise we cannot form this product! On the other hand given this 'f', we can work out the multiplication. As a set, E will be the product $K \times G$, identified with it by the same formulae as in the split case, noting that $pf(g_1, g_2) = 1$, so 'really' we should think of f as ending up in the subgroup K, and then we have

$$(k_1, g_1)(k_2, g_2) = (k_1{}^{s(g_1)}k_2 f(g_1, g_2), g_1 g_2).$$

The product is *twisted* by the pairing f. Of course, we need this multiplication to be associative and, to ensure that, f must satisfy a cocycle condition:

$${}^{s(g_1)}f(g_2, g_3)f(g_1, g_2 g_3) = f(g_1, g_2)f(g_1 g_2, g_3).$$

This is a well known formula from group cohomology, more evidently so if written additively:

$$s(g_1)f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_1, g_2) = 0.$$

Here we actually have various parts of the nerve of G involved in the formula. The group G 'is' a small category (groupoid with one object), which we will, for the moment, denote $G[1]$. The triple $\sigma = (g_1, g_2, g_3)$ is a 3-simplex in $Ner(G[1])$ and its faces are

$$\begin{array}{rcl} d_0\sigma & = & (g_2, g_3), \\ d_1\sigma & = & (g_1g_2, g_3), \\ d_2\sigma & = & (g_1, g_2g_3), \\ d_3\sigma & = & (g_1, g_2). \end{array}$$

This is all very classical. We can use it in the usual way to link $\pi_0(\mathcal{E}xt(G, K))$ with $H^2(G, K)$ and so is the 'modern' version of Schreier's theory of group extensions, at least in the case that K is Abelian.

For a long time there was no obvious way to look at the elements of $H^3(G, K)$ in a similar way. In Mac Lane's homology book, [191], you can find a discussion from the classical viewpoint. In Brown's [55], the link with crossed modules is sketched although no references for the details are given, for which see Mac Lane's [193].

If we have a crossed module $C \stackrel{\partial}{\to} P$, then we saw that $Ker\,\partial$ is central in C and is a $P/\partial C$-module. We thus have a 'crossed 2-fold extension':

$$K \stackrel{i}{\to} C \stackrel{\partial}{\to} P \stackrel{p}{\to} G,$$

where $K = Ker\,\partial$ and $G = P/\partial C$. (We will write $N = \partial C$.)

Repeat the same process as before for the extension

$$N \to P \to G,$$

but take extra care as N is usually not Abelian. Pick a transversal $s : G \to P$ giving $f : G \times G \to N$ as before (even with the same formula). Next look at

$$K \stackrel{i}{\to} C \to N,$$

and lift f to C via a choice of $F(g_1, g_2) \in C$ with image $f(g_1, g_2)$ in N.

The pairing f satisfied the cocycle condition, but we have no means of ensuring that F will do so, i.e. there will be, for each triple (g_1, g_2, g_3), an element $c(g_1, g_2, g_3) \in C$ such that

$$^{s(g_1)}F(g_2, g_3)F(g_1, g_2g_3) = i(c(g_1, g_2, g_3))F(g_1, g_2)F(g_1g_2, g_3),$$

and some of these $c(g_1, g_2, g_3)$ may be non-trivial. The $c(g_1, g_2, g_3)$ will satisfy a cocycle condition correspond to a 4-simplex in $Ner(G[1])$, and one can reconstruct the crossed 2-fold extension up to equivalence from F and c. Here 'equivalence' is generated by maps of 'crossed' exact sequences:

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & C \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & K \arrow[r] & C' \arrow[r] & P' \arrow[r] & G \arrow[r] & 1, \end{tikzcd}$$

but these morphisms need not be isomorphisms. Of course, this identifies $H^3(G, K)$ with π_0 of the resulting category.

What about $H^4(G, K)$? Yes, something similar works, but we do not have the machinery to do it here, yet.

2.3.2 Not really an aside!

Suppose we start with a crossed module, $\mathsf{C} = (C, P, \partial)$. We can build an internal category, $\mathcal{X}(\mathsf{C})$, in *Grps* from it. The group of objects of $\mathcal{X}(\mathsf{C})$ will be P and the group of arrows $C \rtimes P$. The source map

$$s : C \rtimes P \to P \quad \text{is} \quad s(c, p) = p,$$

the target

$$t : C \rtimes P \to P \quad \text{is} \quad t(c, p) = \partial c.p.$$

(That looks a bit strange. That sort of construction usually does not work, multiplying two homomorphisms together is a recipe for trouble! - but it does work here:

$$\begin{aligned} t((c_1, p_1).(c_2, p_2)) &= t(c_1{}^{p_1}c_2, p_1p_2) \\ &= \partial(c_1{}^{p_1}c_2).p_1p_2, \end{aligned}$$

whilst $t(c_1, p_1).t(c_2, p_2) = \partial c_1.p_1.\partial c_2.p_2$, but remember $\partial(c_1{}^{p_1}c_2) = \partial c_1.p_1.\partial c_2.p_1^{-1}$, so they are equal.)

The identity morphism is $i(p) = (1, p)$, but what about the composition. Here it helps to draw a diagram. Suppose $(c_1, p_1) \in C \rtimes P$, then it is an arrow

$$p_1 \overset{(c_1,p_1)}{\longrightarrow} \partial c_1.p_1,$$

and we can only compose it with (c_2, p_2) if $p_2 = \partial c_1.p_1$. This gives

$$p_1 \xrightarrow{(c_1,p_1)} \partial c_1.p_1 \xrightarrow{(c_2,\partial c_1.p_1)} \partial c_2\partial c_1.p_1.$$

The obvious candidate for the composite arrow is (c_2c_1, p_1) and it works!

In fact, $\mathcal{X}(\mathsf{C})$ is an internal groupoid as $(c_1^{-1}, \partial c_1.p_1)$ is an inverse for (c_1, p_1).

Now if we started with an internal category

$$\begin{tikzcd} G_1 \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & G_0 \arrow[l, shift left=2, "i"] \end{tikzcd} ,$$

etc., then set $P = G_0$ and $C = Ker\, s$ with $\partial = t \mid_C$ to get a crossed module.

Theorem 2 *(Brown-Spencer,[70]) The category of crossed modules is equivalent to that of internal categories in Grps.* ■

You have, almost, seen the proof. As beginning students of algebra, you learnt that equivalence relations on groups need to be congruence relations for quotients to work well and that congruence relations 'are the same as' normal subgroups. That is the essence of the proof needed here, but we have groupoids rather than equivalence relations and crossed modules rather than normal subgroups.

Of course, any morphism of crossed modules has to induce an internal functor between the corresponding internal categories and *vice versa*. That is a **good exercise** for you to check that you have understood the link that the Brown-Spencer theorem gives.

This is a good place to mention 2-groups. The notion of 2-category is one that should be fairly clear even if you have not met it before. For instance, the category of small categories, functors and natural transformations is a 2-category. Between each pair of objects, we have not just a set of functors as morphisms but a small category of them with the natural transformations between them as the arrows in this second level of structure. The notion of 2-category is abstracted from this. We will not give a formal definition here (but suggest that you look one up if you have not met the idea before). A 2-category thus has objects, arrows or morphisms (or sometimes '1-cells') between them and then some 2-cells (sometimes called '2-arrows' or '2-morphisms') between them.

Definition: A *2-groupoid* is a 2-category in which all 1-cells and 2-cells are invertible. If the 2-groupoid has just one object then we call it a *2-group*.

Of course, there are also 2-functors between 2-categories and so, in particular, between 2-groups. Again this is for **you to formulate, looking up relevant definitions, etc.**

Internal categories in *Grps* are really exactly the same as 2-groups. The Brown-Spencer theorem thus constructs the *associated 2-group of a crossed module*. The fact that the composition in the internal category must be a group homomorphism implies that the '*interchange law*' must hold. This equation is in fact equivalent via the Brown-Spencer result to the Peiffer identity. (It is **left to you** to find out about the interchange law and to check that it is the Peiffer axiom in disguise. We will see it many times later on.)

Here would be a good place to mention that an internal monoid in *Grps* is just an Abelian group. The argument is well known and is usually known by the name of the *Eckmann-Hilton argument*. This starts by looking at the interchange law, which states that the monoid multiplication must be group homomorphism. From this it derives that the monoid identity must also be the group identity and that the two compositions must coincide. It is then easy to show that the group is Abelian.

2.3.3 Perhaps a bit more of an aside ... for the moment!

This is quite a good place to mention the groupoid based theory of all this. The resulting objects look like abstract 2-categories and are 2-groupoids. We have a set of objects, K_0, a set of arrows, K_1, depicted $x \xrightarrow{p} y$, and a set of two cells

$$\begin{tikzcd} x \arrow[r, bend left, "p"{name=U}] \arrow[r, bend right, "\partial c.p"'{name=D}] & y \arrow[from=U, to=D, phantom, "\Downarrow (c,p)"] \end{tikzcd} .$$

In our previous diagrams, as all the elements of P started and ended at the same single object, we could shift dimension down one step; our old objects are now arrows and our old arrows are 2-cells. We will return to this later.

The important idea to note here is that a 'higher dimensional category' has a link with an algebraic object. The 2-group(oid) provides a useful way of interpreting the structure of the crossed module *and* indicates possible ways towards similar applications and interpretations elsewhere. For instance, a presentation of a monoid leads more naturally to a 2-category than to any analogue of a crossed module, since kernels are less easy to handle than congruences in Mon.

There are other important interpretations of this. Categories such as that of vector spaces, Abelian groups or modules over a ring, have an additional structure coming from the tensor product, $A \otimes B$. They are *monoidal categories*. One can 'multiply' objects together and this is linked to a related multiplication on morphisms between the objects. In many of the important examples the multiplication is not strictly associative, so for instant, if A, B, C are objects there is an isomorphism between $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$, but this isomorphism is most definitely not the identity as the two objects are constructed in different ways. A similar effect happens in the category of sets with ordinary Cartesian product. The isomorphism is there because of universal properties, but it is again *not the identity*. It satisfies some coherence conditions, (a cocycle condition in disguise), relating to associativity of four fold tensors and the associahedron that we gave earlier, is a corresponding diagram for the five fold tensors. (Yes, there is a strong link, but that is not for these notes!) Our 2-group(oid) is the 'suspension' or 'categorification' of a similar structure. We can multiply objects and 'arrows' and the result is a strict 'gr-groupoid', or 'categorical group', i.e. a strict monoidal category with inverses. This is vague here, but will gradually be explored later on. If you want to explore the ideas further now, look at Baez and Dolan, [15].

(At this point, you do not need to know the definition of a monoidal category, but **remember to look it up in the not too distance future**, if you have not met it before, as later on the insights that an understanding of that notion gives you, will be very useful. It can be found in many places in the literature, and on the internet. The approach that you will get on best with depends on your background and your likes and dislikes mathematically, so we will not give one here.)

Just as associativity in a monoid is replaced by a 'lax' associativity 'up to coherent isomorphisms' in the above, gr-groupoids are 'lax' forms of internal categories in groups and thus indicate the presence of a crossed module-like structure, albeit in a weakened or 'laxified' form. Later we will see naturally occurring gr-groupoid structures associated with some constructions in non-Abelian cohomology. There is also a sense in which the link between fibrations and crossed modules given earlier here, indicates that fibrations are like a related form of lax crossed modules. In the notion of fibred category and the related Grothendieck construction, this intuition begins to be 'solidified' into a clearer strong relationship.

2.3.4 Automorphisms of a group yield a 2-group

We could also give this section a subtitle:

The automorphisms of a 1-type give a 2-type.

This is really an extended exercise in playing around with the ideas from the previous two sections. It uses a small amount of categorical language, but, hopefully, in a way that should be easy for even a categorical debutant to follow. The treatment will be quite detailed as it is that detail that provides the links between the abstract and the concrete.

We start with a look at 'functor categories', but with groupoids rather than general small categories as input. Suppose that $\mathcal{G}$ and $\mathcal{H}$ are groupoids, then we can form a new groupoid, $\mathcal{H}^{\mathcal{G}}$, whose objects are the functors, $f : \mathcal{G} \to \mathcal{H}$. Of course, functors in this context are just morphisms of groupoids, and, if $\mathcal{G}$, and $\mathcal{H}$ are $G[1]$ and $H[1]$, that is, two groups, G and H, thought of as one object groupoids, then the objects of $\mathcal{H}^{\mathcal{G}}$ are just the homomorphisms from G to H thought of in a slightly different way.

That gives the objects of $\mathcal{H}^{\mathcal{G}}$. For the morphisms from f_0 to f_1, we 'obviously' should think of natural transformations. (As usual, if you are not sufficiently conversant with elementary categorical ideas, pause and look them up in a suitable text or in Wikipedia.) Suppose $\eta : f_0 \to f_1$ is a natural transformation, then, for each x, an object of $\mathcal{G}$, we have an arrow,

$$\eta(x) : f_0(x) \to f_1(x),$$

in $\mathcal{H}$ such that, if $g : x \to y$ in $\mathcal{G}$, then the square

$$\begin{tikzcd} f_0(x) \arrow[r, "\eta(x)"] \arrow[d, "f_0(g)"'] & f_1(x) \arrow[d, "f_1(g)"] \\ f_0(y) \arrow[r, "\eta(y)"'] & f_1(y) \end{tikzcd}$$

commutes, so η 'is' the family, $\{\eta(x) \mid x \in Ob(\mathcal{G}\}$. Now assume $\mathcal{G} = G[1]$ and $\mathcal{H} = H[1]$, and that we try to interpret $\eta(x) : f_0(x) \to f_1(x)$ back down at the level of the groups, that is, a bit more 'classically' and group theoretically. There is only one object, which we denote $*$, if we need it, so we have that η corresponds to a single element, $\eta(*)$, in H, which we will write as h for simplicity, but now the condition for commutation of the square just says that, for any element $g \in G$,

$$hf_0(g) = f_1(g)h,$$

i.e., that f_0 and f_1 are *conjugate* homomorphisms, $f_1 = hf_0h^{-1}$..

It should be clear, (but **check that it is**), that this definition of morphism makes $\mathcal{H}^{\mathcal{G}}$ into a category, in fact into a groupoid, as the morphisms compose correctly and have inverses. (To get the inverse of η take the family $\{\eta(x)^{-1} \mid x \in Ob(\mathcal{G}\}$ and check the relevant squares commute.)

So far we have 'proved':

Lemma 6 *For groupoids, $\mathcal{G}$ and $\mathcal{H}$, the functor category, $\mathcal{H}^{\mathcal{G}}$, is a groupoid.* ■

We will be a bit sloppy in notation and will write H^G for what should, more precisely, be written $H[1]^{G[1]}$.

We note that it is usual to observe that, for Abelian groups, A, and B, the set of homomorphisms from A to B is itself an Abelian group, but that the set of homomorphisms from one non-Abelian group to another has no such nice structure. Although this is sort of true, the point of the above is that that set forms the set of objects for a very neat algebraic object, namely a groupoid!

If we have a third groupoid, $\mathcal{K}$, then we can also form $\mathcal{K}^{\mathcal{H}}$ and $\mathcal{K}^{\mathcal{G}}$, etc. and, as the objects of $\mathcal{K}^{\mathcal{H}}$ are homomorphisms from $\mathcal{H}$ to $\mathcal{K}$, we might expect to compose with the objects of $\mathcal{H}^{\mathcal{G}}$ to get ones of $\mathcal{K}^{\mathcal{G}}$. We might thus hope for a composition *functor*

$$\mathcal{K}^{\mathcal{H}} \times \mathcal{H}^{\mathcal{G}} \to \mathcal{K}^{\mathcal{G}}.$$

(There are **various things to check**, but we need not worry. We are really working with functors and natural transformations and with the investigation that shows that the category of small categories is 2-category. This means that if you get bogged down in the detail, you can easily find the ideas discussed in many texts on category theory.) This works, so we have that the category, *Grpd* has also a 2-category structure. (It is a '*Grpd*-enriched' category; see later for enriched categories. The formal definition is in section 12.1.1, although the basic idea is used before that. A groupoid-enriched category is a $(2,0)$-category, adopting the same convention as is adopted for ∞-categories, see page 508.)

We need to recall next that in any category, $\mathcal{C}$, the endomorphisms of any object, X, form a monoid, $End(X) := \mathcal{C}(X,X)$. You just use the composition and identities of $\mathcal{C}$ 'restricted to X'. If we play that game with any groupoid enriched category, C, then for any object, X, we will have a groupoid, $\mathsf{C}(X,X)$, which we might write $\mathsf{End}(X)$, (that is, using the same font to indicate 'enriched') and which also has a monoid structure,

$$\mathsf{C}(X,X) \times \mathsf{C}(X,X) \to \mathsf{C}(X,X).$$

It will be a monoid internal to *Grpd*. In particular, for any groupoid, $\mathcal{G}$, we have such an internal monoid of endomorphisms, $\mathcal{G}^{\mathcal{G}}$, and specialising down even further, for any group, G, such an internal monoid, G^G. Note that this is internal to the category of *groupoids* not of groups, as its monoid of objects is the endomorphism monoid of G, not a single element set. Within G^G, we can restrict attention to the subgroupoid on the automorphisms of G. We thus have this groupoid, $\mathsf{Aut}(G)$, which has as objects the automorphisms of G and, as typical morphism, $\eta : f_0 \to f_1$, a conjugation. It is important to note that as η is specified by an element of G and an automorphism, f_0, of G, the pair, (g, f_0), may then be a good way of thinking of it. (Two points, that may be obvious, but are important even if they are, are that the morphism η is not conjugation itself, but conjugates f_0. One has to specify where this morphism starts, its domain, as well as what it does, namely conjugate by g. Secondly, in (g, f_0), we *do* have the information on the codomain of η, as well. It is $gf_0g^{-1} = f_1$.)

Using this basic notation for the morphisms, we will look at the various bits of structure this thing has. (Remember, $\eta : f_0 \to f_1$ and $f_1 = gf_0g^{-1}$, as we will need to use that several times.) We have compositions of these pairs in two ways:

(a) as natural transformations: if

$$\begin{array}{rll} & \eta : f_0 \to f_1, & \eta = (g, f_0), \\ \text{and} & \eta' : f_1 \to f_2, & \eta' = (g', f_1), \end{array}$$

then the composite is $\eta'\sharp_1\eta = (g'g, f_0)$. (That is easy to check. As, for instance, $f_2 = g'f_1(g')^{-1} = (g'g)f_0(g'g)^{-1}$, ..., it all works beautifully). (A word of **warning** here, $(g'g)f_0(g'g)^{-1}$ is the conjugate of the automorphism f_0 by the element $(g'g)$. The bracket does not refer to f_0 applied to the 'thing in the bracket', so, for $x \in G$, $((g'g)f_0(g'g)^{-1})(x)$ is, in fact, $(g'g)f_0(x)(g'g)^{-1}$. This *is* slightly confusing so think about it, so as not to waste time later in avoidable confusion.)

b) using composition, $\sharp_0$, in the monoid structure. To understand this, it is easier to look at that composition as being specialised from the one we singled out earlier,

$$\mathcal{K}^{\mathcal{H}} \times \mathcal{H}^{\mathcal{G}} \to \mathcal{K}^{\mathcal{G}},$$

which is the composition in the 2-category of groupoids. (We really want $\mathcal{G} = \mathcal{H} = \mathcal{K}$, but, by keeping the more general notation, it becomes easier to see the roles of each $\mathcal{G}$.)

We suppose $f_0, f_1 : \mathcal{G} \to \mathcal{H}$, $f_0', f_1' : \mathcal{G} \to \mathcal{H}$, and then $\eta : f_0 \to f_1$, $\eta' : f_0' \to f_1'$. The 2-categorical picture is

$$\begin{tikzcd} \cdot \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\Downarrow\eta"] & \cdot \arrow[r, bend left, "f_0'"] \arrow[r, bend right, "f_1'"'] \arrow[r, phantom, "\Downarrow\eta'"] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, bend left, "f_0'f_0"] \arrow[r, bend right, "f_1'f_1"'] \arrow[r, phantom, "\Downarrow\eta''"] & \cdot \end{tikzcd},$$

with η'' being the desired composite, $\eta'\sharp_0\eta$, but how is it calculated. The important point is the *interchange law*. We can 'whisker' on the left or right, or, since the 'left-right' terminology can get confusing (does 'left' mean 'diagrammatically' or 'algebraically' on the left?), we will often use 'pre-' and 'post-' as alternative prefixes. The terminology may seem slightly strange, but is quite graphic when suitable diagrams are looked at! Whiskering corresponds to an interaction between 1-cell and 2-cells in a 2-category. In 'post-whiskering', the result is the composite of a 2-cell *followed* by a 1-cell:

Post-whiskering:

$$f_0'\sharp_0\eta : f_0'\sharp_0 f_0 \to f_0'\sharp_0 f_1,$$

$$\begin{tikzcd} \cdot \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\Downarrow\eta"] & \cdot \arrow[r, "f_0'"] & \cdot \end{tikzcd}$$

(It is convenient, here, to write the more formal $f_0'\sharp_0 f_0$, for what we would usually write as $f_0'f_0$.) The natural transformation, η is given by a family of arrows in $\mathscr{H}$, so $f_0'\sharp_0\eta$ is given by mapping that family across to $\mathscr{K}$ using f_0'. (Specialising to $\mathcal{G} = \mathscr{H} = \mathscr{K} = G[1]$, if $\eta = (g, f_0)$, then $f_0'\sharp_0\eta = (f_0'(g), f_0'f_0)$, as is easily checked; similarly for $f_1'\sharp_0\eta$.)

Pre-whiskering:

$$\eta'\sharp_0 f_0 : f_0'\sharp_0 f_0 \to f_1'\sharp_0 f_0,$$

$$\begin{tikzcd} \cdot \arrow[r, "f_0"] & \cdot \arrow[r, bend left, "f_0'"] \arrow[r, bend right, "f_1'"'] \arrow[r, phantom, "\Downarrow\eta'"] & \cdot \end{tikzcd}.$$

Here the morphism f_0 does not influence the g-part of η' at all. It just alters the domains. In the case that interests us, if $\eta' = (g', f_0')$, then $\eta'\sharp_0 f_0 = (g', f_0'f_0)$.

The way of working out $\eta'\sharp_0\eta$ is by using $\sharp_1$-composites. First,

$$\eta'\sharp_0\eta : f_0'f_0 \to f_1'f_1,$$

and we can go

$$\eta'\sharp_0 f_0 : f_0'f_0 \to f_1'f_0,$$

and then, to get to where we want to be, that is, $f_1'f_1$, we use

$$f_1'\sharp_0\eta : f_1'f_0 \to f_1'f_1.$$

This uses the $\sharp_1$-composition, so

$$\begin{aligned}
\eta'\sharp_0\eta &= (f_1'\sharp_0\eta)\sharp_1(\eta'\sharp_0 f_0)\\
&= (f_1'(g), f_1'f_0)\sharp_1(g', f_0'f_0)\\
&= (f_1'(g).g', f_0'f_0),
\end{aligned}$$

but $f_1'(g) = g'f_0(g)(g')^{-1}$, so the end results simplifies to $(g'f_0(g), f_0'f_0)$. **Hold on!** That looks nice, but we could have also calculated $\eta'\sharp_0\eta$ using the other form as the composite,

$$\begin{aligned}
\eta'\sharp_0\eta &= (\eta'\sharp_0 f_1)\sharp_1(f_0'\sharp_0\eta)\\
&= (g', f_0'f_1)\sharp_1(f_0'(g), f_0'f_0)\\
&= (g'f_0'(g), f_0'f_0),
\end{aligned}$$

so we did not have any problem. (All the properties of an internal groupoid in *Grps*, or, if you prefer that terminology, 2-group, can be derived from these two compositions. The $\sharp_1$ composition is the 'groupoid' direction, whilst the $\sharp_0$ is the 'group' one.)

We thus have a group of natural transformations made up of pairs, (g, f_0) and whose multiplication is given as above. This is just the semi-direct product group, $G \rtimes Aut(G)$, for the natural and obvious action of $Aut(G)$ on G. This group is sometimes called the *holomorph* of G.

We have two homomorphisms from $G \rtimes Aut(G)$ to $Aut(G)$. One sends (g, f_0) to f_0, so is just the projection, the other sends it to $f_1 = gf_0g^{-1} = \iota_g \circ f_0$. We can recognise this structure as being the associated 2-group of the crossed module, $(G, Aut(G), \iota)$, as we met on page 42. We call $\mathsf{Aut}(G)$, the *automorphism 2-group* of G..

2.3.5 Back to 2-types

From our crossed module, $\mathsf{C} = (C, P, \partial)$, we can build the internal groupoid, $\mathcal{X}(\mathsf{C})$, as before, then apply the nerve construction internally to the internal groupoid structure to get a simplicial group, $K(\mathsf{C})$.

Definition: Given a crossed module, $\mathsf{C} = (C, P, \partial)$, the nerve (taken internally in *Grps*) of the internal groupoid, $\mathcal{X}(\mathsf{C})$, defined by C, will be called the nerve of C or, if more precision is needed, its *simplicial group nerve* and will be denoted $K(\mathsf{C})$.

The simplicial set, $\overline{W}(K(\mathsf{C}))$, or its geometric realisation, would be called the *classifying space* of C.

We need this in some detail in low dimensions.

$$\begin{aligned}
K(\mathsf{C})_0 &= P\\
K(\mathsf{C})_1 &= C \rtimes P \qquad\qquad d_0 = t, d_1 = s\\
K(\mathsf{C})_2 &= C \rtimes (C \rtimes P),
\end{aligned}$$

where $d_0(c_2, c_1, p) = (c_2, \partial c_1.p)$, $d_1(c_2, c_1, p) = (c_2.c_1, p)$ and $d_2(c_2, c_1, p) = (c_1, p)$. The pattern continues with $K(\mathsf{C})_n = C \rtimes (\ldots \rtimes (C \rtimes P)\ldots)$, having n-copies of C. The d_i, for $0 < i < n$, are given by multiplication in C, d_0 is induced from t and d_n is a projection. The s_i are insertions of identities. (We will examine this in more detail later.)

Remark: A word of caution: for G a group considered as a crossed module, this 'nerve' is not the nerve of G in the sense used earlier. It is just the constant simplicial group corresponding to G. What is often called the nerve of G is what here has been called its classifying space. One way to view this is to note that $\mathcal{X}(\mathsf{C})$ has two independent structures, one a group, the other a category, and *this* nerve is of the category structure. The group, G, considered as a crossed module is like a set considered as a (discrete) category, having only identity arrows.

The Moore complex of $K(\mathsf{C})$ is easy to calculate and is just $NK(\mathsf{C})_i = 1$ if $i \geq 2$; $NK(\mathsf{C})_1 \cong C$; $NK(\mathsf{C})_0 \cong P$ with the $\partial : NK(\mathsf{C})_1 \to NK(\mathsf{C})_0$ being exactly the given ∂ of C. (This is left as an exercise. It is a useful one to do in detail.)

Proposition 4 *(Loday, [186]) The category $CMod$ of crossed modules is equivalent to the subcategory of Simp.Grps, consisting of those simplicial groups, G, having Moore complexes of length 1, i.e. $NG_i = 1$ if $i \geq 2$.* ■

This raises the interesting question as to whether it is possible to find alternative algebraic descriptions of the structures corresponding to Moore complexes of length n.

Is there any way of going directly from simplicial groups to crossed modules? Yes. The last two terms of the Moore complex will give us:

$$\partial : NG_1 \to NG_0 = G_0$$

and G_0 acts on NG_1 by conjugation via s_0, i.e. if $g \in G_0$ and $x \in NG_1$, then $s_0(g)xs_0(g)^{-1}$ is also in NG_1. (Of course, we could use multiple degeneracies to make g act on an $x \in NG_n$ just as easily.) As $\partial = d_0$, it respects the G_0 action, so CM1 is satisfied. In general, CM2 will not be satisfied. Suppose $g_1, g_2 \in NG_1$ and examine ${}^{\partial g_1}g_2 = s_0d_0g_1.g_2.s_0d_0g_1^{-1}$. This is rarely equal to $g_1g_2g_1^{-1}$. We write $\langle g_1, g_2\rangle = [g_1, g_2][g_2, s_0d_0g_1] = g_1g_2g_1^{-1}.({}^{\partial g_1}g_2)^{-1}$, so it measures the obstruction to CM2 for this pair g_1, g_2. This is often called the *Peiffer commutator* of g_1 and g_2. Noting that $s_0d_0 = d_0s_1$, we have an element

$$\{g_1, g_2\} = [s_0g_1, s_0g_2][s_0g_2, s_1g_1] \in NG_2$$

and $\partial\{g_1, g_2\} = \langle g_1, g_2\rangle$. This second pairing is called the *Peiffer lifting* (of the Peiffer commutator). Of course, if $NG_2 = 1$, then CM2 is satisfied (as for $K(\mathsf{C})$, above).

We could work with what we will call $M(G, 1)$, namely

$$\overline{\partial} : \frac{NG_1}{\partial NG_2} \to NG_0,$$

with the induced morphism and action. (As $d_0d_0 = d_0d_1$, the morphism is well defined.) This is a crossed module, but we could have divided out by less if we had wanted to. We note that $\{g_1, g_2\}$ is a product of degenerate elements, so we form, in general, the subgroup $D_n \subseteq NG_n$, generated by all degenerate elements.

Lemma 7

$$\overline{\partial} : \frac{NG_1}{\partial(NG_2 \cap D_2)} \to NG_0$$

is a crossed module. ■

This is, in fact, $M(sk_1G, 1)$, where sk_1G is the 1-skeleton of G, *i.e.*, the subsimplicial group generated by the k-simplices for $k = 0, 1$.

The kernel of $M(G, 1)$ is $\pi_1(G)$ and the cokernel $\pi_0(G)$ and

$$\pi_1(G) \to \frac{NG_1}{\partial NG_2} \to NG_0 \to \pi_0(G)$$

represents a class $k(G) \in H^3(\pi_0(G), \pi_1(G))$. Up to a notion of 2-equivalence, $M(G, 1)$ represents the 2-type of G completely. This is an algebraic version of the result of Mac Lane and Whitehead we mentioned earlier. Once we have a bit more on cohomology, we will examine it in detail.

This use of $NG_2 \cap D_2$ and our noting that $\{g_1, g_2\}$ is a product of degenerate elements may remind you of group T-complexes and thin elements. Suppose that G is a group T-complex in the sense of our discussion at the end of the previous chapter (page 39). In a general simplicial group, the subgroups, $NG_n \cap D_n$, will not be trivial. They give measure of the extent to which homotopical information in dimension n on G depends on 'stuff' from lower dimensions., *i.e.*, comparing G with its $(n-1)$-skeleton. (Remember that in homotopy theory, invariants such as the homotopy groups do not necessarily vanish above the dimension of the space, just recall the sphere S^2 and the subtle structure of its higher homotopy groups.)

The construction here of $M(sk_1G, 1)$ involves 'killing' the images of our possible multiple 'D-fillers' for horns, forcing uniqueness. We will see this again later.

Chapter 3

Crossed complexes

Accurate encoding of homotopy types is tricky. Chain complexes, even of G-modules, can only record certain, more or less Abelian, information. Simplicial groups, at the opposite extreme, can encode all connected homotopy types, but at the expense of such a large repetition of the essential information that makes calculation, at best, tedious and, at worst, virtually impossible. Complete information on truncated homotopy types can be stored in the cat^n-groups of Loday, [186]. We will look at these later. An intermediate model due to Blakers and Whitehead, [277], is that of a crossed complex. The algebraic and homotopy theoretic aspects of the theory of crossed complexes have been developed by Brown and Higgins, (cf. [61, 62], etc., in the bibliography and the monograph by Brown, Higgins and Sivera, [64]) and by Baues, [25–27]. We will use them later on in several contexts.

3.1 Crossed complexes: the Definition

We will initially look at reduced crossed complexes, *i.e.*, the group rather than the groupoid based case.

Definition: A *crossed complex*, which will be denoted C, consists of a sequence of groups and morphisms

$$\mathsf{C} : \ldots \to C_n \stackrel{\delta_n}{\to} C_{n-1} \stackrel{\delta_{n-1}}{\to} \ldots \to C_3 \stackrel{\delta_3}{\to} C_2 \stackrel{\delta_2}{\to} C_1$$

satisfying the following:
CC1) $\delta_2 : C_2 \to C_1$ is a crossed module;
CC2) each C_n, $(n > 2)$, is a left $C_1/\delta_1 C_2$-module and each δ_n, $(n > 2)$ is a morphism of left $C_1/\delta_2 C_2$-modules, (for $n = 3$, this means that δ_3 commutes with the action of C_1 and that $\delta_3(C_3) \subset C_2$ must be a $C_1/\delta_2 C_2$-module);
CC3) $\delta\delta = 0$.

The notion of a morphism of crossed complexes is clear. It is a graded collection of morphisms preserving the various structures. We thus get a category, Crs_{red} of reduced crossed complexes.

As we have that a crossed complex is a particular type of chain complex (of non-Abelian groups near the bottom), it is natural to define its homology groups in the obvious way.

Definition: If C is a crossed complex, its n^{th} *homology group* is

$$H_n(\mathsf{C}) = \frac{Ker\,\delta_n}{Im\,\delta_{n+1}}.$$

These homology groups are, of course, functors from Crs_{red} to the category of Abelian groups.

Definition: A morphism $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$ is called a *weak equivalence* if it induces isomorphisms on all homology groups.

There are good reasons for considering the homology groups of a crossed complex as being its homotopy groups. For example, if the crossed complex comes from a simplicial group then the homotopy groups of the simplicial group are the same as the homology groups of the given crossed complex (possibly shifted in dimension, depending on the notational conventions you are using).

The non-reduced version of the concept is only a bit more difficult to write down. It has C_1 as a groupoid on a set of objects C_0 with each C_k, a family of groups indexed by the elements of C_0. The axioms are very similar; see [64] for instance or many of the papers by Brown and Higgins listed in the bibliography. This gives a category, Crs, of (unrestricted) crossed complexes and morphisms between them. This category is very rich in structure. It has a tensor product structure, denoted $\mathsf{C} \otimes \mathsf{D}$ and a corresponding mapping complex construction, $\mathsf{Crs}(\mathsf{C}, \mathsf{D})$, making it into a monoidal closed category. The details are to be found in the papers and book listed above and will be recalled later when needed.

It is worth noting that this notion restricts to give us a notion of *weak equivalence* applicable to crossed modules as well.

Definition: A morphism, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$, between two crossed modules, is called a *weak equivalence* if it induces isomorphisms on π_0 and π_1, that is, on both the kernel and cokernel of the crossed modules.

The relevant reference for π_0 and π_1 is page 46.

3.1.1 Examples: crossed resolutions

As we mentioned earlier, a resolution of a group (or other object) is a model for the homotopy type represented by the group, but which usually is required to have some nice freeness properties. With crossed complexes we have some notion of homotopy around, just as with chain complexes, so we can apply that vague notion of resolution in this context as well. This will give us some neat examples of crossed complexes that are 'tuned' for use in cohomology.

A *crossed resolution* of a group, G, is a crossed complex, C, such that for each $n > 1$, $Im\,\delta_n = Ker\,\delta_{n-1}$ and there is an isomorphism, $C_1/\delta_2 C_2 \cong G$.

A crossed resolution can be constructed from a presentation $\mathscr{P} = (X : R)$ as follows:

Let $C(P) \to F(X)$ be the free crossed module associated with $\mathscr{P}$. We set $C_2 = C(\mathscr{P})$, $C_1 = F(X)$, $\delta_1 = \partial$. Let $\kappa(\mathscr{P}) = Ker(\partial : C(\mathscr{P}) \to F(X))$. This is the module of identities of the presentation and is a left G-module. As the category G-Mod has enough projectives, we can form a free resolution $\mathbb{P}$ of $\kappa(\mathscr{P})$. To obtain a crossed resolution of G, we join $\mathbb{P}$ to the crossed module by setting $C_n = P_{n-2}$ for $n > 3$, $\delta_n = d_{n-2}$ for $n > 3$ and the composite from P_0 to $C(P)$ for $n = 3$.

3.1.2 The standard crossed resolution

We next look at a particular case of the above, namely the standard crossed resolution of G. In this, which we will denote by $\mathsf{C}G$, we have

(i) $C_1G =$ the free group on the underlying set of G. The element corresponding to $u \in G$ will be denoted by $[u]$.

(ii) C_2G is the free crossed module over C_0G on generators, written $[u, v]$, considered as elements of the set $G \times G$, in which the map δ_1 is defined on generators by

$$\delta[u, v] = [uv]^{-1}[u][v].$$

(iii) For $n > 3$, C_nG is the free left G-module on the set G^n, but in which one has equated to zero any generator $[u_1, \ldots, u_n]$ in which some u_i is the identity element of G.

If $n > 2$, $\delta : C_{n+1}G \to C_nG$ is given by the usual formula

$$\begin{aligned} \delta[u_1, \ldots, u_{n+1}] &= {}^{[u_1]}[u_2, \ldots, u_{n+1}] \\ &\quad + \sum_{i=1}^{n}(-1)^i[u_1, \ldots, u_iu_{i+1}, \ldots, u_{n+1}] + (-1)^{n+1}[u_1, \ldots, u_n]. \end{aligned}$$

For $n = 2$, $\delta : C_3G \to C_2G$ is given by

$$\delta[u, v, w] = {}^{[u]}[v, w].[u, v]^{-1}.[uv, w]^{-1}[u, vw].$$

This is the crossed analogue of the inhomogeneous bar resolution, $\mathsf{B}G$, of the group G. A groupoid version can be found in Brown-Higgins, [60], and the abstract group version in Huebschmann, [159]. In the first of these two references, it is pointed out that $\mathsf{C}G$, as constructed, is isomorphic to the crossed complex, $\underline{\pi}(BG)$, of the classifying space of G considered with its skeletal filtration.

For any filtered space, $\underline{X} = (X_n)_{n\in\mathbb{N}}$, its *fundamental crossed complex*, $\underline{\pi}(\underline{X})$, is, in general, a non-reduced crossed complex. It is defined to have

$$\underline{\pi}(\underline{X})_n = (\pi_n(X_n, X_{n-1}, a))_{a\in X_0}$$

with $\underline{\pi}(\underline{X})_1$, the fundamental groupoid $\Pi_1X_1X_0$, and $\underline{\pi}(\underline{X})_2$, the family, $(\pi_2(X_2, X_1, a))_{a\in X_0}$. It will only be reduced if X_0 consists just of one point.

Most of the time we will only discuss the reduced case in detail, although the non-reduced case will be needed sometimes. Following that, we will often use the notation Crs for the category of *reduced* crossed complexes unless we need the more general case. This may occasionally cause a little confusion, but it is much more convenient for most of the time.

There are two useful, but conflicting, conventions as to indexation in crossed complexes. In the topologically inspired one, the bottom group is C_1, in the simplicial and algebraic one, it is C_0. Both get used and both have good motivation. The natural indexation for the standard crossed resolution would seem to be with C_n being generated by n-tuples, i.e. the topological one. (I am not sure that all instances of the other have been avoided, so please be careful!)

G-augmented crossed complexes. Crossed resolutions of G are examples of G-augmented crossed complexes. A *G-augmented crossed complex* consists of a pair (C, φ) where C is a crossed complex and where $\varphi : C_1 \to G$ is a group homomorphism satisfying
(i) $\varphi\delta_1$ is the trivial homomorphism;
(ii) $Ker\,\varphi$ acts trivially on C_i for $i \geq 3$ and also on C_2^{Ab}.

A *morphism*

$$(\alpha, Id_G) : (\mathsf{C}, \varphi) \to (\mathsf{C}', \varphi')$$

of G-augmented crossed complexes consists of a morphism

$$\alpha : \mathsf{C} \to \mathsf{C}'$$

of crossed complexes such that $\varphi'\alpha_0 = \varphi$.

This gives a category, Crs_G, which behaves nicely with respect to change of groups, i.e. if $\varphi : G \to H$, then there are induced functors between the corresponding categories.

3.2 Crossed complexes and chain complexes: I

(Some of the proofs here are given in more detail as they are less routine and are not that available elsewhere. A source for much of this material is in the work of Brown and Higgins, [62], where these ideas were explored thoroughly for the first time; see also the treatment in [64].)

We have introduced crossed complexes where normally chain complexes of modules would have been used. We have seen earlier the bar resolution and now we have the standard crossed resolution. What is the connection between them? The answer is approximately that chain complexes form a category equivalent to a reflective subcategory of Crs. In other words, there is a canonical way of building a chain complex from a crossed one akin to the process of Abelianising a group. The resulting reflection functor sends the standard crossed resolution of a group to the bar resolution. The details involve some interesting ideas.

In section 2.1.1, we saw that, given a morphism, $\theta : M \to N$, of modules over a group G, $\partial : M \to N \rtimes G$, given by $\partial(m) = (\theta(m), 1_G)$ is a crossed module, where $N \rtimes G$ acts on M via the projection to G. That example easily extends to a functorial construction which, from a positive chain complex, D, of G-modules, gives us a crossed complex $\Delta_G(\mathsf{D})$ with $\Delta_G(\mathsf{D})_n = D_n$ if $n > 1$ and equal to $D_1 \rtimes G$ for $n = 1$.

Lemma 8 *$\Delta_G : Ch(G{-}Mod) \to Crs_G$ is an embedding.*

Proof: That Δ_G is a functor is easy to see. It is also easy to check that it is full and faithful, that is it induces bijections,

$$Ch(G{-}Mod)(\mathsf{A}, \mathsf{B}) \to Crs_G(\Delta_G(\mathsf{A}), \Delta_G(\mathsf{B})).$$

The augmentation of $\Delta_G(A)$ is given by the projection of $A_1 \rtimes G$ onto G. ■

We can thus turn a positive chain complex into a crossed complex. Does this functor have a left adjoint? i.e. is there a functor $\xi_G : Crs_G \to Ch(G{-}Mod)$ such that

$$Ch(G{-}Mod)(\xi_G(\mathsf{C}), \mathsf{D}) \to Crs_G(\mathsf{C}, \Delta_G(\mathsf{D}))?$$

If so it would suggest that chain complexes of G-modules are like G-augmented crossed complexes that satisfy some additional equational axioms. As an example of a similar situation think of 'Abelian groups' within 'groups' for which the inclusion has a left adjoint, namely Abelianisation $(G)^{Ab} = G/[G,G]$. Abelian groups are of course groups that satisfy the additional rule $[x,y]=1$. Other examples of such situations are nilpotent groups of a given finite rank c. The subcategories of this general form are called *varieties* and, for instance, the study of varieties of groups is a very interesting area of group theory. Incidentally, it is possible to define various forms of cohomology modulo a variety in some sense. We will not explore that here.

We thus need to look at morphisms of crossed complexes from a crossed complex C to one of form $\Delta_G(\mathsf{D})$, and we need therefore to look at morphisms into a semidirect product. These are useful for other things, so are worth looking at in detail.

3.2.1 Semi-direct products and derivations.

Suppose that we have a diagram

$$\begin{tikzcd} H \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & K \rtimes G \arrow[dl, "proj"] \\ & G & \end{tikzcd}$$

where K is a G-module (written additively, so we write $g.k$ not ${}^g k$ for the action). This is like the very bottom of the situation for a morphism $f : \mathsf{C} \to \Delta_G(\mathsf{D})$.

As the codomain of f is a semidirect product, we can decompose f, as a function, in the form

$$f(h) = (f_1(h), \alpha(h)),$$

identifying its second component using the diagram. The mapping f_1 is not a homomorphism. As f is one, however, we have

$$(f_1(h_1h_2), \alpha(h_1h_2)) = f(h_1)f(h_2) = (f_1(h_1) + \alpha(h_1)f_1(h_2), \alpha(h_1h_2)),$$

i.e. f_1 satisfies

$$f_1(h_1h_2) = f_1(h_1) + \alpha(h_1)f_1(h_2)$$

for all $h_1, h_2 \in H$.

3.2.2 Derivations and derived modules.

We will use the identification of G-modules for a group G with modules over the group ring, $\mathbb{Z}[G]$, of G. Recall that this ring is obtained from the free Abelian group on the set G by defining a multiplication extending linearly that of G itself. (Formally if, for the moment, we denote by e_g, the generator corresponding to $g \in G$, then an arbitrary element of $\mathbb{Z}[G]$ can be written as $\sum_{g\in G} n_g e_g$ where the n_g are integers and only finitely many of them are non-zero. The multiplication is by 'convolution' product, that is,

$$\Big(\sum_{g\in G} n_g e_g\Big)\Big(\sum_{g\in G} m_g e_g\Big) = \sum_{g\in G}\Big(\sum_{g_1\in G} n_{g_1} m_{g_1^{-1}g} e_g\Big).$$

Sometimes, later on, we will need other coefficients than $\mathbb{Z}$ in which case it is appropriate to use the term 'group algebra' of G, over that ring of coefficients.

We will also need the augmentation, $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$, given by $\varepsilon(\sum_{g \in G} n_g e_g) = \sum_{g \in G} n_g$ and its kernel $I(G)$, known as the *augmentation ideal*.

Definitions: Let $\varphi : G \to H$ be a homomorphism of groups. A *φ-derivation*

$$\partial : G \to M$$

from G to a left $\mathbb{Z}[H]$-module, M, is a mapping from G to M, which satisfies the equation

$$\partial(g_1 g_2) = \partial(g_1) + \varphi(g_1)\partial(g_2)$$

for all g_1, $g_2 \in G$.

Such φ-derivations are really all derived from a universal one.

Definition: A *derived module* for φ consists of a left $\mathbb{Z}[H]$-module, D_φ, and a φ-derivation, $\partial_\varphi : G \to D_\varphi$ with the following universal property:

Given any left $\mathbb{Z}[H]$-module, M, and a φ-derivation $\partial : G \to M$, there is a unique morphism

$$\beta : D_\varphi \to M$$

of $\mathbb{Z}[H]$-modules such that $\beta\partial_\varphi = \partial$.

The derivation ∂_φ is called the *universal φ derivation.*

The set of all φ-derivations from G to M has a natural Abelian group structure. We denote this set by $Der_\varphi(G, M)$. This gives a functor from *H-Mod* to *Ab*, the category of Abelian groups. If $(D_\varphi, \partial_\varphi)$ exists, then it sets up a natural isomorphism

$$Der_\varphi(G, M) \cong H{-}Mod(D_\varphi, M),$$

i.e., $(D_\varphi, \partial_\varphi)$ represents the φ-derivation functor.

3.2.3 Existence

The treatment of derived modules that is found in Crowell's paper, [95], provides a basis for what follows. In particular it indicates how to prove the existence of $(D_\varphi, \partial_\varphi)$ for any φ.

Form a $\mathbb{Z}[H]$-module, D, by taking the free left $\mathbb{Z}[H]$-module, $\mathbb{Z}[H]^{(X)}$, on a set of generators, $X = \{\partial g : g \in G\}$. Within $\mathbb{Z}[H]^{(X)}$ form the submodule, Y, generated by the elements

$$\partial(g_1 g_2) - \partial(g_1) - \varphi(g_1)\partial(g_2).$$

Let $D = \mathbb{Z}[H]^{(X)}/Y$ and define $d : G \to D$ to be the composite:

$$G \xrightarrow{\eta} \mathbb{Z}[H]^{(X)} \xrightarrow{quotient} D,$$

where η is "inclusion of the generators", $\eta(g) = \partial g$, thus d, by construction, will be a φ-derivation. The universal property is easily checked and hence $(D_\varphi, \partial_\varphi)$ exists.

We will later on construct $(D_\varphi, \partial_\varphi)$ in a different way which provides a more amenable description of D_φ, namely as a tensor product. As a first step towards this description, we shall give a simple description of D_G, that is, the derived module of the identity morphism of G. More precisely we shall identify (D_G, ∂_G) as being $(I(G), \partial)$, where, as above, $I(G)$ is the augmentation ideal of $\mathbb{Z}[G]$ and $\partial : G \to I(G)$ is the usual map, $\partial(g) = g - 1$.

Our earlier observations give us the following useful result:

Lemma 9 *If G is a group and M is a G-module, then there is an isomorphism*

$$Der_G(G, M) \to Hom/G(G, M \rtimes G)$$

where $Hom/G(G, M \rtimes G)$ is the set of homomorphisms from G to $M \rtimes G$ over G, i.e., *$\theta : G \to M \rtimes G$ such that for each $g \in G$, $\theta(g) = (g, \theta'(g))$ for some $\theta'(g) \in M$.* ■

3.2.4 Derivation modules and augmentation ideals

Proposition 5 *The derivation module, D_G, is isomorphic to $I(G) = Ker(\mathbb{Z}[G] \to \mathbb{Z})$. The universal derivation is*

$$d_G : G \to I(G)$$

given by $d_G(g) = g - 1$.

Proof:

We introduce the following notation: if $\delta : G \to M$ is a derivation, then we write $f_\delta : I(G) \to M$ for the $\mathbb{Z}[G]$-module morphism given by $f_\delta(g - 1) = \delta(g)$ and extended linearly. It is then simple to check the universal property. That this works follows from the fact that $I(G)$, as an Abelian group, is free on the set $\{g - 1 : g \in G\}$ and that the relations in $I(G)$ are generated by those of the form

$$g_1(g_2 - 1) = (g_1 g_2 - 1) - (g_1 - 1).$$

These two facts can be checked independently and are well known[1]. ■

We note a result on the augmentation ideal construction that is not commonly found in the literature.

The proof is easy and so will be omitted.

Lemma 10 *Given groups G and H in $\mathcal{C}$ and a commutative diagram*

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\psi"'] & M \arrow[d, "\varphi"] \\ H \arrow[r, "\delta'"'] & N \end{tikzcd} \qquad (*)$$

where δ, δ' are derivations, M is a left $\mathbb{Z}[G]$-module, N is a left $\mathbb{Z}[H]$-module and φ is a module map over ψ, i.e., *$\varphi(g.m) = \psi(g)\varphi(m)$ for $g \in G$, $m \in M$, then the corresponding diagram*

$$\begin{tikzcd} I(G) \arrow[r, "f_\delta"] \arrow[d, "\psi"'] & M \arrow[d, "\varphi"] \\ I(H) \arrow[r, "f_{\delta'}"'] & N \end{tikzcd} \qquad (**)$$

[1] even being set as exercises early in [56].

is commutative. ■

The earlier proposition has the following corollaries, which we note for future reference:

Corollary 1 *The subset $Im\, d_G = \{g - 1 : g \in G\} \subset I(G)$ generates $I(G)$ as a $\mathbb{Z}[G]$-module. Moreover the relations between these generators are generated by those of the form*

$$(g_1 g_2 - 1) - (g_1 - 1) - g_1(g_2 - 1).$$

■

It is useful to have also the following reformulation of the above results stated explicitly.

Corollary 2 *There is a natural isomorphism*

$$Der_G(G, M) \cong G{-}Mod(I(G), M).$$

■

3.2.5 Generation of $I(G)$.

The first of these two corollaries raises the question as to whether, if $X \subset G$ generates G, does the set $G_X = \{x - 1 : x \in X\}$ generate $I(G)$ as a $\mathbb{Z}[G]$-module.

Proposition 6 *If X generates G, then G_X generates $I(G)$.*

Proof: We know $I(G)$ is generated by the $g - 1$s for $g \in G$. If g is expressible as a word of length n in the generators X then we can write $g - 1$ as a $\mathbb{Z}[G]$-linear combination of terms of the form $x - 1$ in an obvious way. (If $g = w.x$ with w of lesser length than that of g, $g - 1 = w - 1 + w(x - 1)$, so use induction on the length of the expression for g in terms of the generators.) ■

When G is free: If G is free, say, $G \cong F(X)$, *i.e.*, is free on the set X, we can say more.

Proposition 7 *If $G \cong F(X)$ is the free group on the set X, then the set $\{x - 1 : x \in X\}$ freely generates $I(G)$ as a $\mathbb{Z}[G]$-module.*

Proof: (We will write F for $F(X)$.) The easiest proof would seem to be to check the universal property of derived modules for the function $\delta : F \to \mathbb{Z}[G]^{(X)}$, given on generators by

$$\delta(x)(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } y \neq x; \end{cases}$$

then extended using the derivation rule to all of F using induction. This uses essentially that each element of F has a *unique* expression as a reduced word in the generators, X.

Suppose then that we have a derivation $\partial : F \to M$, define $\overline{\partial} : \mathbb{Z}[G]^{(X)} \to M$ by $\overline{\partial}(e_x) = \partial(x)$, extending linearly. Since by construction $\overline{\partial}\delta = \partial$ and is the unique such homomorphism, we are home. ■

Note: In both these proofs we are thinking of the elements of the free module on X as being functions from X to the group ring, these functions being of 'finite support', i.e. being non-zero on only a finite number of elements of X. This can cause some complications if X is infinite or has some topology as it will in some contexts. The *idea* of the proof will usually go across to that situation but details have to change. (A situation in which this happens is in profinite group theory where the derivations have to be continuous for the profinite topology on the group, see [226].)

3.2.6 (D_φ, d_φ), the general case.

We can now return to the identification of (D_φ, d_φ) in the general case.

Proposition 8 *If $\varphi : G \to H$ is a homomorphism of groups, then $D_\varphi \cong \mathbb{Z}[H] \otimes_G I(G)$, the tensor product of $\mathbb{Z}[H]$ and $I(G)$ over G.*

Proof: If M is a $\mathbb{Z}[H]$-module, we will write $\varphi^*(M)$ for the restricted $\mathbb{Z}[G]$-module, i.e. M with G-action given by $g.m := \varphi(g).m$. Recall that the functor φ^* has a left adjoint given by sending a G-module, N to $\mathbb{Z}[H] \otimes_G N$, i.e. take the tensor of Abelian groups, $\mathbb{Z}[H] \otimes N$ and divide out by $x \otimes g.n \equiv x\varphi(g) \otimes n$.

With this notation we have a chain of natural isomorphisms,

$$\begin{aligned} Der_\varphi(G, M) &\cong Der_G(G, \varphi^*(M)) \\ &\cong G{-}Mod(I(G), \varphi^*(M)) \\ &\cong H{-}Mod(\mathbb{Z}[H] \otimes_G I(G), M), \end{aligned}$$

so by universality,

$$D_\varphi \cong \mathbb{Z}[H] \otimes_G I(G),$$

as required. ∎

3.2.7 D_φ for $\varphi : F(X) \to G$.

The above will be particularly useful when φ is the "co-unit" map, $F(X) \to G$, for X a set that generates G. We could, for instance, take $X = G$ as a set, and φ to be the usual natural epimorphism.

In fact we have the following:

Corollary 3 *Let $\varphi : F(X) \to G$ be an epimorphism of groups, then there is an isomorphism*

$$D_\varphi \cong \mathbb{Z}[G]^{(X)}$$

of $\mathbb{Z}[G]$-modules. In this isomorphism, the generator ∂_x, of D_φ corresponding to $x \in X$, satisfies

$$d_\varphi(x) = \partial_x$$

for all $x \in X$. ∎

(You should check that you see how this follows from our earlier results.)

3.3 Associated module sequences

3.3.1 Homological background

Given an exact sequence

$$1 \to K \to L \to Q \to 1$$

of abstract groups, then it is a standard result from homological algebra that there is an associated exact sequence of modules,

$$0 \to K^{Ab} \to \mathbb{Z}[Q] \otimes_L I(L) \to I(Q) \to 0.$$

There are several different proofs of this. Homological proofs give this as a simple consequence of the Tor^L-sequence corresponding to the exact sequence

$$0 \to I(L) \to \mathbb{Z}[L] \to \mathbb{Z} \to 0$$

together with a calculation of $Tor_1^L(\mathbb{Z}[Q], \mathbb{Z})$, but we are not assuming that much knowledge of standard homological algebra. That homological proof also, to some extent, hides what is happening at the 'elementary' level, in both the sense of 'simple' and also that of'what happens to the 'elements' of the groups and modules concerned.

The second type of proof is more directly algebraic and has the advantage that it accentuates various universal properties of the sequence. The most thorough treatment of this would seem to be by Crowell, [95], for the discrete case. We outline it below.

3.3.2 The exact sequence.

Before we start on the discussion of the exact sequence, it will be useful to have at our disposal some elementary results on Abelianisation of the groups in a crossed module. Here we actually only need them for normal subgroups but we will need it shortly anyway in the more general form. Suppose that (C, P, ∂) is a crossed module, and we will set $A = Ker\partial$ with its module structure that we looked at before, and $N = \partial C$, so A is a P/N-module.

Lemma 11 *The Abelianisation of C has a natural $\mathbb{Z}[P/N]$-module structure on it.*

Proof: First we should point out that by "Abelianisation" we mean $C^{Ab} = C/[C, C]$, which is, of course, Abelian and it suffices to prove that N acts trivially on C^{Ab}, since P already acts in a natural way. However, if $n \in N$, and $\partial c = n$, then for any $c' \in C$, we have that ${}^n c' = {}^{\partial c} c' = cc'c^{-1}$, hence ${}^n c'(c')^{-1} \in [C, C]$ or equivalently

$${}^n(c'[C, C]) = c'[C, C],$$

so N does indeed act trivially on C^{Ab}. ∎

Of course N^{Ab} also has the structure of a $\mathbb{Z}[P/N]$-module and thus a crossed module gives one three P/N-modules. These three are linked as shown by the following proposition.

Proposition 9 *Let (C, P, ∂) be a crossed module. Then the induced morphisms*

$$A \to C^{Ab} \to N^{Ab} \to 0$$

form an exact sequence of $\mathbb{Z}[P/N]$-modules.

Proof: It is clear that the sequence

$$1 \to A \to C \to N \to 1$$

is exact and that the induced homomorphism from C^{Ab} to N^{Ab} is an epimorphism. Since the composite homomorphism from A to N is trivial, A is mapped into $Ker(C^{Ab} \to N^{Ab})$ by the composite $A \to C \to C^{Ab}$. It is easily checked that this is onto and hence the sequence is exact as claimed. ∎

Now for the main exact sequence result here:

Proposition 10 *Let*

$$1 \to K \overset{\varphi}{\to} L \overset{\psi}{\to} Q \to 1$$

be an exact sequence of groups and homomorphisms. Then there is an exact sequence

$$0 \to K^{Ab} \overset{\tilde{\varphi}}{\to} \mathbb{Z}[Q]\otimes_L I(L) \overset{\tilde{\psi}}{\to} I(Q) \to 0$$

of $\mathbb{Z}[Q]$*-modules.*

Proof: By the universal property of D_ψ, there is a unique morphism,

$$\tilde{\psi} : D_\psi \to I(Q),$$

such that $\tilde{\psi}\partial_\psi = I(\psi)\partial_L$.

Let $\delta : K \to K^{Ab} = K/[K, K]$ be the canonical Abelianising morphism. We note that $\partial_\psi\varphi : K \to D_\psi$ is a homomorphism (since

$$\begin{aligned} \partial_\psi\varphi(k_1k_2) &= \partial_\psi\varphi(k_1) + \psi\varphi(k_1)\partial_\psi\varphi(k_2) \\ &= \partial_\psi\varphi(k_1) + \partial_\psi\varphi(k_2),) \end{aligned}$$

so let $\tilde{\varphi} : K^{Ab} \to D_\psi$ be the unique morphism satisfying $\tilde{\varphi}\delta = \partial_\psi\varphi$ with K^{Ab} having its natural $\mathbb{Z}[Q]$-module structure.

That the composite $\tilde{\psi}\tilde{\varphi} = 0$ follows easily from $\psi\varphi = 0$. Since D_ψ is generated by symbols $d\ell$ and $\tilde{\psi}(d\ell) = \psi(\ell) - 1$, it follows that $\tilde{\psi}$ is onto. We next turn to "$Ker\ \tilde{\psi} \subseteq Im\ \tilde{\varphi}$".

If we can prove $\alpha : D_\psi \to I(Q)$ is the cokernel of $\tilde{\varphi}$, then we will have checked this inclusion and incidentally will have reproved that $\tilde{\psi}$ is onto.

Now let $D_\psi \to C$ be any morphism such that $\alpha\tilde{\varphi} = 0$. Consider the diagram

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "\delta"'] & L \arrow[r, "\psi"] \arrow[d, "\partial_\psi"] & Q \arrow[d, "\partial_Q"] \\ K^{Ab} \arrow[r, "\tilde{\varphi}"] & D_\psi \arrow[r, "\tilde{\psi}"] \arrow[dr, "\alpha"'] & I(Q) \\ & & C \end{tikzcd}$$

The composite $\alpha\partial_\psi$ vanishes on the image of φ since $\alpha\partial_\psi\varphi = \alpha\tilde{\varphi}\delta$ and $\alpha\tilde{\varphi}$ is assumed zero. Define $d : Q \to C$ by $d(q) = \alpha\partial_\psi(\ell)$ for $\ell \in L$ such that $\psi(\ell) = q$. As $\alpha\partial_\psi$ vanishes on $Im\ \varphi$, this is well defined and

$$\begin{aligned} d(q_1q_2) &= \alpha\partial_\psi(\ell_1\ell_2) \\ &= \alpha\partial_\psi(\ell_1) + \alpha(\psi(\ell_1)\partial_\psi(\ell_2)) \\ &= d(q_1) + q_1d(q_2) \end{aligned}$$

so d factors as $\bar{\alpha}\partial_Q$ in a unique way with $\bar{\alpha} : I(Q) \to C$. It remains to prove that $\alpha = \tilde{\psi}$, but

$$\begin{aligned} \tilde{\psi}\partial_\psi &= I_C(\psi)\partial_L \\ &= \partial_Q\psi \end{aligned}$$

by the naturality of ∂. Now finally note that $\bar{\alpha}\partial_Q = d$ and $d\psi = \alpha\partial_\psi$ to conclude that $\tilde{\psi}\partial_\psi$ and $\alpha\partial_\psi$ are equal. Equality of α and $\bar{\alpha}\tilde{\psi}$ then follows by the uniqueness clause of the universal property of (D_ψ, ∂_ψ).

Next we need to check that $K^{Ab} \to D_\psi$ is a monomorphism. To do this we use the fact that there is a transversal, $s : Q \to L$, satisfying $s(1) = 1$. This means that, following Crowell, [95] p. 224, we can for each $\ell \in L$, $q \in Q$, find an element $q \times \ell$ uniquely determined by the equation

$$\varphi(q \times \ell)) = s(q)\ell s(q\psi(\ell))^{-1},$$

which, of course, defines a function from $Q \times L$ to K. Crowell's lemma 4.5 then shows

$$q \times \ell_1\ell_2 = (q \times \ell_1)(q\psi(\ell_1) \times \ell_2) \quad \text{for} \quad \ell_1, \ell_2 \in L.$$

Now let $M = \mathbb{Z}[Q]^{(X)}$, with $X = \{\partial\ell : \ell \in L\}$, so that there is an exact sequence

$$M \to D_\psi \to 0.$$

The underlying group of $\mathbb{Z}[Q]$ is the free Abelian group on the underlying set of Q. Similarly M, above, has, as underlying group, the free Abelian group on the set $Q \times X$.

Define a map $\tau : M \to K^{Ab}$ of Abelian groups by

$$\tau(a, \partial\ell) = \delta(q \times \ell).$$

We check that if $p(m) = 0$, then $\tau(m) = 0$. Since $Ker\, p$ is generated as a $\mathbb{Z}[Q]$-module by elements of the form

$$\partial(\ell_1\ell_2) - \partial\ell_1 - \psi(\ell_1)\partial\ell_2,$$

it follows that as an Abelian group, $Ker\, p$ is generated by the elements

$$(q, \partial(\ell_1\ell_2)) - (q, \partial\ell_1) - (q\psi(\ell_1), \partial\ell_2).$$

We claim that τ is zero on these elements; in fact

$$\begin{aligned} \tau(q, \partial(\ell_1\ell_2)) &= \delta(q \times (\ell_1\ell_2)) \\ &= \delta(q \times \ell_1) + \delta(q\psi(\ell_1) \times \ell_2) \\ &= \tau(q, \ell_1) + \tau(q\psi(\ell_1), \ell_2). \end{aligned}$$

Thus τ induces a map $\eta : D_\psi \to K^{Ab}$ of Abelian groups.

Finally we check $\eta\tilde{\varphi} =$ identity, so that $\tilde{\varphi}$ is a monomorphism: let $b \in K^{Ab}$, $k \in K$ be such that $\delta(k) = b$, then

$$\begin{aligned} \eta\tilde{\varphi}(b) &= \eta\tilde{\varphi}\delta(k) \\ &= \eta\partial_\psi(k) \\ &= \delta(1 \times \varphi(k)), \end{aligned}$$

but $1 \times \varphi(k)$ is uniquely determined by

$$\varphi(1 \times \varphi(k)) = s(1)\varphi(k)s(1\psi\varphi(k))^{-1} = \varphi(k),$$

since $s(1) = 1$, hence $1 \times \varphi(k) = k$ and $\eta\tilde{\varphi}(b) = \delta(k) = b$ as required. ■

A discussion of the way in which this result interacts with the theory of covering spaces can be found in Crowell's paper already cited. We will very shortly see the connection of this module sequence with the Jacobian matrix of a group presentation and the Fox free differential calculus. It is this latter connection which suggests that we need more or less explicit formulae for the maps $\tilde{\varphi}$ and $\tilde{\psi}$ and hence requires that Crowell's detailed proof be used, not the slicker homological proof.

3.3.3 Reidemeister-Fox derivatives and Jacobian matrices

At various points, we will refer to Reidemeister-Fox derivatives as developed by Fox in a series of articles, see [131], and also summarised in Crowell and Fox, [96]. We will call these derivatives *Fox derivatives.*

Suppose G is a group and M a G-module and let $\delta : G \to M$ be a derivation, (so $\delta(g_1g_2) = \delta(g_1) + g_1\delta(g_2)$ for all $g_1, g_2 \in G$), then, for calculations, the following lemma is very valuable, although very simple to prove.

Lemma 12 *If $\delta : G \to M$ is a derivation, then*
(i) $\delta(1_G) = 0$;
(ii) $\delta(g^{-1}) = -g^{-1}\delta(g)$ for all $g \in G$;
(iii) for any $g \in G$ and $n \geq 1$,

$$\delta(g^n) = (\sum_{k=0}^{n-1} g^k)\delta(g).$$

Proof: As was said, these are easy to prove.
$\delta(g) = \delta(1.g) = \delta(1) + 1\delta(g)$, so $\delta(1) = 0$, and hence (i); then

$$\delta(1) = \delta(g^{-1}g) = \delta(g^{-1}) + g^{-1}\delta(g)$$

to get (ii), and finally induction to get (iii). ■

The Fox derivatives are derivations taking values in the group ring as a left module over itself. They are defined for $G = F(X)$, the free group on a set X. (We usually write F for $F(X)$ in what follows.)

Definition: For each $x \in X$, let

$$\frac{\partial}{\partial x} : F \to \mathbb{Z}F$$

be defined by
(i) for $y \in X$,

$$\frac{\partial y}{\partial x} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } y \neq x; \end{cases}$$

(ii) for any words, $w_1, w_2 \in F$,

$$\frac{\partial}{\partial x}(w_1w_2) = \frac{\partial}{\partial x}w_1 + w_1\frac{\partial}{\partial x}w_2.$$

Of course, a routine proof shows that the derivation property in (ii) defines $\frac{\partial w}{\partial x}$ for any $w \in F$.

This derivation, $\frac{\partial}{\partial x}$, will be called the *Fox derivative with respect to the generator x.*

Example: Let $X = \{u, v\}$, with $r \equiv uvuv^{-1}u^{-1}v^{-1} \in F = F(u, v)$, then

$$\begin{aligned} \frac{\partial r}{\partial u} &= 1 + uv - uvuv^{-1}u^{-1}, \\ \frac{\partial r}{\partial v} &= u - uvuv^{-1} - uvuv^{-1}u^{-1}v^{-1}. \end{aligned}$$

This relation is the typical braid group relation, here in Br_3, and we will come back to these simple calculations later.

It is often useful to extend a derivation $\delta : G \to M$ to a linear map from $\mathbb{Z}G$ to M by the simple rule that $\delta(g+h) = \delta(g) + \delta(h)$.

We have

$$Der(F, \mathbb{Z}F) \cong F\!-\!Mod(IF, \mathbb{Z}F),$$

and that

$$IF \cong \mathbb{Z}F^{(X)},$$

with the isomorphism matching each generating $x-1$ with e_x, the basis element labelled by $x \in X$. (The universal derivation then sends x to e_x.)

For each given x, we thus obtain a morphism of F-modules:

$$d_x : \mathbb{Z}F^{(X)} \to \mathbb{Z}F$$

with

$$\begin{aligned} d_x(e_y) &= 1 \quad \text{if } y = x \\ d_x(e_y) &= 0 \quad \text{if } y \neq x, \end{aligned}$$

i.e., the 'projection onto the x^{th}-factor' or 'evaluation at $x \in X$' depending on the viewpoint taken of the elements of the free module, $\mathbb{Z}F^{(X)}$.

Suppose now that we have a group presentation, $\mathcal{P} = (X : R)$, of a group, G, then we have a short exact sequence of groups

$$1 \to N \overset{\varphi}{\to} F \overset{\gamma}{\to} G \to 1,$$

where $N = N(R)$, $F = F(X)$, *i.e.*, N is the normal closure of R in the free group F. We also have a free crossed module,

$$C \overset{\partial}{\to} F,$$

constructed from the presentation and hence, two short exact sequences of G-modules with $\kappa(\mathcal{P}) = Ker\,\partial$, the module of identities of $\mathcal{P}$,

$$0 \to \kappa(\mathcal{P}) \to C^{Ab} \to N^{Ab} \to 0,$$

and also

$$0 \to N^{Ab} \overset{\tilde{\varphi}}{\to} IF \otimes_F \mathbb{Z}G \to IG \to 0.$$

We note that the first of these is exact because N is a free group, (see Proposition 12, which will be proved shortly), further

$$C^{Ab} \cong \mathbb{Z}G^{(R)},$$

(the proof is left to you to manufacture from earlier results), and the map from this to N^{Ab} in the first sequence sends the generator e_r to $r[N, N]$.

We next revisit the derivation of the associated exact sequence (Proposition 10, page 77) in some detail to see what $\tilde{\varphi}$ does to $r[N, N]$. We have $\tilde{\varphi}(r[N, N]) = \partial_\gamma \varphi(r) = \partial_\gamma(r)$, considering r now as an element of F, and by Corollary 3, on identifying D_γ with $\mathbb{Z}G^{(X)}$ using the isomorphism between IF and $\mathbb{Z}F^{(X)}$, we can identify $\partial_\gamma(x) = e_x$. We are thus left to determine $\partial_\gamma(r)$ in terms of the $\partial_\gamma(x)$, *i.e.*, the e_x. The following lemma does the job for us.

Lemma 13 *Let $\delta : F \to M$ be a derivation and $w \in F$, then*

$$\delta w = \sum_{x \in X} \frac{\partial w}{\partial x} \delta x.$$

Proof: By induction on the length of w. ■

In particular we thus can calculate

$$\partial_\gamma(r) = \sum \frac{\partial r}{\partial x} e_x.$$

Tensoring with $\mathbb{Z}G$, we get

$$\tilde{\varphi}(r[N,N]) = \sum \frac{\partial r}{\partial x} e_x \otimes 1.$$

There is one final step to get this into a usable form:

From the quotient map $\gamma : F \to G$, we, of course, get an induced ring homomorphism, $\gamma : \mathbb{Z}F \to \mathbb{Z}G$, and hence we have elements $\gamma(\frac{\partial r}{\partial x}) \in \mathbb{Z}G$. Of course,

$$\frac{\partial r}{\partial x} e_x \otimes 1 = e_x \otimes \gamma(\frac{\partial r}{\partial x}),$$

so we have, on tidying up notation just a little:

Proposition 11 *The composite map*

$$\mathbb{Z}G^{(R)} \to N^{Ab} \to \mathbb{Z}G^{(X)}$$

sends e_r to $\sum \gamma(\frac{\partial r}{\partial x}) e_x$ and so has a matrix representation given by $J_{\mathscr{P}} = \left(\gamma(\frac{\partial r_i}{\partial x_j})\right)$. ■

Definition: The *Jacobian matrix* of a group presentation, $\mathscr{P} = (X : R)$ of a group G is

$$J_{\mathscr{P}} = \left(\gamma(\frac{\partial r_i}{\partial x_j})\right),$$

in the above notation.

The application of γ to the matrix of Fox derivatives simplifies expressions considerable in the matrix. The usual case of this is if a relator has the form rs^{-1}, then we get

$$\frac{\partial rs^{-1}}{\partial x} = \frac{\partial r}{\partial x} - rs^{-1}\frac{\partial s}{\partial x}$$

and if r or s is quite long this looks moderately horrible to work out! However applying γ to the answer, the term rs^{-1} in the second of the two terms becomes 1. We can actually think of this as replacing rs^{-1} by $r - s$ when working out the Jacobian matrix.

Example: Br_3 revisited. We have $r \equiv uvuv^{-1}u^{-1}v^{-1}$, which has the form $(uvu)(vuv)^{-1}$. This then gives

$$\gamma(\frac{\partial r}{\partial u}) = 1 + uv - v \quad \text{and} \quad \gamma(\frac{\partial r}{\partial v}) = u - 1 - vu,$$

abusing notation to ignore the difference between u, v in $F(u, v)$ and the generating u, v in Br_3.

Homological 2-syzygies: In general we obtain a truncated chain complex:

$$\mathbb{Z}G^{(R)} \xrightarrow{d_2} \mathbb{Z}G^{(X)} \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{d_0} \mathbb{Z} \to 0,$$

with d_2 given by the Jacobian matrix of the presentation, and d_1 sending generator e^1_x to $1 - x$, so $Im\, d_1$ is the augmentation ideal of $\mathbb{Z}G$.

Definition: A *homological 2-syzygy* is an element in $Ker\, d_2$.

A homological 2-syzygy is thus an element to be killed when building the third level of a resolution of G. What are the links between homotopical and homological syzygies? Brown and Huebschmann, [65], show they are isomorphic, as $Ker\, d_2$ is isomorphic to the module of identities. We will examine this result in more detail shortly.

Extended example: Homological Syzygies for the braid group presentations: The Artin braid group, Br_{n+1}, defined using $n + 1$ strands is given by

- generators: y_i, $i = 1, \ldots, n$;
- relations: $r_{ij} \equiv y_i y_j y_i^{-1} y_j^{-1}$ for $i + 1 < j$;
  $r_{ii+1} \equiv y_i y_{i+1} y_i y_{i+1}^{-1} y_i^{-1} y_{i+1}^{-1}$ for $1 \leq i < n$.

We will look at such groups only for small values of n.

By default, Br_2 has one generator and no relations, so is infinite cyclic.

The group Br_3: (We will simplify notation writing $u = y_1$, $v = y_2$.)

This then has presentation $\mathcal{P} = (u, v : r \equiv uvuv^{-1}u^{-1}v^{-1})$. It is also the 'trefoil group', *i.e.*, the fundamental group of the complement of a trefoil knot. If we construct $X(2) = K(\mathcal{P})$, this is already a $K(Br_3, 1)$ space, having a trivial π_2. There are no higher syzygies.

We have all the calculation for working with homological syzygies here. The key part of the complex is the Jacobian matrix as that determines d_2:

$$d_2 = \begin{pmatrix} 1 + uv - v & \quad u - 1 - vu \end{pmatrix}.$$

This has trivial kernel, but, in fact, that comes most easily from the identification with homotopical syzygies.

The group Br_4: simplifying notation as before, we have generators u, v, w and relations

$$\begin{aligned} r_u &\equiv vwvw^{-1}v^{-1}w^{-1}, \\ r_v &\equiv uwu^{-1}w^{-1}, \\ r_w &\equiv uvuv^{-1}u^{-1}v^{-1}. \end{aligned}$$

The 1-syzygies are made up of hexagons for r_u and r_w and a square for r_v. There is a fairly obvious way of fitting together squares and hexagons, namely as a permutohedron, and there is a labelling of such that gives a homotopical 2-syzygy.

The presentation yields a truncated chain complex with d_2

$$\mathbb{Z}G^{(r_u,r_v,r_w)} \xrightarrow{d_2} \mathbb{Z}G^{(u,v,w)}$$

with

$$d_2 = \begin{pmatrix} 0 & 1+vw-w & v-1-wv \\ 1-w & 0 & u-1 \\ 1+uv-v & u-1-vu & 0 \end{pmatrix}$$

and Loday, [187], has calculated that for the permutohedral 2-syzygy, s, one gets another term of the resolution, $\mathbb{Z}G^{(s)}$, and a $d_3 : \mathbb{Z}G^{(s)} \to \mathbb{Z}G^{(r_u,r_v,r_w)}$ given by

$$d_3 = \begin{pmatrix} 1+vu-u-wuv & v-vwu-1-uv-vuwv & 1+vw-w-uvw \end{pmatrix}.$$

For more on methods of working with these syzygies, have a look at Loday's paper, [187], and some of the references that you will find there.

3.4 Crossed complexes and chain complexes: II

(The source for the material and ideas in this section is once again [62].)

3.4.1 The reflection from Crs to chain complexes

It is now time to return to the construction of a left adjoint for Δ_G.

Theorem 3 *(Brown-Higgins, [62] in a slightly more general form.) The functor, Δ_G, has a left adjoint.*

Proof: We construct the left adjoint explicitly as follows:

Let $f. : (\mathsf{C}, \varphi) \to \Delta_G(M.)$ be a morphism in Crs_G, then we have the following commutative diagram

$$\begin{tikzcd} \ldots \arrow[r] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_1 \arrow[r, "\delta_1"] \arrow[d, "f_1"] & C_0 \arrow[r, "\varphi"] \arrow[d, "f_0"] & G \arrow[d, "Id_G"] \\ \ldots \arrow[r] & M_2 \arrow[r, "\delta_2"] & M_1 \arrow[r, "\delta_1"] & M_0 \rtimes G \arrow[r, "pr_G"] & G \end{tikzcd}$$

Since the right hand square commutes, f_0 is given by some formula

$$f_0(c) = (\partial(c), \varphi(c)),$$

where $\partial : C_0 \to M_0$ is a φ-derivation. Thus $\partial = \tilde{f}_0 \partial_\varphi$ for a unique G-module morphism, $\tilde{f}_0 : D_\varphi \to M_0$, and f_0 factors as

$$C_0 \overset{\bar{\varphi}}{\to} D_\varphi \rtimes G \overset{\tilde{f}_0 \rtimes G}{\to} M_0 \rtimes G,$$

where $\bar{\varphi}(c) = (\partial_\varphi(c), \varphi(c))$.

The map $\partial_\varphi \delta_1 : C_1 \to D_\varphi$ is a homomorphism since

$$\begin{aligned} \partial_\varphi \delta_1(c_1c_2) &= \partial_\varphi \partial_1(c_1) + \varphi\partial_1(c_1)\partial_\varphi\partial_1(c_2) \\ &= \partial_\varphi \partial_1(c_1) + \partial_\varphi\partial_1(c_2), \end{aligned}$$

$\varphi\partial_1$ being trivial (because (C,φ) is G-augmented). We thus obtain a map $d : C_1^{Ab} \to D_\varphi$ given by $d(c[C,C]) = \partial_\varphi\partial_1(c)$ for $c \in C_1$. As we observed earlier the Abelian group C_1^{Ab} has a natural $\mathbb{Z}[G]$-module structure making d a G-module morphism.

Similarly there is a unique G-module morphism,

$$\tilde{f}_1 : C_1^{Ab} \to M_1,$$

satisfying

$$\tilde{f}_1(c[C,C]) = f_1(c).$$

Since for $c \in C_1$,

$$(d_1\tilde{f}_1(c), 1) = f_0(\delta_1 c) = (\tilde{f}_0\partial_\varphi(\delta_1 c_1), 1),$$

we have that the diagram

$$\begin{tikzcd} C_1^{Ab} \arrow[r, "\tilde{f}_1"] \arrow[d, "d"'] & M_1 \arrow[d, "d_1"] \\ D_\varphi \arrow[r, "\tilde{f}_0"] & M_0 \end{tikzcd}$$

commutes.

We also note that since $\delta_2 : C_2 \to C_1$ maps into $Ker\ \delta_1$, the composite

$$C_2 \stackrel{\delta_2}{\to} C_1 \stackrel{\text{can}}{\to} C_1^{Ab} \stackrel{d}{\to} D_\varphi,$$

being given by $d(\delta_2(c)[C,C] = \partial_\varphi\delta_1\delta_2(c)$, is trivial and that $\tilde{f}_1\delta_2(c[C,C]) = f_1\delta_2(c) = d_2 f_2(c)$, thus we can define $\xi = \xi_G(\mathsf{C},\varphi)$ by

$$\begin{aligned} \xi_n &= C_n \text{ if } n \geq 2 \\ \xi_1 &= C_1^{Ab}, \\ \xi_0 &= D_\varphi, \end{aligned}$$

the differentials being as constructed. We note that as $Ker\ \varphi$ acts trivially on all C_n for $n \geq 2$, all the C_n have $\mathbb{Z}[G]$-module structures.

That ξ_G gives a functor

$$Crs \to Ch(G{-}Mod)$$

is now easy to check using the uniqueness clauses in the universal properties of D_φ and Abelianisation. Again uniqueness guarantees that the process "f goes to $\tilde{f}$" gives a natural isomorphism

$$Ch(G - Mod)(\xi_G(\mathsf{C},\varphi), \mathsf{M}) \cong Crs_G((\mathsf{C},\varphi), \Delta_G(\mathsf{M}))$$

as required. ■

It is relatively easy to extend the above natural isomorphism to handle morphisms of crossed complexes over different groups. For a detailed treatment one needs a discussion of the way that the change of groups functors work with crossed modules or crossed complexes, that is, if we have a morphism of groups $\theta : G \to H$ then we would expect to get functors between Crs_G and Crs_H induced by θ. These do exist and are very nicely behaved, but they will not be discussed here, see [226] for a full treatment in the more general context of profinite groups.

3.4.2 Crossed resolutions and chain resolutions

One of our motivations for introducing crossed complexes was that they enable us to model more of the sort of information encoded in a $K(G,1)$ than does the usual standard algebraic models, e.g. a chain complex such as the bar resolution. In particular, whilst the bar resolution is very good for cohomology with Abelian coefficients for non-Abelian cohomology the crossed version can allow us to push things further, but then comparison on the Abelian theory is very necessary! It is therefore of importance to see how this $K(G,1)$ information that we have encoded changes under the functor $\xi : Crs \to Ch(G{-}Mod)$.

We start with a crossed resolution determined in low dimensions by a presentation $\mathcal{P} = (X : R)$ of a group, G. Thus, in this case, $C_0 = F(X)$ with $\varphi : F(X) \to G$, the 'usual' epimorphism, and $C_1 \to C_0$ is $C \to F(X)$, the free crossed module on $R \to F(X)$. It is not too hard to show that $C_1^{Ab} \cong \mathbb{Z}[G]^{(R)}$, the free $\mathbb{Z}[G]$-module on R. (The proof is left as an exercise.) This maps down onto $N(R)^{Ab}$, the Abelianisation of the normal closure of R in $F(X)$ via a map

$$\partial_* : \mathbb{Z}[G]^{(R)} \to N(R)^{Ab},$$

given by $\partial_*(e_r) = r[N(R), N(R)]$, where e_r is the generator of $\mathbb{Z}[G]$ corresponding to $r \in R$.

There is also a short exact sequence

$$1 \to N(R) \xrightarrow{i} F(X) \xrightarrow{\varphi} G \to 1$$

and hence by Proposition 10, a short exact sequence

$$0 \to N(R)^{Ab} \xrightarrow{\tilde{i}} \mathbb{Z}[G] \otimes_F I(F) \xrightarrow{\tilde{\varphi}} I(G) \to 0$$

(where we have written $F = F(X)$).

By the Corollary to Proposition 8, we have

$$\mathbb{Z}[G] \otimes_F I(F) \cong \mathbb{Z}[G]^{(X)}.$$

The required map $C_1^{Ab} \to D_\varphi$ is the composite

$$\mathbb{Z}[G]^{(R)} \xrightarrow{\partial_*} N(R)^{Ab} \xrightarrow{\tilde{i}} \mathbb{Z}[G]^{(X)}.$$

We have given an explicit description of ∂_* above, so to complete the description of d, it remains to describe $\tilde{i}$, but $\tilde{i}$ satisfies $\tilde{i}\delta = \partial_\varphi i$, where $\delta : N(R) \to N(R)^{Ab}$, so $\tilde{i}(r[N(R), N(R)]) = d_\varphi(r)$. Thus if r is a relator, *i.e.*, if it is in the image of the subgroup generated by the elements of R, then $\partial(e_r)$ can be written as a finite sum of the form $\sum_x a_x e_x$ and the elements $a_x \in \mathbb{Z}[G]$ are the images of the Fox derivatives.

This operator can best be viewed as the Alexander matrix of a presentation of a group, further study of this operator depends on studying transformations between free modules over group rings, and we will not attempt to study those here.

The rest of the crossed resolution does not change and so, on replacing $I(G)$ by $\mathbb{Z}[G] \to \mathbb{Z}$, we obtain a free pseudocompact $\mathbb{Z}[G]$-resolution of the trivial module $\mathbb{Z}$,

$$\ldots \to \mathbb{Z}[G]^{(R)} \xrightarrow{d} \mathbb{Z}[G]^{(X)} \to \mathbb{Z}[G] \to \mathbb{Z}$$

built up from the presentation. This is the complex of chains on the universal cover, $\widetilde{K(G,1)}$, where $K(G,1)$ is constructed starting from a presentation $\mathcal{P}$.

3.4.3 Standard crossed resolutions and bar resolutions

We next turn to the special case of the standard crossed resolution of G discussed briefly earlier. Of course this is a special case of the previous one, but it pays to examine it in detail.

Clearly in $\xi = \xi(\mathsf{C}G, \varphi)$, we have:
$\xi_0 =$ the free $\mathbb{Z}[G]$-module on the underlying set of G, individual generators being written $[u]$, for $u \in G$;
$\xi_1 =$ the free $\mathbb{Z}[G]$ -module on $G \times G$, generators being written $[u, v]$;
$\xi_n = C_n G$, the free $\mathbb{Z}[G]$ -module on G^{n+1}, etc.

The map $d_2 : \xi_2 \to \xi_1$ induced from δ_2 is given by

$$d_2[u, v, w] = u[v, w] - [u, v] - [uv, w] + [u, vw],$$

and the map $d_1 : \xi_1 \to \xi_0$ by

$$\begin{aligned} d_1([u, v]) &= d_\varphi([uv]^{-1}[u][v]) \\ &= v^{-1}u^{-1}(-[uv] + [u] + u[v]), \end{aligned}$$

a unit times the usual bar resolution formula. Thus, as claimed earlier, the standard crossed resolution is the crossed analogue of the bar resolution.

3.4.4 The intersection $A \cap [C, C]$.

We next turn to a comparison of homological and homotopical syszygies. We have almost all the preliminary work already. The next ingredient is a result that will identify the intersection of the kernel of a crossed module, $A = Ker(C \stackrel{\partial}{\to} P)$ and the commutator subgroup of C.

The kernel of the homomorphism from A to C^{Ab} is, of course, $A \cap [C, C]$ and this need not be trivial. In fact, Brown and Huebschmann ([65], p.160) note that in examples of type $(G, Aut(G), \partial)$, the kernel of ∂ is, of course, the centre ZG of G and $ZG \cap [G, G]$ can be non-trivial, for instance, if G is dicyclic or dihedral.

We will adopt the same notation as previously with $N = \partial C$ etc.

Proposition 12 *If, in the exact sequence of groups*

$$1 \to A \to C \stackrel{p}{\to} N \to 1,$$

the epimorphism from C to N is split (the splitting need not respect P-action), then $A \cap [C, C]$ is trivial.

Proof: Given a splitting $s : N \to C$, (so ps is the identity on N), then the group C can be written as $A \rtimes s(N)$. The commutators in C, therefore, all lie in $s(N)$ since A is Abelian, but then, of course, $A \cap [C, C]$ cannot contain any non-trivial elements. ∎

We used this proposition earlier in the case where N is free. We are thus using the fact that subgroups of free groups are free, in that case. Of course, any epimorphism with codomain a free group is split.

Brown and Huebschmann, [65], p. 168, prove that for an group G with presentation $\mathcal{P}$, the module of identities for $\mathcal{P}$ is naturally isomorphic to the second homology group, $H_2(\tilde{K}(\mathcal{P}))$, of the universal cover of $K(\mathcal{P})$, the 2-complex of the presentation. We can approach this via the algebraic constructions we have.

Given a presentation $\mathcal{P} = \langle X : R\rangle$ of a group G, the algebraic analogue of $K(\mathcal{P})$, we have argued above, is the free crossed module $C(\mathcal{P}) \xrightarrow{d} F(X)$ and the chains on the universal cover of $K(\mathcal{P})$ will be given by ξ_G of this, *i.e.*, by the chain complex

$$\mathbb{Z}[G]^{(R)} \xrightarrow{d} \mathbb{Z}[G]^{(X)}.$$

In general there will be a short exact sequence

$$0 \to \kappa(\mathcal{P}) \cap [C(\mathcal{P}), C(\mathcal{P})] \to \kappa(\mathcal{P}) \to H_2(\xi(C(\mathcal{P})) \to 0.$$

This short exact sequence yields the Brown-Huebschmann result as $N(R)$ will a free group so the epimorphism onto $N(R)$ splits and we can use the above Proposition 12. We thus get

Proposition 13 *If $\mathcal{P} = \langle X : R\rangle$ is a free presentation of G, then there is an isomorphism*

$$\kappa \xrightarrow{\cong} H_2(\xi(C_{\mathcal{C}}(\mathcal{P})) = Ker(d : \mathbb{Z}[G]^R \to \mathbb{Z}[G]^X).$$

■

Note: Here we are using something that will not be true in all algebraic settings. A subgroup of a free group is always free, but the analogous statement for free algebras of other types is not true.

3.5 Simplicial groups and crossed complexes

3.5.1 From simplicial groups to crossed complexes

Given any simplicial group G, the formula,

$$\mathsf{C}(G)_{n+1} = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})},$$

in higher dimensions with, at its 'bottom end', the crossed module,

$$\frac{NG_1}{d_0(NG_2 \cap D_2)} \to NG_0$$

gives a crossed complex with ∂ induced from the boundary in the Moore complex. The detailed proof is too long to indicate here. It just checks the axioms, one by one.

We should have a glance at this formula from various viewpoints, some of which will be revisited later. Once again there is a clear link with the non-uniqueness of fillers for horns in a simplicial group if it is not a group T-complex. We have all those $(NG_n \cap D_n)$ terms involved!

Suppose that we had our simplicial group G and wanted to construct a quotient of it that was a group T-complex. We could do this in a silly way since the trivial simplicial group is clearly a group T-complex, but let us keep the quotient as large as possible. This problem is related to the question of whether the category of group T-complexes forms a reflective subcategory of *Simp.Grps*. The condition $NG \cap D = 1$ looks like some sort of 'equational specification'. Our question can thus really be posed as follows: Suppose we have a simplicial group morphism, $f : G \to H$, and H is a group T-complex. Remember that in group T-complexes, as against the non-algebraic ones, the thin structure is not an added bit of structure. The thin elements are determined by the degeneracies, so whether or not H is or is not a group T-complex is somehow its own affair, and nothing to do with any external factors! Does f factor universally through some 'group T-complexification' of G? Something like

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "proj"'] & & H \\ & G/T(G) \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

with $G/T(G)$ a group T-complex and $\hat{f}$ uniquely determined by the diagram.

One sensible way to look at such a question is to assume, provisionally, that such a factorisation exists and to see what $T(G)$ would have to be. In general, if $f : G \to H$ is any simplicial group morphism (with no restriction on H for the moment), then with a hopefully obvious notation,

$$f_n(NG_n \cap D(G)_n) \subseteq NH_n \cap D(H)_n,$$

since f sends degenerate elements to degenerate elements and preserves products! Back in our situation in which H is a group T-complex, then $f_n(NG_n \cap D(G)_n) = 1$, for the simple reason that the right hand side of that displayed formula is trivial by assumption. We thus have that if some such $T(G)$ exists, then we must have $NG_n \cap D(G)_n \subseteq T(G)_n$ and our first attempt might be to look at the possibility that they should be equal. This is wrong and for fairly trivial reasons. The subgroup $T(G)_n$ of G_n has to be normal if we are to form the quotient by it, and there is no reason why $NG_n \cap D(G)_n$ should be a normal subgroup in general.

We might then be tempted to take the normal subgroup generated by $NG_n \cap D(G)_n$, but that is 'defeatist' in this situation. We might hope to do detailed calculations with the subgroup and if it is specified as a normal closure, we will lose some of our ability to do that, at least without considerable more effort. (Let's be lazy and see if we can get around that difficulty.) If we look again, we find another thing that 'goes wrong' with any attempt to use $NG_n \cap D(G)_n$ as it is. This subgroup would be within NG_n, of course, and we want to induce a map from the Moore complex of G to that of $G/T(G)$. For that to work, we would need not only $NG_n \cap D(G)_n \subseteq T(G)_n$, but the image of $NG_n \cap D(G)_n$ under d_0 to be in $T(G)_{n-1}$. Going up a dimension, we thus need not only $NG_n \cap D(G)_n$, but $d_0(NG_{n+1} \cap D(G)_{n+1}) \subseteq T(G)_n$. We thus need the product subgroup $(NG_n \cap D(G)_n)d_0(NG_{n+1} \cap D(G)_{n+1})$ to be in $T(G)_n$. This looks a bit complicated. Do we need to go any further up the Moore complex? No, because d_0d_0 is trivial. We might thus try

$$T(G)_n = (NG_n \cap D(G)_n)d_0(NG_{n+1} \cap D(G)_{n+1}).$$

You might now think that this is a bit silly because we would still need this product subgroup to be normal in order to form the quotient ... , but it is! The lack of normality of our earlier attempt is absorbed by the image of the next level up. (That is pretty!)

Of course, there are very good reasons why this works. These involve what are sometimes called *Peiffer pairings*. We will see some of these later.

As a consequence of the above discussion, we more or less have:

Proposition 14 *If G is a group T-complex, then NG is a crossed complex.* ■

We certainly have a sketch of

Proposition 15 *The full subcategory of Simp.Grps determined by the group T-complexes is a reflective subcategory.* ■

Of course, the details of the proofs of both of these are left for you to write down. Nearly all of the reasoning for the second result is there for you, but some of the detailed calculations for the first are quite tricky.

The close link between group T-complexes and crossed complexes is evident from these results. You might guess that they form equivalent categories. They do. We will look at the way back from crossed complexes (of groups) to simplicial groups later on, but we need to get back to cohomology.

3.5.2 Simplicial resolutions, a bit of background

We need some such means of going from simplicial groups to crossed complexes so because we can also use simplicial resolutions to 'resolve' a group (and in many other situations). We first sketch in some historical background.

In the 1960s, the connection between simplicial groups and cohomology was examined in detail. The basic idea was that given the adjoint "free-forget" pair of functors between *Groups* and *Sets*, one could generate a free resolution of a group, G, using the resulting comonad (or cotriple) (cf. Mac Lane, [191]). This resolution was not, however, by a chain complex but by a free simplicial group, F, say. It was then shown (Barr and Beck, [20]) that given any G-module, M, and working in the category of groups over G, one could form the cosimplicial G-module, $Hom_{Gps/G}(F, M)$, and hence, by a dual form of the Dold-Kan theorem, a cochain complex $C(G, M)$, whose homotopy type, and hence whose homology, was independent of the choice of F. This homology was the usual Eilenberg-Mac Lane cohomology of G with coefficients in M, but with a shift in dimension (cf. Barr and Beck, [20]).

Other theories of cohomology were developed at about the same time by Grothendieck and Verdier, [9], André, [6, 7], and Quillen, [231, 232]. The first of these was designed for use with "sites", that is, categories together with a Grothendieck topology.

André and Quillen developed, independently, a method of defining cohomology using simplicial resolutions. Their work is best known in commutative algebra, but their methods work in greater generality. Unlike the theory of Barr and Beck (monadic cohomology), they only assume there is enough structure to construct free resolutions; a (co)monad is just one way of doing this. In particular, André, [6, 7], describes a step-by-step, almost combinatorial, process for constructing such resolutions. This ties in well with our earlier comments about using a presentation of a group to construct a crossed resolution and the important link with syzygies. André's method is the simplicial analogue of this.

We will assume for the moment that we have a simplicial resolution, F, of our group, G. Both André and Quillen then consider applying a derived module construction dimensionwise to F, obtaining a simplicial G-module. They then use "Dold-Kan" to give a chain complex of G-modules, which they call the "cotangent complex", denoted L_G or $LAb(G)$, of G (at least in the case of commutative algebras). The homotopy type of $LAb(G)$ does not depend on the choice of resolution and so is a useful invariant of G. We will need to look at this construction in more detail, but will consider a slightly more general situation to start with.

3.5.3 Free simplicial resolutions

Standard theory (cf. Duskin, [107]) shows that if F and F' are free simplicial resolutions of groups, G and H, say, and $f : G \to H$ is a morphism, then f can be lifted to $f' : F \to F'$. The method is the simplicial analogue of lifting a homomorphism of modules to a map of resolutions of those modules, which you should look at first as it is technically simpler. Any two such lifts are homotopic (by a simplicial homotopy).

Of course, f will also lift to a morphism of crossed complexes, $f : \mathsf{C}(F) \to \mathsf{C}(F')$, and any two such lifts will be *homotopic* as crossed complex morphisms. Thus whatever simplicial lift, $f' : F \to F'$, we choose, $\mathsf{C}(f')$ will be a lift in the "crossed" case, and although we do not know at this stage of our discussion of the theory if a homotopy between two simplicial lifts is transferred to a homotopy between the images under C, this does not matter as the *relation* of homotopy is preserved at least in this case of resolutions.

Any group has a free simplicial resolution. There is the obvious adjoint pair of functors

$$\begin{aligned} U &: \mathit{Groups} \to \mathit{Sets} \\ F &: \mathit{Sets} \to \mathit{Groups} \end{aligned}$$

Writing $\eta : Id \to UF$ and $\varepsilon : FU \to Id$ for the unit and counit of this adjunction (cf. Mac Lane, [191, 192]), we have a comonad (or cotriple) on *Groups*, the free group comonad, $(FU, \varepsilon, F\eta U)$. We write $L = FU$, $\delta = F\eta U$, so that

$$\varepsilon : L \to I$$

is the counit of the comonad whilst

$$\delta : L \to L^2$$

is the comultiplication. (For the reader who has not met monads or comonads before, (L, η, δ) behaves as if it was a monoid in the dual of the category of "endofunctors" on *Groups*, see Mac Lane, [192] Chapter VI. We will explore them briefly in section 13.3.4, starting on page 627.)

Now suppose G is a group and set $F(G)_i = L^{i+1}(G)$, so that $F(G)_0$ is the free group on the underlying set of G and so on. The counit (which is just the augmentation morphism from $FU(G)$ to G) gives, in each dimension, face morphisms

$$d_i = L^{n-i}\varepsilon L^i(G) : L^{n+1}(G) \to L^n(G),$$

for $i = 0, \dots, n$, whilst the comultiplication gives degeneracies

$$s_i : L^n(G) \to L^{n+1}(G)$$

$$s_i = L^{n-1-i}\delta L^i,$$

for $i = 0, \dots, n-1$, satisfying the simplicial identities.

Remark: Here we follow the conventions used by Duskin, in his Memoir, [107], so as to aid comparison and 'follow-up'. Later we will also need to look at similar resolutions where the labelling of the faces and degeneracies are reversed, so that the above formula for d_i would be, instead, for d_{n-i}.

This simplicial group, $F(G)$, satisfies $\pi_0(F(G)) \cong G$ (the isomorphism being induced by $\varepsilon(G) : F_0(G) \to G$) and $\pi_n(F(G))$ is trivial if $n \geq 1$. The reason for this is simple. If we apply U once more to $F(G)$, we get a simplicial set and the unit of the adjunction

$$\eta : 1 \to UF$$

allows one to define for each n

$$\eta U(FU)^n : UL^n \to UL^{n+1},$$

which gives a natural contraction of the augmented simplicial set, $UF(G) \to U(G)$, (cf. Duskin, [107]). We will look at this in detail in our later treatment of augmentations, etc. For the moment, it suffices to accept the fact that we do get a resolution, as we do not need to know the details of why this construction works, at least not yet.

If we denote the constant simplicial group on G by $K(G,0)$, the augmentation defines a simplical homomorphism

$$\overline{\varepsilon} : F(G) \to K(G,0)$$

satisfying $U\overline{\varepsilon}.inc = Id$, where $inc : UK(G,0) \to UF(G)$ is the 'inclusion' of simplicial sets given by η, and then these extra maps, $(UF)^n\eta U$, in fact, give a homotopy between $inc.U\overline{\varepsilon}$ and the identity map on $UF(G)$, *i.e.*, $\overline{\varepsilon}$ is a weak homotopy equivalence of simplicial groups. Thus $F(G)$ is a free simplicial resolution of G. It is called the *comonadic free simplicial resolution* of G.

This simplicial resolution has the advantage of being functorial, but the disadvantage of being very big. We turn next to a 'step-by-step' method of constructing a simplicial resolution using ideas pioneered by André, [7], although most of his work was directed more towards commutative algebras, cf. [6].

3.5.4 Step-by-Step Constructions

This section is a brief résumé of how to construct simplicial resolutions by hand rather than functorially. This allows a better interpretation of the generators in each level of the resolution. These are the simplicial analogues of higher syzygies. The work depends heavily on a variety of sources, mainly [6], [179] and [210]. André only treats commutative algebras in detail, but Keune [179] does discuss the general case quite clearly. The treatment here is adapted from the paper by Mutlu and Porter, [215].

Recall of notation: We first recall some notation and terminology, which will be used in the construction of a simplicial resolution. Let $[n]$ be the ordered set, $[n] = \{0 < 1 < \cdots < n\}$. Define the following maps: the injective monotone map $\delta_i^n : [n-1] \to [n]$ is given by

$$\delta_i^n(k) = \begin{cases} k & \text{if} \quad k < i, \\ k+1 & \text{if} \quad k \geq i, \end{cases}$$

for $0 \leq i \leq n \neq 0$. The increasing surjective monotone map $\alpha_i^n : [n+1] \to [n]$ is given by

$$\alpha_i^n(k) = \begin{cases} k & \text{if } k \leq i, \\ k-1 & \text{if } k > i, \end{cases}$$

for $0 \leq i \leq n$. We denote by $\{m, n\}$ the set of increasing surjective maps $[m] \to [n]$.

3.5.5 Killing Elements in Homotopy Groups

Let G be a simplicial group and let $k \geq 1$ be fixed. Suppose we are given a set, Ω, of elements: $\Omega = \{x_\lambda : \lambda \in \Lambda\}$, $x_\lambda \in \pi_{k-1}(\mathsf{G})$, then we can choose a corresponding set of elements $\theta_\lambda \in NG_{k-1}$ so that $x_\lambda = \theta_\lambda\ \partial_k(NG_k)$. (If $k = 1$, then as $NG_0 = G_0$, the condition that $\theta_\lambda \in NG_0$ is immediate.) We want to 'kill' the elements in Ω.

We form a new simplicial group F_n where

1) F_n is the free G_n-group, (*i.e.*, group with G_n-action)

$$F_n = \coprod_{\lambda,t} G_n\{y_{\lambda,t}\} \text{ with } \lambda \in \Lambda \text{ and } \ t \in \{n,k\},$$

where $G_n\{y\} = G_n * < y >$, the co-product of G_n and a free group generated by y.

2) For $0 \leq i \leq n$, the group homomorphism $s_i^n : F_n \to F_{n+1}$ is obtained from the homomorphism $s_i^n : G_n \to G_{n+1}$ with the relations

$$s_i^n(y_{\lambda,t}) = y_{\lambda,u} \quad \text{with} \quad u = t\alpha_i^n, \quad t : [n] \to [k].$$

3) For $0 \leq i \leq n \neq 0$, the group homomorphism $d_i^n : F_n \to F_{n-1}$ is obtained from $d_i^n : G_n \to G_{n-1}$ with the relations

$$d_i^n(y_{\lambda,t}) = \begin{cases} y_{\lambda,u} & \text{if the map} \quad u = t\delta_i^n \quad \text{is surjective,} \\ t'(\theta_\lambda) & \text{if} \qquad\quad u = \delta_k^k t', \\ 1 & \text{if} \qquad\quad u = \delta_j^k t' \quad \text{with} \ \ j \neq k, \end{cases}$$

by extending multiplicatively.

We sometimes denote the F, so constructed by $\mathsf{G}(\Omega)$.

Remark: In a 'step-by-step' construction of a simplicial resolution, (see below), there will thus be the following properties: i) $F_n = G_n$ for $n < k$, ii) $F_k =$ a free G_k-group over a set of non-degenerate indeterminates, all of whose faces are the identity except the k^{th}, and iii) F_n is a free G_n-group on some degenerate elements for $n > k$.

We have immediately the following result, as expected.

Proposition 16 *The inclusion of simplicial groups $G \hookrightarrow F$, where $F = G(\Omega)$, induces a homomorphism*

$$\pi_n(G) \longrightarrow \pi_n(F)$$

for each n, which for $n < k-1$ is an isomorphism,

$$\pi_n(G) \cong \pi_n(F)$$

and for $n = k-1$, is an epimorphism with kernel generated by elements of the form $\bar{\theta}_\lambda = \theta_\lambda \partial_k NG_k$, where $\Omega = \{x_\lambda : \lambda \in \Lambda\}$. ■

3.5.6 Constructing Simplicial Resolutions

The following result is essentially due to André, [6].

Theorem 4 *If G is a group, then it has a free simplicial resolution $\mathbb{F}$.*

Proof: The repetition of the above construction will give us the simplicial resolution of a group. Although 'well known', we sketch the construction so as to establish some notation and terminology.

Let G be a group. The zero step of the construction consists of a choice of a free group F and a surjection $g : F \to G$ which gives an isomorphism $F/Ker\, g \cong G$ as groups. Then we form the constant simplicial group, $F^{(0)}$, for which in every degree n, $F_n = F$ and $d_i^n = \mathrm{id} = s_j^n$ for all i, j. Thus $F^{(0)} = K(F, 0)$ and $\pi_0(F^{(0)}) = F$. Now choose a set, Ω^0, of normal generators of the closed normal subgroup $N = Ker\,(F \xrightarrow{g} G)$, and obtain the simplicial group in which $F_1^{(1)} = F(\Omega^0)$ and for $n > 1$, $F_n^{(1)}$ is a free F_n-group over the degenerate elements as above. This simplicial group will be denoted by $F^{(1)}$ and will be called the *1-skeleton of a simplicial resolution of the group, G.*

The subsequent steps depend on the choice of sets, Ω^0, $\Omega^1, \Omega^2, \ldots, \Omega^k, \ldots$. Let $F^{(k)}$ be the simplicial group constructed after k steps, that is, *the k-skeleton of the resolution*. The set Ω^k is formed by elements a of $F_k^{(k)}$ with $d_i^k(a) = 1$ for $0 \le i \le k$ and whose images $\bar{a}$ in $\pi_k(F^{(k)})$ generate that module over $F_k^{(k)}$ and $F^{(k+1)}$.

Finally we have inclusions of simplicial groups

$$F^{(0)} \subseteq F^{(1)} \subseteq \cdots \subseteq F^{(k-1)} \subseteq F^{(k)} \subseteq \cdots$$

and in passing to the inductive limit (colimit), we obtain an acyclic free simplicial group F with $F_n = F_n^{(k)}$ if $n \le k$. This F, or, more exactly, (F, g), is thus a simplicial resolution of the group G.

The proof of theorem is completed. ■

Remark: A variant of the 'step-by-step' construction gives: *if G is a simplicial group, then there exists a free simplicial group F and a continuous epimorphism $F \longrightarrow G$ which induces isomorphisms on all homotopy groups.* The details are omitted as they should be reasonably clear.

The key observation, which follows from the universal property of the construction, is a freeness statement:

Proposition 17 *Let $F^{(k)}$ be a k-skeleton of a simplicial resolution of G and $(\Omega^k, g^{(k)})$ k-dimension construction data for $F^{(k+1)}$. Suppose given a simplicial group morphism $\Theta : F^{(k)} \longrightarrow G$ such that $\Theta_*(g^{(k)}) = 0$, then Θ extends over $F^{(k+1)}$.*

This freeness statement does not contain a uniqueness clause. That can be achieved by choosing a lift for $\Theta_k g^{(k)}$ to NG_{k+1}, a lift that must exist since $\Theta_*(\pi_k(F^{(k)}))$ is trivial.

When handling combinatorially defined resolutions, rather than functorially defined ones, this proposition is as often as close to 'left adjointness' as is possible without entering the realm of homotopical algebra to an extent greater than is desirable for us here.

We have not talked here about the homotopy of simplicial group morphisms, and so will not discuss homotopy invariance of this construction for which one adapts the description given by André, [6], or Keune, [179]. Of course, the resolution one builds by any means would be homotopicallly equivalent to any other so, for cohomological purposes, it makes no difference how the resolution is built.

Of course, from any simplicial resolution, F, of G, you can get an augmented crossed complex $C(F)$ over G using the formula given earlier and this is a crossed resolution.

3.6 Cohomology and crossed extensions

3.6.1 Cochains

Consider a G-module, M, and a non-negative integer n. We can form the chain complex, $K(M, n)$, having M in dimension n and zeroes elsewhere. We can also form a crossed complex, $\mathsf{K}(M, n)$, that plays the role of the n^{th} Eilenberg-Mac Lane space of M in this setting. We may call it the n^{th} *Eilenberg-Mac Lane crossed complex* of M:

If $n = 0$, $\mathsf{K}(M, n)_0 = M \rtimes G$, $\mathsf{K}(M, n)_i = 0$, $i > 0$.

If $n \geq 1$, $\mathsf{K}(M, n)_0 = G$, $\mathsf{K}(M, n)_n = M$, $\mathsf{K}(M, n)_i = 0$, $i \neq 0$ or n.

One way to view cochains is as chain complex morphisms. Thus on looking at $Ch(\mathsf{B}G, K(M, n))$, one finds exactly $Z^{n+1}(G, M)$, the $(n+1)$-cocycles of the cochain complex $C(G, M)$. We can also view $Z^{n+1}(G, M)$ as $Crs_G(\mathsf{C}G, \mathsf{K}(M, n))$.

In the category of chain complexes, one has that a homotopy from $\mathsf{B}G$ to $K(M, n)$ between 0 and f, say, is merely a coboundary, so that $H^{n+1}(G, M) \cong [\mathsf{B}G, K(M, n)]$, adopting the usual homotopical notation for the group of homotopy classes of maps from the bar resolution $\mathsf{B}G$ to $K(M, n)$. This description has its analogue in the crossed complex case as we shall see.

3.6.2 Homotopies

Let C, C' be two crossed complexes with Q and Q' respectively as the cokernels of their bottom morphism. Suppose λ, $\mu : \mathsf{C} \to \mathsf{C}'$ are two morphisms inducing the same map $\varphi : Q \to Q'$.

A homotopy from λ to μ is a family, $h = \{h_k : k \geq 1\}$, of maps $h_k : C_k \to C'_{k+1}$ satisfying the following conditions:

H1) $h_0 : C_1 \to C'_2$ is a derivation along μ_0 (i.e. for $x, y \in C_0$,

$$h_0(xy) = h_0(x)({}^{\mu_0}h_0(y)),)$$

such that

$$\delta_1 h_0(x) = \lambda_0(x)\mu_0(x)^{-1}, \quad x \in C_0.$$

H2) $h_1 : C_1 \to C'_2$ is a C_0-homomorphism with C_0 acting on C'_2 via λ_0 (or via μ_0, it makes no difference) such that

$$\delta_2 h_1(x) = \mu_1(x)^{-1}(h_0\delta_1(x)^{-1}\lambda_1(x)) \text{ for } x \in C_1.$$

H3) for $k \geq 2$, h_k is a Q-homomorphism (with Q acting on the C'_k via the induced map $\varphi : Q \to Q'$) such that

$$\delta_{k+1}h_k + h_{k-1}\delta_k = \lambda_k - \mu_k.$$

We note that the condition that λ and μ induce the same map, $\varphi : Q \to Q'$, is, in fact, superfluous as this is implied by $H1$.

The properties of homotopies and the relation of homotopy are as one would expect. One finds $H^{n+1}(G, M) \cong [\mathsf{C}G, \mathsf{K}(M, n)]$. Given that in higher dimensions, this is the same set exactly as $[\mathsf{B}G, K(M, n)]$ means that there is not much to check and so the proof has been omitted.

3.6.3 Huebschmann's description of cohomology classes

The transition from this position to obtaining Huebschmann's descriptions of cohomology classes, [159], is now more or less formal. We will, therefore, only sketch the main points.

If G is a group, M is a G-module and $n \geq 1$, a *crossed n-fold extension* is an exact augmented crossed complex,

$$0 \to M \to C_n \to \ldots \to C_2 \to C_1 \to G \to 1.$$

The notion of similarity of such extensions is analogous to that of n-fold extensions in the Abelian Yoneda theory, (cf. Mac Lane, [191]), as is the definition of a Baer sum. We leave the details to you. This yields an Abelian group, $Opext^n(G, M)$, of similarity classes of crossed n-fold extensions of G by M.

Given a cohomology class in $H^{n+1}(G, M)$ realisable as a homotopy class of maps, $f : \mathsf{C}G \to \mathsf{K}(M, n)$, one uses f to form an induced crossed complex, much as in the Abelian Yoneda theory:

$$\begin{tikzcd} & J_n(G) \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\text{pushout}"] & C_n \arrow[r] \arrow[d] & \ldots \arrow[r] & C_1 \arrow[r] \arrow[d] & G \arrow[d, equal] \\ 0 \arrow[r] & M \arrow[r] & M_n \arrow[r] & \ldots \arrow[r] & M_1 \arrow[r] & G \end{tikzcd}$$

where $J_n(G)$ is $Ker(C_nG \to C_{n-1}G)$. (Thus J_nG is also $Im(C_{n+1}G \to C_nG)$ and as the map f satisfies $f\delta = 0$, it is zero on the subgroup $\delta(C_{n+2}G)$ (i.e. is constant on the cosets) and hence passes to $Im(C_{n+1}G \to C_nG)$ in a well defined way.) Arguments using lifting of maps and homotopies show that the assignment of this element of $Opext^n(G, M)$ to $cls(f) \in H^{n+1}(G, M)$ establishes an isomorphism between these groups.

3.6.4 Abstract Kernels.

The importance of having such a description of classes in $H^n(G, M)$ probably resides in low dimensions. To describe classes in $H^3(G, M)$, one has, as before, crossed 2-fold extensions

$$0 \to M \to C_2 \xrightarrow{\partial} C_1 \to G \to 1,$$

where ∂ is a crossed module. One has for any group G, a crossed 2-fold extension

$$0 \to Z(G) \to G \xrightarrow{\partial_G} Aut(G) \to Out(G) \to 1$$

where ∂_G sends $g \in G$ to the corresponding inner automorphism of G. An *abstract kernel* (in the sense of Eilenberg-Mac Lane, [121]) is a homomorphism $\psi : Q \to Out(G)$ and hence provides, by pulling back, a 2-fold extension of Q by the centre, $Z(G)$, of G.

3.7 2-types and cohomology

In classifying homotopy types and in obstruction theory, one frequently has invariants that are elements in cohomology groups of the form $H^m(X, \pi)$, where typically π is the n^{th} homotopy group of some space. When dealing with homotopy types, π will be a group, usually Abelian with a π_1-action, *i.e.*, we are exactly in the situation described earlier, except that X is a homotopy type not a group. Of course, provided that X is connected, we can replace X by a simplicial group, bringing us even nearer to the situation of this section. We shall work within the category of simplicial groups.

3.7.1 2-types

A morphism

$$f : G \to H$$

of simplicial groups is called a *2-equivalence* if it induces isomorphisms

$$\pi_0(f) : \pi_0(G) \to \pi_0(H,)$$

and

$$\pi_1(f) : \pi_1(G) \to \pi_1(H).$$

We can form a quotient category, $Ho_2(Simp.Grps)$, of $Simp.Grps$ by formally inverting the 2-equivalences, then we say two simplicial groups, G and H, have the same *2-type*, (or, more exactly, *homotopy 2-type*), if they are isomorphic in $Ho_2(Simp.Grps)$.

This is, of course, just a special case of the general notion of n-type in which "n-equivalences" are inverted, thus forming the quotient category $Ho_n(Simp.Grps)$.

We recall the following from earlier:

Definition: An *n-equivalence* is a morphism, f, of simplicial groups (or groupoids) inducing isomorphisms, $\pi_i(f)$, for $i = 0, 1, \ldots, n-1$.

Definition: Two simplicial groups, G and H, have the same *n-type* (or, more exactly, *homotopy n-type* if they are isomorphic in $Ho_n(Simp.Grps)$.

Sometimes it is convenient to say that a simplicial group, G, *is* an n-type. This is taken to mean that it represents an n-equivalence class *and* has zero homotopy groups above dimension $n-1$.

3.7.2 Example: 1-types

Before examining 2-types in detail, it will pay to think about 1-types. A morphism f as above is a 1-equivalence if it induces an isomorphism on π_0, *i.e.*, $\pi_0(f)$ is an isomorphism. Given any group G, there is a simplicial group, $K(G,0)$ consisting of G in each dimension with face and degeneracy maps all being identities. Given a simplicial group, H, having $G \cong \pi_0(H)$, the natural quotient map

$$H_0 \to \pi_0(H) \cong G,$$

extends to a natural 1-equivalence between H and $K(\pi_0(H), 0)$.

It is fairly routine to check that

$$\pi_0 : Simp.Grps \to Grps$$

has $K(-,0)$ as an adjoint and that, as the unit is a natural 1-equivalence, and the counit an isomorphism, this adjoint pair induces an equivalence between the category $Ho_1(Simp.Grps)$ of 1-types and the category, $Grps$, of groups. In other words,

groups are algebraic models for 1-types.

3.7.3 Algebraic models for n-types?

So much for 1-types. Can one provide algebraic models for 2-types or, in general, n-types? We touched on this earlier. The criteria that any such "models" might satisfy are debatable. Perhaps ideally, or even unrealistically, there should be an isomorphism class of algebraic "gadgets" for each 2-type. An alternative weaker solution is to ask that a notion of equivalence between the models is possible, and that only equivalence classes, not isomorphism classes, correspond to 2-types, but, in addition, the notion of equivalence is algebraically defined. It is this weaker possibility that corresponds to our aim here.

3.7.4 Algebraic models for 2-types.

If G is a simplicial group, then we can form a crossed module

$$\partial : \frac{NG_1}{d_0(NG_2)} \to G_0,$$

where the action of G_0 is via the degeneracy, $s_0 : G_0 \to G_1$, and ∂ is induced by d_0. (As before we will denote this crossed module by $M(G,1)$.) The kernel of ∂ is

$$\frac{Ker\, d_0 \cap Ker\, d_1}{d_0(NG_2)} \cong \pi_1(G),$$

whilst its cokernel is

$$\frac{G_0}{d_0(NG_1)} \cong \pi_0(G),$$

and so we have a crossed 2-fold extension

$$0 \to \pi_1(G) \to \frac{NG_1}{d_0(NG_2)} \to G_0 \to \pi_0(G) \to 1$$

and hence a cohomology class $k(G) \in H^3(\pi_0(G), \pi_1(G))$.

Suppose now that $f : G \to H$ is a morphism of simplicial groups, then one obtains a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(G) \arrow[r] \arrow[d, "\pi_1(f)"'] & \frac{NG_1}{d_0(NG_2)} \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d, "f_0"] & \pi_0(G) \arrow[r] \arrow[d, "\pi_0(f)"] & 1 \\
0 \arrow[r] & \pi_1(H) \arrow[r] & \frac{NH_1}{d_0(NH_2)} \arrow[r] & H_0 \arrow[r] & \pi_0(H) \arrow[r] & 1
\end{tikzcd}$$

If, therefore, f is a 2-equivalence, $\pi_0(f)$ and $\pi_1(f)$ will be isomorphisms and the diagram shows that, modulo these isomorphisms, $k(G)$ and $k(H)$ are the same cohomology class, i.e. the 2-type of G determines π_0, π_1 and this cohomology class, k in $H^3(\pi_0, \pi_1)$.

Conversely, suppose we are given a group π, a π-module, M, and a cohomology class $k \in H^3(\pi, M)$, then we can realise k by a 2-fold extension

$$0 \to M \to C \xrightarrow{\partial} G \to \pi \to 1$$

as above.

The crossed module, $\mathsf{C} = (C, G, \partial)$, determines a simplicial group $K(\mathsf{C})$ as follows:

Suppose $\mathsf{C} = (C, P, \partial)$ is any crossed module, we construct a simplicial group, $K(\mathsf{C})$, by

$$K(\mathsf{C})_0 = P, \qquad K(\mathsf{C})_1 = C \rtimes P,$$

$$s_0(p) = (1, p),\ d_0^1(c, p) = \partial c.p,\ d_1^1(c, p) = p.$$

Assuming $K(\mathsf{C})_n$ is defined and that it acts on C via the unique composed face map to $K(\mathsf{C})_0 = P$ followed by the given action of P on C, we set

$$\begin{array}{rcl} K(\mathsf{C})_{n+1} & = & C \rtimes K(\mathsf{C})_n; \\ d_0^{n+1}(c_{n+1}, \ldots, c_1, p) & = & (c_{n+1}, ..., c_2, \partial c_1.p); \\ d_i^{n+1}(c_{n+1}, \ldots, c_{i+1}, c_i, \ldots, c_1, p) & = & (c_{n+1}, \ldots, c_{i+1}c_i, \ldots c_1, p) \\ & & \qquad\qquad\text{for } 0 < i < n+1; \\ d_{n+1}^{n+1}(c_{n+1}, \ldots, c_1, p) & = & (c_n, \ldots, c_1, p); \\ s_i^n(c_n, \ldots, c_1, p) & = & (c_n, \ldots, 1, \ldots, c_1, p), \end{array}$$

where the 1 is placed in the i^{th} position.

Clearly $Ker\, d_1^1 = \{(c, p) : p = 1\} \cong C$, whilst $Ker\, d_1^2 \cap Ker\, d_2^2 = \{(c_2, c_1, p) : (c_1, p) = (1, 1) \text{ and } (c_2c_1, p) = (1, 1)\} \cong \{1\}$, hence the "top term" of $M(K(\mathsf{C}), 1)$ is isomorphic to C itself, whilst $K(\mathsf{C})_0$ is P itself. The boundary map ∂ in this interpretation is the original ∂, since it maps $(c, 1)$ to $d_0(c)$, *i.e.*, we have

Lemma 14 *There is a natural isomorphism*

$$\mathsf{C} \cong M(K(\mathsf{C}), 1).$$

■

This construction is the internal nerve of the corresponding internal category in $Grps$, as we noted earlier. All the ideas that go into defining the nerve of a category adapt to handling internal categories, and they produce simplicial objects in the corresponding ambient category. As we have a simplicial group $K(\mathsf{C})$, we might check if it is a group T-complex, but this is more or less immediate as $NK(\mathsf{C})_n = 1$ for $n \geq 2$, whilst $NK(\mathsf{C})_1$ is $\{(c, p) : p = 1\}$ and $s_0(K(\mathsf{C})_0 = \{(c, p) : c = 1\}$.

Suppose now that we had chosen an equivalent 2-fold extension

$$0 \to M \to C' \stackrel{d'}{\to} G' \to \pi \to 1$$

The equivalence guarantees that there is a zig-zag of maps of 2-fold extensions joining it to that considered earlier. We need only look at the case of a direct basic equivalence:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & C \arrow[r, "\partial"] \arrow[d] & G \arrow[r] \arrow[d] & \pi \arrow[r] \arrow[d, "="] & 1 \\ 0 \arrow[r] & M \arrow[r] & C' \arrow[r, "\partial'"] & G' \arrow[r] & \pi \arrow[r] & 1 \end{tikzcd}$$

giving a map of crossed modules, $\varphi : \mathsf{C} \to \mathsf{C}'$, where $\mathsf{C}' = (C', G', \partial')$. This induces a morphism of simplicial groups,

$$K(\varphi) : K(\mathsf{C}) \to K(\mathsf{C}'),$$

that is, of course, a 2-equivalence. If there is a longer zig-zag between C and C', then the intermediate crossed modules give intermediate simplicial groups and a zig-zag of 2-equivalences so that $K(\mathsf{C})$ and $K(\mathsf{C}')$ are isomorphic in $Ho_2(Simp.Grps)$, i.e. they have the same 2-type. This argument can, of course, be reversed.

If G and H have the same 2-type, they are isomorphic within the category $Ho_2(Simp.Grps)$, so they are linked in $Simp.Grps$ by a zig-zag of 2-equivalences, hence the corresponding cohomology classes in $H^3(\pi_0(G), \pi_1(G))$ are the same up to identification of $H^3(\pi_0(G), \pi_1(G))$ and $H^3(\pi_0(H), \pi_1(H))$. This proves the simplicial group analogue of the result of Mac Lane and Whitehead, [195], that we mentioned earlier, giving an algebraic model for 2-types of connected CW-complexes.

Theorem 5 *(Mac Lane and Whitehead, [195]) 2-types are classified by a group π_0, a π_0-module, π_1 and a class in $H^3(\pi_0, \pi_1)$.* ∎

We have handled this in such a way so as to derive an equivalence of categories:

Proposition 18 *There is an equivalence of categories,*

$$Ho_2(Simp.Grps) \cong Ho(CMod),$$

where $Ho(CMod)$ is formed from $CMod$ by formally inverting those maps of crossed modules that induce isomorphisms on both the kernels and the cokernels. ∎

3.8 Re-examining group cohomology with Abelian coefficients

3.8.1 Interpreting group cohomology

We have had

- A definition of group cohomology via the bar resolution: for a group G and a G-module, M:

$$H^n(G, M) = H^n(C(G, M))$$

  together with an identification of $C(G, M)$ with maps from the classifying space / nerve, BG, of G to M, up to shifts in dimension;

- Interpretations

$$\begin{aligned}
H^0(G, M) &\cong M^G, \text{ the module of invariants}\\
H^1(G, M) &\cong Der(G, M)/Pder(G, M)\\
&- \quad \text{by inspection, where } Pder(G, M) \text{ is the submodule of}\\
&\qquad\qquad \text{principal derivations;}\\
H^2(G, M) &\cong Opext(G, M), \text{ i.e. classes of extensions}\\
&\qquad\qquad 0 \to M \to H \to G \to 1
\end{aligned}$$

  and we also have

$$\begin{aligned}
H^n(G, M) &\cong Opext^n(G, M), n \geq 2, \text{ via crossed resolutions}\\
&\cong [\mathsf{C}(G), \mathsf{K}(M, n)]
\end{aligned}$$

Another interpretation, which will be looked at shortly is as $Ext^n(\mathbb{Z}, M)$, where $\mathbb{Z}$ is given the trivial G-module structure. This leads to

$$H^n(G, M) \cong Ext^{n-1}(I(G), M),$$

via the long exact sequence coming from

$$0 \to I(G) \to \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

3.8.2 The Ext long exact sequences

There are several different ways of examining the long exact sequence that we need. We will use fairly elementary methods rather than more 'homologically intensive' one. These latter ones are very elegant and very powerful, but do need a certain amount of development before being used. The more elementary ones have, though, a hidden advantage. The intuitions that they exploit are often related to ones that extend, at least partially, to the non-Abelian case and also to the geometric situations that will be studied later in the notes.

The idea is to explore what happens to an exact sequence of modules

$$\mathcal{E}: \quad 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

over some given ring (we need it for G-modules so there the ring is $\mathbb{Z}[G]$, the group ring of G), when we apply the functor $Hom(-, M)$ for M another module. Of course one gets a sequence

$$Hom(\mathcal{E}, M) : 0 \to Hom(C, M) \xrightarrow{\beta^*} Hom(B, M) \xrightarrow{\alpha^*} Hom(A, M)$$

and it is easy to check that this is exact, but there is no reason why α^* should be onto since a morphism $f : A \to M$ may or may not extend to some g defined over the bigger module B. For instance, if $M = A$, and f is the identity morphism, then f extends if and only if the sequence splits (so $B \cong A \oplus C$). We examine this more closely.

We have

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \\ & M & & & \end{tikzcd}$$

and can form a new diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "\overline{f}"] & C \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & M \arrow[r, "\overline{\alpha}"] & N \arrow[r, "\overline{\beta}"] & C \arrow[r] & 0 \end{tikzcd}$$

where the left hand square is a pushout. You should check that you see why there is an induced morphism $\overline{\beta} : N \to C$ 'emphusing the universal property of pushouts. (This is important as sometimes one wants this sort of construction, or argument, for sheaves of modules and there working with elements causes some slight difficulties.) The existence of this map is guaranteed by the universal property and does not depend on a particular construction of N. Of course this means that the bottom line is defined only up to isomorphism although we can give a very natural explicit model for N, namely it can be represented as the quotient of $B \oplus M$ by the submodule L of elements of the form $(\alpha(a), -f(a))$ for $a \in A$. Then we have $\overline{\beta}(b, m) = \beta(b)$. (Check it is well defined.) It is also useful to have the corresponding formulae for $\overline{\alpha}(m) = (0, m) + L$ and for $\overline{f}(b) = (b, 0) + L$. This gives an extension of modules

$$f^*(\mathcal{E}): \quad 0 \to M \xrightarrow{\overline{\alpha}} N \xrightarrow{\overline{\beta}} C \to 0.$$

If f extends over B to give g, so $g\alpha = f$, then we have a morphism $g' : N \to M$ given by $g'((m, b) + L) = m + g(b)$. (Check that g' is well defined.)

Lemma 15 *f extends over B if and only if $f^*(\mathcal{E})$ is a split extension.*

Proof: We have done the 'only if'. If $f^*(\mathcal{E})$ is split, there is a projection $g' : N \to M$ such that $g'\overline{\alpha}(m) = m$ for all m. Define $g = g'\overline{f}$ to get the extension. ■

We thus get a map

$$Hom(A, M) \xrightarrow{\delta} Ext^1(C, M)$$
$$\delta(f) = [f^*(\mathcal{E})]$$

which extends the exact sequence one step to the right.

Here it is convenient to define $Ext^1(C, M)$ to be the set (actually Abelian group) of extensions of form

$$0 \to M \to ? \to C \to 0$$

modulo equivalence (isomorphism of middle terms with the ends fixed). The Abelian group structure is given by Baer sum (see entry in Wikipedia, or many standard texts on homological algebra).

Important aside: 'Recall' the '*snake lemma*: given a commutative diagram of modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\mu"'] & N \arrow[r] \arrow[d, "\nu"'] & P \arrow[r] \arrow[d, "\psi"'] & 0 \\
0 \arrow[r] & M' \arrow[r] & N' \arrow[r] & P' \arrow[r] & 0
\end{tikzcd}$$

there is an exact sequence

$$0 \to Ker\,\mu \to Ker\,\nu \to Ker\,\psi \xrightarrow{\delta} Coker\,\mu \to Coker\,\nu \to Coker\,\psi \to 0$$

This has as a corollary that if μ and ψ are isomorphisms then so is ν. (Do check that you can construct δ and prove exactness, i.e. using a simple diagram chase.)

Back to extensions: It is fairly easy to show that $Hom(\mathcal{E}, M)$ extends even further to 6 terms with

$$\ldots \xrightarrow{\beta^*} Ext^1(B, M) \xrightarrow{\alpha^*} Ext^1(A, M)$$

Here is how α^* is constructed. Suppose $\mathcal{E}_1 : 0 \to M \to N \to B \to 0$ gives an element of $Ext^1(B, M)$, then we can form a diagram

$$\begin{tikzcd}
\alpha^*(\mathcal{E}_1): & 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & \alpha^{-1}(N) \arrow[r, "p'"] \arrow[d, "\alpha'"'] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\
\mathcal{E}_1: & 0 \arrow[r] & M \arrow[r] & N \arrow[r, "p"'] & B \arrow[r] & 0
\end{tikzcd}$$

by restricting $\mathcal{E}_1$ along α using a pull back in the right hand square. We can give $\alpha^{-1}(N)$ explicitly in the form that the usual construction of pullbacks in categories of modules gives it to us

$$\alpha^{-1}(N) \cong \{(a,n) \mid \alpha(a) = p(n)\}$$

and p' and α' are projections. The construction of β^* is done similarly using pullback along β. It is then easy to check that the obvious extension to $Hom(\mathcal{E}, M)$, mentioned above, is exact, but that there is again no reason why α^* should be onto. (Of course, knowledge of the purely homological way of getting these exact sequence will suggest that there is an $Ext^2(C, M)$ term to come.)

We examine an obstruction to it being so. Suppose given $\mathcal{E}' : 0 \to M \to N_1 \xrightarrow{p'} A \to 0$, giving us an element of $Ext'(A, M)$. If α^* were onto, we would need a $\mathcal{E}_1 : 0 \to M \to N \to B \to 0$ such that $\alpha^{-1}(N) \cong N_1$ leaving M fixed and relating to α as above by a pullback. We can splice $\mathcal{E}'$ and $\mathcal{E}_1$ together to get a suitable looking diagram

$$\begin{tikzcd}[column sep=small] \mathcal{E}' * \mathcal{E}_1 : & 0 \arrow[r] & M \arrow[r] & N' \arrow[rr] \arrow[dr, "p'"'] & & B \arrow[r] & C \arrow[r] & 0 \\ & & & & A \arrow[ur, "\alpha"'] & & & \end{tikzcd}$$

and the row is exact. If we change $\mathcal{E}'$ by an isomorphism than clearly this spliced sequence would react accordingly. If you check up, as suggested, on the Baer sum structure if $Ext^1(A, M)$ and $Ext^2(C, M)$ then you can again check that the above splicing construction yields a homomorphism from the first group to the second. Moreover there is no reason not to extend the splicing construction to a pairing operation on the whole graded family of Ext-groups. This is given in detail in quite a few of the standard books on Homological Algebra, so will not be gone into here.

Two facts we do need to have available are about the structure of $Ext^2(C, M)$. Let $\mathcal{E}xt^2(C, M)$ be the category of 4-term exact sequences

$$0 \to M \to N \to P \to C \to 0$$

and morphisms which are commuting diagrams

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 , \\ 0 \arrow[r] & M \arrow[r] & N' \arrow[r] & P' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

then $Ext^2(C, M)$ is the set of connected components of this category. The important thing to note is that the morphisms are not isomorphisms in general, so two 4-term sequences give the same element in $Ext^2(C, M)$ if they are linked by a zig-zag of intermediate terms of this form. The second fact is that the zero for the Baer sum addition is the class of the 4-term extension

$$0 \to M \to M \xrightarrow{0} C \to C \to 0$$

with 'equals' on the unmarked maps.

Suppose now that the top row in

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & N_1 \arrow[r, "\overline{p}"] \arrow[d, "\alpha'"'] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & M \arrow[r] & N \arrow[r, "p"'] & B \arrow[r] & 0 \end{tikzcd}$$

is obtained by restriction along α from the bottom row. We now form the spliced sequence

$$0 \to M \to N_1 \stackrel{\alpha\overline{p}}{\to} B \to C \to 0.$$

We would hope that this 4-term sequence was trivial, i.e. equivalence to the zero one. We clearly must use the given element in $Ext^1(B, M)$ in a constructive way in the proof that it is trivial, so we form the pushout of $\alpha\overline{p}$ along α' getting us a diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & N_1 \arrow[r, "\alpha\overline{p}"] \arrow[d, "\alpha'"'] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 , \\
0 \arrow[r] & M \arrow[r] & N \arrow[r] & B' \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

with the middle square a pushout. It is now almost immediate that the morphism from B to B' is split, since we can form a commutative square

$$\begin{tikzcd}
N_1 \arrow[r, "\alpha\overline{p}"] \arrow[d, "\alpha'"'] & B \arrow[d, "="] \\
N \arrow[r, "p"'] & B
\end{tikzcd}$$

giving us the required splitting from B' to B. It is now a simple use of the snake lemma, to show that the complementary summand of B in B' is isomorphic to C. We thus have that the bottom row of the diagram above is of the form

$$0 \to M \to N \to B \oplus C \to C.$$

This looks hopeful but to finish off the argument we just produce the morphism:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & M \arrow[r, "0"] \arrow[d] & C \arrow[r] \arrow[d, "incl_2"] & C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & M \arrow[r] & N \arrow[r, "incl_1 p"'] & B \oplus C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

and we have a sequence of maps joining our spliced sequence to the trivial one. (A similar argument goes through in higher dimensions.) Now you should try to prove that if a spliced sequence is linked to a trivial one then it does come from an induced one. That is quite tricky, so look it up in a standard text. An alternative approach is to use the homological algebra to get the trivialising element (coboundary or homotopy, depending on your viewpoint) and then to construct the extension from that. Another thing to do is to consider how the Ext-groups, $Ext^k(A, M)$, vary in M rather than with A. This will be left to you.

3.8.3 From Ext to group cohomology

If we look briefly at the classical homological algebraic method of defining $Ext^K(A, M)$, we would take a projective resolution $\mathsf{P}.$ of A, apply the functor $Hom(-, M)$, to get a cochain complex $Hom(\mathsf{P}., M)$, then take its (co)homology, with $H^n(Hom(\mathsf{P}., M))$ being isomorphic to $Ext^n(A, M)$, or, if you prefer, $Ext^n(A, M)$ being defined to be $H^n(Hom(\mathsf{P}., M))$. This method can be studied in most books on homological algebra (we cite for instance, Mac Lane, [191], Hilton and Stammbach, [152] and Weibel, [274]), so is easily accessible to the reader - and we will not devote much space to it here as a result. We will however summarise some points, notation, definitions of terms etc., some of which you probably know.

First the notion of projective module:

Definition: A module P is *projective* if, given any epimorphism, $f : B \to C$, the induced map $Hom(P, f) : Hom(P, B) \to Hom(P, C)$ is onto. In other words any map from P to C can be lifted to one from P to B.

Any free module is projective.

Of the properties of projectives that we will use, we will note that $Ext^n(P, M) = 0$ for P projective and for any M. To see this recall that any n-fold extension of P by M will end with an epimorphism to P, but such things split as their codomain is projective. It is now relatively easy to use this splitting to show the extension is equivalent to the trivial one.

A resolution of a module A is an augmented chain complex

$$\mathsf{P}. : \ldots \to P_1 \to P_0 \to M$$

which is exact, *i.e.*, it has zero homology in all dimensions. This means that the augmentation induces an isomorphism between $P_0/\partial P_1$ and M. The resolution is projective if each P_n is a projective module.

If $\mathsf{P}.$ and $\mathsf{Q}.$ are both projective resolutions of A, then the cochain complexes $Hom(\mathsf{P}., M)$ and $Hom(\mathsf{Q}., M)$ always have the same homology. (Once again this is standard material from homological algebra so is left to the reader to find in the usual sources.)

An example of a projective resolution is given by the bar resolution, $\mathsf{B}G.$, and the construction $C^n(G, M)$ in the first chaper is exactly $Hom(\mathsf{B}G., M)$. This reolution ends with $BG_0 = \mathbb{Z}[G]$ and the resolution resolves the Abelian group $\mathbb{Z}$ with trivial G-module structure. (This can be seen from our discussion of homological syzygies where we had

$$\mathbb{Z}[G]^{(R)} \to \mathbb{Z}[G]^{(X)} \to \mathbb{Z}[G] \to \mathbb{Z}.$$

In fact we have

$$H^n(G, M) \cong Ext^n(\mathbb{Z}, M)$$

by the fact that $\mathsf{B}G.$ is a projective resolution of $\mathbb{Z}$ and then we can get more information using the short exact sequence

$$0 \to I(G) \to \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

As $\mathbb{Z}[G]$ is a free G-module, it is projective and the long exact sequence for $Ext(-, M)$ thus has every third term trivial (at least for $n > 0$), so

$$Ext^n(\mathbb{Z}, M) \cong Ext^{n-1}(I(G), M)$$

giving another useful interpretation of $H^n(G, M)$.

3.8.4 Exact sequences in cohomology

Of course, the identification of $H^n(G, M)$ as $Ext^n(\mathbb{Z}, M)$ means that, if

$$0 \to L \to M \to N \to 0$$

is an exact sequence of G-modules, we will get a long exact sequence in $H^n(G, -)$, just by looking at the long exact sequence for $Ext^n(\mathbb{Z}, -)$.

What is more interesting - but much more difficult - is to study the way that $H^n(G, M)$ varies as G changes. For a start it is not completely clear what this means! If we change the group in a short exact sequence,t

$$1 \to G \to H \to K \to 1$$

say, then what type of modules should be used fro the 'coefficients', that is to say a G-modules or one over H or K. This problem is, of course, related to the change of groups along an arbitrary homomorphism, so we will look at an group homomorphism $\varphi : G \to H$, with no assumptions as to monomorphism, or normal inclusion, at least to start with.

Suppose given such a φ, then the 'restriction functor' is

$$\varphi^* : H{-}Mod \to G{-}Mod,$$

where, if N is in $H{-}Mod$, $\varphi^*(N)$ has the same underlying Abelian group structure as N, but is a G-module via the action, $g.n := \varphi(g).n$. We have already used that φ^* has a left adjoint φ_* given by $\varphi_*(M) = \mathbb{Z}H \otimes_{\mathbb{Z}G} M$. Now we also need a right adjoint for φ^*.

To construct such an adjoint, we use the old device of assuming that it exists, studying it and then extracting a construction from that study. We have M in $G{-}Mod$ and N in $H{-}Mod$, and we assume a natural isomorphism

$$G{-}Mod(\varphi^*(N), M) \cong H{-}Mod(N, \varphi_\sharp(M)).$$

If we take $N = \mathbb{Z}H$, then, as $H{-}Mod(\mathbb{Z}H, \varphi_\sharp(M)) \cong \varphi_\sharp(M)$, we have a construction of $\varphi_\sharp(M)$, at least as an Abelian group. In fact this gives

$$\varphi_\sharp(M) \cong G{-}Mod(\varphi^*(\mathbb{Z}H), M)$$

and as $\mathbb{Z}H$ is also a right G-module, via $h.g := h.\varphi(g)$, we have a left G-module structure of $\varphi_\sharp(M)$ as expected. In fact, this is immediate from the naturality of the adjunction isomorphism using the left hand position of $G{-}Mod(\varphi^*(\mathbb{Z}H), M)$, as for fixed M, the functor converts the right G-action of $\mathbb{Z}$ to a left one on $\varphi_\sharp(M)$. This allows us to get an explicit elementwise formula for this action as follows: let $m^* : \mathbb{Z}H \to M$ be a left G-module mrphsim This can be specified by what it does to the natural basis of $\mathbb{Z}H$ (as Abelian group), and so is often written $m^* : H \to M$, where the function m^* must satisfy a G-equivariance property: $m^*(\varphi(g).h) = g.m^*(h)$. Any such function can, of course, be extended linearly to a G-module morphism of the earlier form. If $g \in G$, we get a morphism

$$-.\varphi(g) : \varphi^*(\mathbb{Z}H) \to \varphi^*(\mathbb{Z}H)$$

given by 'h goes to $h\varphi(g)$'. This *is* a G-module morphism as the G-module structure is by *left* multiplication, which is independent of this right multiplication. Applying $G{-}Mod(-, M)$, we get $g.m^*$ is given by

$$g.m^*(h) - m^*(h.\varphi(g).$$

This is a *left* G-module structure, although at first that may seem strange. That it is linear is easy to check. What take a little bit of work is to check $(g_1g_2).m^* = g_1(g_2.m^*)$: applying both sides to an element $h \in H$ gives

$$(g_1g_2).m^*(h) = m^*(h\varphi(g_1)\varphi(g_2)),$$

whilst

$$g_1(g_2.m^*)(h) = (g_2.m^*)(h.\varphi(g_1)) = m^*(h\varphi(g_1)\varphi(g_2)).$$

(The checking that $g_1.m^*$ does satisfy the G-equivariance property is left to the reader.)

Remark: There are great similarities between the above calculations and those needed later when examining bitorsors. This is certainly not coincidental!

We built $\varphi_\sharp(M)$ in such a way that it is obviously functorial in M and gives a right adjoint to φ^*. This implies that there is a natural morphism

$$i : N \to \varphi_\sharp\varphi^*(N).$$

We denote this second module by N^*, when the context removes any ambiguity, and especially when φ is the inclusion of a subgroup. The morphism sends n to $n^* : H \to N$, where $n^*(h) = h.n$. (Check that $n^*(\varphi(g).h) = g.n^*(h)$. This reminds us that the codomain of n^* is infact just the *set* N underlying both the H-module N and the G-module $\varphi^*(N)$.)

We examine the cohomology groups $H^n(H, N^*)$. These are the (co)homology groups of the cochain complex $Hom(\mathsf{P}., N^*)$, where $\mathsf{P}.$ is a projective H-module resolution of $\mathbb{Z}$. The adjunction shows that this is isomorphic to $Hom(\varphi^*(\mathsf{P}.), \varphi^*(N))$. If $\varphi^*(\mathsf{P}.)$ is a projective G-module resolution of the trivial G-module $\mathbb{Z}$ then the cohomology of this complex will be $H^n(G, N)$, where N has the structure $\varphi^*(N)$.

The condition that free or projective H modules restrict to free or projective G-modules is satisfied in one important case, namely when G is a subgroup of H, since $\mathbb{Z}H$ is a free Abelian group on the *set* H and H is a disjoint union of right G-cosets, so $\mathbb{Z}H$ splits as a G-module into a direct sum of copies of $\mathbb{Z}G$. This provides part of the proof of Shapiro's lemma

Proposition 19 *If $\varphi : G \to H$ is an inclusion, then for a H-module N, there is a natural isomorphism*

$$H^n(H, N^*) \cong H^n(G, N).$$

■

Corollary 4 *The morphism $i : N \to N^*$ and the above isomorphism yield the restriction morphism*

$$H^n(H, N) \to H^n(G, N).$$

■

This suggest other results. Suppose we have an extension

$$1 \to N \to G \to Q \to 1$$

(so here we replace H by G with N in the old role of G, but in addition, being normal in G).

If we look at $\mathsf{B}N$ and $\mathsf{B}G$ in dimension n, these are free modules over the sets N^n and G^n respectively, with the inclusion between them; G is a disjoint union of N-cosets, indexed by elements of Q, so can we use this to derive properties of the cokernel of $\mathbb{Z}G \otimes_{\mathbb{Z}N} \mathsf{B}N \to \mathsf{B}G$, and to tie them into some resolution of Q, or perhaps, of $\mathbb{Z}$ as a trivial Q-module. The answer must clearly be positive, perhaps with some restrictions such as finiteness, but there are several possible ways of getting to an answer having slightly different results. (You have in the (φ_*, φ^*) and $(\varphi^*, \varphi_\sharp)$ adjunctions, enough of the tools needed to read detailed accounts in the literature, so we will not give them here.)

This also leads to relative cohomology groups and their relationship with the cohomology of the quotient Q. We can also consider the crossed resolutions of the various groups in the extension and work, say, with the induced maps

$$\mathsf{C}(N) \to \mathsf{C}(C)$$

looking at its cokernel or better what should be called its homotopy cokernel.

Another possibility is to examine $\mathsf{C}(N)$ and $\mathsf{C}(Q)$ and the cocycle information needed to specify the extension, and to use all this to try to construct a crossed resolution of G. (We will see something related to this in our examination of non-Abelian cohomology a little later.) A simple case of this is when the extension is split, $G \cong N \rtimes Q$ and using a twisted tensor product for crossed complexes, one can produce a suitable $\mathsf{C}(N) \otimes_\tau \mathsf{C}(Q)$ resolving G, (see Tonks, [264]).

Chapter 4

Syzygies, and higher generation by subgroups

Syzygies are one of the routes to working with resolutions. They often provide insight as to how a presentation relates to geometric aspects of a group, for instance giving structured spaces such as simplicial complexes, or, better, polytopes, on which the group acts. Syzygies extend the role of 'relations' in group presentations to higher dimensions and hence are 'relations between relations ... between relations'. They thus form a very well structured (and thus simpler) case of higher dimensional rewriting. Later we will see relations between this and several important aspects of cohomology. We will also explore some links with ideas from rewriting theory.

4.1 Back to syzygies

There are both homotopical and homological syzygies. We have met homological syzygies earlier and also have:

4.1.1 Homotopical syzygies

We have built a complex, $K(\mathscr{P})$, from a presentation, $\mathscr{P}$, of a group, G. Any element in $\pi_2(K(\mathscr{P}))$ can, of course, be represented by a map from S^2 to $K(\mathscr{P})$ and, by cellular approximation, can be replaced, up to homotopy, by a cellular decomposition of S^2 and a cellular map $\phi : S^2 \to K(\mathscr{P})$. We will adopt the terminology of Kapranov and Saito, [174], and Loday, [187], and say

Definition: A *homotopical 2-syzygy* consists of a cellular subdivision of S^2 together with a map, $\phi : S^2 \to K(\mathscr{P})$, cellular for that decomposition..

Of course, such an object corresponds to an identity among the relations of $\mathscr{P}$, but is a *specific representative* of such an identity. The specification of the cellular decomposition provides valuable combinatorial and geometric information on the presentation.

Definition: A family, $\{\phi_\lambda\}_{\lambda\in\Lambda}$, of such homotopical 2-syzygies is then called *complete* when the homotopy classes $\{[\phi_\lambda]\}_{\lambda\in\Lambda}$ generate $\pi_2(K(\mathscr{P}))$.

In this case, we can use the ϕ_λ to form the next stage of the construction of an Eilenberg-Mac Lane space, $K(G,1)$, by killing this π_2. More exactly, rename $K(\mathscr{P})$ as $X(2)$ and form

$$X(3) := X(2) \cup \bigcup_{\lambda\in\Lambda} e^3_\lambda,$$

by, for each $\lambda \in \Lambda$, attaching a 3-cell, e^3_λ, to $X(2)$ using ϕ_λ. Of course, we then have

$$\pi_1(X(3)) \cong G, \quad \pi_2(X(3)) = 0.$$

Again $\pi_3(X(3))$ may be non-trivial, so we consider *homotopical 3-syzygies*. Such a thing, s, will consist of an oriented polytope decomposition of S^3 together with a continuous map, f_s from S^3 to $X(3)$, which sends the i-skeleton of that decomposition to $X(i)$, $i = 0, 1, 2$.

At this stage we have $X(0) = K(\mathscr{P})_0$, a point, $X(1) = K(\mathscr{P})_1$, and $X(2) = K(\mathscr{P})_2$. One wants enough such 3-syzygies, s, identified algebraically and combinatorially, so that the corresponding homotopy classes, $\{[f_s]\}$ generate $\pi_3(X(3))$.

It is clear, by induction, we get a notion of *homotopical n-syzygy*. We assume $X(n)$ has been built inductively by attaching cells of dimension $\leq n$ along homotopical k-syzygies for $k < n$, so that

$$\pi_1(X(n)) \cong G, \quad \pi_k(X(n)) = 0, \quad k = 2, \ldots, n-1,$$

then a *homotopical n-syzygy*, s, is an oriented polytope decomposition of S^n and a continuous cellular map $f_s : S^n \to X(n)$.

After a choice of a set, $\mathscr{R}_n$, of n-syzygies, so that $\{[s_s] \mid s \in \mathscr{R}_n\}$ generates $\pi_n(X(n))$ as a G-module, we can form $X(n+1)$ by attaching $n+1$-dimensional cells e^{n+1}_s along these f_s for $s \in \mathscr{R}_n$.

If we can do this in a sensible way, for all n, we say the resulting system of syzygies is *complete* and the limit space $X(\infty) = \bigcup X(n)$ is then a cellular model for BG, the classifying space of the group G. We will look at classifying spaces again later.

This construction is, of course, just a homotopical version of the construction of a free resolution of the trivial G-module, $\mathbb{Z}$.

Remark: Some additional aspects of this can be found in Loday's paper [187], in particular the link with the 'pictures' of Igusa, [161, 162].

Example and construction: Given any group, G, we can find a presentation with $\{\langle g\rangle \mid g \neq 1, g \in G\}$ as set of generators and a relation, $r_{g,g'} := \langle g\rangle\langle g'\rangle\langle g'g\rangle^{-1}$, for each pair (g, g') of elements of G. (We write $\langle 1\rangle = 1$ for convenience.) We will have earlier call this the *standard presentation* of the group, G. It is closely related to the nerve of $G[1]$, and also to the various bar resolutions. (There may be a need later to consider a variant in which the identity element of G is not excluded as a generator, however that will still be loosely called the standard presentation. Note that since then $\langle 1\rangle.\langle g\rangle = \langle 1.g\rangle = \langle g\rangle$, the identification $\langle 1\rangle = 1$ is automatic.)

The relation $r_{g,g'}$ gives a triangle

$$\begin{tikzcd} & \cdot \arrow[dr, "g'"] & \\ \cdot \arrow[ur, "g"] \arrow[rr, "g'g"'] & & \cdot \end{tikzcd}$$

and, for each triple (g, g', g''), we get a homotopical 2-syzygy in the form of a tetrahedron.

Higher homotopical syzygies occur for any tuple, $(g_1, \ldots, g_n)$, of non-identity elements of G, by labelling a n-simplex. The limiting cellular space, $X(\infty)$, constructed from this context is just the usual model of the classifying space, BG, as geometric realisation of the *nerve* of G, or if you prefer, of the groupoid $G[1]$ with one object. The corresponding free resolution, $(C_*(G), d)$, is the classical *normalised bar resolution*. Using the bar resolution above dimension 2 together with the crossed module of the presentation at the base, one gets the standard free crossed resolution of the group, G, as we saw in section 3.1.2.

4.1.2 Syzygies for the Steinberg group

(This is adapted from Kapranov and Saito, [174].)

Let R be an associative ring with 1. Recall that the (n^{th} unstable) *Steinberg group*, $St_n(R)$, has generators, $x_{ij}(a)$, labelling the elementary matrices $\varepsilon_{ij}(a)$, having

$$\varepsilon_{ij}(a)_{k,l} = \begin{cases} 1 & \text{if } k = l \\ a & \text{if } (k,l) = (i,j), a \in R \\ 0 & \text{otherwise,} \end{cases}$$

and relations

St1 $\quad x_{i,j}(a)x_{i,j}(b) = x_{i,j}(a+b)$;

St2 $\quad [x_{i,j}(a), x_{k,\ell}(b)] = \begin{cases} 1 & \text{if } i \neq \ell, j \neq k, \\ x_{i,\ell}(ab) & i \neq \ell, j = k \end{cases}$ and in which all indices are positive integers less than or equal to n.

The terminology 'n^{th} unstable' is to make the contrast with the group $St(R)$, the stable version. The unstable version, $St_n(R)$, models 'universal' relations satisfied by the $n \times n$ elementary matrices, whilst, in $St(R)$, the indices, i, j, k etc. are not constrained to be less than or equal to n. We will look at the stable version later.

The identities / homotopical 2-syzygies are built from three types of polygon:

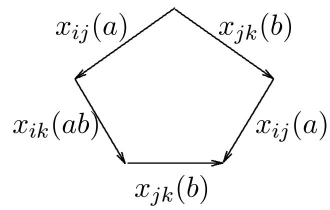
a) a triangle, $T_{ij}(a, b)$ for each i, j, $i \neq j$, coming from St1;

b) a square,

$$\begin{tikzcd} \cdot \arrow[r, "x_{ij}(a)"] \arrow[d, "x_{kl}(b)"'] & \cdot \arrow[d, "x_{kl}(b)"] \\ \cdot \arrow[r, "x_{ij}(a)"'] & \cdot \end{tikzcd}$$

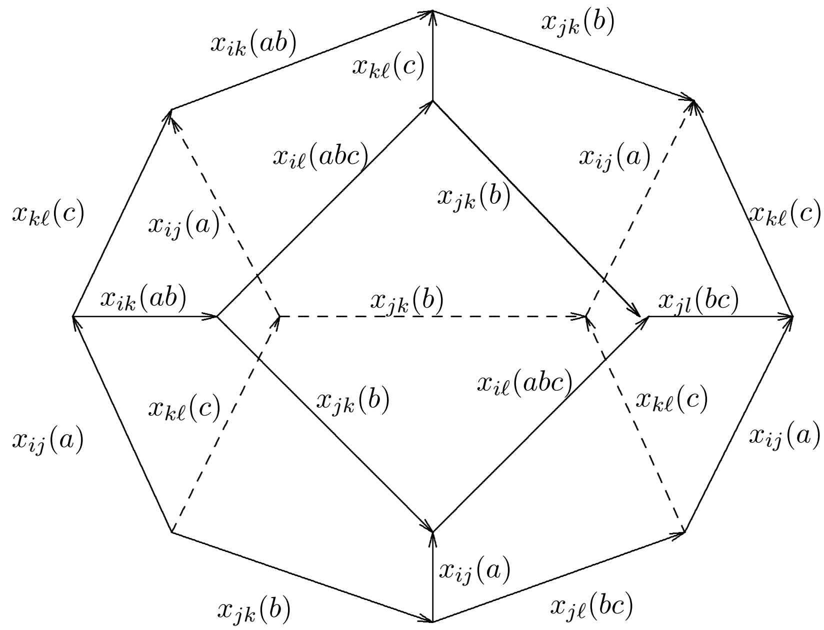
corresponding to the first case of St2 and

c) a pentagon, for the second:



For any pairs (i, j), (k, l), (m, p) with $x_{ij}(a)$, $x_{kl}(b)$, $x_{mp}(c)$, commuting by virtue of St2's first clause, we will, then, have a homotopical syzygy in the form of a labelled cube.

There is also a homotopy 2-syzygy given by the associahedron labelled by generators as shown:



Remark: Kapranov and Saito, [174], have conjectured that the space $X(\infty)$ obtained by gluing labelled higher Stasheff polytopes together, is homotopically equivalent to the homotopy fibre of

$$f : BSt(R) \to BSt(A)^+,$$

where $(-)^+$ denotes Quillen's plus construction. The associahedron is a Stasheff polytope and, by encoding the data that goes to build the identities / syzygies schematically in a 'hieroglyph', Kapranov and Saito make a link between such hieroglyphs and polytopes.

4.2 A brief sideways glance: simple homotopy and algebraic K-theory

The study of the Steinberg group is closely bound up with the development of algebraic K-theory. That subject grew out of two apparently unrelated areas of algebraic geometry and algebraic topology. The second of these, historically, was the development by Grothendieck of (geometric and topological) K-theory based on projective modules over a ring, or finite dimensional vector bundles on a space. (The connection between these is that the space of global sections of a finite dimensional vector bundle on a nice enough space, X, is a finitely generated projective module over the ring of continuous real or complex functions on X. We will look at vector bundles and this link with K-theory a bit more in detail later on; see section **??**. We will be discussing other forms of K-theory in that section as well, so will not give more detail on that more purely topological side of the subject here.)

Algebraic K-theory was initially a body of theory that attempted to generalise parts of linear algebra, notably the theory of dimension of vector spaces, and determinants to modules over arbitrary rings. It has grown into a well developed tool for studying a wide range of algebraic, geometric and even analytic situations from a variety of points of view.

For the purposes here we will give a short description of the low dimensional K-groups of a ring, R, with for initial aim to provide examples for use with the further discussion of rewriting, group presentations, syzygies, and homotopy. The discussion will, however, also look a bit more deeply at various other aspects when they seem to fit well into the overall structure of the notes.

4.2.1 Grothendieck's $K_0(R)$

For our discussion here, it will suffice to say that, given an associative ring, R, we can form the set, $[Proj_{fg}(R)]$ of isomorphism classes of finitely generated projective modules over R. Direct sum gives this a monoid structure. This is then 'completed' to get an Abelian group. We will give a more detailed discussion of this later in Proposition **??**, but here we will just give the formula:

$$K_0(R) := F([Proj_{fg}(R)])/\langle [P] + [Q] - [P \oplus Q] \rangle$$

in which P and Q are finitely generated projective modules, F is the free Abelian group functor and $[P]$ indicates the isomorphism class of P. The relations force the abstract addition in the free Abelian group to mirror the direct sum induced addition on the generators.

4.2.2 Simple homotopy theory

The other area that led to algebraic K-theory was that of simple homotopy theory. J. H. C. Whitehead, following on from earlier ideas of Reidemeister, looked at possible extensions of combinatorial group theory, with its study of presentations of groups, to give a combinatorial homotopy theory; see [278]. This would take the form of an 'algebraic homotopy theory' giving good algebraic models for homotopy types, and would hopefully ease the determination of homotopy equivalences for instance of polyhedra. The 'combinatorial' part was exemplified by his two papers on 'Combinatorial Homotopy Theory' [276, 277], but raised an interesting question. In combinatorial group theory, a major role is played by Tietze's theorem:

Theorem 6 *(Tietze's theorem, 1908, [260]) Given two finite presentations of the same group, one can be obtained from the other by a finite sequence of Tietze transformations.* ■

Proofs of this are easy to find in the literature. For instance, one based on a series of exercises is given in Gilbert and Porter, [133], p.135.

We clearly need to make precise what are the Tietze transformations.

Let $\mathcal{P} = (X : R)$ be a group presentation of a group, G and set $F(X)$ to be the free group on the set X. We consider the following transformations:

T1: Adding a superfluous relation: $(X : R)$ becomes $(X : R')$, where $R' = R \cup \{r\}$ and $r \in N(R)$, the normal closure of the relations in the free group on X, *i.e.*, r is a consequence of R;

T2: Removing a superfluous relation: $(X : R)$ becomes $(X : R')$ where $R' = R - \{r\}$, and r is a consequence of R';

T3: Adding a superfluous generator: $(X : R)$ becomes $(X' : R')$, where $X' = X \cup \{g\}$, g being a new symbol not in X, and $R' = R \cup \{wg^{-1}\}$, where w is a word in the other generators, that is w is in the image of the inclusion of $F(X)$ into $F(X')$;

T4: Removing a superfluous generator: $(X : R)$ becomes $(X' : R')$, where $X' = X - \{g\}$, and $R' = R - \{wg^{-1}\}$ with $w \in F(X')$ and $wg^{-1} \in R$ and no other members of R' involve g.

Definition: These transformations are called *Tietze transformations*.

The question was to ask if there was a higher dimensional version of the Tietze transformations that would somehow generate all *homotopy equivalences*.

Let us imagine the transformation of the complex, $K(\mathcal{P})$, of $\mathcal{P}$ under these moves. The complex is, of course, a simple form of CW-complex, built by attaching cells in dimensions 1 and then 2. If we add a superfluous generator to $\mathcal{P}$ as above (T3), then effectively we add a 2-cell labelled by wg^{-1} and it will be glued on by an attaching map that is defined on a semi-circle in its boundary and on which the path represents the word, w. The other semi-circle yields the loop representing the new generator. This process therefore does not change the homotopy type of $K(\mathcal{P})$. On the other hand, adding a superfluous relation will change the homotopy type of the complex. The new relation corresponds to a 2-cell glued on to $K(\mathcal{P})$, but the attaching map is already null-homotopic in $K(\mathcal{P})$ as it represents a consequence of the relations. The effect is that $K(\mathcal{P}')$ has the homotopy type of $K(\mathcal{P}) \vee S^2$, and the module of identities has an extra free summand.

These thus show both types of behaviour when attaching a cell to a pre-existing complex. In the first, the relation 2-cell is attached by part of its boundary. In the second the new cell is attached by gluing along *all* of its boundary, so will change the homotopy type of $K(\mathcal{P})$. It will not change its fundamental group, just its higher homotopy groups. This raises and interesting question, and that is to mirror these Tietze transformations by higher order ones which do not change the n-type, for some n, but may change the whole homotopy type, but we need to get back towards simple homotopy theory.

Tietze transformations had given a way of manipulating presentations and thus suggested a way of manipulating complexes. The thought behind simple homotopy theory was to produce a way of constructing homotopy equivalences between complexes. This, if it worked, might simplify the task of determining whether two spaces (defined, say, as simplicial complexes) were of the same homotopy type, and if so was it possible to build up the homotopy equivalences between them in some simple way.

The resulting theory was developed initially by Reidemeister and then by Whitehead, culminating in his 1950 paper, [279]. The theory received a further important stimulus with Milnor's classic paper, [204], in which the emphasis was put on elementary expansions.

(A good source for the theory of simple homotopy is Cohen's book, [80].)

We will work here with finite CW-complexes. These are built up by induction by gluing on n-cells, that is copies of $D^n = \{x \in \mathbb{R}^n \mid \sum x_i^2 \leq 1\}$, at each stage. Each D^n has a boundary an $(n-1)$-sphere, $S^{n-1} = \{x \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$. The construction of objects in the category of finite CW-complexes is by attaching cells by means of maps defined on part of all of the boundary of a cell. This will usually change the homotopy type of the space, creating or filling in a 'hole'. The homotopy type will not be changed if the attaching map has domain a hemisphere. We write $S^{n-1} = D^{n-1}_- \cup D^{n-1}_+$, with each hemisphere homeomorphic to a $(n-1)$-cell, and their intersection being the equatorial $(n-2)$-sphere, S^{n-2}, of S^{n-1}.

Given, now, a finite CW-complex, X, we can build a new complex Y, consisting of X and two new cells, e^n and e^{n-1} together with a continuous map, $\varphi : D^n \to Y$ satisfying

(i) $\varphi(D^{n-1}_+) \subseteq X_{n-1}$;

(ii) $\varphi(S^{n-2}) \subseteq X_{n-2}$;

(iii) the restriction of φ to the interior of D^n is a homeomorphism onto e^n;

and

(iv) the restriction of φ to the interior of D^{n-1}_- is a homeomorphism onto e^{n-1}.

There is an obvious inclusion map, $i : X \to Y$, which is called an *elementary expansion*. There is also a retraction map $r : Y \to X$, homotopy inverse to i, and which is called an *elementary contraction*. Both are homotopy equivalences. Can all homotopy equivalences between finite CW-complexes be built by composing such elementary ones? More precisely if we have a homotopy equivalence $f : X \to X'$, is f homotopic to a composite of a finite sequence of elementary expansions and contractions? Such a homotopy equivalence would be called *simple*. Whitehead showed that not all homotopy equivalences are simple and constructed a group of obstructions for the problem with given space X, each non-identity element of the group corresponding to a distinct homotopy class of non-simple homotopy equivalences.

4.2.3 The Whitehead group and $K_1(R)$

We will very briefly sketch how the investigation goes, skimming over the details; for them, see Milnor, [204], or Cohen's book, [80].

Starting with a homotopy equivalence, $f : X \to Y$, we can convert it to a deformation retraction using the mapping cylinder construction. (We will see this in more detail later, but do not need that detail here). This means that we have a CW-pair, (Y, X), with a deformation retraction from Y to X. Classifying the simple homotopy types of X is then transformed into a problem of classifying these. Passing first to their universal covering spaces, $\tilde{Y}$ and $\tilde{X}$, and then to the cellular chain complexes associated to both these, the problem is reduced to examining the *relative* cellular chain complex, $C(\tilde{Y}, \tilde{X})$, obtained from the exact sequence

$$0 \to C(\tilde{X}) \to C(\tilde{Y}) \to C(\tilde{Y}, \tilde{X}) \to 0$$

All of these can be considered as chain complexes of modules over the group ring of $\pi_1 X$. As there are only finitely many cells in X and Y, this chain complex has only finitely many non-zero levels in it. It is also acyclic, *i.e.*, has zero homology because the inclusion of $C(\tilde{X})$ into $C(\tilde{Y})$ induces isomorphism on homology. The cells in $Y - X$ give a preferred basis to the modules concerned.

One further reduction takes the direct sum of the even dimensional $C(\tilde{Y}, \tilde{X})_n$, and similarly that of the odd ones, and the induced boundary from the odds to the evens. (At each stage the reduction is checked to preserve what one want, namely whether or not the inclusion of X into Y is given by some combinations of elementary expansions and contractions. (The last part of this can be examined intuitively by thinking about what happens if you add in an n-cell by a $n-1$-cell in its boundary.)

This reduces the task to one of examining an isomorphism between two based free modules over $\mathbb{Z}\pi_1 X$, and that brings us, finally, to the main point of this section namely the definition of the group $K_1(R)$. (For this original application to simple homotopy theory, one takes $R = \mathbb{Z}\pi_1 X$.)

We will not take a historical order, concentrating on K_1, which was extracted from Whitehead's work, and studied for its own sake by Bass, [22]. Other aspects relating to simple homotopy theory may be looked at later on when we have more tools available.

Let R be an associative ring with 1. As usual $G\ell_n(R)$ will denote the *general linear group* of $n \times n$ non-singular matrices over R. There is an embedding of $G\ell_n(R)$ into $G\ell_{n+1}(R)$ sending a matrix $M = (m_{i,j})$ to the matrix M' obtained from M by adding an extra row and columnof zeros except that $m'_{n+1,n+1} = 1$. This gives a nested sequence of groups

$$G\ell_1(R) \subset G\ell_2(R) \subset \ldots \subset G\ell_n(R) \subset G\ell_{n+1}(R) \subset \ldots$$

and we write $G\ell(R)$ for the colimit (union) of these. It will be called the *stable general linear group over R*

Definition: The group, $K_1(R)$, is $G\ell(R)^{Ab} = G\ell(R)/[G\ell(R), G\ell(R)]$.

This is functorial in R, so that a ring homomorphism, $\varphi : R \to S$ induces $K_1(\varphi) : K_1(R) \to K_1(S)$.

The main initial problem with the above definition of $K_1(R)$ is that of controlling the commutator subgroup of $G\ell(R)$. The key is the *stable elementary linear group*, $E(R)$.

We extend the earlier definition of elementary matrices (on page 127 from the finite dimensional case, *i.e.*, within $G\ell_n(R)$, to being within $G\ell(R)$. Here an *elementary matrix* is of the form $e_{ij}(a) \in G\ell(R)$, for some pair (i, j) of distinct positive integers and which, thus, has an a in the (i, j) position, 1s in every diagonal position and 0 elsewhere. Although there is a small risk of confusion from notational reuse, we will, none-the-less, follow the standard notational convention and write $E_n(R)$ for the subgroup generated by the elementary matrices in $G\ell_n(R)$ and $E(R)$ for the corresponding union of the $E_n(R)$ within $G\ell(R)$. We will call $E_n(R)$ the *elementary subgroup of $G\ell_n(R)$*,

Lemma 16 *If i, j, k are distinct positive integers, then*

$$e_{ij}(a) = [e_{ik}(a), e_{kj}(1)].$$

■

This was already commented on when looking at the Steinberg group, $St_n(R)$, which abstracts the 'generic' properties of the elementary matrices. The following is now obvious.

Proposition 20 *For $n \geq 3$, $E_n(R)$ is a perfect group,* i.e.,

$$[E_n(R), E_n(R)] = E_n(R).$$

■

Now let $M = (m_{ij})$ be any $n \times n$ matrix over R. (It is not assumed to be invertible.)

We note that in $G\ell_{2n}(R)$,

$$\begin{pmatrix} I_n & M \\ 0 & I_n \end{pmatrix} = \prod_{i=1}^{n} \prod_{j=1}^{n} e_{i,j+n}(m_{ij}),$$

so this is in $E_{2n}(R)$. Similarly $\begin{pmatrix} I_n & 0 \\ M & I_n \end{pmatrix} \in E_{2n}(R)$.

Next, let $M \in G\ell_n(R)$ and note

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ M^{-1} - I_n & I_n \end{pmatrix} \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ M - I_n & I_n \end{pmatrix} \begin{pmatrix} I_n & -M^{-1} \\ 0 & I_n \end{pmatrix}$$

(as is easily verified). We thus have

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \in E_{2n}(R),$$

hence it is a product of commutators.

Lemma 17 *If $M, N \in G\ell_n(R)$, then*

$$\begin{pmatrix} [M,N] & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & N^{-1} \end{pmatrix} \begin{pmatrix} (NM)^{-1} & 0 \\ 0 & NM \end{pmatrix},$$

so is in $E_{2n}(R)$.

Proof: Just calculation. ■

Passing to the stable groups, we get the famous *Whitehead lemma*:

Proposition 21

$$[G\ell(R), G\ell(R)] = E(R).$$

■

This was, thus, very easy to prove, but it is crucial for the development of algebraic K-theory. It should be noted that it did depend on having 'enough dimensions', so $[G\ell_n(R), G\ell_n(R)] \subseteq E_{2n}(R)$. For our purposes here, we do not need to question whether 'unstable' versions of this hold, however we will mention that, if $n \geq 3$ and R is a commutative ring, then $[G\ell_n(R), G\ell_n(R)] = E_n(R)$. The proof is given in many texts on algebraic K-theory.

4.2.4 Milnor's K_2

We have already met the definition of $K_2(R)$ (page 43). The stable elementary linear group, $E(R)$, is a quotient of the stable Steinberg group, $St(R)$. (It will help to glance back at the presentation given on page 111 and to check that these are 'generic' relationships between elementary matrices.) This stable Steinberg group is obtained from the various $St_n(R)$ together with the inclusions $St_n(R) \to ST_{n+1}(R)$ obtained by including the generators of the first into the generating set of the second in the obvious way. the colimit of these 'unstable' groups yields the *stable Steinberg group*

As we mentioned early and will prove shortly, there is a central extension:

$$1 \to K_2(R) \to St(R) \xrightarrow{\varphi} E(R) \to 1$$

and thus $\varphi : St(R) \to E(R)$, a crossed module. The group, $G\ell(R)/Im(b)$, is $K_1(R)$, the first algebraic K-group of the ring.

In fact, this is a universal central extension and certain observations about such objects will help interpret what information is contained in $K_2(R)$. We will 'backtrack' a bit so as to keep things relatively self-contained.

Let, as usual, $Z(G)$ denote the centre of a group G.

Lemma 18 *(i)* $Z(E(R)) = 1$*;*
(ii) $Z(St(R)) = K_2(R)$.

Proof: This is elementary, but fun!

Suppose that $N \in Z(E(R))$, then $N \in E_n(R)$ for some n. Within $E_{2n}(R)$,

$$\begin{pmatrix} N & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}\begin{pmatrix} N & 0 \\ 0 & I \end{pmatrix},$$

since N is central in $E(R)$. This works out as

$$\begin{pmatrix} N & N \\ 0 & I \end{pmatrix} = \begin{pmatrix} N & I \\ 0 & I \end{pmatrix},$$

i.e., $N = I$.

Next suppose that $M \in Z(St(R))$, then, as φ is surjective, $\varphi(M) \in Z(E(R))$, so must be trivial, as required. ■

Proposition 22

$$1 \to K_2(R) \to St(R) \xrightarrow{\varphi} E(R) \to 1$$

is a central extension. ■

We next need to examine *universal* central extensions.

Definitions: (i) A central extension

$$1 \to K \xrightarrow{k} H \xrightarrow{\sigma} G \to 1$$

is said to be *weakly universal* if, given any other central extension of G,

$$1 \to L \xrightarrow{k'} E \xrightarrow{\sigma'} G \to 1,$$

there is a homomorphism $\psi : H \to E$ making the diagram

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r, "k"] \arrow[d, "\psi|_K"'] & H \arrow[r, "\sigma"] \arrow[d, "\varphi"'] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & L \arrow[r, "k'"'] & E \arrow[r, "\sigma'"'] & G \arrow[r] & 1
\end{tikzcd}$$

commutes.

(ii) The central extension, as above, of G is *universal* if it is weakly universal and, in the previous definition, the morphism, ψ, is unique with that property.

Proposition 23 *Every group has a weakly universal central extension.*

Proof: Suppose that we have a presentation $(X : R)$ of G, or more usefully for us, a presentation sequence:

$$1 \to K \xrightarrow{k} F \xrightarrow{p} G \to 1,$$

(so $F = F(X)$, the free group on X, and $K = N(R)$ is the kernel of p). The subgroup, $[K, F]$. of F generated by the commutators, $[k(x), y]$, with $x \in K$, and $y \in F$, is normal , as is easily checked and is in K, so we can form an extension

$$1 \to \frac{K}{[K,F]} \to \frac{F}{[K,F]} \to G \to 1.$$

(Note that 'dividing out by this subgroup identifies all $k(x)y$ and $yk(x)$, so should make a central extension. It 'kills' the conjugation action of F on K.)

We will write $H = F/[K, F]$ with $\sigma : H \to G$ for the induced epimorphism, so we now have

$$\mathbb{E} : 1 \to Ker\, \sigma \to H \xrightarrow{\sigma} G \to 1.$$

This is a central extension, as is easily checked (**left to you**).

Now suppose

$$\mathbb{E}' : 1 \to L \xrightarrow{k} E \xrightarrow{\sigma'} G \to 1$$

is another central extension. We have to construct a morphism, $\psi : \mathbb{E} \to \mathbb{E}'$, *i.e.*, $\varphi : H \to E$, compatibly with the projections to G, (and their kernels). As F is free and σ' is an epimorphism, we can find $\tau : F \to E$ such that $\sigma\tau = p$. Now $\sigma'\tau k = 1$, so $\tau k = k'\psi|_K : K \to L$. We examine a commutator $[k(x), y]$ with $x \in K$, $y \in F$. The image of this under τ will be $\tau[k(x), y] = [\tau k(x), \tau(y)] = [k'\tau|_K(x), \tau(y)] = 1$, since $\mathbb{E}'$ is a central extension, so τ induces a $\psi : H \to E$ compatibly with the projections to G, and hence with their kernels. ■

When will G have a *universal* central extension? The answer is: when G is perfect.

Definition: Suppose G is a group, it is *perfect* if $[G, G] = G$, *i.e.*, it is generated by commutators.

Proposition 24 *Every perfect group, G, has a universal central extension.*

Proof: (We can pick up ideas and notation from the previous proof.) As G is perfect, we can restrict $\sigma : H \to G$ to the subgroup $[H, H]$ and still get a surjection. We thus have

$$\begin{tikzcd}
1 \arrow[r] & K \cap [H,H] \arrow[r] \arrow[d] & {[H,H]} \arrow[r, "\sigma"] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & K \arrow[r] & H \arrow[r, "\sigma"'] & G \arrow[r] & 1
\end{tikzcd}$$

It is clear that as the bottom is weakly universal, so is the top one.

We next need a subsidiary result.

Lemma 19 *If* $1 \to Ker\,\sigma \to H \xrightarrow{\sigma} G \to 1$ *is a weakly universal central extension and H is perfect, then G is perfect and the central extension is universal.*

Proof: The first conclusion should be clear, so we are left to prove 'universal'. Suppose we have $\mathbb{E}'$ as before and obtain two morphism φ and φ', from H to E such that $\sigma'\varphi = \sigma'\varphi' = \sigma$. We have, for $h_1, h_2 \in H$, $\varphi(h_1) = \varphi'(h_1)c$, and $\varphi(h_2) = \varphi'(h_2)d$ for some $c, d, \in L$. we calculate that

$$\varphi(h_1 h_2 h_1^{-1} h_2^{-1}) = \varphi'(h_1 h_2 h_1^{-1} h_2^{-1}),$$

since c and d are central in E, but as commutators generate H, $\varphi = \varphi'$ everywhere in H. ■

To complete the proof of the proposition, we show that, back in case $[[H, H]$ is itself perfect. We have

$$[H, H] = \left[\frac{F}{[K, F]}, \frac{F}{[K, F]}\right] = \frac{[F, F]}{[K, F]},$$

now as G is perfect, every element in F can be written in the form $x = ck$ with $c \in [F, F]$ and $k \in K$. (One could say 'F is perfect up to K'.)

Take, now, a $[\overline{x}, \overline{y}] \in [H, H]$, *i.e.*, a commutator of $\overline{x}, \overline{y} \in F/[K, F]$ with $\overline{x}$ denoting the coset $x[K, F]$, etc. Set $x = ck$, $y = d\ell$, $c, d, \in [F, F]$

$$\begin{aligned}
\overline{xyx^{-1}y^{-1}} &= \overline{x}.\overline{y}.\overline{x}^{-1}.\overline{y}^{-1} \\
&= \overline{c}.\overline{d}.\overline{c}^{-1}.\overline{d}^{-1} \\
&= \overline{cdc^{-1}d^{-1}} \in [[H, H], [H, H]]
\end{aligned}$$

since elements of K commute with elements of F mod $[K, F]$. We thus have $[H, H] = [[H, H], [H, H]]$, as claimed. ■

To summarise, suppose we have a group presentation, $G = (X : R)$, of a perfect group, G. This gives us an exact 'presentation sequence'

$$1 \to K \to F \to G \to 1$$

where we abbreviate $N(R)$ to K. There is, then, a short exact sequence:

$$1 \to \frac{K \cap [F, F]}{[K, F]} \to \frac{[F, F]}{[K, F]} \to G \to 1$$

and this is its *universal central extension.*

Remark: The term on the left is the usual formula for the *Schur multiplier* of G and is one of the origins of group *homology.* It gives the Hopf formula for $H_2(G, \mathbb{Z})$, the second homology of G with coefficients in the trivial G-module, $\mathbb{Z}$.

To apply this theory and discussion back to the Steinberg group, $St(R)$, we need to check that $St(R)$ is a perfect group and that the central extension that we have is weakly universal. the first of these is simple.

Lemma 20 *The group $St(R)$ is perfect.*

Proof: We can write any generator $x_{ij}(a)$ as $[x_{ik}(a), x_{kj}(1)]$ for some k other than i or j, so the proof is the same as that $E_n(R)$ is perfect (for $n \geq 3$), that we gave earlier. ∎

This leaves us to check that the central extension

$$1 \to K_2(R) \to St(R) \xrightarrow{\varphi} E(R) \to 1$$

that we saw earlier is weakly universal (as it will then be universal by the previous lemma).

Suppose that we have

$$1 \to L \to E \xrightarrow{\sigma} E(R) \to 1$$

is a central extension. We have to define a morphism $\psi : St(R) \to E$ projecting down to the identity morphism on $E(R)$. As we have $St(R)$ defined by a presentation, the obvious way to proceed is to find suitable images in E for the generators, $x_{ij}(a)$, and then see if the Steinberg relations are satisfied by them.

To start with, for each generator $x_{ij}(a)$ of $St(R)$, we pick an element, $y_{ij}(a)$, in E such that $\sigma(y_{ij}(a)) = e_{ij}(a)$, the corresponding elementary matrix, which is, of course, the image of $x_{ij}(a)$ in $E(R)$. (Note that any other choice of the $y_{ij}(a)$ will differ from this by a family of elements of the kernel, L, and hence by central elements of E.)

We will prove, or note, various useful identities, which will give us what we need.

- $[u, [v, w]] = [uv, w][w, u][w, v]$ for $u, v, w, \in E$;

- for convenience, for $u \in E$, write $\bar{u} = \sigma(u) \in E(R)$, and for $u, v \in E$, write $u \sim v$ if $uv^{-1} \in L$, then note that if $u \sim u'$ and $v \sim v'$, we have $[u, v] = [u', v']$;

- if $u, v, w, \in E$ with $[\bar{u}, \bar{v}] = [\bar{u}, \bar{w}] = 1$, then

  $$[u, [v, w]] = 1.$$

  To see this, put $a = [u, v]$, $b = [u, w]$, so, by assumption, $\bar{a} = \bar{b} = 1$ and $a, b \in L$. We then have $uvu^{-1} = av$, $uwu^{-1} = bw$, and $[av, bw] = [v, w]$, since $a, b \in L$. Next look at

  $$[u, [v, w]] = u[v, w]u^{-1}[v, w]^{-1} = [uvu^{-1}, uwu^{-1}][v, w]^{-1} = 1$$

  by our previous calculation.

We are now ready to look at the $y_{ij}(a)$s and see how nearly they will satisfy the Steinberg relations, (St1 and St2 of page 111). (They will not necessarily satisfy them 'on-the-nose', but we can use them to get another choice that will work.)

- If $i \neq j$, $k \neq \ell$, so the corresponding ys make sense, and further $i \neq \ell, j \neq k$ (to agree with the condition of the first part of the St2) relation), then $[y_{ij}(a), y_{k\ell}(b)] = 1$. To see this we choose n bigger than all the indices involved here, so that we can have $y_{k\ell}(b) \sim [y_{kn}(b), y_{n\ell}(1)]$, as they give the same element when mapped down to $E(R)$. We thus have

$$[y_{ij}(a), y_{k\ell}(b)] = [y_{ij}(a), [y_{kn}(b), y_{n\ell}(1)]] = 1,$$

  by the above, so the ys do go some way towards what we need, (but the other relations need not hold). We will use them, however, to make a better choice.

- Suppose i, j and n are distinct, and, as always, $a \in R$. Set

$$z^n_{ij}(a) = [y_{in}(a), y_{jn}(1)].$$

  It is easy to see that this depends on i, j and a, and, slightly less obviously, that it does not depend on the choice of the $y_{k\ell}$s. Actually it does not depend on n at all. (The details are left **for you to check**, but use the commutator rules above to show $z^n_{ik}(ab) = [y_{ij}(a), y_{jk}(b)]$. That is independent of n.) We write $z_{ij}(a)$ for $z^n_{ij}(a)$, as n is irrelevant, as long as it is sufficiently large. These $z_{ij}(a)$ will do the trick!

We define $\psi : St(R) \to E$ by defining $\psi(x_{ij}(a)) = z_{ij}(a)$ and will check that $z_{ij}(a)$ satisfies the relations of $St(R)$, (as that will mean that this assignment does define a homomorphism by what is sometimes known as von Dyck's Theorem).

Most have been done (and checking this is again **left to you**), except for

$$z_{ij}(a)z_{ij}(b) = z_{ij}(a+b).$$

Clearly their difference is central in E, but that is not enough. We calculate

$$\begin{aligned} z_{ij}(a+b) &= z_{ij}(b+a) \\ &= [z_{ik}(b+a), z_{kj}(1)] \quad \text{with } k \neq i,j \\ &= [z_{ik}(b)z_{ik}(a), z_{kj}(1)] \quad \text{as the 'difference is central'} \\ &= [z_{ik}(b), z_{ij}(a)]z_{ij}(a)z_{ij}(b) \quad \text{using the first commutator identity above} \\ &= z_{ij}(a)z_{ij}(b) \end{aligned}$$

as required.

We have checked, in quite a lot of detail, that

Proposition 25

$$1 \to K_2(R) \to St(R) \xrightarrow{\varphi} E(R) \to 1$$

is a universal central extension. ∎

4.2.5 Higher algebraic K-theory: some first remarks

Milnor's definition of $K_2(R)$ was initially given in a course at Princeton in 1967. The search for higher algebraic K-groups was then intense; see Weibel's excellent history of algebraic K-theory, [275]. The breakthrough was due to Quillen, who in 1969/70, gave the 'plus construction', which was a method of 'killing' the maximal perfect subgroup of a fundamental group, $\pi_1(X)$. Applying this to the classifying space, $BG\ell(R)$, of the stable general linear group, gave a space $BG\ell(R)^+$, whose homotopy groups had the right sort of properties expected of those mysterious higher groups and so were taken to be $K_n(R) := \pi_n(BG\ell(R)^+)$.

Several other constructions of $K_n(R)$ were given in 1971 and were gradually shown to be equivalent to Quillen's. One of these which was based upon the theory of 'buildings' and upper triangular subgroups was by I. Volodin, [272]. We will look at the general construction in the next few sections as it relates closely to our theme of higher syzygies.

We note that there are several other approaches that were developed at about the same time, but will not be looked at in this chapter. There are also generalisations of these ideas.

4.3 Higher generation by subgroups

We now return to more general discussions relating to presentations, syzygies and rewriting, although we will see the link with ideas and methods from K-theory coming in later on.

Often one has a group, G, and a family, $\mathscr{H}$, of subgroups. For example (i) suppose G is given with a presentation, $(X : R)$, then subsets of X yield subgroups of G, and a family of subsets naturally leads to a family of subgroup, or (ii) a group may be a symmetry group of some geometric or combinatorial structure and certain substructures may be fixed by a subgroup, so families of subgroups may correspond to families of substructures. It is common, in this sort of situation, to try to see if information on G can be gleaned from information on the subgroups in $\mathscr{H}$. This will happen to some extent even if it is simply the case that the union of the elements in the subgroups generate G.

A simple example would be if G is generated by three elements, a, b and c, with some relations (possibly not known or not completely known), $\mathscr{H}$ consists of the subgroup generated by a, and that generated by b. There is a possibility that c is not in the subgroup generated by a and b, but how might this become apparent.

It may be that we have, instead of a presentation of G, presentations of the subgroups in $\mathscr{H}$, can we find a presentation of G, and, more generally, suppose we have knowledge of higher (homotopical or homological) syzygies of the presentations of the subgroups in $\mathscr{H}$, can we find not only a presentation of G, but build up knowledge of (at least some of) the syzygies for that presentation?

The key to attacking these problems is a knowledge of the way that the subgroups interact and by building up knowledge of the correspondence between the combinatorics of that interaction and of the induction process of building out from $\mathscr{H}$ to the whole group, G.

Various instances of this process had been studied, notably by Tits, e.g. in [261–263], since, in the situations studied in those papers, the combinatorics leads to the building of a Tits system. They also occur in the work of Behr, [28] and Soulé, [249], but, because of their general approach and the explicit link made to identities among relations, we will use the beautiful paper by Abels and Holz, [2]. This, and some subsequent developments, provides the basis for a way of calculating some syzygies in some interesting situations.

There is also a strong link with Volodin's approach to higher algebraic K-theory, but that will be slightly later in the notes. Here we sketch some of the background and intuition, giving some very elementary examples. When we have more knowledge of how to work with syzygies using both homotopical and homological methods, whether 'crossed' or not, we will return to look in more detail. We will see that this study of 'higher generation' leads in some interesting directions, towards geometric constructions and concepts of use elsewhere.

4.3.1 The nerve of a family of subgroups

We start, therefore, with a group, G, and a family, $\mathscr{H} = \{H_i \mid i \in I\}$ of subgroups of G. Each subgroup, H, determines a family of right cosets, Hg, which cover the *set*, G. Of course, these partition G, so there are no non-trivial intersections between them. If we use *all* the right cosets, $H_i g$, for *all* the H_i in $\mathscr{H}$, then, of course, we expect to get non-trivial intersections.

Remark: There is some disagreement as to which terminology for cosets is the most logical, so we should say exactly what we mean by 'right coset'. A subgroup H of G gives a left action, $H \curvearrowright G$ on the *set*, G, by multiplication on the left, and hence a groupoid whose connected components are the *right* cosets, Hg. The terminology 'right coset' corresponds to the g being on the right. If we considered the right action then we would have left cosets in the corresponding role.

Another notational point is that when writing cosets, we follow the usual rule that there is some informal set of coset representatives being used, or more exactly that the notation looks like that! This can be delicate if we step outside a set based situation, as choosing a set of coset representatives uses the axiom of choice, and in some contexts that would be 'dodgy'.

Let
$$\mathfrak{H} = \coprod_{i \in I} H_i \backslash G = \{H_i g \mid H_i \in \mathscr{H}\},$$
where the g is more as an indicator of right cosets than strictly speaking an index. This is the family of all right cosets of subgroups in $\mathscr{H}$. This covers G and we write $N(\mathfrak{H})$ for the corresponding simplicial complex, which is the *nerve* of this covering.

In many situations, 'nerves' in some form are used to help 'integrate' *local* information into *global*, since they record the way the 'localities' of the information fit together. (We will refer to this type of problem as a 'local-to-global' problem. They occur in many different contexts.) We have met nerves of categories, and will later meet nerves of open covers of topological spaces, but in that latter situation, the topological features of the construction are not central to that construction. We will consider the fairly general case of the nerve of a relation in a while, but for the moment, we will give a working definition, specific to the application that we have in mind here. We will refine and extend that definition later on.

Definition: Let G be a group and $\mathscr{H}$ a family of subgroups of G. Let $\mathfrak{H}$ denote the corresponding covering family of right cosets, $H_i g$, $H_i \in \mathscr{H}$. (We will write $\mathfrak{H} = \mathfrak{H}(G, \mathscr{H})$ or even $\mathfrak{H} = (G, \mathscr{H})$, as a shorthand as well.) The *nerve* of $\mathfrak{H}$ is the simplicial complex, $N(\mathfrak{H})$, whose vertices are the cosets, $H_i g$, $i \in I$, and where a non-empty finite family, $\{H_i g_i\}_{i \in J}$, is a simplex if it has non-empty intersection.

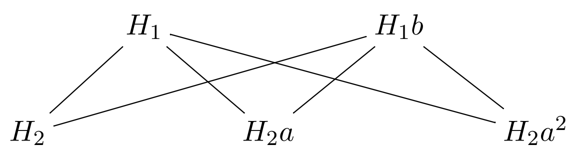
Examples: (i) If $\mathscr{H}$ consists just of one subgroup, H, then $\mathfrak{H}$ is just the set of cosets, $H\backslash G$ and $N(\mathfrak{H})$ is 0-dimensional, consisting just of 0-simplices / vertices.

(ii) If $\mathscr{H} = \{H_1, H_2\}$, (and H_1 and H_2 are not equal!), then any right H_1 coset, H_1g, will intersect some of the right H_2-cosets, for instance, $H_1g \cap H_2g$ always contains g. The nerve, $N(\mathfrak{H})$, is a bipartite graph, considered as a simplicial complex. (If the group G is finite, or more generally, if both subgroups have finite index, the number of edges will depend on the sizes or indcices of H_1, H_2 and $H_1 \cap H_2$.) It is just a graphical way of illustrating the intersections of the cosets, a sort of intersection diagram. (There is an error in [19] in which it is claimed that each coset H_1 will intersect with each of those of H_2.)

As a specific very simple example, consider:

- $S_3 \equiv (a, b : a^3 = b^2 = (ab)^2 = 1)$, (so a denotes, say, the 3-cycle (1 2 3) and b, a transposition (1 2)).
- Take $H_1 = \langle a \rangle = \{1, (1\ 2\ 3), (1\ 3\ 2)\}$, yielding two cosets H_1 and H_1b.
- Similarly take $H_2 = \langle b \rangle = \{1, (1\ 2)\}$ giving cosets H_2, H_2a and H_2a^2.

The covering of S_3 is then $\mathfrak{H} = \{H_1, H_1b, H_2, H_2a, H_2a^2\}$ and has nerve



4.3.2 n-generating families

Abels and Holz, [2], give the following definition:

Definition: A family, $\mathscr{H}$, of subgroups of G is called *n-generating* if the nerve, $N(\mathfrak{H})$, of the corresponding coset covering is $(n-1)$-connected, *i.e.*, $\pi_i N(\mathfrak{H}) = 0$ for $i < n$.

The following results illustrate the idea and motivate the terminology. (They are to be found in [2].)

Proposition 26 *The group, G, is generated by the union of the subgroups, H, in $\mathscr{H}$ if, and only if, $N(\mathfrak{H})$ is connected.*

We will take this apart rather than use the short proof given in [2]. (Hopefully this will show how the idea works and how simple minded the proof can be!)

Proof: Suppose we have that G is generated by the various H in $\mathscr{H}$ and we are given two vertices Hg_1 and Kg_2 for $H, K \in \mathscr{H}$. (The case $H = K$ is allowed here.) Of course, $g_1g_2^{-1} \in G$, so is a product of elements from the various H_is, say, $g_1g_2^{-1} = h_{i_1} \dots h_{i_n}$ with $h_{i_k} \in H_{i_k}$. (This observation suggests an induction on the length of this expression.)

To 'test the water', we assume $g_1g_2^{-1} = h_1 \in H_1$, but then $g_1 \in Hg_1 \cap H_1g_2$ and also $g_2 \in H_1g_2 \cap Kg_2$. (We can indicate this diagrammatically as

$$Hg_1 \overset{g_1}{\overline{\qquad}} H_1g_2 \overset{g_2}{\overline{\qquad}} Kg_2,$$

where each edge is decorated by an element that *witnesses* that the intersection of the two cosets is non-empty.)

If we try next with $g_1g_2^{-1} = h_1h_2$, then $g_1 = h_1h_2g_2$, so we have

$$Hg_1 \overset{g_1}{\text{———}} H_1(h_2g_2) \overset{h_2g_2}{\text{———}} H_2g_2 \overset{g_2}{\text{———}} Kg_2,$$

and the pattern gives the model for an induction on the length of the expression giving $g_1g_2^{-1}$ in terms of elements of the H_is. (Note the link between the expression and the path is very simple.)

Conversely, suppose that $N(\mathfrak{H})$ is connected, then if $g \in G$, we look at Hg and H for some choice of H. There is a sequence of edges in $N(\mathfrak{H})$ joining these two vertices. We examine the length, ℓ, of such an edge path. If $\ell = 1$, there is some $h \in H \cap Hg$, so $g \in H$. If $\ell = 2$,

$$H \overset{x_1}{\text{———}} H'g_1 \overset{x_2}{\text{———}} Hg,$$

and we have $x_1 = h_1 = h_2g_1$ with $h_2 \in H'$, whilst $x_2 = h_3g_1 = h_4g$. We thus obtain $g = h_4^{-1}h_3g_1$ and $g_1 = h_2^{-1}h_1$, so $g = h_4^{-1}h_3h_2^{-1}h_1$, *i.e.*, we have an expansion of g in terms of elements of the various Hs. A proof of the general case is now easy. ■

We next form a diagram, $\mathscr{D}$, consisting of the subgroups, H_i, and all their pairwise intersections, together with the natural inclusions, and we write $H := \sqcup_{\cap}\mathscr{H}$ for $colim\,\mathscr{D}$. (Note that this colimit is within the category of groups.) More exactly, there is a poset $\{H_j, H_j \cap H_k \mid j,k \in I\}$, ordered by inclusion and $\mathscr{D}$ is the inclusion of this diagram into the category of groups. There is a presentation of H with generators x_g, $g \in \bigcup H_j$ and with relations $x_g \cdot x_h = x_{gh}$ if g and h are both in some H_i. (This group, H, is thus a 'coproduct' with amalgamated subgroups.)

There is an obvious homomorphism

$$H = \sqcup_{\cap}\mathscr{H} \to G$$

induced by the inclusions.

Proposition 27 *The family, $\mathscr{H}$, is 2-generating if, and only if, the natural homomorphism,*

$$H = \sqcup_{\cap}\mathscr{H} \to G,$$

is an isomorphism. ■

In fact,

Proposition 28 *There are isomorphisms:*
(a) $\pi_0 N(\mathfrak{H}) \cong G/\langle\bigcup H_j\rangle$;
(b) $\pi_1 N(\mathfrak{H}) \cong Ker(\sqcup_{\cap}\mathscr{H} \to G)$. ■

We almost have shown (a) in our above argument, but will postpone more detailed proofs until later. (They are, in fact, quite easy to give by direct calculation.)

Remark: It is often helpful to take the family, $\mathscr{H}$, of subgroups and to close it up under (finite) intersection and sometimes the inclusion order on the intersections comes in useful as well. This closure operation does not change the homotopy type of the nerve of the corresponding coverings by cosets, in fact, the process of taking intersections corresponds to taking the barycentric subdivision of the original nerve.

4.3.3 A more complex family of examples

An important example of the above situation is in algebraic K-theory. It occurs with the general linear group, $G\ell_n(R)$, of invertible $n\times n$ matrices together with a family of subgroups corresponding to lower triangular matrices, ... but with some subtleties involved.

Let R be an associative ring with identity and n a positive integer.

Let $\Delta = \{(i,j) \mid i \neq j, 1 \leq i,j \leq n\}$ be the set of non-diagonal positions in an $n \times n$ array. We will say that a subset, $\alpha \subseteq \Delta$, is *closed* if

$$(i,j) \in \alpha \text{ and } (j,k) \in \alpha \text{ implies } (i,k) \in \alpha.$$

Note that if $(i,j) \in \alpha$ and α is closed then $(j,i) \notin \alpha$.

Let $\Phi = \{\alpha \subseteq \Delta \mid \alpha \text{ is closed}\}$. There is a reflexive relation $\leq$ on Φ by $\alpha \leq \beta$ if $\alpha \subseteq \beta$. These αs are transitive relations on subsets of the set of integers from 1 to n, so essentially order the elements of the subset. The reason for their use is the following: suppose $(i,j) \in \Delta$ and $r \in R$. The *elementary matrix*, $\varepsilon_{ij}(r)$, is the matrix obtained from the identity $n \times n$ matrix by putting the element r in position (i,j),

$$i.e., \qquad \varepsilon_{ij}(r)_{k,l} = \begin{cases} 1 & \text{if } k = l \\ r & \text{if } (k,l) = (i,j) \\ 0 & \text{otherwise .} \end{cases}$$

Let $G\ell_n(R)_\alpha$, for $\alpha \in \Phi$, denote the subgroup of $G\ell_n(R)$ generated by

$$\{\varepsilon_{ij}(r) \mid (i,j) \in \alpha, r \in R\}.$$

It is easy to see that $(a_{kl}) \in G\ell_n(R)_\alpha$ if and only if

$$a_{k,l} = \begin{cases} 1 & \text{if } k = l \\ \text{arbitrary} & \text{if } (i,j) \in \alpha \\ 0 & \text{if } (i,j) \in \Delta \backslash \alpha. \end{cases}$$

If $\alpha \leq \beta$, then there is an inclusion, $G\ell_n(R)_{\alpha\leq\beta}$ of $G\ell_n(R)_\alpha$ into $G\ell_n(R)_\beta$.

We will consider the $G\ell_n(R)_\alpha$ as forming a family, $\mathcal{G}\ell_n(R)$, of subgroups of $G\ell_n(R)$.

Remark: Although a similar idea is found in Wagoner's paper [273], I actually learnt the idea for this approach to these subgroups from papers by A. K. Bak, [17, 18], and, with others, in [19], and from talks he gave in Bangor and Bielefeld. In these sources, this construction leads on to a discussion of his notion of a global action, and, in the third paper cited, the variant known as a *groupoid atlas*. The motivation, there, is to study the *unstable* algebraic K-theory groups, whilst Volodin's original and Wagoner's approach are more centred on the stable version.

There is a lot more that could be said about these groupoid atlasses, which were introduced to handle the intrinsic homotopy involved in Volodin's definition of a form of algebraic K-theory, [272]. We will not use them explicitly here, but will attempt to show the link between the above and the question of syzygies, higher generation by subgroups, etc.

The nerve of this family would consist of the cosets of these subgroups, linked via their intersections. We need to extract another description of the homotopy type of this simplicial complex and for that will examine the intersections of cosets, and of the subgroups. We will do this in a slightly strange way in as much as we will turn first, or rather after some preparation, to descriptions related to Volodin's version of the higher K-theory of an associative ring. Our approach will be via *Volodin spaces* as used, for instance, in a paper by Suslin and Wodzicki, [255] and then an examination of the various nerves of a relation, before returning to this setting.

4.3.4 Volodin spaces

Let X be a non-empty set, and denote by $E(X)$, the simplicial set having $E(X)_p = X^{p+1}$, so a p-simplex is a $p+1$ tuple, $\underline{x} = (x_0, \ldots, x_p)$, each $x_i \in X$, and in which

$$d_i(\underline{x}) = (x_0, \ldots, \hat{x_i}, \ldots x_p),$$

and

$$s_j(\underline{x}) = (x_0, \ldots, x_j, x_j, \ldots x_p),$$

so d_i omits x_i, whilst s_j repeats x_j.

Lemma 21 *The simplicial set, $E(X)$, is contractible.*

Proof: We thus have to prove that the unique map $E(X) \to \Delta[0]$ is a homotopy equivalence. (That this is the case is well known, but we will none the less give a sketch proof of it as firstly we have not assumed that much knowledge of simplicial homotopy and also as it gives some interesting insights into that subject in a very easy situation.) We pick some $a_0 \in X$ and obtain a map $\Delta[0] \xrightarrow{a_0} E(X)$ by mapping the single 0-simplex of $\Delta[0]$ to the 0-simplex, (a_0) in $E(X)$. We now show that the identity map on $E(X)$ is homotopic to the composite map, $E(X) \to \Delta[0] \xrightarrow{a_0} E(X)$, that 'sends all simplices to a_0'.

We will look at simplicial homotopies in more detail later, (in particular around page 338), but clearly, a homotopy $h : f \simeq g : K \to L$, between two simplicial mapsa $f, g : K \to L$, should be a simplicial map $h : K \times \Delta[1] \to L$, restricting to f and g on the two ends of $K \times \Delta[1]$.. Here we need a homotopy $h : E(X) \times \Delta[1] \to E(X)$ and we look at what this must be on a cylinder over a simplex, $(x_0, \ldots, x_p)$. To see what to do, look at almost the simplest case, $p = 1$, then a schematic representation of h on $(x_0, x_1) \times \Delta[1]$ must look like:

$$\begin{tikzcd}
a_0 \arrow[r] & a_0 \\
x_0 \arrow[u] \arrow[r] \arrow[ur] & x_1 \arrow[u]
\end{tikzcd}$$

More precisely, the two simplices of $E(X) \times \Delta[1]$ that we need have two forms

$$\sigma_1 = ((x_0, 0), (x_1, 0), (x_1, 1))$$

and

$$\sigma_2 = (x_0, 0), (x_0, 1), x_1, 1))$$

being, respectively the bottom right and the top left hand ones. We need $h(\sigma_1) = (x_0 x_1, a_0)$ and $h(\sigma_2) = (x_0, a_0, a_0)$. Now it is easy to see how to set up h, in general, giving the required contracting homotopy. ■

Remark: Any homotopy can be specified by a family of maps, $h_i^n : K_n \to L_{n+1}$, satisfying some rules that will be given later (page 340). It is then easy to specify the $h_i^n : E(X)_n \to E(X)_{n+1}$ generalising the formula we have given above. (We **leave this to you** if you have not seen it before, as it is easy, but also instructive.)

The case we are really interested in is when we replace the general set, X, by the underlying set of a group, G. (As usual, we will not introduce a special notation for the underlying set of G, just writing G for it.) In this case we have the simplicial set, $E(G)$, and the group, G, acts freely on $E(G)$ by

$$g \cdot (g_0, \ldots, g_p) = (gg_0, \ldots, gg_p).$$

(Here we have used a left action of G, and **leave you to check** that the evident right action could equally well be used and we will use it later on.)

The quotient simplicial set of orbits, will be denoted $G\backslash E(G)$. It is often useful to write $[g_1, \ldots, g_p]$ for the orbit of the p-simplex $(1, g_1, g_1g_2, \ldots, g_1g_2 \ldots g_p) \in E(G)_p$.

It is 'instructive' to calculate the faces and degeneracy maps in this notation. We will only look at $[g_1, g_2]$ in detail. This element has representative $(1, g_1, g_1g_2)$. We thus have:

- $d_0(1, g_1, g_1g_2) = (g_1, g_1g_2) \equiv (1, g_2)$, so $d_0[g_1, g_2] = [g_2]$;
- $d_1(1, g_1, g_1g_2) = (1, g_1g_2)$, so $d_1[g_1, g_2] = [g_1g_2]$;
- $d_2(1, g_1, g_1g_2) = (1, g_1)$, so $d_2[g_1, g_2] = [g_1]$.

(That looks familiar!)

For the degeneracies,

- $s_0(1, g_1, g_1g_2) = (1, 1, g_1, g_1g_2)$, so $s_0[g_1, g_2] = [1, g_1, g_2]$;
- $s_1(1, g_1, g_1g_2) = (1, g_1, g_1, g_1g_2)$, so $s_1[g_1, g_2] = [g_1, 1, g_2]$;

and similarly $s_2[g_1, g_2] = [g_1, g_2, 1]$.

The general formulae are now easy to guess and to prove - so they will be **left to you**, and then the following should be obvious.

Lemma 22 *There is a natural simplicial isomorphism,*

$$G\backslash E(G) \xrightarrow{\cong} Ner(G[1]) = BG.$$

■

We thus have that $G\backslash E(G)$ is a 'classifying space' for G.

We note that this shows that $G\backslash E(G)$ is a Kan complex, since we already have that $Ner(G[1])$ is one. It is easy enough to check it directly. Of course, $E(G)$ is Kan as well. Jumping ahead of ourselves, we will sketch that the fundamental group of $G\backslash E(G)$ is $\pi_1(G\backslash E(G)) \cong G$, whilst for $k > 1$, $\pi_k(G\backslash E(G))$ is trivial. (We will have to 'fudge' the details as they either need material that will not be directly handled in these notes (and hence, for which the reader is referred to standard texts on simplicial homotopy theory), or they may depend on ideas that will be only explored later on in the notes, so we will try to sketch enough to whet the appetite!)

First we take on trust that if K is a connected Kan complex, then the k^{th} homotopy group of K can be 'calculated' by looking at homotopy classes of mappings from the boundary of a $k+1$-simplex into K, based at a base point. If you have a map, $\partial\Delta[k+1] \to Ner(G[1])$, then you have *all* the information needed to extend it to a map defined on $\Delta[k+1]$, *i.e.*, the map you started with is null homotopic. (If you want more intuition on this, try looking at the case $k = 2$ and writing down what the various faces in $\partial\Delta[3]$ will give and then see how they determine a 3-simplex in $Ner(G[1])$.)

For dimension 1, the construction of π_1 is, of course, that of the fundamental group(oid), so gives a presentation with set of generators, $\{[g] \mid g \in G\}$, and, for each pair (g_1, g_2), a relation r_{g_1,g_2} corresponding to $[g_1, g_2] \in G\backslash E(G)_2$, and which gives $[g_1][g_2][g_1g_2]^{-1}$, but this was our prime example of a presentation of G, so $\pi_1(G\backslash E(G)) \cong G$.

There is, here, another **useful fact for the reader to check**. The quotient map from $E(G)$ to $G\backslash E(G)$ is a Kan fibration (and this is a **useful example to do in detail** if you are not that conversant with Kan fibrations). The fibre of this quotient map is a constant (or 'discrete') simplicial set with *value* G, so is a $K(G, 0)$. As is well known, and as we will introduce and use later, there is a long exact sequence of homotopy groups for any pointed fibration sequence, $F \to E \to B$, so we can apply this to

$$K(G, 0) \to E(G) \to G\backslash E(G)$$

to get $\pi_i(G\backslash E(G) \cong \pi_{i-1}(K(G, 0))$ and another proof that $G\backslash E(G)$ is an 'Eilenberg Mac Lane space' for G, *i.e.*, a $K(G, 1)$ in the usual notation, (... and yes, this is related to covering spaces ...).

Returning to the construction of what are called 'Volodin spaces' (cf. [255]), we put ourselves back in the context of a group, G, and a family, $\mathscr{H}$, of subgroups of G. We suppose that $\mathscr{H} = \{H_i \mid i \in I\}$ for some indexing set, I. (We may assume extra structure on I, as before, when we get further into the construction.)

Definition: (Suslin-Wodzicki, [255], p. 65.) We denote by $V(G, \mathscr{H})$, or $V(\mathfrak{H})$, the simplicial subset of $E(G)$ formed by simplices, $(g_0, \ldots, g_p)$, that satisfy the condition that there is some $i \in I$ such that, for all $0 \leq j, k \leq p$, $g_jg_k^{-1} \in H_i$.

The simplicial set, $V(G, \mathscr{H})$, will be called the *Volodin space* of $(G, \mathscr{H})$.

Remark: The actual definition given in [255] uses $g_j^{-1}g_k \in H_i$, as there the convention on cosets is gH rather than our Hg.

The subobject, $V(G, \mathscr{H})$, of $E(G)$ is a G-subobject, *i.e.*, it is invariant under the action of G. The corresponding quotient simplicial set $G\backslash V(G, \mathscr{H})$ coincides with the union of the BH_i within the classifying space, BG.

Remark: The construction of $V(G, \mathscr{H})$ is usually ascribed to Volodin in his approach to the higher K-theory groups of a ring, but, in fact, the basic construction is essentially much older, being due to Vietoris in the 1920s, but in a different setting, namely that of a simplicial complex associated to an open cover of a space. This was further studied by Dowker, [106], in 1952, where he abstracted the situation to construct two simplicial complexes from a relation between two sets.

4.3.5 The two nerves of a relation: Dowker's construction

The results of the next few sections are of much more general use than just for a group and a family of its subgroups. We therefore present things in an abstract version.

Let X, Y be sets and R a relation between X and Y, so $R \subseteqq X \times Y$. We write xRy for $(x, y) \in R$.

Fairly generic example: Let X be a set (often a topological space) and Y be a collection of (usually open) subsets of X covering X, *i.e.*, $\bigcup Y = X$. The classical case is when Y is an index set for an open cover of X. The relation is xRy if and only if $x \in y$, or, more exactly, x is in the subset indexed by y.

Returning to the abstract setting, we define two simplicial complexes associated to R, as follows:

(i) $K = K_R$:

(a) the set of vertices is the set X;

(b) a p-simplex of K is a set, $\{x_0, \cdots, x_p\} \subseteq X$, such that there is some $y \in Y$ with x_iRy for $i = 0, 1, \cdots, p$.

(ii) $L = L_R$:

(a) the set of vertices is the set, Y;

(b) p-simplex of K is a set, $\{y_0, \cdots, y_p\} \subseteq Y$, such that there is some $x \in X$ with xRy_j for $j = 0, 1, \cdots, p$.

Clearly the two constructions are in some sense dual to each other. The original motivating example was as above. It had X, a space, and $Y = \mathcal{U} = \{U_\alpha : \alpha \in A\}$, an open cover of X, and, in that case, K_R is the *Vietoris complex* of $\mathcal{U}$, $V(\mathcal{U})$ or $V(X, \mathcal{U})$, of the cover. The 'dual' construction has the open cover, $\mathcal{U}$, or better, the indexing set, A, as its set of vertices, and $\sigma = \langle \alpha_0, \alpha_1, ..., \alpha_p \rangle$, belongs to L_R if and only if the open sets, U_{α_j}, $j = 0, 1, \ldots, p$, have non-empty common intersection. This is the simplicial complex known as the *Čech complex*, *Čech nerve* or simply, *nerve*, of the open covering, $\mathcal{U}$, and it will be denoted $N(X, \mathcal{U})$, or $N(\mathcal{U})$. We will have occasion to repeat this definition later, both when considering Čech non-Abelian cohomology, (starting on page 290), and also when looking at triangulations when examining methods of constructing some simple topological quantum field theories, page **??**.

We will extend the terminology so that for a given relation, R, K_R will be called the *Vietoris nerve* of R, whilst L_R is its *Čech nerve*. (This is rather arbitrary as the Vietoris nerve of R is the Čech nerve of the opposite relation, R^{op}, from Y to X.)

In the situation in this chapter, we have a pair, $(G, \mathcal{H})$, and X is G, whilst Y is the family, $\mathfrak{H}$, of right cosets of subgroups from the family $\mathcal{H}$. The relation is 'xRy if and only if $x \in y$'.

The simplicial complex, K_R, thus has G as its set of vertices and $(g_0, \ldots, g_p)$ is a p-simplex of K_R if, and only if, all the g_ks are in some common right coset, H_ix, in the family $\mathfrak{H}$. It is then just a routine calculation to check that this is the same as saying that the simplex is in $V(\mathfrak{H})$. In other words, the Volodin complex of $(G, \mathscr{H})$ is the same as the Vietoris complex of $\mathfrak{H}$, and it is convenient that both names begin with the letter 'V'! The one difference is that the Vietoris complex is a simplicial complex, whilst the Volodin space is a simplicial set. For each p-simplex $\{g_0, \ldots, g_p\}$, of $V(\mathfrak{H})$, there are $p!$ simplices in the Volodin space.

The corresponding Čech nerve, L_R, is $N(\mathfrak{H})$ as introduced earlier, so, if $\sigma \in N(\mathfrak{H})_p$, $\sigma = \{H_0g_0, \cdots, H_pg_p\}$ with the requirement that $\cap\sigma = \bigcap_{i=0}{}^p H_ig_i \neq \emptyset$.

Before turning to Dowker's result, we will examine barycentric subdivisions as these play a neat role in his proof.

4.3.6 Barycentric subdivisions

Combinatorially, if K is a simplicial complex with vertex set, V_K, then one associates to K the partially ordered set of its simplices. (We avoid our earlier notation of $V(K)$ for the vertex set as being too ambiguous here.) Explicitly we write $S(K)$ (or sometimes S_K), for the set of simplices of K and $(S(K), \subseteq)$ for the partially ordered set with $\subseteq$ being the obvious inclusion. The *barycentric subdivision*, K', of K has $S(K)$ as its set of vertices and a finite set of vertices of K' (*i.e.*, simplices of K) is a simplex of K' if it can be totally ordered by inclusion.) We may sometimes write $Sd(K)$ instead of K'.)

Remark: It is important to note that there is, in general, no natural simplicial map from K' to K. If, however, V_K is given an order in such a way that the vertices of any simplex in K are totally ordered (for instance by picking a total order on V_K), then one can easily specify a map,

$$\varphi : K' \to K,$$

by:
if $\sigma' = \{x_0, \cdots, x_p\}$ is a vertex of K' (so $\sigma' \in S(K)$), let $\varphi\sigma'$ be the least vertex of σ' in the given fixed order.

This preserves simplices, but reverses order so if $\sigma'_1 \subset \sigma'_2$ then $\varphi(\sigma'_1) \geq \varphi(\sigma'_2)$.

If one changes the order, then the resulting map is *contiguous* to the old one:

Definition: Let $\varphi, \psi : K \to L$ be two simplicial maps between simplicial complexes. They are said to be *contiguous* if for any simplex σ of K, $\varphi(\sigma) \cup \psi(\sigma)$ forms a simplex in L.

Contiguity gives a constructive form of homotopy applicable to simplicial maps between simplicial complexes.

If $\psi : K \to L$ is a simplicial map, then it induces $\psi' : K' \to L'$ after subdivision. As there is no way of knowing/picking compatible orders on V_K and V_L in advance, we get that on constructing

$$\varphi_K : K' \to K$$

and

$$\varphi_L : L' \to L$$

that $\varphi_L\psi'$ and $\psi\varphi$ will be contiguous to each other, but rarely equal.

4.3.7 Dowker's lemma

Returning to K_R and L_R, we order the elements of X and Y, then suppose y' is a vertex of L'_R, so $y' = \{y_0, \cdots, y_p\}$, a simplex of L_R and there is an element $x \in X$ with $xRy_i, i = 0, 1, \cdots, p$. Set $\psi y' = x$ for one such x.

If $\sigma = \{y'_0, \cdots, y'_q\}$ is a q-simplex of L'_R, assume y'_0 is its least vertex (in the inclusion ordering)

$$\varphi_L(y'_0) \in y'_0 \subset y' \text{ for each } y_i \in \sigma,$$

hence $\psi y'_i R \varphi_L(y'_0)$ and the elements $\psi y'_0, \cdots, \psi y'_q$ form a simplex in K_R, so $\psi : L'_R \to K_R$ is a simplicial map. It, of course, depends on the ordering used and on the choice of x, but any other choice $\bar{x}$ for $\psi y'$ gives a contiguous map.

Reversing the rôles of X and Y in the above, we get a simplicial map,

$$\bar{\psi} : K'_R \to L_R.$$

Applying barycentric subdivisions again gives

$$\bar{\psi}' : K''_R \to L'_R,$$

and composing with $\psi : L'_R \to K_R$ gives a map

$$\psi\bar{\psi}' : K''_R \to K_R.$$

Of course, there is also a map

$$\varphi_K\varphi'_K : K''_R \to K_R.$$

Proposition 29 *(Dowker, [106] p.88). The two maps $\varphi_K\varphi'_K$ and $\psi\bar{\psi}'$ are contiguous.*

Before proving this, note that contiguity implies homotopy and that $\varphi\varphi'$ is homotopic to the identity map on K_R after realisation, *i.e.*, this shows that

Corollary 5

$$|K_R| \simeq |L_R|.$$

■

The actual homotopy depends on the ordering of the vertices and so is not natural.

Proof of the Proposition:

Let $\sigma''' = \{x''_0, x''_1, \cdots, x''_q\}$ be a simplex of K''_R and as usual assume x''_0 is its least vertex, then for all $i > 0$

$$x''_0 \subset x''_i.$$

We have that φ'_K is clearly order reversing, so $\varphi'_K x''_i \subseteq \varphi'_K x''_0$. Let $y = \bar{\varphi}\varphi'_K x''_0$, then for each $x \in \varphi'_K x''_0$, xRy. Since $\varphi_K\varphi'_K x''_i \in \varphi'_K x''_i \subseteq \varphi'_K x''_0$, we have $\varphi_K\varphi'_K x''_i Ry$.

For each vertex x' of $x_i'', \bar{\psi}x' \in \bar{\psi}'x_i''$, hence as $\varphi_K' x_0'' \in x_0'' \subset x_i'', y = \bar{\psi}\varphi_K' x x_0'' \in \bar{\psi}' x_i''$ for each x_i'', so for each $x_i'', \psi\bar{\psi}' x_i'' R y$, however we therefore have

$$\varphi_k\varphi_K'(\sigma'') \cup \psi\bar{\psi}(\sigma''') = \bigcup \varphi_k\varphi_K'(x_i'') \cup \psi\bar{\psi}; x_i''$$

forms a simplex in K_R, *i.e.*, $\varphi_K\varphi_K'$ and $\psi\bar{\psi}'$ are contiguous. ■

To prove this we had to choose orders on the two sets, and thus we were working with the non-degenerate simplices of the corresponding simplicial sets. (Abels and Holz, [2], use the neat notation of writing $N^{simp}(R)$, etc. for the corresponding simplicial set, either dependent on order or taking all possible orders, *i.e.*, a p-tuple is a simplex in the simplicial set if its underlying set of elements is a simplex in the simplicial complex. Which method is used make essentially no difference much of the time. Their notation can be useful, but we will sometimes tend to ignore the difference as the homotopy groups and homotopy types are independent of which approach one takes. We have briefly discussed this on page 30 and we will revisit it in more detail later in this chapter, in section 4.4.3.)

4.3.8 Flag complexes

The construction of the barycentric subdivision is closely related to that of a flag complex of a poset.

Suppose that $\mathscr{P} = (P, \leq)$ is a partially ordered set (poset), then we can consider is as a category and hence look at its nerve. This is the associated simplicial set of the flag complex of $\mathscr{P}$, which is a simplicial complex, whose construction uses some ideas that can be of use later on, so we will briefly discuss how it relates to our situation.

Definition: A subset, σ, of $\mathscr{P} = (P, \leq)$ is said to be a *flag* if it satisfies, for all $x, y \in P$, either $x \leq y$ or $y \leq x$.

A finite non-empty flag, thus, is a linearly ordered subset of P, *i.e.*, is of the form $\{x_0, \ldots x_p\}$, where $x_0 < \ldots < x_n$ are elements of the set P.

Definition: Let $\mathscr{P} = (P, \leq)$ be a poset. The *flag complex*, $Flag(\mathscr{P})$ of $\mathscr{P}$ is the simplicial complex having the elements of P as its vertices and in which a p-simplex will be a non-empty flag, $x_0 < \ldots x_n$. in $\mathscr{P}$.

This is often also called the *order complex* of the poset.

Lemma 23 *The flag complex construction gives a functor*

$$Flag : Posets \to SimpComp,$$

from the category of partially ordered sets and order preserving maps, to the category of simplicial complexes and simplicial morphisms between them. ■

As a simplicial complex, K, consists of a set, $V(K)$ of vertices and a set $S(K) \subseteq P(V(K)) - \{\emptyset\}$, $S(K)$ can naturally be ordered by inclusion to get a partially ordered set $U(K) = (S(K), \subseteq)$. This gives a functor,

$$U : SimpComp \to Posets.$$

The composite functor,

$$Flag \circ U : SimpComp \to SimpComp,$$

is the *barycentric subdivision* functor, Sd.

If X is a set and $\mathcal{U} = \{U_i \mid i \in I\}$ is a family of subsets of X, we may think of $\mathcal{U}$ as being ordered by inclusion and thus get a poset. (Of course, this will only be significant if there are some inclusions between the U_is, for instance if $\mathcal{U}$ is closed under finite intersection.) This gives a poset, $(\mathcal{U}, \subseteq)$ and we will abbreviate $Flag(\mathcal{U}, \subseteq)$ to $F(\mathcal{U})$.

The links between nerves and flag complexes are strong.

Proposition 30 *(Abels and Holz, [2], p. 312) Suppose given $(X, \mathcal{U})$ as above, and that $\mathcal{U}$ is such that, if U and V are in $\mathcal{U}$ and $U \cap V$ is not empty, then $U \cap V \in \mathcal{U}$, then there is a natural homotopy equivalence,*

$$|N(\mathcal{U})| \simeq |F(\mathcal{U})|.$$

We cannot give a full proof here as it involves a result, namely Quillen's Theorem A, [233], that will not be discussed in these notes. We can however give a sketch (based on the treatment in [2]).

Sketch proof: Abusing notation so as to consider the simplicial complex, $N(\mathcal{U})$, as being the same as the poset of its simplices, we define a mapping:

$$f : N(\mathcal{U}) \to \mathcal{U}$$

sending $\sigma = \{U_0, \ldots, U_p\}$ to $U_\sigma = \cap_{i=0}^{p} U_i$. This is order reversing. (Note that it, of course, needs $\mathcal{U}$ to be closed under pairwise non-empty intersections.) Writing $\mathcal{U}^{op}$ for the poset, $(\mathcal{U}, \supseteq)$, that is with the opposite order, the poset $U \downarrow f$ of objects under some $U \in \mathcal{U}^{op}$ is just $\{\tau \in N(\mathcal{U}) \mid U_\tau \supseteq U\}$, so is a directed poset, and hence is contractible. By Quillen's theorem A, f induces a homotopy equivalence as claimed. ∎

Remark: An interesting variant of these nerve and flag complex constructions combines some aspects of the Vietoris complex construction with the idea of flags to construct a bisimplicial set. A (p, q)-simplex will be pair consisting of a subset $\{x_0, \ldots, x_p\}$ of X together with a flag, $U_0 \subset U_1 \subset \ldots \subset U_q$, such that all the x_i are in U_0. We will not explore this idea here as we have not discussed bisimplicial sets in any detail yet.

Within geometric group theory, the term 'flag complex' is also applied to a closely related, but distinct, concept. These 'flag complexes' are abstract simplicial complexes that satisfy a particular defining property, rather than being defined by how they are constructed. We will see other similar ideas later on in less geometric contexts, but for the moment will give a brief discussion based on the treatment of Bridson and Haefliger, [54], p. 210.

Definition: Let L be a simplicial complex with set of vertices V_L. It satisfies the *no triangles condition* if every finite subset of V_L that is pairwise joined by edges, is a simplex. More precisely, if $\{v_0, \ldots, v_n\}$ is such that for each $i, j \in \{1, \ldots, n\}$, $\{v_i, v_j\}$ is a 1-simplex of L, then $\{v_0, \ldots, v_n\}$ is a simplex of L.

An alternative name for the condition are the '*no empty simplices*' condition. It is also said that in this case L is "*determined by its 1-skeleton*". The point is

Proposition 31 *If simplicial complex, L, is an order complex of some partially ordered set then it is determined by its 1-skeleton.* ■

The proof should be evident.

Geometric group theory contains many other examples of this sort of construction, especially with relation to Coxeter groups. (Perhaps we will return to this later one)

4.3.9 The homotopy type of Vietoris - Volodin complexes

Returning to $V(\mathfrak{H})$, the second complex associated to a pair $(G, \mathcal{H})$, it is possible to extract some homotopy information from it using fairly elementary methods. To go into its structure more deeply we will need to bring more explicitly in the group action of G as well, but that is for later.

The great advantage now is that as we know $N(\mathfrak{H})$ and $V(\mathfrak{H})$ have the same homotopy type (after realisation) so we can use either when working out homotopy invariants. We can also use $N^{simp}(\mathfrak{H})$, or $V^{simp}(\mathfrak{H})$ the corresponding simplicial sets, although, in fact, the Volodin *space* was actually defined as a simplicial set. We will usually leave out the difference between the simplicial complex and the simplicial set as that distinction is largely unnecessary.

If we look at any $gH_i \in \mathcal{H}$, then we have a subcomplex of $V(\mathfrak{H})$ consisting of those $(g_0, \ldots, g_p)$ all of which are in gH_i. In the simplest case, where $g = 1$, this is a copy of $E(H_i)$, and, in general, it is a translated copy of $E(H_i)$, so each forms a contractible subcomplex.

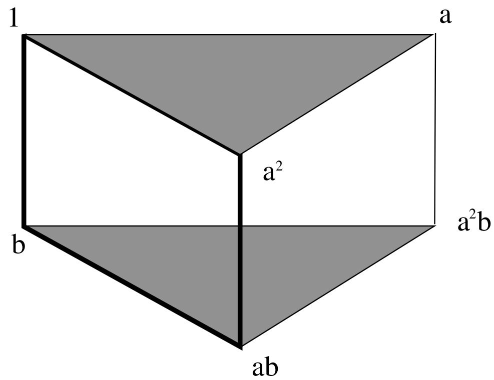
Example: (already considered in section 4.3.1)

$$\begin{aligned}
G &= S_3 = (a, b \mid a^3 = b^2 = (ab)^2 = 1), \text{ with } a = (1,2,3),\ b = (1,2);\\
H_1 &= \langle a \rangle = \{1,\ (1,2,3),\ (1,3,2)\},\\
H_2 &= \langle b \rangle = \{1,\ (1,2)\};\\
\mathcal{H} &= \{H_1, H_2\}
\end{aligned}$$

The intersection diagram given in our earlier look at this example, on page 125, is just the nerve, $N(\mathfrak{H})$, having 5 vertices and 6 edges. The other complex, $V(\mathfrak{H})$, is almost as simple. It has 6 vertices corresponding to the 6 elements of S_3, and each orbit yields a simplex

- $H_1 = \{1, a, a^2\}$ gives a 2-simplex (and three 1-simplices),
- $H_1 b = \{b, ab, a^2 b\}$ also gives a 2-simplex;
- $H_2 = \{1, b\}$ yields a 1-simplex, as do its cosets $H_2 a$ and $H_2 a^2$.

We can clearly see here the contractible subcomplexes mentioned earlier. We have that $V(\mathfrak{H})$ looks like two 2-simplices joined by 1-simplices at the vertices, (see below).



$$V(S_3, \{\langle a\rangle, \langle b\rangle\})$$

As $N(\mathfrak{H})$ is a connected with 5 vertices and 6 edges, we know $\pi_1 N(\mathfrak{H})$ is free on 2 generators. (The number of generators is the number of edges outside a maximal tree.) This same rank can be read of equally easily from $V(\mathfrak{H})$ as that complex is homotopically equivalent to a bouquet of 2 circles, (*i.e.*, a figure eight). The generators of $\pi_1 V(\mathfrak{H})$ can be identified with words in the free product $H_1 * H_2$ (one such being shown in the picture) and relate to the kernel of the natural homomorphism from $H_1 * H_2$ to S_3. The heavy line in the figure corresponds to a loop at 1 given by

$$1 \xrightarrow{(1,b)} b \xrightarrow{(b,ab)} ab \xrightarrow{(ab,a^2)} a^2 \xrightarrow{(a^2,1)} 1$$

We write $g_0 \xrightarrow{(g_0,g_1)} g_1$ as there is an edge, (g_0, g_1) joining g_0 to g_1 in $V(\mathfrak{H})$. We, thus, have that there is a g and an index i such that $\{g_0, g_1\} \in H_i g$, but the index and the elements are not necessarily uniquely determined. We saw that this means that $g_1 g_0^{-1} \in H_i$, so $g_1 = hg_0$ for some $h \in H_i$, and we could equally well abbreviate the notation to $g_0 \xrightarrow{h} g_1$. Note that the only condition required is that h is in some H_i, so the lack of uniqueness mention above is without importance. In our example, we can redraw the diagram corresponding to the heavier loop and we get

$$1 \xrightarrow{b} b \xrightarrow{a} ab \xrightarrow{b} a^2 \xrightarrow{a} 1$$

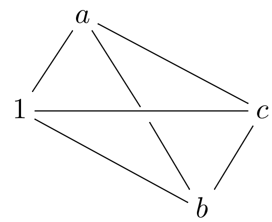
so the loop, representing an element in $\pi_1 N(\mathfrak{H})$, is given by the word $baba \in C_2 * C_3$, which, of course, is in the kernel of the homomorphism from $C_2 * C_3$ to S_3. The reason that this works is clear. Starting at 1, each part of the loop corresponds to a left multiplication either by an element of $H_1 \cong C_3$ or of $H_2 \cong C_2$. We thus get a word in $H_1 * H_2 \cong C_2 * C_3$. As the loop also finishes at 1, we must have that the corresponding word must evaluate to 1 when projected down into S_3.

Note that the two subgroups had simple presentations that combine to give a partial presentation of S_3. The knowledge of the fundamental group, $\pi_1 N(\mathfrak{H})$, then provides information on the 'missing' relations.

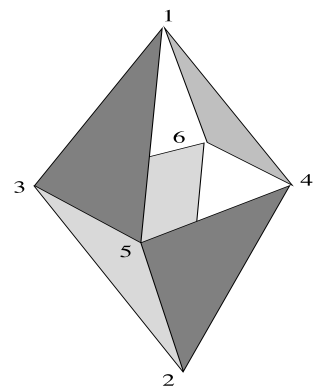
In more complex examples, the interpretation of $\pi_1(V(\mathfrak{H}), 1)$ will be the similar, but sometimes when G has more elements, $N(\mathfrak{H})$ may be easier to analyse than $V(\mathfrak{H})$, but the second may give links with other structure and be more transparent for interpretation. The important idea to retain is that the two complexes give the same information, so either can be used or both together.

Example: $G = K_4$, the Klein 4 group, $\{1, a, b, c\} \cong C_2 \times C_2$, so $a^2 = b^2 = c^2 = 1$ and $ab = c$; $\mathscr{H} = \{H_a, H_b, H_c\}$ where $H_a = \{1, a\}$, etc. Set $\mathfrak{H}_{K4} = (K_4, \mathscr{H})$.

The cosets are $H_a, H_a b, H_b, H_b a, H_c, H_c a$, each with two elements, so $V(\mathfrak{H}_{K4}) \cong$ the 1-skeleton of $\Delta[3]$:



$N(\mathfrak{H}_{K4})$ is "prettier" and a bit more 'interesting': Labelling the cosets from 1 to 6 in the order given above, we have 6 vertices, 12 1-simplices and 4 2-simplices. For instance, $\{1, 3, 5\}$ has the identity in the intersection, $\{1, 4, 6\}$ gives $H_a \cap H_b a \cap H_c a$, so contains a and so on. The picture is of the shell of an octahedron with 4 of the faces removed.



$N(\mathfrak{H}_{K4})$

From either diagram it is clear that $\pi_1 \mathfrak{H}_{K4}$ is free of rank 3. Again explicit representations for elements are easy to give. Using $V(\mathfrak{H})$ and the maximal tree given by the edges 1a, 1b and 1c, a typical generating loop would be

$$1 \to a \to b \to 1,$$

i.e., $(1, a, b, 1)$ as the sequence of points. There is an obvious representative word for this, namely

$$1 \xrightarrow{a} a \xrightarrow{c} b \xrightarrow{b} 1\ .$$

In general, any based path at 1 in an $V(G, \mathscr{H})$ will yield a word in $\sqcup\mathscr{H}$, the free product of the family $\mathscr{H}$. We will think of the path as being represented by a (finite) sequence $(f(n))$ of elements in G, linked by transitions, h_i in the various subgroups. Whether or not that representative is unique depends on whether or not there are non-trivial intersections and "nestings" between the subgroups in the family $\mathscr{H}$, since, for instance, if H_i is a subgroup of H_j, then if $f(n) \to f(n+1)$ using $g \in H_i$, it could equally well be taken to be $g \in H_j$. As we have mentioned before, the characteristic of the Vietoris-Volodin spaces, $V(G, \mathscr{H})$, is that there is only one possible *element* of G linking $f(n)$ to the next $f(n+1)$ namely $f(n+1)f(n)^{-1}$, but this may be in several of the H_i. We thus have a strong link between $\pi_1(V(G, \mathscr{H}))$ and $\underset{\cap}{\sqcup}\mathscr{H}$, the 'amalgamated product' of $\mathscr{H}$ over its intersections, and an analysis of homotopy classes will prove (later) that

$$\pi_1(V(G, \mathscr{H}), 1) \cong \operatorname{Ker}(\underset{\cap}{\sqcup}\mathscr{H} \to G),$$

since a based path $(g_1, g_2, \cdots, g_n)$ ends at 1 if and only if the product $g_1 \cdots g_n = 1$. These identifications will be investigated more fully shortly.

We note that composites of such 'paths' may involve two adjacent transitions between elements being in the same H_i in which case we can use the rewriting system determined by the contractible $E(H_i)$ to simplify the representatives.

Example: The number of subgroups in $\mathscr{H}$ clearly determines the dimension of $N(\mathfrak{H})$, when $\mathfrak{H} = \mathfrak{H}(G, \mathscr{H})$. Here is another 3 subgroup example.

Take $q8 = \{1, i, j, k, -1, -i, -j, -k\}$ to be the quaternion group, so $i^4 = j^4 = k^4 = 1$, and $ij = k$. Set $H_i = \{1, -1, i, -i\}$ etc., so $H_i \cap H_j = H_i \cap H_k = H_j \cap H_k = \{1, -1\}$ and let $\mathscr{H} = \{H_i, H_j, H_k\}$, and $\mathfrak{H}_{q8} = \mathfrak{H}(q8, \mathscr{H})$.

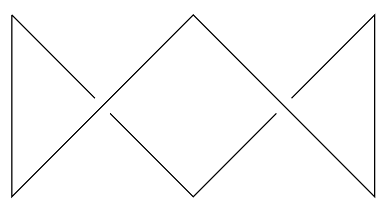
Then $N(\mathfrak{H}_{q8})$ is, as above in Example 4.3.9, a shell of an octahedron with 4 faces missing. Note however that $V(\mathfrak{H}_{q8})$ has 8 vertices and, comparing with $V(\mathfrak{H}_{K4})$, each edge of that diagram has become enlarged to a 3-simplex. It is still feasible to work with $V(\mathfrak{H}_{q8})$ directly, but $N(\mathfrak{H}_{q8})$ gives a clearer indication that

$$\pi_1(\mathfrak{H}_{q8}, 1) \text{ is free of rank 3.}$$

Example: Consider next the symmetric group, S_3, given by the presentation

$$S_3 := (x_1, x_2 \mid x_1^2 = x_2^2 = 1, (x_1x_2)^3 = 1)$$

Take $H_1 = \langle x_1 \rangle$, $H_2 = \langle x_2 \rangle$, so both are of index 3. Each coset intersects two cosets in the other list giving a nerve of form (see below):



so $\pi_1 N(\mathfrak{H}(S_3, \mathscr{H}))$ is infinite cyclic.

Example: The next symmetric group, S_4, has presentation

$$S_4 := (x_1, x_2, x_3 \mid x_1^2 = x_2^2 = x_3^2 = 1, (x_1x_2)^3 = (x_2x_3)^3 = 1, (x_1x_3)^2 = 1).$$

Take $H_1 = \langle x_1, x_2 \rangle$, $H_2 = \langle x_2, x_3 \rangle$, $H_3 = \langle x_1, x_3 \rangle$. H_1 and H_2 are copies of S_3, but H_3 is isomorphic to the Klein 4 group, K_4. Thus there are $4 + 4 + 6$ cosets in all. There are 36 pairwise intersections and each edge is in two 2-simplices. Each vertex is either at the centre of a hexagon or a square, depending on whether it corresponds to a coset of H_1, H_2 or of H_3. There are 24 triangles, and $N(S_4, \mathscr{H})$ is a surface. Calculation of the Euler characteristic gives 2, so this is a triangulation of S^2, the two sphere. (Thanks to Chris Wensley for help with the calculation using GAP.)

The fundamental group of $N(S_4, \mathscr{H})$ is thus trivial and, using the result mentioned above,

$$S_4 \cong \bigsqcup_{\cap} H_i,$$

the coproduct of the subgroups amalgamated over the intersection.

Accepting Proposition 28 for the moment, we can examine an important class of examples.

Example: Some graphs of groups. Let us suppose that $\mathscr{H} = \{H_1, H_2\}$, so just two subgroups of G, then we have

$$H_1 \underset{H_1\cap H_2}{\sqcup} H_2 \to G.$$

This is an isomorphism if and only if $N(\mathfrak{H})$ is a connnected graph which has trivial fundamental group, thus exactly when $N(\mathfrak{H})$ is a *tree*. The vertices of $N(\mathfrak{H})$ are the cosets in $(H_1\backslash G) \sqcup (H_2\backslash G)$ and H_1g_1 and H_2g_2 are connected by an edge if they intersect. This gives us one of the two basic types of a *graph of groups* as defined by Serre, [245, 246],

$$H_1 \underset{H_1\cap H_2}{\overline{\qquad\qquad}} H_2$$

corresponding to a free product with amalgamation. Note this does not give us the other basic type of graph of groups, which corresponds to an HNN-extension. We will explore the connection with this theory in more detail a bit later or, more exactly, we will see a connection with the generalisation *complexes of groups* due to Corson, [53, 90–93] and Haefliger, [145, 146], and developed extensively in the book by Bridson and Haefliger, [54].

We have now seen, somewhat informally, discussions of the low dimensional homotopy invariants of these two nerves, both in examples and, to some extent, in general. We turn now to more formal calculations of those, and in the process will prove Proposition 28.

We will approach the determination of the invariants in an 'elementary' but reasonably formal way. We will repeat some arguments that we have already seen partially to get everything in the same place, but also to impose some more consistent notation.

The set, $\pi_0(V(G,\mathscr{H}))$, of connected components: The vertex set of $V(G,\mathscr{H})$ is the set of elements of G, so we have to work out when two vertices, g and g', are in the same connected component.

Suppose they are connected by a path, that is a sequence of edges, $(\langle g_0,g_1\rangle, \langle g_1,g_2\rangle, \ldots \langle g_{n-1},g_n\rangle)$, in $V(G,\mathscr{H})$ and for some n. We have that an edge such as $\langle g_0,g_1\rangle$ has $d_0\langle g_0,g_1\rangle = g_1$ and $d_1\langle g_0,g_1\rangle = g_0$ and it is an edge because there is some $H_{\alpha_1}\in\mathscr{H}$ and some $x_1\in G$ such that g_0 and g_1 are in the coset $H_{\alpha_1}x_1$. Of course, this means that there are $h_0,h_1\in H_{\alpha_1}$ with $g_0 = h_0x_1$ and $g_1 = h_1x_1$, hence that $g_0g_1^{-1}\in H_{\alpha_1}$. (Conversely if $g_0g_1^{-1}\in H_{\alpha_1}$, then both g_0 and g_1 are in $H_{\alpha_1}g_1$, so $\langle g_0,g_1\rangle$ is an edge.)

We thus have from our path that there are indices $\alpha_1,\ldots,\alpha_n$ such that $g_{i-1}g_i^{-1}\in H_{\alpha_i}$ for each i, whilst $g = g_0$ and $g' = g_n$. We then note that gg'^{-1} is in $\langle\bigcup\mathscr{H}\rangle$, the subgroup generated by the union of the subgroups in the family $\mathscr{H}$, so, if g and g' are in the same component, then $gg'^{-1}\in\langle\bigcup\mathscr{H}\rangle$.

Conversely, suppose $gg'^{-1}\in\langle\bigcup\mathscr{H}\rangle$, then there is a finite sequence of indices, $\alpha_1,\ldots,\alpha_n$ for some n and elements $h_i\in H_{\alpha_i}$ such that $gg'^{-1} = h_1h_2\ldots h_n$. We define $g_0 = g$, $g_i = h_i^{-1}g_{i-1}$ and note that $g_{i-1}, g_i\in H_{\alpha_i}g_i$, thus giving us a path from g to $g_n = h_n^{-1}g_{n-1} = h_n^{-1}\ldots h_1^{-1}g_0 = g'$.

We thus have proved that $\pi_0(V(G,\mathscr{H}))$ is in bijection with $G/\langle\bigcup\mathscr{H}\rangle$, that is the first part of Proposition 28.

The fundamental group, $\pi_1(V(G,\mathscr{H}),1)$, and groupoid, $\Pi_1(V(G,\mathscr{H}))$: Although $V(G,\mathscr{H})$ comes with a natural choice of basepoint, namely 1, and we will eventually be looking at loops at 1, it is more in tune with our just previous discussion to look at the fundamental groupoid, $\Pi_1(V(G,\mathscr{H}))$, rather than the fundamental group $\pi_1(V(G,\mathscr{H}),1)$ of $V(G,\mathscr{H})$ based at 1. We will sometimes abbreviate $\Pi_1(V(G,\mathscr{H}))$ to $\Pi_1\mathfrak{H}$.

The set of objects of this groupoid will be the vertices of $V(G,\mathscr{H})$ and so are the elements of G, and the set of arrows $\Pi_1\mathfrak{H}(g,g')$ will be the set of homotopy classes of paths from g to g'. We saw that a path from, g to g' corresponds to a finite sequence, $\underline{h} = (h_1, h_2, \ldots, h_n)$, of elements from the various subgroups H_{α_i} in $\mathscr{H}$. It is convenient to write

$$g \xrightarrow{(h_1,h_2,\ldots,h_n)} g' = g \xrightarrow{\underline{h}} g',$$

where $h_n^{-1}\ldots h_1^{-1}g = g'$. We can see that given two composable paths

$$g \xrightarrow{\underline{h}} g' \xrightarrow{\underline{h}'} g'',$$

the defining sequence of the composite is given by the concatenation of the two sequences,

$$\underline{hh'} = (h_1, h_2, \ldots, h_n, h'_1, h'_2, \ldots, h'_m).$$

Remark: This notation is not quite accurate. The $\underline{h}$ does not indicate from where the arrow, so labelled, starts. Of course, it is visually clear, but 'really' we should denote the arrows by $(g,\underline{h})$, so then

$$(g,\underline{h})\cdot(\underline{h}^{-1}g,\underline{h}') = (g,\underline{hh'}),$$

or similar. This is clearly a form related to, but not identical, to some sort of 'action groupoid', but that does not quite fit. For a start, it does not give a groupoid as where are the inverses? It does give a category, however. (It is **left for you to check** that $\langle g_0, g_0\rangle$ is the identity at the 'object' g_0.)

The paths between the vertices are not the actual arrows in the fundamental groupoid $\Pi_1\mathfrak{H}$. For that we need to divide out by relations coming from 2-simplices.

For any simplicial complex or simplicial set, K, one can form the fundamental groupoid, (also called in this context the *edge path groupoid*), by taking the free groupoid on the directed graph or *quiver*, given by the 1-skeleton and then dividing out by the 2-simplices. (We will see this several times later; see pages 250, and **??**. It is the classical edge-path groupoid to be found, for instance, in Spanier's book, [250].) The arrows are sequences of concatenated edges and then, if $\langle v_0, v_1, v_2\rangle$ is a 2-simplex, we add a 'relation'

$$\langle v_0, v_1\rangle\langle v_1, v_2\rangle = \langle v_0, v_2\rangle,$$

or if you prefer, rewrite rules:

$$\langle v_0, v_1\rangle\langle v_1, v_2\rangle \Leftrightarrow \langle v_0, v_2\rangle.$$

For $\Pi_1\mathfrak{H}$, a 2-simplex in $V(G,\mathscr{H})$ will, of course, be a triple, (g_0, g_1, g_2), of elements of G contained in some $H_\alpha x$. We explore this in detail as before. There will be three elements, h_0, h_1, h_2 in H_α with $g_i = h_i x$ for $i = 0, 1, 2$ and thus $g_i g_j^{-1} \in H_\alpha$, for each i and j.

Dividing out by these relations has several neat consequences which 'control' the paths and their compositions. For instance, working in the simplicial set version of $V(G, \mathcal{H})$, if we have $\langle g_0, g_1\rangle$ in $V(G, \mathcal{H})$, then $\langle g_1, g_0\rangle$ is there as well, and so is $\langle g_0, g_0\rangle$ and as $\langle g_0, g_1, g_0\rangle$ is in $V(G, \mathcal{H})_2$, we have that

$$\langle g_0, g_1\rangle\langle g_1, g_0\rangle = \langle g_0, g_0\rangle,$$

so $\langle g_0, g_1\rangle$ has $\langle g_1, g_0\rangle$ as its inverse. Another important result of these relations is that it allows simplification of the path labelling sequences. Suppose we have a composite path

$$g_0 \xrightarrow{h_1} g_1 \xrightarrow{h_2} g_2$$

which stays more than one step in a given coset, *i.e.*, both h_1 and h_2 are in some H_α. In this case we can clearly replace that path, up to homotopy, that is, modulo the relations, by

$$g_0 \xrightarrow{h_1 h_2} g_2$$

as $\langle g_0, g_1, g_2\rangle$ is a 2-simplex. This means that every arrow in $\Pi_1\mathfrak{H}$ has a representative whose corresponding sequence $\underline{h}$ corresponds to an element of the coproduct (aka free product), $\sqcup H_i$, of the groups in $\mathcal{H}$. This is still not a unique representative however. We may have a situation

$$g_0 \xrightarrow{h_1} g_1 \xrightarrow{h_2} g_2 \xrightarrow{h_3} g_3$$

where $h_1, h_2 \in H_i$ and $h_2, h_3 \in H_j$, so we will have an overlap with $\langle g_0, g_1\rangle\langle g_1, g_2\rangle\langle g_2, g_3\rangle$ rewriting both to $\langle g_0, g_2\rangle\langle g_2, g_3\rangle$ and to $\langle g_0, g_1\rangle\langle g_1, g_3\rangle$, and so we have to *amalgamate* the coproduct over intersections.

Let us be a bit more precise about this. We form up a diagram of the subgroups H_i in $\mathcal{H}$, together with their pairwise intersections, $H_i \cap H_j$. We write $H = \underset{\cap}{\sqcup}\mathcal{H}$ for its colimit.

Definition: Given a family, $\mathcal{H}$, of subgroups of G, its *free product* or *coproduct amalgamated along the intersections* is the colimit, H, specified above.

This group, H, can be given a simple presentation. Take as set of generators a set, $X = \{x_g \mid g \in \bigcup H_j\}$, in bijection with the elements of the union of the underlying sets of subgroups in $\mathcal{H}$, and for relations all $x_{h_1}x_{h_2} = x_{h_1 h_2}$ where h_1 and h_2 are both in some group, H_i, of the family.

The inclusion of each H_j into G gives a cocone on the diagram of groups, so induces a homomorphism, $p : \underset{\cap}{\sqcup}\mathcal{H} \to G$, which will be essential in our description. This homomorphism, p, thus takes a sequence $\underline{h} = (h_1, \ldots, h_n)$ representing some element of H and evaluates it within G mapping it to the product $h_1 \ldots h_n \in G$.

Clearly we have

Proposition 32 *The fundamental groupoid, $\Pi_1\mathfrak{H}$, has for objects the elements of G and an arrow from g to g' is representable, uniquely, by an element h in $\underset{\cap}{\sqcup}\mathcal{H}$ such that $g = p(h)g'$.* ∎

The proof is by comparison of the two presentations.

Corollary 6 *There is an isomorphism*

$$\pi_1\mathfrak{H} \cong Ker(p : \underset{\cap}{\sqcup}\mathcal{H} \to G)$$

Proof: The group $\pi_1(V(G,\mathcal{H}),1)$ is the vertex group at 1 of the edge path groupoid, so consists of the $h$in H, which evaluate to 1, since here $g = g' = 1$, *i.e.*, the vertex group is just $Ker\, p$. ■

This means that we have $p : H \to G$, whose 'cokernel', $G/p(H)$, 'is' $\pi_0(V(G,\mathcal{H}))$ and whose kernel is $\pi_1(V(G,\mathcal{H}),1)$.

What about $\pi_2 V(\mathfrak{H})$? We will limit ourselves, here, to a special case, and will merely quote a result from the paper of Abels and Holz, [2]. We suppose as always that we are given $(G,\mathcal{H})$ and now assume that we use the standard presentation $\mathcal{P}_j := (X_j : R_j)$ of each H_j. Combining these we get $X = \bigcup X_j$, $R = \bigcup R_j$. We have $\mathcal{H}$ is 2-generating for G if and only if $\mathcal{P} = (X,R)$ is a presentation of G. (That is nice, since it says that there are no hidden extra relations needed, and that corresponds to the intuitions that we were mentioning earlier. There is better to come!) Assuming that $\mathcal{P}$ is a presentation of G, we have a module of identities, $\pi_{\mathcal{P}}$. We also have all the $\pi_{\mathcal{P}_j}$, the identity modules for each of the presentations, $\mathcal{P}_j$. The inclusions of generators and relations induce morphisms of the crossed modules, $C(\mathcal{P}_j) \to C(\mathcal{P})$, and hence of the modules $\pi_{\mathcal{P}_j} \to \pi_{\mathcal{P}}$, although here there is the slight complication that this is a morphism of modules over the inclusion of H_j into G, which we will not look further into here. We let $\pi_{\mathcal{H}}$ be the sub G-module of $\pi_{\mathcal{P}}$ generated by the images of these $\pi_{\mathcal{P}_j}$. We can think of $\pi_{\mathcal{H}}$ as the sub-module of $\pi_{\mathcal{P}}$ consisting of those identities that come from the presentations of the subgroups.

In the above situation, *i.e.*, with standard presentations for the subgroups, we have ([2] Cor. 2.9.)

Proposition 33 *If $\mathcal{H}$ is 2-generating, then there is an isomorphism:*

$$\pi_2(N(\mathfrak{H})) \cong \pi_{\mathcal{P}}/\pi_{\mathcal{H}}.$$

■

We should therefore, and in this case at least, interpret $\pi_2(N(\mathfrak{H}))$ as telling us about the 2-syzygies that are not due to the presentations of the subgroups. We will give shortly a neat example of this but first would note that this does not interpret the 2-type of $V(\mathfrak{H})$ in general, and that somehow is a lack in the theory as developed so far. Abels and Holz do extend this away from the standard presentations of the subgroups, but this requires a bit more than we have available at this stage in the notes so will be 'put on hold' until later.

This gives all the easily available data on these Vietoris-Volodin complexes as far as their elementary homotopy information is concerned. We can, and will, extract more later on, but now want to look at the main example for their original introduction.

4.3.10 Back to the Volodin model ...

Our 'more complex family' of section 4.3.3 leads to a link with higher algebraic K-theory in the version developed initially by Volodin. The usual approach, however, uses a slightly different notation and for some of its details ends up looking different, so here we will give the version of that example nearer to that given by, for instance, Suslin and Wodzicki, [255], or Song, [248]. Let, as before, R be an associative ring, and now let σ be a partial order on $\{1,\ldots,n\}$. If i is less that j in the partial order σ, it is convenient to write $i \overset{\sigma}{<} j$. (Note that this means that some of the elements may only be related to themselves and hence are really not playing a role in such a σ.) We will write $PO(n)$ for the set of partial orders of $\{1,\ldots,n\}$.

Definition: We say an $n \times n$ matrix, $A = (a_{ij})$ is *σ-triangular* if, when $i \overset{\sigma}{\not\leq} j$, $a_{ij} = 0$, and all diagonal entries, a_{ii} are 1.

We let $T_n^\sigma(R)$ be the subgroup of $G\ell_n(R)$ formed by the σ-triangular matrices.

Lemma 24 *If $n \geq 3$, $T_n^\sigma(R)$ has a presentation with generators $x_{ij}(a)$, where $i \overset{\sigma}{<} j$ and $a \in R$, and with relations:*

$$x_{ij}(a)x_{ij}(b) = x_{ij}(a+b) \qquad\qquad i \overset{\sigma}{<} j, \quad a, b \in R$$

and

$$[x_{ij}(a), x_{jk}(b)] = x_{ik}(ab) \qquad\qquad i \overset{\sigma}{<} j \overset{\sigma}{<} k, \quad a, b \in R,$$

$$x_{ij}(a)x_{k\ell}(b) = x_{k\ell}(b)x_{ij}(a), \quad i \neq \ell, j \neq k, \ a, b \in R.$$

■

Remark: In fact, Kapranov and Saito, [174], mention that, not only is this a presentation of $T_n^\sigma(R)$, but with the addition of the syzygies that they describe (and which up to dimension 2 are those given in our section 4.1.2) gives a complete set of syzygies, of dimension 3.

We can 'stablise' the above, since it σ is a partial order on $\{1,\ldots,n\}$, then it extends uniquely to one on $\{1,\ldots,n+1\}$ by specifying that $n+1$ is related to itself in the extended version, but to no other. (The notation and treatment for this is not itself that 'stable' and some sources do not go into a detailed handling of this point, presumably because it is clear what is going on.) We will write $\mathfrak{T}_n = (G\ell_n(R), \mathcal{T}_n)$, where $\mathcal{T}_n = \{T_n^\sigma(R) \mid \sigma \in PO(n)\}$, and then, letting n 'go to infinity' write $\mathfrak{T}$ for the corresponding system based on $G\ell(R)$ with all σ-triangular subgroups for all partial orders having finite 'support', *i.e.*, in which outside some finite set, (its support), the partial order is trivial.

Proposition 34 *For $n \geq 3$, the subgroup of $G\ell_n(R)$ generated by the union of the $T_n^\sigma(R)$ is $E_n(R)$, the elementary subgroup of $G\ell_n(R)$.*

Proof: This should be more or less clear as, by definition, any elementary matrix is σ-triangular for many σ', and conversely, any $T_n^\sigma(R)$ is given as a subgroup of $E_n(R)$. ■

Corollary 7 *The Volodin nerve, $V(\mathfrak{T})$, has*

$$\pi_0 V(\mathfrak{T}) \cong K_1(R).$$

■

The obvious next question to pose is what $\pi_1(V(\mathfrak{T}), 1)$ will be. We know it to be the kernel of $\underset{\cap}{\sqcup} T_n^\sigma(R) \to E(R)$, and the obvious guess would be that it was Milnor's $K_2(R)$. That's right. Proofs are given in several places in the literature, but usually they require a bit more machinery than we have been assuming up to this point in these notes, so we will not give one of those proofs here. The most usual proofs use the natural action of G on $N(\mathfrak{H})$ and a covering space argument.

We will mention this in a bit more detail after we have looked at a sketch proof and will explore aspects of this sort of approach more in a later chapter, but here will attempt to give that sketch proof which, it is hoped, seems more direct and which starts from the descriptions of $\pi_0 V(\mathfrak{T})$ that are consequences of what we have already done above. (We will still need a covering space-type argument, which, since central extensions behave like covering spaces from many points of view, is suggestive of a general approach that is, it seems, nowhere given in the literature with the conceptual simplicity it seems to deserve. Kervaire's treatment of universal central extensions, [178], perhaps goes some way towards what is needed.) We start by looking at paths in $V(\mathfrak{T})$, especially, but not only, those which start at 1. We will be, in part, following Volodin's original treatment in [272] as this is very elementary and 'constructive' in nature. As we said above, he uses covering space intuitions as well, as this seems almost optimal for the identification we need. (Remember that one classical construction of universal covering spaces is from the space of paths that start at the base point, followed by quotienting by fixed end point homotopy as a relation.)

A path in $V(\mathfrak{T})$ as it is of finite length, must live in some $V(\mathfrak{T}_n)$. We thus can represent it by a pair, $(g, \underline{t})$, with $\underline{t} = (t_1, \ldots, t_k)$ for some k, a word with each t_i in some $T_n^{\sigma_i}(R)$, and g in $E_n(R)$ which will be the starting element of the path. (Of course, this representation is not unique, because of the amalgamated subgroups, and we will need to break each t_i up as a product of elementary matrices shortly. The non-uniqueness will be taken account of later on.)

We say that t_i is a *segment* of the path, and that the paths is *elementary* if all the t_is used are elementary matrices.

We now need some 'elementary' linear algebra. We will look at it with respect to the standard maximal linear order on $\{1, \ldots, n\}$ and hence for upper triangular matrices.

Lemma 25 *Let $B = (b_{ij})$ be an upper triangular matrix (with 1s on its diagonal), so b_{ij} is zero if $j < i$. There is a factorisation*

$$B = \prod_{(i,j)} e_{ij}(b_{ij}),$$

with the order of multiplication given by increasing lexicographic order, so $(i, j) > (i_1, j_1)$ if either a) $j > j_1$ or b) $j = j_1$ and $i > i_1$. ■

The proof should be obvious.

We can replace t_k by a path consisting only of elementary matrices (for the ordering σ_i) and with the order of terms given by a lexicographic order in the (i, j)s relative to $\overset{\sigma_i}{<}$. The resulting $t_k = \prod_{(i,j)} e_{ij}(b_{ij})$ and can be 'lifted' to an element

$$\bar{t}_k = \prod_{(i,j)} e_{ij}(x_{ij}) \in St_n(R).$$

This element maps down to the element, t_k, in $E_n(R)$.

Suppose s is a loop, based at 1, in $V(\mathfrak{T})$, but consisting just of elementary matrices in some $T_n^{\sigma_k}(R)$. (We will say s is an elementary loop. We will work with the standard linear order.) As s is a loop at 1, it has a representation as $(1, \underline{s})$, where $\underline{s} = (s_1, \ldots, s_N)$ and the s_ks are in lexicographic order, each s_k is some $e_{ij}(a_i j)$ and, as the path s is a loop, $\prod_{(i,j)} e_{ij}(a_{ij}) = 1$.

Lemma 26 *If s is an elementary loop at 1 in $T_n(R)$, then its lift $\overline{s}$ is $1 \in St_n(R)$.*

Before giving a proof, remember the intuition that seems to be built in Volodin's approach. The $T_n^\sigma(R)$ are seen as patches over which there is a way of lifting paths, so you decompose a long path into bits in the various patches, and then lift them successively. The lifted bits give elements in $St_n(R)$, and 'up there' we have divided out by the homotopy that comes from the relations / rewriting 2-cells. In each patch we expect to get that the lift of s that we are using gives a trivial element (*i.e.*, something like a null-homotopic loop. We thus expect to have to use the presentation of $St(R)$ and, in particular, the embryonic homotopies given by the rewriting 2-cells / relations. As we will see that is exactly what happens.

Proof: We let m be larger than all the i, j involved in the expression for s. (We will generally write $x_{ij}(a)$ etc where a is variable and is really just a 'place marker'.) As $x_{im}(a)x_{kj}(b) = x_{kj}(b)x_{im}(a)$ for $i \neq j$, $k \neq m$, and

$$x_{im}(a)x_{ki}(b) = x_{km}(-ab)x_{ki}(b)x_{im}(a) = x_{ki}(b)x_{km}(-ab)x_{im}(a),$$

we can move all terms of form $x_{im}(a)$ to the right of the product expression for $\overline{s}$. In $St_m(R)$, we thus have

$$\prod_{i<j\leq m} x_{ij}(a) = \prod_{i<j\leq m-1} x_{ij}(a) \cdot \prod_{i<m} x_{im}(a),$$

where, as we said, the a is just a place marker. We thus have that $\overline{s}$ in $St(R)$ can be decomposed as the product of two parts corresponding to loops (down in $E(R)$). These are $\prod_{i<j\leq m-1} x_{ij}(a)$ and $\prod_{i<m} x_{im}(a)$. (As this latter is in the subgroup of $St_m(R)$ generated by the $x_{im}(a)$, this must itself evaluate to 1 as the product does, hence also the other factor must.) Working on the product $\prod_{i<m} x_{im}(a)$ and using the facts firstly that the terms commute with each other by the first rule we recalled above, and then using the first Steinberg relation: $St1 : x_{im}(a)x_{im}(b) = x_{im}(a + b)$, we can now check that this word must itself be trivial as it evaluates to 1.

We now can use backwards induction on m to gradually you get back to the minimal value possible and get the result. ■

Corollary 8 *If s is an elementary loop in some $T_n^\sigma(R)$, then the corresponding lifted word in $St(R)$ is trivial.*

Proof: We have done most of this, except it was in the case of the standard linear order. One can either adapt the above to the general case, or more neatly note that s conjugates, using permutation matrices, to give an element in that linear case. The lifting goes across to $St(R)$ and so the result follows after a bit of checking. ■

Now look at any path in $V(\mathfrak{T})$, starting at 1. Take an elementary representative and examine the initial segment, $1 \xrightarrow{t_1} t_1^{-1}$, so $t_1 \in T_n^{\sigma_1}(R)$. We can lift t_1 to give an element $\overline{t}_1 \in St_n(R)$. This will, in general, depend on the choice of σ_1, but if σ_1' is another possible partial order (*i.e.*, $t_1 \in T_n^{\sigma_1}(R) \cap T_n^{\sigma_1'}(R)$, then the resulting two lifts of t_1 will form a 'loop' $\overline{t}_1 \cdot \overline{t'}_1^{-1}$ in $St_n(R)$, but then this loop must be trivial by the lemma and its corollary. We pass to the next 'node' in the path and continue. The next segment does not start at 1, but the argument adapts easily as the corresponding labelling element in the coproduct with amalgamation is all that is used.

This gives that each path s in $V(\mathfrak{T})$ uniquely determines an element $\overline{s}$ in $St(R)$. It is now fairly clear where the argument has to go. The standard classical construction of a universal covering space is via paths starting at some base point 'modulo' fixed endpoint homotopy, so one checks that homotopic paths lift to the same element of $St(R)$. (This is Volodin's Lemma 3.4 of [272], but it is easy to see how it is to go.) Volodin is using the 'patches' given by the $T_n^\sigma(R)$ to lift a path in $E_n(R)$. (This mix of topological intuition with combinatorics and algebra is the starting point of Bak's theory of global actions, [17, 18], that was mentioned earlier.)

It is now feasible to complete the proof à la Volodin, that the universal cover of $V(E_n(R), \{T_n^\sigma(R)\})$ is 'related to' $St_n(R)$, but that is not really satisfactory as it mixes the categories in which we are working. (A simplicial complex is not a group!) We have a more limited aim, namely to note that if we have an element in $\pi_1(V(\mathfrak{T}), 1)$, then we can pick a loop, s, representing it. We can lift s uniquely by lifting over each 'patch' $T_n^\sigma(R)$ that it uses, to obtain an element in $St(R)$, but as it is a loop its evaluation, back down in $G\ell(R)$ will be trivial. (Topologically its endpoint is over the basepoint!) It is in the kernel of the homomorphism from $St(R)$ to $G\ell(R)$, so determines an element of $K_2(R)$. Finally one reverses the argument to say that if $\overline{s} \in K_2(R)$, then it is in the image of this morphism. We have thus given an idea of how Volodin's theorem, below, can be proved, using fairly elementary ideas.

Theorem 7 *(Volodin, [272], Theorem 2)*

$$\pi_1(V(\mathfrak{T}), 1) \cong K_2(R).$$

■

Remark: The usual proofs of this result given in more recent sources tend to use the classifying spaces, $BT_n^\sigma(R)$ together with the induced mappings to $BG\ell(R)$ to obtain

$$\bigcup BT_n^\sigma(R) \to BG\ell(R),$$

which is then shown to give the 'homotopy fibre' of the map to $BG\ell(R)^+$. This does seem slightly too reliant on spatially based methods from homotopy theory and a more purely combinatorial group theoretic or 'rewriting' analysis of the constructions, related to Volodin's original proof, should be possible.

We hope to return to the study of the Volodin model for higher algebraic K-theory later on, but are near to the limit of what can be done with the limited tools at our disposal here, so will put it aside for the moment.

4.3.11 The case of van Kampen's theorem and presentations of pushouts

The above example / case study coming from algebraic K-theory is very rich in its structure and its applications, but *is* complex, so we will return to a simpler situation to indicate the direction that this theory of 'higher generation by subgroups' can lead us to. To motivate this recall the formulation of the classical form of van Kampen's theorem.

Theorem 8 *(van Kampen) Let $X = U \cup V$, where U, V and $U \cap V$ are non-empty, open and arc-wise connected. Let $x_0 \in U \cap V$ be chosen as a base point, then the diagram*

$$\begin{tikzcd}
\pi_1(U \cap V) \arrow[r, "j_{V*}"] \arrow[d, "j_{U*}"'] & \pi_1(V) \arrow[d, "i_{V*}"] \\
\pi_1(U) \arrow[r, "i_{U*}"'] & \pi_1(X)
\end{tikzcd}$$

is a pushout square of groups, where each fundamental group is based at x_0. ■

Proofs can be found in many places in 'the literature', for instance, in Massey's introduction, [197], or in Crowell and Fox, [96]. A proof of a neat more general form of the result is given in Brown's book, [59]. There the result is given in terms of fundamental groupoids, which is very useful for many applications and several variants are also given there. We may have need for some of these later on, but for the moment what we want is the version in terms of group presentations, cf. [96], page 71, for example. This just translates the above pushout result into one about presentations.

Theorem 9 *(van Kampen: alternative form) Let* $X = U \cup V$, *etc., be as above. Suppose*

- *that* $\pi_1(U, x_0)$ *has a presentation,* $(\mathbf{X} : \mathbf{R})$,
- *that* $\pi_1(V, x_0)$ *has a presentation,* $(\mathbf{Y} : \mathbf{S})$,

and

- *that* $\pi_1(U \cap V, x_0)$ *has one,* $(\mathbf{Z} : \mathbf{T})$,

then $\pi_1(X, x_0)$ *has a presentation,*

$$(\mathbf{X} \cup \mathbf{Y} : \mathbf{R} \cup \mathbf{S} \cup \{(\overline{j_{U*}(z)})(\overline{j_{V*}(z)})^{-1} \mid z \in \mathbf{Z}\}),$$

where $\overline{j_{U*}(z)}$ *is a word in the free group,* $F(\mathbf{X})$ *representing* $j_{U*}(z)$, *and similarly for* $\overline{j_{V*}(z)}$. ■

This form gives a way of calculating a presentation, $\mathscr{P}$, of $\pi_1(X, x_0)$ given presentations of the parts. If we see a presentation as the first part of a recipe to construct a resolution of a group, or alternatively to construct an Eilenberg-Mac Lane space for the group, then this is useful and, of course, is used in courses on elementary algebraic topology to calculate the fundamental groups of surfaces. The obvious points to note are that we take the union of the two generating sets, $\mathbf{X}$ and $\mathbf{Y}$, to be the generating set of $\pi_1(X, x_0)$, but use the generators in $\mathbf{Z}$ to help form *relations* in the pushout presentation, then we use the union of the two sets of relations to give the other relations (which seems sort of natural). This leaves a query. Whatever happened to the relations in the presentation of $\pi_1(U \cap V, x_0)$? To get some idea of what they do, think along the following somewhat vague lines. As those relations correspond to maps of 2-discs into the complex, $K(\mathscr{P})$, of the presentation, $\mathscr{P}$, used to 'kill' the corresponding words, we have two 2-discs with 'the same' boundary and hence map of a 2-sphere into $K(\mathscr{P})$ with no reason for it being homotopically trivial. This suggests that the relations in $\mathbf{T}$ are going to give homotopical 2-syzygies, and this is the case. It also suggests that to build an Eilenberg-MacLane / classifying space from the presentation, $\mathscr{P}$, we could do worse than take the pushout of the complexes of the various other presentations involved.

It is a good idea to abstract this out a bit away from the van Kampen situation for the moment. We suppose that $G = A *_C B$ is a 'free product with amalgamation', so we can describe G by means of a pushout of groups:

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "i"'] & B \arrow[d, "i'"] \\ A \arrow[r, "j'"'] & G \end{tikzcd}$$

It is a standard result that if i and j are injective, then so are i' and j'.

The van Kampen examples can be a bit complex to work through, but we can, in fact, gain some intuition about them from one of the simplest examples of such situations. Consider the trefoil knot group, $G(T_{2,3})$. This has a presentation $(a, b : a^3b^{-2} = 1)$. It is therefore an amalgamated coproduct / pushout of three infinite cyclic groups:

$$\begin{tikzcd}
(z : \emptyset) \arrow[r, "j"] \arrow[d, "i"'] & (b : \emptyset) \arrow[d] \\
(a : \emptyset) \arrow[r] & G(T_{2,3})
\end{tikzcd}$$

where $i(z) = a^3$ and $j(z) = b^2$. We note that all the input presentations are with empty sets of relations, yet $G(T_{2,3})$ has a single non-trivial relation. If we took the complexes of each presentation, we would merely have a circle for each, and that of the presentation of $G(T_{2,3})$ has to have a 2-cell in it, hence we can see that the construction of the presentation of $G(T_{2,3})$ does not just result from a 'pushout of presentations'! (In fact, what is needed is a homotopy pushout, or, in more general situations than the pushout of a diagram of group, a homotopy colimit. We will say a bit more on this shortly.) We now return to our general situation.

Our abstracted situation is that we have presentations, $\mathcal{P}_Q = (X_Q : R_Q)$ for $Q = A, B$ and C, and get the corresponding presentation for G, given by the analogue of that in the above discussion. We take complexes $K(\mathcal{P}_Q$ modelling each of the presentations in turn. The morphisms between the groups in the diagram lift give a diagram

$$\begin{tikzcd}
K(\mathcal{P}_C) \arrow[r, "j_*"] \arrow[d, "i_*"'] & K(\mathcal{P}_B) \arrow[d, "i'_*"] \\
K(\mathcal{P}_A) \arrow[r, "j'_*"'] & K(\mathcal{P}_G)
\end{tikzcd}$$

but as the lifts have to be *chosen*, they are only determined up to homotopy, and this will in general only be a square that is homotopy coherent, *i.e.*, commutative up to a specified homotopy, (see the later discussion in Chapter 11). In fact, as we do not know that i_* and j_* are injective, the result need not be a pushout, so does not tell us much. An alternative is to see what we can construct from the 'corner':

$$\begin{tikzcd}
K(\mathcal{P}_C) \arrow[r, "j_*"] \arrow[d, "i_*"'] & K(\mathcal{P}_B) \\
K(\mathcal{P}_A) &
\end{tikzcd}$$

from this we can take its 'homotopy pushout' which begins to be more like the square we had. We have not met this construction yet; it is a double mapping cylinder. This would form a cylinder on $K(\mathcal{P}_C)$ and use the maps to glue copies of the other spaces to its two ends. In here, we will be getting a cylinder with the discs corresponding to the relations in $\mathcal{P}_C$ and these will to cylindrical 2-cells in that double mapping cylinder and hence to a potential homotopical 2-syzygy. This will be picked up by the crossed module of that space or better still the crossed complex. An analysis of this can be found in Brown-Higgins-Sivera, [64], starting on page 338. This is based on an earlier paper by Brown, Moore, Porter and Wensley, [68]. (As an exercise, it is **worth looking at the trefoil group from this viewpoint** and to draw what intuitively the mapping cylinder must look like ... as much as this is feasible.)

We have used this discussion above for two main reasons, first to suggest that the situation *naturally* leads to having to take the homotopies seriously and that implies a study of (at least some) homotopy coherence theory, and homotopy colimits in particular. The other reason is that it suggests that it provides a key set of concepts, as yet at a vague intuitive level, to understand more fully the theory of 'higher generation by subgroups' of Abels and Holz, [2]. If we get our group G, and a 1-generating family of subgroups, $\mathscr{H}$, and want to work out the 'syzygies of G', *i.e.*, some combinatorial information to enable a (crossed) resolution or a small model of a $K(G,1)$ to be formed, then the idea is that by calculating the syzygies of each of the input groups, the n-syzygies of G should involve those of the H_is, but also the $(n-1)$-syzygies of the pairwise intersections, $H_i \cap H_j$, and then, why not, the $(n-2)$-syzygies of the triple intersections, and so on. We certainly do not have the machinery to pursue this here, and so will leave it vague, at least for now.

(In addition to the above references on the pushout, which use homotopy colimits of crossed complexes over groupoids, the original paper of Abels and Holz, [2], also uses homotopy colimit techniques, but this time with chain complexes. It uses these to prove results on the homological finiteness properties of certain groups. That paper is well worth reading. This use of homotopy colimits is also explored in Stephan Holz's thesis, [157].)

4.4 Group actions and the nerves

Although we will be continuing the theme of the previous section, we will be expanding out our view of the area slightly so as to gain additional tools for the study of the nerve and those Vietoris -Volodin complexes, but also some more interesting potential applications.

Further information on the nerve, $N(\mathfrak{H})$, and the Vietoris-Volodin complex, $V(\mathfrak{H})$, of the coset covering corresponding to a family, $\mathscr{H}$, of subgroups of G, can be obtained by exploiting the natural action of G on these simplicial complexes. This leads to a further connection of these objects to simple examples of key ideas from geometric group theory via the notion of a complex of groups due to Haefliger, [145, 146] and Corson, [90–92].

4.4.1 The G-action on $N(\mathfrak{H})$

Let G be a group, and, as before, $\mathscr{H} = \{H_i \mid i \in I\}$, a family of subgroups of G. Usuall we will assume that $\mathfrak{H} = (G, \mathscr{H})$ is 1-connected, so that G is generated by the union of the subgroups in $\mathscr{H}$. We, again as before, write $N(\mathfrak{H})$ for the nerve of the covering of G formed by the right cosets of the H_is and $V(\mathfrak{H})$ for the corresponding Vietoris complex / Volodin space.

There is an obvious right G-action on $N(\mathfrak{H})$ given as follows:
An n-simplex of $N(\mathfrak{H})$ has the form

$$\sigma = \{H_{\alpha_0}x_0, \dots, H_{\alpha_n}x_n\},$$

where

$$\bigcap \sigma = \bigcap_{k=0}^{n} H_{\alpha_k} x_k$$

is non-empty. If we have $g \in G$, we define $\sigma \cdot g = \{H_{\alpha_0} x_0 g, \ldots, H_{\alpha_n} x_n g\}$ multiplying each coset representative on the right by g.

It is easy to see that if $y \in \bigcap \sigma$, then $yg \in \bigcap(\sigma \cdot g)$, so $\sigma \cdot g$ is an n-simplex of $N(\mathfrak{H})$.

This action has various very nice features. The terminology on these is a bit of a minefield. The action is 'without inversion' in the sense of Haefliger, [145], (but, in his later paper, he changed terminology and redefined that term). That type of action is also called 'regular' by Abels and Holz, [2], but 'regular' is used for another type of condition in several other articles. We will define both these slightly later. (This confusing situation is not that serious for us as the action we have satisfies all the different variants, so for what we need we do not have to worry about which form is being used or if they are equivalent conditions. When we need to consider a more general case, we will take the approach used in Bridson and Haefliger, [54], and so will avoid the potential confusion.)

Proposition 35 *If $\sigma = \{H_{\alpha_0} x_0, \ldots, H_{\alpha_n} x_n\}$ is a simplex in $N(\mathfrak{H})$, there is an element g of G such that $\sigma \cdot g = \sigma_0 := \{H_{\alpha_0}, \ldots, H_{\alpha_n}\}$.*

Proof: As σ is a simplex of $N(\mathfrak{H})$, $\bigcap \sigma$ is non-empty . Let $h \in \bigcap \sigma$, then $h \in H_{\alpha_0} x_0$, so has the form $h = h_0 x_0$ for some $h_0 \in H_{\alpha_0}$. It equally well has the form $h = h_i x_i$ for some $h_i \in H_{\alpha_i}$. Now take $g = h^{-1}$, then we have $H_{\alpha_i} x_i g = H_{\alpha_i}$ for all i, that is, $\sigma \cdot g = \sigma_0$. ■

We will sometimes refer to the σ_0 determined as here, as the basic 'supporting' simplex of σ. We will be using this slightly later on.

One of the definitions of 'regular action' is as follows (cf. Corson, [93]):

Definition: Suppose the G act simplicially on a simplicial complex, K, then the action is *regular* and we say that K is a *regular G-complex* if given elements $g_0, \ldots, g_n \in G$ and a simplex, $\sigma = \{v_0, \ldots, v_n\}$ of K such that $\tau = \{v_0 g_0, \ldots, v_n g_n\}$ is also a simplex of K, then there is a single element, $g \in G$ such that $\sigma \cdot g = \tau$.

We will see, a bit later on, that this definition as stated hides a difficulty in its interpretation. That difficulty does not, however, occur in this case, so we will ignore it for the moment.

Corollary 9 *The action of G on $N(\mathfrak{H})$ is regular.*

Proof: Suppose $\sigma = \{H_{\alpha_0} x_0, \ldots, H_{\alpha_n} x_n\}$, and $g_0, \ldots, g_n$ are in G, then the τ that we get, according to the definition, will be $\{H_{\alpha_0} x_0 g_0, \ldots, H_{\alpha_n} x_n g_n\}$.

Let us write $\sigma_0 = \{H_{\alpha_0}, \ldots, H_{\alpha_n}\}$ for the basic 'supporting' simplex of σ

Using the proposition, we have there is a $h_1 \in G$ such that $\sigma \cdot h_1 = \sigma_0$. Similarly we have $h_2 \in G$ such that $\tau \cdot h - 1 = \sigma_0$, as the basic supporting simplex of τ is once again σ_0 as it involves exactly the same subgroups from $\mathscr{H}$. Now it is easy to take $g = h_1 h_2^{-1}$ to get $\sigma \cdot g = \tau$, as required. ■

We note that if we needed to write down such a g explicitly, we need only pick h_1 and h_2 according to the recipe in the proof of the proposition.

As we said before, Abels and Holz, [2], use 'regular' in a seemingly different way, one that is the same as 'action without inversion' of Haefliger, [145].

Definition: An *action without inversion of a group, G,* on a simplicial complex, K, is an action by simplicial automorphisms such that if $\sigma = \{v_0, \ldots, v_n\}$ and $\sigma \cdot g = \sigma$, then $v_i g = v_i$ for all $i \in \{0, \ldots, n\}$.

Proposition 36 *The action of G on $N(\mathfrak{H})$ is without inversion. In more detail, if*

$$\sigma = \{H_{\alpha_0}x_0, \ldots, H_{\alpha_n}x_n\}$$

is a simplex of $N(\mathfrak{H})$, as before, and $g \in G$ is such that $\sigma \cdot g = \sigma$, then g fixes all the cosets, $H_{\alpha_i}x_i$, that is to say, $H_{\alpha_i}x_i g = H_{\alpha_i}x_i$ for all $i \in I$.

Proof: As we are working with the *simplicial complex*, $N(\mathfrak{H})$, (and not with a corresponding simplicial set), all the H_{α_i} must be distinct (since otherwise the simplex would have empty intersection). Now suppose $\sigma \cdot g = \sigma$, then $H_{\alpha_i}x_i g = H_{\alpha_k}x_k$ for some k, and it is then immediate that H_{α_i} and H_{α_k} must be equal. ■

Stabilisers: One of the ways to understand an action is via its stabiliser subgroups. If we have a simplex, $\sigma = \{H_{\alpha_0}x_0, \ldots, H_{\alpha_n}x_n\}$, it is now quite easy to work out the stabiliser of σ. First, for convenience, recall that

$$Stab_G(\sigma) = \{g \mid \sigma \cdot g = \sigma\}.$$

If $\sigma \cdot g = \sigma$, then $H_{\alpha_i}x_i g = H_{\alpha_i}x_i$, so $g \in x_i^{-1}H_{\alpha_i}x_i$. This does not seem to be that useful, *but* we can take all the x_is to be the sme, and that then gives a complete answer. Let us backtrack a bit. We have that if $a \in \bigcap \sigma$, then $\sigma = \sigma_0 \cdot a$, so if $\sigma \cdot g = \sigma$, $aga^{-1} \in Stab_G(\sigma_0)$. It therefore remains to ask when $H_{\alpha_i}g' = H_{\alpha_i}$ for all i and clearly this is when $g' \in \bigcap \sigma$. We thus have

Proposition 37 *For $\sigma \in N(\mathfrak{H})$ and $a \in \bigcap \sigma$,*

$$Stab_G(\sigma) = a^{-1} \bigcap \sigma a.$$

■

Note that this is independent of the choice of a in $\bigcap \sigma$.

The 'space' of orbits, $N(\mathfrak{H})/G$: Suppose $\mathscr{H}$ is a *finite* family of subgroups of G, then we have a special maximal dimensional simplex in $N(\mathfrak{H})$, namely the family $\mathscr{H}$ itself. If $\mathscr{H}$ has n members then the dimension of this simplex will be $n - 1$. This acts like a fundamental domain of a group action, say on the plane, so we will call it the *fundamental domain simplex* of $N(\mathfrak{H})$. It will usually be denoted σ_0. We have:

Proposition 38 *If $\mathscr{H}$ has n elements, then $N(\mathfrak{H})/G$ is an $(n-1)$-simplex.* ■

Examples: We will look back at some of the examples of nerves that we have given before. In many cases, it is possible to see the group action on the nerve quite clearly and to illustrate the way in which each maximal dimensional simplex is a 'translate' of the fundamental domain simplex.

1. $G = S_3, H_1 = \{1, (1\ 2\ 3), (1\ 3\ 2)\}, H_2 = \{1, (1\ 2)\}$, as on page 125. The nerve $N(\mathfrak{H})$ in this case is the graph given in example 4.3.1 with vertices

$$\begin{array}{ccccc} & H_1 & & H_1b & \\ H_2 & & H_2a & & H_2a^2 \end{array}$$

   (where, as there, $a = (1\ 2\ 3)$, $b = (1\ 2)$). The action is given by: a fixes H_1 and H_1b and permutes the cosets of H_2 in the obvious way; b permutes H_1 and H_1b *and* H_2a and H_2a^2, but fixes H_2 (of course). On 1-simplices, the action follows on an edge is determined by what it does to the two ends. so, for instance,

$$a \in H_1 \cap H_2a \quad \text{so} \quad H_1a^{-1} \cap H_2aa^{-1} = H_1 \cap H_2 \neq \emptyset$$

   and so on. It is thus easy to see directly that $N(\mathfrak{H})/S_3 \cong \Delta[1]$.

   As to the stabilisers: on vertices,

$$Stab_{S_3}(H_1) = H_1;$$
$$Stab_{S_3}(H_1b) = b^{-1}H_1b = H_1,$$

   and note that, in this case, $\sigma = \{H_1b\}$, so $b \in \bigcap \sigma$, and the result is as predicted by the proposition on stabilisers.

   On 1-simplices, as $\sigma_0 = \{H_1, H_2\}$, this has trivial stabiliser, hence so do all 1-simplices.

2. $G = K_4 = \{1, a, b, c\}$ with $\mathscr{H} = \{H_a, H_b, H_c\}$, where $H_x = \langle x\rangle$, (cf. page 138). Here $N(\mathfrak{H}_{K4})$ is the octahedral shell with 4 faces removed.

   Using the same notation as before: a fixes 1 and 2, permutes 3 and 4, and also 5 and 6, so in the diagram in example 4.3.9, a corresponds to a rotation through 180° about the vertical axis. Similarly for b and c, but about the two horizontal axes. The orbit space is $\Delta[2]$ as this example has 3 subgroups.

   We will leave the determination of the stabilisers **to the reader**.

3. $G = q8$, (cf. page 139): $N(\mathfrak{H}_{q8})$ is as in the previous example and has the action of $q8$ given via the quotient homomorphism to K_4 and the action outlined before in 2. Of course, $N(\mathfrak{H}_{q8})/q8$ is again a 2-simplex. (Again the stabilisers are **left to you**. It is interesting to reflect on the relationship between the stabilisers here and in that previous example.)

4. S_4 with three subgroups, $H_1 = \langle(1, 2), (2, 3)\rangle$, $H_2 = \langle(2, 3), (3, 4)\rangle$ and $H_3 = \langle(1, 2), (3, 4)\rangle$, as on page 139. Of course, $N(\mathfrak{H})/S_4$ is a 2-simplex. The stabilisers are not too difficult to calculate.

5. For our last example, for the moment, we will consider a 'generic' one and take up the example of an amalgamated 'free product' / coproduct from page 140. We will write $G = A \sqcup_C B$, where $C = A \cap B$. (The change in notation is for convenience of typing and has no significance.) The

family of subgroups is $\mathscr{H} = \{A, B\}$ or more exactly the images of A and B in G, so the cosets have form Aw_A and Bw_b. If we think about the coset representatives w_A may be assumed to start with a b that is not in C, and similarly in the w_B, we may assume that it starts with an a not in C. The nerve, $N(\mathfrak{H})$, is a bipartite graph and is, in fact, a tree as each w_A or w_B provides a unique direct path back to A or B. For instance,

$$A \longrightarrow Ba_2 \longrightarrow Ab_2a_2 \longrightarrow Ba_1b_2a_2 \longrightarrow Ab_1a_1b_2a_2$$

whilst A is linked to B.

The action is fairly easy to visualise and the orbit space is a 1-simplex, of course.

What about stabilisers? The result we showed earlier reduces the problem to looking at the stabilisers of the vertices of the 'fundamental domain' simplex. These we will denote simply by $v_A = \{A\}$, $v_B = \{B\}$ and, the edge joining them, $\sigma_0 = \{v_a, v_B\}$.

$$Stab_G(v_A) = \{g \mid Ag = A\},$$

so is simply A, and similarly $Stab_G(v_B)$ is B, whilst

$$Stab_G(\sigma_0) = \{g \mid \sigma_0 \cdot g\sigma_0\} = C.$$

If we draw the fundamental domain simply labelled by the corresponding stabilisers, we get (surprise, surprise!)

$$A \underset{A\cap B}{\longrule} B\ ,$$

which is the usual picture for a graph of groups, (cf. Serre, [245, 246]) and which we saw earlier on page 140.

This last example suggests the generic case for more subgroups should be related to some 'simplex of groups' or more generally to 'complexes of groups', and, of course, that is what we will be discussing shortly.

4.4.2 The G-action on $V(\mathfrak{H})$

There is, of course, an equally natural group action of G on $V(\mathfrak{H})$, but its properties are not so 'combinatorial'.

If $\sigma = \{g_0, \ldots, g_n\}$ is an n-simplex of $V(\mathfrak{H})$ and $g \in G$, then the 'obvious' guess for $\sigma \cdot g$ would be $\{g_0g, \ldots, g_ng\}$. (We saw and used the corresponding left G-action earlier when we first introduced Volodin spaces on page 129.) It is immediate that this is a simplex in $V(\mathfrak{H})$, since there is some coset $H_\alpha x$ containing σ as a subset, and $\sigma \cdot g \subseteq H_\alpha xg$.

Where this action is different from that on $N(\mathfrak{H})$ is that it is not '*without inversion*' in general. For instance, if one of the subgroups of $\mathscr{H}$, say $H_1 \subset G$ is a subgroup of order 2, $H_1 = \{1, a\}$ with $a^2 = 1$, then $\sigma = \{1, a\}$ is a 1-simplex of $V(\mathfrak{H})$, but $\sigma \cdot a = \sigma$, whilst a moves both vertices of σ.

This 'irregular' behaviour is not a worry as any simplicial action can be made regular by passing to a barycentric subdivision.

We saw that $V(\mathfrak{H})$ was related to the simplicial set, $E(G)$, which had a G-action whose quotient gave $Ner(G[1])$, the nerve, or simplicial classifying space, BG, of G. We have, for any coset $H_\alpha x$, a sub-simplicial set of $V(\mathfrak{H})$ consisting of those simplices that are within $H_\alpha x$. This is really just a copy of $E(H_\alpha x)$ and is isomorphic to $E(H_\alpha)x$, which, in turn, is isomorphic to $E(H_\alpha)$. What about the action? How does it operate on these parts? For ease of analysis let us suppose we make a choice of a set of coset representatives for H_α in G, then given $g \in G$, we can write $g = hx$ for $h \in H_\alpha$ and for some (chosen) coset representative, x. Now if we have $\sigma \in E(H_\alpha) \subseteq V(\mathfrak{H})$, $\sigma \cdot g = (\sigma \cdot h) cdotx$, so we can think of the action of g as consisting of a part that shifts σ around within $E(H_\alpha)$, followed by a part that 'translates' $E(H_\alpha)$ to $E(H_\alpha)x$. If $\sigma \in E(H_\alpha)y$ to start with then $\sigma \cdot y^{-1} \in \sigma \in E(H_\alpha)$ so we write $yg = hx$ and use the simpler analysis above.

In other words, the action can be thought of as being partially within each $E(H_\alpha)x$ and partially as permuting the different $E(H_\alpha)x$s amongst themselves. This seems, of course, very 'H_α-centric', *i.e.*, seen from the viewpoint of H_α and its cosets, but, as the action is defined the same way irrespective of where one is in $V(\mathfrak{H})$, the different viewpoints are compatible. (Here a detailed treatment would be simpler if $\mathscr{H}$ is closed under intersection, but is not to difficult **to write down** in any case.)

If we now look at $V(\mathfrak{H})/G$, we can easily check that:

Proposition 39 *The inclusion of $V(\mathfrak{H})$ into $E(G)$ induces an isomorphism*

$$V(\mathfrak{H})/G \xrightarrow{\cong} \bigcup E(H_\alpha)/H_\alpha = \bigcup BH_\alpha,$$

between $V(\mathfrak{H})/G$ and the union of the classifying spaces, $BH_\alpha = Ner H_\alpha[1]$, within BG. ■

Remarks: (i) This result is to be found in Suslin and Wodzicki's treatment, [255], of Volodin spaces.

(ii) Just as $\bigcup H_\alpha$ will not usually be a subgroup of G, in general, $\bigcup BH_\alpha$ will not be a Kan complex (within BG which *is* a Kan complex). For instance, in a 2-horn the two given edges may be in different groups of the family, $\mathscr{H}$, so the filler (within BG) will not necessarily be in $\bigcup BH_\alpha$.

4.4.3 Group actions on simplicial complexes

We have been using some ideas on actions of groups on simplicial complexes. For future use, we need this in a bit more depth and generality than merely on the nerves of coverings by cosets. We will not give full details, but need to discuss the regularity conditions that we have already met.

Suppose, as ever, that G is a group and K is a simplicial complex with vertex set V_K and with S_K as its poset of simplices. We have group of simplicial automorphisms, $Aut(K)$ (not to be confused with the *simplicial group* of automorphisms of the simplicial set K^{simp}). An action of G on K is, by definition, a homomorphism from G to $Aut(K)$, so it is a *simplicial action* or an *action by simplicial automorphisms*. The regularity conditions are needed to help ensure that 'obvious' quotienting operations are well behaved.

Remark: From some points of view, some of the problems that we will be examining are partially obscured by our use of group actions rather than converting those actions into some sort of 'action groupoid' as we introduced in section 1.1.1 for the simpler case of a group acting on a set. That viewpoint is highly relevant and will be taken on board in more detail later on, however the links with more 'traditional' viewpoints are also very important not only as the allow transfer of results and ideas between the differently focussed ways of seeing a particular area but also as a source of examples, interpretation and intuition.

Example: The most obvious simple example is the 1-simplex, Δ^1, with the C_2 action that flips the interval about, so let us set this up a bit formally. Let $G = C_2$, the cyclic group of order 2, which we will write as $\{1, a\}$ where, of course, $a^2 = 1$. We take $K = \Delta^1$ so with vertex set, $\{0, 1\}$ and in which the action is given by $0a = 1$, which immediately implies $1a = 0$. The 1-simplex, $\sigma = \{0, 1\}$ is fixed by a, but clearly the individual vertices are not, so this is an action that is not 'without inversion' (see page 152 for the definition). The action is not 'regular' either, and here we meet the slight problem that we mentioned back on page 151 when we introduced the term 'regular'. The problem is one of interpretation. That definition uses the phrasing '*if given elements* $g_0, \ldots, g_n \in G$ *and a simplex,* $\sigma = \{v_0, \ldots, v_n\}$ *of* K *such that* $\tau = \{v_0g_0, \ldots, v_ng_n\}$ *is also a simplex of* K'. In our case, taking $g_0 = 1, g_1 = a$, then are we to take $\{0, 1a\}$ to be a simplex or not? Is it the 0-simplex $\{0\}$? The answer is 'yes', so as to be consistent with the definition of simplicial map. (You may recall or check that we left this to you to workout or look up. The complication is that if $f : K \to L$ is a simplicial map, then it is a map on the vertices, which preserves 'simplexness'. This means that if $\sigma \in S_K$ then $f(\sigma) \in S_L$, but note $f(\sigma)$ is the subset of V_L given by the images of the vertices in σ, hence it may, and usually will, have smaller dimension. (We will look at this point in quite a lot of detail very shortly.) This means that saying $\tau = \{v_0g_0, \ldots, v_ng_n\}$ is a simplex, is not quite accurate as it does not mean that the elements in the listing are distinct. (We will introduce some additional terminology to keep track of this shortly, but for the moment please excuse the slightly sloppy notation.) In our case, this is $\{0\}$. Now it is clear that the action is not regular, as there is no $g \in C_2$ sending $\{0, 1\}$ to $\{0\}$!

This example is simple but quite important as it highlights some weaknesses in both terminology and notation. It also suggests that we should ask the question: 'what are these conditions 'about'? To answer this, we need to look at 'quotienting' and its relationship with the passage from simplicial complexes to simplicial sets and to clarify several issues in the process.

Quotienting operations on simplicial complexes

We will give this in more generality then we actually need. Suppose that K is a simplicial complex with vertex set, V_K, and V_L is a set (not yet of 'vertices' of anything). Suppose we have a surjection $f : V_K \to V_L$. We want to construct a simplicial complex, L from the set V_L and the function, f. There is an obvious way to do it.

Definition: Define a subset $\{w_0, \ldots, w_n\} \subseteq V_L$ to be a simplex if there is a simplex $\{v_0, \ldots, v_n\} \in S_K$ such that $f(v_i) = w_i$ for $i = 0, 1, \ldots, n$. We will call $\{v_0, \ldots, v_n\}$ a *witness* for $\{w_0, \ldots, w_n\}$ in this case.

It should be clear that this *is* a simplicial complex structure on V_L and we will call it the induced simplicial complex along f, or similar terminology. If f is obtained explicitly by some equivalence relation (and, of course, it can always be considered to be given in such a way), then L would be called the *quotient* of K by that equivalence relation.

Lemma 27 *With this structure, f induces a morphism of simplicial complexes, $f : K \to L$.*

Proof: This is more or less obvious, at least at first sight. There is a detail, however, that is worth pointing out. If $\sigma = \{v_0, \ldots, v_n\}$ is a simplex in K, $f(\sigma)$ is a subset of V_L, but it may have fewer than the $n + 1$ elements that σ had, so we cannot, necessarily, use σ as a witness for $f(\sigma)$ being a simplex of L. There will, however, be a subset / face of σ that maps bijectively onto $f(\sigma)$, so $f(\sigma)$ *is* a simplex of L. ■

As an example of this construction, we can return to the 1-simplex, Δ^1, with the C_2-action. To recap, we have $K = \Delta^1$, $V_K = \{0, 1\}$, $0a = 1$, $1a = 0$ and we take $V_L = V_K/C_2 = \{0G\}$, so it is a 1 element set. (In general, here we will write vG for the orbit $\{vg : g \in G\}$ of v under the G-action, as being a fairly self evident notation.) The only simplex in L is thus the 0-simplex, $\{0G\}$. This again illustrates the point in the proof, as $f : K \to L$ sends $\{0, 1\}$ to $\{0G\}$, which is a 0-simplex of L, since, for instance, it is witnessed by $\{0\}$ in K.

We note that this, of course, works well with G-complexes in general, and not just in this simple example. If G acts on the right of K, we can take V_L to be the set of orbits, V_K/G, with $f : V_K \to V_L$, the obvious function assigning the orbit vG to a vertex v. The resulting simplicial complex will then be what we have been calling the *quotient of K by the G-action*. The usual notation will then be K/G for this.

Quotienting operations on simplicial sets

A (right) G-action on a simplicial set, K can be most elegantly defined as a functor $K : G[1]^{op} \to \mathcal{S}$, from the opposite of the category $G[1]$ (with a single object, conveniently denoted by $*$ if needed and with $G[1](*, *) = G$ with composition given by the multiplication) to the category of simplicial sets. Equivalently, we can write the action as a group homomorphism $G^{op} \to Aut(K)$, where $Aut(K)$ is the group of automorphisms of K, or, again equivalently, as a simplicial map

$$K \times K(G, 0) \to K$$

satisfying certain fairly obvious properties, where $K(G, 0)$ is the constant simplicial group with value G in each dimension.

A simplicial set with G action also corresponds to a simplicial G-set, that is, as simplicial object in the category of G-sets.

The quotient simplicial set of orbits is obtained as $(K/G)_n = K_n/G$, the set of G-orbits of n-simplices, with $d_i(\sigma G) = (d_i\sigma)G$, etc. (This works because G is acting via simplicial automorphisms, but that is **left to you to check and to think of other ways of putting it**). If we think of the G-action as a functor (as above) then K/G is the colimit of $K : G[1]^{op} \to \mathcal{S}$; again left to you to check, but this is important when considering the homotopical aspect, such as homotopical syzygies of presentations, as it is then natural that, in such a context, one should replace these colimits by homotopy colimits, ..., but that comes quite a lot later!

From simplicial complexes to simplicial sets

Earlier (page 30) we briefly discussed ways of converting a simplicial complex to a simplicial set. We there concentrated on a construction that was going to give a fairly small simplicial set. This picked a total order, $\leq$, on the vertices of the simplicial complex, K, and then used each $\sigma = \{v_0, \ldots, v_n\}$ by saying that we will put the v_i in order according to the given order. (For ease of notation we will assume that this is the given order we wrote them in, in the 'set' σ, so the corresponding simplex will be $\sigma = \{v_0 \leq \ldots \leq v_n\}$. We thus have, for example, that the simplicial complex, Δ^1, converts to the simplicial set, $\Delta[1]$. (Of course, this is almost cheating as we might have chosen a total order on $\{0, 1\}$ in which $1 \leq 0$, or in the general case have $v_3 \leq v_2 \leq v_0 \leq v_1 \leq v_n \leq \ldots\}$ within σ. In the case of Δ^1, the example is so simple that we get isomorphic simplicial sets whichever way we choose the order.) We will write $\langle v_0, \ldots, v_n\rangle$ for the simplex corresponding to $\{v_0 \leq \ldots \leq v_n\}$. The resulting structure will have nicely behaved face maps but not yet degeneracies. For those we have to add in more simplices and it helps to introduce some terminology, that will be useful in other places as well.

Definition: Let X be a set and $\underline{x} = \langle x_0, \ldots, x_n\rangle$, an $(n+1)$-tuple of elements of X. The *support* or *range* of $\underline{x}$ is the set of components of $\underline{x}$.

As an example, the support of the 5-tuple, $\langle 5, 1, 3, 2, 1\rangle$ is the set $\{1, 2, 3, 5\}$.

Given a total order on V_K, an n-simplex in the associated simplicial set (N.B., 'associated' to the pair $(K, \leq)$, not just to K) will be an $(n+1)$-tuple, $\langle v_0, \ldots, v_n\rangle$, of elements of V_K such that (i) $\{v_0 \leq \ldots \leq v_n\}$ in the order $\leq$ on V_K and (ii) the support of $\langle v_0, \ldots, v_n\rangle$ forms a simplex of K. Faces and degeneracies are defined in a fairly obvious way by deletion of a position and its entry and repetition of a component, so $s_1\langle v_0, v_1, v_2\rangle = \langle v_0, v_1, v_1, v_2\rangle$, for instance.

This construction is very useful as it gives a fairly small simplicial set that models the homotopy type of the simplicial complex, however it is not functorial on simplicial complexes as such because of the choice of order. If one works with simplicial complexes with an order on vertices together with order preserving simplicial maps between them then it will be functorial, but this is not feasible for most situations in which there is a G-action involved. The G-action is very unlikely to preserve the order and so the associated simplicial set will not inherit a G-action. We thus discard this method here, although it can be useful elsewhere.

The second construction does not need a choice of an ordering of the vertex set and so gives us a functor.

Definition: If K is a simplicial complex with vertex set V_K, the *associated simplicial set*, K^{simp}, of K is the simplicial set having $\sigma \in (K^{simp})_n$ if and only if $\sigma \in (V_K)^{n+1}$ and the support of σ is a simplex of K. The face and degeneracy mappings are defined in the usual way via deletions and repetitions.

This is much bigger than the ordered version, for instance if $\{v_0, v_1, v_2\} \in S_K$ is a simplex of K, then not only do we have $\langle v_0, v_1, v_2\rangle \in (K^{simp})_2$, but also $\langle v_1, v_0, v_2\rangle, \langle v_2, v_1, v_0\rangle$, etc. as they all have support $\{v_0, v_1, v_2\}$. Also in K_2 there will be degenerate 2-simplices such as $\langle v_0, v_0, v_1\rangle$, and $\langle v_0, v_2, v_2\rangle$, but that is not all as $\langle v_2, v_1, v_2\rangle$ is there, but is not a degenerate simplex as in degenerate 2-simplices either the first two or the last two vertices will be the same. These all have support that is a non-empty subset of $\{v_0, v_1, v_2\}$, so are simplices in K^{simp}.

With regard to our example, we note that $(\Delta^1)^{simp}$ is *not* $\Delta[1]$, as in addition to $\langle 0, 1\rangle$, we also have $\langle 1, 0\rangle$ in $(\Delta^1)^{simp}_1$.

We record for future use the lemma:

Lemma 28 *$(-)^{simp}$ gives a functor from the category, Simp.Comp, of simplicial complexes, to that of simplicial sets.*

Proof: Although this is really **left to you** to check as being more-or-less 'obvious', it is worth commenting that if $f : K \to L$ is in *Simp.Comp*, we can have $\sigma = \{v_0 \leq \ldots \leq v_n\}$ in K, so there are $(n+1)!$-different simplices in K^{simp} corresponding to σ; now $f(\sigma)$ may contain fewer elements than σ if f is not 1-1 on vertices. Each σ^{simp} that corresponds to σ will be of the form $\sigma^{simp} = \langle v_0, \ldots, v_n\rangle$ for some ordering of the elements of σ, and $f^{simp}(\sigma^{simp})$ will then be

$\langle f(v_0), \ldots, f(v_n)\rangle$, which will be an n-simplex of L^{simp} for the 'obvious' reason. This assignment will, of course, respect the faces and degeneracies, as is easily checked. ■

Corollary 10 *If K ia a simplicial G-complex,* i.e., *a simplicial complex with simplicial G-action, then K^{simp} is a simplicial G-set,* i.e., *the simplcial set, K^{simp}, inherits a simplicial G-action.*

Proof: A (right) G-action on K 'is' a functor $K : G[1]^{op} \to Simp.Comp$, now compose that with $(-)^{simp}$ to get K^{simp}.

If you prefer a more 'hands-on' viewpoint, if $\sigma = \{v_0 \leq \ldots \leq v_n\}$, then $\sigma \cdot g = \{v_0 g \leq \ldots \leq v_n g\}$, and any simplex, $\sigma^{simp} = \langle v_0, \ldots, v_n\rangle$ with support σ gets sent by the action of g to $\langle v_0 g, \ldots, v_n g\rangle$ which has support $\sigma \cdot g$. it then just remains to **check that it all fits together well**. ■

This corollary means that we seem to have *two* ways of getting a simplicial set of orbits from a simplicial G-complex K. We can either form $(K/G)^{simp}$ or K^{simp}/G. They need not be the same.

Example revisited: Again $K = \Delta^1$, $G = C_2$, with $0a = 1$, $1a = 0$.

- K/G is Δ^0, so $(K/G)^{simp}$ is $\Delta[0]$;
- K^{simp} has $K_0^{simp} = \{\langle 0\rangle, \langle 1\rangle$, and the two 0-simplices are swapped by the G-action, $K_1^{simp} = \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 0,1\rangle, \langle 1,0\rangle\}$, and a exchanges the first two of these and also the last two, so $\langle 0,1\rangle a = \langle 1,0\rangle$, etc. and thus

$$(K^{simp}/G)_1 = \{\langle 0,0\rangle G, \langle 0,1\rangle G\},$$

so two orbits. (**You are left** to identify what happens in higher dimensions, and to check that $(K^{simp}/G) \cong S^1 = \Delta[1]/\partial\Delta[1]$, one version of the simplicial circle. Note that this is *not* the simplicial set associated to any simplicial complex.

Another example: There is a similar example with a bit more subtlety. This time take $K = \partial\Delta^2$, the 1-skeleton of the 2-simplex, hence an empty triangle. The vertex set of this is $\{0, 1, 2\}$, whilst the simplices are $\{0, 1\}, \{1, 2\}$, and $\{0, 2\}$, together with the obvious singletons coming from the vertices. Take $G = C_3 = \{1, a, a^2\}$ (and, of course, $a^3 = 1$), with action on K given by $0 \cdot a = 1$, $1 \cdot a = 2$ and, of course, $2 \cdot a = 0$, so a rotation.

- Again K/G has just one vertex, so $(K/G)^{simp}$ is $\Delta[0]$.
- The corresponding K^{simp}/G, has just one vertex, but has two non-degenerate 1-simplices, $\langle 0,1\rangle G$, and $\langle 1,0\rangle G$.
- In fact, it also has some non-degenerate 2-simplices, $\langle 0,1,0\rangle G$ and $\langle 1,0,1\rangle G$
- ... and so on.

This is quite neat. The two non-degenerate 1-simplices are, in some sense, homotopy inverses to each other with the homotopies encoded by the 2-simplices that we have given. Those two homotopies are themselves homotopic by the next level and so on. This is very much typical of a homotopy coherent situation, for the meaning of which see later.

This is an example of an action which is 'without inversion' but is not regular.

Our examples have had $(K/G)^{simp}$ a 0-simplex, and so there is a unique map from K^{simp}/G to $(K/G)^{simp}$ by virtue of that, but such a map always exists.

Proposition 40 *For any simplicial complex, K, with G-action, there is a natural degreewise surjective simplicial morphism,*

$$\varphi : K^{simp}/G \to (K/G)^{simp}.$$

Proof: Suppose that $\langle v_0, \ldots, v_n\rangle \in K_n^{simp}$, take $\varphi(\langle v_0, \ldots, v_n\rangle G) = \langle v_0G, \ldots, v_nG\rangle$, and **check this is a simplicial morphism**. We are left to check it is a degreewise surjective morphism. (Naturality is easy to verify, so is **left to you**. In fact, as K^{simp}/G is the colimit of the functor giving the action, naturality is more-or-less forced to be the case.)

Suppose we have $\sigma = \langle v_0G, \ldots, v_nG\rangle$ is an n-simplex in $(K/G)^{simp}$, then its support is a simplex of K/G. Deleting repeats if any from the list, $(v_0G, \ldots, v_nG)$, we get a sublist, $(v_{i_0}G, \ldots, v_{i_k}G)$, whose elements give the support of σ, so $\{v_{i_0}G, \ldots, v_{i_k}G\} \in S_{K/G}$. By the definition of the simplicial complex structure on K/G, this means that there are elements g_{i_j} of G, for $j = 0, \ldots, k$, such that $\{v_{i_0}g_{i_0}, \ldots, v_{i_k}g_{i_k}\}$ is in S_K. We will write $I = \{i_0, \ldots, i_k\}$.

We can now use this to build a simplex of K^{simp}, as follows:

- examine v_0G; there is some index, which we suppose to be $j_0 \in I$ such that $v_0G = v_{j_0}G$, hence we have some $g_0 \in G$ such that $v_0g_0 = v_{j_0}g_{j_0}$;
- we repeat for v_1; there is some index $j_1 \in I$ such that $v_1G = v_{j_1}G$, hence we have some $g_1 \in G$ such that $v_1g_1 = v_{j_1}g_{j_1}$
- and so on.

This gives us a potential simplex $\langle v_0g_0, \ldots v_ng_n\rangle$. Its support is $\{v_{i_0}g_{i_0}, \ldots, v_{i_k}g_{i_k}\}$, which is in S_K, so $\tau = \langle v_0g_0, \ldots v_ng_n\rangle \in K^{simp}$ and $\varphi(\tau G) = \sigma$, so φ is degreewise surjective. ■

It is a natural question to ask when φ has additional properties and, for us most importantly, when φ is an isomorphism.

Theorem 10 *The natural morphism,*

$$\varphi : K^{simp}/G \to (K/G)^{simp},$$

is an isomorphism if, and only if, K is a regular G-complex.

Proof: Suppose that φ is one-to-one, (and hence is an isomorphism by the above proposition), and now suppose $\{v_0, \ldots, v_n\} \in S_K$ and $g_0, \ldots, g_n \in G$ are such that $\{v_0g_0, \ldots, v_ng_n\}$ is also a simplex of K. We look at $\langle v_0, \ldots, v_n\rangle G$ and $\langle v_0g_0, \ldots, v_ng_n\rangle G$, and note that φ maps then both to $\langle v_0G, \ldots, v_nG\rangle$, hence, as φ is one-to-one, those two simplices must, in fact, be equal. The second of these contains $\langle v_0g_0, \ldots, v_ng_n\rangle$, so there must be a $g \in G$ such that $\langle v_0, \ldots, v_ng\rangle = \langle v_0g_0, \ldots, v_ng_n\rangle$, so the action makes K into a regular G-complex.

Conversely, suppose that K is a regular G-complex and that $\langle v_0, \ldots, v_n\rangle G$ and $\langle v_0', \ldots, v_n'\rangle G$ have the same image under G. This translates to there being an equality

$$\langle v_0G, \ldots, v_nG\rangle = \langle v_0'G, \ldots, v_n'G\rangle,$$

so there are elements $g_0, \ldots, g_n \in G$ such that, for each i, $v_ig_i = v_i'$, We can thus use the condition of regularity to find a single $g \in G$ such that $\langle v_0, \ldots, v_n\rangle g = \langle v_0', \ldots, v_n'\rangle$, but that implies that in fact

$$\langle v_0, \ldots, v_n\rangle G = \langle v_0', \ldots, v_n'\rangle G,$$

so φ is one-to-one, hence an isomorphism. ∎

We could continue looking at the various conditions of G-complexes for instance, the 'without inversion' one that we mentioned earlier, but will rather leave that to the reader to follow up, for instance, in Prasolov's book, [230], or in Bredon's notes, [47], and will start on some of the simpler ideas of the theory of Complexes of Groups.

4.5 Complexes of groups

This situation that we saw with the group actions on the nerve, $N(\mathfrak{H})$, is a simple form of a general one considered by Haefliger (cf. [54, 145, 146]) and Corson (cf. [90–92]). They consider a simplicial complex (or more generally a simplicial cell complex, cf. Haefliger, [145] or a scwol (small category without loops), cf. Bridson and Haefliger, [54]) on which a group G acts 'without inversion' or, in the variant used by Corson, with a regular G-complex. Their work introduced complexes of groups, a notion generalising that of graphs of groups as in Bass-Serre theory, [245, 246] and also, [23], but developed into a central part of geometric group theory later on. We will give definitions shortly, but first need to revise some of the more detailed notation and terminology relating to barycentres, barycentric subdivisions, etc. extending our discussion in section 4.3.6. Here we will be limiting ourselves initially to the simpler form of the ideas, but will generalise later.

These complexes of groups are important not only for discussion of properties relating to syzygies, but because they provide fairly simple examples of orbifolds, and topological stacks, both of which are ideas that we will encounter (much) later on in these notes.

4.5.1 Simplicial complexes, barycentres and scwols

If K is a simplicial complex, we can encode the information in K in a simply way by considering K as a partially ordered set. The elements of this partially ordered set are the elements of S_K, the set of simplices of K, ordered by inclusion. As we mentioned earlier, the barycentric subdivision of K is then just the (categorical) nerve of the poset $(S_K, \subseteq)$. We will follow Haefliger [145] in orienting the edges of K' in the following way:

The vertices of $K'(= Sd(K))$ are the simplices of K. An (unoriented) edge of K' consists of a pair (σ, τ) with either $\sigma \subset \tau$ or $\tau \subset \sigma$. If a is an edge of K' contained in a simplex, σ, of K, then the *initial point* $i(a)$ of a will be the barycentre of σ, (*i.e.*, σ as a vertex of K') and its *terminal point*, $t(a)$, will the barycentre of some smaller simplex, τ. We write $i(a) = \sigma$, $t(a) = \tau$ and so have $a = (\tau, \sigma)$, with $\tau \subset \sigma$. (This is perhaps the opposite order from that which seems natural, but it avoids considering dual posets later.)

Examples:

(i) The simplest case is for the 1-simplex, Δ^1, which has, as we have said earlier, vertex set $\{0, 1\}$ or if you prefer, $\{1, 2\}$, (to keep the same notation as Haefliger), and all non-empty subsets are simplices. This gives, a slightly abbreviated notation,

$$\{1\} \longleftarrow \{1, 2\} \longrightarrow \{2\}$$

(ii) For the 2-simplex, considered as the simplicial complex of non-empty subsets of $\{1, 2, 3\}$, this gives

$$\begin{tikzcd}
 & & \{2\} & & \\
 & \{1,2\} \arrow[ur] \arrow[dl] & & \{2,3\} \arrow[ul] \arrow[dr] & \\
 & & \{1,2,3\} \arrow[uu] \arrow[ul] \arrow[ur] \arrow[dll] \arrow[d] \arrow[drr] & & \\
\{1\} & & \{1,3\} \arrow[ll] \arrow[rr] & & \{3\}
\end{tikzcd}$$

It is quite usual to consider partially ordered sets as categories, so we could just leave things like this and use this partially ordered set of simplices in K as the categorical model of subdivision. Haefliger, however, wanted a more general type of complex than merely simplicial complexes, so introduces a specific construction of a small category associated to K, (cf. [145]), extracts an abstraction of the key properties that he needs from this subdivision category and the uses that abstraction (a *scwol*) as a means to build the generalisation he wanted. We will follow his construction explicitly, including the notation, as his conventions are not always identical to those we have used earlier (e.g., because of the graph-theoretic link with graphs of groups, the terminology *initial* and *terminal* instead of *source* and *target* for the two ends of an arrow in a small category, are used. This results in a use of i and t, instead of s and t as notation. This should not be too confusing, but otherwise the comparison and cross referencing to the original sources would be difficult.) Before giving the precise definition of a small category without loops, or *scwol*, we will given the example of the scwol associated to a simplicial complex.

Example: Given a simplicial complex, define a category, $C(K)$, with set of objects, S_K, the set of vertices of the barycentric subdivision, K', of K and with arrows, $Arr(C(K)) = E_{K'} \sqcup S_K$, the set of edges of K' together with S_K. (Of course, the vertices are being considered as identity arrows at themselves.) Two edges a and b are considered composable if $i(a) = t(b)$ and the composite is $c = ba$ such that a, b, c form the boundary of a 2-simplex in K':

$$\begin{tikzcd}
 & \cdot \\
\cdot \arrow[ur, "c"] \arrow[r, "b"'] & \cdot \arrow[u, "a"']
\end{tikzcd}$$

This category, $C(K)$, is an example of a *small category without loops* as introduced by Haefliger [54, 145]. In general, in this section, we shall consider a small category, C, to consist of a set, $V(C)$, of vertices or objects (denoted here by Greek letters, τ, σ, etc.) and a set, $E(C)$, of edges (denoted by Latin letters, a, b, $\ldots$), together with maps:

- identity, $id : V(C) \to E(C)$;
- $i : E(C) \to V(C)$, the *initial vertex* or map,

- $t : E(C) \to V(C)$, the *terminal vertex* or *target* map,
  and a composition:

- $E^{(2)}(C) \to E(C)$,
  where $E^{(2)}(C) = \{(a, b) \in E(C) \times E(C) : i(a) = t(b)\}$,

together with rules requiring associativity of composition, correct behaviour of the identities, so $i(id(v) = t(id(v)) = v$, etc., and the rules $i(ba) = i(b)$, $t(ba) = t(a)$ for ba, the composite of a and b.

Definition: A small category, C, is a *small category without loops*, or *scwol*, if for all a in $E(C)$, $i(a) \neq t(a)$.

Remark: Haefliger's definition of a small category without loops in [54] (p. 521) is optimised for the statement of the no loops condition, but actually omits to define composition of an arbitrary arrow with an identity at a vertex. This is handled correctly (p.573) in an appendix. This does not influence the later development.

In general, it is clear that scwols need not be posets, and so are not restricted to come from simplicial complexes. Scwols do have associated simplicial cell complexes, essentially obtained by taking their nerve, (cf. page 29), and then taking the geometric realisation so this is the *classifying space of the scwol*. This is sometimes considered, however, together with explicit orderings on the cells that result, retaining in this way the important amount of 'directionality' that is within the scwol, but not in the classifying space as such. In this case, the term *ordered simplicial cell complex* is used by Haefliger, [145, 146]. Some neat examples of ordered simplicial cell complexes are given by Bridson and Haefliger, starting on page 524 of [54].

For the moment, we will move attention back to the 'geometric' situation and the definition of a complex of groups and will pretend that we have a simplicial (cell) complex, K. We will later make the necessary changes to get a complex of groups defined directly on a scwol and will give, somewhat later in the noted, categorical interpretations of what the 'geometry' is handling here.

4.5.2 Complexes of groups: introduction

As we said above, we will start by giving a 'geometric' form of the notion of a complex of groups, here. Our aim is not to explore all the resulting theory, so we will restrict attention to those aspects that seem to have evident uses as examples, etc., later on.

Definition: A *complex of groups*, $G(K)$, on K is specified by the data, $(\{G_\sigma\}, \{\psi_a\}, \{g_{a,b}\})$ given by

1) a group, G_σ, for each simplex, σ, of K;

2) an injective homomorphism,
$$\psi_a : G_{i(a)} \to G_{t(a)},$$
for each edge, $a \in E_K$, of the barycentric subdivision of K;

3) for each pair of composable edges, a and b, in E_K, an element $g_{a,b} \in G_{t(a)}$ is given such that

$$g_{a,b}^{-1}\psi_{ba}(\_)g_{a,b} = \psi_a\psi_b$$

and such that the "cocycle condition"

$$g_{a,cb}\psi_a(g_{b,c}) = g_{ab,c}g_{a,b}$$

holds.

(If the dimension of K is less than 3, this last condition is trivially satisfied, since the existence of a triple of composable (non-identity) edges implies that there would be some 3-simplices in K.)

The groups, G_σ, are sometimes called the local groups of the complex of groups.

This definition is quite 'bare hands' and so we will, of course, need some examples. Later we will generalise and through that generalisation obtain a neater, more elegant, formulation as well. We will give some simple examples shortly, but before that there is a construction giving a 'generic' example, or almost.

Almost generic example: Developable complexes of groups.

Suppose we have a simplicial complex, $\tilde{K}$, with a right G-action, which is "without inversion" or is regular, both are used. Write $K = \tilde{K}/G$ for the quotient complex. We will specify a complex of groups, $G(K)$, on K:
Set $p : \tilde{K} \to K$ to be the quotient mapping.

For a simplex, σ, of K, pick a $\tilde{\sigma} \in K$ with $p(\tilde{\sigma}) = \sigma$. We say $\tilde{\sigma}$ is the *chosen lift* of σ. Set

$$\begin{aligned} G_\sigma =& G_{\tilde{\sigma}}, \text{ the stabiliser subgroup of } \tilde{\sigma}, \\ =& \{g : \tilde{\sigma}g = \tilde{\sigma}\}. \end{aligned}$$

For each $a \in E_K$ with $i(a) = \sigma$, let $\tilde{a}$ be the edge in $\tilde{\sigma}$, whose projection is a, *i.e.*, $p(\tilde{a}) = a$ and $i(\tilde{a}) = \tilde{\sigma}$. There is then some $h_a \in G$ with $t(\tilde{a}.h_a) = \tilde{\tau}$, where $\tilde{\tau}$ is the chosen lift of $\tau = t(a)$. (If $t(\tilde{a}) = \tilde{\tau}$ already, we agree to take H_a to be the identity of G.)

Define

$$\psi_a : G_{i(a)} \to G_{t(a)}$$

by

$$\psi_a(g) = h_a^{-1}gh_a \quad \text{ for } g \in G_{i(a)}.$$

Given two composable edges a and b, we have a configuration such as

$$\begin{tikzcd} & \cdot \\ \cdot \arrow[ur, "ba"] \arrow[r, "b"'] & \cdot \arrow[u, "a"'] \end{tikzcd}$$

and hence a diagram

$$\begin{tikzcd} & G_{t(a)} \\ G_{i(b)} \arrow[ur, "\psi_{ba}"] \arrow[r, "\psi_b"'] & G_{i(a)} \arrow[u, "\psi_a"'] \end{tikzcd}$$

but there is no reason why it should be commutative, in fact:

$$\psi_a\psi_b(g) = h_a^{-1}h_b^{-1}gh_bh_a,$$

whilst

$$\psi_{ba} = h_{ba}^{-1}gh_{ba},$$

so we take

$$g_{a,b} = h_{ba}^{-1}h_bh_a.$$

Verification of conditions (Although easy to do, this helps the intuition.):

(i) Suppose $g \in G_{i(a)}$, then $\tilde{a} = (\tilde{\tau}h_a^{-1}, \tilde{\sigma})$ or $\tilde{a}.h_a = (\tilde{\tau}, \tilde{\sigma}.h_a)$. As $\tilde{\sigma}g = \tilde{\sigma}$, and $\tilde{\tau}h_a^{-1} \subset \tilde{\sigma}$, we have

$$\tilde{\tau}h_a^{-1}g = \tilde{\tau}h_a^{-1}$$

and $h_a^{-1}gh_a \in G_{t(a)}$, *i.e.*, $\psi_a(g) \in G_{t(a)}$.

(ii) It is clear that $g_{a,b}$ as defined above does the job as it was chosen to do so!

(iii) There remains the cocycle condition:

$$g_{a,cb}\psi_a(g_{b,c}) = h_{cba}^{-1}h_{cb}h_a.h_a^{-1}h_{cb}^{-1}h_ch_bh_a$$

$$g_{a,cb}.g_{a,b} = h_{cba}^{-1}h_ch_{ba}.h_{ba}^{-1}h_bh_a,$$

so it all does check out correctly.

In the case of a group, G, acting on the nerve of a family of subgroups, $\mathscr{H}$, where $\mathscr{H} = \{H_1, \cdots, H_n\}$ with $H_i < G$, then $N(\mathfrak{H})/G \cong \Delta^{n-1}$. Suppose $\sigma \in S_{\Delta^{n-2}}$ then if $\sigma = \{\alpha_1, \cdots, \alpha_r\}$, we can always choose $\tilde{\sigma} = \{H_{\alpha_1}, \cdots, H_{\alpha_r}\}$. If a is an edge of $Sd(\Delta^{n-1})$ then, for $i(a) = \sigma$ and $t(a) = \tau, \tilde{\tau} \subset \tilde{\sigma}$, hence

$$G_\tau = G_{\tilde{\tau}} = \bigcap\{H_i \mid i \in \tilde{\tau}\},$$
$$G_\sigma = G_{\tilde{\sigma}} = \bigcap\{H_i \mid i \in \tilde{\sigma}\},$$

so there is no need to have $h_a \neq 1$. Because of this, ψ_a is simply an inclusion of a subgroup and $g_{a,b}$ can be chosen to be 1. The Abels-Holz situation, thus, leads to simplices of groups of a particularly simple kind.

In general, not all complexes of groups are *developable*. Shortly we will give Haefliger's characterisation of the developable ones. All graphs of groups are developable and we turn to them next.

4.5.3 Graphs of groups

The notion of a complex of groups was a natural development of that of a graph of groups due to Bass and Serre, [245, 246] and also, [23]. It seems a good idea to give some definitions of some main elementary ideas from that theory as they provide some insight into the generalised form. (We will adapt the definition as given by Corson in [90].)

We first introduce some notation. Let Γ be a graph and e an edge of Γ. We will choose an orientation for each edge, and e together with that chosen orientation will be denote either by e, itself, or if more precision is needed by e^+. That edge with the opposite orientation will be denoted e^-. As we will be using both, this does not mean that we have a directed graph, merely that, for convenience we need to be able to talk of each edge together with both possible orientations and this is one way of handling that need. We will adopt the notation $i(e)$ for the initial vertex of $e = e^+$, and $t(e)$ for the 'other end'. Of course, $i(e^-) = t(e^+)$, etc. Note that we can also think of this as being a directed graph, or quiver,

$$E_\Gamma \underset{t}{\overset{i}{\rightrightarrows}} V_\Gamma \ ,$$

together with an involution on the edges,

$$^- : E_\Gamma \to E_\Gamma$$

called *edge reversal*, satisfying some obvious properties. This is particularly useful for various definitions slightly later on.

Definition: A *graph of groups*, $\mathcal{G}$, is a pair, $\mathcal{G} = (\Gamma, G)$, consisting of an (abstract) connected graph, Γ and an assignment, G, which assigns to each vertex, v of Γ, a group G_v, and to each *oriented edge* (*i.e.*, an edge of Γ together with a direction on it), a group G_e, such that $G_e = G_{e^-}$ and an *monomorphism*, $\mu_e : G_e \to G_{i(e)}$.

If we consider Γ as a 1-dimensional simplicial complex, and work with its associated poset, S_Γ, then the above gives a functor from the opposite category, S_Γ^{op} to the category of groups and monomorphisms between them. This category, S_Γ^{op}, has objects the vertices and the edges and for each edge there is one morphism $e \to i(e)$, (and, of course, another $e \to i(e^-)$).

It should be fairly obvious that a graph of groups is a simple example of a complex of groups. We leave the detailed checking to the diligent reader. (Note the exposition here is adapted from various sources on graphs of groups, so there will be some minor things to check, in particular that the extra structure given in the case of complexes of groups has no content in this simple case.)

Examples: Suppose that T is a tree, and therefore, in particular, a graph, and there is a group π acting on the right on T (an action, which is assumed to be without inversions). The orbit graph $\Gamma = T/\pi$ supports a natural structure of a graph of groups. In this, the vertex groups, G_v, are the vertex stabilisers of the actions, so

$$G_v = \{g \in \pi \mid v.g = v\},$$

and the edge group of an edge, e, is

$$G_e = \{g \in \pi \mid e.g = v\}.$$

As any automorphism of T that fixes e must fix both $i(e)$ and $t(e)$, the group G_e is a subgroup of both $G(i(e))$ and $G(t(e))$. This gives the information necessary for a graph of groups based on Γ.

In fact, given any graph of groups, one can find a tree, T, and a group acting on it, but for this we need the idea of the fundamental group of a graph of groups.

4.5.4 The fundamental group(oid) of a graph of groups

The fundamental group of a graph of groups, (Γ, G), can be defined in several equivalent ways. There are basically two approaches one topological and the other algebraic. In the algebraic one the usual starting point is to choose a maximal tree in Γ. This seems a bit counter to our approach so we will, instead, first define the fundamental *groupoid* of (Γ, G), a definition first given Higgins in [150] and an equivalent one is given in [209], where the formulation is optimised for computational uses. We will explicitly use the description of Γ as having an involution, $^{-}$, which 'reverses' arrows.

Definition: Given a graph of groups, $\mathcal{G} = (\Gamma, G)$, its *fundamental groupoid*, Π_1, $\Pi_1(\mathcal{G})$ or, if more detail is needed, $\Pi_1(\Gamma, G)$, is the groupoid specified as follows:

- the objects of Π_1 are the vertices of Γ;
- a generating graph of arrows for Π_1 is given by Γ, together with the elements of all the groups, G_v, for $v \in V_\Gamma$, where
- if $v \in V_\Gamma$ and $g \in G_v$, both the source and target of g are equal to v, *i.e.*, for each $g \in G_v$ we have a loop labelled g at the vertex v in this generating graph.

The defining relations are

(i) if $v \in V_\Gamma$, and $a, b, c \in G_v$ satisfy $ab = c$, then $ab = c \in \Pi_1$;

(ii) if $e \in E_\Gamma$ and $a \in G_e$, then

$$\mu_e(a) = e\mu_{\bar{e}}(a)\bar{e}.$$

Remarks: (i) As a consequence of the last relation, we get $\bar{e} = e^{-1}$, so the algebraic inverse within the groupoid coincides with the more 'geometric' inverse obtained by the edge reversal involution.

(ii) Again in this last relation, it is worth taking this apart a bit as it is here that the edge groups interact with the vertex groups, whilst it is only the vertex groups that give generators. (It is worth comparing this situation with our earlier discussion on the van Kampen theorem and presentations of pushouts of groups in section 4.3.11, as that was a closely related situation, although, as there, we still do not have enough machinery to do it justice, and to explain 'what is going on'.)

We have $e \in E_\Gamma$ is an edge, going from $i(e)$ to $t(e)$, there is an edge, $\bar{e}$, going in the reverse direction. We therefore have two injections, $\mu_e : G_e \to G_{i(e)}$ and $\mu_{\bar{e}} : G_e \to G_{t(e)}$. We also have two generators in Π_1, $e : i(e) \to t(e)$ and $\bar{e} : t(e) \to i(e)$. If $a \in G_e$, we have $\mu_e(a) \in G_{i(e)}$ and the composite $e\mu_{\bar{e}}(a)\bar{e}$ is also in this vertex group, $G_{i(e)}$. (We are reading off the composite as

$$i(a) \xrightarrow{e} t(e) \xrightarrow{\mu_{\bar{e}}(a)} t(a) \xrightarrow{\bar{e}} i(e),$$

so in 'concatenation order'.)

Definition: The *fundamental group* of (Γ, G) at a vertex v, is the vertex group, $\Pi_1(\mathcal{G})(v)$, of $\Pi_1(\mathcal{G})$ at v.

As Γ is connected, the fundamental groups at any two vertices are isomorphic. This is slightly deceptive, however, as they may be isomorphic by many different isomorphisms corresponding to different paths between those vertices. This is essentially the same point as saying that a presentation of a group has, *really*, to be given together with an explicit isomorphism to the group, although for many (most?) purposes this is not useful information.

If a presentation of $\Pi_1(\mathcal{G})(v)$ is desired, it can be obtained by choosing a maximal tree, T, in the graph Γ.

Proposition 41 *Given a maximal tree, T, in Γ, the fundamental group, $\Pi_1(\mathcal{G})(v)$, has a presentation, $(X : R)$, where X is the disjoint union of the G_vs and the set E_Γ, of edges of Γ, and the relations, R, are the relations are*

(i) if $v \in V_\Gamma$, and $a, b, c \in G_v$ satisfy $ab = c$, then $ab = c \in \Pi_1(\mathcal{G})(v)$;

(ii) if $e \in E_\Gamma$ and $a \in G_e$, then

$$\mu_e(a) = e\mu_{\overline{e}}(a)\overline{e},$$

and

(iii) $e = 1$ if $e \in T$. ■

4.5.5 A graph of 2-complexes

If we have a graph of groups, $\mathcal{G} = (\Gamma, G)$, then one way to obtain a 'presentation' of $\mathcal{G}$ is via a graph of 2-complexes. The ideas is easily accessible in Corson, [90], but is also discussed in the lecture notes of Scott and Wall, [242], where graphs of spaces, in more generality, are introduced. We will return to this later.

Clearly the graph of groups $\mathcal{G}$ can be considered as a functor, $G : S_\Gamma^{op} \to Grp$, with the proviso that each morphism of S_Γ is sent to a monomorphism of groups. This viewpoint will be useful very shortly.

Definition: (i) A graph of 2-complexes, (Γ, X) is a functor, $X : S_\Gamma^{op} \to CW$, from S_Γ^{op} to the category of CW-complexes such that, for each vertex, v, (resp. each edge, e) of Γ, the space X_v, (resp. X_e) is a (pointed connected) 2-complex, and, for each edge, e, the maps $X_e \to X_{i(e)}$ (and $X_e \to X_{t(e)}$) are cellular and preserve base points.

(ii) The graph of 2-complexes, (Γ, X), is a *presentation* of a graph of groups, $\mathcal{G} = (\Gamma, G)$, if there are (given) isomorphisms

- $\pi_1(X_v, *_v) \cong G_v$;
- $\pi_1(X_e, *_e) \cong G_e$,

which are compatible with the edge monomorphisms, so

$$\begin{tikzcd}
\pi_1(X_e, *_e) \arrow[r, "\cong"] \arrow[d] & G_e \arrow[d, "\mu_e"] \\
\pi_1(X_{i(e)}, *_{i(e)}) \arrow[r, "\cong"'] & G_{i(e)}
\end{tikzcd}$$

commutes; similarly for $\mu_{\bar{e}}$.

Put more succinctly, applying the fundamental group functor, π_1, to (Γ, X) gives (Γ, G) up to natural isomorphism.

Of course, the various X_v and X_e are essentially given by a presentation of the corresponding groups (except that we do not state that the 2-complexes will be reduced), so it is natural to extend our previous discussion of higher syzygies to this case. We mention that in the case of a pushout of groups, a double mapping cylinder allowed one to write out the 2-syzygies in at least a simple case. In this more general case, we have an analogous construction, which generalises that and which we introduce next. (We will later on see this as a simple example of a homotopy colimit.)

Definition: Given a graph of 2-complexes, (Γ, X), its *total space* is the space constructed as follows:

- take the coproduct (so disjoint union) of the spaces X_v for $v \in V_\Gamma$ together with the spaces $X_e \times [0,1]$ for $e \in E_\Gamma$;
- identify along the following maps:
  - $X_e \times [0,1] \to X_{\bar{e}} \times [0,1]$ sending (x,t) to $(x, 1-t)$;
  - on the subspace, $X_e \times \{0\}$, of $X_e \times [0,1]$, use $X_e \times \{0\} \cong X_e \to X_{i(e)}$, given by the structure map of $X : S_\Gamma^{op} \to CW$, (and similarly for $\bar{e}$).

We will denote the resulting space by $Tot(\Gamma, X)$, or sometimes simply X_Γ.

Example: Take Γ to be the graph with two vertices, 0 and 1, and one edge, $(0,1)$, joining them, then a graph of 2-complexes is given by a 'span' diagram

$$X_0 \longleftarrow X_{0,1} \longrightarrow X_1 \ .$$

The resulting total space is given by the colimit of the diagram:

$$\begin{tikzcd}
 & X_{0,1} \arrow[r] \arrow[d, "e_1"] & X_1 \\
X_{0,1} \arrow[r, "e_0"] \arrow[d] & X_{0,1} \times [0,1] & \\
X_0 & &
\end{tikzcd}$$

where $e_i : X_{0,1} \to X_{0,1} \times [0,1]$ sends x to (x, i) for $i = 0, 1$, so is the appropriate double mapping cylinder.

Proposition 42 *If (Γ, X) is a graph of 2-complexes that presents (Γ, G), then the fundamental group of $Tot(\Gamma, X)$ is isomorphic to the fundamental group of (Γ, G) (based at any vertex).* ■

We will not give a proof. One is given in [242] and we will later see a generalisation of it, so including one here seems inessential. A proof can be given using the van Kampen theorem in a more general form than we have quoted above.

The above strays out of our usual, more algebraic, territory as it uses topological methods. An intermediate approach which uses the algebraic ideas combined with some combinatorial constructs can be usefully obtained by looking at the corresponding construction within the category of groupoids. This is taken, here, from Emma Moore's thesis, [209]. As usual, $\mathcal{I}$ denotes the interval groupoid, which has two objects 0 and 1 and morphisms $\iota : 0 \to 1$ and its inverse, together with, of course, the identity arrows at each object, also, if H is a group, $H[1]$ denotes the one object groupoid to which it corresponds. (Reminder we will be working within the category of groupoids in the following definition, so, in particular, coproduct, $\sqcup$, has to be interpreted accordingly.)

Definition: Given a graph of groups, $\mathcal{G} = (\Gamma, G)$, the *total groupoid*, $Tot(\mathcal{G})$ is defined as the quotient of $(\bigsqcup\{G_v[1] \mid v \in V_\Gamma\}) \sqcup (\bigsqcup(\{G_e[1] \times \mathcal{I} \mid e \in E_\Gamma\})$ by the relations corresponding to

- $G_e[1] \times \mathcal{I} \overset{\cong}{\leftrightarrow} G_{\bar{e}}[1] \times \mathcal{I}$ by $(g, \iota) \leftrightarrow (g, \iota^{-1})$,
 and
- $G_e[1] \times \{0\} \to G_{i(e)}[1]$ given by $(g, 0) \leftrightarrow \mu_e(g)$.

Example: For a graph of groups, $\mathcal{G} = (\Gamma, G)$, where Γ has just two vertices and one edge, $Tot(\mathcal{G})$, is the groupoid double mapping cylinder.

Theorem 11 *The total groupoid of a graph of groups, $\mathcal{G}$, is isomorphic to the fundamental groupoid of $\mathcal{G}$.*

Sketch proof: First of all we note that the set of objects of $\Pi_1\mathcal{G}$ is V_Γ, whilst that of $Tot(\mathcal{G})$ is obtained by quotienting from a set made up as the disjoint union of V_Γ with two copies of E_Γ, one labelled $E_\Gamma \times \{0\}$, the other $E_\Gamma \times \{1\}$. The first relation identifies each $(e, 0)$ with the corresponding $(\bar{e}, 1)$, so after that there is but one copy of each 'edge vertex', then that edge vertex, $(e, 0)$, is identified with the vertex of $G_{i(e)}$. We thus have $Ob(Tot(\mathcal{G}))$ is bijective with V_Γ, and as each equivalence class of objects contains exactly one element naturally identify the two sets without risk of losing naturality.

We next construct a morphism from $Tot(\mathcal{G})$ to $\Pi_1\mathcal{G}$. To do this, we use the natural morphisms from the coproduct used to construct $Tot(\mathcal{G})$ to $\Pi_1\mathcal{G}$, and then check that the relations / identifications are consistent with the relations in the presentation of $\Pi_1\mathcal{G}$. This is easy to check at an intuitive level, but needs a bit of care for the detail. (We will see generalisations of such results later, so **leave these details for your consideration**.) ■

This does not *directly* help in our search for machinery to calculate syzygies, but note that for calculations with graphs of 2-complexes should be mirrored by calculations with some sort of graph of crossed modules, and this is in part suggested by the result of Abels and Holz that we mentioned earlier, (Proposition 33, page 143), and also our comments on the 2-syzygies of a pushout presentation, discussed in section 4.3.11, however these idea would seem to be better explored in the more general context of complexes of groups, so we will put them aside for the moment.

4.5.6 A brief glance at the Bass-Serre theory

We will, very briefly, now turn to some aspects relating to the formulation and (sketch) proof of one of the main theorems of Serre's theory of graphs of groups. (We will, in part, use ideas from the discussion in K. Brown's book, [56], on cohomology of groups, as it links the theory into standard material on equivariant homology, and for that is thoroughly to be recommended. The relevant sections are at the end of Chapter II, starting on page 52, and then section 9 of Chapter VII, page 178.)

Suppose G acts on a *tree*, Γ, and let $e \in E_\Gamma$ be an edge of Γ, having vertices v and w.

Definition: The edge, e, is called a *fundamental domain* for the G-action if (i) given any edge, e' of Γ, there is a $g \in G$ such that $e \cdot g = e'$, and (ii) every vertex of Γ is equivalent, modulo the action, to eith v or w, but not both.

This second condition, of course, implies that, if we have some $v' \in V_\Gamma$ and $g \in G$ such that $v' = v \cdot g$, then $\{h \mid h \in G, v' = w \cdot h\}$ is empty. We thus have that the subgraph given just by e, v and w maps isomophically to Γ/G under the projection from Γ to Γ/G.

Lemma 29 *Suppose e is a fundamental domain for the action of G on Γ, then*

$$G_e = G_v \cap G_w.$$

Proof: First note that as Γ is a tree, $G_e \supseteq G_v \cap G_w$, since there is only the edge, e between v and w, so, if $v \cdot g = v$ and $w \cdot g = w$, then the only possibility is that $e \cdot g = e$.

On the other hand, v and w are not in the same G-orbit, by the fact that e is a fundamental domain, hence, if $e \cdot g = e$, then g must fix both v and w, which proves the opposite inclusion. ∎

Theorem 12 *(Serre) Let G be a group acting on a tree Γ in such a way that there is a fundamental domain, (which is an edge, e, as above), then G is a 'free product with amalgamation' $G \cong G_v \sqcup_{G_e} G_w$,* i.e.*, there is a pushout square,*

$$\begin{array}{ccc} G_e & \longrightarrow & G_v \\ \downarrow & & \downarrow \\ G_w & \longrightarrow & G \end{array}$$

in which the top horizontal and left vertical arrows are inclusions of subgroups, (and hence the other two arrows are monomorphisms).

Conversely given a pushout,

$$\begin{array}{ccc} A & \longrightarrow & G_1 \\ \downarrow & & \downarrow \\ G_2 & \longrightarrow & G \end{array}$$

in which $A \to G_1$ and $A \to G_2$ are monomorphisms, there is a tree on which G acts, as above, with G_1, G_2 and A being G_v, G_w and G_e for the obvious notational choice, and in which e is a fundamental domain.

Sketch proof: There is clearly a morphism from $G_v \sqcup_{G_e} G_w$ to G since the square given in the statement of the result clearly commutes. We have to show this is an isomorphism.

We form the graph of groups corresponding to the G-action on Γ, which will be

$$G_v \underset{G_e}{\longrule} G_w \ ,$$

or, in the notation we have used earlier,

$$G_v \longleftarrow G_e \longrightarrow G_w \ ,$$

and which has $G_v \sqcup_{G_e} G_w$ as its fundamental group. If, now, $g \in G$, we can look at the unique path in the *tree*, Γ, from v to $v \cdot g$. This must start something like the following

$$v \overset{e}{\longrule} w \overset{eg_1}{\longrule} vg_1 \overset{eg_1g_2}{\longrule} wg_1g_2 \longrule \dots ,$$

but then, as $wG_1 = w$, $g_1 \in G_w$, and as $g_2 \in G_{vg_1}$, $g_1g_2g_1^{-1} \in G_v$, and you continue on in this form until eventually you get to $v \cdot g$. Hence g can be written as a word in the elements of G_v and G_w. This shows that the natural map in onto. (There is ambiguity in the word one gets but the amalgamation handles that. That also handles the injectivity of the natural map. We **leave the details for you to complete**. They are very similar to the arguments we looked at when examining $\pi_1(N(\mathfrak{H}))$. The similarity is not by accident as is fairly clear it is hoped.)

We now turn to the converse. Given that $G = G_1 \sqcup_A G_2$, there is virtually no choice as to how to construct a graph, Γ which will 'undo' the construction above. We must take the set of vertices to be $(G_1\backslash G) \sqcup (G_2\backslash G)$, and the set of edges to be $A\backslash G$, in each case the set of right cosets of the respective subgroup. As $A \subseteq G_1$, there is a natural map $A\backslash G \to G_1\backslash G$, and similarly $A\backslash G \to G_2\backslash G$, giving the source and target maps of the graph, Γ, that we are constructing, so $Ag \in A\backslash G$ joins G_1g to G_2g. The group, G, acts on the right on Γ with any edge a fundamental domain and with G_1, G_2 and A as the corresponding stabilisers. The graph Γ is connected by an argument similar to that above, whilst it can have no non-trivial reduced loops, so is a tree. ■

Remark: The observant reader will, of course, have noticed that, as G_1 and G_2 are subgroups of G, we can form $N(\mathfrak{H})$, for $\mathscr{H} = (G_1, G_2)$. This means, as we have stated before when discussing families of groups and their nerves, that $N(\mathfrak{H})$ has as vertices the elements of $(G_1\backslash G) \sqcup (G_2\backslash G)$. What about the 1-simplices?

We have $\sigma = \langle G_1g, G_2h\rangle$ is a 1-simplex of $N(\mathfrak{H})$ if there is an $x \in G_1g \cap G_2h$, and, as we saw earlier, then $\sigma x^{-1} = \langle G_1, G_2\rangle$. This edge of $N(\mathfrak{H})$ is $A1$, the coset of A with representative 1, and hence the basic edge of the graph, Γ, going between the cosets / vertices G_11 and G_21. We can thus define

$$N(\mathfrak{H}) \to \Gamma$$

by $\langle G_ig\rangle$ goes to G_ig, of course, and $\langle G_1g, G_2h\rangle = \langle G_1, G_2\rangle x$ goes to Ax in Γ. This is a simplicial isomorphism (or as both are 1-dimensional simplicial complexes, *i.e.*, graphs, an isomorphism of graphs). There is thus a considerable overlap between the Abels-Holz theory and (part of) the theory of graphs and complexes of groups. It is only a part of that theory, however, since the above result, and these remarks, only handle the case of actions having a fundamental domain.

If the action of G has no fundamental domain then some edges would have both ends within the same G-orbit. This means that the action would not be regular. Examples of this occur with

HNN-extensions. We have some group, H, with a subgroup, A, together with a monomorphism, $\theta : A \to H$. The HNN-extension, $H*_{(A,\theta)}$, is obtained by adjoining an element t to H, subject to the relations:

$$t^{-1}at = \theta(a)$$

for all $a \in A$. To see the relationship with graphs of groups examine the fundamental group of the graph of groups, $\mathcal{G}$, with underlying graph the graph with one vertex, v, and one edge, e, and nothing else. Take the vertex group, G_v, to be H, the edge group, G_e, to be A, and the two morphisms from G_e to Gv are the inclusion of A into H and the given monomorphism, θ. Clearly $\Pi_1(\mathcal{G})(v) \cong H*_{(A,\theta)}$.

The interesting question is: if we have (higher) presentation data on A and H, can we get similar information on $H_{(A,\theta)}$?

Remark: There is a neat way of considering HNN-extensions via pushouts of groupoids, (rather than of groups) that ties in nicely with our glance at total groupoids above. As before, we let H be a group and A a subgroup together with a monomorphism, $\theta : A \to H$. We form a pushout of groupoids:

$$\begin{tikzcd}
\{0,1\} \times A[1] \arrow[r, "k"] \arrow[d, "i"'] & H[1] \arrow[d] \\
\mathcal{I} \times A[1] \arrow[r] & \mathcal{G}
\end{tikzcd}$$

where, as ever, $\mathcal{I}$ is the interval groupoid, $k(0,1) = a$, $k(1,a) = \theta(a)$, and i is the inclusion of the two ends of the cylinder. As $H[1]$ is a 1-object group so is $\mathcal{G}$, so it is $G[1]$ for some group, G. This group, G can be written as a factor group

$$C_\infty * H/\{(t^{-1}a^{-1}t)\theta(a) \mid a \in A\},$$

where $C_\infty = \langle t \mid \emptyset\rangle$, is the infinite cyclic group generated by an element t, so G is $H*_{(A,\theta)}$ in our earlier notation.

For HNN-extensions as here, there is also a tree, Γ, with an action of the group $H*_{(A,\theta)}$ in this case, but it is not as easy to describe. In general, given any graph of groups, $\mathcal{G}$ with fundamental group $G = \Pi_1(\mathcal{G})(v)$, there is a tree, T, with an action of G on it, such that the graph of groups that results from that action is isomorphic to $\mathcal{G}$. The idea of the construction is an analogue of the construction of a universal covering space using homotopy classes of paths based at a base point. We will not give it here, as we will see generalisations later on.

4.5.7 Fundamental group(oid) of a complex of groups

We now will go back to the higher dimensional situation, since, apart from any other reason, the Abels - Holz context with a family of n subgroups naturally leads to a $(n-1)$-simplex of groups, so graphs of groups are not general enough for their study. In fact, as we mentioned earlier, general complexes of groups are closely related to orbifolds and more generally to topological stacks, so the families of groups case is just one example of a situation leading to their study.

We now need to extend the definition of a fundamental group from applying to graphs of groups to the more general case.

Let $G(K) = (K, G_\sigma, \psi_a, g_{a,b})$ be a complex of groups as before, and, for convenience, let $E_K^\pm$ denote the set of edges of the barycentric subdivision, K', with an orientation, $a^+ = a$, and a^- to be a with the opposite orientation, so $i(a^-) = t(a^+)$, etc.

First define $FG(K)$ to be the group generated by

$$\bigsqcup\{G_\sigma : \sigma \in V_K\} \cup E_K^\pm$$

subject to the relations

- the relations of each G_σ,
- $(a^+)^{-1} = a^-$ and $(a^-)^{-1} = a^+$,
- $\psi_a(g) = a^- g a^+$ for $g \in G_{i(a)}$,
- $(ba)^+ g_{a,b} = b^+ a^+$ for composable a, b.

The image of G_σ in $FG(K)$ will be denoted $\bar{G}_\sigma$.

Haefliger defines $\pi_1(G(K), \sigma_0)$ in two equivalent ways:

Definition: Version 1: If $\sigma_0, \sigma_1 \in V_K$, the vertices of K, a $G(K)$*-path*, c, from σ_0 to σ_1 is a sequence, $(g_0, e_1, g_1, \cdots, e_n, g_n)$, where $(e_1, \cdots, e_n)$ is an edge path in K' from $i(e_1) = \sigma_0$ to $t(e_n) = \sigma_1$, $e_i \in E_K^\pm$, for $i = 1, \cdots, n$, and where $g_k \in G_{t(e_k)} = G_{i(e_{k+1})}$.

Such a $G(K)$-path, c, represents $g_0 e_1 \cdots e_n g_n \in FG(K)$. Two such paths from σ_0 to σ_1 are said to be *homotopic* if they represent the same element of $FG(K)$. We set $\Pi_1(G(K), \sigma_0, \sigma_1)$ equal to the subset of $FG(K)$ represented by $G(K)$-paths from σ_0 to σ_1. These can be used to form a *fundamental groupoid* of $G(K)$. When $\sigma_0 = \sigma_1$, we write

$$\pi_1(G(K), \sigma_0) = \Pi_1(G(K), \sigma_0, \sigma_0).$$

This is a subgroup of $F(G)$ and is called the *fundamental group* of $G(K)$.

Definition: Version 2: Assume K is connected and pick a maximal tree, T, in the 1-skeleton of $Sd(K) = K'$. Let $N(T)$ be the normal subgroup of $FG(K)$ generated by $\{a^+ : a \in T\}$, then

$$\pi_1(G(K), T) \cong FG(K)/N(T),$$

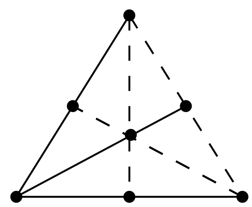
and hence has a presentation:

- generators	$\sqcup G_\sigma \sqcup E_K$	
- relations :	- $g_1 \cdot g_2 = g_1 g_2$	within any particular G_σ
	- $\psi_a(g) = a^{-1} g a$	$g \in G_{i(a)}$,
	- $(ba) g_{a,b} = b.a$	if $a, b \in E_K$ are composable
	- $a = 1$	if $a \in T$.

Clearly if we just have a complex of groups which is a graph of groups then the above gives the fundamental group of that in the previous sense.

If we restrict back to the Abels-Holz situation of group with a family of subgroups we get the following:

Example: Suppose $\mathfrak{H} = (G, \mathscr{H})$, $\tilde{K} = N(\mathfrak{H})$, $\mathscr{H} = \{H_1, \cdots, H_n\}$, so $K = \Delta^{n-1}$. Pick the maximal tree with edges radiating out from the vertex $\{H_1\}$, e.g. if $n = 3$, we get figure 4.5.7



Barycentric Subdivision of Δ^2 with the chosen maximal tree shown.

There is an obvious collapse of Δ^{n-1} to T. We have already noted that all the $g_{a,b}$ are trivial in these examples so we can prove (inductively via the collapsing order) that if a is any edge in $Sd(\Delta^{n-1})$, the fact that $\alpha = 1$ for $\alpha \in T$ implies that $a = 1$ in $\pi_1(G(K), 1)$. Thus $\pi_1(G(K), 1)$ has a presentation with

- generators $\sqcup G_\sigma$
- relations : - $g_1 \cdot g_2 = g_1 g_2$ within any particular G_σ
  - $\psi_\alpha(g) = g$ for $g \in G_{i(\alpha)}$

As $G_\sigma = \bigcap\{H_i \mid i \in \sigma\}$, we have

$$\pi_1(G(K), 1) \cong \underset{\cap}{\sqcup} H_i,$$

the coproduct of the H_i amalgamated over the intersection.

It is noticeable that there is, as before, a homomorphism

$$\pi_1(G(K), 1) \to G$$

with kernel $\pi_1 N(\mathfrak{H})$.

4.5.8 Haefliger's theorem on developable complexes of groups

Theorem 13 *Let K be a connected simplicial (cell) complex and $G(K)$ be a complex of groups on K. Assume that for each σ, the natural homomorphism from G_σ to $FG(K)$ is injective, then $G(K)$ is developable, (*i.e.*, one can construct a $\tilde{K}$ with a group G acting on it, with $\tilde{K}/G$ isomorphic to K and $G(K)$ isomorphic to the associated complex of groups of the G-action).* ■

Instead of a detailed proof we will just give the construction, **leaving you to check through the details** or to **look them up** for instance in [54].

Construction: For simplicity, we will assume that K is a simplicial complex. The more general case of it being a simplicial cell complex and thus determined by a scwol is handled in [54]. Choose a maximal tree, T in the 1-skeleton of $K' = Sd(K)$, the barycentric subdivision of K, then there is a simply connected $\tilde{K}$ and an action of $G = \pi_1 G(K)$ (without inversion) on $\tilde{K}$ and a projection $p : \tilde{K} \to K$ inducing $\tilde{K}/G \cong K$. The complex $\tilde{K}$ is specified via its associated category, $C(\tilde{K})$, as follows:

- the set, $V(\tilde{K})$, of simplices of $\tilde{K}$ is the set,

$$\{(G_\sigma g, \sigma) \mid \sigma \in S_K = V_{Sd(K)}, g \in G = \pi_1 G(K)\};$$

  these form the objects of $C(\tilde{K})$, then $p((G_\sigma g, \sigma) = \sigma$;

- the set, $E(\tilde{K})$, of 'edges' will be $\{(G_\sigma g, a) \mid a \in E(X), i(a) = \sigma\}$, then $p(G_\sigma g, a) = a$, whilst $i(G_\sigma g, a) = (G_\sigma g, i(a))$ and $t(G_\sigma g, a) = (G_{t(a)} a g, t(a))$.

This is well defined for $h \in G_{i(a)}$, since $ahg = \psi_a(h)ag$ in $FG(K)$.

The action of G is obvious and there is an obvious lifting of T to a maximal tree, $\tilde{T}$ in $\tilde{K}$ via a 'choice', $\tilde{\sigma} = (G_\sigma, \sigma)$, $\tilde{a} = (G_{i(a)}, a)$, which yields an associated complex of groups isomorphic to $G(K)$.

The following is a continuation of the same theme:

Theorem 14 *If $G(K)$ is obtained from a group, $\bar{G}$, acting on some complex, $\bar{K}$, with $K = \bar{K}/\bar{G}$, and there is given a lifting, $\bar{T}$, of the maximal tree, T, then there is a surjective homomorphism,$\varphi : G \to \bar{G}$, and a φ-equivariant covering map $f : \tilde{K} \to \bar{K}$ mapping $\tilde{T}$ to T over K. The kernel of φ is the fundamental group of $\bar{K}$, viewed as the 'algebraic' Poincaré-Galois group of $\bar{K}$.* ■

By the 'algebraic' Poincaré-Galois group, here we are meaning the analogue here of the group of deck transformations in covering space theory, as in Grothendieck's theory of the fundamental group.

4.5.9 Over to scwols

As we said earlier, the use of simplicial complexes is unduly restrictive in the theory in the previous section, 4.5.8 and in the development of that theory, both Haefliger and Corson used various types of cell complex which were more general than merely simplicial complexes. Eventually it has become clear that what is essential in all these was the *combinatorial data* relating to the face inclusions of the cell complexes. As an example, we have used the partially ordered set, $(S_K, \supseteq)$, of simplices of a simplicial complex, K, where $\sigma \leq \tau$ if $\sigma \supseteq \tau$. (This was used to form the barycentric subdivision of the complex, K.) We note that this means that in the small category, $C(K)$, associated to this poset if there is an arrow from σ to τ, then τ has to be a subset of σ, so has at most the same number of elements as does σ, so morphisms in $C(K)$ 'drop dimension' (except for the identities of course). It is thus clear for several reasons that $C(K)$ is 'without loops'. This is obvious because it is a partially ordered set, but also since, if a non-identity arrow drops dimension, it cannot give one a loop. This gives an argument that is slightly more general and has some intuitive additional geometric content.

We saw (page 173) that HNN-extensions corresponded, in Bass-Serre theory, to a graph of groups whose underlying graph was not a poset. The basic graph had one vertex and one edge,

$$\begin{tikzcd} v\cdot \arrow[loop right, "e"] \end{tikzcd}$$

and hence $C(K)$ was

$$\begin{tikzcd} e \arrow[r, bend left] \arrow[r, bend right] & v \end{tikzcd}$$

so is most definitely *not* a poset! It is, however, 'without loops'. 'Small categories without loops' (*i.e.*, scwols), therefore seem a good generalisation of both the simplicial complex based theory and one based on graphs. We gave the definition of a scwol earlier (on page 163) and so will not repeat it here. We did mention that most of the theory that has been developed above generalises without difficulty from the partially order set (and thus simplicial complex) case to that using scwols. This latter theory is described in detail in Bridson and Haefliger, [54], so we will not develop it here, except in so much as it overlaps with the themes of these notes. We will however give a few of the definitions to show how, in going from the simplicial complex / poset case to that involving scwols, one can adapt ideas quite easily. (It may help to think of simplicial cell complexes as they are geometric realisations of scwols, just as simplicial complexes are for posets. Again these definitions and properties are discussed in Chapter III of [54].)

First we look at encoding the 'action without inversion' and 'regular actions' type of condition in terms of scwols. We suppose given a scwol, $\mathcal{X}$, with vertex set, $V(\mathcal{X})$, and edge / arrow set, $E(\mathcal{X})$. As usual, a group G acts on $\mathcal{X}$ if there is a given homomorphism from G to the automorphism group of $\mathcal{X}$.

When G acts on $\mathcal{X}$, it will be assumed that this is 'without inversion':

Definition: The action of G on $\mathcal{X}$ is *without inversion* if

(i) for all $a \in E(\mathcal{X})$ and $g \in G$, $i(a) \cdot g \neq t(a)$ (so no g 'flips' an edge);

(ii) for all $g \in G$ and $a \in E(\mathcal{X})$, if $i(a) \cdot g = i(a)$ then $a \cdot g = a$.

Remarks: (a) The second condition means that the stabiliser of $i(a)$ contains that of a.

(b) If $\mathcal{X}$ is *finite dimensional* in the sense that it contains no infinite sequence of composable arrows, then condition (i) will automatically be satisfied since, if $i(a) \cdot g = t(a)$, we could form $a, a \cdot g, a \cdot g^2, \ldots)$ an infinite sequence since it cannot loop as $\mathcal{X}$ has no loops.

(c) Suppose that condition (ii) is satisfied and that v is a vertex / object of $\mathcal{X}$, such that $v \cdot g = v$, then g fixes any simplex, $\sigma = (a_1, a_2, \ldots, a_n) \in Ner(\mathcal{X})$ for which $i(a_1) = v$, by induction on n.

4.5.10 Quotient of a scwol by an action

We suppose that G acts on $\mathcal{X}$, and try to organise the data in $V(\mathcal{X})/G$ and $E(\mathcal{X})/G$ to form a category $\mathcal{Y}$, which, hopefully will be a scwol. We take $V(\mathcal{Y}) = V(\mathcal{X})/G$, and $E(\mathcal{Y}) = E(\mathcal{X})/G$. If a is an edge of $\mathcal{X}$, then the orbit aG is an edge from $i(a)G$ to $t(a)G$. We thus have at least a graph. If aG and bG are composable edges in that graph, then $t(a)G = i(b)G$, so $t(a) = i(b) \cdot g$ for some $g \in G$. There is thus an edge $b' = b \cdot g$ such that ab' is defined in $\mathcal{X}$. We define $aG.bG = (ab')G$, then **check this is independent of the choices made**. Associativity is easy to check as also are the identities, as $1_{vG} = 1_v G$, etc. We thus have a category (and did not use either condition for that). We write $\mathcal{Y} = \mathcal{X}/G$.

Lemma 30 *If the group action satisfies condition (i), then $\mathcal{Y}$ is a scwol.*

Proof: Suppose $a \in E(\mathcal{X})$, and let us explore the consequences of the equation $i(a)G = t(a)G$, *i.e.*, that aG is a loop in $\mathcal{Y}$, then $t(a) = i(a) \cdot g$ for some $g \in G$. If condition (i) is satisfied, this does not happen, which proves the lemma. ■

It is clear that the right notion of a morphism of scwols should be just that of a functor.

We next need some ideas from the algebraic approach to category theory given by Higgins, [151].

Definition: Given a small category, $\mathcal{X}$, and an object v of $\mathcal{X}$, the *star* of v is the set,

$$Star_n(\mathcal{X}) = \{a \in E(\mathcal{X}) \mid i(a) = v\}.$$

The *costar*, $Costar_v(\mathcal{X})$, is similarly defined with the condition $t(a) = v$.

The costar of v in $\mathcal{X}$ is, of course, the same as the star of v in $\mathcal{X}^{op}$.

Definition: If $f : \mathcal{X} \to \mathcal{X}'$ is a morphism of scwols, (with $\mathcal{X}$ non-empty), we say f is *star bijective* if for all $v \in V(\mathcal{X})$, f induces, by restriction, a bijection

$$Star_v(\mathcal{X}) \to Star_{f(v)}(\mathcal{X}').$$

We may also use related terminology: *star injective*, *star surjective*, *costar injective*, *costar surjective* and *costar bijective.*

Remark: Bridson and Haefliger, [54], p. 526, use the term 'non-degenerate' for what we have called 'star bijective'. We have chosen this latter term since 'non-degenerate' is a much overused term, but also because the terminology of 'star bijective', etc., was already used by Higgins in 1971, as was mentioned, but also is current in groupoid theory, cf. Brown, [59], which is a rewrite of his earlier books, (cf. [58]). There 'star bijective' and 'costar bijective' coincide as, in a groupoid, all arrows are invertible. The connection with covering spaces, covering groupoids, etc., is examined in these latter references and provides some insight as to the usefulness of the idea here.

If we have a group, G, acting on a scwol $\mathcal{X}$ and $\mathcal{Y} = \mathcal{X}/G$, the obvious assignment of vG to v and aG to a defines a morphism,

$$p : \mathcal{X} \to \mathcal{Y},$$

called the *projection morphism.*

Lemma 31 *If action satisfies condition (ii), above, the projection morphism is star bijective.*

Proof: We suppose that v is an object of $\mathcal{X}$ and $aG \in Star_{vG}(\mathcal{Y})$, then $i(a) \in vG$ so there is some g such that $i(a) = v \cdot g$. We thus have $i(a \cdot g^{-1}) = v$ and $a \cdot g^{-1} \in Star_v(\mathcal{X})$ has image aG, *i.e.*, p is start surjective.

We now suppose $a, a' \in Star_v(\mathcal{X})$ are such that $p(a) = p(a')$. We thus have $a = a' \cdot g$ for some $g \in G$. We knew $i(a) = i(a') = v$ and now have $i(a) = i(a') \cdot g$ as well. If condition (ii) holds, then we can conclude that $a = a'$, so p is star injective as well. ■

The above favours the star rather than the costar, but, of course, there is a dual condition to (ii) that can be put on the action namely : if $t(a) \cdot g = t(a)$, then $a \cdot g = a$. We could call this 'condition (ii)*' as it is the dual of condition (ii) above, and is true of p if condition (ii) is true of p^{op}. It is then clear that the dual of the lemma applies and if condition (ii)* holds then p is costar bijective. If both (ii) and (ii)* are satisfied then, of course, p will be both star and costar bijective. This leads to the following:

4.5.11 Coverings of small categories

Definition: A functor, $f : \mathcal{X} \to \mathcal{X}'$, between small categories is a *covering* if it is surjective on objects, star and costar bijective.

Note on terminology: Bridson and Haefliger, [54], make the definition for connected categories and with $\mathcal{X}$ non-empty, otherwise one can get to check the star and costar bijectivity conditions at no vertices, and, of course, they are then both satisfied at all the vertices that one checks! For a similar reason, for the non-connected case, they require surjectivity as otherwise there might be a component of $\mathcal{X}'$ which has no pre-image in $\mathcal{X}$ which would not seem to correspond to some of the intuitions on coverings. The terminology in Higgins, [151], is different and he uses covering to mean star bijective, with costar bijective being referred to as co-covering. He also does not insist that coverings be surjective.

Corollary 11 *If $\mathcal{X}$ is a connected scwol, and G acts on $\mathcal{X}$ without inversion, and satisfies (ii)*, then $p : \mathcal{X} \to \mathcal{X}/G$ is a covering. In particular, any free action of G on $\mathcal{X}$ yields a covering $p : \mathcal{X} \to \mathcal{X}/G$.*

Proof: This is just a question of collecting up and repackaging some of the earlier stuff, except for the last statement, but, if G acts freely, then the condition $i(a) \cdot g = i(a)$ implies $g = 1_G$, and similarly for that involving $t(a)$, thus both (i) and (ii)* are both automatically satisfied. ■

Definition: A *Galois covering* of a scwol, $\mathcal{Y}$, with Galois group, G, is a covering, $p : \mathcal{X} \to \mathcal{Y}$, together with a *free* action of G on $\mathcal{X}$ such that p induces an isomorphism between $\mathcal{X}/G$ and $\mathcal{Y}$.

In particular, if $p : \mathcal{X} \to \mathcal{Y}$ is a Galois covering, as above, then G must act in a simply transitive way, *i.e.*, given $x, x' \in V(\mathcal{X})$ with $p(x) = p(x')$, there is a *unique* $g \in G$ such that $x \cdot g = x'$, and similarly on edges. This is a particular case of a *principal G-bundle*, or *G-torsor*, over $\mathcal{Y}$ in this context of small categories. We will be seeing various variants of this idea of a G-torsor later on in the notes.

4.5.12 Fundamental group(oid) of a scwol

All this mention of coverings is reminiscent of the corresponding theory in the topological context (and even more for the 'groupoidal' context), and so there should be some analogue, one might guess, to some fundamental group or groupoid of the scwols involved. The fundamental group or groupoid of a scwol is fairly simple to define as a scwol is a small category. It therefore has a nerve (or if you prefer 'classifying space'), as defined in section 1.3.1. This will be a simplicial set, but usually is not a Kan complex, so that in general its homotopy groups are slightly less easy to define, or rather to write down, than otherwise. The fundamental groupoid of any simplicial set is very easy to define, however, as we can mimic the 'classical' edge-path definition for simplicial complexes. We saw this earlier on page 141, so just recall it below.

If K is a simplicial set, its vertices and 1-simplices form a directed graph, the 1-skeleton of K. We form the fundamental groupoid, $\Pi_1 K$, by taking the free groupoid on this directed graph and then dividing out by the 2-simplices, explicitly for each $\sigma \in K_2$, we have a relation

$$d_2(\sigma)d_0(\sigma) \Leftrightarrow d_1(\sigma).$$

We have also seen this recently in the context of a complex of groups, since a complex of groups with all its local groups trivial is nothing more than the 'underlying' complex or scwol. We have a definition of the fundamental group(oid) of a complex of groups, so have one of that underlying scwol. This gives us a presentation. (In fact, we might seem to be a bit lax here because the definitions we gave were officially for the case of a simplicial complex and its poset (scwol) of simplices, however a glance at that definition will show that it does not in anyway depend on that fact, so does fit the bill here as well.) We thus get a presentation of the fundamental groupoid, $\Pi_1\mathcal{X}$, of a scwol, $\mathcal{X}$, by defining it to be $\Pi_1 Ner(\mathcal{X})$ and then by means of this recipe. We will shortly see how to define a 'classifying space' for a complex of groups, $\mathcal{G}$, which will be the nerve of a small category associated with $\mathcal{G}$ and whose fundamental groupoid, according to this recipe again, will be the fundamental groupoid of the complex of groups.

If we pick a basepoint, v_0 in $\mathcal{X}$, or assume that it is connected, or restrict attention to a single chosen component, then we can define $\pi_1(\mathcal{X}, v_0)$, the fundamental group of $\mathcal{X}$ based at v_0, to be the vertex group, $\Pi_1\mathcal{X}(v_0)$. (The usual results as to this not depending on the choice of v_0 within its connected component, of course, hold true. It is worth noting, however, that this always hides a quite important fact. If v_0 and v_1 are two different choices of base point within the same component of $\mathcal{X}$, then yes, there is an isomorphism between $\pi_1(\mathcal{X}, v_0)$ and $\pi_1(\mathcal{X}, v_1)$. The isomorphism will depend on the path chosen between v_0 and v_1, so it is not always sufficient just to say the two groups are isomorphic, and that is why, quite often, it is good to use $\Pi_1\mathcal{X}$ instead of $\pi_1(\mathcal{X}, v_0)$, even when $\mathcal{X}$ is connected.)

We can also give a presentation of $\pi_1(\mathcal{X}, v_0)$ by choosing a maximal tree, T, in $\mathcal{X}$ and then setting any generating edge that is in T to be equal to 1. We can also use the terminology of edge paths and homotopies between them to build a useful equivalent description of $\pi_1(\mathcal{X}, v_0)$ and this can, also, be a useful intuition to have to hand, so we will next describe that briefly.

As usual, given $\mathcal{X}$, an *edge path* joining objects / vertices v_0 and v_1, is a sequence, $\mathsf{e} = (e_1, \ldots, e_k)$, of elements of $E^{\pm}(\mathcal{X})$, *i.e.*, e_i is either in $E(\mathcal{X})$ or its reversal $e^- \in E(\mathcal{X})$. These elements satisfy $i(e_1) = v_0$, $i(e_j) = t(e_{j-1})$, $j = 1, \ldots, k$, $t(e_k) = v_1$, (and where $i(e^-) = t(e)$, etc., as always).

Remark: As we have mentioned before, the empty sequence causes a bit of a nuisance here. We 'really' need to specify an edge path to be a sequence $(x_0, e_1, x_1, \ldots, e_k, x_k)$ with the $x_i \in V(\mathcal{X})$ and $e_i \in E^{\pm}(\mathcal{X})$ having $i(e_j) = x_{j-1}$, $t(e_j) = x_j$ for $j = 1, \ldots, k$, then the identity at x_0 is the sequence (x_0). We will usually abuse notation however and omit the intermediate objects.

The description of $\Pi_1\mathcal{X}$ or of $\pi_1(\mathcal{X}, v_0)$, then proceeds via a notion of homotopy of edge paths, where homotopy is determined by reduction rules: $e \cdot e^- \Leftrightarrow 1_{i(e)}$ together the composition rules corresponding to composition within $\mathcal{X}$, (since up to this point we have not used that $\mathcal{X}$ is a small *category*, so has a composition). We will look at this in more detail shortly.

As we might expect, coverings have nice lifting properties with respect to paths. Suppose that we have a covering, $f : \mathcal{X} \to \mathcal{Y}$, a path, $\mathsf{e} = (e_1, \ldots, e_k)$, starting at some $y_0 \in \mathcal{Y}$, and an object, x_0, of $\mathcal{X}$ such that $f(x_0) = y_0$. For ease of exposition, let us first assume $e_1 \in E(\mathcal{X})$ with $i(e_1) = y_0$. As f is star bijective, it induces a bijection $Star_{x_0}(\mathcal{X}) \to Star_{y_0}(\mathcal{Y})$, so there is a unique edge $\bar{e}_1$ in $Star_{x_0}(\mathcal{X})$ such that $f(\bar{e}_1) = e_1$. If e_1 was the reverse of an edge of $\mathcal{Y}$, then we would use the bijective correspondence between $Costar_{x_0}(\mathcal{X})$ and $Costar_{y_0}(\mathcal{Y})$ instead. A simple proof by induction on the length of e then gives:

Proposition 43 *Any covering map, $f : \mathcal{X} \to \mathcal{Y}$, has unique path lifting.* ■

There is also a result about lifting homotopies, but this will be **left to you**. (This is similar to the above, but it would probably be helpful if we gave some more details on homotopies of edge paths, packaged in a way that was more conducive for that result!)

Homotopies of edge paths: It will help to be a bit more explicit about homotopy of edge paths in $\mathcal{X}$, not only for the above, but so as to able to mimic the topological construction of simply connected / universal coverings.

Let $\mathsf{e} = (e_1, \ldots, e_n)$ be an edge path in $\mathcal{X}$, joining v_0 to v_1. We can clearly perform the following operations on e (corresponding to the rewrites / relations that we used when presenting the fundamental groupoid of $\mathcal{X}$).

(A) Assume that, for some j, $e_j = e$ and $e_{j+1} = e^-$, for some $e \in E(\mathcal{X}$, (or more generally for $e \in E^{\pm}(\mathcal{X})$), then we can rewrite e to e', where e' is obtained from e by deleting the two entries, e_j and e_{j+1} and 'closing up'. (Note this also includes the case $e_j = e^-$ and $e_{j+1} = e$, since $(e^-)^- = e$.) We can also use the reverse operation, inserting an edge and its reverse in any suitable place.

(B) If we have $e_j = e$, $e_{j+1} = e'$ with both of $e, e' \in E(\mathcal{X})$, (*i.e.*, they are both 'original edges'), then we can replace e by $\mathsf{e}' = (e_1, \ldots, e_j e_{j+1}, \ldots, e_n)$, where $e_j e_{j+1}$ is the composite in $\mathcal{X}$ of the two arrows. We also have the reverse of this operation, factorising an arrow within $\mathcal{X}$ and inserting the two factors in place of their composite in the string of edges.

(C) If an edge, e_j, is an identity arrow at some $v = i(e_j)$, then we can replace e by the contracted path leaving out e_j. We can also insert appropriate identities.

These moves or rewrites may sometimes be called *elementary homotopies.* Note they do not change the start or end vertex of a path. A homotopy between two edge paths from v_0 to v_1 will be a sequence of such elementary homotopies (and their inverses). The usual terminology and related notation will be used such as 'homotopy classes', $[\mathsf{e}]$, denoting the homotopy class of e, and so on.

4.5.13 Constructing a simply connected covering of a connected category

Continuing that use of the usual terminology, we will say that a connected small category, $\mathcal{X}$, is *simply connected* if $\pi_1(\mathcal{X}, v)$ is the trivial group.

The obvious way to construct a covering of a small category, $\mathcal{X}$, is by mimicking the standard construction of a universal covering from topology. This construction for categories is given in detail in [54], p. 580, so we will just sketch what is needed here.

We pick a vertex / object, v_0, in $\mathcal{X}$, to act as a base-point and form a category, $\mathcal{X}'$, in which the objects are the homotopy classes, $[\mathsf{e}]$, of paths starting at v_0. The functor, $p : \mathcal{X}' \to \mathcal{X}$, sends an object, $[\mathsf{e}]$, to the 'other end', $t(e_n)$, of $[\mathsf{e}]$. As before, let $\mathsf{e} = (e_1, \ldots, e_n)$ be an edge path. An arrow starting at $[\mathsf{e}]$ is a pair, $([\mathsf{e}], e')$, where $e' \in E(\mathcal{X})$ with $i(e') = t(e_n) = p([\mathsf{e}])$; the start vertex of $([\mathsf{e}], e')$ will be $[\mathsf{e}]$, as we said, whilst the target vertex will be $[\mathsf{e} \cdot e']$, the homotopy class of the path obtained by concatenating e with the edge e', so if $\mathsf{e} = (e_1, \ldots, e_n)$, $\mathsf{e} \cdot e' = (e_1, \ldots, e_n, e')$.

Composition is the final structure to define on $\mathcal{X}'$, and it is obtained in the obvious way so as to make p a functor. Associativity etc. are then **easy to check**.

Lemma 32 *The functor, $p : \mathcal{X}' \to \mathcal{X}$, is a covering.* ■

This is just a question of **checking** star and costar bijectivity. The first is more or less immediate; the second only slightly less so.

As in the topological situation (or the simplicial one, or the groupoid one, etc., cf. Gabriel and Zisman, [132], or Brown, [59]), this covering is a universal covering, is a simply connected covering, and has lots of nice properties. We will start making this more precise and giving (sketch) proofs.

Definition: A covering, $p : \mathcal{X}' \to \mathcal{X}$, of a connected $\mathcal{X}$ is said to be *universal* if $\mathcal{X}'$ is connected and, for every covering $q : \mathcal{X}'' \to \mathcal{X}$ of $\mathcal{X}$ together with objects v' of $\mathcal{X}'$ and v'' of $\mathcal{X}''$ satisfying $p(v') = q(v'')$, there is a functor, $f : \mathcal{X}' \to \mathcal{X}''$, over $\mathcal{X}$ (so $qf = p$) such that $f(v') = v''$.

It is usual to expect uniqueness on such a universal property, but here it is a consequence of the other properties.

Lemma 33 *(i) Any functor, $f : (\mathcal{X}', p) \to (\mathcal{X}'', q)$, over $\mathcal{X}$ between (connected) coverings is itself a covering.*

(ii) If f and f' both satisfy the properties in the definition, then they are equal.

Proof: (i) First we note that the result does not depend on $(\mathcal{X}', p)$ being constructed as above, and that connectedness is only needed to avoid some obvious silly counterexamples. We thus start with arbitrary connected $(\mathcal{X}', p)$ and $(\mathcal{X}'', q)$ and a morphism f between them (so a functor with $qf = p$). Let v' be an object of $\mathcal{X}'$. We look at f restricted to $Star_{v'}(\mathcal{X}')$. We have to show that this is a bijection. We know $p : Star_{v'}(\mathcal{X}') \to Star_{p(v')}(\mathcal{X})$ is a bijection as is $q : Star_{f(v')}(\mathcal{X}'') \to Star_{q(f(v'))}(\mathcal{X}) = Star_{p(v')}(\mathcal{X})$, (where we have omitted using a different notation for the restrictions to the stars in each case). We therefore have that f restricted to $Star_{v'}(\mathcal{X}')$ is a bijection (with inverse $p^{-1}q$ after suitable restrictions). Of course, the same argument applies to costars with minor changes. We thus have that f is a covering.

(ii) We have $qf = qf' = p$ and $f(v') = f'(v')$, so we look at f and f' restricted to $Sta_{v'}(\mathcal{X}')$. Here they coincide as both are $q^{-1}p$; similarly on the costars. If we want to see what they do to some object or arrow of $\mathcal{X}'$, then we can find a 'zig-zag' of arrows joining v' to that object or arrow, since $\mathcal{X}'$ is connected. we use 'continuation' along this zig-zag to show that f and f' must agree on the given object or arrow. The details are **left to you** to write down. ■

Proposition 44 *The covering, $(\mathcal{X}', p)$, constructed above is a universal covering.*

Sketch proof: Suppose we have some connected covering, $q : \mathcal{X}'' \to \mathcal{X}$, together with an object $v' = [\mathsf{e}]$ in $\mathcal{X}'$ and another, v'', in $\mathcal{X}''$ satisfying $q(v'') = p(v')$.

We first look at the simple case, where $v' = [(v_0)]$, *i.e.*, the homotopy class of the constant path at v_0. (We write v'_0 for this and will use it later.) If $w' = [\mathsf{e}']$ is another object of $\mathcal{X}'$, then e' is a path from v_0 to $p(w')$, which we lift to a path, $\tilde{\mathsf{e}'}$ in $\mathcal{X}''$ starting at v'', our given object in $\mathcal{X}''$. (Remember, as we are in the case $v' = v'_0$, $q(v'') = v_0$, so this does work.) We define $f(w') = t(\tilde{\mathsf{e}'})$. This *does* give $qf(w') = p(w')$. We can see this works as a functor, since, if $([\mathsf{e}'], e')$ is an arrow starting at w', then the end of $([\mathsf{e}'], e')$ is $t(\mathsf{e}' \cdot e') = t(e')$, and unique path lifting give a path from $f(w')$ to $f(t(e'))$ in $\mathcal{X}''$. This will be $f([\mathsf{e}'], e')$. This not only gives a functor, but shows that f *must* be constructed like this as the value of $f(w')$ is determined bit-by-bit by the path, e', to which it corresponds.

Next we go to the general case with $v' = [\mathsf{e}]$, v'' in $\mathcal{X}''$, etc. If $w' = [\mathsf{e}']$ is in $\mathcal{X}'$, then $\mathsf{e}^{-1} \cdot \mathsf{e}'$ is a path from $p(v')$ to $p(w')$, which can be uniquely lifted to $\mathcal{X}''$ to a path, $\tilde{\mathsf{e}}^{-1} \cdot \tilde{\mathsf{e}'}$ starting at v', and we try $f(w') = t(\tilde{\mathsf{e}}^{-1} \cdot \tilde{\mathsf{e}'})$. It remains only to check that everything works, but **that is left to you**. (We note the extent to which unique path and homotopy lifting is used in this.) ■

Other facets of the construction and the properties of universal coverings go across from the classical topological case to this categorical one in a fairly easy manner. We will give a few of these, chosen, in the main, for links with ideas that will be developed later on in the notes. First some more general information about coverings and path lifting.

Proposition 45 *Let $q : \mathcal{Y} \to \mathcal{X}$ be a covering of a connected category, $\mathcal{X}$. Let x_0 in $\mathcal{X}$ and y_0 in $\mathcal{Y}$ be objects such that $q(y_0) = x_0$, then the induced homomorphism,*

$$q_* : \pi_1(\mathcal{Y}, y_0) \to \pi_1(\mathcal{X}, x_0),$$

is an injection. ■

This is a consequence of the fact that unique path lifting holds and that homotopies lift.

Proposition 46 *(i) For any covering, $q : \mathcal{Y} \to \mathcal{X}$, there is a functor, $\mathsf{L}(q) : \Pi_1\mathcal{X}^{op} \to Sets$, defined by $\mathsf{L}(q) : (x) = q^{-1}(x)$, the 'fibre' over x, and, if $[\mathsf{e}] : x \to x'$ in $\Pi_1\mathcal{X}$, then, for $y \in q^{-1}(x)$, $\mathsf{L}(q) : ([\mathsf{e}])(y) = t(\tilde{\mathsf{e}})$, where $\tilde{\mathsf{e}}$ is the unique path in $\mathcal{Y}$, starting at y and covering e.*

(ii) Let $\mathsf{R}/\mathcal{X}$ denote the category of coverings of $\mathcal{X}$, then there is a functor,

$$\mathsf{L} : \mathsf{R}/\mathcal{X} \to Sets^{\Pi_1\mathcal{X}^{op}},$$

given by assigning $\mathsf{L}(q)$ to q.

Proof: (i) This is mostly just checking that things are functorial, so that part of the proof will be **left to you, the reader**. For part (ii), it is worth noting that $\mathsf{R}/\mathcal{X}$ is a subcategory of the category of small categories over $\mathcal{X}$, and that $Sets^{\Pi_1\mathcal{X}^{op}}$ is the category of functors from $\Pi_1\mathcal{X}^{op}$ to the category of sets. (These are sometimes called *local systems*, hence the letter L.) If $f : (\mathcal{Y}_1, p_1) \to (\mathcal{Y}_2, p_2)$ is a morphism of coverings, we have $\mathsf{L}(p_i) : \Pi_1\mathcal{X}^{op} \to Sets$, for $i = 1, 2$ and need a natural transformation, $\mathsf{L}(f) : \mathsf{L}(p_1) \to \mathsf{L}(p_2)$,between them, and that means, for each x in $\mathcal{X}$, we need $\mathsf{L}(f)(x) : p_1^{-1}(x) \to p_2^{-1}(x)$. The obvious simplest thing to try is simply f restricted to the fibre over x, and then compatibility with paths, etc., follows again through unique path lifting and lifting of homotopies. The remaining checking can again be **left to you**. ■

4.5.14 The groupoid case

Although it has special features which mean that the picture there simplifies, these also give an indication of what a general theory of coverings in these categorical contexts may involve. The theory in some generality is handled in Gabriel and Zisman, [132], p. 140, and various other sources. A slightly different approach has been given in [226], as well, adapting the Gabriel-Zisman treatment to fit in with the general theory of coverings put forward by Grothendieck in SGA1, [141], which in part explains the use of R for covering as it reflects the initial letter of 'revêtement', which is the French for 'covering' in this sense. It is also important to note that coverings of groupoids are special cases of *fibrations of groupoids* in which lifts exist but need not be unique; see [57] and [59] and later here when we consider fibred categories / fibrations of *categories* in some detail in Chapter 9.

To any covering $p : \mathcal{R} \to \mathcal{G}$ of a (connected) groupoid $\mathcal{G}$, we can assign the local system, $L(p) : \mathcal{G}^{op} \to Sets$, which sends an object x in $\mathcal{G}$ to the fibre over x, that is, $p^{-1}(x)$. Conversely given any $L : \mathcal{G}^{op} \to Sets$, we can assign a covering, $\mathsf{R}(L) = (p_L : \mathcal{R}(L) \to \mathcal{G})$, and we turn to this next.

The above description of the construction of a 'universal' covering category was 'bare hands', *i.e.*, the building of $\mathcal{X}'$ was given without much machinery and with minimal motivation that it would work. The description of $p_L : \mathcal{R}(L) \to \mathcal{G}$, given in [132] is more 'elegant' and starts towards general categorical considerations that will be examined later on, in particular, the Grothendieck construction (section 9.2) and the homotopy colimit, section 14.6.1. This construction is therefore very important as part of the 'categorification' process, although it is still fairly non-technical. It is clearly a categorical version of a semi-direct product construction.

Given a local system, $L : \mathcal{G}^{op} \to Sets$, on $\mathcal{G}$, the set of objects of $\mathcal{R}(L)$ is the disjoint union, $\bigsqcup\{L(x) \mid x \in Ob(\mathcal{G})\}$. (We actually need L to be non-empty if what follows is to work without glitches.) As usual, it is useful to write elements of a disjoint union as pairs, one part indicating an index the other an element in the set corresponding to that index. The order does not matter, but here we will take (y, x), where x is an index, in this case an object of $\mathcal{G}$, and $y \in L(x)$ is simply an element. (It is worth recalling that this care is important, for instance, if L was a constant functor, since the same y might occur in many different sets of the family, $\{L(x) \mid x \in Ob(\mathcal{G})\}$.) It is sometimes useful to write this ordered pair in the form, $y \otimes x$, or $x \otimes y$, depending on the conventions, as if we had a tensor product, even if we are in a non-additive situation. The functor, p_L, on objects, will map (y, x) to x.

A morphism in $\mathcal{R}(L)$ from (y, x) to (y', x') will consist of a morphism $f : x \to x'$. This will induce $L(f) : L(x') \to L(x)$, since L is contravariant. We need, then to specify that $y = L(f)(y')$. Again it is often useful to write the morphism as (y', f) as otherwise the same f will occur as the name of many morphisms. (This is the same point we made back on page 17 when talking of action groupoids. In fact, if a group G acts *on the right* on a set X, then this corresponds to a local system $L : G[1]^{op} \to Sets$, and our $\mathcal{R}(L)$ in this case is just the action groupoid of that action, adjusted for the right actions rather than the left action as given back there.) In this notation, $(y', f) : (L(f)(y'), x) \to (y', x')$. The 'functor', $p_L : \mathcal{R}(L) \to \mathcal{G}$, then sends (y', f) to f. We say 'functor' since we have not yet got a composition in the structure that we have denoted by $\mathcal{R}(L)$. (Beware, this is where the details get a bit confusing, because we are using both the function composition and the algebraic/concatenation conventions in the same structure, so we advise **take your time and check the details!**) Composition in $\mathcal{R}(L)$ is defined as follows: if $f : x \to x'$ and $g : x' \to x''$, then $fg : x \to x''$. That looks simple, but we need to see this in the more detailed notation. We have $(y', f) : (L(f)(y'), x) \to (y', x')$ and, say, $y'' : (L(g)(y''), x') \to (y'', x'')$, so the composite actually requires that $y' = L(g)(y'')$. We then have

$$L(f)L(g)(y'') \to (L(g)(y''), x') \to (y'', x'').$$

If things fit together as they 'should', this would correspond to a morphism from $(L(fg)(y''), x)$ to (y'', x''), but L is 'contravariant', so does this really work? Yes, it does. Look again at $L(fg)$ as a composite

$$L(x'') \xrightarrow{L(g)} L(x') \xrightarrow{L(f)} L(x),$$

so, using ordinary function notation, we can apply this to $y'' \in L(x'')$. This will give $L(f)L(g)(y'')$, so it did work: $L(fg)(y'') = L(f)L(g)(y'')$. Taking a bit of care, it is then easy to show that $\mathcal{R}(L)$ is a groupoid, and that p_L is a functor.

Remark: In fact we did nothing here that required $\mathcal{G}$ to be a groupoid and we could have started with $\mathcal{X}$ a small category with $L : \mathcal{X}^{op} \to Sets$ and would have obtained $\mathcal{R}(L)$ with $p_L : \mathcal{R}(L) \to \mathcal{X}$.

Returning to the case of a connected groupoid, $\mathcal{G}$, we note that L and R define an equivalence of categories between $\mathsf{R}/\mathcal{G}$ and $Sets^{\mathcal{G}^{op}}$. (This is quite **routine to check**.) This means that coverings in the groupoid case are more or less the same as local systems of sets, *i.e.*, functors from $\mathcal{G}^{op}$ to $Sets$. We would expect, therefore, that properties of such functors should correspond to properties of coverings. They do and, in at least one case, give useful new insights on both sides of the equivalence.

Recall from elementary category theory, that a functor $F : \mathcal{C} \to Sets$ is said to be *representable* if there is an object, c of $\mathcal{C}$ and a natural isomorphism of functors between F and the functor $\mathcal{C}(c, -) : \mathcal{C} \to Sets$. (We, in fact need the case $\mathcal{C} = \mathcal{G}^{op}$, so have a functor of the form, $\mathcal{G}(-, x)$, here. It is important to think 'geometrically' here as this functor is an analogue of the costar of x in $\mathcal{G}$, recording the arrows that *end* at x, as a family indexed by their domains.)

Suppose we start with a representable local system, $L : \mathcal{G}^{op} \to Sets$, (which will eventually be identified with $L(p)$ for some groupoid covering, $p : \mathcal{R} \to \mathcal{G}$.) There is, then, by representability, some object x_0 in $\mathcal{G}$ such that $L(x) \cong \mathcal{G}(x, x_0)$, naturally in x. We look at $\mathcal{R}(L)$, or more exactly $\mathcal{R}(\mathcal{G}(-, x_0))$. This groupoid will have, as its objects, pairs, (g, x), with $x \in Ob(\mathcal{G})$ and $g \in \mathcal{G}(x, x_0)$, so it seems that $\mathcal{R}(L) \cong \mathcal{G}/x_0$, the 'slice' groupoid of 'elements over x_0'. We have to check the morphisms. A morphism from (g, x) to (g', x') will be a morphism $h : x \to x'$ such that $g = L(h)(g')$, but $L(h) : L(x') \to L(x)$ is just pre-composition with h, *i.e.*, $L(h)(g') = hg'$. We thus do have $\mathcal{R}(L) \cong \mathcal{G}/x_0$, but better than that we can see that $\mathcal{R}(L)$ must be a simply connected groupoid, *i.e.*, $\mathcal{R}(L)$ has only trivial vertex groups, since, if we have any (g, x) and look for an $h : x \to x$ such that h gives a loop at (g, x), we have to have $hg = g$, so h will be the identity. (Note that this argument *does* use that $\mathcal{G}$ is a groupoid as we have used that g was invertible.)

Suppose now we start with a covering, $p : \mathcal{R} \to \mathcal{G}$, in which $\mathcal{R}$ is a simply connected and connected groupoid. In fact, (R, p) must be a universal covering of $\mathcal{G}$. To see this, suppose $(\mathcal{S}, q)$ is another covering and that we have x an object of $\mathcal{R}$, y an object of $\mathcal{S}$ such that $p(x) = q(y)$. As $\mathcal{R}$ is simply connected, it is *codiscrete*, *i.e.*, there is exactly one arrow between any two objects of $\mathcal{R}$; see back on page 17. Suppose that x' and $r : x' \to x$ are respectively an object of $\mathcal{R}$ and the unique morphism from it to x, then $p(r) : p(x') \to p(x)$ is an arrow ending at $p(x) = q(y)$, so we use costar bijectivity to lift $p(r)$ to a uniquely defined arrow, which we will denote $f(r) : f(x') \to y$. Uniqueness ensures that this is a functor.

Finally any simply connected covering is representable. (This is now not too difficult to see, so is **left to the reader**.)

In the wider context of small categories rather than groupoids, it is easy to check that simply connected coverings are universal and conversely, but the link with representability does not go through so easily. (The point is that representability favours the costar and ignores the star. For groupoids, invertibility of arrows eliminates the problem. There are interesting problems in handling the various analogues of covering spaces in these situations as it depends on the intended use of the theory and hence on the exact definition of covering that is used. The one we have given is not the only possibility as we have already mentioned as the question of the directed nature of paths etc. can be useful to add in as in discussions of directed homotopy. For the moment we will leave the general discussion at that.)

In both the small category and the groupoid case, there *is* a Galois theory of coverings, yielding equivalences of categories between $\mathsf{R}/\mathcal{X}$ and $Sets^{\Pi_1\mathcal{X}^{op}}$. As $\mathcal{G}$ is assumed to be connected, this latter category is equivalent to $Sets^{\pi_1\mathcal{X}^{op}}$, which is a category of right $\pi_1\mathcal{X}$-sets. We will not follow this up for the moment as it would lead us a bit too far away from complexes of groups, at least as we have been exploring them here. One final observation is that, if $\mathcal{X}$ is a connected scwol, with base point v_0, and $p : \tilde{\mathcal{X}} \to \mathcal{X}$ is a covering, (so $\tilde{\mathcal{X}}$ is also a scwol), then $\pi_1(\mathcal{X}, v_0)$ acts on $\tilde{\mathcal{X}}$, so that $p : \tilde{\mathcal{X}} \to \mathcal{X}$ is a Galois covering with $\pi_1(\mathcal{X}, v_0)$ as its Galois group. This corresponds to the local system given by the action of $\pi_1(\mathcal{X}, v_0)$ on itself by multiplication on the right.

4.6 Complexes of groups on a scwol

Earlier we pretended to use a simplicial complex as the main 'base' on which to build complexes of groups, and said that it was easy to make the transition to the more general one defined on a simplicial cell complex, and thus on a scwol. It *is* easy and could be 'left to you', but it is also convenient to now have the definition given *explicitly* before going further, so here it is. (The main reference for this is, as usual, Bridson and Haefliger, [54], p. 535.)

4.6.1 General complexes of groups revisited

Definition: Let $\mathcal{Y}$ be a scwol. A *complex of groups*, $G(\mathcal{Y}) = \{\{G_v\}, \{\psi_a\}, \{g_{a,b}\}\}$, over $\mathcal{Y}$ is given by the following data:

(1) a family, $\{G_v \mid v \in V(\mathcal{Y})$, of groups, indexed by the objects of $\mathcal{Y}$, the group G_v being called the *local group at* v;

(2) for each edge $a \in E(\mathcal{Y})$, a monomorphism,

$$\psi_a : G_{i(a)} \to G_{t(a)};$$

if a is an identity arrow in $\mathcal{Y}$, ψ_a is to be the identity homomorphism;

(3) for each pair of composable arrows, $(a, b) \in E^{(2)}(\mathcal{Y})$, a *twisting element*, $g_{a,b} \in G_{t(a)}$,

$$g_{a,b}^{-1}\psi_{ba}(\_)g_{a,b} = \psi_a\psi_b$$

and such that the "cocycle condition" holds, *i.e.*, that for each triple in $E^{(3)}(\mathcal{Y})$ of composable elements,

$$g_{a,cb}\psi_a(g_{b,c}) = g_{ab,c}g_{a,b}.$$

(This definition does need checking for the composition conventions it uses.)

If the twisting elements of $G(\mathcal{Y})$ are trivial, then we say the complex of groups is *simple*.

It is sometimes useful to write $Ad(g)$ for the (right action by) conjugation by g, and then the first condition for the ψ_as becomes

$$Ad(g_{a,b})\psi_{ba} = \psi_a\psi_b.$$

Remark: As usual in this chapter, $E^{(k)}(\mathcal{Y})$ denoted the set of composable k-tuples of elements of edges / arrows of $\mathcal{Y}$, and so is just another notation for $Ner(\mathcal{Y})_{k-1}$, the set of $(k-1)$-simplices of the nerve of $\mathcal{Y}$. This simple observation will allow us, later on, to give other descriptions of complexes of groups and to put them into a much wider context, which can be useful both for their area of geometric group theory, but for suggesting analogous higher dimensional categorified analogues of complexes of groups.

The cocycle condition, of course, reminds one of 'cohomology' and the cocycle condition for the factor set of an extension (cf. page 56), and so one should expect there to be some form of coboundary around, and there is. Suppose that we have, for each edge $a \in E(\mathcal{Y})$, an element $g_a \in G_{t(a)}$, then we can form a new complex of groups, $G'(\mathcal{Y})$ over $\mathcal{Y}$ with $G'_v = G_v$, $\psi'_a = Ad(g_a)\psi_a$ and $g'_{a,b} = g_a\psi_a(g_b)g_{a,b}g_{ba}^{-1}$(needs checking that it works) and we say that $G'(\mathcal{Y})$ is obtained from $G(\mathcal{Y})$ by *deformation* using the coboundary, $\{g_a\}$, or that it is *deduced* from $G(\mathcal{Y})$ by that coboundary.

4.6.2 Morphisms of complexes of groups

Let $G(\mathcal{Y})$ and $G(\mathcal{Y}')$ be two complexes over scwols $\mathcal{Y}$ and $\mathcal{Y}'$. (We will use a superfix 'prime' on the data for $G(\mathcal{Y}')$.)

Let $f : \mathcal{Y} \to \mathcal{Y}'$ be a functor.

Definition: A *morphism of complexes of groups*, $\phi : G(\mathcal{Y}) \to G(\mathcal{Y}')$, over f consists of

(a) a family, $\{\phi_v : G_v \to G'_{f(v)} \mid v \in V(\mathcal{Y}\}$, of homomorphisms, and

(b) a family, $\{\phi(a) \in G_{t(f(a))} \mid a \in E(\mathcal{Y})\}$, of elements of local groups, indexed by the edges of $\mathcal{Y}$,

such that

(i) $Ad(\phi(a))\psi'_{f(a)}\phi_{i(a)} = \phi_{t(a)}\psi_a$ for each $a \in E(\mathcal{Y})$,

and for all $(a,b) \in E^{(2)}(\mathcal{Y})$,

(ii) $\pi_{t(a)}(g_{a,b})\phi(ab) = \phi(a)\psi'_{f(a)}(\phi(b))g_{f(a),f(b)}$.

This definition will probably look obscure. It is the 'bare hands' / minimal technology version of something that we will see later and in that latter version is much simpler.

4.6.3 Homotopy of morphisms of complexes of groups

If we think of groups as groupoids, then we can use the natural homotopy notions of groupoids to talk about homotopy between homomorphisms of groups. This, as you probably know, corresponds firstly to natural transformations between the morphisms (which are being considered as being 'just functors'), and one step further this gives 'conjugation' as corresponding to 'homotopy'. We can already see that the definition of complex of groups itself involves a certain amount of 'homotopy' of this sort, as also does the definition of morphisms between complexes of groups, but we can also use that to obtain a notion of homotopy between morphisms of complexes of groups. To gain a bit of intuition about the following, think 'natural transformation', hence a family of elements of the objects satisfying some conditions.

Definition: Let $\phi, \phi' : G(\mathcal{Y}) \to G(\mathcal{Y}')$ be two morphisms over a given $f : \mathcal{Y} \to \mathcal{Y}'$. A *homotopy*, h, from ϕ to ϕ' is given by a family of elements, $\{h_v \mid v \in V(\mathcal{Y}), h_v \in G_v\}$, such that for all $v \in V(\mathcal{Y})$,

$$\phi'_v = Ad(h_v)\phi_v,$$

(*i.e.*, for $g \in G_v$, $\phi'_v(g) = h_v\phi_v(g)h_v^{-1}$), and, for all $v \in V(\mathcal{Y})$ and $a \in E(\mathcal{Y})$,

$$\phi'(a) = h_{t(a)}\phi(a)\psi_{f(a)}h_{i(a)}^{-1}.$$

Morphisms of complexes of groups compose in a fairly obvious way, and composition behaves nicely with respect to homotopy.

4.6.4 The category associated to a complex of groups

We associate to a complex of groups, $G(\mathcal{Y})$, a small category, $CG(\mathcal{Y})$. This has $V(\mathcal{Y})$ as its set of objects and its set of arrows are the pairs, (g, α) with $\alpha \in \mathcal{Y}$ and $g \in G_{t(\alpha)}$. The source and target maps of $CG(\mathcal{Y})$ are given by

$$i(g, \alpha) = i(\alpha) \qquad\qquad t(g, \alpha) = t(g).$$

(The composition in $CG(\mathcal{Y})$ is very like that in an extension given by a cocycle, see section 2.3.1, page 56.)

The composite of (g, α) and (h, β) is defined if $t(\alpha) = i(\beta)$ and then it is

$$(g, \alpha)(h, \beta) = (g\psi_\alpha(h)g_{\alpha,\beta}, \alpha\beta).$$

Lemma 34 *The conditions on ψ_α and $g_{\alpha,\beta}$ imposed by the definition of a complex of groups are equivalent to the associativity of this composition.*

Proof: This is very similar to the associativity of the middle term of a group extension in the Schreier theory (as on page 56) being equivalent to a cocycle condition. It therefore should be safe to **leave it as an exercise for the reader**. (The **adventurous reader may try to extend** some of the other methods of the Schreier theory to this context!) ■

We note that, clearly, the assignment: $(g, \alpha) \mapsto \alpha$ is a functor from $CG(\mathcal{Y})$ to $\mathcal{Y}$.

This construction helps to explain the slightly obscure formulation of morphism. Let $\varphi : G(\mathcal{Y}) \to G(\mathcal{Y}')$ be a morphism over $f : \mathcal{Y} \to \mathcal{Y}'$ and define $C(\varphi) : CG(\mathcal{Y}) \to CG(\mathcal{Y}')$ on objects by f and on arrows by $C(\varphi)(g, \alpha) = (\varphi_{t(\alpha)}(g)\varphi(g), f(\alpha))$.

Proposition 47 *If $\varphi : G(\mathcal{Y}) \to G(\mathcal{Y}')$ is a morphism over f, then $C(\varphi) : CG(\mathcal{Y}) \to CG(\mathcal{Y}')$ is a functor. Conversely, any functor $\varphi : CG(\mathcal{Y}) \to CG(\mathcal{Y}')$ that projects down to f, defines a morphism from $G(\mathcal{Y})$ to $G(\mathcal{Y}')$ over f.*

Sketch proof: The forward direction is fairly easy to check. For the 'converse' direction, the key is that the φ_v and $\varphi(a)$ are determined by $\varphi(g, 1_v) = (\varphi_v(g), 1_{f(v)})$ and $\varphi(1_{t(a)}, a) = (\varphi(a), f(a))$.■

Because of this, composition is now easy to define. (No prizes for guessing to what 'homotopy' corresponds!)

4.7 Orbifolds, orbihedra and groupoids

We have already looked in section 4.5.2, at complexes of groups as generalisations of the result of a group acting on a scwol or simplicial cell complex. This was, there, for the purpose of finding szyzgies, but there are similar ideas coming from other areas, and these lead on, eventually, in the direction of 'stacks'. One of these is in the theory of orbifolds. Orbifolds are a generalisation of manifolds, examples of which include the orbit spaces of any proper action by a (discrete) group on a manifold, and, hence, they include moduli space constructions, and also the leaf space of any foliation with compact leaves and finite holonomy. (The meaning of 'moduli space' and 'foliation' here will not be expanded on, merely being used as illustrations of external links for the moment, so, if you have not met there ideas before, it does not matter for this chapter.)

A useful source on orbifolds with a perspective near to the one here is Moerdijk and Pronk's [207].

4.7.1 Orbifolds

We will, more-or-less, assume that the reader has met the idea of a manifold, at least to have seen a manifold as being given by local charts, which relate its local structure to the local structure of a model sapce, which is often taken to be $\mathbb{R}^n$. Orbifolds are defined similarly, but have different local models. (We should note that orbifolds have been called V-manifolds and Satake manifolds. They were introduced by Satake, [239, 240], and then studied extensively by Thurston, [259], who, it seems, 'coined' the term 'orbifold'.)

We fix an integer, $n > 0$ and let M be a space.

Definition: (i) An *orbifold chart* on M is given by a connected open subset, $\tilde{U} \subseteq \mathbb{R}^n$, together with a finite group, G, of smooth (*i.e.*, C^∞-) automorphisms of $\tilde{U}$, a map $\varphi : \tilde{U} \to M$, so that φ is G invariant (so $\varphi \circ g = \varphi$ for all $g \in G$), and a homeomorphism of $\tilde{U}/G$ onto an open subset $U = \varphi(\tilde{U})$ of M.

(ii) An *embedding*, $\lambda : (\tilde{U}, G, \varphi) \hookrightarrow (\tilde{V}, H, \psi)$, is a smooth embedding $\lambda : \tilde{U} \hookrightarrow \tilde{V}$ such that $\psi \circ \lambda = \varphi$.

(iii) An *orbifold atlas* on M is a family, $\mathcal{U} = \{(\tilde{U}, G, \varphi)\}$, of orbifold charts, which cover M and which are *locally compatible*, in the sense that:

given any two charts, $(\tilde{U}, G, \varphi)$ *and* $(\tilde{V}, H, \psi)$ *for* $U = \varphi(\tilde{U})$, $V = \psi(\tilde{V}) \subseteq M$ *having non-empty intersection, and any* $x \in U \cap V$, *there is an open neighbourhood,* $W \subseteq U \cap V$, $x \in W$, *and chart* $(\tilde{W}.K, \chi)$, *with* $\chi(\tilde{W}) = W$, *and embeddings,* $(\tilde{W}.K, \chi) \hookrightarrow (\tilde{U}, G, \varphi)$ *and* $(\tilde{W}.K, \chi) \hookrightarrow (\tilde{V}, H, \psi)$.

This clearly generalises the notion of chart as used in the classical definition of manifold. The idea is that one has a covering of M by patches that look like a quotient of an open set of $\mathbb{R}^n$ by a group action. To complete the orbifold definition, we will need notions of refinement and equivalence of orbifold atlases.

Definition: (iv) An orbifold atlas, $\mathcal{U}$, is said to *refine* another, $\mathcal{V}$, if for every chart in $\mathcal{U}$, there is an embedding into some chart of $\mathcal{V}$.

(v) Two orbifold atlases, $\mathcal{U}$ and $\mathcal{V}$ are said to be *equivalent* if they have a common refinement.

Finally:

Definition: An *orbifold* (of dimension n) is a paracompact Hausdorff space, M, equipped with an equivalence class of orbifold atlases.

(More to go here)

Chapter 5

Beyond 2-types

The title of this chapter promises to go beyond 2-types and in particular, we want to model what is there algebraically. We have so far only done this with the crossed complexes. These do give all the homotopy groups of a simplicial group, but the homotopy types they represent are of a fairly simple type, as they have vanishing Whitehead products.

We will return to crossed complexes later on, but will first look at the general idea of n-types, going into what was said earlier in more detail.

5.1 n-types and decompositions of homotopy types

We will start with a fairly classical treatment of the ideas behind the idea of n-types of topological spaces.

5.1.1 n-types of spaces

We earlier (starting in section 3.7.1) discussed 'n-equivalences' and 'n-types'. As homotopy types are enormously complex in structure, we can try to study them by 'filtering' that information in various ways, thus attempting to see how the information at the n^{th}-level depends on that at lower levels. The informational filtration by n-type is very algebraic and very natural. It has two very satisfying interacting aspects. It gives complete models for a subclass of homotopy types, namely those whose homotopy groups vanish for all high enough n, but, at the same time, gives a set of approximating notions of equivalence that, on all 'spaces', give useful information on weak equivalences.

We start with one version of the topological notion:

Definition: Given a cellular mapping, $f : (X, x_0) \to (Y, y_0)$, between connected pointed spaces, f is said to be an *n-equivalence* if the induced homomorphisms, $\pi_k(f) : \pi_k(X, x_0) \to \pi_k(Y, y_0)$, for $1 \leq k \leq n$, are all isomorphisms. More generally, on relaxing the connectedness requirements on the spaces, a cellular mapping, $f : X \to Y$, is an *n-equivalence* if it induces a bijection on π_0, that is, $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection, and for each $x_0 \in X$ and $1 \leq k \leq n$, $\pi_k(f) : \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$ is an isomorphism.

Remark: It is important to note that here the mappings are cellular, not just continuous. We will see consequences of this later.

There are alternative descriptions and these can be useful. We recall them next, emphasising certain facts and viewpoints that perhaps have not yet been stressed enough in our earlier treatments, but can be useful for our use of these ideas here.

We start by recalling some standard notions of classical homotopy theory. We let CW be the category of all CW-complexes and *cellular* maps, and CW_{c*} be the corresponding category of pointed connected complexes, again with cellular maps. (The notions below generalise easily to the non-connected multi-pointed case.) If X is such a CW-complex, here we will write X^n for its *n-skeleton*[1], that is, the union of all the cells in X of dimension at most n. We say that X has *dimension n* if $X = X^n$.

It is important to remember that the homotopy type of X^n is *not* an invariant of the homotopy type of X. (Just think about subdivision if you are in doubt about this.) It was partially to handle this that Whitehead introduced the notion of N-type, as this does give such invariants. The two ways of viewing n-types, which we have already mentioned, are both important. We recall that in one, they are certain equivalence classes of CW-complexes, whilst in the other, they are homotopy types of certain spaces with special characteristics. (Useful sources for this topic include Baues' Handbook article on 'Homotopy Types', [27].)

Let CW^{n+1}_{c*} be the full subcategory of CW_{c*} consisting of complexes of dimension $\leq n+1$. (To emphasise where we are working, we will sometimes write X^{n+1}, Y^{n+1}, etc. for objects here.) Let $f, g : X^{n+1} \to Y^{n+1}$ be two maps in CW^{n+1}_{c*} and $f|_{X^n}, g|_{X^n} : X^n \to Y^{n+1}$ their restrictions to the n-skeleton of X. (Note that the codomain is still the $(n+1)$-skeleton of Y.)

Definition: We say f, and g, as above, are *n-homotopic* if $f|_{X^n} \simeq g|_{X^n}$ (that is, within Y^{n+1}). We write $f \simeq_n g$ in this case.

It can be useful to remember that f and g, in this, need only be defined on the $(n+1)$-skeleton of X. (This statement is true, but is deliberately silly. We, in fact, assumed that X had dimension $\leq n+1$, but what we said is still useful, since if we have any complex, X, we can restrict to its $(n+1)$-skeleton, X^{n+1}, yet do not need f or g to be defined on *all* of X, merely on X^{n+1}.)

Our first version of (connected) n-types, in this approach, is obtained by taking $CW^{n+1}_{c*}/\simeq_n$, that is, taking the complexes of dimension $\leq n+1$ and the cellular maps between them, and then dividing out the hom-sets by the equivalence relation, $\simeq_n$. From this perspective, we have:

Definition: (Whitehead) A connected *n-type* is an isomorphism class in the category, $CW^{n+1}_{c*}/\simeq_n$.

That sets up, a bit more formally, the first type of description of n-types. If we have a connected CW-complex, X, then we assign to it the isomorphism class of X^{n+1} in $CW^{n+1}_{c*}/\simeq_n$ (for any choice of base point) to get its n-type. From this viewpoint, we get a notion of n-equivalence from the notion of n-homotopy:

[1] An alternative notation, which will also sometimes be used for the n-skeleton, is $X_{\leq n}$. This is useful, in particular when the superfix notation can conflict with the notation for powers of a basic object, X. This alternative notation is, more-or-less, self explanatory.

Definition: A cellular map, $f : X \to Y$, between CW-complexes is an *n-equivalence* if $f^{n+1} : X^{n+1} \to Y^{n+1}$ gives an isomorphism in $CW_{c*}^{n+1}/\simeq_n$.

This is also called *n-homotopy equivalence*, with the earlier version, that based on the homotopy groups, then called *n-weak equivalence*. It amounts to f^{n+1} having a *n-homotopy inverse*, $g^{n+1} : Y^{n+1} \to X^{n+1}$, so $f^{n+1}g^{n+1} \simeq_n 1_{Y^{n+1}}$ $g^{n+1}f^{n+1} \simeq_n 1_{X^{n+1}}$. Here it is *not* claimed that there is some $g : Y \to X$ that extends g^{n+1} to the whole of Y, merely there is a map, g, defined on the $(n+1)$-skeleton.

(These are stated for connected spaces, but as usual the extension to non-connected complexes is easy to do.)

Let us take these ideas apart one stage more. Suppose that P is a CW-complex of dimension $\leq n$, and $f : X \to Y$ is a n-equivalence in the above sense. We note that, as we are looking at cellular maps and cellular homotopies, the inclusion $i^{n+1} : X^{n+1} \to X$ induces a bijection

$$[P, i^{n+1}] : [P, X^{n+1}] \to [P, X],$$

but then it is clear that

$$[P, f] : [P, X] \to [P, Y]$$

is also a bijection. (Note that if we had required P to have dimension $n+1$, then $[P, i^{n+1}] : [P, X^{n+1}] \to [P, X]$ might not be *injective* as two non-homotopic maps with image in X^{n+1} may be homotopic within the whole of X. That being so $[P, i^{n+1}]$ will be surjective, but just not a bijection. The same would be true for $[P, f]$.)

So much for the first viewpoint, i.e., as equivalence classes of objects in CW_{c*}. For the second approach, that is, n-types as homotopy types of certain spaces delineated by conditions, we work in the bigger category of (pointed connected) CW-complexes and *all continuous maps*, i.e., not just the cellular ones (although, remember, the classical cellular approximation theorem tells us that any (general continuous) map is homotopic to a cellular one). We will temporarily call this category 'spaces', (following the treatment in Baues' Handbook article, [27]). We form $\mathsf{spaces}/\simeq$, the quotient category of 'spaces' and homotopy classes of maps.

Definition: The subcategory, $\mathsf{n{-}types}$, of $\mathsf{spaces}/\simeq$, is the full subcategory consisting of spaces, X, with $\pi_i(X) = 0$ for $i > n$. Such spaces, or their homotopy types, may also be called *n-types*. The generalisation to the non-connected case should be clear.

We now have two different definitions of n-type of CW-complexes (and that is without mentioning n-types of simplicial sets, simplicial groups $\mathcal{S}$-groupoids, etc.). We need to check on the relationship between them. For this, we introduce *Postnikov functors* and in a later section will study the related *Postnikov tower* that decomposes a homotopy type. Note the Postnikov functors are usually defined so as to be functorial at the level of the homotopy categories, not at the level of the spaces and maps, although this is possible. We will comment on this a bit more later on, but let us describe the main ideas first as these directly relate to the comparison of the two ways of approaching n-types.

Definition: The n^{th} *Postnikov functor*,

$$P_n : CW_{c*}/\simeq \to \mathsf{n\!-\!types}$$

is defined by killing homotopy groups above dimension n, that is, we *choose* a CW-complex, P_nX, with

$$(P_nX)^{n+1} = X^{n+1},$$

and, by attaching cells to X in dimensions $> n$, with $\pi_i(P_nX) = 0$ for $i > n$. If $f : X \to Y$ is a cellular map, we choose a map $P_nf : P_nX \to P_nY$, so that $(P_nf)^{n+1} = f^{n+1}$. The functor P_n takes the homotopy class, $[f]$, to $[P_nf]$.

The first point to note is that the choices are absorbed by the homotopy. To examine this more deeply we make several:

Remarks: (i) First a word about 'killing homotopy groups'. (This is very like the construction of resolutions of a group.)

Suppose that we have a space, X, and a set of representatives, $\varphi_g : S^{n+1} \to X$, of generators, g, of the homotopy group, $\pi_{n+1}(X)$, then we form

$$X(1) := X \sqcup_{\{\varphi_g\}} \bigsqcup_g D^{n+2},$$

i.e., we glue $(n+2)$-dimensional discs to X, along their boundaries, using the representing maps. We now take $\pi_{n+2}(X(1))$ and a generating set for that, form $X(2)$ by the same sort of construction, and continue to higher dimensions.

If $f : X \to Y$, then each $f(\varphi_g) : S^{n+1} \to Y$ defines an element of $\pi_{n+1}(Y)$, and this will be 'killed' within $\pi_{n+1}(Y(1))$. There is thus a null homotopy for that map within $Y(1)$. We choose one such and use it to extend f over the disc attached by φ_g. Doing this for each generator, we extend f to $f(1) : X(1) \to Y(1)$, and so on.

This is unbelievably non-canonical and non-functorial at the level of spaces, but the different choices can fairly easily be shown to yield homotopy equivalent spaces and homotopic maps. This is discussed in many of the standard algebraic topology textbooks, see, for instance, Hatcher, [147].

The basis of these constructions is a simple extension lemma, (cf. Hatcher, [147], lemma 4.7, p.350, for instance).

Lemma 35 *Given a CW pair, (X, A), and a map, $f : A \to Y$, with Y path connected, then f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X - A$ has cells in dimension n.*■

(ii) Things are clearer when working with simplicial sets as we will see shortly. In that case, there is a good functorial 'Postnikov tower' of Postnikov functors, defined at the level of simplicial sets, and morphisms and not merely at the homotopy level. That works beautifully for what we need, but at the slight cost of moving from 'spaces' to simplicial sets, there using Kan complexes (which is no real bother, as singular complexes are Kan), and finally taking geometric realisations to get back to the spaces. As we said, we will look at this shortly.

There are inclusion maps, $P_n(X) : X \to P_nX$, whose homotopy classes give a natural transformation from the identity to P_n. (This is defined on the homotopy categories of course.) For

$f : X \to Y$ in CW_{c*}, then $P_n f$ can be chosen to make the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_n(X)"'] & Y \arrow[d, "p_n(Y)"] \\ P_n X \arrow[r, "P_n f"'] & P_n Y \end{tikzcd}$$

commutative 'on the nose'. We note that these maps make each $(P_{n+1}X, X)$ into a CW-pair and, as $P_{n+1}X - X$ has only cells of dimension $n+3$ or greater, and $\pi_i(P_n X) = 0$ in those dimensions, we can apply the extension lemma to the map, $p_n(X) : X \to P_n X$ and thus extend it to $P_{n+1}X$, giving $p_n^{n+1}(X) : P_{n+1}X \to P_n X$, and this satisfies $p_n^{n+1}(X) \cdot p_{n+1}(X) = p_n(X)$. These map, $p_n^{n+1}(X)$ fit into a tower diagram with a 'cone' of maps from X:

$$\begin{tikzcd} & \vdots \arrow[d] \\ & P_3 X \arrow[d, "p_2^3(X)"] \\ & P_2 X \arrow[d, "p_1^2(X)"] \\ X \arrow[r, "p_1(X)"'] \arrow[ur, "p_2(X)"] \arrow[uur, "p_3(X)"] \arrow[uuur, dashed] & P_1(X) \end{tikzcd}$$

The limit of the tower is isomorphic to X itself. This is known as a *Postnikov tower* for X. We will return to such towers in section 5.1.3.

It is useful to refer to $X \to P_n X$, or more loosely to $P_n X$ as a *Postnikov section* of X, or as the n^{th}*-Postnikov section* of X, even though it is only determined up to homotopy equivalence.

We return to the n^{th} Postnikov functor, P_n, and can use it to define n-equivalences in a different way.

Definition: A map, $f : X \to Y$, is called a P_n-equivalence if the induced morphism, $[P_n f]$, in $\mathsf{n-types}$ is an isomorphism.

Of course, we expect these P_n-equivalences to just be n-equivalences under another name. To examine this, we look again at P_n.

We had the Postnikov functor:

$$P_n : CW_{c*}/\simeq \to \mathsf{n-types}.$$

If we look at $CW_{c*}^{n+1}/\simeq_n$, we need to see that a P_n construction adapts to give a functor

$$P_n : CW_{c*}^{n+1}/\simeq_n \to \mathsf{n-types},$$

as this does not follow trivially from the previous case. Suppose X and Y are $(n+1)$-dimensional connected pointed CW complexes and $f \simeq_n g : X \to Y$, then $f|_{X^n} \simeq g|_{X^n}$. We have to check that $P_n f \simeq P_n g$.

We have some $h : f|_{X^n} \simeq g|_{X^n} : X_n \times I \to Y^{n+1} \hookrightarrow P_n Y$, and also have the map from $P_n X \times \{0,1\}$ to $P_n Y$ given by $P_n f$ and $P_n g$. These are compatible so define a map from the subcomplex, $X_n \times I \cup P_n X \times \{0,1\}$ of $P_n X \times I$, to $P_n Y$. The cells in $P_n X \times I$ that are not in that subcomplex, all have dimension $n+3$ or greater, since P_n is obtained from X^{n+1} by adding cells. We have $\pi_i(P_n Y) = 0$ for $i > n$, so an application of the extension lemma gives us an extension ver $P_n X \times I$ giving a homotopy between $P_n f$ and $P_n g$, as required. This proves

Lemma 36 *P_n give a functor from $CW^{n+1}_{c*}/\simeq_n$ to* $\mathsf{n-types}$. ■

We claim that this functor is an equivalence of categories, which will show, after a bit more checking, that the two notions of n-equivalence coincide and will relate the main notions of (topological) homotopy n-types.

To prove that P_n is an equivalence of categories, it is, perhaps, easiest to look for a functor in the opposite sense that might serve as a 'quasi-inverse'. If we have that X is a (connected, pointed) CW-complex with $\pi_i(X) = 0$ for $i > n$, then we can take its $(n+1)$-skeleton, X^{n+1} to get something in CW^{n+1}_{c*}. This is not quite a functor, since not all the morphisms in spaces are cellular. Each continuous map between such complexes is *homotopic* to a cellular map, but, whilst taking the $(n+1)$-skeleton *is* a functor with respect to cellular maps, we have to verify that if we choose two cellular approximations for some $f : X \to Y$, then their $(n+1)$-skeletons are, at least, n-homotopic.

Suppose that $f_0, f_1 : X \to Y$ are two cellular maps between n-types (to be thought of, in the first instance, as two 'rival' cellular approximations to some $f : X \to Y$). We assume they are homotopic by a homotopy $h : f_0 \simeq f_1$, which again using cellular approximation, can be assumed to be a cellular homotopy. We take f_0^{n+1} and f_1^{n+1} and see if they are n-homotopic.- Yes they are. They may not be homotopic, since h may use $n+2$-cells in the process of 'homotoping' between f_0^{n+1} and f_1^{n+1} within Y, but $F_0|_{X^n}$ and $f_1|_{X^n}$ are homotopic via h restricted to $X_n \times I$, i.e., exactly what is needed.

We have checked not only that our idea of taking $(n+1)$-skeletons is compatible with the cellular approximations, but also that that assignment induces a functor from $\mathsf{n-types}$ to $CW^{n+1}_{c*}/\simeq_n$. (Of course, in fact, this is the restriction of a functor from spaces to $CW_{c*}/\simeq_n$, as we nowhere use that X and Y were n-types.)

Theorem 15 *The n^{th} Postnikov functor, P_n, gives an equivalence of categories between $CW_{c*}/\simeq_n$ and* $\mathsf{n-types}$. *A quasi inverse is given by the $(n+1)$-skeleton functor.*

Proof: We examine the two composite functors.

If X is in CW_{c*}, then $(P_n X)^{n+1} = X^{n+1}$, by definition. The inclusion of X^{n+1} into X gives an isomorphism in $CW_{c*}/\simeq_n$, since $\simeq_n$ uses nothing in X above dimension $n+1$.

The other composite starts with an n-type, Y, say, takes Y^{n+1}, then forms $P_n(Y^{n+1})$. The inclusion of Y^{n+1} into Y extends by the extension lemma, to a map $P_n(Y^{n+1}) \to Y$, which induces isomorphisms on all homotopy groups, so is a weak homotopy equivalence, and thus, as we are handling CW-complexes, is a homotopy equivalence, i.e., an isomorphism in $\mathsf{n-types}$, which completes the proof. ■

Remark: It is worth noting that, in the above, we have 'naturally' defined maps from X *to* $(P_nX)^{n+1}$ and from $P_n(Y^{n+1})$ *to* Y, which suggests an adjointness behind the equivalence. In fact, we actually did *not* assume that X was in $CW_{c*}^{n+1}/\simeq_n$, so, in some sense, proved that $\mathsf{n-types}$ was equivalent to a homotopically reflective subcategory of CW_{c*}. (Of course, connectedness has nothing to do with the picture and was for convenience only.)

We thus have a fairly complete picture of homotopy n-types and n-equivalence in the topological case. If $f : X \to Y$ is such that $[P_nf]$ is an isomorphism in $\mathsf{n-types}$, then $[f^{n+1}]$ is an isomorphism in $CW_{c*}^{n+1}/\simeq_n$, hence an n-equivalence á lá Whitehead.

If X and Y are (connected, pointed) $(n+1)$-dimensional CW-complexes, and $f : X \to Y$ is cellular, then f is an n-equivalence if, and only if, it induces isomorphisms on all π_i for $i \leq n$. In general, i.e., with no dimensional constraint, as we have defined it, f is an n-equivalence if, and only if f^{n+1} is an n-equivalence in this more restricted sense.

We write $Ho_n(Top)$ for the category of CW-complexes (or more generally, topological spaces, after inverting the n-equivalences. If we are just considering the CW-complexes, this is just the same as $\mathsf{n-types}$ up to equivalence and n-types are just isomorphism classes of objects in this category. (If considering spaces other than those having the homotopy types of CW-complexes, then this is better thought of as the singular n-types, but we will not usually need this level of generality in our development.) It seems that, in his original thoughts on algebraic homotopy theory, Whitehead hoped to find algebraic models for n-types, that is, to find algebraic descriptions of isomorphism classes of spaces within $Ho_n(Top)$. Classifying 1-types is 'easy' as they have models that are just groups, so classification reduces to classifying groups up to isomorphism. This is still not an easy task, but there are a wide range of tools available for it. As was previously mentioned, Mac Lane and Whitehead, [195], gave a complete algebraic model for 2-types. (Note: their 3-types are modern terminology's 2-types.) The model they proposed was the crossed module and we have seen the extension of their result to n-types given by Loday.

It should be pointed out that, although n-equivalence is defined in terms of the π_k, $0 \leq k \leq n$, the interactions between the various π_ks mean that not every sequence $\{\varphi_k : \pi_k(X) \to \pi_k(Y)\}_{0\leq k\leq n}$ can be realised as the induced morphisms coming from some $f : X \to Y$, even if the φ_k are all isomorphisms.

One approach that we will be looking at in our exploration of the basics of Whitehead's idea of Algebraic Homotopy and its implications and developments, is to convert the problems to ones in the study simplicial groups or, more generally, in $\mathcal{S}$-groupoids. For this we will need a knowledge of the corresponding theory for n-types of simplicial sets. This is very elegant, so would, in any case, be worth looking at in some detail.

5.1.2 n-types of simplicial sets and the coskeleton functors

(Sources for this section include, at a fairly introductory level, the description of the coskeleton functors in Duskin's Memoir, [107], his paper, [110], and Beke's paper, [29]. There is also a description of the skeleton and coskeleton constructions in the nLab, [221], (search on 'simplicial skeleton'). The original introduction of this construction would seem to be by Verdier in SGA4, [9], with an early use being in Artin and Mazur's *Étale homotopy*, Lecture Notes, [11].)

First let us summarise some basic ideas. For simplicial sets and simplicially enriched group(oid)s, the definitions of n-equivalence are analogous, and we give them now for convenience:

Definition: For $f : G \to H$ a morphism of $\mathcal{S}$-groupoids, f is an *n-equivalence* if $\pi_0 f : \pi_0 G \to \pi_0 H$ is an equivalence of the fundamental groupoids of G and H and for each object $x \in Ob(G)$ and each k, $1 \leq k \leq n$,

$$\pi_k f : \pi_k(G\{x\}) \to \pi_k(H\{f(x)\})$$

is an isomorphism.

We write $Ho_n(\mathcal{S}-Grpd)$ for the corresponding category of n-types, i.e., $\mathcal{S}-Grpd(\Sigma_n^{-1})$, where Σ_n is the class of all n-equivalences of $\mathcal{S}$-groupoids. An *n-type of $\mathcal{S}$-groupoids* is an isomorphism class within $Ho_n(\mathcal{S}-Grpd)$.

Cautionary note: If K is a simplicial set, then as $\pi_k(K) \cong \pi_{k-1}(GK)$, the n-type of K corresponds to the $(n-1)$-type of GK.

We need to look at simplicial n-types, in general, and in some more detail, and will start by the theory for simplicial sets. On a first reading the above summary may suffice.

The theory sketched out in the previous section uses the $(n+1)$- and n-skeletons of a CW-complex in a neat way. If we go over to simplicial sets as models for homotopy types then skeletons are easy to define, but some points do need making about them.

The n-skeleton of a CW-complex is the union of all cells of dimension less than or equal to n, so the set of higher dimensional cells in an n-skeleton is, clearly, empty. On the other hand, a simplicial set, K, has in addition to the simplices in each dimension, the face and degeneracy operators, i.e., the various $d_i : K_n \to K_{n-1}$ and $s_j : K_n \to K_{n+1}$, so to get the n-skeleton of K, we cannot *just* take the k-simplices for $k \leq n$, throwing away everything in higher dimensions, and hope to get a simplicial set. If $\sigma \in K_n$, then the $s_j\sigma$ are in K_{n+1}, so K_{n+1} cannot be empty. The point is rather that, in the n-skeleton, *all* simplices in dimensions greater than n will be degenerate.

Our first task, therefore, is to set this up more abstractly and categorically. A simplicial set, K is a functor, $K : \mathbf{\Delta}^{op} \to Sets$ and we want to restrict attention to those parts of K in dimensions less than or equal to n, discarding, initially, all higher dimensional simplices, before reinstating those that we need.

(We will introduce the ideas for simplicial sets, but we can, and will later, extend the discussion to simplicial groups, and, in general, to simplicial objects in a category, $\mathcal{A}$. The latter situation will require some conditions on the existence of various limits and colimits in $\mathcal{A}$, but we will introduce these as we go along. The ability to use more general categories is a great simplification for later developments.)

Recall that the category, $\mathbf{\Delta}$, consists of all finite ordinals and all order preserving maps between them. Given any natural number n, we can form a full subcategory, $\mathbf{\Delta}[0,n]$, with objects the ordinals $[0],\ldots,[n]$, and all order preserving maps between *them*. As the category of simplicial sets is $\mathcal{S} = Sets^{\mathbf{\Delta}^{op}}$, there is a restriction functor, call *n-truncation* or, more fully, *simplicial n-truncation*,

$$tr^n : \mathcal{S} \to Sets^{\mathbf{\Delta}[0,n]^{op}},$$

which, to a simplicial set, K, assigns the n-truncated simplicial set, $tr^n(K)$, with the same data in dimensions less than $n+1$, but which forgets all information on higher dimensions. A functor, $K : \mathbf{\Delta}[0,n]^{op} \to Sets$ is equivalent to a system, $K = \{(K_k)_{0\leq k\leq n}, d_i, s_j\}$, of sets and functions, (or more generally of objects and arrows of $\mathcal{A}$). These are to be such that the d_i and s_j verify the simplicial identities wherever they make sense.

Remark: Setting up notation and terminology for the more general case, we have a category $Tr^nSimp.\mathcal{A} = \mathcal{A}^{\mathbf{\Delta}[0,n]^{op}}$ of n-truncated simplicial objects in $\mathcal{A}$. The category of n-truncated simplicial sets is then $Tr^nSimp.Sets = Tr^n\mathcal{S} = Sets^{\mathbf{\Delta}[0,n]^{op}}$. Back in the general case, the analogue of the above restriction functor gives us a restriction functor:

$$tr^n : Simp.\mathcal{A} \to Tr^nSimp.\mathcal{A}.$$

If the category $\mathcal{A}$ has finite colimits, then this functor, tr^n has a left adjoint, which we will denote sk^n, and which is called the *n-skeleton* of the truncated simplicial object. The proof that this left adjoint exists is most neatly seen by using the theory of Kan extensions, for which see Mac Lane, [192], here with a discussion starting in section 13.3.1, or the nLab, [221], (search on 'Kan extension'.)

The *idea* of the construction of that left adjoint is, however, quite simple and is just an encoding of the intuitive idea that we sketched out above. We first look at it in the case of a simplicial set. We have K in $Tr^n\mathcal{S}$, and want $(sk^nK)_{n+1}$, that is the first missing level, (after that we can presumably repeat the idea to get the higher levels of sk^nK). We clearly need degenerate copies of all simplices in K_n and that suggests, (slightly incorrectly), that we take this $(sk^nK)_{n+1}$ to be the disjoint union of sets, $s_i(K_n) = \{s_i(x) \mid x \in K_n\}$. (The elements $s_i(x)$ are just copies of x indexed by the degeneracy mapping. If you prefer another notation, use pairs (x, s_i) as this corresponds more to one of the usual models of disjoint unions.) This is not right, since these $s_i(x)$ are not independent of each other. If x is already a degenerate element, say $x = s_jy$ then $s_ix = s_is_jy$ and, as we will need the simplicial identities to hold in the end result, this must be *the same element* as $s_{j+1}s_iy$, (this is if $i \leq j$). In other words, we should not use a disjoint union of these sets, $s_i(K_n)$, but will have to identify elements according to the simplicial identities, that is, we must form some sort of colimit. In fact, one forms a diagram consisting of copies of K_n and K_{n-1}, and then forms its colimit to get $(sk^nK)_{n+1}$. The next task is to define the face and degeneracy maps linking the new level with the old ones, so as to get an $(n+1)$-truncated simplicial sets. (It is a **good idea** to try this out in some simple cases such as for $n = 1$ and 2 and then to **look up** a 'slick' version, as then you will, more easily, see what makes the slick version work.)

Of course, the use of simplicial *sets* here is not crucial, but if working with simplicial objects in some $\mathcal{A}$, then we will need, as we mentioned earlier, that $\mathcal{A}$ has finite colimits so as to be able to form $(sk^nK)_{n+1}$. The process is then repeated as we now have a $(n+1)$-truncated object.

Remark: Shortly we will be using skeletons (and coskeletons) of simplicial *groups*. In such a context, it should be noted that not all elements in $(sk^nG)_m$, for $m > n$, need be, themselves, degenerate. For instance, we might have g, and g', in G_n, so have for two different indices, i, j, elements s_ig and s_jg' in $(sk^nG)_{n+1}$, but, more often than not, their product $s_ig.s_jg'$, will not be a degenerate element. This fact is crucial and is one reason why, in homotopy theory, it is possible to have non-trivial homotopy groups above the dimension of a space.

If we are considering simplicial sets, or, more generally, simplicial objects in $\mathcal{A}$, where $\mathcal{A}$ has finite *limits*, the truncation functor, tr^n, has a *right* adjoint, which will be denoted $cosk^n$. This is called the *n-coskeleton functor*. (Warning: this term will also be used for the *composite* $cosk^n \circ tr^n$, from $Simp.\mathcal{A}$ to itself as it is too useful to 'waste' on the more restrictive situation! Usually no confusion will arise, especially as we will use a slightly different notation.)

The fact that $cosk^n$ is right adjoint to tr^n means that, at least in the case of simplicial sets, $cosk^n$ has a very simple description. If K is a simplicial set and L is an n-truncated simplicial set, then we have

$$Tr^n\mathcal{S}(tr^n(K), L) \cong \mathcal{S}(K, cosk^nL).$$

Taking $K = \Delta[m]$, the simplicial m-dimensional simplex, we get

$$(cosk^nL)_m = \mathcal{S}(\Delta[m], cosk^nL) \cong Tr^n\mathcal{S}(tr^n(\Delta[m]), L),$$

giving us a recipe for the simplices of $cosk^nL$ in all dimensions. As $tr^n\Delta[m]$ is an n-dimensional shell of a m-dimensional simplex, we can think of it intuitively as being a family of n-simplices stuck together along lower dimensional bits in some neat way (governed by the simplicial identities). We thus would expect $cosk^L_m$ to be made up of compatible families of n-simplices of L, and this suggests a 'limit' - which makes sense as sk^nL was thought of as a colimit.

As with the left adjoint of tr^n, the right adjoint can be described as a Kan extension, which would give an explicit 'end' formula and also a limit formula that we could take apart. At this stage in the notes, it is not being assumed that those parts of categorical toolbag are available to us. (They *are* discussed later with Kan extension starting on page 615 and with ends (and coends) discussed in section 13.4.) Because of this it seems better to adopt a fairly 'barehands' approach, which is more elementary and nearer the initial intuition of what is needed, but the way to go beyond the limitations of this approach is to understand Kan extensions fully. (The approach that we will use will be adapted from Duskin's memoir, [107].)

For a category, $\mathcal{A}$, with finite limits, we suppose given an n-truncated simplicial object, $L \in Tr^nSimp.\mathcal{A}$ and we consider all the face maps at level n

$$d_0, \ldots, d_n : L_n \to L_{n-1}.$$

Definition: An object, K_{n+1}, together with morphisms $p_0, \ldots, p_{n+1} : K_{n+1} \to L_n$ is said to be the *simplicial kernel* of $(d_0, \ldots, d_n)$ if the family $(p_0, \ldots, p_{n+1})$ satisfies the simplicial identities with respect to the d_is and, moreover, has the following universal property: given any family, $x_0, \ldots, x_{n+1}$ of morphisms from some object, T, to L_n, which satisfy the simplicial identities with respect to the face morphisms, $d_0, \ldots, d_n$ (so that for $0 \leq i < j \leq n+1$, $d_i x_j = d_{j+1} x_i$), there is a unique morphism $x = \langle x_0, \ldots, x_{n+1} \rangle : T \to K_{n+1}$ such that for each i, $p_i x = x_i$.

This is clearly just a special type of limit. We would expect to get this K_{n+1}, together with the projections, p_i, as some sort of multiple pullback, corresponding to the 'naive' description we gave above. (To gain intuition on this point, **look at** the case $n = 1$, so we have $d_0, d_1 : L_1 \to L_0$ and want K_2 with maps $p_0, p_1, p_2 : K_2 \to L_1$, and these must satisfy the simplicial identities. It is **worth your while**, if you have not seen this before, to draw a diagram, consisting of some copies of L_1 and L_0, and the face maps built from $d_0, d_1 : L_1 \to L_0$, so that the limit of the diagram is K_2.) In general, the $(n+1)^{st}$ simplicial kernel, as above, is the object that is made up of all the potential boundaries of simplices, so that it is made of families of n-dimensional simplices which 'match' together as if they were the faces of something in dimension $n+1$. If L is a simplicial object, we will write $\Delta[n+1](L)$ for this $(n+1)^{st}$ simplicial kernel.

If the simplicial kernel is to do the job, we should be able to use it to take $(cosk^n L)_{n+1} = K_{n+1}$, that is to form a $(n+1)$-truncated simplicial objects from it having the right properties. We, first, need face and degeneracy morphism defined in a natural way. As the p_i were to satisfy the face simplicial identities, they are the obvious candidates for the face morphisms. We will, then, need to define for each j between 0 and n, a morphism $s_j : L_n \to K_{k+1}$. The universal property of K_{n+1} gives that such a morphism will be of the form

$$s_j = \langle s_{j,0}, \ldots, s_{j,n+1} \rangle,$$

for $s_{j,k} : L_n \to L_n$, and, of course, in this notation $d_i : K_{n+1} \to L_n$ will be the i^{th} projection, p_i. This gives us the recipe for determining the $s_{j,k}$ as we must have, for instance, if $k < j$,

$$s_{j,k} = d_k s_j = s_{j-1} d_k,$$

so as to make sure that the s_j satisfy the simplicial identities. (**It is useful** to list the various cases yourself.) It is now clear that the following holds:

Lemma 37 *The data $((cosk^n L)_k, (d_i), (s_j))$, where*

(i) $(cosk^n L)_k$ is equal to L_k for $k \leq n$ and $(cosk^n L)_{n+1} = K_{n+1}$, the simplicial kernel (as above),

(ii) the d_i are the structural limit cone projections, and

(iii) the s_j are defined by the universal property and the simplicial identities,

defines an $(n+1)$-truncated simplicial object. ■

We denote this by $tr^{n+1} cosk^n L$, as it is the next step in the construction of $cosk^n L$.

We have as a consequence the following:

Proposition 48 *Suppose given a simplicial object, T, and a morphism, $f : tr^n T \to L$, then there is a unique morphism,*

$$\tilde{f} : tr^{n+1} T \to tr^{n+1} cosk^n L,$$

that extends f in the obvious sense. ■

We may now construct $cosk^n L$ by successive simplicial kernels in the obvious way, and, generalising the above proposition to each successive dimension, prove that the result gives a right adjoint to tr^n.

Remarks: (i) The n-skeleton functor, that we saw earlier, can be given by an analogous simplicial cokernel construction using the degeneracy operators instead of the faces to give a universal object, and then applying the universal property to obtain the face morphisms. The object $sk^n(L)$ is then obtained by iterating that construction. (This is a **good exercise to follow up on** as it sheds useful light on what the skeleton will be in other situations where our intuitions are less strong than for simplicial sets.)

(ii) We are often, in fact, usually, interested more bby the composites

$$sk_n := sk^n \circ tr^n,$$

and

$$cosk_n := cosk^n \circ tr^n,$$

which will be called the n-skeleton and n-coskeleton functors on $Simp.\mathcal{A}$. (The superfix / suffix notation is just to distinguish them and no special significance should be read into it.)

Proposition 49 *(i) If $p \geq q$, then $cosk_p cosk_q = cosk_q$.*
(ii) If $p \leq q$, then $cosk_p cosk_q = cosk_p$.

Proof: This is a simple **exercise** in the definition, or, alternatively, in the constructions, so is **left to the reader** to work out or check up on in the literature. ■

A similar result holds for skeletons, and this is, again, **left to you** to investigate.

So far in this section we have just looked at the skeleton and coskeleton functors, but we are wanting these for a discussion of simplicial n-types. If we adopt the view that an n-type is a homotopy type with vanishing homotopy groups above dimension n, this goes across without pain to the context of simplicial sets, and, in fact, to many other situations such as simplicial sheaves on a space or simplicial objects in a (Grothendieck) topos, $\mathcal{E}$.

Aside: A good reason for briefly looking at this is that it introduces several useful concepts and the linked terminology. These in the main are due to Jack Duskin, who developed them for the study of simplicial objects in a topos. We will give the definitions and subsequent discussion within the classical setting of *Sets*, but this is really only because we have not given a thorough and detailed treatment of toposes earlier. The basic point is that if the arguments used in the development are 'constructive' then, usually with some minor changes, the theory will generalise from a category of sets, to one of sheaves, and eventually to any Grothendieck topos. To make that statement more precise would require quite a lot more discussion, and would take us away from our main themes, so investigation is **left to you**.

We start with a slight variant of the Kan fibration definition that we met earlier, (see page 36). We recall that $\Lambda^i[n]$ is the (n, i)-*horn* or (n, i)-*box*, obtained by discarding the top dimensional n-simplex and its i^{th} face and all the degeneracies of those simplices.

Definition: A simplicial map $p : E \to B$ is a *Kan fibration*, or *satisfies the Kan lifting condition*, *in dimension* n if, in every commutative square (of solid arrows) of form

$$\begin{tikzcd} \Lambda^i[n] \arrow[r, "f_1"] \arrow[d, "inc"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "f_0"'] \arrow[ur, dashed, "f"] & B \end{tikzcd}$$

a diagonal map (indicated by the dashed arrow) exists, i.e., there is an $f : \Delta[n] \to E$ such that $pf = f_0$, $f.inc = f_1$, so f lifts f_0 and extends f_1.

We thus have that p is a Kan fibration if it is one in *all dimensions*. We can refine the above (following Duskin, [108]).

Definition: A simplicial map $p : E \to B$ *satisfies the exact Kan lifting condition in dimension* n if, in every commutative square (as above), precisely one diagonal map f exists.

Starting with the Kan fibration condition, we singled out the Kan complexes as being those simplicial sets for which the unique map $K \to \Delta[0]$ was a Kan fibration. We clearly can do a similar thing here.

Definition: A simplicial set K is an *exact* n*-type*, or n*-hypergroupoid*, if $K \to \Delta[0]$ is a Kan fibration that is exact in dimensions greater than n.

The definition of n-hypergroupoid used by Glenn, [136], is slightly different from this as it only requires the (exact) Kan condition in dimensions greater than n, so not requiring K to 'be' a Kan complex in lower dimensions. The n-hypergroupoid terminology is due to Duskin, [108], whilst 'exact n-type' is Beke's, [29].

If X is a simplicial object in some category $\mathcal{A}$, we recall that $\Delta[n](X)$ denotes the object given by the relevant simplicial kernel in dimension n. We thus have for a simplicial set, X,

$$\Delta[n](X) = \{(x_0, \ldots, x_k) \mid x_i \in X_{k-1} \text{ and, whenever } i < j,\ d_i x_j = d_{j-1} x_i\}.$$

We will also use a notation $\Lambda^i[n](X)$, for the corresponding object with x_i omitted, (interpreted internally within $\mathcal{A}$). This is the object of (n, i)-horns in X.

If we need a version of these ideas in $Simp(\mathcal{E})$ or $Simp.\mathcal{A}$, then we can easily adapt our earlier discussion of horns and Kan objects in that context. For instance:

Proposition 50 *If $\mathcal{A}$ is a finite limit category, a morphism, $p : E \to B$, in $Simp.\mathcal{A}$ is an exact Kan fibration in dimension n if, and only if, the natural maps $E_n \to \Lambda^k[n](E) \times_{\Lambda^k[n](B)} B_n$ are all isomorphisms in $\mathcal{A}$.* ■

Corollary 12 *In $Simp.\mathcal{A}$, an object, K, is an exact n-type (or n-hypergroupoid) if, and only if, the natural map, $K_k \to \Lambda^j[k](K)$, is an epimorphism for $k \leq n$ and an isomorphism for $k > n$.* ■

To begin to take 'exact n-types' apart, we will need to look again at look at the coskeleton functors. It is very useful for our purposes to have a description of when a simplicial set, K, is isomorphic to its own n-coskeleton. The following summary is actually adapted from Beke's paper, [29], but is quite well known and moderately easy to prove, so the proof will be **left as an exercise**.

Proposition 51 *For a simplicial set, K, the following are equivalent:*

1) K is isomorphic to an object in the image of $cosk_n$.

2) The natural morphism $K \to cosk_n(K)$ is an isomorphism.

3) The natural 'boundary' map $b_k(x) = (d_0x, \ldots, d_kx)$, from K_k to $\Delta[k](K)$ is a bijection for all $k > n$.

4) The natural map, $K_k \to Sets^{\mathbf{\Delta}[0,n]^{op}}(tr^n\Delta[k], tr^n(K))$, which sends a k-simplex x of K, considered as its 'name', $\ulcorner x \urcorner : \Delta[k] \to K$, to the n-truncation, of $\ulcorner x \urcorner$, is a bijection for all $k > n$.

5) For any $k > n$, and any pair of (solid) arrows

$$\begin{tikzcd} \partial\Delta[k] \arrow[r] \arrow[d] & K \\ \Delta[k] \arrow[ur, dashed] & \end{tikzcd}$$

there is precisely one *(dotted) arrow making the diagram commute.* ■

As we said, the proof is **left to you**, as it is just a question of translating between different viewpoints.

Definition: If K satisfies any, and hence all, of the above conditions, it is called *n-coskeletal*.

The first two conditions can be transferred verbatim for simplicial objects in any category with finite limits, and thus for simplicial objects in a topos. Condition 3 can also be handled in those contexts, using iterated pullbacks to construct $\Delta[k](K)$. Condition 4) can also be used if the category of simplicial objects has finite cotensors (see the discussion of tensors and cotensors in simplicially enriched categories in section 11.3.2, page 505). A similar comment may be made about 5), since using cotensors allows one to 'internalise' the condition - but it ends up then being 3) in an enriched form. The details will not be needed in our later discussion, so are **left to you if you need them**.

We use this notion of n-coskeletal object in the following way

Proposition 52 *(cf. Beke, [29], Proposition 1.3) (i) If K satisfies the exact Kan condition above dimension n, then K must be $(n+1)$-coskeletal.*
(ii) If K is n-coskeletal, then it satisfies the exact Kan condition above dimension $n+1$.
(iii) If K is an n-coskeletal Kan complex, then it has vanishing homotopy groups in dimensions n and above.
(iv) An exact n-type has vanishing homotopy groups above dimension n.

Before we prove this, it needs noting that there is an internal version in $Simp(\mathcal{E})$ for $\mathcal{E}$ a topos, see [29]. We have refrained from giving it only to avoid the need to define the homotopy groups of such an object internally.

Proof: (i) Suppose we are given a map $b : \partial\Delta[k] \to K$ for $k > n+1$, then we can omit d_0b to get a $(k,0)$-horn in K. By assumption, this horn has a filler, $f : \Delta[k] \to K$, so we consider both d_0f and d_0b. As they have the same boundary and since K satisfies the exact Kan condition above dimension n, they must coincide. We have thus that f is a filler for b. By exactness, we have that it is unique.

(ii) If $m > n+1$, $tr_n(\Lambda^k[m]) \to tr_n(\Delta[m])$ is fairly obviously an isomorphism. Now $cosk_n(K)$ satisfies the exact Kan condition in dimension m if, and only if, for any horn, $\underline{x} : \Lambda^k[m] \to K$, there is a diagram

$$\begin{tikzcd}
\Lambda^k[m] \arrow[r, "\underline{x}"] \arrow[d, "inc"'] & cosk_n K \arrow[d] \\
\Delta[k] \arrow[r] \arrow[ur, dashed, "\exists!"] & 1
\end{tikzcd}$$

with unique diagonal. Using the adjunction, this gives a diagram

$$\begin{tikzcd}
tr_n\Lambda^k[m] \arrow[r, "\overline{x}"] \arrow[d, "inc"'] & K \arrow[d] \\
tr_n\Delta[k] \arrow[r] \arrow[ur, dashed, "?"] & 1
\end{tikzcd}$$

and we have noted that the left hand side is an isomorphism if $m > n+1$.

(iii) If K is Kan, the topological description of homotopy groups goes over to K, i.e., as the group of homotopy classes of maps from $\partial\Delta[n]$ to K mapping a vertex to chosen basepoint. Such a map will fill in dimensions $k \geq n$, so all the $\pi_k(K)$ will be trivial for any base point. (**You should fill in the details of this argument.**)

(iv) This just combines (i) and (iii). ∎

We note that (iv) above says that exact n-types are n-types!

5.1.3 Postnikov towers

In the topological case, we saw above that given any (connected) CW-complex, X, we could construct a sequence of Postnikov sections, P_nX, and maps between them, $P_{n+1}X \to P_nX$. We referred to this as a *Postnikov tower* for X. In the simplicial case, we found that the coskeletons gave us a corresponding construction, (and we will shortly see an alternative, if related, one). It is often useful to demand a bit more structure in the tower, structure that is always potentially there but which is usually not in its 'optimal form'. To make them more 'useful', we first review the definition of Postnikov towers and some of their properties. (We refer the reader, who wants a slightly more detailed introduction, to Hatcher's book, [147], p. 410.) First a redefinition, (adapted to our needs from [147]).

Definition: A *Postnikov tower* for a (connected) space X is a commutative diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_3 \arrow[d, "p_2^3(X)"] \\
 & X_2 \arrow[d, "p_1^2(X)"] \\
X \arrow[r, "p_1(X)"'] \arrow[ur, "p_2(X)"] \arrow[uur, "p_3(X)"] \arrow[uuur, dashed] & X_1
\end{tikzcd}$$

such that

(i) the map $X \to X_n$ induces an isomorphism on π_i for $i \leq n$;

(ii) $\pi_i(X_i) = 0$ for $i > n$.

Remark: A Postnikov tower for X always exists by our discussion in section 5.1.1 and, hidden in that discussion is the information that shows that the tower is unique up to a form of homotopy equivalence for towers.

If we convert each maps $X_n \to X_{n-1}$ into a fibration (in the usual way be pulling back the pathspace fibration on X_{n-1} along this map, see the discussion of the corresponding construction for chain complexes, in section 8.2.1, where the term *mapping cocone* is used), then its fibre (which is, then, the *homotopy fibre* of the original map), will be an Eilenberg-Mac Lane space, $K(\pi_n X, n)$, as the difference between the homotopy groups of X_n and X_{n-1} is exactly $\pi_n(X)$ in dimension n. (More exactly, we should look at the long exact homotopy sequence for this fibration, but we do not have this available within the notes so far so if you need more precision on this refer to Hatcher, [147], or other texts on homotopy theory.)

Definition: A *fibrant Postnikov tower* for X is a Postnikov tower (as above) in which each $X_n \to X_{n-1}$ is a fibration.

The discussion above shows that any Postnikov tower can be replaced, up to homotopy equivalence, by a fibrant one. There is here a technical remark that is worth making, but requires that the reader has met the theory of model categories. (It can safely be ignored if you have not yet met this.) On the category of towers of spaces (or or simplicial sets, etc.), there is a model category structure in which these fibrant towers are exactly the fibrant objects.

Moving over to the simplicial case, we restrict attention to Kan complexes, as they are much better behaved, homotopically, than arbitrary ones. We have the n^{th} coskeleton, $cosk_n K$ of a Kan complex, K, and the first query is whether it is a Kan complex itself. Certainly in dimensions lower than n, as it agrees with K there, any k-horn will have a filler. We thus look at an $(n+1)$-horn, $x_0, \ldots, \hat{x}_i, \ldots, x_{n+1}$, corresponding to the map, $\underline{x} : \Lambda^i[n+1] \to cosk_n K$, (using the usual convention with a 'hat' indicating the missing face). All the faces, x_k, are in $(cosk_n K)_n = K_n$, so all toegther they form a $(n+1)$-horn in K, which, of course, can be filled by some $y \in K_{n+1}$ We have its naming map $\ulcorner y \urcorner : \Delta[n+1] \to K$, which we restrict to $sk_n\Delta[n+1]$ to get a filler for our original $\underline{x}$. We thus do have that $cosk_n K$ satisfies the Kan filler condition in dimension $n+1$.

We look, next, at dimension $n = 2$ (expecting, of course, that the situation there will tell us how to handle the general case in higher dimensions). In fact, we have already seen the argument that we will use above.

Suppose $\underline{x} : \Lambda^i[n+2] \to cosk_n K$, then $\underline{x}$ corresponds, under the adjunction to a map, $\overline{x} : sk_n\Lambda^i[n+2] \to K$, but, and this is the neat argument we saw before, $sk_n\Lambda^i[n+2] = sk_n\Delta[n+2]$ (or, if you want to be precise, the inclusion of $\Lambda^i[n+2]$ into $\Delta[n+2]$ restricts to the 'identity' isomorphism on the n-skeletons). This means that $\overline{x}$ is already in $(cosk_n K)_{n+2}$. (Of course, dotting i's and crossing t's, that statement is also not true, but means $\Lambda^i[\ell] \to \Delta[\ell]$ induces a bijection

$$\mathcal{S}(\Delta[\ell], cosk_n K) \xrightarrow{\cong} \mathcal{S}(\Lambda^i[\ell], cosk_n K)$$

for $\ell = n+2$, and, in fact, for all $\ell \geq n+2$, so $sk_n\Lambda^i[n+2] \xrightarrow{\cong} sk_n\Delta[n+2]$ for all $\ell \geq n+2$.) We summarise this in a proposition for possible later use.

Proposition 53 *If K is a Kan complex, then so is $cosk_n K$.* ∎

We next glance at the canonical map

$$p_n^{n+1} : cosk_{n+1}K \to cosk_n K.$$

This does not seem to be a fibration, but that is not too worrying since (i) we can replace is by a fibration as in the topological case, and (ii) we will see there is a subtower of this *cosk* tower which is fibrant and very neat and we turn to it next. Its beauty is that it adapts well to many other simplicial settings, such as that of simplicial groups, without much adjustment, and it is functorial.

The canonical map, $p_n = \eta(K) : K \to cosk_n K$, which is the unit of the adjunction, can be very easily described in combinatorial terms, since $(cosk_n K)_m = \mathcal{S}(sk_n\Delta[m], K)$. If x is a m-simplex in K, then its 'name' $\ulcorner x \urcorner : \Delta[m] \to K$ determines it precisely and conversely, (by the Yoneda lemma and the equation $\ulcorner x \urcorner \iota_n = x$). There is an inclusion, $i_m : sk_n\Delta[m] \to \Delta[m]$, and $\ulcorner x \urcorner \circ i_m$ is an m-simplex in $cosk_n K$. This is $\eta(x)$.

In $(cosk_n K)_m$, there can be simplices that are not restrictions of m-simplices in K and these are, for instance, simplices that, together, 'kill' the homotopy groups (above dimension n, that is.) As K is Kan, $\pi_m(K) \cong [S^m, K]$, the set of pointed homotopy classes of pointed maps from $S^m = \partial\Delta[m+1]$, or alternatively, $S^m = \Delta[m]/\partial\Delta[m]$. (Both identifications are useful and we can go from one to the other since they are weakly homotopy equivalent.) We note that, for instance, $sk_{m-1}S^m = sk_{m-1}\Delta[m]$, so any m-sphere in K has a canonical filler in $cosk_{m-1}K$. Other cases are slightly more tricky, but can be **left to you**, as, in any case, when we consider these more formally slightly later on we will use a slightly different argument.

The image of $\eta(K)$ is, in each dimension m, obtained by dividing K_m by the equivalence relation determined by $\eta(K)_m$, i.e., define $\sim_n$ on K_m by $x \sim_n y$ if, and only if, the representing maps, $x, y : \Delta[m] \to K$ agree on $sk_n\Delta[m]$. (We will dispense with the 'name' notation, $\ulcorner x \urcorner$, here, as it tends to clutter the notation and is not needed, if no confusion is likely to occur. We are thus pretending that $K_m = \mathcal{S}(\Delta[m], K)$, rather than merely being naturally isomorphic.)

We write $[x]_n$ for the $\sim_n$-equivalence class of x. We note that if $m \leq n$ then $\sim_n$ is simply equality as the n-skeleton of $\Delta[m]$ is all of $\Delta[m]$.

Definition: The simplicial set, $K(n) := K/\sim_n$ is called the n^{th} *Postnikov section* of K.

That $\sim_m$ is compatible with the face and degeneracy maps is **easy to check**, so $K(n)$ is a simplicial set and , equally simply, the natural quotient, $q_n : K \to K(n)$, so $q_n(x) = [x]_n$, is simplicial. (It is the codomain restriction of $p_n = \eta(K)$.) This is best seen using the fact that is is induced from the *cosimplicial* inclusions $sk_n\Delta[m] \to \Delta[m]$. The cosimplicial viewpoint also gives that the inclusions $sk_n\Delta[m] \to sk_{n+1}\Delta[m]$ induce the quotient maps, $q_n^{n+1} : K(n+1) \to K(n)$, (which are the restrictions of the p_n^{n+1}), and that $q_n^{n+1}q_{n+1} = q_n$.

Lemma 38 *For a (connected) Kan complex, K, and for each n:*
(i) The map $q_n : K \to K(n)$ is a Kan fibration, and $K(n)$ is a Kan complex.
(ii) The map, $q_n^{n+1} : K(n+1) \to K(n)$, is a Kan fibration.
(iii) The map, q_n, induces an isomorphism on π_i for $0 \leq i \leq n$.
(iv) The homotopy groups of $K(n)$ are trivial above dimension n, $K(n)$ is an n-type.

Proof: (i) Suppose we have a commutative diagram

$$\begin{tikzcd} \Lambda^i[m] \arrow[r, "(x_0,\ldots,\hat{x_i},\ldots,x_m)"] \arrow[d] & K \arrow[d, "q_n"] \\ \Delta[m] \arrow[r, "{[y]_n}"'] & K(n) \end{tikzcd}$$

where we have written the i-horn as an $(m+1)$-tuple of $(m-1)$-simplices, with a gap at the 'hat'. We need to lift $[y]_n$ to some y agreeing with the x_ks, i.e., $d_k y = x_k$.

If $m \leq n$, there is no problem as q_n the identity in those dimensions.

For $m = n+1$, we have if y is a representative of $[y]_n$, then as $\sim_n$ is the identity relation in dimension n, $d_k y = x_k$ for $k \neq i$, so y is a suitable lift.

For $m > n+1$, we use that K is Kan to find a filler $x \in K_n m+1$ for the (m,i)-horn, so $d_k x = x - k$ for $k \neq i$. Now $sk_n\Lambda^i[m] = sk_n\Delta[m]$, as we have used before, and so $q_n(x) = [x]_n = [y]_n$.

In general, if $p : K \to L$ is a surjective Kan fibration and K is a Kan complex, then L is Kan, so the last part of (i) follows.

(ii) Look at $K(n+1)$ and form $K(n+1)(n)$, i.e. divide it out by $\sim_n$. This gives $K(n)$ with the quotient being just q_n^{n+1}. By (i), this will be a fibration.

We next pick a base vertex, $v \in K_0$ and look at the various $\pi_m(K,v)$ and $\pi_m(K(n),[v]_n)$. Clearly, as q_n 'is the identity' in dimensions $m \leq n$, the induced morphisms $\pi_m(q_n)$ 'is the identity' in dimensions $m < n$. For (iii), we have, thus, only to examine $\pi_n(q_n)$. Suppose $f : \Delta[n] \to K$ sends $\partial\Delta[n]$ to $\{v\}$, i.e., represents an element of $\pi_n(K)$, and that $q_n f$ is null-homotopic, then $q_n f$ extends to a map,$\overline{F} : \Delta[n+1] \to K(n)$ such that $q_n f = d_0\overline{F}$, and $d_i\overline{F} = v$ for $i \neq 0$. We can lift $\overline{F}$ to a map $F : \Delta[n+1] \to K$, since q_n is surjective *and* the n-dimensional faces are mapped by the identity. We thus have that f itself was null-homotopic, so $\pi_n(q_n)$ is a monomorphism. As $\pi_n(q_n)$ is cearly an epimorphism, this handles (iii).

(iv) Any map $f : \Delta[m] \to K(n)$ is determined by its restriction,$f| : sk_n\Delta[m] \to K$, but

$$sk_n\partial\Delta[m] \to sk_n\Delta[m]$$

is the identity if $m > n$, and $f|_{\partial\Delta[m]}$ is constant with value v, so $\pi_m(K(n)) = 0$ if $m > n$. ■

We thus have proved the connected case of the following:

Theorem 16 *If K is a Kan complex, $(K(n), q_n^{n+1}, q_n)$, forms a (functorial) fibrant Postnikov tower for K.* ■

The non-connected case is a simple extension of this connected one involving disjoint unions, so ....

Of course, the inclusion of $K(n)$ into $cosk_n K$ is a weak equivalence.

Remarks: (i) A note of caution seems in order. Some sources tend to confuse $K(n)$ and $cosk_n K$, and whilst, for many homotopical purposes, this is not critical, for certain purposes the use of one is prefereable to that of the other, so it seems better to keep the distinction.

(ii) The study of Postnikov complexes, which abstract the properties of the $K(n)$, is important in the study of coskeletal simplicial sets and nerves of higher categories, for which see the important paper of Duskin, [110].

(iii) Putting a naturally defined model category structure on the category of n-types (and on the corresponding simplicial presheaves and sheaves) has been done using these Postnikov sections, see Biedermann, [33]. He notes that his construction depends on using the Postnikov section approach that we have just outlined, rather than the coskeleton, as that latter one disturbs some of the necessary structure.

(iv) If you need more on Postnikov towers in simplicial sets, a good source is Goerss and Jardine, [137], Chapter 6, whilst Duskin's paper, [110], mentioned above, gives some powerful tools for manipulating them and also coskeletons.

5.1.4 Whitehead towers

Postnikov towers approximate a homotopy type by its tower of n-types, that is, by 'n-co-connected' spaces. The Whitehead tower of a homotopy type produces a sequence of n-connected approximations to it. Before we look at this in detail, let us consider what this should mean. (As sources, we will initially use Hatcher, [147], p. 356 in the topological case, before looking at the simplicial case. Another useful source is the nLab page on 'Whitehead towers', ([221], and search on 'Whitehead tower').)

What we would expect from a naive dualisation of Postnikov tower for a pointed space, X, would be a diagram,

$$\begin{tikzcd} \ldots \arrow[r] \arrow[drrr, dashed] & Z_2 \arrow[r] \arrow[drr] & Z_1 \arrow[r] \arrow[dr] & Z_0 \arrow[d] \\ & & & X \end{tikzcd}$$

with Z_n an n connected space, (so $\pi_i(Z_n) = 0$ for $i \leq n$), and the composite map $Z_n \to X$ inducing an isomorphism on all homotopy groups, π_i for $i > n$. The space Z_0 would be path connected and homotopy equivalent to the component of X containing the base point. The next space, Z_1 would be simply connected and would have the homotopy properties of the universal cover of Z_0. We would then think of $Z_n \to X$ as an 'n-connected cover' of the (pointed connected component, Z_0, of the)space, X.

Definition: The *Whitehead tower* of a pointed space, (X, x) is a sequence of fibrations

$$\ldots \to X\langle n\rangle \to \ldots \to X\langle 1\rangle \to X\langle 0\rangle \to X$$

where each $X\langle n\rangle \to X\langle n-1\rangle$ induces isomorphisms on the homotopy groups, π_i, for $i > n$ and such that $X\langle n\rangle$ is n-connected, so $\pi_k(X\langle n\rangle)$ is trivial for all $k \leq n$.

The problem of constructing such a tower was posed by Hurewicz and solved by George Whitehead in 1952. We will assume that we have chosen a Postnikov tower for a CW-complex, X, so giving a map $p_n : X \to P_n X$.

We next to form the *homotopy fibre* or *mapping cocone* of this map, over the basepoint, x_0, of $P_n X$. We have already seen this idea, page 45, so will just briefly review how it is constructed. We first form the pullback

$$\begin{tikzcd} M^{p_n} \arrow[r, "\pi^{p_n}"] \arrow[d, "j^{p_n}"'] & (P_n X)^I \arrow[d, "e_0"] \\ X \arrow[r, "p_n"'] & P_n X \end{tikzcd}$$

so M^{p_n} consists of pairs, (x, λ), where $x \in X$ and $\lambda : I \to P_n X$ is a path with $\lambda(0) = p_n(x)$. We set $i^{p_n} = e_1 \circ \pi^{p_n}$, so that $i^{p_n}(x, \lambda) = \lambda(1)$. The fact that $i^{p_n} : M^{p_n} \to P_n X$ is a fibration is standard, as is that $j^{p_n} : M^{p_n} \to X$ is a homotopy equivalence. (If you want a proof of these, **after trying to give one yourself**, there are proofs in many standard textbooks, such as that of Hatcher, and the abstract setting of such results is discussed in Kamps and Porter, [171]. This all fits well into a 'homotopical' context, and that is explored more on the nLab, [221], search under 'mapping cocone' and follow the links.) For brevity, we will write $\overline{X}$ for M^{p_n}, $\overline{p_n} : \overline{X} \to P_n X$ for i^{p_n}. The homotopy fibre of p_n is then the fibre of $\overline{p_n}$ over the base point of $P_n X$. It is $F^h(p_n) = \{(x, \lambda) \mid \lambda(1) = x_0\}$.

We thus have a fibration sequence,

$$F^h(p_n) \to \overline{X} \to P_n X,$$

and, hence, by standard homotopy theory, a long exact sequence of homotopy groups,

$$\ldots \to \pi_k(F^h(p_n)) \to \pi_k(\overline{X}) \to \pi_k(P_n X) \to \pi_{k-1}(F^h(p_n)) \to \ldots$$

Note that $\pi_k(\overline{X}) \cong \pi_k(X)$, since j^{p_n} is a homotopy equivalence. (If you have not met this long fibration exact sequence before, **check it up, briefly** in any standard book on homotopy theory. We will look at it, and also the dual situation in cohomology, in more detail later on, starting in section 8.2.)

If we look at this long exact sequence, below the value $k = n$, the homomorphism $\pi_k(\overline{X}) \to \pi_k(P_n X)$ is an isomorphism, so $\pi_k(F^h(p_n)) = 0$ in that range, whilst as $\pi_k(P_n X) = 0$ if $k > n$, there $\pi_k(F^h(p_n)) \to \pi_k(\overline{X})$ is an isomorphism. Thus the homotopy fibre, $F^h(p_n)$ is n-connected.

This looks good, as this is a functorial construction (or, more exactly, any lack of functoriality is due to a lack of functoriality of the Postnikov tower). We have a composite map $F^h(p_n) \to \overline{X} \to X$. This sends (x, λ) to x, of course. We will write $X\langle n\rangle := F^h(p_n)$, in the expectation that it will form part of a 'Whitehead tower'.

The next ingredient that we need will be a map

$$X\langle n+1\rangle \to X\langle n\rangle.$$

We do have a (chosen) map $p_n^{n+1} : P_{n+1}X \to P_nX$, which is compatible with the 'projections' $p_n : X \to P_nX$, so $p_n^{n+1}p_{n+1} = p_n$. This induces a map from the homotopy fibre of p_{n+1} to that of p_n. (This is **left to you to check**[2].)

We note that the fibre of $X\langle n+1\rangle \to X\langle n\rangle$ is a $K(\pi_n(X), n)$.

Remarks: (i) The above hides slightly the fact that the construction of a Whitehead tower is only really 'natural' up to homotopy as that was already the case for the Postnikov tower in the topological case.

(ii) For the simplicial case, we can use either the coskeleton based tower or, better, the Postnikov section one, as that is already fibrant as we saw. As the p_n and p_n^{n+1} are fibrations in that case, we can replace the homotopy pullbacks by pullbacks, and the homotopy fibres by fibres, thus gaining more insight into the relationship of the objects in the corresponding Whitehead tower to the Kan complex being 'resolved'. (The detailed description is **left to you**.)

(iii) The theory and constructions adapt well to other simplicial contexts such as that of simplicial groups, where, as fibrations are simply degreewise epimorphisms, many of the constructions take on a much simpler algebraic aspect.

The case of a topological group, G: In this case, one can find a topological model for each $G\langle n\rangle$ which is a topological group, and, as there is a topological Abelian group model for the $K(\pi, n)$s occurring as the fibres in the tower, there is a short exact sequence

$$1 \to K(\pi_n(G), n) \to G\langle n+1\rangle \to G\langle n\rangle \to 1.$$

Example: The Whitehead tower of the orthogonal group, $O(n)$.

For large n, the orthogonal group, $O(n)$, has the following homotopy groups:

i	0	1	2	3	4	5	6	7
$\pi_i(O(n))$	C_2	C_2	0	C_∞	0	0	0	C_∞

There are then periodicity results for higher dimensions giving $\pi_{k+8}(O(n)) \cong \pi_k(O(n))$. The first space of the Whitehead tower of $O(n)$ is, of course, $O(n)\langle 0\rangle = SO(n)$, as it is the (0-)connected component of the identity element.

The next space is the group, $O(n)\langle 1\rangle = Spin(n)$, (which we will look at in more detail later; see section **??**). There is a short exact sequence:

$$1 \to C_2 \to Spin(n) \to SO(n) \to 1.$$

The next homotopy group is trivial and $O(n)\langle 2\rangle = O(n)\langle 3\rangle = String(n)$. This is a very interesting group, but we have not yet the machinery to do it justice. (For more on it in our sort of setting, see, for instance, Jurco, [170], Schommer-Pries, [241]. We will return to it later.)

[2] The usual proof uses the functoriality of $(-)^I$ and the naturality of the various mappings, and then the universal property of pullbacks. Everything is being 'chosen up to homotopy', so there are subtleties that **do** need thinking about, and it is a good idea to try to get a reasonably homotopy 'coherent' argument going on behind the proof. The construction is a 'homotopy pullback' and the property you are looking for is the analogue of the universal property of pullbacks to this more structured setting. It is, in the long term, important to get used to this sort of situation as well as to the sort of geometric / higher categorical picture that it corresponds to, as this is needed for generalisations.

5.2 Crossed squares

We next turn back to algebraic models of these n-types that we have now introduced more formally. We have already seen models for 2-types, namely the crossed modules that we looked at earlier, now we turn to 3-types. There are several different types of model here. We start with one that is relatively simple in its apparent structure.

5.2.1 An introduction to crossed squares

We saw earlier that crossed modules were like normal subgroups except that the inclusion map is replaced by a homomorphism that need not be a monomorphism. We even noted that all crossed modules are, up to isomorphism, obtainable by applying π_0 to a simplicial "inclusion crossed module".

Given a pair of normal subgroups M, N of a group G, we can form a square

$$\begin{tikzcd} M\cap N \arrow[r] \arrow[d] & N \arrow[d] \\ M \arrow[r] & G \end{tikzcd}$$

in which each morphism is an inclusion crossed module and there is a commutator map

$$h : M \times N \to M \cap N$$

$$h(m, n) = [m, n].$$

This forms a *crossed square* of groups, in fact, it is a special type of such that we will call an *inclusion crossed square*. Later we will be dealing with crossed squares as crossed n-cubes, for $n = 2$. Here we will give an interim definition of crossed squares. The notion is due to Guin-Walery and Loday, [143], and this slightly shortened form of the definition is adapted from Brown-Loday, [67].

5.2.2 Crossed squares, definition and examples

Definition: (First version) A *crossed square* (more correctly *crossed square of groups*) is a commutative square of groups and homomorphisms

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"] & P \end{tikzcd}$$

together with actions of the group P on L, M and N (and hence actions of M on L and N via μ and of N on L and M via ν) and a function $h : M \times N \to L$. This structure is to satisfy the following axioms:
(i) the maps λ , λ' preserve the actions of P, furthermore, with the given actions, the maps μ, ν and $\kappa = \mu\lambda = \mu'\lambda'$ are crossed modules;
(ii) $\lambda h(m, n) = m^n m^{-1}$, $\lambda' h(m, n) = {}^m n n^{-1}$;
(iii) $h(\lambda\ell, n) = \ell^n \ell^{-1}$, $h(m, \lambda'\ell) = {}^m\ell\ell^{-1}$;

(iv) $h(mm', n) = {}^m h(m', n)h(m, n)$, $h(m, nn') = h(m, n)^n h(m, n')$;
(v) $h({}^p m, {}^p n) = {}^p h(m, n)$;
for all $\ell \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$.

There is an evident notion of morphism of crossed squares, just preserve all the structure, and we obtain a category Crs^2, the *category of crossed squares*.

Examples

In addition to the above class of examples, we have the following:

(a) Given any simplicial group, G, and two simplicial normal subgroups, M and N, the square

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & N \arrow[d] \\ M \arrow[r] & G \end{tikzcd}$$

with inclusions and with $h = [\ ,\ ] : M \times N \to G$ is a simplicial "inclusion crossed square" of simplicial groups. Applying π_0 to the diagram gives a crossed square and, in fact, all crossed squares arise in this way (up to isomorphism).

b) Any simplicial group, G, yields a crossed square, $M(G, 2)$, defined by

$$\begin{tikzcd} \dfrac{NG_2}{d_0(NG_3)} \arrow[r] \arrow[d] & Ker\, d_1 \arrow[d] \\ Kerd_2 \arrow[r] & G_1 \end{tikzcd}$$

for suitable maps. This is, in fact, part of the construction that shows that all connected 3-types are modelled by crossed squares.

Another way of encoding 3-types is using the truncated simplicial group and Conduché's notion of 2-crossed module.

5.3 2-crossed modules and related ideas

5.3.1 Truncations.

Definition: Given a chain complex, (X, ∂), and an integer n, the *truncation of X at level n* is the complex $t_{n]}X$ defined by

$$(t_{n]}X)_i = \begin{cases} 0 & \text{for } i > n \\ X_n / Im\, \partial_n & \\ X_i & \text{for } i < n. \end{cases}$$

For $i < n$, the differential of $t_{n]}X$ is the same as that of X, whilst the n^{th}-differential is induced by ∂.

(For more on truncations see Illusie [163, 164]). Truncation is, of course, functorial.

Remark on terminology: There are several schools of thought on the terminology here. The problem is whether this should be 'truncation' or 'co-truncation'. To some extent both are 'wrong' as n-truncated chain complexes should not have any information available in dimensions greater than n, if the model of simplicial sets was to be followed. This would then be expected to have right and left adjoints, which would correspond, approximately to the coskeleton and skeleton functors of simplicial set theory that we have already seen. At the moment the 'jury' seems to be out and the terminological conventions fairly lax. (We may thus decide to change this later on if convincing arguments are presented.)

This construction will work for chain complexes of groups provided each $Im\,\partial$ is a *normal* subgroup of the corresponding X, i.e., provided X is a *normal chain complex of groups.*

Proposition 54 *There is a truncation functor $t_{n]} : Simp.Grps \to Simp.Grps$ such that there is a natural isomorphism*

$$t_{n]}NG \cong Nt_{n]}G,$$

where N is the Moore complex functor from Simp.Grps to the category of normal chain complexes of groups.

Proof: We first note that $d_0(NG_{n+1})$ is contained in G_n as a normal subgroup and that all face maps of G vanish on it. We can thus take

$$\begin{aligned} (t_{n]}G)_i &= G_i \text{ for all } i < n \\ (t_{n]}G)_n &= G_n/d_0(NG_{n+1}) \end{aligned}$$

and for $i > n$, we take the semidirect decomposition of G_i, which we will see shortly, given by Proposition 67, delete all occurrences of NG_k for $k > n$ and replace any NG_n by $NG_n/d_0(NG_{n+1})$. The definition of face and degeneracy is easy as is the verification that $t_{n]}N$ and $Nt_{n]}$ are the same and that the various actions are compatible. ■

This truncation functor has nice properties. (In the chain complex case, these are discussed in Illusie, [163].)

Proposition 55 *Let $T_{n]}$ be the full subcategory of Simp.Grps defined by the simplicial groups whose Moore complex is trivial in dimensions greater than n and let $i_n : T_{n]} \to Simp.Grps$ be the inclusion functor.*

a) The functor $t_{n]}$ is left adjoint to i_n. (We will usually drop the i_n and so also write $t_{n]}$ for the composite functor.)

b) The natural transformation, η, co-unit of the adjunction, is a natural epimorphism which induces an isomorphism on π_i for $i \leq n$. The unit of the adjunction is isomorphic to the identity transformation, so $T_{n]}$ is a reflective subcategory of Simp.Grps.

c) For any simplicial group G, $\pi_i(t_{n]}G) = 0$ if $i > n$.

d) To the inclusion, $T_{n]} \to T_{n+1]}$, there corresponds a natural epimorphism η_n from $t_{n+1]}$ to $t_{n]}$. If G is a simplicial group, the kernel of $\eta_n(G)$ is a $K(\pi_{n+1}(G), n+1)$, i.e., has a single non-zero homotopy group in dimension $n+1$, that being $\pi_{n+1}(G)$, i.e., is an 'Eilenberg-MacLane space' of type $(\pi_{n+1}(G), n+1)$. ■

As each statement is readily verified using the Moore complex and the semidirect product decomposition, the proof of the above will be left to you, however you will need Proposition 67, page 251.

Definition: We will say that a simplicial group, G, is *n-truncated* if $NG_k = 1$ for all $k > n$.

Of course, $T_{n]}$ is the category of n-truncated simplicial groups.

A comparison of these properties with those of the *coskeleton functors* (cf., above, section 5.1.2, page 198, or for an 'original' source, Artin and Mazur, [11]) is worth making. We will not look at this in detail here, but will just summarise the results. We will meet them again later on from time to time.

Given any integer $k \geq 0$, there is a functor, $cosk_k$, defined on the category of simplicial sets, which is the composite of a truncation functor (differently defined) and its right adjoint. The n-simplices of $cosk_k\, X$ are given by $Hom(sk_k\Delta[n], X)$, the set of simplicial maps from the k-skeleton of the n-simplex, $\Delta[n]$, to the simplicial set, X. There is a canonical map from X to $cosk_k\, X$, whose homotopy fibre is $(k-1)$-connected. The canonical map from $cosk_k\, X$ to $cosk_{k-1}\, X$ thus has homotopy fibre an Eilenberg-MacLane 'space' of type $(\pi_k(X), k)$.

This k-coskeleton is constructed using finite limits and there is an analogue in any category of simplicial objects in a category, $\mathscr{D}$, provided only that $\mathscr{D}$ has finite limits, thus in particular in $Simp.Grps$. Conduché, [81], has calculated the Moore complex of $cosk_{k+1}\, G$ for a simplicial group, G, using a construction described in Duskin's Memoir, [107]. His result gives

$$\begin{aligned} N(cosk_{k+1}G)_r &= 0 \qquad \text{if } r > k+2 \\ N(cosk_{k+1}G)_{k+2} &= Ker(\partial_{k+1} : NG_{k+1} \to NG_k), \end{aligned}$$

and

$$N(cosk_{k+1}G)_r = NG_r \qquad \text{if } r \leq k+1.$$

There is an epimorphism from $cosk_{n+1}G$ to $t_{n]}G$, which, on passing to Moore complexes, gives

$$\begin{tikzcd} 0 \arrow[r] & Ker\,\partial_{k+1} \arrow[r] \arrow[d] & NG_{k+1} \arrow[r] \arrow[d] & NG_k \arrow[r, "\partial_{k+1}"] \arrow[d] & NG_{k-1} \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & NG_k/Im(\partial_{k+1}) \arrow[r] & NG_{k-1} \end{tikzcd}$$

This epimorphism of chain complexes thus has a kernel with trivial homology. The epimorphism therefore induces an isomorphism on all homotopy groups and hence is a weak homotopy equivalence. We may thus use either $t_{n]}G$ or $cosk_{n+1}G$ as a model of the n-type of G.

5.3.2 Truncated simplicial groups and the Brown-Loday lemma

The theory of crossed n-cubes that we have hinted at above is not the only way of encoding higher n-types. Another method would be to use these truncated simplicial groups as suggested above. A detailed study of this is complicated in high dimension, but feasible for 3-types and, in fact, reveals some interesting insights into crossed squares in the process.

As a first step to understanding truncated simplicial groups a bit more, we will give a variant of an argument that we have already seen. We will look at a 1-truncated simplicial group. The analysis is really a simple use of the sort of insights given by the Brown-Loday lemma.

Proposition 56 *(The Brown-Loday lemma) Let N_2 be the normal subgroup of G_2 generated by elements of the form*

$$F_{(1),(0)}(x,y) = [s_1x, s_0y][s_0y, s_0x]$$

for $x, y \in NG_1 = Ker\, d_1$, then $NG_2 \cap D_2 = N_2$ and consequently

$$\partial(NG_2 \cap D_2) = [Ker\, d_0, Ker\, d_1].$$

■

Note the link with group T-complex type conditions through the intersection, $NG_2 \cap D_2$.

The form of this element, $F_{(1),(0)}(x,y)$, is obtained by taking the two elements, x and y, of degree 1 in the Moore complex of a simplicial group, G, mapping them up to degree 2 by complementary degeneracies, and then looking at the component of the result that is in the Moore complex term, NG_2. (It is easy to show that G_2 is a semidirect product of NG_2 and degenerate copies of lower degree Moore complex terms.) The idea behind this pairing can be extended to higher dimensions. It gives the *Peiffer pairings*,

$$F_{\alpha,\beta} : NG_p \times NG_q \to NG_{p+q}.$$

In general, these take $x \in NG_p$ and $y \in NG_q$ and (α, β) a complimentary pair of index strings (of suitable lengths), and sends (x, y) to the component in NG_{p+q} of $[s_\alpha x, s_\beta y]$; see the series of papers [213–217]. This again uses the Conduché decomposition lemma, [81], that we will see later on, cf. page 251. It is also worth noting that the Peiffer pairing ends up in $NG_{p+q} \cap D_{p+q}$, so would all be zero in a group T-complex.

A very closely related notion is that of hypercrossed complex as in Carrasco and Cegarra, [76, 77]. There one uses the component of $s_\alpha x.s_\beta y$ in NG_{p+q} to give a pairing and adds cohomological information to the result to get a reconstruction technique for G from NG, *i.e.*, an *ultimate Dold-Kan theorem*, thus hypercrossed complexes generalise 2-crossed modules and 2-crossed complexes to all dimensions.

5.3.3 1- and 2-truncated simplicial groups

Suppose that G is a simplicial group and that $NG_i = 1$ for $i \geq 2$. This leaves us just with

$$\partial : NG_1 \to NG_0.$$

We make $NG_0 = G_0$ act on NG_1 by conjugation as before

$${}^g c = s_0(g)cs_0(g)^{-1} \text{ for } g \in G_0, c \in NG_1,$$

and, of course, $\partial({}^g c) = g.\partial c.g^{-1}$. The first crossed module axiom is, thus, satisfied. For the other one, we note that $F_{(1),(0)}(c_1, c_2) \in NG_2$, which is trivial, so

$$\begin{aligned} 1 &= d_0([s_1c_1, s_0c_2][s_0c_2, s_0c_1]) \\ &= [s_0d_0c_1, c_2][c_2, c_1] = ({}^{\partial c_1}c_2)(c_1c_2c_1^{-1})^{-1}, \end{aligned}$$

so the Peiffer identity holds as well. Thus $\partial : NG_1 \to NG_0$ is a crossed module. As we have already seen that the functor $\mathcal{G}$ provides a way to construct a simplicial group from a crossed module and that the result has Moore complex of length 1, we have the following slight reformulation of earlier results:

Proposition 57 *The category of crossed modules is equivalent to the subcategory $T_{1]}$ of 1-truncated simplicial groups.* ∎

The main reason for restating and proving this result in this form is that we can glean more information from the proof for examining the next level, 2-truncated simplicial groups.

If we replace our 1-truncated simplicial group by an arbitrary one, then we have already introduced the idea of a Peiffer commutator of two elements, and there we used the term 'Peiffer lifting' without specifying what particular interest the construction had. We recall that here: Given a simplicial group, G, and two elements $c_1, c_2 \in NG_1$ as above, then the *Peiffer commutator* of c_1 and c_2 is defined by

$$\langle c_1, c_2\rangle = ({}^{\partial c_1}c_2)(c_1c_2c_1^{-1})^{-1}.$$

We met earlier, $F_{(1),(0)}$, which gives the *Peiffer lifting* denoted

$$\{-,-\} : NG_1 \times NG_1 \to NG_2,$$

where

$$\{c_1, c_2\} = [s_1c_1, s_0c_2][s_0c_2, s_0c_1]$$

and we noted

$$\partial\{c_1, c_2\} = \langle c_1, c_2\rangle.$$

These structures come into their own for a 2-truncated simplicial group. Suppose that G is now a simplicial group, which is 2-truncated, so its Moore complex looks like:

$$\ldots 1 \to NG_2 \xrightarrow{\partial_2} NG_1 \xrightarrow{\partial_1} NG_0.$$

For the moment, we will concentrate our attention on the morphism ∂_2.

The group NG_1 acts on NG_2 via conjugation using s_0 or s_1. We will use s_0 for the moment, so that if $g \in NG_1$ and $c \in NG_2$,

$${}^{g}c = s_0(g)cs_0(g)^{-1}.$$

It is once again clear that $\partial_2({}^{g}c) = g.\partial_2(c).g^{-1}$ and, as before, we consider, for $c_1, c_2 \in NG_2$ this time, the Peiffer pairing given by

$$[s_1c_1, s_0c_2][s_0c_2, s_0c_1],$$

which is, this time, the component of $[s_1c_1, s_0c_2]$ in NG_3. However that latter group is trivial, so this element is trivial, and hence, so is its image in NG_2. The same calculation as before shows that, with this s_0-based action of NG_1 on NG_2, (NG_2, NG_1, ∂_2) is a crossed module.

We also know that there is a Peiffer lifting

$$\{-,-\} : NG_1 \times NG_1 \to NG_2,$$

which measures the obstruction to $NG_1 \to NG_0$ being a crossed module, since $\partial\{-,-\}$ is the Peiffer commutator, whose vanishing is equivalent to $NG_1 \to NG_0$ being a crossed module. We do not have yet in our investigation a detailed knowledge of how the two structures interact, nor any other distinguishing properties of $\{-,-\}$. We will not give such a detailed derivation here, but from it we can obtain the following:

Proposition 58 *Let G be a 2-truncated simplicial group. The Peiffer lifting,*

$$\{-,-\} : NG_1 \times NG_1 \to NG_2,$$

has the following properties:
(i) it is a map such that if $m_0, m_1 \in NG_1$,

$$\partial\{m_0, m_1\} = {}^{\partial m_0}m_1.(m_0 m_1 m_0^{-1})^{-1};$$

(ii) if $\ell_0, \ell_1 \in NG_2$,

$$\{\partial\ell_0, \partial\ell_1\} = [\ell_0, \ell_1];$$

(iii) if $\ell \in NG_2$ and $m \in NG_1$, then

$$\{m, \partial\ell\}\{\partial\ell, m\} = {}^{\partial m}\ell.\ell^{-1};$$

(iv) if $m_0, m_1, m_2 \in NG_1$, then
a) $\{m_0, m_1 m_2\} = \{m_0, m_1\}^{(m_0 m_1 m_0^{-1})}\{m_0, m_2\},$
b) $\{m_0 m_1, m_2\} = {}^{\partial m_0}\{m_1, m_2\}\{m_0, m_1 m_2 m_1^{-1}\};$
(v) if $n \in NG_0$ and $m_0, m_1 \in NG_1$, then

$${}^{n}\{m_0, m_1\} = \{{}^{n}m_0, {}^{n}m_1\}.$$

■

The above can be encoded in the definition of a 2-crossed module.

5.3.4 2-crossed modules, the definition

Definition: A *2-crossed module* is a normal complex of groups

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N,$$

together with an action of N on all three groups and a mapping

$$\{-,-\} : M \times M \to L$$

such that

(i) the action of N on itself is by conjugation, and ∂_2 and ∂_1 are N-equivariant;

(ii) for all $m_0, m_1 \in M$,
$$\partial_2\{m_0, m_1\} = {}^{\partial_1 m_0}m_1.m_0 m_1^{-1} m_0^{-1};$$

(iii) if $\ell_0, \ell_1 \in L$, then
$$\{\partial_2\ell_0, \partial_2\ell_1\} = [\ell_0, \ell_1];$$

(iv) if $\ell \in L$ and $m \in M$, then
$$\{m, \partial\ell\}\{\partial\ell, m\} = {}^{\partial m}\ell.\ell^{-1};$$

(v) for all $m_0, m_1, m_2 \in M$,

(a) $\{m_0, m_1m_2\} = \{m_0, m_1\}\{\partial\{m_0, m_2\}, (m_0m_1m_0^{-1})\}\{m_0, m_2\}$;

(b) $\{m_0m_1, m_2\} = {}^{\partial m_0}\{m_1, m_2\}\{m_0, m_1m_2m_1^{-1}\}$;

(vi) if $n \in N$ and $m_0, m_1 \in M$, then

$$^n\{m_0, m_1\} = \{{}^nm_0, {}^nm_1\}.$$

The pairing $\{-,-\} : M \times M \to L$ is often called the *Peiffer lifting* of the 2-crossed module.

The only one of these axioms that looks 'daunting' is (v)a). Note that we have not specified that M acts on L. We could have done that as follows: if $m \in M$ and $\ell \in L$, define

$$^m\ell = \{\partial\ell, m\}\ell.$$

Now (v)a) simplifies to the expression

$$\{m_0, m_1m_2\} = \{m_0, m_1\}\,^{(m_0m_1m_0^{-1})}\{m_0, m_2\}.$$

We denote such a 2-crossed module by $\{L, M, N, \partial_2, \partial_1\}$, or similar, only adding in notation for the actions and the pairing if explicitly needed for the context. A *morphism of 2-crossed modules* is, fairly obviously, given by a diagram

$$\begin{tikzcd} L \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & M \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & N \arrow[d, "f_0"'] \\ L' \arrow[r, "\partial_2'"'] & M' \arrow[r, "\partial_1'"'] & N' \end{tikzcd},$$

where $f_0\partial_1 = \partial_1' f_1$,$f_1\partial_2 = \partial_2' f_2$,

$$f_1({}^nm) = {}^{f_0(n)}f_1(m), \quad f_2({}^n\ell) = {}^{f_0(n)}f_2(\ell),$$

and

$$\{-,-\}(f_1 \times f_1) = f_2\{-,-\},$$

for all $\ell \in L$, $m \in M$, $n \in N$.

These compose in an obvious way giving a category which we will denote by $2{-}CMod$.

The following should be clear.

Theorem 17 *The Moore complex of a 2-truncated simplicial group is a 2-crossed module. The assignment is functorial.* ■

We will denote this functor by $\mathsf{C}^{(2)} : T_{2]} \to 2{-}CMod$. It is an equivalence of categories.

5.3.5 Examples of 2-crossed modules

Of course, the construction of 2-crossed modules from simplicial groups gives a generic family of examples, but we can do better than that and show how these new crossed gadgets link in with others that we have met earlier.

Example 1: Any crossed module gives a 2-crossed module, since if (M, N, ∂) is a crossed module, we need only add a trivial $L = 1$, and the resulting sequence

$$L \to M \to N$$

with the 'obvious actions' is a 2-crossed module! This is, of course, functorial and $CMod$ can be considered to be a full subcategory of $2-CMod$ in this way. It is a reflective subcategory since there is a reflection functor obtained as follows:

If

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

is a 2-crossed module, then $Im\,\partial_2$ is a normal subgroup of M and we have (with a small abuse of notation):

Proposition 59 *If $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$ is a 2-crossed module then there is an induced crossed module structure on*

$$\partial_1 : \frac{M}{Im\,\partial_2} \to N.$$

■

But we can do better than this:

Example 2: Any crossed complex of length 2, that is one of form

$$\ldots \to 1 \to 1 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$$

gives us a 2-crossed complex on taking $L = C_2$, $M = C_1$ and $N = C_0$, with $\{m, m'\} = 1$ for all $m, m' \in M$. We will check this in a moment, but note that this gives a functor from $Crs_{2]}$ to $2-CMod$ extending the one we gave in Example 1.

Of course, (i) crossed complexes of length 2 are the same as 2-truncated crossed complexes.

5.3.6 Exploration of trivial Peiffer lifting

Suppose we have a 2-crossed module

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N,$$

with the extra condition that $\{m_0, m_1\} = 1$ for all $m_0, m_1 \in M$. The obvious thing to do is to see what each of the defining properties of a 2-crossed module give in this case.

(i) There is an action of N on L and M and the ∂s are N-equivariant. (This gives nothing new in our special case.)

(ii) $\{-,-\}$ is a lifting of the Peiffer commutator - so if $\{m_0, m_1\} = 1$, the Peiffer identity holds for (M, N, ∂_1), i.e., that is a crossed module;

(iii) if $\ell_0, \ell_1 \in L$, then $1 = \{\partial_2\ell_0, \partial_2\ell_1\} = [\ell_1, \ell_0]$, so L is Abelian
and,

(iv) as $\{-,-\}$ is trivial ${}^{\partial m}\ell = \ell$, so ∂M has trivial action on L.

Axioms (v) and (vi) vanish.

We leave the reader, if they so wish, to structure this into a formal proof that the 2-crossed module is precisely a 2-truncated crossed complex.

Our earlier discussion should suggest:

Proposition 60 *The category $Crs_{2]}$ of crossed complexes of length 2 is equivalent to the full sub-category of $2{-}CMod$ given by those 2-crossed complexes with trivial Peiffer lifting.* ■

We leave the proof of this to the reader.

A final comment is that in a 2-truncated simplicial group, G, one obviously has that it satisfies the thin filler condition (cf. page 40) in dimensions greater than 2, since $NG_k = 1$ for all $k > 2$ and if the Peiffer lifting is trivial in the corresponding 2-crossed module, G satisfies it in dimension 2 as well. (As D_1 is $s_0(G_0)$, any simplicial group satisfies the thin filler condition in dimension 1.)

In the next section we will give other examples of 2-crossed modules, those coming from crossed squares.

5.3.7 2-crossed modules and crossed squares

We now have several 'competing' models for homotopy 3-types. Since we can go from simplicial groups to both crossed square and 2-crossed modules, there should be some link between the latter two situations. In his work on homotopy n-types, Loday gave a construction of what he called a 'mapping cone' for a crossed square. Conduché later noticed that this naturally had the structure of a 2-crossed module. This is looked at in detail in a paper by Conduché, [82].

Suppose that

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

is a crossed square, then its mapping cone complex is

$$L \overset{\partial_2}{\to} M \rtimes N \overset{\partial_1}{\to} P,$$

where $\partial_2\ell = (\lambda\ell^{-1}, \lambda'\ell)$ and $\partial_1(m,n) = \mu(m)\nu(n)$.

We first note that the semi-direct product $M \rtimes N$ is formed by making N act on M *via* P, i.e.

$${}^{n}m = {}^{\nu(n)}m,$$

where the P-action is the given one. The fact that (λ^{-1}, λ') and $\mu\nu$ are homomorphisms is an interesting and instructive, but easy, exercise:

i) $(m,n)(m',n') = (m^{\nu(n)}m', nn')$, so

$$\begin{aligned} \partial_1((m,n)(m',n')) &= \mu(m^{\nu(n)}m').\nu(nn') \\ &= \mu(m)\nu(n)\mu(m')\nu(n)^{-1}\nu(n)\nu(n') \\ &= (\mu(m)\nu(n))(\mu(m')\nu(n')); \end{aligned}$$

(ii) if $\ell, \ell' \in L$, then, of course,

$$\begin{aligned} \partial_1(\ell\ell') &= (\lambda(\ell\ell')^{-1}, \lambda'(\ell\ell')) \\ &= (\lambda(\ell')^{-1}\lambda(\ell)^{-1}, \lambda'(\ell)\lambda'(\ell')). \end{aligned}$$

whilst

$$\begin{aligned}\partial_1(\ell)\partial_1(\ell') &= (\lambda(\ell)^{-1}, \lambda'(\ell))(\lambda(\ell')^{-1}, \lambda'(\ell')) \\ &= (\lambda(\ell)^{-1}.{}^{\nu\lambda'(\ell^{-1})}\lambda(\ell')^{-1}, \lambda'(\ell\ell')),\end{aligned}$$

thus the second coordinates are the same, but, as $\nu\lambda' = \mu\lambda$, the first coordinates are also equal.

These elementary calculations are useful as they pave the way for the calculation of the Peiffer commutator of $x = (m, n)$ and $y = (c, a)$ in the above complex:

$$\begin{aligned}\langle x, y\rangle &= {}^{\partial x}y.xy^{-1}x^{-1} \\ &= {}^{\mu m.\nu n}(c, a).(m, n)({}^{a^{-1}}c^{-1}, a^{-1})({}^{n^{-1}}m^{-1}, n^{-1}) \\ &= ({}^{\mu m\nu n}c, {}^{\mu m\nu n}a)(m^{\nu(na^{-1})}c^{-1}.{}^{\nu(na^{-1}n^{-1})}m^{-1}, na^{-1}n^{-1}),\end{aligned}$$

which on multiplying out and simplifying is

$$({}^{\nu(na^{-1}n^{-1})}m.m^{-1}, {}^{\mu m}(nan^{-1}).(na^{-1}n^{-1})).$$

(Note that any dependence on c vanishes!)

Conduché defined the Peiffer lifting in this situation by

$$\{x, y\} = h(m, nan^{-1}).$$

It is immediate to check that this works

$$\begin{aligned}\partial_2\{x, y\} &= (\lambda h(m, nan^{-1}), \lambda' h(m, nan^{-1})) \\ &= ({}^{\nu(na^{-1}n^{-1})}m.m^{-1}, {}^{\mu m}(nan^{-1}).(na^{-1}n^{-1}),\end{aligned}$$

by the axioms of a crossed square.

We will not check all the axioms for a 2-crossed module for this structure, but will note the proofs for one or two of them as they illustrate the connection between the properties of the h-map and those of the Peiffer lifting.

2CM(iii) : $\{\partial\ell_0, \partial\ell_1\} = [\ell_1, \ell_0]$. As $\partial\ell = (\lambda\ell^{-1}, \lambda'\ell)$, this needs the calculation of

$$h(\lambda\ell_0^{-1}, \lambda'(\ell_0\ell_1\ell_0^{-1})),$$

but the crossed square axiom :

$$h(\lambda\ell, n) = \ell.{}^{n}\ell^{-1}, \text{ and } h(m, \lambda'\ell) = {}^{m}\ell.\ell^{-1},$$

together with the fact that the map $\lambda : L \to M$ is a crossed module, give

$$\begin{aligned}h(\lambda\ell_0^{-1}, \lambda'(\ell_0\ell_1\ell_0^{-1})) &= {}^{\mu\lambda(\ell_0^{-1}}(\ell_0\ell_1\ell_0^{-1}).\ell_0\ell_1^{-1}\ell_0^{-1}) \\ &= [\ell_1, \ell_0].\end{aligned}$$

We need $\{(m, n), (\lambda\ell^{-1}, \lambda'\ell)\}\{(\lambda\ell^{-1}, \lambda'\ell), (m, n)\}$ to equal ${}^{\mu(m)\nu(n)}\ell.\ell^{-1}$, but evaluating the initial expression gives

$$\begin{aligned}h(m, n.\lambda'\ell.n^{-1})h(\lambda\ell^{-1}, \lambda'\ell.n.\lambda'\ell^{-1}) &= h(m, \lambda'({}^{n}\ell))h(\lambda\ell^{-1}, \lambda'\ell.n.\lambda'\ell^{-1}) \\ &= {}^{\mu(m)\nu(n)}\ell.{}^{\nu(n)}\ell^{-1}.\ell^{-1}.{}^{\nu\lambda'(\ell).\nu(n).\nu\lambda'\ell^{-1}}\ell,\end{aligned}$$

and this does simplify as expected to give the correct results.

We thus have two ways of going from a simplicial group, G, to a 2-crossed module:
(a) directly to get

$$\frac{NG_2}{\partial NG_3} \to NG_1 \to NG_0;$$

(b) indirectly via $M(G,2)$ and then by the above construction to get

$$\frac{NG_2}{\partial NG_3} \to Ker\, d_0 \rtimes Ker\, d_1 \to G_1$$

and they clearly give the same homotopy type. More precisely G_1 decomposes as $Ker\, d_0 \rtimes s_0 G_0$ and the $Ker\, d_0$ factor in the middle term of (b) maps down to that in this decomposition by the identity map, thus d_0 induces a quotient map from (b) to (a) with kernel isomorphic to

$$1 \to Ker\, d_0 \overset{=}{\to} Ker\, d_0,$$

which is acyclic/contractible.

5.3.8 2-crossed complexes

(These were not discussed in the lectures in Buenos Aires due to lack of time.) Crossed complexes are a useful extension of crossed modules allowing not only the encoding of an algebraic model for the 2-type, but also information on the 'chains on the universal cover', e.g. if G is a simplicial group, earlier, in section 3.5.1, we had $\mathsf{C}(G)$, the crossed complex constructed from the Moore complex of G, given by

$$C(G)_n = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})},$$

in higher dimensions and having at its 'bottom end' the crossed module,

$$\frac{NG_1}{d_0(NG_2 \cap D_2)} \to NG_0.$$

For a crossed complex, $\pi(X)$, coming from a CW-complex (as a filtered space, filtered by its skeleta), these groups in dimensions ≥ 3 coincide with the corresponding groups of the complex of chains on the universal cover of X. In general, the analogue of that chain complex can be extracted functorially from a general crossed complex; see [64] or [226]. The tail on a crossed complex allows extra dimensions, not available just with crossed modules, in which homotopies can be constructed. The category Crs is very much better structured than is $CMod$ itself and so 'adding a tail' would seem to be a 'good thing to do', so with 2-crossed modules, we can try and do something similar, adding a similar 'tail'.

We have an obvious normal chain complex of groups that ends

$$\ldots \to C(G)_3 \to \frac{NG_2}{d_0(NG_3 \cap D_3)} \to NG_1 \to NG_0.$$

Here there are more of the structural Peiffer pairings of the Moore complex NG that survive to the quotient, but it should be clear that, as they take values in the $NG_n \cap D_n$, in general these will again be almost all trivial if the receiving dimension, n, is greater than 2. For $n \leq 2$, these pairings are those that we have been using earlier in this chapter. The one exceptional case that is important here, as in the crossed complex case, is that which gives the action of NG_0 on $C_n(G)$ for $n \geq 3$, which, just as before, gives $C_n(G)$ the structure of a $\pi_0 G$-module. Abstracting from this gives the definition of a 2-crossed complex.

Definition: A *2-crossed complex* is a normal complex of groups

$$\ldots \to C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots \longrightarrow C_0,$$

together with a 2-crossed module structure given on $C_2 \to C_1 \to C_0$ by a Peiffer lifting function $\{-,-\} : C_1 \times C_1 \to C_2$, such that, on writing $\pi = Coker(C_1 \to C_0)$,

(i) each C_n, $n \geq 3$ and $Ker\,\partial_2$ are π-modules and the ∂_n for $n \geq 4$, together with the codomain restriction of ∂_3, are π-module homomorphisms;

(ii) the π-module structure on $Ker\,\partial_2$ is the action induced from the C_0-action on C_2 for which the action of $\partial_1 C_1$ is trivial.

A 2-crossed complex morphism is defined in the obvious way, being compatible with all the actions, the pairings and Peiffer liftings. We will denote by $2{-}Crs$, the corresponding category.

There are reduced and unreduced versions of this definition. In the discussion and in the notation we use, we will quietly ignore the groupoid based non-reduced version, but it is easy to give simply by replacing simplicial groups by simplicially enriched groupoids, and making fairly obvious changes to the definitions.

Proposition 61 *The construction above defines a functor, $\mathsf{C}^{(2)}$, from Simp.Grps to $2{-}Crs$.* ■

There are no prizes for guessing that the simplicial groups whose homotopy types are accurately encoded in $2{-}Crs$ by this functor are those that satisfy the thin condition in dimensions greater than 3. In fact, the construction of the functor $\mathsf{C}^{(2)}$ explicitly kills off the intersection $NG_k \cap D_k$ for $k \geq 3$.

We have noted above that any 2-crossed module,

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N,$$

gives us a short crossed complex by dividing L by the subgroup $\{M, M\}$, the image of the Peiffer lifting. (We do not need this, but $\{M, M\}$ is easily checked to be a normal subgroup of L.) We also discussed those 2-crossed complexes that had trivial Peiffer lifting. They were just the length 2 crossed complexes. This allows one to show that crossed complexes form a reflective subcategory of $2{-}Crs$ and to give a simple description of the reflector:

Proposition 62 *There is an embedding*

$$Crs \to 2{-}Crs,$$

which has a left adjoint, L say, compatible with the functors defined from Simp.Grps to $2{-}Crs$ and to Crs, i.e. $\mathsf{C}(G) \cong \mathsf{L}\mathsf{C}^{(2)}(G)$. ■

5.4 Cat^n-groups and crossed n-cubes

5.4.1 Cat^2-groups and crossed squares

In the simplest examples of crossed squares, μ and μ' are normal subgroup inclusions and $L = M \cap N$, with h being the conjugation map. Moreover this type of example is almost 'generic' since, if

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r] & G \end{tikzcd}$$

is a simplicial crossed square constructed from a simplicial group, G, and two simplicial normal subgroups, M and N, then applying π_0, the square gives a crossed square and, up to isomorphism, all crossed squares arise in this way.

Although when first defined by D. Guin-Walery and J.-L. Loday, [143], the notion of crossed squares was not linked to that of cat^2-groups, it was in this form that Loday gave their generalisation to an n-fold structure, cat^n-groups (see [186] and below).

Definition: A *cat^1-group* is a triple, (G, s, t), where G is a group and s, t are endomorphisms of G satisfying conditions
(i) $st = t$ and $ts = s$.
(ii) $[Ker\, s, \; Ker\, t] = 1$.

A cat^1-group is a reformulation of an internal groupoid in $Grps$. (The interchange law is given by the $[Ker, Ker]$ condition; left for you to check) As these latter objects are equivalent to crossed modules, we expect to be able to go between cat^1-groups and crossed modules without hindrance, and we can:

Setting $M = Ker\, s$, $N = Im\, s$ and $\partial = t|M$, then the action of N on M by conjugation within G makes $\partial : M \to N$ into a crossed module. Conversely if $\partial : M \to N$ is a crossed module, then setting $G = M \rtimes N$ and letting s, t be defined by

$$s(m, n) = (1, n)$$

and

$$t(m, n) = (1, \partial(m)n)$$

for $m \in M$, $n \in N$, we have that (G, s, t) is a cat^1-group. Again this is one of those simple, but key calculations that are well worth doing yourself.

For a cat^2-group, we again have a group, G, but this time with two independent cat^1-group structures on it. Explicitly:

Definition: A *cat^2-group* is a 5-tuple (G, s_1, t_1, s_2, t_2), where (G, s_i, t_i), $i = 1, 2$, are cat^1-groups and

$$s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i$$

for $i, j = 1, 2, \quad i \neq j$.

There is an obvious notion of morphism between cat^2-groups and with this we obtain a category, $Cat^2(Grps)$.

Theorem 18 *[186] There is an equivalence of categories between the category of cat^2-groups and that of crossed squares.*

Proof: The cat^1-group (G, s_1, t_1) will give us a crossed module with $M = Ker\, s_1$, $N = Im\, s_1$, and $\partial = t|M$, but, as the two cat^1-group structures are independent, (G, s_2, t_2) restricts to give cat^1-group structures on both M and N and makes ∂ a morphism of cat^1-groups as is easily checked. We thus get a morphism of crossed modules

$$\begin{tikzcd} Ker\, s_1 \cap Ker\, s_2 \arrow[r] \arrow[d] & Im\, s_1 \cap Ker\, s_2 \arrow[d] \\ Ker\, s_2 \cap Im\, s_1 \arrow[r] & Im\, s_1 \cap Im\, s_2, \end{tikzcd}$$

where each morphism is a crossed module for the natural action, i.e., conjugation in G. It remains to produce an h-map, but this is given by the commutator within G, since, if $x \in Ker\, s_2 \cap Im\, s_1$ and $y \in Im\, s_2 \cap Ker\, s_1$, then $[x, y] \in Ker\, s_1 \cap Ker\, s_2$. It is easy to check the axioms for a crossed square. The converse is left as an exercise. ■

5.4.2 Interpretation of crossed squares and cat^2-groups

We have said that crossed squares and cat^2-groups give equivalent categories and we will see that, similarly, for the crossed n-cubes and cat^n-groups, which will be introduced shortly. The simplest case of that general situation is one that we have already already met namely that of crossed modules and cat^1-groups, and there we earlier saw how to interpret a crossed modules as being the essential data for a 2-group(oid).

We thus have, you may recall (combining ideas from pages 58 and 225), that a crossed module, (C, P, ∂), gives us a cat^1-group / 2-group, $(C \rtimes P, s, t)$, with $s(c, p) = p$ being the source of an element (c, p) and $t(c, p) = \partial c.p$ being its target. The definition of cat^2-group does not explicitly use the language of 'internal categories', we mentioned that the $[Ker\, s, Ker\, t] = 1$ condition is a version of the interchange law, and that a cat^{1}group can be interpreted as an internal category in *Grps*. This leads to pictures such as

$$p_1 \xrightarrow{(c_1,p_1)} \partial c_1.p_1,$$

(cf. section 2.3.2, page 58) indicating that (c, p) interprets as an arrow having source and target as indicated. We could equally well use the 2-category or 2-group(oid) style diagram:

$$\begin{tikzcd} \cdot \arrow[r, bend left, "p"] \arrow[r, bend right, "\partial c.p"'] \arrow[r, phantom, "\Downarrow (c,p)"] & \cdot \end{tikzcd}$$

as we discussed earlier in section 2.3.3.

If we start with a cat^1-group, (G, s, t), then the picture is

$$s(g) \xrightarrow{g} t(g).$$

It thus looks that the source and target are 'objects' of the category structure that we know to be there. Where do they live? Clearly in $Im\, s$ or $Im\, t$, or both. Life is easy on us however. We note that $Im\, s = Im\, t$, since $st = t$ implies that $Im\, t \subseteq Im\, s$, whilst we also have $ts = s$, giving the other inclusion. The subgroup, $Im\, s$, corresponds to the group P of the crossed module, considered as a subgroup of the 'big group' $C \rtimes P$.

It is sometimes more convenient to write an internal category in the form

$$\begin{tikzcd} G_1 \arrow[r, shift left=2, "\sigma"] \arrow[r, "\tau"'] & G_0 \arrow[l, shift left=3, "\iota"] \end{tikzcd} ,$$

so that G_1 is an object of arrows and G_0 the object of objects, in our case, the 'group of objects'. The cat^1-group notation replaces the source, target and identity maps by the composites $s = \iota\sigma$ and $t = \iota\tau$. This, of course, gives endomorphisms of G_1, which are simpler to handle than having a 'many sorted' picture with two separate groups. The downside of that simplicity is that the object of objects is slightly hidden. Of course, it is this subgroup, $Im\, s$, and the inclusion of that subgroup into $G = G_1$ is the morphism denoted ι. It is therefore reasonable to draw the 'objects' as blobs or points rather than as elements of G, e.g., as loops on the single real object of the group thought of as a single object groupoid. The resulting pictures *are* easier to draw! and to interpret.

A cat^2-group is similarly a category-like structure, internal to cat^1-groups, so is a double category internal to the category of groups, as the two category structures are independent of each other. This is emphasised if we look at the elements of a cat^2-group in an analogous way to the above. First suppose that (G, s_1, t_1, s_2, t_2) is a cat^2-group, then we might draw, for each $g \in G$, a square diagram:

$$\begin{tikzcd} \cdot \arrow[r, "t_2 g"] & \cdot \\ \cdot \arrow[u, "s_1 g"] \arrow[r, "s_2 g"'] \arrow[ur, phantom, "g"] & \cdot \arrow[u, "t_1 g"'] \end{tikzcd}$$

Now the left vertical arrow is in the subgroup, $Im\, s_1 = Im\, t_1$. (We can refer to $s_1 g$ as the 1-source, and $t_1 g$ as the 1-target, of g, and similarly for 2-source, and so on.) The square is a schema consistent the the equations: $s_1 t_2 = t_2 s_1$, and the three other similar ones. The element $s_1 t_2 g$ is the 1-source of the 2-target of g, so is the vertex at the top left of the square. It is also the 2-target of the 1-source of g, of course.

Such squares compose horizontally and vertically, provided the relevant sources and targets match, but how does this relate to the group structure on G?

Looking back, once more, to a cat^1-group, (G, s, t) and a resulting composition

$$s(g) \xrightarrow{g} t(g) = s(g') \xrightarrow{g'} t(g'),$$

it is not immediately clear how the composite is to be studied, but look back to the corresponding crossed module based description and it becomes clearer. We had in section 2.3.2,

$$p \xrightarrow{(c,p)} \partial c.p \xrightarrow{(c\prime,\partial c.p)} \partial c' \partial c.p,$$

and the composition was given as $(c', \partial c.p) \star (c, p) = (c'c, p)$. Back in cat^1-group language, this corresponds to $g' \star g = g' s(g')^{-1} g$. (We can check that $s(g' s(g')^{-1} g) = s(g)$ and that $t(g' s(g')^{-1} g) = t(g')$, as we would expect.)

We can extend this to cat^2-groups giving a way of composing the squares that we have in this context. For instance, for horizontal composition, we have

$$\begin{tikzcd}
\cdot \arrow[r] & \cdot \arrow[r] & \cdot & & \cdot \arrow[r] & \cdot \\
\cdot \arrow[r] \arrow[u] \arrow[ur, phantom, "g"] & \cdot \arrow[r] \arrow[u] \arrow[ur, phantom, "g'"] & \cdot \arrow[u] & = & \cdot \arrow[r] \arrow[u] \arrow[ur, phantom, "g's_1(g')^{-1}g"] & \cdot \arrow[u]
\end{tikzcd}$$

and similarly for vertical composition, replacing s_1 by s_2.

That gives a double category interpretation for a cat^2-group, but how does this relate to a crossed square,

$$\begin{tikzcd}
L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\mu'"'] & P
\end{tikzcd}$$

with h-map $h : M \times N \to L$. The construction hinted at earlier is first to form the cat^1-groups of the two vertical crossed modules, giving

$$\partial : L \rtimes N \to M \rtimes P, \text{ with } \partial(\ell, n) = (\lambda(\ell), \nu(n)),$$

with ∂ the induced map. There is an action of $M \rtimes P$ on $L \rtimes N$ (which will be examined shortly) giving a crossed module structure to the result. This action is non-trivial to define (or discover), so here is a way of thinking of it that may help.

We 'know' that a crossed square is meant to be a crossed module of crossed modules, so, if the above ∂ and action does give a crossed module, we will then be able to form a 'big group', $(L \rtimes N) \rtimes (M \rtimes P)$, with a cat^2-group structure on it. The action of $M \rtimes P$ on $L \rtimes N$ will need to correspond to conjugation within this 'big group' as the idea of semi-direct products is, amongst other things, to realise an action: if G acts on H, $H \rtimes G$ has multiplication given by $(h_1, g_1)(h_2, g_2) = (h_1 {}^{g_1}h_2, g_1g_2)$. In particular, it is easy to **work out**

$$(h, g)^{-1} = ({}^{g^{-1}}h^{-1}, g^{-1}),$$

so

$$(1, g)(h, 1)(1, g)^{-1} = ({}^{g}h, 1).$$

In our situation, we thus can work out the conjugation,

$$((1, 1), (m, p))((\ell, n), (1, 1))((1, 1), ({}^{p^{-1}}m^{-1}, p^{-1})) = ({}^{(m,p)}(\ell, n), (1, 1)).$$

Now this looks as if we are getting nowhere, but let us remember that any crossed square is isomorphic to the π_0 of an 'inclusion crossed square' of simplicial groups, (this was mentioned on page 213). This suggests that we first look at a group G, and a pair of normal subgroups M, N, and the inclusion crossed square

$$\begin{tikzcd}
M \cap N \arrow[r] \arrow[d] & N \arrow[d] \\
M \arrow[r] & G
\end{tikzcd}$$

with $h(m,n) = [m,n]$. If we track the above discussion of the action and the definition of ∂ in this example, we get the induced map, ∂, is the inclusion of $(M \cap N) \rtimes N$ into $M \rtimes G$. Here, therefore, there is, '*gratis*', an action of $M \rtimes G$ on $(M \cap N) \rtimes N$, namely by inner automorphisms / conjugation:

$$\begin{aligned}(m,g)(\ell,n)({}^{g^{-1}}m^{-1}, g^{-1})) &= (m,g)(\ell.n.{}^{g^{-1}}m.n^{-1}, ng) \\ &= (m.{}^{g}\ell.{}^{g}n.m.{}^{g}n^{-1}, gmg^{-1}),\end{aligned}$$

which can conveniently be written

$$({}^{mg}\ell.[m, {}^{g}n], {}^{g}n).$$

This suggests a formula for an action in the general case

$$\begin{aligned}{}^{(m,p)}(\ell,n) &= {}^{m}({}^{p}\ell, {}^{p}n) \\ &= ({}^{\mu(m)p}\ell.h(m, {}^{p}n), {}^{p}n).\end{aligned}$$

If we start with a simplicial inclusion crossed square, and form its 'big simplicial group' simplicially using the previous formula, then this *will* give the action of $M \rtimes P$ on $L \rtimes N$ in the general case, so our guess looks as if it is correct. Note that in both the particular case of the inclusion crossed square and this general case, we can derive $h(m,n)$ as a commutator within the 'big group'. (Of course, for the first of these, the h-map was defined as a commutator within G.)

We could go on to play around with other facets of this construction. This would be **well worthwhile** - but is better **left to the reader**. For instance, one obvious query is that $(L \rtimes N) \rtimes (M \rtimes P)$ should not be dependent on thinking of a crossed square as a morphism of (vertical) crossed modules. It is also a morphism of horizontal crossed modules, so this 'big group', if it is to give a useful object, should be isomorphic to $(L \rtimes M) \rtimes (N \rtimes P)$. It is, but what is a specific natural isomorphism doing the job. As somehow M has to 'pass through' N, we should expect to have to use the h-map.

There are other 'games to play'. Central extensions gave an instance of crossed modules, so what is their analogue for crossed squares. Double central extensions have been introduced by Janelidze in [165] and have been further studied by others, [125, 138, 238]. They provide a related idea. It is **left to you** to explore any connections that there are.

If we start with a crossed square, as above, what is the analogue of the picture

$$p_1 \xrightarrow{(c_1,p_1)} \partial c_1.p_1,$$

representing an element of the 'big group' of a crossed module. Suppose (ℓ, n, m, p) is such an element, then it is easy to see the 2-cell that corresponds to it must be:

$$\begin{tikzcd}
\nu(n)p \arrow[rr, "\lambda(\ell).{}^{\nu(n)}m{,}\nu(n)p)"] & & \mu(\lambda(\ell))\nu(n)\mu(m)p \\
& (\ell,n,m,p) & \\
p \arrow[uu, "(n{,}p)"] \arrow[rr, "(m{,}p)"'] & & \mu(m)p \arrow[uu, "\lambda'(\ell)n{,}\mu(m)p)"']
\end{tikzcd}$$

The details of how to compose, etc. are again **left to you**. It is, however, worth just checking the way in which the two edges on the top and on the right do match up. The right hand edge will clearly end at $\nu(\lambda'(\ell))\nu(n)\mu(m)p$, which, as $\nu\lambda' = \mu\lambda$, gives the expression on the top right vertex. Of more fun is the top edge. This ends at

$$\mu(\lambda(\ell)).\mu({}^{\nu(n)}m).\nu(n).p = \mu(\lambda(\ell)).\nu(n)\mu(m)\nu(n)^{-1}\nu(n)p,$$

so is as required, using the fact that μ is a crossed module.

In such a square 2-cell, the square itself is in the 'big group', the edges are in the cat^1-groups corresponding to vertical and horizontal crossed modules of the crossed square, and the vertices are in P.

Particularly interesting is the case of two crossed modules, $\mu : M \to P$ and $\nu : N \to P$, together with the corresponding $L = M \otimes N$, the Brown-Loday tensor product of the two, (cf. [66, 67]). Approximately, $M \otimes N$ is the universal codomain for an h-map based on the two given sides of the resulting crossed square. (A treatment of this construction has been included in the notes, [226], - please ignore the profinite conditions if using it 'discretely'.)

5.4.3 Cat^n-groups and crossed n-cubes, the general case

Of the two notions named in the title of this section, the first is easier to define.

Definition: A *cat^n-group* is a group G together with $2n$ endomorphisms $s_i, t_i, (1 \leq i \leq n)$ such that

$$s_it_i = t_i, \text{ and } t_is_i = s_i \text{ for all } i,$$

$$s_is_j = s_js_i, \quad t_it_j = t_jt_i, \quad s_it_j = t_js_i \text{ for } i \neq j$$

and, for all i,

$$[Ker\, s_i, Ker\, t_i] = 1.$$

A cat^n-group is thus a group with n independent cat^1-group structures on it.

As a cat^1-group can also be reformulated as an internal groupoid in the category of groups, a cat^n-group, not surprisingly, leads to an internal n-fold groupoid in the same setting.

The definition of crossed n-cube as an n-fold crossed module was initially suggested by Ellis in his thesis. The only problem was to determine the sense in which one crossed module should act on another. Since the number of axioms controlling the structure increased from crossed modules to crossed squares, one might fear that the number and complexity of the axioms would increase drastically in passing to higher 'dimensions'. The formulation that resulted from the joint work, [123], of Ellis and Steiner showed how that could be avoided by encoding the actions and the h-maps in the same structure.

We write $\langle n\rangle$ for the set $\{1, \ldots, n\}$.

Definition: A *crossed n-cube*, M, is a family of groups, $\{M_A : A \subseteq \langle n\rangle\}$, together with homomorphisms, $\mu_i : M_A \to M_{A-\{i\}}$, for $i \in \langle n\rangle, A \subseteq \langle n\rangle$, and functions, $h : M_A \times M_B \to M_{A\cup B}$, for $A, B \subseteq \langle n\rangle$, such that if ${}^ab$ denotes $h(a, b)b$ for $a \in M_A$ and $b \in M_B$ with $A \subseteq B$, then for $a, a' \in M_A$, $b, b' \in M_B$, $c \in M_C$ and $i, j \in \langle n\rangle$, the following axioms hold:

(1) $\mu_i a = a$ if $i \notin A$
(2) $\mu_i \mu_j a = \mu_j \mu_i a$
(3) $\mu_i h(a, b) = h(\mu_i a, \mu_i b)$
(4) $h(a, b) = h(\mu_i a, b) = h(a, \mu_i b)$ if $i \in A \cap B$
(5) $h(a, a') = [a, a']$
(6) $h(a, b) = h(b, a)^{-1}$
(7) $h(a, b) = 1$ if $a = 1$ or $b = 1$
(8) $h(aa', b) = {}^a h(a', b) h(a, b)$
(9) $h(a, bb') = h(a, b)^b h(a, b')$
(10) ${}^a h(h(a^{-1}, b), c)^c h(h(c^{-1}, a), b)^b h(h(b^{-1}, c), a) = 1$
(11) ${}^a h(b, c) = h({}^a b, {}^a c)$ if $A \subseteq B \cap C$.

A morphism of crossed n-cubes

$$\{M_A\} \to \{M'_A\}$$

is a family of homomorphisms, $\{f_A : M_A \to M'_A \mid A \subseteq \langle n \rangle\}$, which commute with the maps, μ_i, and the functions, h. This gives us a category, Crs^n, equivalent to that of cat^n-groups.

Remarks: 1. In the correspondence between cat^n-groups and crossed n-cubes (see Ellis and Steiner, [123]), the cat^n-group corresponding to a crossed n-cube, (M_A), is constructed as a repeated semidirect product of the various M_A. Within the resulting "big group", the h-functions interpret as being commutators. This partially explains the structure of the h-function axioms.

2. For $n = 1$, these eleven axioms reduce to the usual crossed module axioms. For $n = 2$, they give a crossed square:

$$\begin{tikzcd} M_{\langle 2 \rangle} \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & M_{\{1\}} \arrow[d, "\mu_1"] \\ M_{\{2\}} \arrow[r, "\mu_2"'] & M_\emptyset \end{tikzcd},$$

with the h-map, that was previously specified, being $h : M_{\{1\}} \times M_{\{2\}} \to M_{\langle 2 \rangle}$. The other h-maps in the above definition correspond to the various actions as explained in the definition itself.

Theorem 19 *[123] There are equivalences of categories*

$$Crs^n \simeq Cat^n(Grps),$$

■

5.5 Loday's Theorem and its extensions

In 1982, Loday proved a generalisation of the MacLane-Whitehead result that stated that connected homotopy 2-types (they called them 3-types) were modelled by crossed modules. The extension used cat^n-groups, and, as cat^1-groups 'are' crossed modules, we should expect cat^n-groups to model connected $(n+1)$-types (if the MacLane-Whitehead result is to be the $n = 1$ case, see page 213).

We have mentioned that 'simplicial groupoids' model all homotopy types and had a construction of both a crossed module $M(G, 1)$ and a crossed square, $M(G, 2)$ from a simplicial group, G. These are the $n = 1$ and $n = 2$ cases of a general construction of a crossed n-cube from G that we will give in a moment First we note a rather neat result.

We saw early on in these notes, (Lemma 4, page 42), that if $\partial : C \to P$ was a crossed module, then $\partial C \lhd P$, i.e. is a normal subgroup of P. A crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

can be thought of as a (horizontal or vertical,) crossed module of crossed modules:

$$\begin{tikzcd} L \arrow[d] & & M \arrow[d] \\ N & \longrightarrow & P \end{tikzcd}$$

(λ, ν) gives such a crossed module with domain (L, N, λ') and codomain (M, P, μ) and so on. (Working out the precise meaning of 'crossed module of crossed modules' and, in particular, what it should mean to have an action of one crossed module on another, is a very useful exercise; try it!) The image of (λ, ν) is a normal sub-crossed module of (M, P, μ), so we can form a quotient

$$\overline{\mu} : M/\lambda L \to P/\nu N,$$

and this is a crossed module. (This is not hard to check. There are lots of different ways of checking it, but perhaps the best way is just to show how $P/\nu N$ acts on $M/\lambda L$, in an obvious way, and then to check the induced map, $\overline{\mu}$, has the right properties - just by checking them. This gives one a feeling for how the various parts of the definition of a crossed square are used here.)

Another result from near the start of these notes, (Lemma 5), is that $Ker\,\partial$ is a central subgroup of C and ∂C acts trivially on it, so $Ker\,\partial$ has a natural $P/\partial C$-module structure. Is there an analogue of this for a crossed square? Of course, referring again to our crossed square, above, the kernel of (λ, ν) would be $\lambda' : Ker\,\lambda \to Ker\,\nu$ (omitting any indication of restriction of λ' for convenience). Both $Ker\,\lambda$ and $Ker\,\nu$ are Abelian, as they themselves are kernels of crossed modules, so $Ker\,\lambda$ is a $M/\lambda L$-module and $Ker\,\nu$ is a $P/\nu N$-module. (It is left to the diligent reader to work out the detailed structure here and to explore crossed modules that are modules over other ones.)

We had, for a given simplicial group, G, the crossed square

$$\begin{tikzcd} \dfrac{NG_2}{d_0(NG_3)} \arrow[r] \arrow[d] & Ker\,d_1 \arrow[d] \\ Kerd_2 \arrow[r] & G_1 \end{tikzcd}$$

which was denoted $M(G, 2)$. (The top horizontal and left vertical maps are induced by d_0.) Let us examine the horizontal quotient and kernel.

First the quotient, this has NG_1/d_0NG_2 as its 'top' group and $G_1/Ker\,d_0 \cong G_0$, as its bottom one. Checking all the induced maps shows quite quickly that the quotient crossed module is $M(G, 1)$, up to isomorphism.

What about the kernel? Well, the bottom horizontal map is an inclusion, so has trivial kernel, whilst the top is induced by d_0, and so the kernel here can be calculated to be $Ker\, d_0 \cap NG_2$, divided by $d_0(NG_3)$, but that is $Ker\, \partial / Im\, \partial$ in the Moore complex, so is $H_2(NG)$ and thus is $\pi_2(G)$. We thus have, from previous calculations, that for $M(G,1)$, there is a crossed 2-fold extension

$$\pi_1(G) \to \frac{NG_1}{\partial NG_2} \to NG_0 \to \pi_0(G)$$

and for $M(G,2)$, a similar object, a crossed 2-fold extension of crossed modules:

$$\begin{tikzcd}
1 \arrow[r] & \pi_2(G) \arrow[r] \arrow[d] & Ker\, d_1 \arrow[r] \arrow[d] & NG_2/d_0(NG_3) \arrow[r] \arrow[d] & Ker\, d_1 NG_1/d_0(NG_2) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & 1 \arrow[r] & Ker\, d_0 \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & 1
\end{tikzcd}$$

'Obviously' this should give an element of '$H^3(M(G,2),(\pi_2(G) \to 1))$', but we have not given any description of what that cohomology group should be. It can be done, but we will not go in that direction for the moment. Rather we will use the route via simplicial groups.

5.5.1 Simplicial groups and crossed n-cubes, the main ideas

We have that simplicial groups yield crossed squares by the $M(G,2)$ construction, and that, from $M(G,2)$, we can calculate $\pi_0(G)$, $\pi_1(G)$, and $\pi_2(G)$. If G represents a 3-type of a space (or the 2-type of a simplicial group), then we would expect these homotopy groups to be the only non-trivial ones. (Any simplicial group can be truncated to give one with these π_i as the only non-trivial ones.) This suggests that going from 3-types to crossed squares in a nice way should be just a question of combining the functorial constructions

$$\mathsf{Spaces} \overset{Sing}{\longrightarrow} \mathsf{Simplicial\ Sets}$$

$$\mathsf{Simplicial\ Sets} \overset{G(\ )}{\longrightarrow} \mathcal{S}-\mathsf{Groupoids}$$

$$\mathcal{S}-\mathsf{Groupoids} \overset{M(\ ,2)}{\longrightarrow} \mathsf{Crossed\ squares}.$$

Of course, we would need to see if, for $f : X \to Y$ a 3-equivalence (so f induces isomorphisms on π_i for $i = 0,1,2,3$), what would be the relationship between the corresponding crossed squares. We would also need to know that each crossed square was in sense 'equivalent' to one of the form $M(G,2)$ for some G constructed from it, in other words to reverse, in part, the last construction. (The other constructions have well known inverses at the homotopy level.)

We will use a 'multinerve' construction, generalising the nerve that we have already met. We will denote this by $E^{(n)}(\mathsf{M})$ for M a crossed n-cube.

For $n = 1$, $E^{(1)}$ is just the nerve of the crossed module, so if $\mathsf{M} = (C, P, \partial)$, we have $E^{(1)}(\mathsf{M}) = K(\mathsf{M})$ as given already on page 64.

For $n = 2$, i.e., for a crossed square, M, we form the 'double nerve ' of the associated cat^2-group of M. From M, we first form the 'crossed module of cat^1-groups'

$$L \rtimes N \overset{(\lambda,\nu)}{\longrightarrow} M \rtimes P,$$

where, for instance, in $M \rtimes P$ the source endomorphism is $s(m,p) = (1,p)$ and the target is $t(m,p) = (1, \partial m.p)$. (We could repeat in the horizontal direction to form $(L \rtimes N) \rtimes (M \rtimes P)$, which is the 'big group' of the cat^2-group associated to M, but, in fact, will not do this except implicitly, as it is easier to form a simplicial crossed module in this situation. This,

$$E^{(1)}(L \stackrel{\lambda'}{\to} N) \longrightarrow E^{(1)}(M \stackrel{\mu}{\to} P),$$

is obtained by applying the $E^{(1)}$ construction to the vertical crossed modules. The two parts are linked by a morphism of simplicial groups induced from (λ, ν) and which is compatible with the action of the right hand simplicial group on the left hand one. (This action is not that obvious to write down - unless you have already done the previously suggested 'exercises'. It uses the h-maps from $M \times N$ to L, etc. in an essential way, and is, in some ways, best viewed within $(L \rtimes N) \rtimes (M \rtimes P)$ as being derived from conjugation. Details are, for instance, in Porter, [226] or [227] as well as in the discussion of the equivalence between cat^n-groups and crossed n-cubes in the original, [123].)

With this simplicial crossed module, we apply the nerve in the second horizontal direction to get a bisimplicial group, $\mathcal{E}^{(2)}(\mathsf{M})$. (Of course, if we started with a crossed n-cube, we could repeat the application of the nerve functor n-times, one in each direction to get an n-simplicial group $\mathcal{E}^{(n)}(\mathsf{M})$.)

There are two ways of getting from a bisimplicial set or group to a simplicial one. One is the diagonal, so if $\{G_{p,q}\}$ is a bisimplicial group, $\mathsf{diag}(G_{\bullet,\bullet})_n = G_{n,n}$ with fairly obvious face and degeneracy maps. The other is the *codiagonal* (also sometimes called the 'bar construction'). This was introduced by Artin and Mazur, [10]. It picks up related terms in the various $G_{p,q}$ for $p + q = n$. (An example is for any simplicial group, G, on taking the nerve in each dimension. You get a bisimplicial set whose codiagonal is $\overline{W}(G)$, with the formula given later in these notes.) We will consider the codiagonal in some detail later on, (starting on page 650). The two constructions give homotopically equivalent simplicial groups. Proofs of this can be found in several places in the literature, for instance, in the paper by Cegarra and Remedios, [78]. Here we will set $E^{(n)}(\mathsf{M}) = \mathsf{diag}\mathcal{E}^{(n)}(\mathsf{M})$.

At this stage, for the reader trying to understand what is going on here, it is worth calculating the Moore complex of these simplicial groups. This is technically quite tricky as it is easy to make a slip, but it is not hard to see that they are 'closely related' to the 2-crossed module / mapping cone complex:

$$L \to M \rtimes N \to P$$

that we met earlier, (page 221), that is due to Loday and Conduché, see [82]. Of course, such detailed calculations are much harder to generalise to crossed n-cubes and other techniques are used, see [227] or the alternative version based on the technology of cat^n-groups due to Bullejos, Cegarra and Duskin, [73].

In any of these approaches from a crossed n-cube or cat^n-group, you either extract a n-simplicial group and then a simplicial group, by diagonal or codiagonal, or going one stage further, applying the nerve functor to the n-simplicial group to get a $(n+1)$-simplicial set, which is then 'attacked' using the diagonal or codiagonal functors to get out a simplicial set. This end result is the simplicial model for the crossed n-cube and has the same homotopy groups as M. It is known as the *classifying space of the crossed n-cube or cat^n-group*. (That term is usual, but it actually gives rise to an interesting obvious question, which has a simple answer in some ways but not if one looks at it thoroughly. That question is : *what does this classifying space classify?* That question will to some extent return to haunt us later one. The simple answer would be certain types of simplicial fibre bundles with fibre a $n+1$-type, but that throws away all the hard work to get the crossed n-cube itself, so ... .

Returning to the simplicial group approach, one applies the $M(-,n)$-functor, that we have so far seen only for $n=1$ and 2, to get back a new crossed n-cube. This is not M itself in general, but is 'quasi-isomorphic' to it.

Definition: A morphism, $f : \mathsf{M} \to \mathsf{N}$, of crossed n-cubes will be called a *trivial epimorphism* if $\mathcal{E}^{(n)}(f) : \mathcal{E}^{(n)}(\mathsf{M}) \to \mathcal{E}^{(n)}(\mathsf{N})$ is an epimorphism (and thus a fibration) of simplicial groups having contractible kernel.

Starting with the category, Crs^n, of crossed n-cubes, inverting the trivial epimorphisms gives a category, $Ho(Crs^n)$, and f will be called a *quasi-isomorphism* if it gives an isomorphism in this category.

Remark: Any trivial epimorphism of crossed modules is a *weak equivalence* in the sense of section 3.1, page 68. This follows from the long exact fibration sequence. Conversely any such weak equivalence is a quasi-isomorphism.

We can now state Loday's result in the form given in [227]:

Theorem 20 *The functor*

$$M(-,n) : Simp.Grps \to Crs^n$$

induces an equivalence of categories

$$Ho_n(Simp.Grps) \stackrel{\simeq}{\to} Ho(Crs^n).$$

∎

As yet we have not actually given the definition of $M(G,n)$ for $n>2$ so here it is:

Definition: Given a simplicial group, G, the crossed n-cube, $M(G,n)$, is given by:
(a) for $A \subseteq \langle n\rangle$,

$$M(G,n)_A = \frac{\bigcap\{Ker\, d_j^n \,:\, j \in A\}}{d_0(Ker\, d_1^{n+1} \cap \bigcap\{Ker\, d_{j+1}^{n+1} \,:\, j \in A\})};$$

(b) if $i \in \langle n\rangle$, the homomorphism $\mu_i : M(G,n)_A \to M(G,n)_{A\setminus\{i\}}$ is induced from the inclusion of $\bigcap\{Ker\, d_j^n \,:\, j \in A\}$ into $\bigcap\{Ker\, d_j^n \,:\, j \in A\setminus\{i\}\}$;
(c) representing an element in $M(G,n)_A$ by $\overline{x}$, where $x \in \bigcap\{Ker\, d_j^n \,:\, j \in A\}$, (so the overbar denotes a coset), and, for $A, B \subseteq \langle n\rangle$, $\overline{x} \in M(G,n)_A$, $\overline{y} \in M(G,n)_B$,

$$h(\overline{x},\overline{y}) = \overline{[x,y]} \in M(G,n)_{A\cup B}.$$

Where this definition 'comes from' and why it works is a bit to lengthy to include here, so we refer the interested reader to [226]. From its many properties, we will mention just the following one, linking $M(G, n)$ with $M(G, n-1)$ in a similar way to that we have examined for $n = 2$.

We will use the following notation: $M(G, n)_1$ will denote the crossed $(n-1)$-cube obtained by restricting to those $A \subseteq \langle n \rangle$ with $1 \in A$ and $M(G, n)_0$ that obtained from the terms with $A \subseteq \langle n \rangle$ with $1 \notin A$.

Proposition 63 *Given a simplicial group G and $n \geq 1$, there is an exact sequence of crossed $(n-1)$-cubes:*

$$1 \to K \to M(G,n)_1 \stackrel{\mu_1}{\to} M(G,n)_0 \to M(G,n-1) \to 1,$$

where, if $B \subseteq \langle n-1 \rangle$ and $B \neq \langle n-1 \rangle$, then $K_B = \{1\}$, whilst $K_{\langle n-1 \rangle} \cong \pi_n(G)$. ■

There are some special cases of crossed n-cubes, or the associated cat^n-groups that are worth looking at. For instance in [225], Paoli gives a new perspective on cat^n groups. It identifies a full subcategory of them (which are called *weakly globular*) which is sufficient to model connected $n+1$-types, but which has much better homotopical properties than the general ones. This, in fact, gives a more transparent algebraic description of the Postnikov decomposition and of the homotopy groups of the classifying space, and it also gives a kind of minimality property. Using weakly globular cat^n groups one can also describe a comparison functor to the Tamsamani model of $n+1$-types (cf. Tamsamani, [257]) which preserves the homotopy type.

5.5.2 Squared complexes

We have met crossed squares and 2-crossed modules and the different ways they encode the homotopy 3-type. We have extended 2-crossed modules to 2-crossed complexes, so it is natural curiosity to try to extend crossed squares to a 'cube' formulation. We will see this is just the start of another hierarchy which is in some ways simpler than that suggested by the hypercrossed complexes, and their variants, etc. The first step is the following which was introduced by Ellis, [122].

Definition: A *squared complex* consists of a diagram of group homomorphisms

$$\begin{tikzcd} & & & & N \arrow[dr, "\mu"] & \\ \ldots \arrow[r] & C_4 \arrow[r, "\partial_4"] & C_3 \arrow[r, "\partial_3"] & L \arrow[ur, "\lambda'"] \arrow[dr, "\lambda"'] & & P \\ & & & & M \arrow[ur, "\mu'"'] & \end{tikzcd}$$

together with actions of P on L, N, M and C_i for $i \geq 3$, and a function $h : M \times N \longrightarrow L$. The following axioms need to be satisfied.

(i) The square $\left(\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & N \arrow[d, "\mu"] \\ M \arrow[r, "\mu'"'] & P \end{tikzcd}\right)$ is a crossed square;

(ii) The group C_n is Abelian for $n \geq 3$

(iii) The boundary homomorphisms satisfy $\partial_n \partial_{n+1} = 1$ for $n \geq 3$, and $\partial_3(C_3)$ lies in the intersection

$Ker\,\lambda \cap Ker\,\lambda'$;
(iv) The action of P on C_n for $n \geq 3$ is such that μM and $\mu' N$ act trivially. Thus each C_n is a π_0-module with $\pi_0 = P/\mu M \mu' N$.
(v) The homomorphisms ∂_n are π_0-module homomorphisms for $n \geq 3$.

This last condition does make sense since the axioms for crossed squares imply that $Ker\,\mu' \cap Ker\mu$ is a π_0-module.

Definition: A *morphism of squared complexes*,

$$\Phi : \left(C_*, \left(\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & N \arrow[d, "\mu"] \\ M \arrow[r, "\mu'"'] & P \end{tikzcd}\right)\right) \longrightarrow \left(C'_*, \left(\begin{tikzcd} L' \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & N' \arrow[d, "\mu"] \\ M' \arrow[r, "\mu'"'] & P' \end{tikzcd}\right)\right)$$

consists of a morphism of crossed squares $(\Phi_L, \Phi_N, \Phi_M, \Phi_P)$, together with a family of equivariant homomorphisms Φ_n for $n \geq 3$ satisfying $\Phi_L \partial_3 = \partial'_3 \Phi_L$ and $\Phi_{n-1}\partial_n = \partial'_n \Phi_n$ for $n \geq 4$. There is clearly a category $SqComp$ of squared complexes.

A squared complex is thus a crossed square with a 'tail' attached.

Any simplicial group will give us such a gadget by taking the crossed square to be $M(sk_2G, 2)$, that is,

$$\begin{tikzcd} \dfrac{NG_2}{d_0(NG_3 \cap D_3)} \arrow[r] \arrow[d] & Ker\,d_1 \arrow[d] \\ Kerd_2 \arrow[r] & G_1 \end{tikzcd}$$

and then, for $n \geq 3$,

$$C_n(G) = \frac{NG_n}{(NG_n \cap D_n) d_0(NG_{n+1} \cap D_{n+1})}.$$

The above complex contains not only the information for the crossed square $M(G, 2)$ that represents the 3-type, but also the whole of $\mathsf{C}^{(2)}(G)$, the 2-crossed complex of G and thus the crossed complex and the 'chains on the universal cover' of G.

The advantage of working with crossed squares or squared complexes rather than the more linearly displayed models is that they can more easily encode 'non-symmetric' information. We will show this in low dimensions here but will later indicate how to extend it to higher ones. For instance, one gets a building process for homotopy types that reflects more the algebra. In examples, given two crossed modules, $\mu : M \to P$ and $\nu : N \to P$, there is a universal crossed square defining a 'tensor product' of the two crossed modules. We have

$$\begin{tikzcd} M \otimes N \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

is a crossed square and hence represents a 3-type. It is universal with regard to crossed squares having the same right-hand and bottom crossed modules, (see [66, 67] for the original theory and [226] for its connections with other material).

Equivalently we could represent its 3-type as a 2-crossed module

$$M \otimes N \longrightarrow M \rtimes N \xrightarrow{\mu\nu} P$$

or

$$M \otimes N \longrightarrow \frac{(M \rtimes N)}{\sim} \longrightarrow \frac{P}{\mu M},$$

where $\sim$ corresponds to dividing out by the μM action. However, of these, the crossed square lays out the information in a clearer format and so can often have some advantages.

5.6 Crossed $\mathbb{N}$-cubes

5.6.1 Just replace n by $\mathbb{N}$?

We have already suggested (page 216) how one might model all homotopy types using hypercrossed complexes, i.e. by adding more of the potential structure to the Moore complex of a simplicial group. We also saw how crossed modules (which are, from this viewpoint, 1-truncated hypercrossed complexes) generalised to crossed complexes, which have a better structured homotopical and homological algebra. We have seen earlier the transition from 2-crossed modules (= 2-truncated hypercrossed complexes) to 2-crossed complexes and briefly in the previous section, how crossed squares generalised to give squared complexes.

We will end this progression by looking at an elegant theoretical treatment of a generalisation of both crossed complexes and squared complexes. These gadgets are related to the "Moore chain complexes of order $(n+1)$ of a simplicial group", as briefly studied by Baues in [26], but have some of the advantages of crossed squares over 2-crossed modules, namely they can be 'non-symmetric', and hence are easily specified by, say, an 'inclusion crossed n-cube' consisting of a simplicial group and n simplicial normal subgroups. This allows for extra freedom in constructions. Also the axioms are very much simpler!

The definition of a crossed n-cube involves the set $\langle n\rangle = \{1, 2, \ldots, n\}$. One obvious way to extend this, eliminating dependence on n, is to try replacing $\langle n\rangle$ by $\mathbb{N} = \{1, 2, \ldots\}$ and taking the subsets A, B, C, in that definition to be finite, a condition previously automatic. This gives the notion of a crossed $\mathbb{N}$-cube:

Definition: A *crossed* $\mathbb{N}$-*cube*, M, is a family of groups,

$$\{M_A \mid A \subset \mathbb{N}, A \text{ finite}\},$$

together with homomorphisms, $\mu_i : M_A \to M_{A-\{i\}}$, $(i \in \mathbb{N},\ A \subset_{fin} \mathbb{N})$, and functions, $h : M_A \times M_B \to M_{A\cup B}, (A, B \subset_{fin} \mathbb{N})$, such that if ${}^a b$ denotes $h(a, b)b$ for $a \in M_A$ and $b \in M_B$ with $A \subseteq B$, then for $a, a' \in M_A$, $b, b' \in M_B$, $c \in M_C$ and $i, j \in \mathbb{N}$, the following axioms hold:

(1) $\mu_i a = a$ if $i \notin A$
(2) $\mu_i\mu_j a = \mu_j\mu_i a$
(3) $\mu_i h(a, b) = h(\mu_i a, \mu_i b)$
(4) $h(a, b) = h(\mu_i a, b) = h(a, \mu_i b)$ if $i \in A \cap B$
(5) $h(a, a') = [a, a']$
(6) $h(a, b) = h(b, a)^{-1}$
(7) $h(a, b) = 1$ if $a = 1$ or $b = 1$

(8) $h(aa', b) = {}^a h(a', b) h(a, b)$
(9) $h(a, bb') = h(a, b)^b h(a, b')$
(10) ${}^a h(h(a^{-1}, b), c)^c h(h(c^{-1}, a), b)^b h(h(b^{-1}, c), a) = 1$
(11) ${}^a h(b, c) = h({}^a b, {}^a c)$ if $A \subseteq B \cap C$.

(We have written $A \subset_{fin} \mathbb{N}$ as a shorthand for $A \subset \mathbb{N}$ with A finite.) Of course, these are formally identical to those given previously except in as much as there is no bound on the size of the finite sets A, B, C involved.

Examples: The first example is somewhat obvious, the second slightly surprising.

(i) As, for any n, $\langle n\rangle \subset \mathbb{N}$, if M is a crossed n-cube, then we can extend it trivially to an crossed $\mathbb{N}$-cube by defining $M_A = M_A$ if $A \subseteq \langle n\rangle$, and $M_A = 1$ otherwise. The h-maps, $M_A \times M_B \to M_{A\cup B}$, are then clearly determined by those of the original crossed n-cube.

(ii) Suppose $\mathsf{M} = \{M_A, \mu_i, h\}$ is a crossed $\mathbb{N}$-cube, which is such that M_A is trivial unless A is of form $\langle n\rangle$ for some n, (where we interpret $\emptyset$ as being $\langle 0\rangle$, and so $M_\emptyset$ is not required to be trivial). We will write $C_n = M_{\langle n\rangle}$ and $\partial_n : C_n \to C_{n-1}$ for the morphism $\mu_n : M_{\langle n\rangle} \to M_{\langle n-1\rangle}$.

We note that $\partial_{n-1}\partial_n$ is trivial as it factorises via the trivial group:

$$\begin{tikzcd}
M_{\langle n\rangle} \arrow[r] \arrow[d] & M_{\langle n-1\rangle} \arrow[d] \\
M_A \arrow[r] & M_{\langle n-2\rangle}
\end{tikzcd}$$

where $A = \langle n\rangle - \{n-1\}$, so $M_A = 1$. We thus have that (C_n, ∂_n) is a complex of groups.

There is a pairing

$$C_0 \times C_n \to C_n$$

given by $h : M_\emptyset \times M_{\langle n\rangle} \to M_{\langle n\rangle}$, and thus an action

$${}^a b = h(a, b)b,$$

whilst $\partial({}^a b) = {}^a\partial b$, since $\mu_n h(a, b) = h(\mu_n a, \mu_n b)$, which is $h(a, \mu_n b)$, since $n \notin \emptyset$!

The map $\partial_1 : C_1 \to C_0$ is a crossed module by exactly the proof that a crossed 1-cube is a crossed module.

If $a = \partial_1 b$, then for $c \in C_n$, $n \geq 2$,

$$\begin{aligned} ac &= h(\partial_1 b, c)c \\ &= h(b, \mu_1 c)c, \end{aligned}$$

since $1 \in \langle 1\rangle \cap \langle n\rangle$, but $\mu_1 c \in M_{\langle n\rangle - \{1\}}$, the trivial group so

$${}^a c = c.$$

We will not systematically check *all* the axioms, but clearly (C_n, ∂) is a crossed complex. (The detailed checking *is* best left to the reader.) Conversely any crossed complex gives a crossed $\mathbb{N}$-cube.

These examples show that both crossed n-cubes, for all n, and crossed complexes are examples of crossed $\mathbb{N}$-cubes. The obvious question, given our previous discussion, is to try to put Ellis' squared complex in the same framework. There is an obvious method to try out, and it works! One takes $M_A = 1$ unless $A = \langle n\rangle$ for some $n \in \mathbb{N}$ or if $A \subseteq \langle 2\rangle$. This does it, but it also indicates an effective way of encoding higher dimensional analogues of these squared complexes.

To do this, given $n \geq 1$, we have a subcategory of the category of crossed $\mathbb{N}$-cubes specified by the crossed n-cube complexes, that is, by $M_A = 1$ unless $A = \langle m\rangle$ for some $m \in \mathbb{N}$ or if $A \subseteq \langle n\rangle$ for the given n.

As we are going to explore these gadgets in a bit of detail, we introduce some notation.

$Crs^{\mathbb{N}}$ will denote the category of crossed $\mathbb{N}$-cubes of groups; $Crs^n.Comp$ will denote the subcategory of $Crs^{\mathbb{N}}$ determined by the crossed n-cube complexes. Thus, for instance, $Crs^1.Comp$ becomes an alternative notation for the category of crossed complexes.

5.6.2 From simplicial groups to crossed n-cube complexes

To show how these gadgets relate to ordinary 'bog-standard' models of homotopy types, we will show how to obtain a crossed n-cube complex from a simplicial group G.

To obtain a crossed n-cube complex from a simplicial group G, one analyses the constructions giving crossed complexes and crossed square complexes. For crossed complexes, one used the relative homotopy groups of G, so that the base crossed module is

$$\frac{NG_1}{(NG_1 \cap D_1)d_0(NG_2 \cap D_2)} \to G_0,$$

but $NG_1 \cap D_1 = 1$ since D_1 is generated by the $s_0(g)$ with $g \in G_0$.

For an arbitrary simplicial group, H, the crossed module $M(H, 1)$ was given by

$$\frac{NH_1}{d_0(NH_2)} \to H_0,$$

so the earlier crossed module was $M(sk_1G, 1)$, as $N(sk_1G)_2 = NG_2 \cap D_2$.

Similarly for the crossed square complex associated to G, we explicitly took the 'base' crossed square to be $M(sk_2G, 2)$.

Proposition 64 *Let G be a simplicial group and $n \in \mathbb{N}$. Define a family M_A, $A \subset \mathbb{N}$, A finite, by*
(i) if $A = \langle m\rangle$ and $m > n$, then

$$M_A = \frac{NG_m}{(NG_m \cap D_m)d_0(NG_{m+1} \cap D_{m+1})};$$

(ii) if $A \subseteq \langle n\rangle$,

$$\begin{aligned} M_A &= M(sk_nG, n)_A \\ &= \frac{\bigcap\{Ker\, d^n_j : j \in A\}}{d_0(Ker\, d^{n+1}_1 \cap \bigcap\{Ker\, d^{n+1}_{j+1} : j \in A\} \cap D_{n+1})} : \end{aligned}$$

(iii) if A is otherwise, then M_A is trivial.

Further define $\mu_i : M_A \to M_{A-\{i\}}$ by
(iv) if $i \in A$, then μ_i is the identity morphism;
(v) if $A = \langle m\rangle$, with $m > n$ and $i = m$, then μ_m is induced by d_0, and is trivial if $i \neq m$;

(vi) if $A \subseteq \langle n\rangle$, then μ_i is induced by the inclusions of intersections (i.e. as in $M(sk_nG, n)$);
(vii) otherwise μ_i is trivial.

Finally define $h : M_A \times M_B \to M_{A\cup B}$ by
(viii) if $A = \emptyset$ and $B = \langle m\rangle$ with $m > n$ then as $M_\emptyset = G_{n-1}$ and $M_B = C(G)_m$, if $a \in M_\emptyset$ and $b \in M_B$,

$$h(a, b) = [s_0^{m-n+1}(a), b] \in M_B;$$

similarly if $A = \langle m\rangle$ and $B = \emptyset$;
(ix) if $A, B \subseteq \langle n\rangle$, h is defined as in $M(sk_nG, n)$;
(x) otherwise h is trivial.

This data defines a crossed $\mathbb{N}$-cube which is, in fact, a crossed n-cube complex.

Proof: Much of this can be safely 'left to the reader'. It uses results from earlier parts of the notes. Note, however, that (viii) and (x) effectively say that it is only the $s_0^{n-1}G_0$ part of G_{n-1} that acts on any $M_{\langle m\rangle}$ and even then the image of $d_0 : NG_1 \to G_0$ acts trivially. To see this note that any $a \in G_{n-1}$ that is in some $Ker\, d_i$ is in the image of some μ_i, hence $a = \mu_i x$ say, but then

$$\begin{aligned} h(a, b) &= h(\mu_i x, b) \\ &= h(x, \mu_i b) \\ &= 1, \end{aligned}$$

by necessity if the structure is to be crossed $\mathbb{N}$-cube. Thus to check that the h-maps, and, in particular, those involved with part (viii) of the definition, satisfy the axioms, it suffices to use the methods mentioned earlier for checking that $\mathsf{C}(G)$ was a crossed complex, see [226]. ■

We might denote this crossed n-cube complex by $\mathsf{C}(G, n)$, as it combines both the technology of the $M(G, n)$ and the $\mathsf{C}(G)$. These models have yet to be explored in any depth, but see [226] and below for some preliminary results.

5.6.3 From n to $n - 1$: collecting up ideas and evidence

We noted earlier that given $M(G, n)$, the quotient crossed $(n - 1)$-cube was $M(G, n - 1)$. Is a similar result true here? Is there an epimorphism from $\mathsf{C}(G, n)$ to $\mathsf{C}(G, n-1)$? In fact this is linked with another problem. We have a nested sequence of full categories of $Crs^{\mathbb{N}}$,

$$Crs^1.Comp \subset Crs^2.Comp \subset \ldots \subset Crs^n.Comp \subset \ldots \subset Crs^{\mathbb{N}}.$$

Does the inclusion of $Crs^{n-1}.Comp$ into $Crs^n.Comp$ have a left adjoint, in other words, is $Crs^{n-1}.Comp$ a reflective subcategory of $Crs^n.Comp$? We investigate this question here only for $n = 2$ as this is at the same time easiest to see and also one of the most useful cases.

In this case, the crossed square complexes can be neatly represented as

$$\mathsf{C} := \quad \begin{tikzcd} \ldots \arrow[r] & C_3 \arrow[r, "\mu_3"] & C_{\langle 2\rangle} \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & C_{\langle 1\rangle} \arrow[d, "\mu_1"] \\ & & C_{\{2\}} \arrow[r, "\mu_2"'] & C_\emptyset \end{tikzcd} ,$$

whilst those corresponding to crossed complexes look like

$$\begin{tikzcd} \mathsf{D} := & \ldots \arrow[r] & D_3 \arrow[r, "\mu_3"] & D_{\langle 2\rangle} \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & D_{\langle 1\rangle} \arrow[d, "\mu_1"] \\ & & & 1 \arrow[r, "\mu_2"'] & D_{\emptyset} \end{tikzcd} .$$

A map φ in $Crs^2.Comp$ from C to D, clearly, must kill off $C_{\{2\}}$ and hence must also kill off $\mu_2(C_{\{2\}})$, which is normal in $C_{\emptyset}$. That is not all. If $a \in C_{\{2\}}$, $b \in C_{\{1\}}$ or $C_{\langle 2\rangle}$, then

$$\varphi(h(a,b)) = h(\varphi a, \varphi b) = 1,$$

and $\varphi a = 1$, thus φ must kill off the action of $C_{\{2\}}$ on $C_{\langle 2\rangle}$, and all elements of this form, $h(a,b)$ with $a \in C_{\{2\}}$, $b \in C_{\{1\}}$ or $C_{\langle 2\rangle}$.

Example: To illustrate what is happening let us examine the case of an inclusion crossed square. Suppose G is a group and M, N normal subgroups, then

$$\mathsf{C} = \left(\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r] & G \end{tikzcd}\right)$$

is a crossed square. Any 2-truncated crossed complex also gives a crossed square

$$\mathsf{D} = \left(\begin{tikzcd} D_2 \arrow[r] \arrow[d] & D_1 \arrow[d] \\ 1 \arrow[r] & D_0 \end{tikzcd}\right),$$

and any map from C to D factors through

$$\begin{tikzcd} \frac{M\cap N}{[M,N]} \arrow[r] \arrow[d] & \frac{M}{[M,N]} \arrow[d] \\ 1 \arrow[r] & G/N \end{tikzcd}$$

Proposition 65 *The inclusion of $Crs^1.Comp$ into $Crs^2.Comp$ has a left adjoint, denoted L. This left adjoint is a reflection, fixing the objects of the subcategory.* ∎

The proof should be fairly obvious so we will leave it as an exercise.

From $\mathsf{C}(G,2)$ to $\mathsf{C}(G,1)$: What happens if we apply this L to $\mathsf{C}(G,2)$? The answer is not that much of a surprise!

Proposition 66 *If G is a simplicial group, then there is a natural isomorphism*

$$\mathsf{L}(\mathsf{C}(G,2)) \cong \mathsf{C}(G,1).$$

∎

(Of course, the 'crossed 1-cube complex', $\mathsf{C}(G, 1)$, is just the crossed complex $\mathsf{C}(G)$ under another name.)

This does generalise to higher dimensions. We thus have a series of crossed approximations to homotopy types, each one reflecting nicely down to the previous one, but what do these crossed gadgets tell us about the spaces being modelled? To explore that we must go back to crossed modules and their classifying spaces. There is a two way process here, algebraic gadgets tell us information about spaces, but conversely spaces can inform us about algebra.

Chapter 6

Classifying spaces, and extensions

We will first look in detail at the construction of classifying spaces and their applications for the non-Abelian cohomology of *groups*. This will use things we have already met. Later on we will need to transfer some of this to a sheaf theoretic context to handle 'gerbes' and to look at other forms of non-Abelian cohomology.

6.1 Non-Abelian extensions revisited

We again start with an extension of groups:

$$\mathcal{E}: \quad 1 \to K \to E \xrightarrow{p} G \to 1.$$

From a section, s, we constructed a factor set, f, but this *is* a bit messy. What do we mean by that? We are working in the category of groups, but neither s nor f are group morphisms. For s, there is an obvious thing to do. The function s induces a homomorphism, k_1, from $C_1(G)$, the free group on the set, G, to E and

$$\begin{tikzcd} C_1(G) \arrow[r] \arrow[d, "k_1"] & G \arrow[d, "="] \\ E \arrow[r, "p"] & G \end{tikzcd}$$

commutes. One might be tempted to do the same for f, but f is partially controlled by s, so we try something else. When we were discussing identities among relations (page 47), we looked at the example of taking $X = \{\langle g\rangle \mid g \neq 1, g \in G\}$ and a relation $r_{g,g'} := \langle g\rangle\langle g'\rangle\langle gg'\rangle^{-1}$ for each pair (g, g') of elements of G. (Here we will write $\langle g_1, g_2\rangle$ for r_{g_1,g_2}.)

We can use this presentation $\mathcal{P}$ to build a free crossed module

$$C(\mathcal{P}) := C_2(G) \to C_1(G).$$

We noted earlier that the identities were going to correspond to tetrahedra, and that, in fact, we could continue the construction by taking $C_n(G) =$ the free G-module on $\langle g_1, \ldots, g_n\rangle$, $g_i \neq 1$, i.e. the normalised bar resolution. This is very nearly the usual bar resolution coming from the nerve of G, but we have a crossed module at the base, not just some more modules.

We met this structure earlier when we were looking at syzygies, and later on with crossed n-fold extensions, but is it of any use to us here?

We know $pf(g_1, g_2) = 1$, so $f(g_1, g_2) \in K$, and $C_2(G)$ is a free crossed module ... . Also, $K \to E$ is a normal inclusion, so is a crossed module ... . Thinking along these lines, we try

$$k_2 : C_2(G) \to K$$

defined on generators by f, *i.e.*, $i(k_2(\langle g_1, g_2\rangle)) = f(g_1, g_2)$. It is fairly easy to check this works, that

$$\partial k_2(\langle g_1, g_2\rangle) = k_1\partial(\langle g_1, g_2\rangle),$$

and that the actions are compatible, *i.e.*, $\mathbf{k} : C(\mathscr{P}) \to \mathscr{E}$, where will write $\mathscr{E}$ also for the crossed module, (K, E, i).

In other words, it seems that the section and the resulting factor set give us a morphism of crossed modules, $\mathbf{k}$. We note however that f satisfies a cocycle condition, so what does that look like here? To answer this we make the boundary, $\partial_3 : C_3(G) \to C_2(G)$, precise.

$$\partial_3\langle g_1, g_2, g_3\rangle = {}^{\langle g_1\rangle}\langle g_2, g_3\rangle\langle g_1, g_2g_3\rangle\langle g_1g_2, g_3\rangle^{-1}\langle g_1, g_2\rangle^{-1}$$

and, of course, the cocycle condition just says that $k_2\partial_3$ is trivial.

We can use the idea of a crossed complex as being a crossed module with a tail which is a chain complex, to point out that $\mathbf{k}$ gives a morphism of crossed complexes:

$$\begin{array}{ccccccccc}
\mathsf{C}(G) : \ldots & \longrightarrow & C_3(G) & \longrightarrow & C_2(G) & \longrightarrow & C_1(G) & \longrightarrow & G \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle k_2} & & \downarrow{\scriptstyle k_1} & & \downarrow \\
\mathscr{E} : \ldots & \longrightarrow & 1 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & G
\end{array}$$

where the crossed module $\mathscr{E}$ is thought of as a crossed complex with trivial tail.

Back to our general extension,

$$\mathscr{E} : \quad 1 \to K \to E \overset{p}{\to} G \to 1,$$

we note that the choice of a section, s, does not allow the use of an action of G on K. Of course, there is an action of E on K by conjugation and hence s does give us an action of $C_1(G)$ on K. If we translate 'action of G on a group, K', to being a functor from the *groupoid*, $G[1]$, to *Grps* sending the single object of $G[1]$ to the object K, then we can consider the 2-category structure of *Grps* with 2-cells given by conjugation, (so that if K and L are groups, and $f_1, f_2 : K \to L$ homomorphisms, a 2-cell $\alpha : f_1 \Longrightarrow f_2$ will be given by an element $\ell \in L$ such that

$$f_2(x) = \ell f_1(x)\ell^{-1}$$

for all $x \in K$). With this categorical perspective, s *does* give a lax functor from $G[1]$ to *Grps*. We essentially replace the action $G \to Aut(K)$, when s is a splitting, by a lax action (see Blanco, Bullejos and Faro, [34]);

$$\begin{array}{ccccc}
\longrightarrow & C_2(G) & \longrightarrow & C_1(G) \\
 & \downarrow{\scriptstyle k_2} & & \downarrow{\scriptstyle k_1} \\
 & K & \longrightarrow & E \\
 & \downarrow{\scriptstyle =} & & \downarrow \\
 & K & \longrightarrow & Aut(K).
\end{array}$$

Using this lax action and $\mathbf{k}$, we can reinterpret the classical reconstruction method of Schreier as forming the semidirect product $K \rtimes C_1(G)$, then dividing out by all pairs,

$$(k_2(\langle g_1, g_2\rangle), \partial_2(\langle g_1, g_2\rangle)^{-1}).$$

(We give Brown and Porter's article, [69], as a reference for a discussion of this construction.)

By itself this reinterpretation does not give us much. It just gives a slightly different viewpoint, however two points need making. This formulation is nearer the sort of approach that we will need to handle the classification of gerbes and the use of $K \to Aut(K)$ to handle the lax action of G reveals a problem and also a power in this formulation.

Dedecker, [104], noted that any theory of non-Abelian cohomology of groups must take account of the variation with K. Suppose we have two groups, K and L, and lax actions of G on them. What should it mean to say that some homomorphism $\alpha : K \to L$ is compatible with the lax actions?

A lax action of G on K can be given by a morphism of crossed modules / complexes, $Act_{G,K} : \mathsf{C}(G) \to \mathsf{Aut}(K)$, but $\mathsf{Aut}(K)$ is not functorial in K, so we do not automatically get a morphism of crossed modules, $\mathsf{Aut}(\alpha) : \mathsf{Aut}(K) \to \mathsf{Aut}(L)$. Perhaps the problem is slightly wrongly stated. One might say α is compatible with the lax G-actions if such a morphism of crossed modules existed and such that $Act_{G,L} = \mathsf{Aut}(\alpha)Act_{G,K}$. It is then just one final step to try to classify extensions with a finer notion of equivalence.

Definition: Suppose we have a crossed module, $\mathsf{Q} = (K, Q, q)$. An extension of K by G of the type of Q is a diagram:

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\omega"] & G \arrow[r] & 1 \\
 & K \arrow[r, "q"'] & Q & &
\end{tikzcd}$$

where ω gives a morphism of crossed modules.

There is an obvious notion of equivalence of two such extensions, where the isomorphism on the middle terms must commute with the structural maps ω and ω'. The special case when $Q = Aut(K)$ gives one the standard notion. In general, one gets a set of equivalence classes of such extensions $Ext_{K\to Q}(G, K)$ and this can be related to the cohomology set $H^2(G, K \to Q)$. This can also be stated in terms of a category $\mathcal{E}xt_{\mathsf{Q}}(G)$ of extensions of type Q, then the cohomology set is the set of components of this category.

This latter object can be defined using any free crossed resolution of G as there is a notion of homotopy for morphisms of crossed complexes such that this set is $[\mathsf{C}(G), \mathsf{Q}]$. Any other free crossed resolution of G has the same homotopy as $\mathsf{C}(G)$ and so will do just as well. Finding a complete set of syzygies for a presentation of G will do.

Example:

$$G = (x, y \mid x^2 = y^3)$$

This is the trefoil group. It is a one relator presentation and has no identities, so $C(\mathscr{P})$ is already a crossed resolution. A morphism of crossed modules, $\mathbf{k} : C(\mathscr{P}) \to \mathsf{Q}$, is specified by elements

$q_x, q_y \in Q$, and $a_r \in K$ such that $\mathbf{k}(a_r) = (q_x)^2(q_y)^{-3}$. Using this one can give a presentation of the E that results.

Remark: Extensions correspond to 'bitorsors' as we will see. These in higher dimensions then yields gerbes with action of a gr-stack and a corresponding cohomology. In the case of gerbes, as against extensions, a related notion was introduced by Debremaeker[1], [99–102]. This has recently been revisited by Milne, [202], and Aldrovandi, [4], who consider the special case where both K and Q are Abelian and the action of Q is trivial. This links with various important structures on gerbes and also with Abelian motives and hypercohomology. In all these cases, Q is being viewed as the coefficients of the cohomology and the gerbes / extensions have interpretations accordingly. Another very closely related approach is given in Breen, [48, 50]. We explore these ideas later in these notes.

We can think of the canonical case $K \to Aut(K)$ as being a 'natural' choice for extensions by K of a group, G. It is the structural crossed module of the 'fibre'. The crossed modules case says we can restrict or, alternatively, lift this structural crossed module to Q. This may, perhaps, be thought of as analogous to the situation that we will examine shortly where geometric structure corresponds to the restriction or the lifting of the natural structural group of a bundle. Both restricting to a subgroup and lifting to a covering group are useful and perhaps the same is true here.

6.2 Classifying spaces

The classifying spaces of crossed modules are never far from the surface in this approach to cohomology and related areas. They will play a very important role in the discussion of gerbes, as, for instance, in Larry Breen's work, [48–50], and later on here.

Classifying spaces of (discrete) groups are well known. One method of construction is to form the nerve, $Ner(G)$, of the group, G, (considered as a small groupoid, $\underset{\sim}{G}$ or $G[1]$, as usual). The classifying space is obtained by taking the geometric realisation, $BG = |Ner(G)|$.

To explore this notion, and how it relates to crossed modules, we need to take a short excursion into some simplicially based notions.

A classifying space of a group classifies principal G-bundles (G-torsors) over a space, X, in terms of homotopy classes of maps from X to BG, using a universal principal G-bundle $EG \to BG$.

This is very topological! If possible, it is useful to avoid the use of geometric realisations, since (i) this restricts one to groups and groupoids and makes handling more general 'algebras' difficult and (ii) for algebraic geometry, the topology involved is not the right kind as a sheaf-theoretic, topos based construction would be more appropriate. The classifying space is, therefore, often replaced by the nerve, as in Breen, [50].

How about classifying spaces for crossed modules? Given a crossed module, $\mathsf{M} = (C, G, \theta)$, say, we can form the associated 2-group, $\mathcal{X}(\mathsf{M})$. This gives a simplicial group by taking the nerve of the groupoid structure, then we can form $\overline{W}$ of that to get a simplicial set, $Ner(\mathsf{M})$. To reassure ourselves that this *is* a good generalisation of $Ner(G)$, we observe that if C is the trivial group, then $Ner(\mathsf{M}) = Ner(G)$. But this raises the question:

What does this 'classifying space' classify?

[1]An Englsih translation of [99] was made available in 2017 as [103]

To answer that we must digress to provide more details on the functors G and $\overline{W}$, we mentioned earlier.

6.2.1 Simplicially enriched groupoids

We denote the category of simplicial sets by $\mathcal{S}$ and that of simplicially enriched groupoids by $\mathcal{S}-Grpd$. This latter category includes that of simplicial groups, but it must be remembered that a simplicial object in the category of groupoids will, in general, have a non-trivial simplicial set as its 'object of objects', whilst in $\mathcal{S}-Grpd$, the corresponding simplicial object of objects will be constant. This corresponds to a groupoid in which each collection of 'arrows' between objects is a simplicial set, not just a set, and composition is a simplicial morphism, hence the term 'simplicially enriched'. We will often abbreviate the term 'simplicially enriched groupoid' to '$\mathcal{S}$-groupoid', but the reader should note that in some of the sources on this material the looser term 'simplicial groupoid' is used to describe these objects, usually with a note to the effect that this is not a completely accurate term to use.

Remark: Later, in section 11.2.1, we will need to work with $\mathcal{S}$-categories, *i.e.*, simplicially enriched categories. Some brief introduction can be found in [171], in the notes, [229] and the references cited there. We *will* give a fairly detailed discussion of the main parts of the elementary theory of $\mathcal{S}$-categories later.

The loop groupoid functor of Dwyer and Kan, [113], is a functor

$$G : \mathcal{S} \longrightarrow \mathcal{S}-Grpd,$$

which takes the simplicial set K to the simplicially enriched groupoid GK, where $(GK)_n$ is the free groupoid on the directed graph

$$K_{n+1} \overset{s}{\underset{t}{\rightrightarrows}} K_0\ ,$$

where the two functions, s, source, and t, target, are $s = (d_1)^{n+1}$ and $t = d_0(d_2)^n$ with relations $\overline{s_0 x} = id$ for $x \in K_n$. (Here we will try to use the convention that if $x \in K_n$, then the corresponding generator of $(GK)_{n-1}$ will be written $\overline{x}$[2].)

The face and degeneracy maps are given on generators by

$$\begin{aligned} s_i^{GK}(\overline{x}) &= \overline{s_{i+1}^K(x)}, \\ d_i^{GK}(\overline{x}) &= \overline{d_{i+1}^K(x)}, \text{for } \ x \in K_{n+1}, 1 < i \le n \end{aligned}$$

and

$$d_0^{GK}(\overline{x}) = (\overline{d_0^K(x)})^{-1}(\overline{d_1^K(x)}).$$

This loop groupoid functor has a right adjoint, $\overline{W}$, called the *classifying space* functor. The details as to its construction will be given shortly. It is important to note that if K is reduced, *i.e.*, has just one vertex, then GK will be a simplicial group, so is a well known type of object. This helps when studying these gadgets as we can often use simplicial group constructions, suitable adapted, in the $\mathcal{S}$-groupoid context. The first we will see is the Moore complex.

[2] It may sometimes occur that the overline is (unintentionally) left off the symbol and once the reader is familiarised with the idea of the construction, that convention may be informally relaxed and the overline omitted.

Definition: Given any $\mathcal{S}$-groupoid, G, its Moore complex, NG, is given by

$$NG_n = \bigcap_{i=1}^{n} Ker(d_i : G_n \longrightarrow G_{n-1})$$

with differential $\partial : NG_n \longrightarrow NG_{n-1}$ being the restriction of d_0. If $n \geq 1$, this is just a disjoint union of groups, one for each object in the object set, O, of G. If we write $G\{x\}$ for the simplicial group of elements that start and end at $x \in O$, then at object x, one has

$$NG\{x\}_n = (NG_n)\{x\}.$$

In dimension 0, one has $NG_0 = G_0$, so the $NG_n\{x\}$, for different objects x, are linked by the actions of the 0-simplices, acting by conjugation via repeated degeneracies.

The quotient, $NG_0/\partial(NG_1)$, is a groupoid, which is the *fundamental groupoid* of the simplicially enriched groupoid, G. We can also view this quotient as being obtained from the $\mathcal{S}$-enriched category G by applying the 'connected components' functor π_0 to each simplicial hom-set $G(x, y)$. If $G = G(K)$, the loop groupoid of a simplicial set K, then this fundamental groupoid is exactly the fundamental groupoid, ΠK, of K and we can take this as defining that groupoid if we need to be more precise later. This means that ΠK is obtained by taking the free groupoid on the 1-skeleton of K and then dividing out by relations corresponding to the 2-simplices: if $\sigma \in K_2$, we have a relation

$$\overline{d_2(\sigma)}.\overline{d_0(\sigma)} \equiv \overline{d_1(\sigma)}.$$

(**You are left** to explore this a bit more, justifying the claims we have made. You may also like to review the treatment in the book by Gabriel and Zisman, [132].)

For simplicity in the description below, we will often assume that the $\mathcal{S}$-groupoid is *reduced*, that is, its set O, of objects is just a singleton set $\{*\}$, so G is just a simplicial group.

Suppose that NG_m is trivial for $m > n$.

If $n = 0$, then NG_0 is just the group G_0 and the simplicial group (or groupoid) represents an Eilenberg-MacLane space, $K(G_0, 1)$.

If $n = 1$, then $\partial : NG_1 \longrightarrow NG_0$ has a natural crossed module structure.

Returning to the discussion of the Moore complex, if $n = 2$, then

$$NG_2 \xrightarrow{\partial} NG_1 \xrightarrow{\partial} NG_0$$

has a 2-crossed module structure in the sense of Conduché, [81] and above section 5.3. (These statements are for groups and hence for connected homotopy types. The non-connected case, handled by working with simplicially enriched groupoids, is an easy extension.)

In all cases, the simplicial group will have non-trivial homotopy groups only in the range covered by the non-trivial part of the Moore complex.

Now relaxing the restriction on G, for each $n > 1$, let D_n denote the subgroupoid of G_n generated by the degenerate elements. Instead of asking that NG_n be trivial, we can ask that $NG_n \cap D_n$ be. The importance of this is that the structural information on the homotopy type represented by G includes structure such as the Whitehead products and these all lie in the subgroupoids $NG_n \cap D_n$. If these are all trivial then the algebraic structure of the Moore complex is simpler, being that of a crossed complex, and $\overline{W}G$ is a simplicial set whose realisation is the *classifying space of that crossed complex*, cf. [63]. The simplicial set, $\overline{W}G$, is isomorphic to the *nerve* of the crossed complex.

Notational warning. As was mentioned before, the indexing of levels in constructions with crossed complexes may cause some confusion. The Dwyer-Kan construction is essentially a 'loop' construction, whilst $\overline{W}$ is a 'suspension'. They are like 'shift' operators for chain complexes. For example G decreases dimension, as an old 1-simplex x yields a generator in dimension 0, and so on. Our usual notation for crossed complexes has C_0 as the set of objects, C_1 corresponding to a relative fundamental groupoid, and C_n abstracting its properties from $\pi_n(X_n, X_{n-1}, p)$, hence the natural topological indexing *has* been used. For the $\mathcal{S}$-groupoid $G(K)$, the set of objects is separated out and $G(K)_0$ is a groupoid on the 1-simplices of K, a dimension shift. Because of this, in the notation being used here, the crossed complex $\mathsf{C}(G)$ associated to an $\mathcal{S}$-groupoid, G, will have a dimension shift as well: explicitly

$$C(G)_n = \frac{NG_{n-1}}{(NG_{n-1} \cap D_{n-1})d_0(NG_n \cap D_n)} \quad \text{for } n \geq 2,$$

$C(G)_1 = NG_0$, and, of course, C_0 is the common set of objects of G. In some papers where only the algebraic constructions are being treated, this convention is not used and C is given without this dimension shift relative to the Moore complex. Because of this, care is sometimes needed when comparing formulae from different sources.

6.2.2 Conduché's decomposition and the Dold-Kan Theorem

The category of crossed complexes (of groupoids) is equivalent to a reflective subcategory of the category $\mathcal{S}-Grpd$ and the reflection is defined by the obvious functor : take the Moore complex of the $\mathcal{S}$-groupoid and divide out by the $NG_n \cap D_n$, see [115, 116]. We will denote by $C : \mathcal{S}-Grpd \longrightarrow Crs$ the resulting composite functor, Moore complex followed by reflection. Of course, we have the formula, more or less as before, (cf. page 87),

$$C(G)_{n+1} = \frac{NG_n}{(NG_n \cap D_n)\ d_0(NG_{n+1} \cap D_{n+1})}.$$

The Moore complex functor itself is part of an adjoint (Dold-Kan) equivalence between the category $\mathcal{S} - Grpd$ and the category of hypercrossed complexes, [77], and this restricts to the Ashley-Conduché version of the Dold-Kan theorem of [13].

In order to justify the description of the nerve, and thus the related classifying space, of a crossed complex C, we will specify the functors involved, namely the Dold-Kan inverse construction and the $\overline{W}$. (We will leave **the reader** to chase up the detailed proof of this crossed complex form of the Dold-Kan theorem. The functors will be here, but the detailed proofs that they do give an equivalence will be left to you to give or find in the literature.) This will also give us extra tools for later use. We will first need the Conduché decomposition lemma, [81].

Proposition 67 *If G is a simplicial group(oid), then G_n decomposes as a multiple semidirect product:*

$$G_n \cong NG_n \rtimes s_0NG_{n-1} \rtimes s_1NG_{n-1} \rtimes s_1s_0NG_{n-2} \rtimes s_2NG_{n-1} \rtimes \ldots s_{n-1}s_{n-2}\ldots s_0NG_0$$

■

The order of the terms corresponds to a lexicographic ordering of the indices $\emptyset$; 0; 1; 1,0; 2; 2,0; 2,1; 2,1,0; 3; 3,0; ... and so on, the term corresponding to $i_1 > \ldots > i_p$ being $s_{i_1} \ldots s_{i_p} NG_{n-p}$.

The proof of this result is based on a simple lemma, which is easy to prove.

Lemma 39 *If G is a simplicial group(oid), then G_n decomposes as a semidirect product:*

$$G_n \cong Ker\ d_n^n \rtimes s_{n-1}^{n-1}(G_{n-1}).$$

■

We next note that in the classical (Abelian) Dold-Kan theorem, (cf. [97]), the equivalence of categories is constructed using the Moore complex and a functor K constructed via the original direct sum / Abelian version of Conduché's decomposition, cf. for instance, [97].

For each non-negatively graded chain complex, $\mathsf{D} = (D_n, \partial)$. in Ab, $K\mathsf{D}$ is the simplicial Abelian group with

$$(K\mathsf{D})_n = \oplus_a (D_{n-\sharp(a)}, s_a),$$

the sum being indexed by all descending sequences, $a = \{n > i_p \geq \ldots \geq\ i_1 \geq 0\}$, where $s_a = s_{i_p} \ldots s_{i_1}$, and where $\sharp(a) = p$, the summand D_n corresponding to the empty sequence.

The face and degeneracy operators in $K\mathsf{D}$ are given by the rules:
(1) if $d_i s_a = s_b$, then d_i will map (D_{n-p}, s_a) to $(D_{(n-1)-(p-1)}, s_b)$ by the identity on D_{n-p}; its components into other direct summands will be zero;
(2) if $d_i s_a = s_b d_0$, then d_i will map (D_{n-p}, s_a) to (D_{n-p-1}, s_b) as the homomorphism $\partial_{n-p} : D_{n-p} \to D_{n-p-1}$; its components into other direct summands will be zero;
(3) if $d_i s_a = s_b d_j$, $j > 0$, then $d_i(D_{n-p}, s_a) = 0$;
(4) if $s_i s_a = s_b$, then s_i maps (D_{n-p}, s_a) to $(D_{(n+1)-(p+1)}, s_b)$ by the identity on D_{n-p}; its components into other direct summands will be zero.

This suggests that we form a functor

$$K : Crs \to \mathcal{S} - Grpd$$

using a semidirect product, but we have to take care as there will be a dimension shift, our lowest dimension being C_1:
if C is in Crs, set

$$K(\mathsf{C})_n = C_{n+1} \rtimes s_0 C_n \rtimes s_1 C_n \rtimes s_1 s_0 C_{n-1} \rtimes \cdots \rtimes s_{n-1} s_{n-2} \ldots s_0 C_1.$$

The order of terms is to be that of the proposition given above. The formation of the semidirect product is as in the proof we hinted at of that proposition, that is the bracketing is inductively given by

$$(C_{n+1} \ldots \rtimes s_{n-2} \ldots s_0 C_2) \rtimes (s_{n-1} C_n \rtimes \ldots \rtimes s_{n-1} \ldots s_0 C_1);$$

each $s_\alpha(C_{n+1-\sharp(\alpha)})$ is an indexed copy of $C_{n+1-\sharp(\alpha)}$; the action of

$$s_{n-1} C_{n-1} \rtimes \ldots \rtimes s_{n-1} \ldots s_0 C_0 \ \ (\cong s_{n-1} K(\mathsf{C})_{n-1})$$

on $C_{n+1} \rtimes \ldots s_{n-2} \ldots s_0 C_1$, is given componentwise by the actions of each C_i and as C is a crossed complex, these are all via C_0. This implies, of course, that the majority of the components of these actions are trivial.

To see how this looks in low dimensions, it is simple to give the first few terms of the simplicial group(oid). As we are taking a reduced crossed complex as illustration, the result is a simplicial group, $K(\mathsf{C})$, having

- $K(\mathsf{C})_0 = C_1$
- $K(\mathsf{C})_1 = C_2 \rtimes s_0(C_1)$
- $K(\mathsf{C})_2 = (C_3 \rtimes s_0C_2) \rtimes (s_1C_2 \rtimes s_1s_0C_1)$
- $K(\mathsf{C})_3 = (C_4 \rtimes s_0C_3 \rtimes s_1C_3 \rtimes s_1s_0C_2) \rtimes (s_2C_3 \rtimes s_2s_0C_2 \rtimes s_2s_1C_2 \rtimes s_2s_1s_0C_1)$,

and so on.

The face and degeneracy maps are determined by the obvious rules adapting those in the Abelian case, so that if $c \in C_k$, the corresponding copy of c in $s_\alpha C_k$ will be denoted $s_\alpha c$ and a face or degeneracy operator will usually act just on the index. The exception to this is if, when renormalised to the form $s_\beta d_\gamma$ using the simplicial identities, γ is non-empty. If $d_\gamma = d_0$ then $d_\gamma c$ becomes $\delta_k c \in C_{k-1}$, otherwise $d_\gamma c$ will be trivial.

Lemma 40 *The above defines a functor*

$$K : Crs \to \mathcal{S} - Grpd$$

such that $\mathsf{C}K \cong Id$. ■

This extends the functor $K : CMod \to Simp.Grps$, given earlier, to crossed complexes as there $C_k = 1$ for $k > 2$.

One obvious question, given our earlier discussion of group T complexes, and its fairly obvious adaptation to groupoid T-complexes, is if we start with a crossed complex C and construct this simplicially enriched groupoid $K(\mathsf{C})$, is this a groupoid T-complex? As the thin filler condition for groupoid T-complexes involves the Moore complex, it is enough to look at the single object simplicial group case. We have the following:

Proposition 68 *If* C *is a crossed complex, then* $K\mathsf{C}$ *is a group* T*-complex.*

Proof: We have to check that $NK(\mathsf{C})_n \cap D_n = 1$. We suppose $g \in NK(\mathsf{C})_n$ is a product of degenerate elements, then, using the semidirect decomposition, we can write g in the form

$$g = s_1(g_1) \dots s_{n-1}(g_{n-1}). \qquad (*)$$

The only problem in doing this is handling any element that comes from C_0, but this can be done via the action of C_0 on the C_i.

As $g \in Ker\, d_n$, we have

$$1 = d_n g = s_1 d_{n-1}(g_1) \dots s_{n-2} d_{n-1}(g_{n-2}).g_{n-1},$$

so we can replace g_{n-1} by a product of degenerate elements and use $s_{n-1}s_i = s_i s_{n-2}$ and rewriting to obtain a new expression for g in the form (*), but with no s_{n-1} term. Repeating using d_{n-1} on this new expression yields that the new g_{n-2} is also in D_{n-1} and so on until we obtain

$$g = s_0(g^{(1)})$$

where $g^{(1)} \in D_{n-1}$, writing $g^{(1)}$ in the form (*) gives

$$g = s_0 s_0({g_1}^{(1)} \dots s_0 s_{n-2}({g_{n-2}}^{(1)}),$$

but $d_1d_ng = 1$, so $g_{n-2}{}^{(1)} \in D_{n-2}$. Repeating we eventually get $g = s_0s_0(g^{(n)})$ with $g^{(2)} \in D_{n-2}$. This process continues until we get $g = s_0{}^{(n)}(g^{(n)})$ with $g^{(n)} \in K(\mathsf{C})_0$, but $d_1 \dots d_ng = g^{(n)}$ and $d_1 \dots d_ng = 1$, so $g = 1$ as required. ■

Note that this proof, which is based on Ashley's proof that simplicial Abelian groups are group T-complexes (cf., [13]), depends in a strong way on being able to write g in the form $(*)$, *i.e.*, on the triviality of almost all the actions together with the explicit nature of the action of C_0.

Collecting up the pieces we have all the main points in the proof of the following Dold-Kan theorem for crossed complexes.

Theorem 21 *There is an equivalence of categories*

$$Grpd.T\!-\!comp. \overset{\simeq}{\longleftrightarrow} Crs.$$

■

Checking that we do have all the parts necessary and providing any missing pieces is a good exercise, so will be **left to you**. A treatment more or less consistent with the conventions here can be found in [226].

6.2.3 $\overline{W}$ and the nerve of a crossed complex

We next need to make explicit the $\overline{W}$ construction. The simplicial / algebraic description of the nerve of a crossed complex, C, is then as $\overline{W}(K(\mathsf{C}))$. We first give this description for a general simplicially enriched groupoid.

Let H be an $\mathcal{S}$-groupoid, then $\overline{W}H$ is the simplicial set described by

- $(\overline{W}H)_0 = ob(H_0)$, the set of objects of the groupoid of 0-simplices (and hence of the groupoid at each level);
- $(\overline{W}H)_1 = arr(H_0)$, the set of arrows of the groupoid H_0:

and for $n \geq 2$,

- $(\overline{W}H)_n = \{(h_{n-1}, \dots, h_0) \mid h_i \in arr(H_i) \text{ and } s(h_{i-1}) = t(h_i), 0 < i < n\}$.

Here s and t are generic symbols for the domain and codomain mappings of all the groupoids involved. The face and degeneracy mappings between $\overline{W}(H)_1$ and $\overline{W}(H)_0$ are the source and target maps and the identity maps of H_0, respectively; whilst the face and degeneracy maps at higher levels are given as follows:

The face and degeneracy maps are given by

- $d_0(h_{n-1}, \dots, h_0) = (h_{n-2}, \dots, h_0)$;
- for $0 < i < n$, $d_i(h_{n-1}, \dots, h_0) = (d_{i-1}h_{n-1}, d_{i-2}h_{n-2}, \dots, d_0h_{n-i}h_{n-i-1}, h_{n-i-2}, \dots, h_0)$;

and

- $d_n(h_{n-1}, \dots, h_0) = (d_{n-1}h_{n-1}, d_{n-2}h_{n-2}, \dots, d_1h_1)$;

whilst

- $s_0(h_{n-1}, \dots, h_0) = (id_{dom(h_{n-1})}, h_{n-1}, \dots, h_0)$;

and,

- for $0 < i \leq n$, $s_i(h_{n-1}, \ldots, h_0) = (s_{i-1}h_{n-1}, \ldots, s_0 h_{n-i}, id_{cod(h_{n-i})}, h_{n-i-1}, \ldots, h_0)$.

Remarks: (i) We note that if H is a constant simplicial groupoid, $\overline{W}(H)$ is the same as the nerve of that groupoid for the algebraic composition order. Later on, when re-examining the classifying space construction, we may need to rework the above definition in a form using the functional composition order.

(ii) None of the formulae use the invertibility of any of the h_i, and so the description gives a simplicial set even if G is merely an $\mathcal{S}$-category.

To help understand the structure of the nerve of a (reduced) crossed complex, C, we will calculate $Ner(\mathsf{C}) = \overline{W}(K(\mathsf{C}))$ in low dimensions. This will enable comparison with formulae given earlier. The calculations are just the result of careful application of the formulae for $\overline{W}$ to $H = K(\mathsf{C})$:

- $Ner(\mathsf{C})_0 = *$, as we are considering a *reduced* crossed complex - in the general case, this is C_0;

- $Ner(\mathsf{C})_1 = C_1$, as a *set* of 'directed edges' or arrows - we will avoid using a special notation for 'underlying set of a group(oid)';

- $Ner(\mathsf{C})_2 = \{(h_1, h_0) \mid h_1 = (c_2, s_0(c_1)), h_0 = c'_1, \text{ with } c_2 \in C_2, c_1, c'_1 \in C_1\}$, and such a 2-simplex has faces given as in the diagram

$$\begin{tikzcd}
& {} \arrow[dr, "c'_1"] \arrow[d, phantom, "(h_1{,}h_0)"] & \\
{} \arrow[ur, "c_1"] \arrow[rr, "\delta c_2.c_1.c'_1"'] & {} & {}
\end{tikzcd}$$

  Note that $h_1 : c_1 \longrightarrow \delta c_2.c_1$ in the internal category corresponding to the crossed module, (C_2, C_1, δ), so the formation of this 2-simplex corresponds to a right whiskering of that 2-cell (in the corresponding 2-groupoid) by the arrow c'_1;

- $Ner(\mathsf{C})_3 = \{(h_2, h_1, h_0) \mid h_1 = (c_3, s_0c_2^0, s_1c_2^1, s_1s_0c_1), h_1 = (c'_2, s_0(c'_1)), h_0 = c''_1\}$ in the evident notation. Here the faces of the 3-simplex (h_2, h_1, h_0) are as in the diagrams, (in each of which the label for the 2-simplex itself has been abbreviated):

$$d_3 : \quad \begin{tikzcd}
& {} \arrow[dr, "c'_1"] \arrow[d, phantom, "c_2^1"] & \\
{} \arrow[ur, "c_1"] \arrow[rr, "\delta c_2^1.c_1.c'_1"'] & {} & {}
\end{tikzcd} \quad ; \quad d_2 : \quad \begin{tikzcd}
& {} \arrow[dr, "\delta c'_2.c'_1.c''_1"] \arrow[d, phantom, "c_2^0c_2^1"] & \\
{} \arrow[ur, "c_1"] \arrow[rr, "\delta(c_2^0c_2^1).c_1.\delta c'_2.c'_1.c''_1"'] & {} & {}
\end{tikzcd}$$

$$d_1: \quad \begin{tikzcd} & {} \arrow[dr, "c_1''"] \arrow[d, phantom, "\delta c_3.c_2^0.{}^{\delta c_2^1.c_1}c_2'"] & \\ {} \arrow[ur, "\delta c_2^1.c_1.c_1'"] \arrow[rr, "\delta(c_2^0c_2^1).c_1.\delta c_2'.c_1'.c_1''"'] & {} & {} \end{tikzcd} \quad ; \quad d_0: \quad \begin{tikzcd} & {} \arrow[dr, "c_1''"] \arrow[d, phantom, "c_2'"] & \\ {} \arrow[ur, "c_1'"] \arrow[rr, "\delta c_2'.c_1'.c_1''"'] & {} & {} \end{tikzcd}$$

The only face where any real thought has to be used is d_1. In this the d_1 face has to be checked to be consistent with the others. The calculation goes like this:

$$\begin{aligned} \delta(\delta c_3.c_2^0.{}^{\delta c_2^1.c_1}c_2').(\delta c_2^1.c_1.c_1').c_1'' &= \delta c_2^0.(\delta c_2^1.c_1.\delta c_2'.c_1^{-1}.(\delta c_2^1)^{-1}).\delta c_2^1.c_1.c_1'.c_1'' \\ &= \delta(c_2^0c_2^1).c_1.\delta c_2'.c_1'.c_1'' \end{aligned}$$

This uses (i) $\delta\delta c_3$ is trivial, being a boundary of a boundary, and (ii) the second crossed module rule for expanding $\delta({}^{\delta c_2^1.c_1}c_2')$ as $\delta c_2^1.c_1.\delta c_2'.c_1^{-1}.(\delta c_2^1)^{-1}$.

This diagrammatic representation, although useful, is limited. A recursive approach can be used as well as the simplicial / algebraic one given above. In this, $Ner(\mathsf{C})$ is built up via its skeletons, specifying a simplex in $Ner(\mathsf{C})_n$ as an element of C_n, together with the empty simplex that it 'fills', i.e. the set of compatible $(n-1)$-simplices. This description is used by Ashley, ([13], p. 37). More on nerves of crossed complexes can be found in Nan Tie, [218, 219]. There is also a very neat 'singular complex' description, $Ner(\mathsf{C})_n = Crs(\pi(n), \mathsf{C})$, where $\pi(n)$ is the free crossed complex on the n-simplex, $\Delta[n]$. We will have occasion to see this in more detail later.

This singular complex description shows another important feature. If we have an n-simplex $f : \pi(n) \to \mathsf{C}$, we will say it is *thin* if the image $f(\iota_n)$ of the top dimensional generator in $\pi(n)$ is trivial. The nerve together with the filtered set of thin elements forms a T-complex in the sense of section 1.3.6. This is discussed in Ashley, [13], and Brown-Higgins, [63].

6.3 Simplicial Automorphisms and Regular Representations

The usual enrichment of the category of simplicial sets is given by :
for each $n \geq 0$, the set of n-simplices is

$$\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L),$$

together with obvious face and degeneracy maps.
Composition : for $f \in \underline{\mathcal{S}}(K, L)_n$, $g \in \underline{\mathcal{S}}(L, M)_n$, so $f : \Delta[n] \times K \to L$, $g : \Delta[n] \times L \to M$,

$$g \circ f := (\Delta[n] \times K \xrightarrow{diag \times K} \Delta[n] \times \Delta[n] \times K \xrightarrow{\Delta[n] \times f} \Delta[n] \times L \xrightarrow{g} M);$$

Identity : $id_K : \Delta[0] \times K \xrightarrow{\cong} K$.

Definition: The simplicial set, $\underline{\mathcal{S}}(K, L)$, defined above, is called the *simplicial mapping space of maps from K to L.*

This **clearly** is functorial in both K and L. (Of course, with differing 'variance'. It is 'contra-variant' in K, so that $\underline{\mathcal{S}}(-, L)$ is a functor from $\mathcal{S}^{op}$ to $\mathcal{S}$, but $\underline{\mathcal{S}}(K, -) : \mathcal{S} \to \mathcal{S}$. In the category,

$\mathcal{S}$, each of the functors 'product with K' for K a simplicial set, has a right adjoint, namely this $\underline{\mathcal{S}}(K,-)$. Technically $\mathcal{S}$ is a *Cartesian closed category*, a notion we will explore briefly in the next section. In any such setting we can restrict to looking at endomorphisms of an object, and, here we can go further and get a simplicial group of automorphisms of a simplicial set, K, analogously to our construction of the automorphism 2-group of a group (recall from section 2.3.4).

Explicitly, for fixed K, $\underline{\mathcal{S}}(K,K)$ is a simplicial monoid, called the *simplicial endomorphism monoid of* K and $\mathsf{aut}(K)$ will be the corresponding simplicial group of invertible elements, that is the *simplicial automorphism group of* K.

If $f : K \times \Delta[n] \longrightarrow L$ is an n-simplex, then we can form a diagram

$$\begin{tikzcd} K \times \Delta[n] \arrow[rr, "(f,p)"] \arrow[dr] & & L \times \Delta[n] \arrow[dl] \\ & \Delta[n] & \end{tikzcd}$$

in which the two slanting arrow are the obvious projections, (so $(f,p)(k,\sigma) = (f(k,\sigma),\sigma)$). Taking $K = L$, $f \in \mathsf{aut}(K)$ if and only if (f,p) is an isomorphism of simplicial sets.

Given a simplicial set K, and an n-simplex, x, in K, there is a representing map,

$$\mathbf{x} : \Delta[n] \longrightarrow K,$$

that sends the top dimensional generating simplex of $\Delta[n]$ to x.

As was just said, the mapping space construction, above, is part of an adjunction,

$$\mathcal{S}(K \times L, M) \cong \mathcal{S}(L, \underline{\mathcal{S}}(K,M)),$$

in which, given $\theta : K \times L \longrightarrow M$ and $y \in L_n$, the corresponding simplicial map

$$\bar{\theta} : L \longrightarrow \underline{\mathcal{S}}(K,M)$$

sends y to the composite

$$K \times \Delta[n] \xrightarrow{K \times \mathbf{y}} K \times L \xrightarrow{\theta} M .$$

In a simplicial group G, the multiplication is a simplicial map, $\#_0 : G \times G \longrightarrow G$, and so, by the adjunction, we get a simplicial map

$$G \longrightarrow \underline{\mathcal{S}}(G,G)$$

and this is a simplicial monoid morphism. This gives the *right regular representation* of G,

$$\rho = \rho_G : G \longrightarrow \mathsf{aut}(G).$$

We will look at this idea of *representations* in more detail later.

This morphism, ρ, needs careful interpretation. In dimension n, an element $g \in G_n$ acts by multiplication on the right on G, but even in dimension 0, this action is not as simple as one might think. (NB. Here $\mathsf{aut}(G)$ is the simplicial group of 'simplicial automorphisms of the underlying simplicial set of G' as, of course, multiplication by an element does not give a mapping that respects the group structure.) Simple examples are called for:

In general, 0-simplices give simplicial maps corresponding to multiplication by that element, so that for $g \in G_0$, and $x \in G_n$,

$$\rho(g)(x) = x\#_0 s_0^{(n)}(g).$$

Suppose, now, $g \in G_1$, then $\rho(g) \in \mathsf{aut}(G)_1 \subset \underline{\mathcal{S}}(G,G)_1 = \mathcal{S}(G \times \Delta[1], G)$. In other words, $\rho(g)$ is a homotopy between $\rho(d_1 g)$ and $\rho(d_0 g)$. Of course, it is an invertible element of $\underline{\mathcal{S}}(G,G)_1$ and this will have implications for its properties as a homotopy, and, to use a geometric term, we will loosely refer to it as an *isotopy*.

In dimension 1, we, thus, have that elements give isotopies, and in higher dimensions, we have 'isotopies of isotopies', and so on.

Of course, the existence of these automorphism simplicial groups, $\mathsf{aut}(K)$, leads to a notion of a *(permutation) representation for a simplicial group*, G, as being a simplicial group morphism from G to $aut(K)$ for some simplicial set K. Likewise, if we have a simplical vector space, V, then we can construct a group of its automorphisms and thus consider linear representations as well. We will return to this later so give no details here.

6.4 Simplicial actions and principal fibrations

We saw, back in the first chapter, (page 17), the idea of a group, G, acting on a set, X. This is clearly linked to what was discussed in the previous section. A group action was given by a map,

$$a : G \times X \to X,$$

(and we may write $g.x$, or simply gx, for the image $a(g,x)$), satisfying obvious conditions such as an 'associativity' rule $g_2.g_1).x = g_2.(g_1.x)$ and an 'identity' rule $1_G.x = x$, both for all possible gs and xs. Of course, this 'action by g' gives a permutation of X, that is, a bijection form X to itself.

6.4.1 More on 'actions' and Cartesian closed categories

We know that the behaviour we have just been using for simplicial sets is also 'there' in the much simpler case of *Sets*, *i.e.*, given sets X, Y and Z, there is a natural isomorphism

$$Sets(X \times Y, Z) \cong Sets(X, Sets(Y,Z)),$$

given by sending a 'function of two variables', $f : X \times Y \to Z$, to $\tilde{f} : X \to Sets(Y,Z)$, where $\tilde{f}(x) : Y \to Z$ sends y to $f(x,y)$. (We often write Z^Y for $Sets(Y,Z)$, since, for instance, if $Y = \{1,2\}$, a two element set, $Sets(Y,Z) \cong Z \times Z = Z^2$, in the usual sense.) Technically, this is saying that $- \times Y$ has an adjoint given by $Sets(Y,-)$.

Definition: A category, $\mathcal{C}$, is *Cartesian closed*, or is a *ccc*, if it has all finite products and for any two objects, Y and Z, there is an *exponential*, Z^Y, in $\mathcal{C}$, so that $(-)^Y$ is right adjoint to $- \times Y$.

Recall or note: To say that $\mathcal{C}$ has all products says that, for any two objects X and Y in $\mathcal{C}$, their product $X \times Y$ is also there, and that there is a *terminal object*, and conversely. If you have not really met 'terminal objects' explicitly before an object, $\top$, (sometimes read as 'top'), is *terminal* if, for any X in $\mathcal{C}$, there is a *unique* morphism, $!_X$, from X to $\top$. The simplest examples to think about are (i) any one element (singleton) set is terminal in *Sets*, (ii) the trivial group is terminal in *Groups*, and so on. The dual notion is *initial object*. An object, $\perp$, is *initial* if there is a unique morphism from $\perp$ to X, again for all X in $\mathcal{C}$. The empty set is initial in *Sets*; the trivial group is initial in *Groups*.

If you have not formally met these, now is a good time to check up in texts that give an introduction to category theory and categorical ideas. In particular, it is worth thinking about why the terminal object in a category, if it exists, is the 'empty product', *i.e.*, the product of an empty family of objects. This can initially seem strange, but is a *very useful insight* that will come in later, when we discuss sheaves.

We can use this property of *Sets*, and $\mathcal{S}$, or more generally for any ccc, to give a second description of a group action. The function $a : G \times X \to X$ gives, by the adjunction, a function

$$\tilde{a} : G \to Sets(G, G).$$

This set, $Sets(G, G)$, is a monoid under composition, and we can pick out $Perm(X)$ or, if you prefer the notations, $Symm(X)$ or $Aut(X)$, the subgroup of self bijections or permutations of G. In this guise, an action of G on X is a group homomorphism from G to $Perm(X)$. (You might like to **consider how this selection of the invertibles** in the 'internal' monoid, $\mathcal{C}(X, X)$, could be done in a general ccc.)

As we mentioned, the category, $\mathcal{S}$, is also Cartesian closed, and we can use the above observation, together with our identification of the simplicial group of automorphisms, $\mathsf{aut}(Y)$, of a simplicial set Y from our earlier discussion, to describe the action of a simplicial group, G, on a simplicial set, Y. A *simplicial action* would thus be, equivalently, a simplicial map,

$$a : G \times Y \to Y,$$

satisfying associativity and identity rules, or a morphism of simplicial groups,

$$\tilde{a} : G \to \mathsf{aut}(Y).$$

We thus have the well known equivalence of 'actions' and 'representations'. This will be another recurring theme throughout these notes with embellishments, variations, etc. in different contexts. it is sometimes the 'aut'-object version that is easiest to give, sometimes not, and for some contexts, although $\mathcal{C}(X, X)$ will always be a monoid internal to some base category, the automorphisms may be hard to 'carve' out of it. (The structure may only be 'monoidal' not 'Cartesian' closed, for instance.) For this reason it pays to have both approaches.

We can identify various properties of group actions for a special mention. Some of these have already been 'met in passing' earlier, but now we need the details. Here G may be a group or a simplicial group (or often more generally, but we do not need that yet) and X will be a set respectively a simplicial set, etc. (We choose a slightly different form of condition, than we will be using later on. The links between them can be **left to you**.)

Definition: (i) A left group action

$$a : G \times X \to X,$$

is said to be *effective* (or *faithful*) if $gx = x$ for all $x \in X$ implies that $g = 1_G$.

(ii) The G-action is said to be *free* (or sometimes, *principal*, cf. May, [198]) if $gx = x$ for some $x \in X$ implies $g = 1_G$.

(iii) If $x \in X$, the *orbit* of x is the set $\{g.x \mid g \in G\}$.

Clearly (i) can be, more or less equivalently, stated as, if $g \neq 1_G$, then there is an $x \in X$ such that $gx \neq x$. This is a form sometimes given in the literature. Whether or not you consider it equivalent depends on your logic. The use of negation means that in some context this formulation of the condition is less easy to use than the former.

For future use, it will be convenient to also have slightly different, but equivalent, ways of viewing these simplicial actions. For these we need to go back again to the simplicial mapping space, $\underline{\mathcal{S}}(K, L)$ and the composition, (see page 256). Suppose we have, as there, three simplicial sets, K, L and M, and the composition:

$$\underline{\mathcal{S}}(K, L) \times \underline{\mathcal{S}}(L, M) \to \underline{\mathcal{S}}(K, M).$$

(The product is symmetric so this is equivalent to

$$\underline{\mathcal{S}}(L, M) \times \underline{\mathcal{S}}(K, L) \to \underline{\mathcal{S}}(K, M).$$

The former is the viewpoint of the 'algebraic' concatentation composition order, the latter is the 'analytic' and 'topological' one. Of course, which you choose is up to you. We will tend to use the second, but sometimes .... .)

We want to look at the situation where $K = \Delta[0]$. As $\Delta[0]$ is the terminal object in $\mathcal{S}$, $\Delta[0] \times \Delta[n] \cong \Delta[n]$, so $\underline{\mathcal{S}}(\Delta[0], L) \cong L$. If we substitute from this back into the previous composition, we get

$$eval : L \times \underline{\mathcal{S}}(L, M) \to M.$$

(It is equally valid, to write the product around the other way, giving

$$eval : \underline{\mathcal{S}}(L, M) \times L \to M,$$

which correspond better to the 'analytic' Leibniz composition order. We will often use this form as well.) In either notational form, this is the simplicially enriched evaluation map, the analogue of $eval(x, f) = f(x)$ in the set theoretic case. (We will usually write *eval* for this sort of map.) Of course, if $L = M$, this situation is exactly that of the simplicial action of the simplicial monoid of self maps of L on L itself.

We can take the simplicial version apart quite easily, to see what makes it work.

Going back one stage, if $g \in \underline{\mathcal{S}}(K, L)_n$ and $f \in \underline{\mathcal{S}}(L, M)_n$, we can form their composite using the trick we saw earlier, in the discussion in section 6.3, page 256. We can replace $g : K \times \Delta[n] \to L$, by a map over $\Delta[n]$, given by $\overline{g} = (g, p_2) : K \times \Delta[n] \to L \times \Delta[n]$, and then compose with $f : L \times \Delta[n] \to M$ to get the composite $f \circ g \in \underline{\mathcal{S}}(K, M)_n$, or use the 'over $\Delta[n]$ version to get $\overline{f \circ g} = \overline{f}\overline{g} : K \times \Delta[n] \to M \times \Delta[n]$. We note

$$\overline{f \circ g}(k, \sigma) = (f(g(k, \sigma), \sigma), \sigma),$$

(yes, we do need all those σs!).

Next we try the formulae with $K = \Delta[0]$ and '$g = \ulcorner x \urcorner$', the 'naming' map for an n-simplex, x, in L. That is not quite right, and to make things 'crystal clear', we had better be precise. The naming map for x has domain $\Delta[n]$ and we need the corresponding map, g, defined on $\Delta[0] \times \Delta[n]$. (Here the notation is getting almost 'silly', but to track things through it is probably necessary to do this, at least once! It shows how the details are there and can be taken out from the abstract packaging if and when we need them.) This map g is defined by $g(s_0^{m)}\iota_0, \sigma) = \ulcorner x \urcorner(\sigma)$, and this is 'really' given by $g(s_0^{(n)}(\iota_0), \iota_n)$ as that special case determines the others by the simplicial identities, so that, for $\sigma \in \Delta[n]_m$, so $\sigma : [m] \to [n]$, $g(s_0^{m)}\iota_0, \sigma) = L_\sigma g(s_0^{(n)}(\iota_0), \iota_n)$. (It may help here to think of σ as one of the usual face inclusions or degeneracies, at least to start with.) We have not yet used what g is, but $g(s_0^{(n)}(\iota_0), \iota_n) = x$, that is all! We can now work out (with all the identifications taken into account),

$$eval(x, f) = \overline{f \circ g}(s_0^{(n)}\iota_0, \iota_n) = f(x, \iota_n).$$

We might have guessed that this was the formula, ... what else could it be? This derivation, however, obtains it consistently with the natural 'action' formula, without having to check any complicated simplicial identities.

We will use this formula in the next chapter when discussing the structure of fibre bundles in the simplicial context.

6.4.2 G-principal fibrations

Specialising down to the simplicial case for now, suppose that G is a simplicial group acting on a simplicial set, E, then we can form a quotient complex, B, by identifying x with $g.x$, $x \in E_q$, $g \in G_q$. In other words the q-simplices of B are the orbits of the q-simplices of E, under the action of G_q. We note that this works (for **you to check**).

Lemma 41 *(i) The graded set, $\{B_q\}_{q\geq 0}$ forms a simplicial set with induced face and degeneracy maps, so that, if $[x]_G$ denotes the orbit of x under the action of G_q, then $d_i^B[x]_G = [d_i^E x]_G$, and similarly $s_i^B[x]_G = [s_i^E x]_G$.*

(ii) The graded function, $p : E \to B$, $p(x) = [x]_G$, is a simplicial map. ∎

Definition: A map of the form $p : E \to B$, as above, is called a *principal fibration*, or, more exactly, *G-principal fibration* if we need to emphasise the simplicial group being used.

A morphism between two such objects will be a simplicial map over B, which is G-equivariant for the given G-actions.

(Any such morphism will be an isomorphism; **for you to check**.)

We will denote the set of isomorphism classes of G-principal fibrations on B by $Princ_G(B)$.

This definition really only makes sense if such a p is a fibration. Luckily we have:

Proposition 69 *Any map $p : E \to B$, as above, is a Kan fibration.*

Proof: Suppose $p : E \to B$ is a principal fibration. We assume that we have (cf. page 36) a commutative diagram

$$\begin{tikzcd} \Lambda^i[n] \arrow[r, "f_1"] \arrow[d, "inc"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "f_0"'] & B \end{tikzcd}$$

and will write $b = f_0(\iota_n)$ for the corresponding n-simplex in B, and $(x_0, \ldots, x_{i-1}, -, x_{i+1}, \ldots, x_n)$ a compatible set of $(n-1)$-simplices up in E, in other words, a (n,i)-horn in E and a filler, b, for its image down in B.

Pick a $x \in E_n$ such that $p(x) = b$, then as $d_j p(x) = p(x_j)$, we have there are unique elements $g_j \in G_{n-1}$ such that $d_j x = g_j x_j$. ('Uniqueness' comes from the assumed properties of the action.) It is easy to **check** (again using 'uniqueness') that the g_js give a (n,i)-horn in G, which, since G is a 'Kan complex', has a filler (use the algorithm in section 1.3.4). Let g be the filler and set $y = g^{-1}x$. It is now **easy to check** that $d_k y = x_k$ for all $k \neq i$, *i.e.*, that y is a suitable filler. ■

We need to investigate the class of these principal fibrations (for some fixed G). (We will tend to omit specific mention of the simplicial group G being used if, within a context, it is 'fixed', so, for instance, if we are not concerned with a 'change of groups' context.)

Let us suppose that $p : E \to B$ is a principal fibration and that $f : X \to B$ is any simplicial map. We can form a pullback fibration

$$\begin{tikzcd} E_f \arrow[r, "f'"] \arrow[d, "f^*(p)"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B. \end{tikzcd}$$

Is this pullback a G-principal fibration? Or to use terminology that we introduced earlier (section 1.3.4), is the class of principal fibrations pullbacks stable?

There are several proofs of the result that it is, some of which are very neat, but here we will use the trusted method of 'brute force and ignorance', using as little extra machinery as possible. We have a reasonable model for E_f, so we should expect to be able to give it an explicit G-action in a fairly obvious natural way. We then can see what the orbits look like. That sounds a simple plan and it in fact works nicely.

We will model E_f as $E \times_B X$. (Previously, we had it around the other way as $X \times_B E$, but the two are isomorphic and this way is marginally easier notationally.) Recall the n-simplices in $E \times_B X$ are pairs (e, x) with $e \in E_n$, $x \in X_n$ and $p(e) = f(x)$. The G-action is staring at us. It surely must be

$$g \cdot (e, x) = (g \cdot e, x),$$

but does this work? We note $p(e) = [e]_G$, the G-orbit of e, so $p(g \cdot e) = p(e) = f(x)$, so we end up in the correct object. (**You are left** to check that this *is* a G-action and that it is free and effective.) What are the orbits?

We have (e, x) and (e', y) will be in the same orbit provided that there is a g such that $(g \cdot e, x) = (e', y)$, but that means that $x = y$ and that e and e' are in the same G-orbit within E. This has various consequences, which you are **left to explore**, but it is clear that, up to isomorphism, the map $f^*(p)$, which is projection onto the x component, *is* the quotient by the action. We have verified (except for the bits **left to you**:

Proposition 70 *If $p : E \to B$ is a G-principal fibration, and $f : X \to B$ is a simplicial map, then $(E_f, X, f^*(p))$ is a G-principal fibration.* ∎

Of particular interest is the case when $X = \Delta[n]$, so that f is a 'naming' map, (cf. page 29), $\ulcorner b \urcorner$, for some n-simplex, $b \in B_n$. We can, in this case, think of E_f as being the 'fibre' over b, although b is in dimension n.

This is very useful because of the following:

Lemma 42 *If $p : E \to \Delta[n]$ is a G-principal fibration, then $E \cong \Delta[n] \times G$, with p corresponding to the first projection.*

Before launching into the proof, it should be pointed out that here $\Delta[n] \times G$, should really be written $\Delta[n] \times U(G)$, where $U(G)$ is the underlying simplicial set of G. Of course there is a natural free and effective G-action on $U(G)$, with exactly one orbit. We have suppressed the U as this is a common 'abuse' of notation.

Proof: We have a single non-degenerate n-simplex in $\Delta[n]$, namely ι_n, which corresponds to the identity map in $\Delta[n]_n = \mathbf{\Delta}([n], [n])$. We pick any $e_n \in p^{-1}(\iota_n)$ and get a map, $\ulcorner e_n \urcorner : \Delta[n] \to E$, naming e_n. Of course, the composite, $p \circ \ulcorner e_n \urcorner$, is the identity on $\Delta[n]$. (This means that the fibration is 'split', in a sense we will see several times later on.)

Suppose $e \in E_m$, then $p(e) = \mu \in \Delta[n]_m = \mathbf{\Delta}([m], [n])$. We have another possibly different element in $p^{-1}(\mu)$, since $\mu : [m] \to [n]$ induces $E(\mu) : E_n \to E_m$, and so we have an element $E(\mu)(e_n)$. (**You can easily check** that, as p is a simplicial map, $p(E(\mu)(e_n)) = \mu$, i.e. $E(\mu)(e_n) \in p^{-1}(\mu)$, but therefore there is a unique element $g_m \in G_m$ such that $g_m \cdot E(\mu)(e_n) = e$. Starting with e, we got a unique pair $(\mu, g_m) \in (\Delta[n] \times G)_m$ and, from that pair, we can retrieve e by the formula. (You are **left to check** that this yields a simplicial isomorphism over $\Delta[n]$.) ∎

We will see this sort of argument several times later. We have a 'global section,' here $\ulcorner e_n \urcorner$, of some G-principal 'thing' (fibration, bundle, torsor, whatever) and the conclusion is that the 'thing' is trivial' that is, a product 'thing'.

6.4.3 Homotopy and induced fibrations

A key result that we will see later is that, if you use homotopic maps to pullback something like a fibration, or its more structured version, a fibre bundle, then you get 'related' pullbacks. Here we will look at the simplest, least structured, case, where we are forming pullbacks of *fibrations*. As this is a very important result, we will include quite a lot of detail.

As $\Delta[1]_0 = \Delta([0], [1])$, it has two elements, which we will write as e_0 and e_1, where $e_i(0) = i$, for $i = 0, 1$. (We will use this simplified notation several times later in the notes and should point out that e_0 corresponds to δ_1, and so induces d_1 if passing to simplicial notation, whilst e_1*is* δ_0, corresponding to d_1, which is the 'face opposite 1', hence is 0. This is slightly confusing, but the added intuition of $K \times \Delta[1]$ being a cylinder with $K \times \ulcorner e_0 \urcorner : K \cong K \times \Delta[0] \to K \times \Delta[1]$ being inclusion at the bottom end is too good to pass by!)

In what follows, we will quietly write e_i instead of $\ulcorner e_i \urcorner$, as it is a lot more convenient.

Proposition 71 *Let $p : E \to B$ be a Kan fibration and let $f, g : A \to B$ be homotopic simplicial maps, with $F : f \simeq g$, a specific homotopy, then there is a homotopy equivalence* over *A between $f^*(p) : E_f \to A$ and $g^*(p) : E_g \to A$.*

Proof: We first write $f = F \circ (A \times e_0)$, then we form E_f in two stages, by forming two pullbacks:

$$\begin{tikzcd} E_f \arrow[r, "i_f"] \arrow[d, "f^*(p)"'] & E_F \arrow[r] \arrow[d, "F^*(p)"] & E \arrow[d, "p"] \\ A \arrow[r, "A\times e_0"'] & A \times \Delta[1] \arrow[r, "F"'] & B \end{tikzcd}$$

A similar construction works, of course, for E_g using $A \times e_1$.

We have, from Lemma 2, that, as $F^*(p)$ is a Kan fibration, so is $q_f := \underline{S}(E_f, F^*(p))$, and so also is $q_g := \underline{S}(E_g, F^*(p))$. These maps just compose with $F^*(p)$, so

$$q_f(i_f) = f^*(p) \times e_0.$$

Next we note that $f^*(p) \times \Delta[1] : E_f \times \Delta[1] \to A \times \Delta[1]$, so is in $\underline{S}(E_f, A \times \Delta[1])_1$ and $f^*(p) \times e_0 = d_1(f^*(p) \times \Delta[1])$. We now have a (1,1)0-horn, $(-, i_f)$ in $\underline{S}(E_f, E_F)$, whose image $(-, q - f(i_f))$ in $\underline{S}(E_f, A \times \Delta[1])$ has a filler, namely $f^*(p) \times \Delta[1]$. We can thus lift that filler to one y_f, say, in $\underline{S}(E_f, E_F)_1$, with $d_1(y_f) = i_f$, and, of course, $q_f(y_f) = f^*(p) \times \Delta[1]$. What is the other end, $d_0(y_f)$?

This is also in $\underline{S}(E_f, E_F)_0$, so is a simplicial map from E_f to E_F. This suggests it might be a map of fibrations. Does

$$\begin{tikzcd} E_f \arrow[r, "d_0(y_f)"] \arrow[d, "f^*(p)"'] & E_F^{F^*(p)} \\ A \arrow[r, "A\times e_1"'] & A \times \Delta[1] \end{tikzcd}$$

commute? We calculate,

$$\begin{aligned} F^*(p) d_0(y_f) &= q_F(d_0(y_f)) \\ &= d_0(q_f(y_f)) \\ &= d_0(f^*(p) \times \Delta[1]) \\ &= (A \times e_1) \circ f^*(p), \end{aligned}$$

so it is, but this means that, as bottom 'right-hand corner' of the square, had E_g as its pullback, we get a map, $\alpha : E_f \to E_g$, over A, so that $f^*(p) = g^*(p)\alpha$, and $d_0(y_f) = i_g\alpha$. This gives us the first part of our homotopy equivalence.

Reversing the roles of f and g, we get a y_g in $\underline{S}(E_g, E_F)_1$ with $d_0(y_g) = i_g$, then $q_g(y_g) = g^*(p) \times \Delta[1]$, and we get a $\beta : E_g \to E_f$ such that $f^*(p)\beta = g^*(p)$ and $i_f\beta = d_1(y_g)$.

We now have to look at the composites $\alpha\beta$ and $\beta\alpha$, and to show they are homotopic (over A) to the identities. Of course, we need only produce one of these as the other will follow 'similarly', on reversing the roles of f and g.

Considering $s_0(\alpha) \in \underline{\mathcal{S}}(E_f, E_g)_1$ and $y_g \in \underline{\mathcal{S}}(E_g, E_F)_1$, we have a composite (really a composite homotopy), that we will denote by $\xi \in \underline{\mathcal{S}}(E_f, E_F)_1$. We can check (**for you to do**) that $d_0(\xi) = d_0(y_f)$ and $d_1(\xi) = d_i(y_g)\alpha = i_f\beta\alpha$. We thus have a horn

$$h = \begin{tikzcd} & i_f\beta\alpha \arrow[dr, "\xi"] & \\ i_f \arrow[rr, "y_f"'] & & d_0(y_f) \end{tikzcd}$$

in $\underline{\mathcal{S}}(E_f, E_F)$. We look at its image in $\underline{\mathcal{S}}(E_f, A\times\Delta[1])$, and **check** it can be filled by $s_0(f^*(p)\times\Delta[1])$, that means that, as $F^*(p)$ is a Kan fibration, we can find a filler, z, for h, so set $w := d_2(z)$. (This is a composite homotopy, as if it was topologically 'y_f followed by the reverse of ξ.') this homotopy, w, is in $\underline{\mathcal{S}}(E_f, E_F)$, not in $\underline{\mathcal{S}}(E_f, E_f)$, but otherwise does the right sort of thing.

To get a homotopy with E_f as codomain, we use the left hand pullback square of the above double pullback diagram, so have to work out $F^*(p)(w)$. This is just our $q_f(w)$ and that, by the description of z as a filler is $d_2s_0(f^*(p) \times \Delta[1]) = s_0d_1(f^*(p) \times \Delta[1]) = f^*(p).pr_{E_f}.(A \times e_0)$, so we have a map $w' : E_f \times \Delta[1] \to E_f$, as in the diagram

$$\begin{tikzcd} E_f \times \Delta[1] \arrow[dr, "w'"] \arrow[drr, bend left, "w"] \arrow[ddr, bend right, "f^*(p).pr_{E_f}"'] & & \\ & E_f \arrow[r, "i_f"'] \arrow[d, "f^*(p)"] & E_F \arrow[d, "F^*(p)"] \\ & A \arrow[r, "A\times e_0"'] & A \times \Delta[1], \end{tikzcd}$$

where $pr_{E_f} : E_f \times \Delta[1] \to E_f$ is the projection. Note that w' is a homotopy *over* A, so is 'in the fibres'.

This w' certainly goes between the right objects, but is it the required homotopy. We check

$$i_f.w'.e_1 = w.e_1 = i_f\beta\alpha,$$

but i_f is the induced map from $A \times e_0$, which is a (split) monomorphism, so i_f is itself a monomorphism, and so $w'.e_1 = \beta\alpha$. Similarly $w'.e_0 = id_{E_f}$, so w' does what was hoped for.

We reverse the roles of α and β, and of f and g, to get the last part of the proof. ■

6.5 $\overline{W}$, W and twisted Cartesian products

Suppose we have simplicial sets, Y, a potential 'fibre' and B, a potential 'base', which will be assumed to be pointed by a vertex, $*$. Inspired by the sort of construction that works for the construction of group extensions, we are going to try to construct a fibration sequence,

$$Y \longrightarrow E \longrightarrow B.$$

Clearly the product $E = B \times Y$ will give such a sequence, but can we somehow *twist* this Cartesian product to get a more general construction? We will try setting $E_n = B_n \times Y_n$ and will change as little as possible in the data specifying faces and degeneracies. In fact we will take all the degeneracy maps to be exactly those of the Cartesian product, and all but d_0 of the face maps likewise. This leaves just the zeroth face map.

In, say, a covering space considered as a fibration with discrete fibre, the fundamental group(oid) of the base acts by automorphisms / permutations on the fibre, and the fundamental group(oid) is generated by the edges, hence by elements of dimension one greater than that of the fibre, so we try a formula for d_0 of form

$$d_0(b, y) = (d_0 b, t(b)(d_0 y)),$$

where $t(b)$ is an automorphism of Y, determined by b in some way, hence giving a function $t : B_n \longrightarrow \mathsf{aut}(Y)_{n-1}$. Note here Y is an arbitrary simplicial set, not the underlying simplicial set of a simplicial group as was previously the case when we considered aut, but this makes no difference to the definition.

Of course, with these tentative definitions, we must still have that the simplicial identities hold, but it is easy to check that these will hold exactly if t satisfies the following equations

$$\begin{aligned} d_i t(b) &= t(d_{i+1} b) \quad \text{for} \quad i > 0, \\ d_0 t(b) &= t(d_1 b) \#_0 t(d_0 b)^{-1}, \\ s_i t(b) &= t(s_{i+1} b) \quad \text{for} \quad i \geq 0, \\ t(s_0 b) &= *. \end{aligned}$$

A function, t, satisfying these equations will be called a *twisting function*, and the simplicial set E, thus constructed, will be called a *regular twisted Cartesian product* or *T.C.P.* We write $E = B \times_t Y$.

It is often useful to assume that the twisting function is 'normalised', so that $t(*)$ is the identity automorphism. We usually will tacitly make this assumption if the base is pointed.

If this construction is to make sense, then we really need also a 'projection' from E to B and Y should be isomorphic to its fibre over the base point, $*$. The obvious simplicial map works, sending (b, y) to b. It *is* simplicial and clearly has a copy of Y as its fibre.

Of course, a twisting function is not a simplicial map, but the formulae it satisfies look closely linked to those of the Dwyer-Kan loop group(oid) construction, given earlier, page 249. In fact:

Proposition 72 *A twisting function, $t : B \longrightarrow \mathsf{aut}(Y)$, determines a unique homomorphism of simplicial groupoids, $t : GB \to \mathsf{aut}(Y)$, and conversely.* ■

Of course, since G is left adjoint to $\overline{W}$, we could equally well note that t gave a simplicial morphism $t : B \longrightarrow \overline{W}(\mathsf{aut}(Y))$, and conversely.

Of course, we could restrict attention to a particular class of simplicially enriched groupoids such as those coming from groups (constant simplicial groups), or nerves of crossed modules, or of crossed complexes, etc. We will see some aspects of this in the following chapter, but we will be generalising it as well.

This adjointness gives us a 'universal' twisting function for any simplicial group, H. We have the general natural isomorphism,

$$\mathcal{S}(B, \overline{W}H) \cong Simp.Grpd(G(B), H),$$

so, as usual in these situations, it is very tempting to look at the special case where $B = \overline{W}H$ itself and hence to get the counit of the adjunction from $G\overline{W}(H)$ to H corresponding to the identity simplicial map from $\overline{W}H$ to itself. By the general properties of adjointness, this map 'generates' the natural isomorphism in the general case.

From our point of view, the two natural isomorphic sets are much better viewed as being $\mathsf{Tw}(B,H)$, the set of twisting functions $\tau : B \to H$, so the key case will be a 'universal' twisting function, $\tau_H : \overline{W}H \to H$ and hence a universal twisted Cartesian product $\overline{W}H \times_{\tau_H} H$. (Notational point: the context tells us that the fibre H is the underlying simplicial set of the simplicial group, H, but no special notation will be used for this here.) This universal twisted Cartesian product is called the *classifying bundle for H* and is denoted WH. We can unpack its definition from its construction, but will not give the detailed derivation (which is suggested as a **useful exercise**). Clearly

$$(WH)_n = H_n \times_t \overline{W}(H)_n,$$

so from our earlier description of $\overline{W}(H)$, we have

$$WH_n = H_n \times H_{n-1} \times \ldots \times H_0.$$

The face maps are given by

$$d_i(h_n, \ldots, h_0) = (d_i h_n, \ldots, d_0 h_{n-i}.h_{n-i-1}, h_{n-i-2}, \ldots, h_0)$$

for all i, $0 \leq i \leq n$, whilst

$$s_i(h_n, \ldots, h_0) = (s_i h_n, \ldots s_0 h_{n-i}, 1, h_{n-i-1}, \ldots, h_0).$$

(It is noteworthy that $d_0(h_n, \ldots, h_0) = (d_0 h_n.h_{n-1}, h_{n-2}, \ldots, h_0)$ so the universal twist, τ_H, must somehow be built in to this. In fact τ_H is an 'obvious' map as one would hope. We have $\overline{W}(H)_n = H_{n-1} \times \ldots \times H_0$ and we need $(\tau_H)_n : \overline{W}(H)_n \to H_{n-1}$, since it is to be a twisting map and so has degree -1. The obvious formula to try is that τ_H is the projection map - and it works. The details are left to you. A glance back at the formula for the general d_0 in a twisted Cartesian product will help.)

We start by showing that $p : W(H) \to \overline{W}(H)$ is a principal fibration. This simplicial map just is the projection onto the second factor in the T.C.P. To prove this is such a principal fibration, we first examine $W(H)$ more closely and then at an obvious action. The simplicial set, $W(H)$, contains a copy of (the underlying simplicial set of) H as the fibre over the element $(1, 1, \ldots, 1) \in \overline{W}(H)$. There is then a fairly obvious action of H on $W(H)$, given by, in dimensions n,

$$h'.(h_n, \ldots, h_0) = (h'h_n, \ldots, h_0).$$

In other words, just using multiplication on the first factor. As multiplication is a simplicial map, $H \times H \to H$, or simply glancing at the formulae, we have that this *is* a simplicial action.

That action is *free*, since the regular representation is free as an action. (After all, this is just saying that, if $gx = x$ for some $x \in H$, then $g = 1$, so is obvious!) The action is also faithful / effective, for similar reasons. What are the orbits? As the action only changes the first coordinate, and does that freely and faithfully, the orbits coincide with the fibres of the projection map from $W(H)$ to $\overline{W}(H)$, so that p is also the quotient map coming from the action. It follows that

Lemma 43 *The simplicial map*

$$W(H) \to \overline{W}(H),$$

is a principal fibration. ∎

The following observations now are either corollaries of this, simple to check or should be looked up in 'the literature'.

1). The simplicial set, $W(H)$, is a Kan complex.

2). $W(H)$ is contractible, *i.e.*, is homotopy equivalent to $\Delta[0]$.

3). The simplicial map,

$$W(H) \to \overline{W}(H),$$

is a Kan fibration with fibre the underlying simplicial set of H, (so the long exact sequence of homotopy groups together with point 2) shows that $\pi_n(\overline{W}H) \cong \pi_{n-1}(H)$).

4). If $p : E \to B$ is a principal H-bundle, that is, E is $H \times_t B$ for some twisting function, $t : B \to H$, then we have a simplicial map

$$f_t : B \to \overline{W}(H)$$

given by $f_t(b) = (t(b), t(d_0 b), \ldots, t(d_0^{n-1} b))$, and we can pull back $(W(H) \to \overline{W}(H))$ along f_t to get a principal H-bundle over B

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] & W(H) \arrow[d] \\ B \arrow[r, "f_t"'] & \overline{W}(H). \end{tikzcd}$$

We can, of course, calculate E' and p' precisely:

$$\begin{aligned} E' &\cong \{((h_n, h_{n-1}, \ldots, h_0), b) \mid h_{n-1} = t(b), \ldots h_0 = t(d_0^{n-1} b)\} \\ &\cong \{(h_n, b) \mid h_n \in H_n, b \in B_n\} \\ &= H_n \times B_n. \end{aligned}$$

It should come as no surprise to find that $E' \cong H \times_t B$, so is E itself up to isomorphism, and that p' is p in disguise.

The assignment of f_t to t gives a one-one correspondence between the set, $Princ_H(B)$, of H-equivalence classes of principal H-bundles with base B, and the set, $[B, \overline{W}(H)]$, of homotopy classes of simplicial maps from B to $\overline{W}(H)$.

An important thing to remember is that not all T.C.Ps are principal fibrations. To get a T.C.P., we just need a fibre Y, a base, B, and a simplicial group, G, acting on Y, together with our twisting function, $t : B \to \overline{W}(G)$. From B and t, we can build a principal fibration which is, of course, a T.C.P. but has fibre the underlying simplicial set of G. To build the T.C.P., $B \times_t Y$, we need the *additional* information about the *representation* $G \to \mathsf{aut}(Y)$, that is, the action of G on the fibre, and, of course, that representation need not be an isomorphism. In general, we have: 'fibre bundle = principal fibration plus representation', as a rule of thumb. This is not just in the simplicial case. (We will consider fibre bundles and similar other structures in a lot more detail in the next chapter.)

A good introduction to simplicial bundle theory can be found in Curtis' classical survey article, [97] section 6, or, for a thorough treatment, May's book, [198]. For full details, you are invited to look there, at least to know what is there. We have not gone into all the detail here. We will revisit the overall theory several times later on, drawing parallels and comparisons that will, it is hoped, shed light both on it and on geometrically related theories elsewhere in the area.

6.6 More examples of Simplicial Groups

We have already seen several general constructions of simplicial groups, for instance, the simplicial resolutions of a group, the loop group on a reduced simplicial set, the internal nerve of a crossed module / cat^1-group, and so on. The previous few sections give some ideas for other construction leading to simplicial groups. We will concentrate on two such.

Let G be a topological (or Lie) group (so a group internal to 'the' category of topological spaces - whichever one is most appropriate for the situation). The singular complex functor, $Sing : Top \to \mathcal{S}$, preserves products,

$$Sing(X \times Y) \cong Sing(X) \times Sing(Y),$$

so it follows that, as the multiplication on G is continuous, there is an induced simplicial map,

$$Sing(G) \times Sing(G) \to Sing(G).$$

With the map induced from the maps that picks out the identity element and that give the inverse, this makes $Sing(G)$ into a simplicial group. This gives a large number of interesting simplicial groups, corresponding to general linear, orthogonal, and other topological (or Lie) groups of various dimension. Of course, the homotopy groups of these simplicial groups correspond to those of the groups themselves.

A closely related construction involves a similar idea to the $\mathsf{aut}(K)$ simplicial group, that we used when discussing simplicial bundles, twisted Cartesian products, etc., a few sections ago. We had a simplicial set, K, and hence a simplicial monoid, $\underline{\mathcal{S}}(K, K)$, of endomorphisms of K. The simplicial group, $\mathsf{aut}(K)$, was the corresponding simplicial group of simplicial automorphisms of K. We had a representation of such an $f : K \times \Delta[k] \to K$ as $(f, p) : K \times \Delta[k] \to K \times \Delta[k]$ and this was an automorphism *over* $\Delta[k]$, (look back to page 257).

This sort of construction will work in any situations where the basic category being studied is 'simplicially enriched', i.e. the usual hom-sets of the category form the vertices of simplicial hom-sets and the composition maps between these are simplicial. We will formally introduce this idea later, (see Chapter 11, and in particular section 11.2, page 495). Here we will give some examples of this type of idea in situations that are useful in geometric and topological contexts.

We will assume that X is a (locally finite) simplicial complex. In applications X is often $\mathbb{R}^n$, or S^n or similar. We think of the product, $\Delta^k \times X$, as a 'bundle over the k-simplex, Δ^k, or, if working in the piecewise linear (PL) setting, a PL bundle over Δ^k. The simplicial group, $\mathcal{H}(X)$, is then the simplicial group having $\mathcal{H}(X)_k$ being the set of homeomorphisms of $\Delta^k \times X$ over Δ^k, or, alternatively, the (PL) bundle isomorphisms of $\Delta^k \times X$. As a variant, if $A \subset X$ is a subcomplex, one can restrict to those bundle isomorphisms that fix $\Delta^k \times A$ pointwise.

Various examples of this were used to study the problem of the existence and classification of triangulations and smoothings for manifolds. The construction occurs, for instance, in Kuiper and Lashof, [180, 181]. Later on starting in section **??**, we will look at another variant of these examples concerning microbundle theory, (see Buoncristiano, [74, 75]), as it gives a nice interpretation of some simplicial bundles in a geometric setting.

Chapter 7

Non-Abelian Cohomology: Torsors, and Bitorsors

One of the problems to be faced when presenting the applications of crossed modules, etc., is that such is the breadth of these applications that they may safely be assumed to be potentially of interest to mathematicians of very differing backgrounds, algebraists of many different hues, geometers both algebraic and differential, theoretical physicists and, of course, algebraic topologists. To make these notes as useful as possible, some part of the more basic ‘intuitions’ from the background material from some of these areas has been included at various points. This cannot be ‘all inclusive’ nor ‘universal’ as different groups of potential readers have different needs. The real problems are those of transfer of ‘technology’ between the areas and of explanation of the differing terminology used for the same concept in different contexts. Often, essentially the same idea or result will appear in several places. This repetition is not just laziness on the authors behalf. The introduction of a concept bit-by-bit from various angles almost necessitates such a treatment.

For the background on bundle-like constructions (sheaves, torsors, stacks, gerbes, 2-stacks, etc.), the geometric intuition of ‘things over X’ or X-parametrised ‘things’ of various forms, does permeate much of the theory, so we will start with some fairly basic ideas, and so will, no doubt, for some of the time, be ‘preaching to the converted’, however that ‘bundle’ intuition is so important for this and later sections that something more than a superficial treatment is required.

(In the original lectures at Buenos Aires, I did assume that that intuition was understood, but in any case concentrated on the ‘group extension’ case rather than on ‘gerbes’ and their kin. By this means I avoided the need to rely too heavily on material that could not be treated to the required depth in the time available. However I cannot escape the need to cover some of that material here!)

Initially crossed modules, etc., will not be that much in evidence, *but* it is important to see how they do enter in ‘geometrically’ or their later introduction can seem rather artificial.

We start by looking at descent, *i.e.*, the problem of putting ‘local’ bits of structure into a global whole.

7.1 Descent: Bundles, and Covering Spaces

(Remember, if you have met ‘descent’ or ‘bundles’, then you should ‘skim’ this section only / anyway.)

We will look at these structures via some ‘case studies’ to start with.

7.1.1 Case study 1: Topological Interpretations of Descent.

Suppose A and B are topological spaces and $\alpha : A \to B$ is a continuous map (sometimes called a 'space over B' or loosely speaking a 'bundle over B', although that can also have a more specialised meaning later). The space, B, will usually be called the *base*, whilst A is the *total space* of the bundle, α.

An obvious and important example is a product, $A = B \times F$, with α being the projection. We call this a *trivial bundle* on B.

If $U \subset B$ is an open set, then we get a restriction $\alpha_U : \alpha^{-1}(U) \to U$. If $V \subset B$ is another open set, we, of course, have $\alpha_V : \alpha^{-1}(V) \to V$ and over $U \cap V$ the two restrictions 'coincide', *i.e.*, if we form the pullbacks

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & \alpha^{-1}(U) \arrow[d] \\ U \cap V \arrow[r] & U \end{tikzcd} \qquad \begin{tikzcd} ? \arrow[r] \arrow[d] & \alpha^{-1}(V) \arrow[d] \\ U \cap V \arrow[r] & V \end{tikzcd}$$

the resulting spaces over $U \cap V$ are 'the same'. (We have to be a bit careful, since we formed them by pullbacks so they are determined only 'up to isomorphism' and we should take care to interpret 'the same' as meaning 'being isomorphic' as spaces over $U \cap V$. This care will be important later.) Now assume that for each $b \in B$, we choose an open neighbourhood $U_b \subset B$ of b. We then have a family

$$\alpha_b : A_b \to U_b \qquad\qquad b \in B,$$

where we have written A_b for $\alpha^{-1}(U_b)$, and we know information about the behaviour over intersections.

Can we reverse this process? More precisely, can we start with a family,

$$\{\alpha_b : A_b \to U_b : b \in B\},$$

of maps (with A_b now standing for an arbitrary space) and add in, say, information on the 'compatibility' over the intersections of the cover, $\{U_b : b \in B\}$, so as to rebuild a space over B, $\alpha : A \to B$, which will restrict to the given family?

We will need to be more precise about that 'compatibility', but will leave it aside until a bit later. Clearly, indexing the cover by the elements of B is a bit impractical as usually we just need, or are given, some (open) cover, $\mathcal{U}$, of B, and then can choose, for each $b \in B$, some set of the cover containing b. This way we do not repeat sets unless we expressly need to. Thinking like this we have a cover $\mathcal{U}$ and for each U in $\mathcal{U}$, a space over U, $\alpha_U : A_U \to U$. To encode the condition on compatibility on intersections, we need some (temporary) notation: If $U, U' \in \mathcal{U}$, write $(A_U)_{U'}$ for the restriction of A_U over the intersection $U \cap U'$, similarly $(\alpha_U)_{U'}$ for the restriction of α_U to $U \cap U'$. This is given by the further pullback of α_U along the inclusion of $U \cap U'$ into U, so we also get a map

$$(\alpha_U)_{U'} : (A_U)_{U'} \to U \cap U'.$$

We noted that if the family $\{\alpha_U \mid U \in \mathcal{U}\}$ did come from a single $\alpha : A \to B$, then the α_Us agreed up to isomorphism on the intersections, *i.e.*, we needed homeomorphisms

$$\xi_{U,U'} : (A_U)_{U'} \xrightarrow{\cong} (A_{U'})_U$$

over $U \cap U'$ if we were going to give an adequate description. (These are sometimes called the *transition functions* or gluing *cocycles*.) This, of course, means that

$$(\alpha_{U'})_U \circ \xi_{U,U'} = (\alpha_U)_{U'}.$$

Clearly we should require
1. $\xi_{U,U}$ = identity,
but also if U'' is another set in the cover, we would need
2. $\xi_{U',U''} \circ \xi_{U,U'} = \xi_{U,U''}$
over the triple intersection $U \cap U' \cap U''$.

(This condition 2. is a *cocycle condition*, similar in many ways to ones we have met earlier in apparently very different contexts.)

These two conditions are inspired by observation on decomposing an original bundle. They give us 'descent data', but are our 'descent data' enough to construct and, in general, to classify such spaces over B? The obvious way to attempt construction of an α from the data $\{\alpha_U; \xi_{U,U'}\}$ is to 'glue' the spaces A_U together using the $\xi_{U,U'}$. 'Gluing' is almost always a colimiting process, but as that can be realised using coproducts (disjoint union) and coequalisers (quotients by an equivalence relation), we will follow a two step construction

Step 1: Let $C = \sqcup_{U \in \mathcal{U}} A_U$ and $\gamma : C \to \sqcup_{U \in \mathcal{U}} U$, the induced map. Thus if we consider a specific U in $\mathcal{U}$, we will have inclusions of A_U into C and U into $\sqcup U$ and a diagram

$$\begin{tikzcd} A_U \arrow[r, hook] \arrow[d, "\alpha_U"'] & C = \sqcup A_U \arrow[d, "\gamma"] \\ U \arrow[r, hook] & \sqcup U \end{tikzcd}$$

Remember that a useful notation for elements in a disjoint union is a pair, (element, index), where the index is the index of the set in which the element is. We write (a, U) for an element of C, then $\gamma(a, U) = (\alpha_U(a), U)$, since $a \in A_U$.

Step 2: We relate elements of C to each other by the rule:

$$(a, U) \sim (a', U')$$

if and only if
(i) $\alpha_U(a) = \alpha_{U'}(a')$,
and
(ii) we want to glue corresponding elements in fibres over the same point of B, so need something like $\xi_{U,U'}(a) = a'$. Although intuitively correct, as it says that if a and a' are over the same point of $U \cap U'$ then they are to be 'related' or 'linked' by the homeomorphism, $\xi_{U,U'}$, a close look at the formula shows it does not quite make sense. Before we can apply $\xi_{U,U'}$ to a, we have to restrict a to be in $(A_U)_{U'}$ and the result will be in $(A_{U'})_U$. Perhaps the neatest way to present this is to look at another disjoint union, this time $\sqcup_{U,U'}(A_U)_{U'}$, and to map this to $C = \sqcup_{U \in \mathcal{U}} A_U$ in two ways. The first of these, $\mathbf{a}$, say, takes the component $(A_U)_{U'}$ and injects it into C via the injection of A_U. The second map, $\mathbf{b}$, first sends $(A_U)_{U'}$ to $(A_{U'})_U)$ using $\xi_{U,U'}$ then sends that second component to $(A_{U'})$ and thus into C. We thus get the correct version of the formula for (ii) to be:

there is an $x \in \sqcup_{U,U'}(A_U)_{U'}$ *such that* $\mathbf{a}(x) = a$ *and* $\mathbf{b}(x) = a'$.

The two conditions on the homeomorphisms ξ readily imply that this is an equivalence relation and that the α_U together define a map

$$\alpha : A = C/_\sim \ \to B$$

given by

$$\alpha[(a,U)] = \alpha_U(a),$$

on the equivalence class, $[(a,U)]$ of (a,U). For this to be the case, we only needed $\alpha_U(a) = \alpha_{U'}(a')$ to hold. Why did we impose the second condition, *i.e.*, the cocycle condition? Simply, if we had not, we would risked having an equivalence relation that crushed C down to B. Each fibre $\alpha^{-1}(b)$ might have been a single point since each $\alpha_U^{-1}(a)$ could have been in a single equivalence class.

We now have a space over B, $\alpha : A \to B$ (with A having the quotient topology, which ensures that α will be continuous).

If we had started with such a space, decomposed over $\mathcal{U}$, then had constructed a 'new space' from that data, would we have got back where we started? Yes, up to isomorphism (*i.e.*, homeomorphism over B). To discuss this, it helps to introduce the category, Top/B, of spaces over B. This has continuous maps $\alpha : A \to B$ (often written (A,α)) as its objects, whilst a map from (A,α) to $\alpha' : A' \to B$ will be a continuous map $f : A \to A'$ making the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & A' \arrow[dl, "\alpha'"] \\ & B & \end{tikzcd}$$

commutative. This, however, raises another question.

If we have such an f and an (open) cover $\mathcal{U}$ of B, we restrict f to $\alpha^{-1}(U)$ to get

$$f_U : A_U \to A'_U$$

which, of course, is in Top/U. If we have data,

$$\{\alpha_U : A_U \to U, \{\xi_{U,U'}\}\}$$

for (A,α) and similarly for (A',α'), and morphisms

$$\{f_U : A_U \to A'_U\},$$

when can we 'rebuild' $f : A \to A'$? We would expect that we would need a compatibility between the various f_U and the $\xi_{U,U'}$ and $\xi'_{U,U'}$. The obvious condition would be that whenever we had U, U' in $\mathcal{U}$, the diagram

$$\begin{tikzcd} (A_U)_{U'} \arrow[r, "(f_U)_{U'}"] \arrow[d, "\xi_{U,U'}"'] & (A'_U)_{U'} \arrow[d, "\xi'_{U,U'}"'] \\ (A_{U'})_U \arrow[r, "(f_{U'})_U"] & (A'_{U'})_U \end{tikzcd}$$

should commute, where we have extended our notation to use $(f_U)_{U'}$ for the restriction of f_U to $\alpha^{-1}(U \cap U')$. To codify this neatly, we can form each category, Top/U, for $U \in \mathcal{U}$, then form the category, D, consisting of families of objects, $\{\alpha_U : U \in \mathcal{U}\}$, of $\prod Top/U$ together with the extra structure of the $\xi_{U,U'}$. Morphisms in D are families $\{f_U\}$ as above, compatible with the structural isomorphisms $\xi_{U,U'}$.

Remark: For any specific pair consisting of a family, $\mathcal{A} = \{(A_U, \alpha_U) : U \in \mathcal{U}\}$ and the extra $\xi_{U,U'}$s is a *set of descent data for* $\mathcal{A}$. We will look at both this construction and its higher dimensional relatives in quite a lot of detail and generality later on. The category of these things and the corresponding morphisms can be called the *category of descent data relative to the cover*, $\mathcal{U}$.

The reason for the use of the word 'descent' is that, in many geometric situations, structure is easily encoded on some basic 'patches'. This structure, that is locally defined, 'descends' to the space giving it a similar structure. In many cases, the A_U have the fairly trivial form, $U \times F$, for some fibre F. This fibre often has extra structure and the $\xi_{U,U'}$ have then to be structure preserving automorphisms of the space, F. The term 'bundle' is often used in general, but some authors restrict its use to this *locally trivial* case. The classic case of a locally trivial bundle is a Möbius band as a bundle over the circle. Locally, on the circle, the band is of form $U \times [-1, 1]$, but globally one has a twist. A bit more formally, and for use later, we will define:

Definition: A bundle, $\alpha : A \to B$, is said to be *locally trivial* if there is an open cover, $\mathcal{U}$, of B, such that, for each U in $\mathcal{U}$, A_U is homeomorphic to $U \times F$, for some fibre F, compatibly with the projections, α_U and $p_U : U \times F \to U$.

We will gradually build up more precise intuitions about what 'compatibly' means, and as we do so, the above definition will gain in precision and strength.

7.1.2 Case Study 2: Covering Spaces

This is a classic case of a class of 'spaces over' another space. It is also of central importance for the development of possible generalisations to higher 'dimensions', (cf. Grothendieck's *Pursuit of Stacks*, [140].) We have a continuous map

$$\alpha : A \to B$$

and for any point $b \in B$, there is an open neighbourhood U of b such that $\alpha^{-1}(U)$ is the disjoint union of open subsets of A, each of which is mapped homeomorphically onto U by α. The map α is then called a *covering projection*. On such a U, $\alpha^{-1}(U)$ is $\sqcup U_i$ over some index set which can be taken to be $\alpha^{-1}(b) = F_b$, the fibre over b. Then we may identify $\alpha^{-1}(U)$ with $U \times F_b$ for any $b \in U$. This F_b is 'the same' up to isomorphism for all $b \in U$. If B is connected then for any b, $b' \in B$, we can link them by a chain of pairwise intersecting open sets of the above form and hence show that $F_b \cong F_{b'}$. We can thus take each $\alpha^{-1}(U) \cong U \times F$ and F will be a discrete space provided B is nice enough. The descent data in this situation will be the local covering projections

$$\alpha_U : U \times F \to U$$

together with the homeomorphisms

$$\xi_{U,U'} : (U \cap U') \times F \to (U \cap U') \times F$$

over $(U \cap U')$. Provided that $(U \cap U')$ is connected, this $\xi_{U,U'}$ will be determined by a permutation of F.

We often, however, want to allow for non-connected $(U \cap U')$. For instance, take B to be the unit circle S^1, $F = \{-1, 1\}$,

$$U_1 = \{\underline{x} \in S^1 \mid \underline{x} = (x, y), x > -0.1\}$$

$$U_2 = \{\underline{x} \in S^1 \mid \underline{x} = (x, y), x < 0.1\}.$$

The intersection, $U_1 \cap U_2$, is not connected, so we specify ξ_{U_1,U_2} separately on the two connected components of $U_1 \cap U_2$. We have

$$U_1 \cap U_2 = \{(x, y) \in S^1 \mid |x| < 0.1, y > 0\} \cup \{(x, y) \mid |x| < 0.1, y < 0\}.$$

Let $\xi_{U_1,U_2}((x,y),t) = \begin{cases} ((x,y),t) & \text{if } y > 0 \\ ((x,y),-t) & \text{if } y < 0, \end{cases}$
so on the part with negative y, ξ exchanges the two leaves. The resulting glued space is either viewed as the edge of the Möbius band or as the map,

$$S^1 \to S^1$$

$$e^{i\theta} \mapsto e^{i2\theta}.$$

Remark: The well known link between covering spaces and actions of the fundamental group $\pi_1(B)$ on Sets is at the heart of this example.

A neat way to picture the n-fold covering spaces of S^1 for $n \geq 2$ is to consider a knot on the surface of a torus, $S^1 \times S^1$, for instance the trefoil. The projection to the first factor of $S^1 \times S^1$ gives a covering, as does the second projection. It is **also instructive** to consider the covering space $\mathbb{R}^2 \to S^1 \times S^1$, working out what the various transitions are for a cover. We note the way that quotients of $\mathbb{R}^n$ by certain geometrically defined group actions, yields neat examples of coverings (although some may be 'ramified', an area we will not stray into here.)

In general, when we have a local product structure, so $\alpha^{-1}(U) \cong U \times F$, the homeomorphisms $\xi_{U,U'}$ have a nicer description than the general one, since being 'over' the intersection, they have to have the form that interprets at the product levels as being $\xi_{U,U'}(x,y) = (x, \xi'_{U,U'}(x)(y))$ where $\xi'_{U,U'} : U \cap U' \to Aut(F)$. In the case of covering spaces F is discrete, so $\xi'_{U,U'}(x)$ will give a permutation of F.

7.1.3 Case Study 3: Fibre bundles

The examples here are to introduce / recall how torsors / principal fibre bundles are defined topologically and also to give some explicit instances of how fibre bundles arise in geometry.

(Often in this context, the terminology 'total space' is used for the source of the bundle projection.)

First some naturally occurring examples.

(i) Let S^n denote the usual n-sphere represented as a subspace of $\mathbb{R}^{n+1}$,

$$S^n = \{\underline{x} \in \mathbb{R}^{n+1} \big| \; \|\underline{x}\| = 1\},$$

where $\|\underline{x}\| = \sqrt{\langle \underline{x} \mid \underline{x} \rangle}$ for $\langle \underline{x} \mid \underline{y} \rangle$, the usual Euclidean inner product on $\mathbb{R}^{n+1}$. The *tangent bundle* of S^n, τS^n is the 'bundle' with total space,

$$TS^n = \{(\underline{b}, \underline{x}) \bigm| \langle \underline{b} \mid \underline{x} \rangle = 0\} \subset S^n \times \mathbb{R}^{n+1}.$$

We thus have a projection

$$p : TS^n \to S^n$$

given by $p(\underline{b}, \underline{x}) = \underline{b}$, as a space over S^n.

Similarly the *normal bundle*, νS^n, of S^n is given with total space,

$$NS^n = \{(\underline{b}, \underline{x}) \bigm| \underline{x} = k\underline{b} \text{ for some } k \in \mathbb{R}\} \subset S^n \times \mathbb{R}^{n+1}.$$

The projection map $q : NS^n \to S^n$ gives, as before, a space over S^n, $\nu S^n = (NS^n, q, S^n)$.

Another example extends this to a geometric context of great richness.

(ii) First we need to introduce generalisations, the Grassmann varieties, of projective spaces and in order to see what topology it is to have, we look at a related space first. The *Stiefel variety* of k-frames in $\mathbb{R}^n$, denoted $V_k(\mathbb{R}^n)$, is the subspace of $(S^{n-1})^k$ such that $(v_1, \ldots, v_k) \in V_k(\mathbb{R}^n)$ if and only if each $\langle v_i \mid v_j \rangle = \delta_{i,j}$, so that it is 1 if $i = j$ and is zero otherwise. Note $V_1(\mathbb{R}^n) = S^{n-1}$.

The *Grassmann variety* of k-dimensional subspaces of $\mathbb{R}^n$, denoted $G_k(\mathbb{R}^n)$, is the set of k-dimensional subspaces of $\mathbb{R}^n$. There is an obvious function,

$$\alpha : V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n),$$

mapping $(v_1, \ldots, v_k)$ to $span_{\mathbb{R}}\langle v_1, \ldots, v_k \rangle \subseteq \mathbb{R}^n$, that is, the subspace with $(v_1, \ldots, v_k)$ as basis. We give $G_k(\mathbb{R}^n)$ the quotient topology defined by α. (For $k = 1$, we have $G_1(\mathbb{R}^n)$ is the real projective space of dimension $n - 1$.)

This geometric setting also produces further important examples of 'bundles', this time on these Grassmann varieties.

Consider the subspace of $G_k(\mathbb{R}^n) \times \mathbb{R}^n$ given by those (V, x) with $x \in V$. Using the projection $p(V, x) = V$ gives the bundle,

$$\gamma_k^n = (\gamma_k^n, p, G_k(\mathbb{R}^n)).$$

This is *canonical k-dimensional vector bundle on $G_k(\mathbb{R}^n)$.*

Similarly the *orthogonal complement bundle*, ${}^*\gamma_k^n$, has total space consisting of those (V, x) with $\langle V \mid x \rangle = 0$, *i.e.*, x is orthogonal to V.

All of these 'bundles' have vector space structures on their fibres. They are all *locally trivial* (so in each case $\alpha^{-1}(U) \cong U \times F$ for suitable open subsets U of the base), and the resulting $\xi_{U,U'}$ have form

$$\xi_{U,U'}(x, t) = (x, \xi'_{U,U'}(x))(t)$$

where $\xi'_{U,U'} : U \cap U' \to G\ell_M(\mathbb{R})$ for suitable M. (As usual, $G\ell_M(\mathbb{R})$, which may sometimes also be denoted $G\ell(M, \mathbb{R})$, is the *general linear group of non-singular $M \times M$ matrices over* $\mathbb{R}$. Here it is considered as a topological group. It also has a smooth structure and is an important example of a *Lie group*.) Such *vector bundles* are prime examples of the situation in which the fibres have extra structure.

We will see, use and study vector bundles in more detail later on, for the moment, we introduce the example of a *trivial vector bundle* in addition to those geometrically occurring ones above. We will work over the real numbers as our basic field, but could equally well use $\mathbb{C}$ or more generally.

Definition: A *trivial (real) vector bundle* of dimension m, on a space B is one of the form $\mathbb{R}^m \times B \to B$, the mapping being, naturally, the projection. We will denote this by ε_B^m.

Even more structure can be encoded, for instance, by giving each fibre an inner product structure with the requirement that the $\xi'_{U,U'}$ take values in $O_M(\mathbb{R})$, or $O(M, \mathbb{R})$, the *orthogonal group*, hence that they preserve that extra structure. Abstracting from this we have a group, G, which acts by automorphisms on the space, F, and have our descent data isomorphisms $\xi_{U,U'}$ of the form $\xi_{U,U'}(x,t) = (x, \xi'_{U,U'}(x))(t)$ for some continuous $\xi'_{U,U'} : U \cap U' \to G$.

As usual, if G is a (topological) group, by a *G-space*, we mean a space X with an action (left action):

$$G \times X \to X,$$

$$(g, x) \to g.x.$$

The action is *free* if $g.x = x$ implies $g = 1$. The action is *transitive* if given any x and y in X there is a $g \in G$ with $g.x = y$. Let X^* be the subspace

$$X^* = \{(x, g.x) : x \in X, g \in G\} \subseteq X \times X,$$

(cf. our earlier discussion of action groupoids on page 17).

There is a function (called the *translation function*)

$$\tau : X^* \to G$$

such that $\tau(x, x')x = x'$ for all $(x, x') \in X^*$. We note

(i) $\tau(x, x) = 1$,

(ii) $\tau(x', x'')\tau(x, x') = \tau(x, x'')$,

(iii) $\tau(x', x) = \tau(x, x')^{-1}$

for all $x, x', x'' \in X$.

A G-space, X, is called *principal* provided X is a free, transitive G-space with *continuous* translation function $\tau : X^* \to G$.

Proposition 73 *Suppose X is a principal G-space, then the mapping*

$$G \times X \to X \times X$$

$$(g, x) \to (x, g.x)$$

is a homeomorphism.

Proof: The mapping is continuous by its construction. Its inverse is (τ, pr_1), which is also continuous. ■

This is often taken as the definition of a principal G-space, so you could try to prove the converse. We, in fact, need a fibrewise version of this.

Given any G-space, X, we can form a quotient X/G with a continuous map $\alpha : X \to X/G$. A bundle $\mathsf{X} = (X, \alpha, B)$ is called a *G-bundle* if X has a G-action, so that B is homeomorphic to X/G compatibly with the projections from X. The bundle is a *principal G-bundle* if X is a principal G-space *over* B. What does this mean? In a G-bundle, as above, the fibres of α are orbits of the G-action, so the action is 'fibrewise'. We can replace G by $\underline{G} = G \times B$ and, thinking of it as a space over B, perhaps rather oddly, write the action within the category Top/B. We replace the product in Top by that in Top/B, which is just the pullback along projections in Top. The action is thus

$$\underline{G} \times_B X \to X$$

over B, or just $\underline{G} \times \mathsf{X} \to \mathsf{X}$ in the notation valid in Top/B. Now 'principalness' will say that the action is free and transitive, and that the translation function is a continuous map *over* B. A neater way to handle this is to use the above proposition and to define X to be a principal G-bundle if the corresponding morphism over B,

$$\underline{G} \times \mathsf{X} \to \mathsf{X} \times \mathsf{X}$$

is an isomorphism in Top/B. We will not explore this more here as that *is*, more or less, the way we will define G-torsors later on, except that we will be using a bundle or sheaf of groups rather than simply $\underline{G}$.

We note that if $\xi = (X, p, B)$ is a principal G-bundle then the fibre $p^{-1}(b)$ is homeomorphic to G for any point $b \in B$. It is usual in topological situations to require that the bundle be locally trivial. For the moment, we can summarise the idea of principal G-bundle as follows:

A principal G-bundle is a fibre bundle $p : X \to B$ together with a continuous left action $G \times X \to X$ by a topological group G such that G preserves the fibers of p and acts freely and transitively on them.

Later we will see other more categorical views of principal G-bundles. As we have mentioned, they will reappear as 'G-torsors' in various settings. For the moment we need them to provide the link to the general notion of fibre bundle.

For F, a (right) G-space with action $G \times F \to F$, we can form a quotient, X_F, of $F \times X$ by identifying (f, gx) with (fg, x). The composite

$$F \times X \xrightarrow{pr_2} X \to X/G$$

factors via X_F to give $\beta : X_F \to X/G$, where $\beta(f, x)$ is the orbit of x, *i.e.*, the image of x in X/G. The earlier examples of 'bundles' were all examples of this construction. The resulting (X_F, β, B) is called a *fibre bundle* over B $(= X/G)$.

Note: The theory of fibre bundles was developed by Cartan and later by Ehresmann and others from the 1930s onwards. Their study arose out of questions on the topology and geometry of manifolds. In 1950, Steenrod's book, [252], gave what was to become the first reasonably full treatment of the theory. Atiyah, Hirzebruch and then, in book form, Husemoller, [160] in

1966 linked this theory up with K-theory, which had come from algebraic geometry. The books contain much of the basic theory including the local coordinate description of fibre bundles which is most relevant for the understanding of the descent theory aspects of this area (cf. Chapter 5 of Husemoller, [160]). The restriction of looking at the local properties relative to an open cover makes this treatment slightly too restrictive for our purposes. It *is* sufficient, it seems, for many of the applications in algebraic topology, differential geometry and topology and related areas of mathematical physics, however as Grothendieck points out (SGA1, [141], p.146), in algebraic geometry *localisation of properties*, although still linked to certain types of "base change" (as here with base change along the map

$$\sqcup\, \mathcal{U} \to B$$

for $\mathcal{U}$ an open cover of B), needs to consider other families of base change. These are linked with some problems of commutative algebra that are interesting in their own right and reveal other aspects of the descent problem, see [39]. For these geometric applications, we need to replace a purely topological viewpoint by one in which *sheaves* take a front seat role.

(The Wikipedia entries for principal G-space, principal bundle and 'fiber' bundle are good places to start seeing how these concepts get applied to problems in geometry. For a picture of how to build a fibre bundle out of wood, see http://www.popmath.org.uk/sculpmath/pagesm/fibundle.html.)

7.1.4 Change of Base

This is a theme that we will revisit several times. Suppose that we have a good knowledge of 'bundles' over some space, B', but want bundles over another space, B. We have a continuous map, $f : B \to B'$, and hope to glean information on bundles on B by comparing them with those on B', using f in some way. (We could be looking to transfer the information the other way as well, but this way will suffice for the moment!)

What we have used when restricting to open subsets of a base space was pullback and that works here as well. Suppose $p' : A' \to B'$ is a principal G-bundle over B', then we form the pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

Categorically the pullback, as it is characterised by a universal property, is only determined up to isomorphism, but we can pick a definite model for A in the form

$$A' \times_{B'} B = \{(a, b) \mid p'(a) = f(b)\},$$

with $a \in A'$ and $b \in B$. The projection of A onto B is given by sending (a, b) to b and the map from A to A' by the obvious other projection. As we have an action of G on the left of A' it is tempting to see if there is one on A and the obvious thing to attempt is $g.(a, b) = (g.a, b)$. Does this make sense? Yes, because $p'(g.a) = p'(a)$, since B' is the space of orbits of the action of G on A'. Is $A \to B$ then a principal G-bundle? Again the answer is yes. To gain some idea why look at the fibres. We know the fibres of a principal G bundle are copies of the *space* G, and fibres of the pullback are the same as fibres of the original. The action is concentrated in the fibres as the orbit space of the action *is* the base.

The one question is whether the map

$$\underline{G} \times_B A \to A \times_B A$$

is an isomorphism. You can see that it is in two ways. The elements of A are pairs (a, b), as above. The map is $((g, b), (a, b)) \mapsto ((a, b), (g.a, b)$ and this is clearly in the fibres as the second component in each pair is the same. It has an inverse surely, (since an element in $A \times BA$, has the form $((a_1, b), (a_2, b))$ and since A' is a principal bundle we can continuously find g such that $a_2 = g.a_1$). The alternative approach is to note that the map fits into a diagram with lots of pull back squares and to note that is is induced from the corresponding map for (A', B', p').

We thus have, it would seem, that $f : B \to B'$ induces a 'functor' from the category of principal G-bundles over B' to the corresponding one over B. (The word 'functor' is given between inverted commas since we have not discussed morphisms between bundles of this form. That is left to you both to formulate the notion and to check that the inverted commas can be removed. In any case we will be considering this in the more general setting of G-torsors slightly later in this chapter.)

We thus have induced bundles, $f^*(\mathsf{A}')$, but different maps, f, can lead to isomorphic bundles. More precisely, suppose f and g are two maps from B to B', then if f and g are homotopic (under mild compactness conditions on the spaces) it is fairly easy to prove that for any (principal) bundle A' on B', the two bundles $f^*(\mathsf{A}')$, and $g^*(\mathsf{A}')$, are isomorphic. We will not give the details here as they are in most text books on the area, (see, for instance, [160], or [176]), but the idea is that if $H : B \times I \to B'$ is a homotopy between f and g, we get a bundle $H^*(\mathsf{A})$ with base $B \times I$. You now use local triviality of the bundle to cover $B \times I$ by open sets over which this bundle trivialises. Using compactness of B, we get a sequence of points t_i in I and an open cover of $B \times I$ made up of open sets of the form $U \times (t_i, t_{i+2})$. Now we work our way up the cylinder showing that the bundle over each slice, $B \times \{t_i\}$, is isomorphic to that on the previous slice. (There are lots of details left vague here and you should look them up if you have not seen the result before.)

This result shows that categories of principal bundles over homotopically equivalent spaces will be equivalent, and, in particular, that over any contractible space, all principal bundles are isomorphic to each other and hence are all isomorphic to the product principal bundle. It also shows that if we can cover B with an open cover made up of contractible open sets that all bundles trivialise over that cover.

Remarks: In many different theories of bundle-like objects there is an *induced bundle* construction given by pullback along a continuous map on the 'bases'. In *most* of those cases, it seems, homotopic maps induce isomorphic 'bundles', again with possibly a compactness requirement of some sort on the bases.. This happens with vector bundles, (as follows from the result on principal bundles mentioned above.) In these cases, the only bundles of that type on a contractible space will be product bundles. (We will keep this vague directing the reader to the literature as before.)

7.2 Descent: simplicial fibre bundles

To understand topological descent, as in the theory of fibre bundles as sketched out above, it is useful to see the somewhat simpler simplicial theory. This has aspects that are not so immediately obvious as in the topological case, yet some of these will be very useful when we get further in our study handling sheaves and later on stacks.

The basics of simplicial fibre bundle theory were developed in the 1950s and early 1960s, the start being in a paper by Barratt, Gugenheim and Moore, [21]. We have already discussed several of the features of this theory. A useful survey is given by Curtis, [97], and a full description of the theory are available in May's book, [198], with many aspects also treated in Goerss and Jardine, [137].

7.2.1 Fibre bundles, the simplicial viewpoint

We earlier saw how, in the simplicial setting, the G-principal fibrations, when pulled back over any simplex of their base, gave a trivial product fibration. It is this feature that we abstract to get a working notion of simplicial fibre bundle.

Definition: A *(simplicial) fibre bundle* with fibre, Y, over a simplicial set, B, is a simplicial map, $f : E \to B$ such that for any n-simplex, $b \in B_n$, (for any n), the pullback over the representing ('naming') map, $\ulcorner b \urcorner : \Delta[n] \to B$, is a trivial bundle, that is, isomorphic to a product of Y with $\Delta[n]$ together with its projection onto $\Delta[n]$.

We thus have a diagram

$$\begin{tikzcd}
Y \times \Delta[n] \arrow[r] \arrow[d, "p_2"'] & E \arrow[d, "f"] \\
\Delta[n] \arrow[r, "\ulcorner b \urcorner"'] & B,
\end{tikzcd}$$

which is a pullback.

It is worthwhile just thinking about the comparison between this and what we have been looking at for topological bundles. The role played there by the open cover is taken by the family of *all* simplices of the base. (From this one can build a neat category, and in a very similar way from a plain classical open cover you can form all finite (non-empty) intersections, add them into the cover and build a category from these and the inclusions between them. It will pay to retain that thought for when we launch into discussion of sheaves, and, in particular, stacks, etc.)

It is, thus, important to note that in any simplicial fibre bundle, we have fibres over all simplices, not just the 'vertices'. The 'fibre' over an n-simplex, b, of the base, is given by the pullback

$$\begin{tikzcd}
E \times_B \Delta[n] \arrow[r] \arrow[d, "p_2"'] & E \arrow[d, "f"] \\
\Delta[n] \arrow[r, "\ulcorner b \urcorner"'] & B,
\end{tikzcd}$$

The usual notion of 'fibre' then corresponds to the case where $n = 0$. We will sometimes write $E(b) = E \times_B \Delta[n]$, since $E \times_B \Delta[n]$ as a notation, does not actually record the b being considered. For instance, given $e \in E_n$, we have the fibre through e will be $E(p(e))$.

Examples of fibre bundles: (i) **Trivial product bundles**:

Lemma 44 *The trivial product bundle, $p_B : Y \times B \to B$, is a fibre bundle in this sense.*

Proof: To see this, we pick an arbitrary, $\ulcorner b \urcorner : \Delta[n] \to B$, and embed it in the commutative diagram:

$$\begin{tikzcd} Y \times \Delta[n] \arrow[r, "Y\times\ulcorner b\urcorner"] \arrow[d, "p_2"'] & Y \times B \arrow[r] \arrow[d, "p_B"] & Y \arrow[d] \\ \Delta[n] \arrow[r, "\ulcorner b\urcorner"'] & B \arrow[r] & \Delta[0], \end{tikzcd}$$

where the two arrows with codomain $\Delta[0]$ are the unique such maps, (since $\Delta[0]$ is terminal in $\mathcal{S}$). This means that both the right-hand square and the outer rectangle are pullbacks, and then it is an elementary (standard) exercise of category theory to show that the left hand square is also a pullback, which completes the proof. ∎

(ii) **Any G-principal fibration** is a fibre bundle, since we saw, Lemma 42, that the fibre bundle condition was satisfied. The fibre in this case is the underlying simplicial set of the simplicial group, G.

7.2.2 Atlases of a simplicial fibre bundle

The idea of atlases originally emerged in the theory of manifolds. manifolds are specified by local 'charts' and, of course, a collection of charts makes, yes you guessed, ... Here we will see how that idea can be adapted to a simplicial setting.

Let (E, B, p) be a fibre bundle with fibre, Y, then we see that, for any $b \in B_n$, there is an isomorphism

$$\alpha(b) : Y \times \Delta[n] \to E \times_B \Delta[n],$$

given by the diagram:

$$\begin{tikzcd} Y \times \Delta[n] \arrow[r, "\alpha(b)", "\cong"'] \arrow[dr, "p_2"'] & E \times_B \Delta[n] \arrow[r, "p_1"] \arrow[d, "p_2"] & E \arrow[d, "p"] \\ & \Delta[n] \arrow[r, "\ulcorner b\urcorner"'] & B \end{tikzcd}$$

using the universal property of pullbacks. Set $a(b) : Y \times \Delta[n] \to E$ to be the composite $p_1\alpha(b)$.

Remark: If we think of b as a 'patch' over which (E, B, p) trivialises, then $\alpha(b)$ is the trivialising isomorphism identifying E 'restricted to the patch b' with a product. A face of b may be shared with another n-simplex, so we can expect interactions / transitions between the different descriptions / trivialisations.

Definition: The family $\alpha = \{\alpha(b) \mid b \in B\}$ (or, equivalently, $\mathbf{a} = \{a(b) \mid b \in B\}$) will be called an *atlas* for (E, B, p).

That α determines $\mathbf{a}$ is obvious, but we have also $\alpha(b)(y, \sigma) = (a(b)(y, \sigma), \sigma)$, so $\mathbf{a}$ also determines α. We should also point out that in the definition, we are using $b \in B$ as a convenient shorthand for $b \in \bigsqcup_n B_n$.

It is often useful to think of $\alpha(b)$ as an element of $\underline{\mathcal{S}}(Y, E \times_B \Delta[n])_n$ and $a(b) \in \underline{\mathcal{S}}(Y, E)_n$, since this makes the following idea very clear.

Suppose we consider the automorphism simplicial group, $\mathsf{aut}(Y)$, (cf. page 257) and a subsimplicial group, G, of it. Pick a family $\mathbf{g} = \{g(b) \mid b \in B\}$, of elements of G, where, if $b \in B_n$, $g(b) \in G_n$. There is a new atlas $\alpha \cdot \mathbf{g} = \{\alpha(b)g(b) \mid b \in B\}$ obtained by 'precomposing' with $\mathbf{g}$. (We can also use $\mathbf{a} \cdot \mathbf{g}$ with the obvious definition.)

Definition: Two atlases, α and α', are said to be *G-equivalent* is $\alpha' = \alpha \cdot \mathbf{g}$ for some family, $\mathbf{g}$, of elements from G.

So far, there has been no requirement on the atlas α to respect faces and degeneracies in any way. In fact, we do not really *want* to match faces, since, even in such a simple case as the Möbius band, strict preservation of faces (something like $a(d_ib) = d_i(a(b))$, perhaps) would not allow the 'twisting' that we would need.) On the other hand, if we have $a(b)$ defined for a non-degenerate simplex, b, then we already have a suitable $a(s_ib)$ around, namely $s_ia(b)$, so why not take that! (You may like to **investigate** this with regard to theuniversal property that we used to define the $\alpha(b)$s.)

Definition: An atlas, $\mathbf{a}$, is *normalised* if, for each $b \in B$, $a(s_ib) = s_ia(b)$ in $\underline{S}(Y,E)$.

Lemma 45 *Given any atlas, $\mathbf{a}$, there is a normalised atlas, $\mathbf{a}'$, that agrees with $\mathbf{a}$ on the non-degenerate simplices of B.* ■

The proof, which is simply a question of making a definition, then verifying that it works is **left to you**.

Turning to the face maps, as we said, we do not necessarily have $a(d_ib) = d_ia(b)$, but we might expect the two sided to be linked by an automorphism of the fibre, of some type. We know

$$d_i(\alpha(b)) = (Y \times \Delta[n-1] \stackrel{Y\times\delta_i}{\to} Y \times \Delta[n] \stackrel{\alpha(b)}{\to} E \times_B \Delta[n],$$

is an isomorphism onto its image. The i^{th} face inclusion $\delta : \Delta[n-1] \to \Delta[n]$ also induces

$$E \times \delta_i : E \times_B \Delta[n-1] \to E \times_B \Delta[n],$$

which we will call θ, and which element-wise is given by $\theta(e,\sigma) = (e, \delta_i \circ \sigma)$, and the image of $\theta \circ \alpha(d_ib)$ is the same as that of $d_i(\alpha(b))$, namely elements of the form $(e, \delta_i \circ \sigma)$. We thus obtain an automorphism, $t_i(b)$, of $Y \times \Delta[n-1]$ with

$$\alpha(d_ib) \circ t_i(b) = d_i(\alpha(b)).$$

('Corestricting' $\alpha(d_ib)$ and $d_i(\alpha(b))$ to that image, we have $t_i(b) = \alpha(d_ib)^{-1} \circ d_i(\alpha(b))$, so $t_i(b)$ is uniquely determined.)

This 'corestriction' argument is reasonably clear as an element based level, but it leaves a lot to check. It is useful to give an equivalent more categorical construction of t, which gets around the verification, for instance, that $t_i(b)$ is a simplicial map - which was 'swept under the carpet' in the above - and is more 'universally valid' as it shows what categorical and simplicial properties are being used.

Let us go back a stage, therefore, and take things apart as 'pullbacks' and in quite some detail. This is initially a bit tedious perhaps, but it is worth doing.

- $\ulcorner d_i b \urcorner$ is the composite
$$\Delta[n-1] \xrightarrow{\delta_i} \Delta[n] \xrightarrow{\ulcorner b \urcorner} B,$$
and so $\alpha(d_i b)$ fits in a diagram:
$$\begin{tikzcd} Y \times \Delta[n-1] \arrow[r, "\alpha(d_ib)", "\cong"'] \arrow[dr, "p_2"'] & E \times_B \Delta[n-1] \arrow[r, "E\times_B\delta_i"] \arrow[d, "p_2"] & E \times_B \Delta[n] \arrow[r, "p_1"] \arrow[d, "p_2"] & E \arrow[d, "p"] \\ & \Delta[n-1] \arrow[r, "\delta_i"'] & \Delta[n] \arrow[r, "\ulcorner b \urcorner"'] & B \end{tikzcd}$$

- We have $\alpha(b) : Y \times \Delta[n] \to E \times_B \Delta[n]$ and want to obtain a restriction of it to the i^{th} face, *i.e.*, to $Y \times \Delta[n-1]$ along $Y \times \delta_i$, *and*, at the same time, that 'corestriction' to $E \times_B \Delta[n-1]$. We want to form the square diagram
$$\begin{tikzcd} Y \times \Delta[n-1] \arrow[r, "\widetilde{d_i(\alpha(b))}"] \arrow[d, "Y\times\delta_i"'] & E \times_B \Delta[n-1] \arrow[d, "E\times_B\delta_i"] \\ Y \times \Delta[n] \arrow[r, "\alpha(b)", "\cong"'] & E \times_B \Delta[n], \end{tikzcd}$$
where the top horizontal arrow, $\widetilde{d_i(\alpha(b))}$, is 'induced from' $\alpha(b)$. We should check how exactly it is built. As it is goinginto an object specified by a pullback, we need only specify its two components, that is, the projections onto E and $\Delta[n-1]$. (Of course, this is exactly what we did in in the element-wise description.) The component going to E is just found by going the other way around the square and folowing that composite by p_1 down to E. The component to $\Delta[n-1]$ is just the projection, p_2. (To see what is going on **draw a diagram yourself**.) We have to verify that the square commutes. This uses the pullback 'uniqueness' clause for $E \times_B \Delta[n]$.

- We note that the corestriction, $\widetilde{d_i(\alpha(b))}$, is a monomorphism, as its composite with $E \times_B \delta_i$ is one. We claim it is an isomorphism. It remains to show, for instance, that it is a split epimorphism. (That is relative easy to try, so is a good place to attack what is needed.)

  First note that
$$\begin{tikzcd} Y \times \Delta[n-1] \arrow[r, "Y\times\delta_i"] \arrow[d] & Y \times \Delta[n] \arrow[d] \\ \Delta[n-1] \arrow[r, "\delta_i"'] & \Delta[n] \end{tikzcd}$$
is a pullback, as is also
$$\begin{tikzcd} E \times_B \Delta[n-1] \arrow[r, "E\times_B\delta_i"] \arrow[d] & E \times_B \Delta[n] \arrow[d] \\ \Delta[n-1] \arrow[r, "\delta_i"'] & \Delta[n]. \end{tikzcd}$$
(In each case, you can put an obvious pullback square to the right, so that the composite 'rectangle' is again a pullback - that same argument again.) We build the inverse to $\tilde{d} :=$

$\widetilde{d_i(\alpha(b))}$, using the first of these two squares. The component of that inverse going to $\Delta[n-1]$ is the obvious one, whilst to $Y \times \Delta[n]$, we use $\alpha(b)$. (You are **left to check commutativity**.) To check then that this map we have constructed, does split $\tilde{d}$, we use the uniqueness clause for the second of these pullbacks.

The final step in proving that $\tilde{d}$ is an isomorphism is the 'usual' proof that if a morphism is both a monomorphism and a split epimorphism then the splitting is, in fact, the inverse for the original monomorphism (which is thus an isomorphism). (**If you have not seen this before**, first **check** the categorical meaning of monomorphism, then **work out a proof** of the fact.)

We, therefore, have

$$Y \times \Delta[n-1] \xrightarrow[\cong]{\alpha(d_i b)} E \times_B \Delta[n-1]$$

and

$$Y \times \Delta[n-1] \xrightarrow[\cong]{\tilde{d}} E \times_B \Delta[n-1] ,$$

both over $\Delta[n-1]$, as you **easily check** from the above. We thus get

$$t_i(b) = \alpha(d_i b)^{-1}.\tilde{d},$$

and this is in $\mathsf{aut}(Y)_{n-1}$. We note that these elements are completely determined by the normalised atlas.

Definition: The automorphisms, $t_i(b)$, for $b \in B$ are called the *transition elements* of the atlas, α.

If the transition elements all lie in a subgroup, G, of $\mathsf{aut}(Y)$, then we say α, (or, equivalently, a), is a *G-atlas.*

An atlas, α, is *regular* if, for $i > 0$, its transition elements, $t_i(b)$, are all identities.

We thus have that, in a regular normalised atlas, we just need to specify the $t_0(b)$, as these may be non-trivial. (To see where this theory is going at this point, you may find it helps to think $t =$ 'twisting', as well as, $t =$ 'transition', and to look back at our discussion of T.C.P.s (section 6.5, page 265).)

Lemma 46 *Every (normalised) G-atlas is G-equivalent to a (normalised) regular G-atlas.*

Proof: We start with a G-atlas, which we will assume normalised. (The unnormalised case is more or less identical.) We will use it in the form $\mathbf{a}$, rather than α, but, of course, this really makes no difference. We will build, by induction, a G-equivalent regular one, $\mathbf{a}'$.

On vertices, we take $a'(b) = a(b)$. That gets us going, so we now assume $a'(b)$ is defined for all simplices of dimension less than n, and that $\mathbf{a}'$ is regular and G-equivalent to $\mathbf{a}$, to the extent that this makes sense. We next want to define $a'(b)$ for b, a (non-degenerate) n-simplex. (The degenerate ones are handled by the normalisation condition.)

We look at the $(n, 0)$-horn in B corresponding to b, *i.e.*, made up of all the $d_i b$ for $i \neq 0$. We have elements $g_i(b)$ such that

$$a'(d_i b) = a(d_i b) g_i(b),$$

since $\mathbf{a}'$ is G-equivalent to $\mathbf{a}$ in this dimension, then, using

$$a(d_i b)t_i(b) = d_i(a(b)),$$

we get $a'(d_i b) = d_i(a(b)).t_i(b)^{-1}.g_i(b) = d_i(a(b)).h_i$, where we have set $h_i = t_i(b)^{-1}.g_i(b)$. Since $\mathbf{a}'$, so far defined is regular, we have, for $0 < i \leq j$, after a bit of simplicial identity work (**for you**), that

$$d_i d_j(a(b))d_i h_j = d_i d_j(a(b))d_{j-1}h_i,$$

which implies that $d_i h_j = d_{j-1}h_i$, the the hs form a $(n,0)$-horn in G. we now wheel out our method for filling horns in G to get a $h \in G_n$ with $d_i h = h_i$, for $i > 0$, and we set $a'(b) = a(b)h$. we heck

$$\begin{aligned} d_i a'(b) &= d_i a(b))d_i h \\ &= d_i a(b)h_i \\ &= a'(d_i b). \end{aligned}$$

The resulting $\mathbf{a}'$, is now defined up to and including dimension n, is normalised and regular, and G-equivalent to $\mathbf{a}$. We get this in all dimensions by induction. ■

7.2.3 Fibre bundles are T.C.P.s

We saw earlier that G-principal fibrations were locally trivial and hence are fibre bundles, and that twisted Cartesian products (T.C.Ps) are principal fibrations. We now have regular atlases, yielding structures that look like twisting functions. This suggests that the various ideas are really 'the same'. We will not complete all the details that show that they are, since that theory is in various texts (for instance, May's book, [198]), but will more-or-less complete our *sketch* of the interrelationships.

There remains, for our sketch, an investigation of the transition elements for simplicial fibre bundles and a 'sketch proof' that fibre bundles are just T.C.Ps.

Suppose we have some simplicial fibre bundle and a normalised regular G-atlas, $\mathbf{a} = \{a(b) \mid b \in B\}$, giving as the only possibly non-trivial transition elements, the $t(b) := t_0(b)$. We thus have

$$d_0 a(b) = a(d_0 b).t(b).$$

(To avoid looking back all the time to the definition of twisting function, we repeat it here for convenience and also to adjust conventions. We had:

a function, t, satisfying the following equations will be called a *twisting function*:

$$\begin{aligned} d_i t(b) &= t(d_{i+1}b) \quad \text{for} \quad i > 0, \\ d_0 t(b) &= t(d_0 b)^{-1} t(d_1 b), \\ s_i t(b) &= t(s_{i+1}b) \quad \text{for} \quad i \geq 0, \\ t(s_0 b) &= *. \end{aligned}$$

(**Warning:** The version on page 266 corresponded to the 'algebraic' diagrammatic composition order, and here we have used the 'Leibniz' composition order so we have adjusted the second equation accordingly.)

Lemma 47 *The transition elements, $t(b)$, above, define a twisting function.*

Proof: We use the defining equation (above) for the $t(b)$ and, in particular, the uniqueness of these elements with this property, (together with the 'regular' and 'normalised' conditions for $\mathbf{a}$). We leave the majority of the cases **to you**, since conce you have seen one or two of these, the others are easy.

(We wil do a very easy one as a 'warm up', then the important, and more tricky, one relating toe d_0 and d_1, *i.e.*, the twist.)

Applying the equation above to s_0b, we get

$$d_0a(s_0b) = a(d_0s_0b).t(s_0b) = a(b).t(s_0b),$$

but $\mathbf{a}$ is normalised, so $a(s_0b) = s_0(b)$ and the left hand side is thus just $a(b)$. we can thus conclude that $t(s_0b)$ is the identity. (That was easy!)

We now turn to the relation involving $t(d_0b)$ and $t(d_1b)$, etc.:

$$d_0a(d_1b) = a(d_0d_1b).t(d_1b),$$

but we also have

$$d_0a(d_0b) = a(d_0d_0b).t(d_0b),$$

and, of course, $d_0d_1b) = d_0d_0b$.

We next apply d_0 to the 'master equation', simply giving

$$d_0d_0a(b) = d_0a(d_0b).d_0t(b),$$

and to $d_1a(b) = a(d_1b)$ to get

$$d_0d_1a(b) = d_1a(d_1b).$$

Again using the simplicial identity $d_0d_1 = d_0d_0$, we rearrange terms algebraically to get

$$d_0t(b) = t(d_0b)^{-1}t(d_1b),$$

as expected.

The other equations are **left to you**. (You just mix applying a d_i or s_i to the 'master equation' inside (i.e, on b) and outside, then use normalisation, regularity and the simplicial identities.) ■

It is thus possible to use E to find $\mathbf{a}$ and thus t, and thence to form $B \times_t Y$. We need now to compare $B \times_t Y$ with E.

To start with we will do something that looks as if it is 'cheating'. We have, for $b \in B_n$ that $a(b) \in \underline{S}(Y,E)$, so *do* have a *graded* map

$$\mathbf{a} : B \to \underline{S}(Y,E).$$

Our assumptions about $\mathbf{a}$ being regular, normalised, etc., imply that this is very nearly a simplicial map. (The only thing that goes wrong is the d_0-face compatibility.)

If $\mathbf{a}$ was simplicial, we could 'fli[it' through the adjunction to get $\xi : B \times Y \to E$. We know how to do this. We form the composite

$$B \times Y \xrightarrow{\mathbf{a}\times Y} \underline{S}(Y,E) \times Y \xrightarrow{eval} E,$$

where *eval* is the map we met earlier (page 261), and which, as you will recall, we worked hard to get a complete description of. For $y \in Y_n$, and $f : Y \times \Delta[n] \to E \in \underline{S}(Y, E)_n$, we had that

$$eval(f, y) = f(y, \iota_n),$$

where, as always, ι_n is the unique non-degenerate n-simplex in $\Delta[n]$, corresponding to the identity map on $[n]$ in the description $\Delta[n] = \mathbf{\Delta}(-, [n])$. We can pretend that $\mathbf{a}$ is simplicial, see what ξ is given by and then see how much it is or is not simplicial. We can read off, if $y \in Y_n$ and $b \in B_n$,

$$\xi(b, y) = a(b)(y, \iota_n).$$

This map ξ is 'as simplicial as is $\mathbf{a}$'. We will check this, or part of it, by hand, but although it follows from generalities on the adjunction process, verifying the conditions needs care.

First we note that if $f : Y \times \Delta[n] \to E$, then $d_i f = f \circ (Y \times \Delta[\delta - i])$, where $\delta_i : [n-1] \to [n]$ is the i^{th} face inclusion (so we get $\Delta[\delta_i] : \Delta[n-1] \to \Delta[n]$). We examine the evaluation map in detail as it is the key to the calculation. By its construction, it is bound to be simplicial, but we need also to see what that means at this 'elementary' level. We have

$$\begin{array}{ccc}
\underline{S}(Y,E)_n \times Y_n & \xrightarrow{eval} & E_n \\
\downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} \\
\underline{S}(Y,E)_{n-1} \times Y_{n-1} & \xrightarrow{eval} & E_{n-1}
\end{array}$$

and, for $i > 0$,

$$\begin{aligned}
d_i\xi(b, y) &= d_i(a(b)(y, \iota_n) = eval(d_i a(b), d_i y) \\
&= eval(a(d_i b), d_i y) = \xi(d_i b, d_i y) = \xi d_i(b, y).
\end{aligned}$$

Similarly, we have, for s_i that $s_i\xi = \xi s_i$. That just leaves $d_0\xi$ and, of course

$$d_0\xi(b, y) = eval(a(d_0 b).t(b), d_0 y),$$

by the same sort of argument, and then this is $a(d_0(b))(t(b)d_0 y, \iota_{n-1}) = \xi(d_0 b, t(b)d_0 y)$. (You may want to check this last bit for yourself. You need to translate to-and-fro between a G-actions on Y as being $a : G \times Y \to Y$ and the adjoint $a : G \to \mathsf{aut}(Y)$, again using *eval*.)

This gives us that, if we *define* a new d_0 on this product by *twisting* it using t (and, of course, this is just giving us $B \times_t Y$ as we have already seen it, on page 265) with, explicitly,

$$d_0(b, y) = (d_0 b, t(b)(d_0 y)),$$

then we actually obtain

$$\xi : B \times_t Y \to E$$

as a simplicial map. We note that $p\xi = p_B$, the projection onto B of the T.C.P., so ξ is 'over B'.

Proposition 74 *This map, ξ, is an isomorphism (over B).*

Proof: We start by constructing, for each $b \in B_n$, a map $\nu(b) : E(b) \to Y$, where, as before, $E(b) = E \times_B \Delta[n]$, the pullback of E along $\ulcorner b \urcorner$, so is the 'fibre over b'. We have $\alpha(b) : Y \times \Delta[n] \to E(b)$ is an isomorphism, and so we can form $\nu(b) := pr_Y \alpha(b)^{-1} : E(b) \to Y$. Using this we send and n-simplex e to $(p(e), \nu(p(e))(e, \iota_n))$, where $(e, \iota_n) \in E(p(e))$ This gives us something in $B \times_t Y$ and ξ is then easily seen to send that n-simplex back to e. That the other composite is the identity ia also easy (for **you to check**). ■

We thus have a pretty full picture of how principal fibrations are principal fibre bundles, given by twisted Cartesian products of a particular type, that principal H-fibre bundles are classified by $\overline{W}(H)$, since $Princ_H(B) \cong [B, \overline{W}(H)]$, that general fibre bundles in the simplicial context are T.C.P.s and so correspond to a principal bundle and a representation of the corresponding group, and probably some other things as well. As these have been spread over different chapters, since we wanted to make use of the ideas as we went along, **you may find it helpful** to now read one of the texts, such as [198] or the survey, [97], that give the whole theory in one go. We will periodically be recalling part of this, making comparisons with other ideas and methods, and possibly pushing this theory on new directions (as this is 'classical').

7.2.4 ...and descent in all that?

In earlier sections, we looked at descent in a topological context. There we used an open cover, $\mathcal{U}$, of the base space and had transitions, $\xi_{U,U'}$, on intersections of these open patches, with a condition on triple intersections. The idea was to take the A_U for the various open sets, U, of the cover $\mathcal{U}$, and to glue them together, using the $\xi_{U,U'}$ to get the right amount of 'twisting' from patch to patch, with the cocycle condition to ensure the different gluings are compatible.

That somehow looks initially very different from what we have been doing in our discussion of simplicial fibre bundles. We would not expect to have 'open sets', but what takes their place in the simplicial context. We will look at this only briefly, but from several directions. The ideas that we would use for a full treatment will be studied in more depth in the following chapters. This therefore is a 'once over lightly' treatment of just a few of the ideas and insights. The ideas will be recalled, and treated in some depth in later chapters, but not always from the same perspective.

We start by looking at the open cover from a simplicial viewpoint. We have already seen the construction as we met it when discussing the nerve of a relation in section 4.3.5, but here we will be taking it in a different direction and so, for convenience we repeat the definition. In fact we will need to re-repeat the definition further on in the notes, as we will need to explore some of its geometric links with triangulations, see page **??**.

Definition: The *Čech complex*, *Čech nerve* or simply, *nerve*, of the open cover, $\mathcal{U}$, is the simplicial complex, $N(\mathcal{U})$, specified by:

- Vertex set : the collection of open sets in $\mathcal{U} = \{U_a \mid a \in A\}$ (alternatively, the set, A, of labels or indices of $\mathcal{U}$);

- Simplices : the set of vertices, $\sigma = \langle \alpha_0, \alpha_1, ..., \alpha_p \rangle$, belongs to $N(\mathcal{U})$ if and only if the open sets, U_{α_j}, $j = 0, 1, \ldots, p$, have non-empty common intersection.

As usual, if we choose an order on the indexing set, *i.e.*, the set of vertices of $N(\mathcal{U})$, then we can construct a neat simplicial set out of this, so that $\langle U_0, U_1 \rangle \in N(\mathcal{U})_1$ means $U_0 \cap U_1 \neq \emptyset$ and

U_0 is listed before U_1 in the chosen order. (We could, of course, not bother about the order and just consider all possible simplices. For instance, $\langle U_0, U_0, U_1\rangle$ would be $s_0\langle U_0, U_1\rangle$, but apparently the same simplex, $\langle U_1, U_0, U_0\rangle = s_1\langle U_1, U_0\rangle$, will also be there. This gives a larger simplicial set, but does have the advantage of being constructed without involving an order. **You are left to investigate** if this second construction gives something really different from the other. It is larger, but does it retract to the other form, for instance.) For the moment we will not look at what happens when passing to a finer cover of the space. We will return to this later however when discussing homotopy coherence.

(For simplicity of exposition, we will assume local triviality, so $A_U = U \times F$, for some 'fibre' F.) Returning to 'descent', looking at our transition functions, $\xi_{U,U'}$, they assign elements of the group, G, which acts on F, to these 1-simplices, $\langle U, U'\rangle$. (We assume G is a discrete group, not one of the more complex topological groups that also occur in this context.) Taking the group, G, we can form the constant simplicial group $K(G, 0)$, which has G in all dimensions and identity maps for all face and degeneracy morphism. This, then, gives a simplicial map from $N(\mathcal{U})$ to $\overline{W}K(G, 0)$. (You can check this if you wish, *but* we will be looking at it in great detail later on anyway.) We thus get a twisted Cartesian product $N(\mathcal{U}) \times_t K(G, 0)$. That gives us one way of seeing simplicial fibre bundles as being generalisations of the topological ones. They replace a very simple constant simplicial group by an arbitrary one, so have 'higher order transitions' acting as well. Untangling the complex intuitions and interpretations of this simple idea will be one of the themes from now on, not constantly 'up front', but quietly increasing in importance as we go further.

Another way of thinking of descent data is as 'building plans' for the fibre bundle given the bits, $A_U \cong U \times F$. We took the disjoint union, $\sqcup_U A_U$, then 'quotiented' by the gluing instructions encoded in the descent data, (see section 7.1.1). This is a fairly typical simple example of a colimit construction. We will study the categorical notion of colimit (and limit) later in some detail and will use it, and generalisations, many times. (These notes are intended to be reasonably accessible to people who have not had much formal contact with the theory of categories, although some basic knowledge of terminology is assumed as has been mentioned several times already. If you have not met 'colimits' formally, then **do** look up the definition. It may initially not 'mean' much to you, but it will help if you have some intuition. Something like: colimits are 'gluing' processes. You form a 'disjoint union' (coproduct), putting pieces out ready for use in the construction, then 'divide out' by an equivalence relation given, or at least, generated, by some maps between the different pieces.) We will see, more formally, the way that topological descent fits into this colimit / gluing intuition later on, but it is clearly also here in this simplicial context.

We have our basic pieces, $Y \times \Delta[n]$, and we glue them together using the 'combinatorial' information encoded in the simplicial set B. One way to view that is by using a neat construction of a category from a simplicial set.

Suppose we have a simplicial set, B, then we can form a small category $Cat(B)$ (also denoted (Yon, B), as it is an example of a *comma category*[1]). This has as its set of objects the simplices, b,

[1] We assume some acquaintance with such comma categories, for which, if you are not sure of the detailed meaning, you should check in standard categorical sources. We will variously use (F, G), and $F \downarrow G$ for such comma categories. The particular cases in which one or other of the two functors is constant will give an '*over category*' or an '*under category*', both of which we will meet several times. The typical special notation for these would be $\mathcal{C}/X$ and $X/\mathcal{C}$ respectively, but, although those forms will be used, this can cause confusion as X/G also would be used for the quotient of a G-object by the G-action, so care has to be taken. Some ambiguity is potentially there, but is relatively easy to resolve in any given case. Because of this we will sometimes use $\mathcal{C} \downarrow X$ instead. In any case, as all of these forms are in the literature, the reader is advised to get familiar with all of them!

of B, or, more usefully, their representing maps, such as $\ulcorner b \urcorner : \Delta[n] \to B$. If $\ulcorner b \urcorner$ and $\ulcorner c \urcorner : \Delta[m] \to B$ are two such, not necessarily of the same dimension, then a morphism in $Cat(B)$ from $\ulcorner b \urcorner$ to $\ulcorner c \urcorner$ 'is' a diagram:

$$\begin{tikzcd} \Delta[n] \arrow[rr, "\Delta[\mu]"] \arrow[dr, "\ulcorner b \urcorner"'] & & \Delta[m] \arrow[dl, "\ulcorner c \urcorner"] \\ & B & \end{tikzcd}$$

i.e., $\mu : [n] \to [m]$ is a morphism in $\mathbf{\Delta}$, so is a 'monotone map' which induces $\Delta[\mu]$ as shown. Saying that the diagram commutes says, of course, that $\ulcorner b \urcorner = \ulcorner c \urcorner \circ \Delta[\mu]$. Again, of course, $b \in B_n$ and $c \in B_m$ and μ induces a map $B_\mu : B_m \to B_n$. The obvious relationship corresponding to 'commutative' is that $B_\mu(c) = b$ and this holds. (You can take this, in the definition of morphism, to replace commutativity of the triangle as it is equivalent, then it comes out as saying 'a morphism $\mu : \ulcorner b \urcorner \to \ulcorner c \urcorner$ is a $\mu : [n] \to [m]$ such that $B_\mu(c) = b$, but it is very worth while checking through the above at a categorical level as well.)

If now you look back at our discussion of the reconstruction of (E, B, p) from the various patches, $Y \times \Delta[n]$, which corresponded to an n-simplex b in B, the process of gluing these together is completely analogous to our earlier discussion. It is again a 'colimit'. (You may, quite rightly ask, 'how come we get a twisted Cartesian *product* from a disjoint union type construction?' This is neat - and, of course, you may have seen it before. Looking just at sets A and B, if we form $A \times B$, then $A \times B = \coprod \{\{a\} \times B \mid a \in A\}$, so we can write a product as a disjoint union of (identical) labelled copies of the second set, each indexed by an element of the first one. (First and second here are really interchangeable of course.) We will see this type of construction several times later on. For instance if G is a simplicial groupoid and K is a simplicial set, we can form a new simplicial groupoid $K \otimes G$ with $(K \otimes G)_n$ being a disjoint union (coproduct) of copies of G_n indexed by the n-simplices of K. We will see this in detail later on, so this mention is 'in passing', but it is hopefully suggestive as to the sort of viewpoint we can use and adapt later.

The structure of simplicial fibre bundles is thus closely linked to the same intuitions and techniques used in the topological case. We now turn to sheaves, and will see those same ideas coming out again, with of course, their own flavour in the new context.

7.3 Descent: Sheaves

(As with previous sections, this should be 'skimmed' if you have met the subject matter, here sheaves, before. A good accessible account and brief introduction to this is Ieke Moerdijk's Lisbon notes, [206]. These also are useful for alternative developments of later material and are thoroughly to be recommended.)

7.3.1 Introduction and definition

Sheaves provide a useful alternative to bundles when handling 'local-to-global' constructions. The intuition is, in many ways, the same as that of bundles. We have a space B and for each $b \in B$, a 'fibre' over b, *i.e.*, a set F_b, and we want to have F_b varying in some continuous way as we vary b continuously. In other words, naively a sheaf is a continuously varying family of 'sets'.

That is much too informal to use as a definition as it has employed several terms that have not been defined. Before seeing how that intuition might be encoded more exactly, we will return to the 'spaces over B'. Let $\alpha : A \to B$ be a space over B as before, and, once again, let $U \subset B$ be an open set. This time we will not consider $\alpha^{-1}(U)$, but will look at *local sections of* α *over* U. A *(local) section* of α, over U is a continuous map $s : U \to A$ such that, for all $x \in U$, $\alpha s(x) = x$, that is, $s(x)$ is always in the fibre over x. We write $\Gamma_A(U)$ for the set of such local sections, although this notation *does not* record the all important map, α, in it.

If $V \subset U$ is another open set of B and $s : U \to A$ is a local section of α over U, then the restriction, $s|_V$, of s to V is a local section of α over V. We thus get, from $V \subset U$, an induced 'restriction' map

$$\mathsf{res}^U_V : \Gamma_A(U) \to \Gamma_A(V).$$

Of course, if $W \subset V$ is another such

$$\mathsf{res}^U_V \circ \mathsf{res}^V_W = \mathsf{res}^U_W.$$

There is a little teasing problem here. Suppose V is empty. Of course, the empty set is a subset of all the other open sets, so what should $\Gamma_A(\emptyset)$ be? The empty space is the initial object in the category of spaces so there is a unique map from it to A and, of course, this is a local section! (You can either check the condition at all points of the domain or argue that composition of this empty local section with the projection p yields the unique map from $\emptyset$ into B, as required.)

Back to the generalities, there is, again of course, a neat, and well known, categorical description of this setting.

Let *Open*(B) denote the partially ordered set of open sets of B with the usual order coming from inclusion, and consider it as a category in the usual way. The above construction just gave a functor

$$\Gamma_A : Open(B)^{op} \to Sets,$$

a *presheaf* on B. Any functor $F : Open(B)^{op} \to Sets$ is called a *presheaf*, but not all presheaves come from 'spaces over B' by the local sections construction, as it is fairly clear that Γ_A has some special properties, for instance, we saw that such a presheaf must send $\emptyset$ to the singleton set, but we also have the gluing property:

Suppose $s_1 \in \Gamma_A(U_1)$ and $s_2 \in \Gamma_A(U_2)$ are two local sections and

$$\mathsf{res}^{U_1}_{U_1\cap U_2}(s_1) = \mathsf{res}^{U_2}_{U_1\cap U_2}(s_2),$$

so these local sections agree on the intersection of their domains, then define

$$s : U_1 \cup U_2 \to A$$

by

$$s(x) = \begin{cases} s_1(x) & \text{if } x \in U_1 \\ s_2(x) & \text{if } x \in U_2. \end{cases}$$

It is easy to prove that s is continuous and so gives a local section over $U_1 \cup U_2$. We need not stop with just two local sections. If we have any family of local sections, over a family of open sets, that coincide on pairwise intersections, then they can be glued together, just as above, to give a unique local section on the union of those open sets, restricting to the given ones with which we started on their original domains. This gluing property is the defining property of the sheaves amongst the presheaves on B:

Definition: A presheaf, $F : Open(B)^{op} \to Sets$, is a *sheaf* if given any family, $\mathcal{U}$, of open sets of B, say $\mathcal{U} = \{U_i\}_{i\in I}$, and elements $s_i \in F(U_i)$ for $i \in I$, such that for $i, j \in I$ $\mathsf{res}^{U_i}_{U_i\cap U_j}(s_i) = \mathsf{res}^{U_j}_{U_i\cap U_j}(s_j)$, there is a *unique* $s \in F(U)$, for $U = \bigcup U_j$, such that $\mathsf{res}^{U}_{U_i}(s) = s_i$ for all i.

Query: Does this gluing property imply the normalisation condition that $F(\emptyset)$ is a singleton? **It does, but this is for you to investigate!**

Example and Definition: Let $\alpha : A \to B$ be a 'bundle', then, for U open in B, take $\Gamma_\alpha(U) = \{s : U \to A \mid \alpha s(x) = x \text{ for all } x \in U\}$, defines a presheaf on B. It is a sheaf. The functions, s, are called *local sections*, as before, and Γ_α is called the *sheaf of local sections of* α. (We will sometimes, as above, slightly abuse notation and write Γ_A instead of Γ_α, if the map α is unambiguous in the context.)

For later purposes and comparisons, we will note that a *compatible family, s_i, of local elements*, as above, gives an element $\underline{s}$ in the product set $\prod\{F(U_i) : i \in I\}$. Not just any family of elements however. We also have a product of the parts over the intersections. We write $U_{ij} = U_i \cap U_j$ and get a product $\prod\{F(U_{i,j}) : i, j \in I\}$. There are two functions, which we will call a and b for convenience only, defined from $\prod\{F(U_i) : i \in I\}$ to $\prod\{F(U_{ij}) : i, j \in I\}$. To specify these we see how they project onto the factors $F(U_{ij})$. (Technically, we have maps $\prod F(U_{ij}) \xrightarrow{p_{ij}} F(U_{ij})$, being the $\{ij\}^{th}$ projection of the product.) The specifications are

$$p_{ij}a(\underline{s}) = res^{U_i}_{U_{ij}}(s_i),$$

whilst

$$p_{ij}b(\underline{s}) = res^{U_j}_{U_{ij}}(s_i).$$

We can now give the compatibility condition as $\underline{s}$ is a compatible family of local elements exactly if $a(\underline{s}) = b(\underline{s})$:

$$\begin{tikzcd} Eq(a,b) \arrow[r] & \prod F(U_j) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \prod F(U_{ij}) \end{tikzcd},$$

i.e., $\underline{s}$ is in the *equaliser* $Eq(a, b)$ of a and b. This equaliser is sometimes called the *set of descent data for the presheaf relative to the cover*. It may be denoted $Des(\mathcal{U}, F)$.

From this perspective, we note that the restriction maps give a map,

$$c : F(U) \to \prod F(U_i),$$

with $p_i des(s) = res^U_{U_i}(s)$ and we know $a.c = b.c$. We thus get a function, des, from $F(U)$ to $Des(\mathcal{U}, F)$ assigning $des(s) := c(s)$ to s. We have F is a sheaf exactly when this map, des, is a bijection; it is a *separated presheaf* when this map is one-one, see below.

This scenario is quite useful for sheaves, but it really comes into its own when we look at higher dimensional analogues such as stacks.

We will note quite a lot of facts about sheaves and presheaves, but will not give a detailed development, since here is not a suitable place to give a lengthy treatment of sheaf theory.

7.3.2 Presheaves and sheaves

The category, $Sh(B)$, of sheaves on a space, B, is a reflective subcategory of the category, $Presh(B) = [Open(B)^{op}, Sets]$, of presheaves on B.

We first note a half-way house between general presheaves and sheaves.

A presheaf, F, is *separated* if there is at most one $s \in F(U)$ such that $res^U_{U_i}(s) = s_i$ for all i. ('Sheafness' would also require this, but, in addition, asks for the existence of such an s, not just uniqueness if it exists.) In fact:

The functors,

$$Sh(B) \to Sep.Presh(B) \to Presh(B),$$

have left adjoints.

If F is a presheaf, we will write $s(F)$ for the corresponding separated presheaf and $a(F)$ for the associated sheaf. We can give explicit constructions of $s(F)$ and $a(F)$.

- Define an equivalence relation $\sim_U$ on $F(U)$, where, if $a, b \in F(U)$, then $a \sim b$ if and only if $res^U_{U_i}(a) = res^U_{U_i}(b)$ for all i, then $s(F)$ given by $s(F)(U) = F(U)/\sim_U$ is a separated presheaf. (For you to check the presheaf structure.)

- Suppose F is separated (if not replace it by $s(F)$ and rename!) Form $F_{\mathcal{U}}$, the set of compatible families (relative to $\mathcal{U}$) of elements in the $F(U_i)$. If $\mathcal{V} < \mathcal{U}$ is a finer cover of U, (so for each $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ with $V \subseteq U$), then there is a function $res^{\mathcal{U}}_{\mathcal{V}} : F_{\mathcal{U}} \to F_{\mathcal{V}}$ where $res^{\mathcal{U}}_{\mathcal{V}}(\underline{s})_j = res^{U_i}_{V_j}(s_i)$ if $V_j \subseteq U_i$. (**Check** it is well defined.)

  Varying $\mathcal{U}$, we get a diagram of sets and form

  $$a(F)(U) = colim_{\mathcal{U}} F_{\mathcal{U}}.$$

  Explicitly we generate an equivalence relation on the union of the $F_{\mathcal{U}}$s by

  $$\underline{s}_{\mathcal{U}} \sim \underline{s}_{\mathcal{V}}$$

  if $\mathcal{V} < \mathcal{U}$ and $res^{\mathcal{U}}_{\mathcal{V}}(\underline{s}_{\mathcal{U}}) = \underline{s}_{\mathcal{V}}$, and then form the quotient.

(The details are well known and, if you have not met them before, should be checked or looked up, e.g. in a related context, [39], p. 268. The sort of constructions used will be useful throughout this chapter. It is a good idea to try to rewrite this in terms of the equaliser description given earlier, to see what is happening there.)

7.3.3 Sheaves and étale spaces

The category, $Sh(B)$, is equivalent to the category of étale spaces over B.

A continuous map, $f : X \to Y$, between topological spaces is *étale* if, for every $x \in X$, there is an open neighbourhood U of x in X and an open neighbourhood, V, of $f(x)$ in Y such that f restricts to a homeomorphism $f : U \to V$. We also say that X is an *étale space over* Y.

Given a presheaf, F, on B and $b \in B$, let

$$F_b = colim_{b \in U} F(U).$$

and $\mathsf{germ}_b : F(U) \to F_b$, be the natural map. The colimit is constructed using a disjoint union followed by using an equivalence relation. This germ map just send an element to its equivalence class. More precisely: the set, F_b, is the 'stalk' of F at b. It is made up of equivalence classes of 'germs' of *locally defined elements*, *i.e.*, (U, b, x), where b is the point at which we are looking, U is an open set with $b \in U$ and $x \in F(U)$. If (U, b, x_U) and (V, b, x_V) are two such germs, they are equivalent if there is a $W \subset U \cap V$, again open in B, such that

$$res^U_W(x_U) = res^V_W(x_V),$$

i.e., x_U and x_V agree 'near to b'. Now let $E(F) = \bigsqcup_{b\in B} F_b$ be the disjoint union with $\pi : E(F) \to B$, the obvious projection.

The topology on $E(F)$ is given by basic open sets: if $x \in F(U)$, $B(x) = \{\mathsf{germ}_b(x) \mid b \in U\}$ is to be open. (The idea is that we make x into a continuous local section of $E(F)$ over U by this means.) This makes $(E(F), \pi)$ an étale space over B.

We could construct $a(F)$ as $\Gamma_{E(F)}$, *i.e.*, the sheaf of local sections of $E(F)$.

7.3.4 Covering spaces and locally constant sheaves

A covering space is an étale space which is locally trivial, and it then corresponds to a locally constant sheaf on B.

For any set S, there is a constant sheaf on B, defined by the presheaf $F(U) = S$ for all $U \in Open(B)$. The corresponding étale space is $B \times S$ with its projection onto B, and where S is given the discrete topology. A sheaf is *locally constant* if for each $b \in B$, there is an open set, U_b, containing b such that the restriction of F to U_b is a constant sheaf or, more strictly speaking, is isomorphic to a constant sheaf. The category of locally constant sheaves on B and all sheaf morphisms between them will be denoted $LCSh(B)$.

We can rephrase this in a neat way that introduces viewpoints that will be useful later on. The open sets, U_b, give us an open cover of B, so we could pick a subcover with the same trivialising property. We thus assume that we have a cover $\mathcal{U}$ and form a space, $\bigsqcup \mathcal{U}$, by taking the disjoint union of the open sets in $\mathcal{U}$. (Recall that a convenient way of working with $\bigsqcup \mathcal{U}$ is to denote its elements by pairs (b, U) with $b \in U$ and $U \in \mathcal{U}$. We then have a copy of each b for each open set from the cover of which it is an element.) There is an obvious projection map

$$p : \bigsqcup \mathcal{U} \to B,$$

which is $p(b, U) = b$, and this is, fairly obviously, an étale map. We pull back F along p to get a sheaf on $\bigsqcup \mathcal{U}$ and, of course, this pulled back sheaf is constant.

This trick of turning a (topological) open cover into a map is very important. It forms the basis of the theory of Grothendieck topologies. In that theory, one replaces $Open(B)$ by a category, $\mathcal{C}$, so a presheaf on $\mathcal{C}$ is just a functor $F : \mathcal{C}^{op} \to Sets$. The sheaf condition is adapted to this setting by specifying what (families of) morphisms in $\mathcal{C}$ are to be considered 'covers' with an axiomatisation of their desired properties. For instance, for an open cover, $\mathcal{U}$ of B, if for each $U \in \mathcal{U}$, we pick an open cover of it and then combine these open covers together we get an open cover of B. That is mirrored by a condition on the covering families in the Grothendieck topology.

We will not treat Grothendieck topologies in great detail here as, once again, that might take us too far away from the 'crossed menagerie' and the related issues of cohomology. We will give a definition shortly. It *will* be necessary, however, to have such a definition of a Grothendieck topos, *i.e.*, the category of sheaves for such a Grothendieck topology and we will attempt to show how it relates to some of the topics we are considering. For greater detail from a very approachable viewpoint, the approach from Borceux and Janelidze's book, [39], is suggested, but we warn the reader that they also avoid very lengthy discussions of the topic, as their aim is not topos theory *per se*, but generalised Galois theory.

7.3.5 A siting of Grothendieck toposes

Definition: A *Grothendieck topos* is a category, $\mathcal{E}$, which is equivalent to a full reflective subcategory,

$$\mathcal{E} \underset{}{\overset{a}{\rightleftarrows}} [\mathcal{C}^{op}, Sets]$$

of a presheaf category, $Presh(\mathcal{C}) = [\mathcal{C}^{op}, Sets]$, where the left adjoint, a, preserves finite limits.

The reflective nature of this category means that when considering morphisms *from* a (pre)sheaf to a sheaf, it is enough to give them at the presheaf level, since they will automatically be sheafified.

We had early on in our discussion of sheaves, the statement:

The category, $Sh(B)$, of sheaves on a space, B, is a reflective subcategory of the category, $Presh(B) = [Open(B)^{op}, Sets]$, of presheaves on B.

We can now rephrase this as a proposition:

Proposition 75 *The category, $Sh(B)$, of sheaves on a space, B, is a Grothendieck topos.* ■

In addition to the category of sheaves on a space, B, we also have several other important examples of the notion.

Example: (i) For any $\mathcal{C}$, the presheaf category, $Presh(\mathcal{C})$, is itself a full reflective subcategory of itself! It thus is a Grothendieck topos.

In particular, the category, $\mathcal{S}$, of simplicial sets is a Grothendieck topos (by taking $\mathcal{C} = \mathbf{\Delta}$). Later we will consider sheaves and bundles of groups, *i.e.*, group objects in the topos of sheaves on a (base) space B. Equally well, we could look at group objects in presheaf toposes such as $[\mathcal{C}^{op}, Sets]$, and these are the group valued presheaves, and thus, in particular, *Simp.Grps* is just the category of presheaves of groups on $\mathbf{\Delta}$.

We can take this 'analogy' further. If we have an étale space, $\alpha : A \to B$, over B, then a local section is a map $s : U \to A$ for $U \in Open(B)$, such that $\alpha s(x) = x$ for all $x \in U$. A presheaf, $F : Open(B)^{op} \to Sets$, is thought of as having $F(U)$ as being the local sections over U of 'something' over B. That does not quite give an idea which is wholly expressed within the category of (pre)sheaves itself, as we needed to talk about U itself as well, but, from U, we can get a presheaf, much as above, namely the representable presheaf,

$$\hat{U} = Open(B)(-, U).$$

This presheaf takes value a singleton on V if $V \subseteq U$ and is empty otherwise. The inclusion of U into B is the étale map that corresponds to this, so our local section $s : U \to A$ is the analogue of, (in fact, corresponds exactly to), a map of presheaves,

$$s : \hat{U} \to \Gamma_A$$

and if $F : Open(B)^{op} \to Sets$ is arbitrary, $F(U) = Presh(B)(\hat{U}, F)$ by the Yoneda lemma, with each presheaf morphism φ from $\hat{U}$ to F yielding an element $\varphi_U(id_U) \in F(U)$. (Remember presheaf morphisms are merely natural transformations between the corresponding functors.)

Example: (ii) Another very important example of a presheaf topos, as above, comes from any group, G. We can, as we have done several times already, consider G as a one object groupoid, $G[1]$. It is then a suitable instance of a small category, which can be fed into the machine of the previous example. The category, $Presh(G[1])$, will be a Grothendieck topos, but what is the interpretation of these objects? From a straightforward perspective, they are set valued functors on $G[1]^{op}$. Suppose that $X : G[1]^{op} \to Sets$ is one such, then, abusing notation like mad, write $X = X(*)$ for the image of the single object, $*$, of $G[1]^{op}$, and if $g \in G$, and $x \in X$, write $X(g)(x) = x.g$, then (and this is *left to you*) we can easily check that X is a *right G-set*. Conversely any right G-set, gives a presheaf on $G[1]$ and this sets up an equivalence of categories. (You should also check on morphisms.) If you prefer left G-sets, replace G by the opposite group, G^{op}.

This example is important as it provides the bridge between the cohomology of groups and the cohomology of spaces via a cohomology of toposes. We will see the above argument several times in what follows. (Following the idea that the reader should be able to 'dip' into these notes, we may repeat the point again and again!)

Example: (iii) Any category with a Grothendieck topology on it leads to a Grothendieck topos. We need a definition.

Definition: A *Grothendieck topology* on a category, $\mathcal{C}$, is an assignment of families of 'coverings', $\{U_\alpha \to U\}_\alpha$ for each object U in $\mathcal{C}$ such that

- If $\{U_\alpha \to U\}_\alpha$ and $\{U_{\alpha\beta} \to U_\alpha\}_\beta$ are coverings, so is $\{U_{\alpha\beta} \to U\}_{\alpha\beta}$, *i.e.*, 'coverings of coverings are coverings';
- If $\{U_\alpha \to U\}_\alpha$ is a covering family and $V \to U$ is a morphism in $\mathcal{C}$, then the pullback family $\{U_\alpha \times_U \to V\}_\alpha$ is a covering family for V, *i.e.*, 'coverings are pullback stable';
- If $\{V \xrightarrow{\cong} U\}$ is an isomorphism, then this singleton family is a covering family.

A category together with a Grothendieck topology is called a *site*.

Given a site based on $\mathcal{C}$, a presheaf $F : \mathcal{C}^{op} \to Sets$ is called a *sheaf* on the site if for any object U and covering family $\{U_\alpha \to U\}_\alpha$, the sequence

$$F(U) \longrightarrow \prod F(U_\alpha) \rightrightarrows \prod F(U_\alpha \times_U U_\beta) \ ,$$

is an equaliser. (If the left hand morphism is merely injective then F will be a 'separated presheaf' in this context'.) The category of sheaves for a given site gives a Grothendieck topos.

Returning to the general case of $[\mathcal{C}^{op}, Sets]$, the Yoneda lemma shows the importance of the representable presheaves. In our key example with $\mathcal{C} = \mathbf{\Delta}$, these representable presheaves are just the simplices, $\Delta[n] = \mathbf{\Delta}(-, [n])$. Our observations above point out that if K is a simplicial set, $K_n = K[n] \cong \mathcal{S}(\Delta[n], K)$ and this is the analogue of $F(U)$, *i.e.*, the analogue of the set of local sections of F. Of course, there is no notion of topological continuity in the classical sense here, and as, in the 'presheaf topos' $\mathcal{S}$, all presheaves are sheaves, we have that in some sense 'all sections are as if they were continuous'. (The topological language is being pushed to breaking point here, so the corresponding intuitions would need refining if we were to follow them up properly. One *can* do this with the language of Grothendieck topologies, but we will not explore that further here. To some extent this is done in [39] with a different end point in mind. Here our purpose is to explain loosely why $\mathcal{S}$ is a topos, and why that may be useful and, reciprocally, what do the simplicial ideas, seen from that presheaf / sheaf viewpoint, suggest about general toposes.)

One further fact worth noting is that if $\mathcal{E}$ is a topos and B is an object in $\mathcal{E}$, then the 'slice category', $\mathcal{E}/B$, is also a topos. It thus is Cartesian closed, *i.e.*, not only does it have finite limits, but the functor $- \times A : \mathcal{E} \to \mathcal{E}$, which sends an object X to $X \times A$ for some fixed object A, has a right adjoint $(-)^A$ thought of as being the object of maps from A to whatever. General results can be found in the various books on topos theory, which give very general constructions of these mapping space objects in settings such as the slice toposes. We will need some elementary ideas about Cartesian closed categories later.

7.3.6 Hypercovers and covers

It is sometimes necessary to mention 'hypercovers', instead of 'covers' when looking at generalisations of sheaves.

In any topos $\mathcal{E}$, there is a precise sense in which $\mathcal{E}$ behaves like a generalisation of the category of sets, but with a logic that replaces the two truth values $\{0, 1\}$ of ordinary Boolean logic by a more general object of truth values. In the topos, $Sh(B)$, of sheaves on a space B, this truth value object is the lattice of open sets, $Open(B)$. This may seem a bit weird, but in fact works beautifully[2]. This allows one to do things like simplicial homotopy theory *within* $\mathcal{E}$. This replaces the category, $\mathcal{S}$, of simplicial sets by $Simp(\mathcal{E})$ and if $\mathcal{E} = Sh(B)$, then the objects are just simplicial sheaves on B, *i.e.*, sheaves of simplicial sets on B.

Any open cover $\mathcal{U}$ of a space B yields $\bigsqcup \mathcal{U}$, as before, and one can take repeated pullbacks to construct a simplicial sheaf on B from that cover. It is fun to view this in another way as it illustrates some of the ideas working within the topos $\mathcal{E}$ and, in particular, within $Sh(B)$.

Firstly, in *Sets*, there is a terminal object, 1, 'the one point set'. In a topos $\mathcal{E}$, there is a terminal object, $1_\mathcal{E}$, and, for $\mathcal{E} = Sh(B)$, this is the constant sheaf with value the one point set. Viewed as an étale space, it is just the identity map, $B \stackrel{id}{\to} B$. (This multitude of viewpoints may initially seem to lead to confusion, but it does give a beautifully rich context in which to work, with different intuitions and analogies interacting and combining.)

Within $\mathcal{E}$, we have a product, so if A_1, $A_2 \in \mathcal{E}$, we can form $A_1 \times A_2$. What does this looks like for $\mathcal{E} = Sh(B)$? The A_i gives étale spaces $\alpha_i : A_i \to B$, $i = 1, 2$ and $A_1 \times A_2$ corresponds to the pullback

$$A_1 \times_B A_2 \to B.$$

In particular, if $\mathcal{U}$ is an open cover of B, write $U \to 1$ for $\mathcal{U}$ viewed as a sheaf / étale space, $\bigsqcup \mathcal{U} \to B$, within $Sh(B)$, then the product

$$\begin{tikzcd} U \times U \arrow[r, shift left=2] \arrow[r, shift right=2] & U \arrow[l] \end{tikzcd}$$

[2] The logic is non-Boolean in general, so occasionally you need to take care with classical arguments.

makes U into a groupoid / equivalence relation within $\mathcal{E} = Sh(B)$. The simplicial object defined by multiple pullbacks is just the nerve of this groupoid, which will be denoted $N(U)$, or more often $N(\mathcal{U})$. In low dimensions[3] , this looks like

$$N(U): \quad \ldots \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} U \times \cdots \times U \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \cdots \overset{d_0}{\underset{d_2}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}}} U \times U \overset{d_0}{\underset{d_1}{\rightrightarrows}} U \xrightarrow{p} 1.$$

In the case when B is a manifold and $\mathcal{U}$ is an open cover by contractible open sets such that all the finite intersections of sets from $\mathcal{U}$ are also contractible (sometimes called a 'Leray cover', cf. [189]), the groupoid above is called a 'Leray groupoid', see the same cited paper.

In cases where B is not such a 'locally nice space', or if we replace $Sh(B)$ by a more general topos, the simplicial sheaf given by $\mathcal{U}$ is too far away from being an internal Kan complex and so we have to replace the nerve of a cover by a 'hypercover', which is a 'Kan' simplicial sheaf, K, with an 'augmentation map' $K \to 1$, which is a 'weak homotopy equivalence'. (Look up papers on hypercovers for a much more accurate treatment of them than we have given here.) Of course, this is very like the situation in group cohomology, where one starts with a 'resolution' of G. This is a resolution of B, or, better, of 1 by a simplicial object.

It will be useful later on to give a 'down-to-earth' description of the various levels of $N(\mathcal{U})$. The zeroth level $N(\mathcal{U})_0$ is just the sheaf $\mathcal{U} = \sqcup\{U : U \in \mathcal{U}\}$, or rather the local sections of this over B. A point in this étale space can be represented by a pair (b, U) where $b \in U$, *i.e.*, the point b of B indexed by U. The projection to B, of course, just sends (b, U) to b. This notation is one way of labelling points in a disjoint union, namely the point and an index labelling in which of the sets of the collection is it being considered to be for that part of the disjoint union. Now a point of the pullback over B will be a pair of such points with the same b, so is easily represented as (b, U_0, U_1) where (b, U_0) and (b, U_1) are both points in the above sense. This however implies that $b \in U_0 \cap U_1$, and here, and in higher levels, this idea works: a point in the multiple pullback occurring at level n is of the form $(b, U_0, \ldots, U_n)$, where $b \in \bigcap_{i=0}^{n} U_i$.

There is yet another useful point to make about this multiple way of considering an open cover as a sheaf (or a family or a simplicial sheaf or groupoid or étale space). It tells us what a morphism between open covers might be and hence what the category of open covers of a space B 'is'.

We will take a naive viewpoint (as that is often a good place to start), and then may refine it slightly *if*, or *as*, we hit problems. An open cover of a space B is a *family*, $\mathcal{U} = \{U_i \mid i \in I(\mathcal{U})\}$, of open sets of B, where we refer to $I(\mathcal{U})\}$ as the index set of the family. Of course, we need $\bigcup \mathcal{U} = B$ as well.

If $\mathcal{V}$ is another such covering family, then we would expect a map of coverings, $\alpha : \mathcal{V} \to \mathcal{U}$, to be a map of families. Here it will help to have a formal definition of the category of families in an abstract category, $\mathbb{A}$. (A good reference for this notion is chapter 6 of the book by Borceux and Janelidze, [39], that we have mentioned several times before.)

Definition: Let $\mathbb{A}$ be a category. A *family*, $\mathcal{A}$, of objects of $\mathbb{A}$ is a function, $\mathcal{A} : I(\mathcal{A}) \to Ob(\mathbb{A})$, from the index set, $I(\mathcal{A})$, of the family to the collection of objects of the category, $\mathbb{A}$. For a set, I, we say that $\mathcal{A}$ is an I*-indexed family* if $I(\mathcal{A}) = I$.

[3]In terms of étale spaces over B, you just replace $\times$ by $\times_B$ and 1 by B.

A morphism, $\alpha : \mathcal{A} \to \mathcal{B}$, of families consists of a map $I(\alpha) : I(\mathcal{A}) \to I(\mathcal{B})$ and an $I(\mathcal{A})$-indexed family of morphisms, $\{\alpha_i : A_i \to B_{I(\alpha)(i)}\}$. The category, $\mathsf{Fam}(\mathbb{A})$, is the category of such families and the morphisms between them.

An open covering, $\mathcal{U}$, of a space, B, is then a family in the category, $Open(B)$, of open sets of B and inclusions between them satisfying the condition $\bigcup \mathcal{U} = B$. This leads to a category, $Cov(B)$, of open coverings of B.

Remark: The above definition is very closely related to the idea of refinement of open coverings that one finds in classical treatments of Čech homology and cohomology, for instance, see Spanier, [250], and which we will look at in some more detail later on here (section 11.5.9). It is notable that to handle the constructions of these well, one has to take the relation of 'finer than' and chose a 'refinement map' which realises the relation in a more 'constructive' way. (The relations says that there is a function 'doing the job', the refinement map picks out one of the possible ones.) This is very like a situation we will meet many times later on. The classical approach asks for the *existence* of something, the more modern approach needs that something to be specified.

We have each open cover, $\mathcal{U}$, of our space B gives a sheaf, namely the *sheaf of local sections* of the étale space, $\bigsqcup \mathcal{U} \to B$. We note the following:

Lemma 48 *If $\mathcal{V}$ and $\mathcal{U}$ are open covers of a space B, then a morphism, α, from $\mathcal{V}$ to $\mathcal{U}$, induces a map of the corresponding étale spaces over the base B:*

$$\begin{tikzcd}
\bigsqcup \mathcal{V} \arrow[rr, "\alpha"] \arrow[dr] & & \bigsqcup \mathcal{U} \arrow[dl] \\
& B &
\end{tikzcd}$$

■

Of course, as you would expect, any such morphism will induce a morphism of the corresponding groupoids or simplicial sheaves.

We have to be a bit careful here, since if the sets in the covers are not connected, we could get maps between these étale spaces that did not correspond to morphisms of the covers. We will **leave you to explore this**, but also suggest looking at [39].

7.3.7 Simplicial approaches to descent data

For the sake of generalisations later in these pages, we will look at the relationship, for an open cover, $\mathcal{U}$, between $N(\mathcal{U})$ and $Des(\mathcal{U}, F)$. We will revisit this from time to time and, no doubt, will repeat ideas in the discussion, each time with a bit more generality or a bit more detail.

Suppose we have an open cover, $\mathcal{U}$, of a space, X, and a presheaf, $F : Open(X)^{op} \to Sets$ (or sometimes to a more general category, $\mathbb{A}$, say), then $\mathcal{U}$ gives us the simplicial (pre)sheaf $\mathsf{U} \to 1$ in $Sh(X)$, whilst F gives a presheaf of simplicial sets by considering each $F(U)$ as a constant simplicial set. We will continue to use F for this simplicial presheaf, although $K(F, 0)$ might be a more accurate notation.

As we now have two simplicial presheaves, it is a question of natural curiosity to look at the morphisms between them and, of the two possible directions for a morphism, one from $N(\mathcal{U})$ to F

would seem the more significant. Suppose $a : N(\mathcal{U}) \to F$ is a morphism of simplicial presheaves, then for an open subset, U, of X, we will have a map of simplicial sets,

$$a_U : N(\mathcal{U})_U \to F(U),$$

which we need to investigate. We adopt the idea of starting with 'elements' that we know *must* be in $N(\mathcal{U})_U$, and seeing where they are going to be mapped by a_U. Images of other simplices will, in fact, be determined by these.

If U is an open set, U_i, actually in the open cover, $\mathcal{U}$, then there is an evident 0-simplex in $N(\mathcal{U})_U$, namely the section, $s_U : x \mapsto (x, U)$. If U is a subset of some U_0 in $\mathcal{U}$, then we know that $(N(\mathcal{U})_U)_0$ will contain $s_U := res_U^{U_0}(s_{U_0})$. Such 0-simplices then 'generate' all the information on $(N(\mathcal{U})_U)_0)$, since if, for instance, $U \subseteq U_0 \cap U_1$, for $U_0, U_1 \in \mathcal{U}$, there are at least two such 0-simplices, $res_U^{U_0}(s_{U_0})$ and $res_U^{U_1}(s_{U_1})$. These simplices are, however, linked by a 1-simplex namely the restriction, to U, of the section $x \mapsto (x, U_0, U_1)$. Of course, if $U \subseteq U_0 \cap \ldots \cap U_n$, with the $U_i \in \mathcal{U}$, then there will be at least $n+1$ vertices, $res_U^{U_i}(s_{U_i})$, $i = 0, \ldots, n$ in $N(\mathcal{U})_U$, and they will form an n-simplex, namely the restriction to U of the local section, $x \mapsto (x, U_0, \ldots, U_n)$, of $N(\mathcal{U})_n$, defined on $U_0 \cap \ldots \cap U_n$.

We can now look at a_U and we will mirror our discussion above. For an open set, $U = U_0$ in the open cover $\mathcal{U}$, $a_{U_0}(s_{U_0})$ is an element of $F(U_0)$, whilst if $U \subseteq U_0$, then $a_U(s_U) = res_U^{U_0}(a_{U_0}(s_{U_0}))$. In the situation that $U \subseteq U_0 \cap U_1$, the intersection of two sets in $\mathcal{U}$, then we claim $res_U^{U_0}(a_{U_0}(s_{U_0})) = res_U^{U_1}(a_{U_1}(s_{U_1}))$, so $a_U(s_U)$ is unambiguous. The justification for our claim is that the 1-simplex joining $res_U^{U_0}(s_{U_0})$ and $res_U^{U_1}(s_{U_1})$ is sent by a_U to a 1-simplex of $F(U)$. That 1-simplex will be degenerate as $F(U)$ is a constant simplicial set, so its two ends are equal. We thus have that a_U gives us a compatible family of local sections , *i.e.*, , a descent datum. We can 'run this argument backwards' as well to show that $Des(\mathcal{U}, F)$ is naturally isomorphic to the set, $SimpPresh(X)(N(\mathcal{U}), F)$, of morphisms of simplicial presheaves from $N(\mathcal{U})$ to F. (In fact, this is the zeroth level of the *simplicial set*[4], $\underline{SimpPresh}(X)(N(\mathcal{U}), F)$, where

$$\underline{SimpPresh}(X)(N(\mathcal{U}), F)_n := SimpPresh(X)(N(\mathcal{U}) \times \Delta[n], F).$$

As, here, F is a constant simplicial presheaf, this is a constant simplicial set, just repeating the information in $SimpPresh(X)(N(\mathcal{U}), F)$, so it is not that interesting, *but* the corresponding simplicial set in the case in which we replace F by a possibly non-constant simplicial presheaf contains more information; see sections 7.4.5 and 7.5.5 for examples.)

7.3.8 Base change at the sheaf level

Changing the base induces a pair of adjoint functors.

It is often necessary to examine what happens when we 'change the base space' for our sheaves. Suppose X is a space and $Sh(X)$ the corresponding category of sheaves on X. We might have a subspace, A, of X, and ask for the relationship between $Sh(X)$ and $Sh(A)$, for instance: Is there an induced functor? In which direction? If so, when does it have nice properties? and so on. More generally, if $f : X \to Y$ is a continuous map, then we might seek to have some 'induced functors' between $Sh(X)$ and $Sh(Y)$.

[4]... and gives us an example of a simplicially enriched category

First take a look at presheaves in this spatial context[5], and so naturally we need to look at the behaviour of f on open sets. The partially ordered sets, $Open(X)$ and $Open(Y)$, can be thought of as categories as we already have done, and since continuity of f is just : if V is open in Y, then $f^{-1}(V)$ is open in X, f corresponds to a functor

$$f^t : Open(Y) \to Open(X),$$

the superfix, ^{t}, standing for 'transpose'. (You should **check functoriality**. It is routine.)

As a presheaf, F, on X is just a functor $F : Open(X)^{op} \to Sets$, we can precompose with $(f^t)^{op}$ to get a presheaf on Y, *i.e.*, we have a presheaf, $f_*(F)$. This is then given by $f_*(F)(V) = F(f^{-1}(V))$. If $\mathcal{V} = \{V_i\}$ is an open cover of V, then $f^{-1}(\mathcal{V}) = \{f^{-1}(V_i)\}$ is an open cover of $f^{-1}(V)$, so it is easy to check that, if F is a sheaf on X, $f_*(F)$ is a sheaf on Y. (An interesting exercise is to consider the inclusion, f, of a subspace, A, into Y and a sheaf F on A. What is the value of $f_*(F)(V)$ if $A \cap V = \emptyset$ and why?) The sheaf $f_*(F)$ is often called the *direct image* of F under f, but in some ways this is not that good a name for it as it is not really an 'image'.

The construction gives a functor,

$$f_* : Sh(X) \to Sh(Y),$$

and, clearly, if $g : Y \to Z$ as well, then $(gf)_* = g_* f_*$, whilst $(Id_X)_* = Id_{Sh(X)}$. (Note we are saying that f_* is a functor, but also that writing $Sh(f)$ for f_* would give us a 'sheaf category functor' going from spaces to categories. That is more or less true, but things are, in fact, richer and more complex than just this.) The richness of the situation is that f also induces a functor going in the other direction, that is from $Sh(Y)$ to $Sh(X)$. This is easier to see if we change our view of sheaves back from special presheaves to étale spaces over the base.

Suppose we have a space over Y, $p : A \to Y$, then we can form the pullback, $X \times_Y A$. This is, in fact, 'only specified up to isomorphism' as it is defined by a universal property[6]. (You should check up on this point if you are unsure, although we will discuss it in some more detail as we go along.) There is a 'usual construction' of it namely as a subspace of the product $X \times A$:

$$X \times_Y A = \{(x, a) \mid f(x) = p(a)\},$$

but note again this is not 'the' pullback, just a choice of representing object within the class of isomorphic objects satisfying the specifying universal pullback property - and we also need the structural maps $p_X : X \times_Y A \to X$ and $X \times_Y A \to A$ in order to complete the picture. Of course, for instance, $p_X(x, a) = x$. There is no canonical choice of pullback possible and the resulting coherence situation is the source of much of the higher dimensional structure that we will be meeting later.

We will find it useful to use the universal property more or less explicitly, so it may be good to recall it here:

We have a square

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "p_X"'] & A \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

[5] We will look at this initially for presheaves on spaces, but will need the greater generality of presheaves on small categories for later use.

[6] so perhaps saying 'a pullback' would emphasise that point, ..., but we will not do that.

such that (i) it commutes: $pf' = fp_X$, and (ii) given any object B and maps $q : B \to A$ such that $pg = qf$, then there is a *unique* morphism $\alpha : B \to P$ such that $p_X\alpha = q$ and $f'\alpha = g$.

We repeat that this property determines P, p_X and f' up to isomorphism only. Our construction of P as $X \times_Y A$ for the situation in the category of spaces shows that such a P exists, but does not impose any odour of 'canonisation' on the object constructed.

We next look at local sections of (P, p_X). We have an open set, U, in X, and a section, $s : U \to P$ of p, over U , so that $p_X s(x) = x$ for all $x \in U$. This means that s determines, and is determined by, a map from U to A, namely $f' \circ s$, such that $f(x) = pf's(x)$ for all $x \in U$. This looks a bit like a local section of $A \xrightarrow{p} Y$ over $f(U)$, but we do not know if $f(U)$ is open in Y. To make things work, we can take $f^*(F)(U) = colim\{F(V) : V \text{ open in } Y, f(U) \subseteq V\}$, so we have the elements of $f^*(F)(U)$ are germs of local sections of F, whose domain contains $f(U)$. (This seems intuitively the right ideas, but you should check that it works in that it gives us a sheaf on X, and, moreover, that it is functorial, giving us a functor,

$$f^* : Sh(Y) \to Sh(X),$$

which is the sheaf version of the pullback functor on presheaves. **See why it works yourself, but looks up the details in a sheaf theory textbook or in, say, the Stacks Project, chapters 6 and 7, [251, Tag 008C] and [251, Tag 00VC]**. Of course, warned by previous comments, you will want to check that if $g : Y \to Z$, $(gf)^*$ and f^*g^* will be naturally isomorphic, (but usually not 'equal'). This will be very important later on.

If $F \in Sh(X)$, the sheaf we have just constructed is variously called the *pullback of F along f*, the *inverse image sheaf* or, if f is the inclusion of a subspace into Y, the *restriction of F to X*. This construction is also said to lead to *induced sheaves* or sometimes *co-induced sheaves* depending on the style of terminology being used.

Now suppose $f : X \to Y$ and so we have

$$f_* : Sh(X) \to Sh(Y),$$

and

$$f^* : Sh(Y) \to Sh(X).$$

These functors must be related somehow! In fact if $F \in Sh(Y)$ and $G \in Sh(X)$, then

$$Sh(X)(f^*(F), G) \cong Sh(Y)(F, f_*(G)).$$

We sketch a bit of this, leaving the details to be looked for. Suppose $\varphi : F \to f_*(G)$ in $Sh(Y)$, then for an open set V in Y, we have

$$\varphi_V : F(V) \to G(f^{-1}(V)).$$

Now suppose U is open in X and $V \supseteq f(U)$, then $f^{-1}(V) \supseteq U$, so we have

$$F(V) \xrightarrow{\varphi} G(f^{-1}(V)) \to G(U),$$

and, passing to the colimit, we get a map from $f^*(F)(U)$ to $G(U)$. The other way around is similar, so is left for you to worry out for yourselves.

Of course, the above natural isomorphism says 'f^* is left adjoint to f_*', and this implies a lot of nice properties that are often used. In fact this passage from continuous maps to pairs of adjoint functors on the corresponding sheaf categories, is abstracted in the notion of *geometric morphism* of toposes.. As usual, we will not follow this up here, as there are excellent introductions and in depth treatments of geometric morphisms available 'in the literature' and 'online'.

There is one more '(very) useful fact' about f^*. It is left exact, *i.e.*, it preserves finite limits[7]. This is, again, not that hard to prove, but is **left to the reader to prove or to check up on in the literature.**

This makes for quite a lot of 'facts' about sheaves and their uses, but there is one more observation to make before passing to other things. Often geometric information is encoded by a sheaf, sometimes 'of rings', sometimes 'of modules' or 'of chain complexes'. For instance, on a differential manifold, one has a sheaf of differential functions and also the de Rham complex, which is a sheaf of differential graded algebras. In algebraic geometry, the usual basic object is a scheme, which is a space together with a sheaf of commutative rings on it that is 'locally' like the prime spectrum of a commutative ring. There are many other examples. We will also be looking at sheaves of groups and sheaves of crossed modules.

It would have been nice to show how a sheaf theoretic viewpoint provides the link between covering space theory and Galois theory, but again this would take us a little too far afield, so we refer to Borceux and Janelidze, [39], and the references therein.

This is part of a much wider perspective, which is related to the passage from spaces to more general settings such as (Grothendieck) toposes[8]. We have already seen the idea of a presheaf on a small category $\mathcal{C}$, as being a functor $S : \mathcal{C}^{op} \to Sets$, and we set $Presh(\mathcal{C})$ to signify the category of functors, so denoted $Sets^{\mathcal{C}^{op}}$, or $[\mathcal{C}^{op}, Sets]$. Of course, the arguments we have given above adapt to show that any functor $F : \mathcal{C} \to \mathcal{D}$ will induce a functor

$$F^* : Presh(\mathcal{D}) \to Presh(\mathcal{C})$$

by $F^*(S) = S \circ (F^{op})$. In the next two sections, we will see that F^* has adjoints on both sides, given by the Kan extensions along F.

7.3.9 Pullback and pushforward in the spatial case

Let us put ourselves initially in a more general setting, so as to give a fairly elementary discussion of pullbacks and 'pushforwards'. We will work in a fairly general category, $\mathcal{C}$, but with the proviso that the constructions we will use exist whenever they are used[9]! Suppose we have a morphism, $f : X \to Y$ and two objects $p_A : A \to X$ and $p_B : B \to Y$, then we have a set, $\mathcal{C}/f(p_A, p_B)$, of morphisms, $f' : A \to B$, which 'cover' f, *i.e.*, so that the square,

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "p_A"'] & B \arrow[d, "p_B"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

[7]This is an essential part of the notion of 'geometric morphism' that we just mentioned.
[8]and eventually to ∞-groupoids.
[9]The existence of finite limits is typically all that is needed.

commutes,and so $p_B f' = f p_A$. We immediately have that $\mathcal{C}/f(p_A, p_B)$ can also be described as $\mathcal{C}/Y(fp_A, p_B)$, *i.e.*, we can think of the above as

$$\begin{tikzcd} A \arrow[rr, "f'"] \arrow[dr, "fp_A"'] & & B \arrow[dl, "p_B"] \\ & Y & \end{tikzcd}$$

More or less dually, we could take the pullback of p_B along f to get an object, $f^{-1}(B) \xrightarrow{f^{-1}(p_B)} X$, over X and

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "p_A"'] & & f^{-1}(B) \arrow[dl, "f^{-1}(p_B)"] \\ & X & \end{tikzcd}$$

uniquely determining $A \to f^{-1}(B)$, so a description of $\mathcal{C}/f(p_A, p_B)$ as $\mathcal{C}/X(p_A, f^{-1}(p_B))$. We thus have two functors

(i) $f_* : \mathcal{C}/X \to \mathcal{C}/Y$, given by 'post-composition with f',

and

(ii) $f^* : \mathcal{C}/Y \to \mathcal{C}/X$, given by pullback, *i.e.*, by 'pullback' along f,

and f^* is left adjoint to f_*. The proof of that last statement is constructed by just putting the two descriptions of $\mathcal{C}/f(p_A, p_B)$ together.

If we now shift attention to presheaves[10] on X and Y, we can not immediately apply the above as presheaves need not correspond to local sections of 'spaces over' some fixed space, but we saw that, as $Presh(X)$ is just $Sets^{Open(X)^{op}}$, the 'preimage functor':

$$f^t : Open(Y) \to Open(X),$$

$$f^t : V \mapsto f^{-1}(V) = \{x \mid f(x) \in V\},$$

induces, by pre-composition, a functor,[11]

$$f_* : Presh(X) \to Presh(Y),$$

so $f_* = Sets^{f^{t\,op}}$. This 'push-forward' functor has a left adjoint, f^{-1}, given by *left Kan extension* along f^t. For the moment, we will not give a detailed account of Kan extensions. Good introductions are available in standard category theory texts such as Mac Lane's [192], or Borceux, [37], section 3.7. We will look at them a little in the next section and then later, starting in section 13.3.1, page 615, but, at present, it will suffice to say that given a presheaf, F, on Y, there is a diagram:

$$\begin{tikzcd} Open(Y)^{op} \arrow[r, "F"] \arrow[d, "f^t"'] & Sets \\ Open(X)^{op} \arrow[ur, dashed, "Lan_{f^t}F"'] & \end{tikzcd}$$

[10] of sets, but really that choice is just for the purposes of the exposition,

[11] More generally, of course, any functor, $F : \mathbb{A} \to \mathbb{B}$, of (small) categories induces $Sets^{F^{op}} : Sets^{\mathbb{B}^{op}} \to Sets^{\mathbb{A}^{op}}$.

and the left Kan extension is a functor, $Lan_{f^t}F$, as shown, together with a natural transformation, $\eta_F : F \Rightarrow (Lan_{f^t}F) \circ f^t$, having nice universal properties. Such a $Lan_{f^t}F$ always exist (and we will see generalisations of this later, together with explicit means of constructing such things using colimits, in the next section, and their generalisation to 'coends') and we take $f^{-1} : Presh(Y) \to Presh(X)$, defined by $f^{-1}F = Lan_{f^t}F$. It is standard that f^{-1} is left adjoint to f_*. We have

$$Presh(X)(f^{-1}F, G) \cong Presh(Y)(F, f_*G).$$

Now if G is a sheaf on X, then, after checking that f_* preserves sheaves, the inverse image, or pull-back functor f^{-1} *on sheaves* is clearly going to be 'left Kan extension' followed by 'sheafification'. The detailed argument is worth following up in the literature cited above[12].

That description as 'left Kan extension followed by sheafification' is exactly the one we gave earlier of $f^*(F)$ in terms of a colimit.

We will usually be a bit sloppy with notation here, but sometimes it may be necessary to distinguish between the left Kan extension (giving a presheaf) and the 'left Kan extension' followed by sheafification'. If that is the case, we will write $f_p^{-1}(F)$ for the presheaf version, with $f^{-1}(F)$ as the sheafified one; see the Stacks Project, [251, Tag 008C].

Important special cases: Some special cases of this sort of adjoint pair will be quite important later on[13].

Recall that, for $f : X \to Y$, we have the adjoint functors

$$f_* : Sh(X) \to Sh(Y),$$

and

$$f^* : Sh(Y) \to Sh(X),$$

and for U an open set of X, $f^*(F)(U) = colim\{F(V) : V \text{ open in } Y, f(U) \subseteq V\}$, so we have the elements of $f^*(F)(U)$ are germs of local sections of F, whose domain contains $f(U)$.

(i) Suppose we have a space, X, a point $x \in X$ and, eventually, a sheaf, F, on X. The point, x, can be thought of as a map, $x : \top \to X$, from a singleton space, $\top = \{*\}$, to X, picking out or 'naming' the point x. This gives the pair of adjoint functors:

$$x_* : Sh(\top) \to Sh(X),$$

$$x^* : Sh(X) \to Sh(\top).$$

A sheaf on $\top$ is really just a set, so we have $x_* : Sets \to Sh(X)$, which sends a set to the 'skyscraper sheaf' with that set as its value above (the closure of) $\{x\}$, and is a singleton set otherwise[14].

For the inverse image functor, we have

$$x^* : Sh(X) \to Sets,$$

[12]It is also given in the article on 'inverse image' in the nLab, [221].

[13]We will give these for sheaves, although they can also be important just at the presheaf level.

[14]In the spaces that we will be considering points will 'usually be closed', *i.e.*, each singleton $\{x\}$ will be a closed set, but in general this may not be the case. For more on skyscraper sheaves, see the nLab, [221], or the Stacks Project, [251, Tag 0099].

and our description of $f^*(F)$, in general, gives that the elements of $x^*(F)(U)$ will be germs of local sections of F, whose domains contain $x(U)$. Here U is an open set in $\top$, so is either empty and in that case its image certainly does not contain x, or is the whole space $\top$, and then $x(U)$ is just the singleton set, $\{x\} \subseteq X$ and hence the *set* corresponding to $x_*(F)$ in $Sh(\top)$ is the *stalk*, F_x, of F at x.

(ii) This time, again, consider the terminal space[15], $\top = \{*\}$ and a general space X, but reverse the roles, so we look at the unique continuous map, $!_X : X \to \top$. We get adjoint functors,

$$(!_X)_* : Sh(X) \to Sh(\top) \cong Sets,$$

$$(!_X)^* : Sh(\top) \to Sh(X).$$

As you might guess, $(!_X)_*$ is the functor giving for a sheaf, S, on X, the set, $S(X)$, of *global sections*[16] of S, whilst $(!_X)^*$ assigns to a set, A, the constant sheaf on X having value A; cf. section 7.3.4.

(iii) Suppose now that $f : A \to X$ is the inclusion of an open subspace. Looking at the adjoint functors, $f_* : Sh(A) \to Sh(X)$ and $f^* : Sh(X) \to Sh(A)$, the first thing to note is that, for S, a sheaf on A, and V an open set of X

$$f_*(S)(V) = S(f^{-1}(V)) = S(A \cap V).$$

We note that this works independently of whether or not A is open in X. we thus have $f_*(S)(V) = S(V)$ if $V \subseteq A$, but is a singleton if $V \cap A$ is empty. (This latter fact might not be the case if S had been a presheaf that was not a sheaf, but remember that $S(\emptyset)$ is always a singleton set[17].)

Next consider a (pre)sheaf, T, on X, then $F^*(T)$ will be the presheaf obtained by taking, for U in $Open(A)$, the colimit of the $T(V)$ as varies amongst the open sets of X that contain U, but, as A is itself open in X so is U and that colimit reduces just to $T(U)$, *i.e.*,

- $f^*(T)$ is the presheaf given by $f^*(T)(U) = T(U)$, so is clearly, and by definition, the restriction, $T|_A$, of T to A. If T is a sheaf, so is $T|_A$, and, for any point $a \in A$, the stalk of $T|_A$ at a is 'the same as' that of T at a (considered as a point of X).

- The composite functor, $f^* f_*$ is the identity in both the presheaf and the sheaf cases.

If $f : A \to X$ is an open inclusion, we thus have, not only, functors

$$f_* : Sh(A) \to Sh(X),$$

and

$$f^* : Sh(X) \to Sh(A),$$

with $f^* \dashv f_*$, but that $Sh(A)$ behaves as a reflective subcategory of $Sh(X)$. In fact, there is another functor $f_! : Sh(A) \to Sh(X)$, which is left adjoint to the restriction functor f^*, so

$$f_! \dashv f^* \dashv f_*.$$

[15] This is the terminal object in the category of topological spaces.

[16] In terms of the 'sheaves as étale spaces' viewpoint; cf. section 7.3.3, $S(X)$ is exactly the set of sections, $s : X \to E(S)$, that is, 'local' sections with domain the whole space.

[17] or more generally a terminal object if one is looking at $\mathbb{A}$-valued sheaves.

This works for both presheaves and sheaves and we will distinguish those cases by denoting[18] the presheaf case by $f_{p!}$ and the sheaf case by $f_!$, omitting the suffix, p.

Suppose that T is a presheaf (of sets) on A, the *extension of T by the empty set* is a presheaf given by

$$f_{p!}(T)(V) = \begin{cases} \emptyset & \text{if } V \nsubseteq A, \\ T(V) & \text{if } V \subseteq A. \end{cases}$$

When we are considering sheaves then $f_{p!}(T)$ will need sheafifying to get the correct object, $f_!(T)$.

In fact, this is still relevant when looking at general presheaves, but of course the description in terms of 'open inclusions' will need reinterpreting.

7.3.10 Change of base for general presheaves

To continue the discussion from the previous section, and, as we have said several times, the study we made of set valued presheaves on spaces and their interaction with continuous functions *between* spaces, this is, of course, just a special case of a much more general situation in which $Open(X)$ is replaced by a general (small) category, $\mathcal{C}$, say, a continuous map, $f : X \to Y$, by a general functor, $F : \mathcal{C} \to \mathcal{D}$, and in which a presheaf on $\mathcal{C}$ is a functor, $S : \mathcal{C}^{op} \to Sets$, or, eventually, one with codomain a category, $\mathbb{A}$, which is 'suitably structured'. Although the spatial context is important, the general case of $Presh(\mathcal{C}) = [\mathcal{C}^{op}, Sets]$ is fundamental when going beyond spaces to toposes[19], but first we must go back to the basic simple categorical setting. We will give a fairly standard treatment of base change for set-valued presheaves. To keep the context together we may repeat some ideas from earlier for convenience. The description will be quite brief, but some detail is needed as it will form the basis for base change of fibred categories slightly later on and it is useful to have a place in the notes where this is given.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor with $\mathcal{C}$ and $\mathcal{D}$, small categories, then pre-composition gives a functor

$$F^* : Presh(\mathcal{D}) \to Presh(\mathcal{C}),$$

in which, for S in $Presh(\mathcal{D})$, $F^*(S) = S \circ F : \mathcal{C}^{op} \to Sets$. (We note that often in the literature, as here, the superfix 'op' is left off the symbol for a functor if there is little chance of confusion, so 'really' $F^*(S) = S \circ F^{op}$. We will often follow this tradition for convenience of type-setting, but it can sometimes lead to a slight degree of confusion!)

This functor has both a left and a right adjoint, given by Kan extensions. We will concentrate, initially, on the left adjoint which is given by a colimit. (Think back to the previous section and the spatial case.) If $V \in Ob(\mathcal{D})$, consider the comma category, denoted V/F or $V \downarrow F$, so the objects are pairs, (U, φ), where $U \in Ob(\mathcal{C})$ and $\varphi : V \to F(U)$ and

$$(V \downarrow F)((U, \varphi), (U', \varphi')) = \{f : U \to U' \mid F(f) \circ \varphi = \varphi'\}.$$

Remark: In many geometric situations, and in particular types of quite general settings[20] as well, the category $V \downarrow F$ is *filtering*, *i.e.*, it behaves very much like a directed poset. This is so, for

[18]following the convention of, say, the Stacks Project

[19]... and even further to intuitions behind ∞-toposes and other ∞-categorical analogues of these structures.

[20]see below

instance, in the spatial case we looked at earlier as the category $V \downarrow F$ is then that of the open sets containing $f(U)$ and not only is that a poset under inclusion, but, given two such, their intersection if another such. This is very useful if one needs that the various constructions preserve structure[21]. This important aspect is examined in the Stacks Project's treatment of this theory; see [251, Tag 00VC]. We will not need this, at least not yet, and will keep things more general. We would remind readers that the theory of Kan extensions, which is what we will be using, is revisited in section 13.3.1, starting on page 615, and, in any case, is handled thoroughly at about the level that we need, in standard category theory texts such as Mac Lane's [192], and Borceux, [37], section 3.7. as well as in Cordier and Porter, [89], where it is applied in a related context to that which we will be studying.

The category, $V \downarrow F$, comes with a functor, $cod_V : V \downarrow F \to \mathcal{C}$, given by $cod_V(U, \varphi) = U$, etc., and, moreover, if $g : V' \to V$ is a morphism in $\mathcal{D}$, then it induces a functor, $g \downarrow F : V \downarrow F \to V' \downarrow F$, by pre-composition, so sending (U, φ) to $(U, \varphi \circ g)$, which is clearly compatible with the codomain functors, cod_V and $cod_{V'}$.

Given a presheaf, S, on $\mathcal{C}$, we define $F_*(S) : \mathcal{D}^{op} \to Sets$ by, of V in $\mathcal{D}$,

$$F_*(S)(V) = colim\big((V \downarrow F)^{op} \xrightarrow{cod_V} \mathcal{C}op \xrightarrow{S} Sets,$$

and similarly on morphisms, using the universal property[22] for colimits[23].

That F_* is left adjoint to F^* is a consequence of the Kan extension context, so we refer to the categorical sources already mentioned for this. Of course, just to repeat, if $f : X \to Y$ is a continuous map, then $f^t : Open(Y) \to Open(X)$ can be taken as a suitable F and we obtain the theory and constructions we saw earlier[24] .

7.3.11 Change of index and geometric morphisms

Referring to $[\mathcal{C}^{op}, Sets]$ as 'presheaves in $\mathcal{C}$', although perfectly correct, presupposes that that presheaf viewpoint is the most appropriate one, yet often one is handling just 'diagrams of sets. or other objects, indexed by some small category, $\mathcal{C}$. If, for instance, $\mathcal{C}$ is discrete, and so is 'just a set' then the presheaf point of view is not in the foreground, yet the idea of 'change of base' still makes sense, although, as the objects in $[\mathcal{C}^{op}, Sets]$ are just $\mathcal{C}$-indexed families of sets, 'change of indexing' might be a better term to use. Even in that discrete case, this change of indexing along a function can be a very useful operation. More generally one thinks of $[\mathcal{C}, Sets]$ as a category of $\mathcal{C}$-indexed diagrams of sets. This is an important area on its own related to the idea of indexed category theory[25].

We will start the section however with the definition of geometric morphism, which is important when looking at Grothendieck toposes, before looking at two results the first of which is the general form of 'changing the indexing on a category of diagrams', whilst the second looks at a particular case which gives geometric morphisms between presheaf categories.

[21] such as exact sequences in categories of presheaves of modules

[22] Of course, $F_*(S) = Lan_F S$, the left Kan extension of S along F.

[23] or for left Kan extensions!

[24] Confusion about which functor is with an asterisk as superfix and which as a suffix is handled, in the usual way, by the rule of thumb that superfices relate to contravariant / morphism reversing constructions with the suffices relating to (covariant) constructions that preserve the direction of morphisms.

[25] In general, an indexed category is a pseudo-functor from $\mathcal{C}^{op}$ to Cat, and so gives a fibration on $\mathcal{C}$, but the perspective of indexed category theory concentrates on a slightly different set of results.

We have seen that a continuous map, $f : X \to Y$, induces an adjoint pair, (f^*, f_*), of functors on the categories of sheaves. We have noted that a Grothendieck topos is a generalisation of a category of sheaves on a space, so the following defines the corresponding generalisation of a continuous map.

Definition: Let $\mathcal{E}$ and $\mathcal{F}$ be toposes, then a *geometric morphism*, $f : \mathcal{E} \to \mathcal{F}$, is a pair of adjoint functors, $f_* : \mathcal{E} \to \mathcal{F}$, $f^* : \mathcal{F} \to \mathcal{E}$, $f^* \dashv f_*$, such that the left adjoint preserves finite limits.

Extending terminology from the case of sheaves on spaces, we say that f_* is the *direct image functor*, whilst f^* is the *inverse image functor* of the geometric morphism, f.

Remark: We have seen that f^* sometimes itself has a left adjoint $f_!$ and in this case f is said to be an *essential geometric morphism.*

We next record the general situation with diagrams of sets[26].

Let $\mathcal{A}$ and $\mathcal{B}$ be two small categories, $[\mathcal{A}, Sets]$ and $[\mathcal{B}, Sets]$ be the corresponding categories of $\mathcal{A}-$ (resp. $\mathcal{B}-$) indexed diagrams. Given a functor, $f : \mathcal{A} \to \mathcal{B}$, there is the usual induced functor, $f^* : [\mathcal{B}, Sets] \to [\mathcal{A}, Sets]$, induced by pre-composition and which we have met many times now. We have also hinted at the following:

Proposition 76 *Every such functor, f induces an adjoint triple:*

$$f_! \dashv f^* \dashv f_*.$$

Proof: Categories of set valued diagrams have all limits and colimits, so both left and right Kan extensions along f exist. We take $f_! = Lan_f$ and $f_* = Ran_f$.

(As before, we **leave it up to the reader** to search out the definitions, and the basic theory of Kan extensions. We will visit that again later (see page 615). Standard detailed references for this theory in the literature include Mac Lane, [192], and Borceux, [37], section 3.7, and the relevant pages of the nLab, [221].) ■

As we have said, we have recorded this result in the language of diagrams indexed by the two categories, but, of course, one of the main applications is obtained when the two categories are thought of as being the duals of some other categories as then we are back in the domain of presheaves, so taking $\mathcal{A} = \mathcal{C}^{op}$, etc.

If $\mathcal{C}$ and $\mathcal{D}$ have an extra property, namely that they have finite limits, and we also have that $f : \mathcal{C} \to \mathcal{D}$ preserves those finite limits, then between the presheaf categories, $[\mathcal{C}^{op}, Sets]$ and $[\mathcal{D}^{op}, Sets]$, the adjoint triple induced by f has nice properties. We can relate this back to the topological situation of an inclusion, $A \xrightarrow{f} X$, and in the construction of $f_!$, the objects of interest are the open sets, V containing $f(A)$. The intersection of two such open sets is another such, and the intersection acts like a binary product on $Open(X)$. That category also has a terminal object, namely X itself. We quote the result without proof (as there are many detailed proofs in the literature).

[26] although it is easy to extend to a more general case with diagrams in a 'suitably structured' category.

Proposition 77 *Let $\mathcal{C}$ and $\mathcal{D}$ have finite limits and $f : \mathcal{C} \to \mathcal{D}$ preserve those finite limits, then in the adjoint triple, $f_! \dashv f^* \dashv f_*$, the left Kan extension $f_!$ also preserves finite limits and so f^* is the direct image part of a geometric morphism*

$$f_! \dashv f^* : Presh(\mathcal{D}) \to Presh(\mathcal{C}).$$

■

7.3.12 Change of base and descent data

It is natural to ask how 'change of base' interacts with descent data. Suppose, as before, we have $f : X \to Y$ and a presheaf, F on X. If $\mathcal{V}$ is an open cover of Y, then (i) we can form $f^{-1}(\mathcal{V})$, which will be an open cover of Y, and we can look at $Des(f^{-1}(\mathcal{V}), F)$, but, alternatively, (ii) we have $f_*(F) = F \circ (f^{-1})^{op}$, which is a presheaf on Y, and so could form $Des(\mathcal{V}, f_*(F))$. How are these two sets of descent data related?

In the classical treatment of Čech homology, a continuous map, $f : X \to Y$, induces a morphism of simplicial sets, $N(X, f^{-1}(\mathcal{V})) \to N(Y, \mathcal{V})$, given by

$$N(f) = N(f, \mathcal{V}) : \langle (f^{-1}(V_0), \ldots, f^{-1}(V_n) \rangle \mapsto \langle V_0, \ldots, V_n \rangle.$$

Note that this makes sense since if $\langle (f^{-1}(V_0), \ldots, f^{-1}(V_n) \rangle$ *is* an n-simplex of $N(X, f^{-1}(\mathcal{V}))$, then there is some $x \in \bigcap_{i=0}^n f^{-1}(V_i)$, which then implies that $f(x) \in \bigcap_{i=0}^n V_i$, so what is 'on the right' does make sense.

We, of course, need the simplicial *sheaf* version of this. For this, it pays to look at $\mathcal{V}$ as $\mathsf{V} \to 1$ *i.e.*, as the corresponding morphism of (pre)sheaves, in the notation we have been using *within* $Sh(Y)$, or as $V = \coprod \mathcal{V} \to Y$, thinking of 'sheaves on Y' as 'étale spaces over Y'. The situation is thus

$$\begin{tikzcd} f^{-1}V \arrow[r, dashed] \arrow[d, dashed] & V \arrow[d, "p_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

in which the square is a pullback. We perhaps need to explore $f^{-1}V$ a little to see how it corresponds to $\coprod f^{-1}\mathcal{V}$. Of course, a pullback is determined only up to isomorphism, but we do have a clear representing object namely $X \times_Y V$, *i.e.*, a fibred product', thus identifying $f^{-1}V$ as consisting of pairs, $(x, (y, V_0))$ with $f(x) = p_V(y, V_0) = y$, that is, $f^{-1}(V_0)$ is given by pairs, (x, V_0), where $f(x) \in V_0$. This is to be compared with $\coprod f^{-1}(\mathcal{V})$, in which the elements are pairs, $(x, f^{-1}(V_0))$, with $x \in f^{-1}(V_0)$, so ...!

The corresponding simplicial sheaf, $N(f^{-1}(\mathcal{V}))$ is thus the obvious one to use. If we note, however, that $N(\mathcal{V})$ is a (simplicial) sheaf on Y, then we could also think about using $f^*(N(\mathcal{V}))$, applying f^* to each level of this simplicial sheaf. We note that, on reverting to the 'internal notation' of $Sh(Y)$, in which we use $\mathsf{V} \to 1$ for $\coprod \mathcal{V} \to Y$, etc., then, again within $Sh(Y)$, $N(\mathsf{V})_n = \mathsf{V}^{(n+1)}$, the $(n+1)$-fold power of V, we can note that f^* is left exact, so not only is $f^*(\mathsf{V}) \to 1$ the correct internal notation for $\coprod f^{-1}(\mathcal{V}) \to X$, but $f^*(\mathsf{V}^{(n+1)}) \cong (f^*(\mathsf{V}))^{(n+1)} \cong (f^{-1}(\mathsf{V}))^{(n+1)}$ for all n, as f^* preserves finite limits. It is thus easy to check that $N(f^{-1}(\mathsf{V}))$ and $f^*(N(\mathsf{V}))$ are isomorphic simplicial sheaves. Given our earlier identification of $Des(\mathcal{U}, F)$ in terms of morphisms of simplicial sheaves, this has as an immediate consequence that

$$\begin{aligned} Des(\mathcal{V}, f_*(F)) &\cong SimpPresh(Y)(N(\mathcal{V}, f_*(F)) \\ &\cong SimpPresh(X)(f^*N(\mathcal{V}), F) \\ &\cong SimpPresh(X)(N(f^{-1}(\mathcal{V})), F) \\ &\cong Des(f^{-1}(\mathcal{V}), F), \end{aligned}$$

so the two options for the set of descent data are in natural bijective correspondence.

The situation for $Des(\mathcal{U}, f^*(G))$ would be much more difficult to give in general as f^* is usually *not* a right adjoint, so easy 'playing with' the various adjunction isomorphisms, pushing around symbols, is not available as a method of analysis. Things are simpler sometimes if f has special properties, for instance, if it is an open map, as $f(U)$ will then be open for each U in $\mathcal{U}$, but these situations are a bit too far away from our current lines of exploration to look at here although they are closely related to cohomology in algebraic geometric settings.

7.4 Descent: Torsors

(Some of the best sources for the material in this section are in the various notes and papers of Breen, [48, 49] and, in particular, his Astérisque monograph, [50] and his Minneapolis notes, [51].)

The demands of algebraic geometry mean that principal G-bundles for G, a (topological) group, are not sufficient to handle all that one would like to do with such things. One generalisation is to vary G over a base. This may be to replace G by a sheaf of groups, or by a group object in Top/B, *i.e.*, a group bundle. (This is the topological analogue of a group scheme.) The situation that we considered earlier then corresponds to a constant sheaf of groups or the group bundle $G_B := (B \times G \to B)$ given by projection from the product. It also includes the vector bundles that we briefly saw earlier. The more general case, however, does not change things much. We have a parametrised family of groups G_b, $b \in B$, acting on a parametrised family of spaces, X_b, $b \in B$. The sheaf of groups viewpoint corresponds to an étale space on B and thus to a group bundle on B with each G_b discrete as a topological group. We will let, in the following, G be a bundle of groups on a space B. (We will on occasion abuse notation and write G instead of G_B for the 'constant G' example.)

Technically we will need to be working in a setting where we can talk of a bundle of locally defined maps from one bundle to another. This is fine in the sheaf theoretic setting, and will be assumed to be the case in the general case of a suitable category of bundles within the ambient category, Top/B. It corresponds to the functor $- \times A$ always having a right adjoint $(-)^A$, the function bundle of locally defined maps from A to whatever. Terminologically, we are assuming that our category of bundles on B, Bun/B is a *Cartesian closed category.*

7.4.1 Torsors: definition and elementary properties

Definition: A left *G-torsor* on B is a space $P \xrightarrow{\pi} B$ over B together with a left group action

$$G \times_B P \to P$$

$$(g,p) \longmapsto g.p$$

such that the induced morphism

$$\phi : G \times_B P \to P \times_B P$$

$$(g,p) \longmapsto (g.p, p)$$

is an isomorphism. In addition we require that there exists a family of local sections, $s_i : U_i \to P$, for some open cover, $\mathcal{U} = (U_i)_{i\in I}$, of B.

A right G-torsor is defined similarly with a right G-action. If P is a left G-torsor, there is an associated right G-torsor, P^o, with action $p.g = g^{-1}.p$.

When we refer to a G-torsor, without mentioning 'left' or 'right', we will mean a left G-torsor.

The connection with our earlier definition of principal G-bundle can be made more evident if we note that, on writing $\theta = \phi^{-1} : P \times_B P \to G \times_B P$, then the analogue of the *translation function* of page 278, is the *translation morphism*, $\tau : P \times_B P \to G$, given by $pr_1 \circ \theta$. The morphism θ then equals (τ, pr_2).

The effect of the requirement that local sections exist is to ensure that the bundle $P \xrightarrow{\pi} B$ is locally trivial, *i.e.*, locally like $G \to B$. This is a consequence of the following lemma.

Lemma 49 *Suppose $P \xrightarrow{\pi} B$ is a G-torsor for which there is a global section*

$$s : B \to P$$

of π, then there is an isomorphism

$$G \xrightarrow{f} P$$

of spaces over B.

Proof: Define a function $f : G \to P$ by $f(g) = (g.s(b))$, where $g \in G_b$. As the projection of the group bundle G is continuous, f is continuous. To get an inverse for f, consider the map

$$P \xrightarrow{\pi} B \xrightarrow{s} P.$$

For any $p \in P$, $s\pi(p)$ is in the same fibre as p itself, so we get a continuous map

$$P \xrightarrow{(id,s\pi)} P \times_B P \xrightarrow{\cong} G \times_B P$$

on composing with the inverse of the torsor's structural isomorphism. Finally projecting on to G gives a map $h : P \to G$. This is continuous and checking what it does on fibres shows it to be the required inverse for f. ∎

This does not, of course, transfer a group structure to P, but says that P is like G with 'an identity crisis'. It no longer knows what its identity is!

The group bundle, $G \to B$, considered as a space over B is naturally a G-torsor with multiplication on the left giving the G-action. Check the conditions. It has a global section, since we required it to be a group object in Top/B, so there is a continuous map, e, over B from the terminal object of Top/B to G, which plays the role of the identity. As that terminal object is (isomorphic to) the identity on B, $B \to B$, this splits $G \to B$,

$$\begin{tikzcd} B \arrow[rr, "e"] \arrow[dr, "="'] & & G \arrow[dl] \\ & B & \end{tikzcd}$$

This trivial G-torsor will be denoted T_G.

Applying this to a general G-torsor, the local section $s_i : U_i \to P$ makes $P_{U_i} = \pi^{-1}(U_i)$, the restricted torsor over the open set U_i, into the trivial G_{U_i}-torsor over U_i, so P is *locally trivial.* It is important to note again that this means that P looks locally like G, (but if G is not a product bundle, P will not be locally a product, so need not be locally trivial in the stronger sense used in topological situations). The way that P differs globally from G is measured by cohomology. (An important visual example is, once again, the boundary circle of the Möbius band, *i.e.*, the double cover of the circle, S^1, that twists as you go around that base circle. It is locally a product $U \times \{-1, 1\}$, but not globally so.)

The next observation is very important for us as it shows how the language of G-torsors starts to interact with that of groupoids. First an obvious definition.

Definition: If P and Q are two left G-torsors, then a *morphism, $f : P \to Q$, of G-torsors* (over B) is a continuous map over B such that $f(g.p) = g.f(p)$ for all $g \in G$, $p \in P$.

Here and elsewhere, it is to be understood that we only write $g.p$ if $g \in G_b$ and $p \in P_b$ for the same b. This avoids our constantly repeating mention of the base space and its points. If working with sheaves on a site, *i.e.*, a category $\mathcal{C}$, with a Grothendieck topology, the g and p correspond to locally defined 'elements' in some $G(C)$ and $P(C)$ respectively, so the same (abusive) notation suffices.

Lemma 50 *Any morphism, $f : P \to Q$, is an isomorphism.*

Proof: We have trivialising covers, $\mathcal{U}$ for P, and $\mathcal{V}$ for Q, on which local sections are known to exist. By taking intersections, or any other way, we can get a mutual refinement on which both P and Q trivialise, so we can assume $\mathcal{U} = \mathcal{V}$. We thus are looking at a morphism, f, and local sections, $s : U \to P$, $t : U \to Q$, which (locally) determine isomorphisms to T_G over U. We thus have reduced the problem, at least initially, to showing that $f : T_G \to T_G$ is always an isomorphism, but

$$f(1_G) = g.1_G$$

for some $g \in G_B$, *i.e.*, for some global element of G. Moreover g is uniquely determined by f. Now it is clear that the morphism sending 1_G to $g^{-1}.1_G$ is inverse to f. (Although it is probably an obvious comment, we should point out that saying where a single global element goes determines the morphism, and, within T_G, any (locally defined) element is given by multiplication of the global section, 1_G, by that element, but now regarded as an element of G itself.)

Back to our original $f : P \to Q$, on each U, we have $f_U : P_U \to Q_U$, its restriction to the parts of P and Q over U, is an isomorphism, so we construct the inverse locally and then glue it into a single f^{-1}. ∎

Remark on descent of morphisms: Although we have not yet completed the proof, it is instructive to go into this in a bit more detail, since it introduces methods and intuitions that here should be more or less clear, but later, in more 'lax' or 'categorified' settings will need both good intuition and the ability to argue in detail with (generalisations of) local sections.

If we use s and t, then with respect to these local sections over U, every local element of P_U has the form $g_U.s_U$ for some unique locally defined $g_U : U \to G$ (or in sheaf theoretic notation $g_U \in G(U)$). Similarly in Q_U, local elements looks like $g_U.t_U$, but then

$$f(g_U.s_U) = g_U.f(s_U),$$

so we only need to look at $f(s_U)$. As $f(s_U) \in Q_U$, it determines some unique local element $h_U \in G(U)$ with

$$f(s_U) = h_U.t_U,$$

and checking for behaviour when composing morphisms, it is then clear that

$$f_U^{-1}(t_U) = h_U^{-1}.s_U$$

with continuity of f^{-1} handled by the continuity of inversion, that of t and of multiplication.

As the construction of f_U^{-1} is done using maps defined locally over U, f_U^{-1} is in Top/U (or alternatively, is a map of sheaves on U). We now have to check that this locally defined morphism 'descends' from $\bigsqcup \mathcal{U}$ to B.

Of course, it is 'clear' that it must do so! Each h_U is uniquely defined so ... . That *is* true, but when we go to higher dimensional situations we will often not have uniqueness, merely uniqueness up to isomorphism, or equivalence, so we will spell things out in all the 'gory detail'.

We need to check what happens on intersection $U_1 \cap U_2$ of local patches in our trivialising cover, $\mathcal{U}$. Write $f_i = f_{U_i}$, $i = 1, 2$, etc. for simplicity. The local sections s_1 and s_2 (resp. t_1 and t_2) will not, in general, agree on $U_1 \cap U_2$, so we have

$$f_1(s_1) = h_1.t_1,$$

$$f_2(s_2) = h_2.t_2,$$

but the key local elements $h_1|_{U_1 \cap U_2}$ and $h_2|_{U_1 \cap U_2}$ need not agree. A bit more notation will probably help. Let us denote by s_{12} the restriction of $s_1 : U_1 \to P$ to the intersection $U_1 \cap U_2$ and similarly $s_{21} = s_2|_{U_1 \cap U_2}$, extending this convention to other maps when needed.

We then have some $g_{12} \in G_{U_1 \cap U_2}$ for which

$$s_{21} = g_{12}.s_{12}, \text{ (and } s_{12} = g_{21}.s_{21}, \text{ so } g_{12} = g_{21}^{-1}),$$

but then, over $U_1 \cap U_2$,

$$f(s_{21}) = g_{12}.f(s_{12}).$$

We thus have

$$t_{21} = h_{21}^{-1} g_{12} h_{12} t_{12}.$$

Now turning to f^{-1}, defined locally by $f_i^{-1} : Q_{U_i} \to P_{U_i}$, $i = 1, 2$ with

$$f_i^{-1}(t_i) = h_i^{-1}.s_i,$$

then over $U_1 \cap U_2$, $f_{ij}^{-1}(t_{ij}) = h_{ij}^{-1}s_{ij}$, but we also have $f_j^{-1}(t_{ji}) = h_{ji}^{-1}s_{ji}$ and we have to check that on $Q_{U_i \cap U_j}$, $f_{ij}^{-1} = f_{ji}^{-1}$. To do this, it is sufficient to calculate $f_{ji}^{-1}(t_{ij})$ and to compare it with $f_{ij}^{-1}(t_{ij})$ as both are defined on the same generating local section and so extend via their G-equivariant nature. We have

$$\begin{aligned}
f_{ji}^{-1}(t_{ij}) &= f_{ji}^{-1}(h_{ij}^{-1}g_{ji}h_{ji}t_{ji}) \\
&= h_{ij}^{-1}g_{ji}h_{ji}f_{ji}^{-1}(t_{ji}) \\
&= h_{ij}^{-1}g_{ji}h_{ji}h_{ji}^{-1}.s_{ji} \\
&= h_{ij}^{-1}g_{ji}g_{ij}s_{ij} \\
&= h_{ij}^{-1}s_{ij} \\
&= f_{ij}^{-1}(t_{ij}),
\end{aligned}$$

so the two restrictions *do* agree over the intersection and hence *do* give a morphisms from Q to P inverse to f. (This last point is easy to check.) ■

If we denote the category of left G-torsors on B by $Tors(B, G)$ (or $Tors(G)$ if B is understood), then we have

Proposition 78 *$Tors(B, G)$ is a groupoid.* ■

7.4.2 Torsors and Cohomology

In the above discussion, we saw how a choice of local sections $s_i : U_i \to P$ gave rise to a map $g_{ij} : U_{ij} \to G$. (Here we will again abbreviate: $U_i \cap U_j = U_{ij}$. This notation will be extended to give $U_{ijk} = U_i \cap U_j \cap U_k$, etc.)

The maps g_{ij} are to satisfy

$$s_i = g_{ij}s_j$$

on U_{ij} and for all indices i, j. The map, g_{ij}, gives the translation from the description using s_i to that using s_j. Of course, as g_{ij} is invertible, it can also translate back again. These elements are uniquely determined by the sections, so over a triple intersection, U_{ijk}, we have the 1-cocycle equation,

$$g_{ij}g_{jk} = g_{ik}.$$

If we use different local sections, say s'_i, assumed to be on the same open cover, there will be local elements, $g_i : U_i \to G$, such that $s'_i = g_i.s_i$ for all $i \in I$. The corresponding cocycles g_{ij} and g'_{ij} will be related by a coboundary relation

$$g'_{ij} = g_i g_{ij} g_j^{-1}.$$

These equations will determine an equivalence relation on the set, $Z^1(\mathcal{U}, G)$, of 1-cocycles for $\mathcal{U}$, as before, the (fixed) open cover. The set of equivalence classes will be denoted $H^1(\mathcal{U}, G)$. To remove the dependence on the open cover, one passes to the limit on finer covers to get the Čech non-Abelian cohomology set, $\check{H}^1(B, G) = colim_{\mathcal{U}} H^1(\mathcal{U}, G)$, which, by its construction classifies isomorphism classes of G-torsors on B. The trivial left G-torsor, T_G, gives a natural distinguished element to $\check{H}^1(B, G)$.

This looks quite good. We have started with a torsor and seem to have classified it, up to isomorphism, by cocycles. The one deficiency is that we need to know that cocycles give torsors, *i.e.*, a (re)construction process of P from the cocycle (g_{ij}), but without prior knowledge of P itself.

The method we will use will take the basic ingredients of the group bundle, G, and will twist them using the g_{ij}. First if we have $\gamma \in \check{H}^1(B, G)$, by the basic construction of colimits, we can pick an open cover $\mathcal{U}$ and a $g_{\mathcal{U}} = (g_{ij})$, whose cohomology class represents γ in the colimit. Next taking this $\mathcal{U} = \{U_i\}$, and g_{ij}, let

$$P = \bigsqcup_i G(U_i)/\sim .$$

As we are once again using a disjoint union, we will give our points an index, (g, i), and, of course,

$$(g, i) \sim (gg_{ij}, j).$$

We have a projection $P \to B$ induced from the bundle projections $G(U) \to B$. (For you to check that it works.) This is continuous if P is given the quotient topology. Moreover the multiplications

$$G(U) \times G(U) \to G(U)$$

give a left action

$$G \times P \to P$$

making P into a left G-torsor as hoped for.

To sum up:

Theorem 22 *The set, $\check{H}^1(B, G)$, is in one-one correspondence with the set of isomorphism classes of G-torsors on B, that is, with the set $\pi_0 Tors(B; G)$ of connected components of the groupoid, $Tors(B; G)$.* ■

The relationship for isomorphisms is **left for you to check**.

7.4.3 Change of base

This link with cohomology suggests that we should see what might happen if we changed the base space B in the above. As cohomology is about maps *out of* the space, we should expect that if $f : B \to B'$ is a continuous map then we would get an induced map going from $\check{H}^1(B', G)$ to $\check{H}^1(B, f^*(G))$, but what would this look like through the G-torsors perspective? Suppose we have a G-torsor, Q, over B', then Q is a sheaf on B', so we have an induced sheaf $f^*(Q)$ on B given by pullback, as above, page 304. Strictly speaking as G is a sheaf or bundle of groups on B', $f^*(Q)$ cannot be a G-torsor, but might be a $f^*(G)$-torsor.

We have checked some of what has to be examined before, in the simpler case of principal G-bundles. We will repeat some of the results, but with slightly more categorical proofs as the very element based approach we used is fine for that topological setting, but is here beginning to be less optimal with a sheaf of groups as coefficients. (We will not, however, go to a elegant, fully categorical proof as we have not treated geometric morphisms of toposes.)

First we need an action of $f^*(G)$ on $f^*(Q)$. We have the action of G on Q. There is a quick derivation of this which we will sketch. The functor f^* is a left adjoint and so preserves colimits ..., which is useless to us in this situation! It is also a right adjoint of another functor which we have not discussed. It therefore preserves products and thus actions. A way to see that

$f^*(G \times_{B'} Q) \cong f^*(G) \times_B f^*(Q)$, without producing the left adjoint of f^* is via the étale space description of sheaves. In that description, $f^*(G)$, etc., are all given by pullbacks. We draw a diagram:

$$\begin{tikzcd}
 & f^*(G \times_{B'} Q) \arrow[rrr] \arrow[dr] \arrow[ddl] & & & G \times_{B'} Q \arrow[dr] \arrow[ddl] & \\
 & & f^*(Q) \arrow[rrr] \arrow[ddl] & & & Q \arrow[ddl] \\
f^*(G) \arrow[rrr] \arrow[dr] & & & G \arrow[dr] & & \\
 & B \arrow[rrr] & & & B' &
\end{tikzcd}$$

Each face of the resulting cube is a pullback, as is the vertical square given by the diagonals of the two ends plus the top and bottom maps, but the same would be true of the equivalent diagram with $f^*(G \times_{B'} Q)$ replaced by $f^*(G) \times_B f^*(Q)$, so these two objects are isomorphic.

If we now look at what happens to the action then the original action of G on Q induces one of $f^*(G)$ on $f^*(Q)$ as hoped for. (The detailed verification is left to you as usual.) As the first condition of the definition of torsor again involves pullbacks, it is now fairly routine to check it for $f^*(Q)$. The other condition is the existence of local sections and we have to use a slightly different approach for this. We know that there is an open cover $\mathcal{U}$ of B' over which local sections exist, say, $s_i : U_i \to Q$, $U_i \in \mathcal{U}$. The obvious open cover for B is $f^{-1}(\mathcal{U})$, so we look for sections $f^{-1}(U_i) \to f^*(Q)$. As $f^*(Q)$ is given by a pullback, we will get such a map if we specify maps $f^{-1}(U_i) \to Q$ and $f^{-1}(U_i) \to B$ making the obvious square commute. The map $f^{-1}(U_i) \to B$ 'must' be the inclusion ... what else could it be, so we will try that. Composing that with f gives a map $f^{-1}(U_i) \to B'$, which can also be written as the composite of f restricted to $f^{-1}(U_i)$ followed by the inclusion of U_i into B', so we can compose that restriction of f with s_i to get a map to Q. Since s_i is a section over U_i of the map $Q \to B'$, it is now easy to check that the 'obvious square' commutes. (**Left to you**.) We have built a local section over $f^{-1}(U_i)$. We thus have

Proposition 79 *If Q is a G-torsor over B', then $f^*(Q)$ is a $f^*(G)$-torsor over B.* ■

The new torsor $f^*(Q)$ would here loosely be called the *induced torsor of Q along f*.

We have a cocycle description of torsors. If we have one for Q, what will be the one for $f^*(Q)$? In a sense, we know what the answer is without doing any calculation. The cocycle description of Q gives a class in $H^2(B', G)$ and the induced map from that to $H^2(B, f^*(G))$ must surely be given by composition with f. The fact that the coefficients change as well as the space should come out 'in the wash'. We would, from this perspective, also expect the maps induced from homotopic maps to be the same. We know what to expect but what about the details!

Suppose we pick local sections s_i for Q over the various U_i in a cover $\mathcal{U}$ of B', and we get the $g_{ij} \in G(U_{ij})$ as above. These satisfy

$$s_i = g_{ij} s_j.$$

We have just seen that suitable local sections over the $f^{-1}(U_i)$ are given by the pairs of maps $(s_i f, inc) : f^{-1}U_i \to Q \times_{B'} B$, but these are determined just be the first component. Likewise the sections g_{ij} over pairwise intersections of G, correspond by composition to the corresponding elements $g_{ij} f$ over the pairwise intersections of $f^{-1}(\mathcal{U})$, and, of course, these are the transition cocycles for the $s_i f$. That they are cocycles follows since the g_{ij} satisfy the cocycle condition.

To summarise: the cocycle data for $f^*(Q)$ can be derived from that for Q merely by precomposing by the relevant restrictions of f to the sets of the cover $f^{-1}(\mathcal{U})$ and their intersections. Just as we expected.

Having seen that homotopic maps induced isomorphic principal bundles in an earlier section, it is natural to expect the same thing to happen here. It does, but rather than explore that here we will put it aside for a little while until we have a simplicial description of torsors in sections 7.4.5 and 7.5.5. That will make life a lot easier.

We have changed the base, what about changing the 'coefficients'?

7.4.4 Contracted Product and 'Change of Groups'

In Abelian cohomology, one would expect the cohomology 'set' (there a group) to vary nicely with the coefficient sheaf of groups, G. Something like that occurs here as well and determines some essential structure on the torsors. Suppose $\varphi : G \to H$ is a homomorphism of sheaves of groups, then one expects there to be induced functors between $Tors(G)$ and $Tors(H)$ in one direction or the other. Thinking of the better known case of a ring homomorphism, $\varphi : R \to S$, and modules over R or S, then we could, for an S-module, M, form an R-module by restriction along φ. The analogue works for an H-set X as one gets a G-set by defining $g.x = \varphi(g).x$, but there is no reason to expect the resulting G-set to be principal, so this does not look so feasible for torsors. There is, however, another module construction. Suppose that N is a left R-module, and make S into a *right* R-module, S_R by $s.r = s\varphi(r)$, then we can form $S_R \otimes_R N$, and the left S-action by multiplication is nicely behaved. The point is that S is behaving here as a two sided module over itself, and also as a (S, R)-bimodule. The corresponding idea in torsor theory is that of a bitorsor, explored in depth by Breen in [48], which we will examine later in this chapter.

Before looking at this in a bit more detail, we will look at the contracted product, which replaces the tensor product here. Suppose we have a category, $\mathcal{C}$, and an internal group, G, in $\mathcal{C}$. Here we have various examples in mind. If $\mathcal{C} = Sh(B)$, G will be a sheaf of groups; if $\mathcal{C}$ is the category of groupoids, G will be an internal group in that category, *i.e.*, a *(strict) gr-groupoid*, and will correspond to a crossed module, and, if we combine the two ideas, $\mathcal{C}$ is a category of sheaves of groupoids, so G is a sheaf of gr-groupoids, corresponding to a sheaf of crossed modules, and so on in various variants.

A left G-object in $\mathcal{C}$ is an object X together with a morphism, (left action),

$$\lambda : G \times X \to X,$$

satisfying obvious rules. Similarly a right G-object Y comes with a morphism, (right action),

$$\rho : Y \times G \to Y.$$

The *contracted product* of Y and X is, intuitively, formed from $Y \times X$ by dividing by an equivalence relation

$$(y.g, g^{-1}.x) \equiv (y, x).$$

The usual notation is $Y \wedge^G X$, *but* this is often inadequate as it assumes X, (resp. Y), stands for the object *and* the G-object, unambiguously, whilst, of course, X really stands for (X, λ) and Y for (Y, ρ). It is sometimes useful, therefore, to add the action into the notation, but only when confusion would occur otherwise, so $Y_\rho \wedge^G {}_\lambda X$ is the full notation, but variants such as $Y_\rho \wedge^G X$ would be used if it was clear what λ was, etc.

We gave an element based description of $Y \wedge^G X$, but how can we adapt this to work within our general $\mathcal{C}$? There are obvious maps

$$Y \times G \times X \underset{(Y,\lambda)}{\overset{(\rho,X)}{\rightrightarrows}} Y \times X\ ,$$

and we can form their coequaliser. (As usual, we assume that the category $\mathcal{C}$ has all limits and colimits that we need to make constructions, and to enable definitions to make sense, but we do not constantly remind the reader of these hidden conditions!) Of course, we met this construction earlier when considering a left principal G-bundle and a right G-space (fibre), F, forming the fibre bundle $X_F = F \wedge^G X$; it was also at the heart of the regular twisted Cartesian product construction from our discussion of simplicial twisting maps.

Example: Suppose $\varphi : G \to H$ is a morphism of group bundles on B, then we can give H a right G-action by

$$H \times_B G \xrightarrow{H\times\varphi} H \times_B H \to H$$

where the second map is multiplication. If P is a G-object such as a G-torsor, we have a contracted product $H_\varphi \wedge^G P$.

Lemma 51 *If P is a G-torsor, then $H_\varphi \wedge^G P$ is an H-torsor.*

Proof: Writing $Q = H_\varphi \wedge^G P$, we check the usual map,

$$H \times_B Q \to Q \times_B Q,$$

is an isomorphism. This is merely checking that the 'obvious' fibrewise formula is well defined. This sends a pair $([h, p], [h_1, p])$ to $(hh_1^{-1}, [h_1, p])$. That verification is **left to the reader**. (That all elements in $Q \times_B Q$ can be written in this form follows from the fact that **P** is a **G**-torsor, and is again **left to the reader.**)

Local sections of P immediately yield local sections of Q, so Q *is* an H-torsor. ■

A group homomorphism

$$\varphi : G \to H$$

thereby gives us a functor

$$\varphi_* : Tors(G) \to Tors(H) \qquad \varphi_*(P) = H_\varphi \wedge^G P.$$

Of course, there are still some details (**for you**) to check, namely relating to behaviour on morphisms of G-torsors. (These are probably 'clear', but **do need checking**.)

Another point from this calculation is that we could work with 'elements' as if in a G-set. This can be thought of either as working, carefully, in each fibre of the torsor or using local sections or as a heuristic to obtain a formula that is then encoded purely in terms of the structural maps. All of these viewpoints are valid and all are useful.

Now suppose $\mu, \nu : G \to H$ are two group homomorphisms, thus giving us two functors,

$$\mu_*, \nu_* : Tors(G) \to Tors(H).$$

When is there a natural transformation $\eta : \mu_* \to \nu_*$? The answer is neat and very useful.

Lemma 52 *(cf. Breen, [50], Lemma 1.5)*
A natural transformation $\eta : \mu_* \to \nu_*$ is determined by a choice of a section h of H such that

$$\nu = h^{-1}\mu h.$$

Proof: Suppose that P is a G-torsor, then $\mu_*(P) = H_\mu \wedge^G P$, similarly for $\nu_*(P)$ and $\eta_P : H_\mu \wedge^G P \to H_\nu \wedge^G P$.

If we look locally

$$\eta_P([\mu(g), p]) = h.[\nu(g), p]$$

for some h, since $\eta_P(\mu(g), p)$ is of form $[h_1, p]$ for some h_1 and as $\nu_*(P)$ is an H-torsors, etc.

(Unfortunately we need to know h does not depend on g, and is defined globally, so this suggests looking at the special case where global sections *do* exist, *i.e.*, $P = T_G$, the trivial G-torsor. There we can assume $g = 1_G$, so

$$\eta_{T_G}([1_H, p]) = h.[1_H, p],$$

giving us a possible h. We know that η_P is H-equivariant and natural as well as being 'well-defined'. We use these properties as follows:

If $g \in G$,

$$\begin{aligned} \eta_{T_G}[\mu(g), p] &= \eta_{T_G}[1_H, g.p] \\ &= h[1_H, g.p] \\ &= h[\nu(g), p] \\ &= h.\nu(g)[1_H, p], \end{aligned}$$

whilst also

$$\begin{aligned} \eta_{T_G}[\mu(g), p] &= \eta_{T_G}(\mu(g).[1_H, p]) \\ &= \mu(g)\eta_{T_G}[1_H, p] \\ &= \mu(g)h[1_H, p], \end{aligned}$$

using that η_{T_G} is H-equivariant. We thus have a globally defined h with

$$\mu(g)h = h\nu(g)$$

for all $g \in G$,

$$\text{or} \quad \mu = i_h \circ \nu \quad \text{or} \quad \nu = i'_h \circ \mu,$$

where i_h is inner automorphism by h and i'_h, that by h^{-1}.

Conversely given such an h, we can *define* η by our earlier formula, extending it by H-equivariance and naturality. Checking well definition is quite easy, but instructive, and so is left to you. ■

Recall from section 2.3.4 that for any groupoids G, H, the functor category H^G has groupoid morphisms as its objects and that the natural transformations can be seen to be 'conjugations'. In particular, if $G = H$ is a group, the full subcategory $\mathsf{Aut}(G)$ of G^G given by the automorphisms of G is an internal group object in the category of groupoids, so corresponds to a crossed module. What crossed module? What else, $i : G \to Aut(G)$.

Two automorphisms μ, ν are related by a natural transformation if and only if there is a g such the $\mu = i_g \circ \nu$, where i_g is inner automorphism by g. The similarity with our current setting is *not* coincidental and can be exploited!

Another fairly obvious result is that, if P is a G-torsor, then

$$G \wedge^G P \cong P,$$

since locally we have each representative (g,p) is equivalent to $(1_G, g.p)$. The details are **left as an almost trivial exercise**.

This notation is 'dangerous' however, as we pointed out earlier. We are using the right multiplication of G on itself to give us the contracted product, but we could also make G act on itself by conjugation *on the right*: for $g \in G$, $x \in G$, with G being considered as a bundle,

$$x.g = g^{-1}xg.$$

We will write this action as i', for 'inner', so have $G_{i'} \wedge^G P$ as well. This is, in fact, a very useful object. It is related to automorphisms of P in the following way:

Suppose that $\alpha : P \to P$ is a locally defined automorphism of G-torsors, *i.e.*, a local section of $Aut_G(P)$. Continuing to work locally, pick a section (local element) p. As α is 'fibrewise',

$$\alpha(p) = g_p.p$$

for some local elements g_p of G, and as α is G-equivariant,

$$\alpha(g.p) = g\alpha(p) = gg_p.p.$$

Assigning, to each pair (g,p) in $G \times P$. the automorphism given by

$$\alpha(g_1, p) = g_1 g.p$$

gives a map

$$\lambda : G \times P \to Aut_G(P), \quad \lambda(g,p)(p) = g.p,$$

and this is an epimorphism by our previous argument. 'Obviously'

$$\lambda(g,p) = \lambda(gg', (g')^{-1}p),$$

so the map λ passes to the quotient $G \wedge^G P$ **-or does it**? We have not actually examined the definition of $\lambda(g,p)$ that closely.

Look at this from another direction. Examine $\lambda(g, g'p)$ as an automorphism of P. To work out $\lambda(g, g'p)(p)$, we have first to convert p:

$$\lambda(g, g'p)(p) = \lambda(g, g'p)((g')^{-1}g'.p),$$

as $\lambda(g, g'p)$ is specified by what it does to its basic P-part. Now

$$\lambda(g, g'p)((g')^{-1}g'.p) = (g')^{-1}\lambda(g, g'p)(g'.p)$$

by G-equivariance, and so equals

$$(g')^{-1}gg'.p,$$

which is $\lambda((g')^{-1}gg', p)(p)$.

Thus our initial impulse was hasty. We do have $Aut_G(P)$ as a contracted product, $G \wedge^G P$, but not with right multiplication as the action of G on itself, rather it uses right conjugation. We have proved

Lemma 53 *For any G-torsor P, there is an isomorphism*

$$\lambda : G_{i'} \wedge^G P \xrightarrow{\cong} Aut_G(P),$$

where $i' : G \to Aut(G)^o$, $i'(g)(g') = g^{-1}g'g$, yielding the right conjugation action of G on itself. ■

Perhaps something more needs to be said about $Aut_G(P)$ here. We are working with sheaves or bundles and so have an essentially Cartesian closed situation, in other words function objects exist. For each pair of sheaves, X, Y on B, $Hom(X, Y)$ is a sheaf. In particular $End(X)$ is a sheaf and $Aut(X)$ a subsheaf of it. It thus makes basic sense to have that $Aut_G(P)$ is a G-torsor. Of course, it is also a group object, since automorphisms (gauge transformations) of P are invertible. This group is sometimes written P^{ad}. It is the group (bundle) of G-equivariant fibre preserving automorphisms of P; it is also called the *gauge group* of P. (The precise origin in the thoughts of Hermann Weyl of the use of 'Gauge' are fun to look up, but they make me think that the term is very much over used in mathematical physics, as Weyl's use seems to have been beautifully simple and down to earth, whilst the mystique of the modern use by comparison may be tending to obscure the simple idea from a simple minded mathematician's viewpoint.)

In the isomorphic $G_{i'} \wedge^G P$ version, it is instructive to explore the group structure, but this is left for you to do. This group operates on the *right* of P, by the rule

$$p.\alpha = \alpha^{-1}(p),$$

and makes P into a right P^{ad}-torsor. (Exploration of these statements is well worth while and is **left as an exercise**. It, of course, presupposes that P^{ad} is seen as a bundle /sheaf of groups, which itself needs 'deconstructing' before you start. The overall intuition should be fairly clear *but* the technicalities, detailed verifications, etc., **do need mastering**.)

A cohomological perspective on change of groups. We have that $\check{H}^1(B, G)$ is the set of isomorphism classes of G-torsors on B, *i.e.*, $\pi_0 Tors(G)$, the set of connected components of the groupoid $Tors(G)$. We have now seen that if $\varphi : G \to H$ is a homomorphism of group bundles and P is a G-torsor, then $H_\varphi \wedge^G P = \varphi_*(P)$ is an H-torsor and that this gives a functor $\varphi_* : G \to H$. This will, of course, induce a function on sets of connected components and hence, as one might expect, an induced function

$$\varphi : \check{H}^1(B, G) \to \check{H}^1(B, H).$$

There is another obvious way of inducing such a function, as the elements of $\check{H}^1(B, G)$ are classes of cocycles, (g_{ij}), and so composing with φ sends $[(g_{ij})]$ to $[\varphi(g_{ij})]$. It is standard to check that this does induce a function from $H^1(\mathcal{U}, G)$ to $H^1(\mathcal{U}, H)$ and, by its independence from $\mathcal{U}$, it is then routine to check that it induces a corresponding map on Čech non-Abelian cohomology.

It is easy to see that these two induced maps are the same. (It would be surprising if they were not!) Pick a set of local sections, $\{s_i\}$, for P over a trivialising cover, $\mathcal{U}$, and we get $\{[1, s_i]\}$ is a set of local sections for $H_\varphi \wedge^G P$. Changing patches, $s_i = g_{ij}s_j$, and so

$$[1, s_i] = [1, g_{ij}sj] = [\varphi(g_{ij}).1, s_j] = \varphi(g_{ij})[1, s_j],$$

and the transition functions for $\varphi_*(P)$ are exactly as expected. (The rest of the details are **left as an exercise**.) The important thing for later use is the identification of the cocycles for $\varphi_*(P)$. This will be especially important when discussing G-bitorsors in the next section.

7.4.5 Simplicial Description of Torsors

As usual we look at a sheaf or bundle of groups, G, on a space, B, and suppose P is a G-torsor. We then know there is an open cover, $\mathcal{U}$, of B and trivialising local sections, $s_i : U_i \to P$, over the various different open sets U_i of $\mathcal{U}$. We have seen that over the intersections U_{ij}, the restrictions of the two local sections s_i and s_j must be related and this gives us transition cocycles $g_{ij} : U_{ij} \to G$ such that

$$s_i = g_{ij} s_j,$$

where, over triple intersections, the 1-cocycle condition

$$g_{ij} g_{jk} = g_{ik}$$

must be satisfied.

The information on intersections in $\mathcal{U}$ is neatly organised in the simplicial sheaf, $N(\mathcal{U})$, (cf. page 300 in section 7.3.6). We also know that from a sheaf of groups we can construct various simplicial sheaves. Is there a way of viewing the cocycles g_{ij} from this simplicial perspective?

From a group, G, (no sheaves for the moment), we earlier saw the uses of models for the classifying space, BG, of G. We could use the nerve of G as a group or rather its nerve as a single object groupoid, $G[1]$. We could alternatively take the constant simplicial group, $K(G, 0)$ (so $K(G, 0)_n = G$ for all $n \geq 0$, with all face and degeneracies, being the identity isomorphism of G). If we then formed $\overline{W}(K(G, 0))$, we get $Ner(G[1])$ back.

These different approaches all yield a simplicial set (and if you really want a space, you just take its geometric realisation). This simplicial set will be denoted BG, even though that notation is often restricted to that corresponding space. We have to be a bit careful about the order of composition in the groupoid, $G[1]$, if it is to be consistent with the construction K, which was the nerve of an internal groupoid in the category of groups. We also have to be careful about our use of *left* actions and the assumption that that makes about the order of composition being 'functional' rather than algebraic (which latter order works best with right actions). That being said, we have

- BG_0 = a singleton set, $\{*\}$;
- $BG_1 = G$, as a set, and in general,
- $BG_n = \underbrace{G \times \ldots \times G}_{n}$

Writing $\mathbf{g} = (g_n, \ldots, g_1)$ for an n-simplex of BG, we have

$$\begin{aligned} d_0\mathbf{g} &= (g_n, \ldots, g_2), \\ d_i\mathbf{g} &= (g_n, \ldots, g_{i+1}g_i, \ldots, g_0), \quad 0 < i < n, \\ d_n\mathbf{g} &= (g_{n-1}, \ldots, g_1), \end{aligned}$$

with the degeneracy maps, s_j, given by insertion of 1_G in the j^{th} place, shifting later entries one place to the right. (Warning: multiple use of the label s_j here *may* cause some confusion, but each use is the natural one in that context!)

We have already seen this several times (but repetition *is* useful). The key diagram is usually that indicating a 2-simplex, $\mathbf{g} = (g_2, g_1)$, namely

$$\begin{tikzcd} & * \arrow[dr, "g_2"] & \\ * \arrow[ur, "g_1"] \arrow[rr, "g_2 g_1"'] & & * \end{tikzcd}$$

Back to G being a sheaf of groups, and we get BG will be a sheaf of simplicial sets. We now have two simplicial sheaves, $N(\mathcal{U})$ and BG. Curiosity alone should suggest that we compare these via a simplicial morphism and, for our purposes, it should be a simplicial sheaf map, $f : N(\mathcal{U}) \to BG$.

Looking back at $N(\mathcal{U})$ and its construction (page 300), the zero simplices are formed by the open sets and as BG_0 is trivial, f_0 is not much of interest!

At the next level, $f_1 : N(\mathcal{U})_1 \to BG_1$, so consists - yes, of course, - of local sections over the intersections U_{ij}, hence g_{ij} in $G(U_{ij})$ or G_{ij}. Over triple intersections U_{ijk}, f_2 will give a 2-simplex, as above, so $g_{ij}g_{jk} = g_{ik}$, given by $f_2 : U_{ijk} \to G \times G$, $f_2 = (g_{jk}, g_{ij})$.

We thus have our 1-cocycle condition is automatic from the simplicial structure.

What about change of the choice of local sections of P, *i.e.*, $s_i : U_i \to P$. If we change these, we get elements $g_i \in G_i$ such that $s' = g_i s_i$ and the new g'_{ij} are related to the old by a sort of conjugacy rule:

$$g'_{ij} = g_i g_{ij} g_j^{-1},$$

which can be visualised as a square

$$\begin{tikzcd} \cdot \arrow[r, "g_{ij}"] \arrow[d, "g_j"'] & \cdot \arrow[d, "g_i"] \\ \cdot \arrow[r, "g'_{ij}"'] & \cdot \end{tikzcd}$$

This is reminiscent of a homotopy, and, in fact, defines one from our f (relative to the $\{s_i\}$) to f' (relative to the $\{s'_i\}$). In other words, we are identifying isomorphism classes of G-torsors that trivialise over $\mathcal{U}$ with homotopy classes, *i.e.*, elements of $[N(\mathcal{U}), BG]$. We will return to this later when we discuss passing to refinements of $\mathcal{U}$ to get a homotopy description of all G-torsors, so we will not give the details here.

Several questions should come to mind at this stage. Given our recent description of 'change of groups', an obvious thing to do is to view that from a simplicial perspective. Suppose $\varphi : G \to H$ is a homomorphism of sheaves of groups. It is easy to see that φ induces a map of simplicial sheaves, $B\varphi : BG \to BH$, so we get, for given $\mathcal{U}$, an induced map

$$[N(\mathcal{U}), B\varphi] : [N(\mathcal{U}), BG] \to [N(\mathcal{U}), BH].$$

If we start off with a G-torsor, P, and use our change of groups methods above, what is the link between $\varphi_*(P)$ and the image of the isomorphisms class of P as represented by some map from $N(\mathcal{U})$ to BG. Of course, we have just seen that if $\{g_{ij}\}$ represents P then $\{\varphi(g_{ij})\}$ represents $\varphi_*(P)$ - but this is exactly the image under $[N(\mathcal{U}), B\varphi]$. There is thus yet another good way of interpreting the change of groups functor from $Tors(G)$ to $Tors(H)$, namely as a simplicial induced map from BG to BH. (Later we will see that $Tors(G)$ is the stack completion of BG or equivalently of $G[1]$ and this yields a variant of this simplicial viewpoint.)

Picking up an earlier problem, what about change of base. If we have the above simplicial description of isomorphism classes of those G-torsors on a base B that trivialise over some open cover $\mathcal{U}$, in terms of homotopy classes of maps from $N(\mathcal{U})$ to BG, and then we change the base along a continuous map, how does this look from a simplicial viewpoint?

To start with we rename some objects to get things into line with our earlier discussion. We will consider two spaces B and B' and a continuous map $f : B \to B'$. We have a sheaf or bundle of groups G on B' and hence an induced pullback sheaf $f^*(G)$ on B. We assume given some open cover $\mathcal{U}$ of B', and hence an open cover $f^{-1}(\mathcal{U})$ of B, and will be interested in those $f^*(G)$-torsors that trivialise over $f^{-1}(\mathcal{U})$ and which are induced from G-torsors that trivialise over $\mathcal{U}$.

7.4.6 Torsors and exact sequences

One classical method of analysing the cohomology, and, in so doing, of providing interpretations of cohomology classes, is to vary the coefficients within an exact sequence. For instance, if

$$1 \to L \xrightarrow{u} M \xrightarrow{v} N \to 1$$

is an exact sequence of sheaves of groups, then one might try to relate torsors over L, M and N. The usual techniques would then be to see what is the likelihood of having something like a long exact sequence of the cohomology 'sets' or groups. Where should it start?

We will, to start with, look at the Abelian case, but will try not to use commutativity so as to get as general a result as possible. Sheaf cohomology with coefficients in sheaves of Abelian groups, etc., is considered as measuring the non-exactness of the global sections functor. Given a sheaf, L, of Abelian groups on B, $\Gamma_B(L)$ is one of several notations used for the Abelian group of global sections of L. Another is $L(B)$, of course. If the exact sequence above had been of Abelian sheaves, we would have had a long exact sequence

$$0 \to L(B) \to M(B) \to N(B) \to \check{H}^1(B, L) \to \check{H}^1(B, M) \to \check{H}^1(B, N) \to \check{H}^2(B, L) \to \ldots,$$

and so on. It is to be noted that the induced map, $v_* : M(B) \to N(B)$, need not be onto, so $\check{H}^1(B, L)$ picks up the obstruction to 'lifting' a global section of N to one of M. This is particularly interesting to us here since we have linked $\check{H}^1(B, L)$ with L-torsors in the general situation - and, of course, that interpretation is also valid in the Abelian case.

To see how $\check{H}^1(B, L)$ arises naturally in this situation, suppose given a global section h of N. As our exact sequence above was of sheaves, we have to examine what that means. This can be viewed from several angles. An exact sequence of sheaves may not be exact as a sequence of presheaves. The functor that forgets that sheaves are sheaves has a left adjoint namely 'sheafification', so will itself be 'left exact', e.g., will preserve monomorphisms. (If you do not know of this type of result, try to prove it yourself.) It need not preserve epimorphisms. Sheafification itself will preserve epimorphisms, but not all epimorphisms need be the sheafification of an epimorphism at the presheaf level. An epimorphism of sheaves will give an epimorphism on stalks. (We are thinking here of sheaves on a space, B, rather than more general topos centred results.) This means epimorphisms are locally defined. Suppose we have a point $b \in B$, then if x is in the stalk of N above b, it means that x is representable as a pair (x_U, U), where $b \in U$, U is an open set and $x_U \in N(U)$, the group of local sections of N over U. (Recall, from page 296, section 7.3.3, that the stalk of a sheaf N at a point b is a colimit of the $N(U)$ for $b \in U$.) The morphism v being an epimorphism, there is an element y in the stalk of M at b, say $y = [(y_V, V)]$, such that over some open set $W \subseteq U \cap V$, $v(y_W) = x_W$.

Now start, not with an element in a stalk, but rather with a global section x of N. This does give an element in each stalk and we can find an open cover $\mathcal{U}$ such that over each U_i in $\mathcal{U}$, we can find a local section, y_i, mapping down to the restriction, x_i, of x to U_i, (but remember that different global sections will most likely need different covers, etc.). There is no reason these y_i should be compatible on intersections U_{ij}, so there will be (unique) elements, $\ell_{ij} \in L_{ij} = L(U_{ij})$, such that

$$y_i = u(\ell_{ij})y_j,$$

since both y_i and y_j map to x_{ij} over U_{ij}. As u is a monomorphism, these ℓ_{ij} will satisfy the cocycle condition,

$$\ell_{ij}\ell_{jk} = \ell_{ik}$$

and, as you no doubt now expect, if we change the local sections y_i within the L_i-coset of possible choices, then $y_i' = u(\ell_i)y_i$ and the ℓ_i define a coboundary.

In other words, there is an L-torsor, $P(x)$, which is constructed from the global section x of N, and which is trivial exactly when the y_i can be chosen compatibly, *i.e.*, when there is a global section y mapping down to x. We can thus think of $P(x)$ as being the obstruction to lifting x to a global section of M. (Of course, the choices made have to be checked not to matter, up to isomorphism of $P(x)$ - but that can be safely '**left to the reader**'.)

There is thus an extension of the earlier sequence to

$$0 \to L(B) \to M(B) \to N(B) \to \pi_0(Tors(L)),$$

where the last term corresponds to $\check{H}^1(B, L)$. (The notation π_0 is, you may recall, to designate the set of connected components of a groupoid, simplicial set or space and $Tors(L)$ is a groupoid as we have seen.)

The next two terms in the long exact sequence, $\check{H}^1(B, M)$ and $\check{H}^1(B, N)$, are easy to handle geometrically. They give $\pi_0(Tors(M))$ and $\pi_0(Tors(N))$ respectively, and, of course, the induced maps are those given by the 'change of groups' along u and v. Exactness of the result is then routine to check, but

$$v_* : \pi_0(Tors(M)) \to \pi_0(Tors(N))$$

will not, in general, be onto. (You would not expect it to be as the standard homological machinery gives a $\check{H}^2(B, L)$ term.) Of course, none of the above depended on the sheaves involved being Abelian, but if they are not, $\check{H}^1(B, L)$ is not an Abelian group, it is just a pointed set. It is still given by $\pi_0(Tors(L))$, and $Tors(L)$ *is* always a groupoid, so there is a second layer that is hidden by the homological approach namely the automorphisms of the different objects in this groupoid.

7.5 Bitorsors

The fact that the left G-torsor is also a right P^{ad}-torsor suggests the notion of a bitorsor, the analogue of a left R-, right S-module for our non-Abelian setting. (Our basic reference for this will be Breen's Grothendieck Festschrift paper, [48] and his beautiful 'Notes on 1- and 2-gerbes', [51], based on his Minneapolis lectures.)

7.5.1 Bitorsors: definition and elementary properties

Definition: Let G, H be two bundles of groups on B or more generally two group objects in a topos, $\mathcal{E}$. A (G, H)-*bitorsor* on B is a space P over B together with fibre preserving left and right actions of G and H, respectively, on P, which commute with each other,

$$(g.p).h = g.(p.h),$$

and which define both a left G-torsor and a right H-torsor structure on P. If $G = H$, we say *G-bitorsor* rather than (G, G)-bitorsor.

There is an obvious extension of the notion to that of a (G, H)-bitorsor in a topos. We leave the exact formulation to you.

A family of local sections s_i of a (G, H)-bitorsor defines a local identification of P as *the* trivial left G-torsor and *the* trivial right H-torsor. It therefore determines a family of local isomorphisms $u_i : H_{U_i} \to G_{U_i}$, given by the rule $s_i h = u_i(h)s_i$, for $h \in H_{U_i}$. It is important to note that this does not mean that G and H are *globally* isomorphic.

Examples: a) The trivial (left) G-torsor T_G is also a right G-torsor (using right multiplication) and has a G-bitorsor structure.

b) Any left G-torsor, P, is a (G, P^{ad})-bitorsor, as above. Any G-torsor, P, is a (G, H)-bitorsor if and only if $H \cong P^{ad}$.

c) Let

$$1 \to G \xrightarrow{i} H \xrightarrow{j} K \to 1$$

be an exact sequence of bundles of groups on B. Form $G_K = G \times_B K$, which is again a bundle of groups, then H is a G_K-bitorsor over K. This needs a bit of working through. For a start, K is a bundle of groups so has a (hidden) structural projection, $K \to B$. Thinking of this as a cover as we have done previously, then G_K is the induced bundle of groups on K (as a space), so we have transferred attention from Top/B to Top/K or from $Sh(B)$ to $Sh(K)$. There are actions of G_K on H,

$$h \star (g, k) = hi(g),$$

(but note that requires us to use $H \xrightarrow{j} K$, as the structural projection of H over K, again, going to bundles on K,

$$(g, k).h = i(g).h,$$

but is only defined if $j(h) = k$, as we are 'over K,' in this equation).

This is somewhat simplified if we have $B = 1$, when it is simply an exact sequence of groups, G_K is $G \times K$ as a group over K, via projection, and so on.

There is an obvious notion of morphism of bitorsors and thus various categories, $Bitors(G, H)$, $Bitors(G) := Bitors(G, G)$, ... It should come as no surprise that if P is a (G, H)-bitorsor and Q is a (H, K)-bitorsor, both on B, then $P \wedge^H Q$ is a (G, K)-bitorsor. Moreover, P gives a (H, G)-bitorsor, P^o, (o for 'opposite') by reversing the two actions. (**For you to check out.**) We thus have that a (G, H)-bitorsor will induce a functor

$$Tors(H) \to Tors(G)$$

and that, for a given bundle of groups G, the category of G-bitorsors has a monoidal structure given by $P \wedge^G Q$ and with T_G as unit object. The opposite construction acts like an inverse,

$$P \wedge^G P^o \cong T_G \cong P^o \wedge^G P,$$

but note that these are isomorphisms *not* equality.

Lemma 54 *The category, $Bitors(G)$, with contracted product is a group-like monoidal category, with the bitorsor T_G as unit and P^o, an inverse for P.*

Proof: This is **left as an exercise**, but here is a suggestion for the above isomorphisms: use local sections to send any $[p, p']$ in $P^o \wedge^G P$ to an element of G, now show independence of that element on the choice of local section. It is also necessary to check through the group-like monoidal category axioms, which are left for you to find in detail. ■

A group-like monoidal category is often called a *gr-category*. We have already (essentially introduced on page 60) seen that strict gr-categories are 'the same as' crossed modules, so once again that crossed structure is lurking around just beneath the surface. It is interesting and useful (*i.e.*, an **exercise left to the reader!**) to examine the above structure when G is a sheaf of *Abelian* groups, for instance to show that the monoidal structure is symmetric.

A very useful result, akin to Lemma 53 above, gives a similar interpretation of $Isom_G(P, Q)$, where P is a (G, H)-bitorsor and Q a left G-torsor. As P is thus also a left G-torsor and $Tors(G)$ is a groupoid, $Isom_G(P, Q)$ is just the sheaf of G-equivariant torsor maps from P to Q, all of which are invertible. The following lemma identifies this as a contracted product.

Lemma 55 *Let P be a (G, H)-bitorsor and Q a left G-torsor, then there is an isomorphism*

$$Isom_G(P, Q) \overset{\cong}{\to} P^o \wedge^G Q.$$

Proof: We start by noting a morphism in the other direction. Suppose we take a local element in $P^o \wedge^G Q$ given by $(p, q) \in P^o \times Q$, defined over an open set U. We have

$$(p, q) \equiv (p.g^{-1}, g.q),$$

but as $p \in P^o$, $p.g^{-1} = q.p$ with the original left G-action on P. We assign to (p, q) the isomorphism, $\alpha_{(p,q)}$, from P to Q defined over U, which sends p to q. Of course, $\alpha_{(p,q)}$ is to be extended to a G-equivariant map, $\alpha_{(p,q)}(g.p) = g.q$, but we effectively knew that fact already since

$$\alpha_{(p,q)} = \alpha_{(p.g^{-1}, g.q)},$$

so it sends $p.g^{-1}$ to $g.q$. Of course, if $\beta : P_U \to Q_U$ is a local morphism defined over some U, then we can assume P_U has a local section p and that $\beta(p) = q$ for some local section q of Q. (If not, refine U by an open cover on which P trivialises and work on the open sets of that finer open cover.) However then we can assign $[p, q]$ in $P^o \wedge^G Q$ to the morphism β. The rest of the details should now be easy to check. ■

7.5.2 Bitorsor form of Morita theory (First version):

Within the theory of modules and more generally of Abelian categories, there is a very important set of results known as Morita theory, describing equivalences between categories of modules. The idea is that if R and S are rings, then we can use a homomorphism as above to induce a right R, left S module structure on S itself and this is what induces, via tensor product, a functor from $Mod(S)$ to $Mod(R)$. We have seen the corresponding idea with torsors above. Not all functors between $Mod(R)$ and $Mod(S)$ are induced by morphisms at the ring level in this way however, but provided we look at equivalences between categories, this bimodule idea allows us to describe the equivalences precisely - and this does go across to the torsor context.

The first essential is to recall the definition of an equivalence of categories.

Definition: A functor $F : \mathcal{C} \to \mathcal{D}$ between two categories is an *equivalence* if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms, $\eta : GF \Rightarrow Id_{\mathcal{C}}$ and $\eta' : FG \Rightarrow Id_{\mathcal{D}}$. We say G is *(quasi-)inverse* to F.

Proposition 80 *A (G, H)-bitorsor Q on B induces an equivalence*

$$Tors(H) \overset{\Phi_Q}{\to} Tors(G)$$

$$M \longmapsto Q \wedge^H M$$

between the corresponding categories of left torsors on B. In addition if P is a (H, K)-bitorsor on B, then there is a natural isomorphism of functors

$$\Phi_{Q\wedge^H P} \cong \Phi_Q \circ \Phi_P,$$

and, in particular, the equivalence Φ_{Q^o} is quasi-inverse to Φ_Q.

Proof: The last part follows from the statement on composites, which should be clear by construction and, of course, $T_H \wedge^H Q \cong Q$, as we saw earlier. This proof is thus just a compilation of earlier ideas - and so will be **left to the reader**! ■

In fact it is now easy to give a weak version of the torsor Morita theorem.

Proposition 81 *If*

$$\Phi : Tors(H) \to Tors(G)$$

is an equivalence of categories, then there is a (G, H)-bitorsor, Q, which itself induces such an equivalence.

Proof: We will limit ourselves to pointing out that we can take $Q = \Phi(T_H)$. This inherits its right H-action from the right action of H on T_H. (You should **check** that it is a right H-torsor for this action.) ■

It is, in fact, the case that Φ is equivalent to the equivalence induced by Q, but this is more relevant in a later context, so will be revisited then.

7.5.3 Twisted objects:

Continuing our study of torsors and bitorsors, as such, we should mention the analogue of fibre bundles in this context.

Let P be a left G-torsor on B and E a space over B on which G acts on the right. We can again use the contracted product construction to form $E^P := E \wedge^G P$ over B. In this context we call E^P the *P-twisted form* of E.

Choice of a local section s of P over an open set U determines an isomorphism $\varphi_P : E^P_{|U} \cong E_U$, so E^P is locally isomorphic to E. (Beware, especially if you are used to the case where E is a product space over B, so $E = F \times B$, say. In that case E^P is locally trivial in a very strong sense, but this need not be so in general).

Suppose E_1 is now a space over B and there is an open cover $\mathcal{U}$ of B over which E_1 is locally isomorphic to E, then the sheaf or bundle $Isom_B(E_1, E)$ is a left torsor on B for the action of the bundle of groups, $G := Aut_B(E)$. This gives us a G-torsor and a space, E, on which G acts on the right.

These two constructions are inverse to each other.

In particular, if we are given G and have a second bundle of groups, H, on B, which is locally isomorphic to G, then $P := Isom_B(H, G)$ is a $Aut_B(G)$-torsor. It is worth pausing to think out the components of this fact. The object $Isom_B(H, G)$ exists, as before, because of the Cartesian closed assumption about our categories of bundles over B, (e.g. if we are interpreting bundles as sheaves, $Isom_B(H, G)$ is a subsheaf of the function sheaf, $Sh(B)(H, G)$, but although it would always have an action of $Aut_B(G)$, we need the 'H is locally isomorphic to G' condition to ensure the existence of local sections and hence to ensure it is a $Aut_B(G)$-torsor).

Look now at $G \wedge^{Aut(G)} P$ and the map

$$G \wedge^{Aut(G)} P \to H$$

$$(g, u) \mapsto u^{-1}(g).$$

(We make $Aut_B(G)$ act on the right of G, via the obvious left action.) This map is an isomorphism and so H is the P-twisted form of G for this right $Aut_B(G)$-action.

On the other hand, if G is a bundle of groups on B and P is a left G-torsor, $H := G \wedge^{Aut(G)} P$ is a bundle of groups on B locally isomorphic to G and this identifies P with the left $Aut_B(G)$-torsor, $Isom_B(H, G)$.

This provides a torsor's-eye-view of our examples on fibre bundles given in section 7.1.3, (Case study, page 276). We will sketch in a few more details:

A vector bundle, V, of rank n on B is locally isomorphic to $\mathbb{R}^n_B := \mathbb{R}^n \times B$. The group of automorphisms of this is the trivial bundle of groups, $G\ell(n, \mathbb{R})_B := Gl(n, \mathbb{R}) \times B$. The left $G\ell(n, \mathbb{R})_B$-torsor on B associated to V is $Isom(V, \mathbb{R}^n_B)$ and this is just the *frame bundle*, P_V, of V. The vector bundle V is a bundle of groups, so the above discussion applies, showing it to be the P_V-twist of $\mathbb{R}^n_B$. Conversely for any $G\ell(n, \mathbb{R})_B$-torsor P on B, the twisted object $V = \mathbb{R}^n_B \wedge^{G\ell(n,\mathbb{R})_B} P$ is the rank n vector bundle associated to P and its frame bundle P_V is canonically isomorphic to P. (If you have not explored vector bundles and differential manifolds, a brief excursion into that area may be well worthwhile, as it reinforces the geometric origins and intuitions behind this area of cohomology.)

7.5.4 Cohomology and Bitorsors

Earlier, (page 317), we saw how local sections, s, of a torsor, P, over an open cover, $\mathcal{U}$, led to 'transition maps', or 'cocycles', $g_{ij} : U_{ij} \to G$, on the intersections. Changing local sections to $s'_i : U_i \to P$, $s'_i = g_i s_i$, we have that the corresponding cocycles g'_{ij} are related via the coboundary relation

$$g'_{ij} = g_i g_{ij} g_j^{-1},$$

to the earlier ones. This led to the set of equivalence classes, $H^1(\mathcal{U}, G)$, and eventually to the cohomology set $\check{H}^1(B, G)$, which classified isomorphism classes of G-torsors on B.

What would be the additional structure available if P was a (G, H)-bitorsor? The family of local sections $s_i : U_i \to P$ then would also determine a family of local isomorphisms $u_i : H_{U_i} \to G_{U_i}$, where

$$u_i(h)s_i = s_i.h.$$

Remark: This formula needs a bit of thought. That u_i *is* a bijection is clear, as it follows from the fact that P is a G-torsor, but that it *is* a homomorphism needs a bit more care. The defining equation is specifically using the local section s_i so, for instance, on a more general element $g.s_i$ we have to extend the formula using G-equivariance, (remember the two actions are independent), so $(g.s_i).h = g.u_i(h).s_i$. In particular, if h_1 and h_2 are two local section of H over U_i, then $s_i.(h_1h_2) = u_i(h_1).s_i.h_2 = u_i(h_1)u_i(h_2).s_i$, so $u_i(h_1h_2)$ does equal $u_i(h_1)u_i(h_2)$.

Over an intersection U_{ij} of the cover, $s_i = g_{ij}s_j$, so

$$u_i = i_{g_{ij}}u_j$$

with as usual, i the inner automorphism homomorphism from G to $Aut_B(G)$, sending g to i_g. The (u_i, g_{ij}) therefore satisfy the *cocycle conditions*

$$g_{ik} = g_{ij}g_{jk}$$

and

$$u_i = i_{g_{ij}}u_j.$$

Changing the local sections to $s'_i = g_is_i$ in the usual way determines coboundary relations

$$g'_{ij} = g_ig_{ij}g_j^{-1}$$

and

$$u'_i = i_{g_i}u_i.$$

Isomorphism classes of (G, H)-bitorsors on B with given local trivialisation over $\mathcal{U}$, thus are classified by the set of equivalence classes of such cocycle pairs (g_{ij}, u_i) modulo coboundaries. In the most important case of G-bitorsors, the u_i are locally defined automorphisms of the G_{U_i} and so are local sections of $Aut(G)$.

We thus have from a G-bitorsor, P, a fairly simple way to get a piece of descent data, $\{(g_{ij}, u_i)\}$, with the right sort of credentials to hope for a 'reconstruction' process. We needed P to trivialise over the open cover $\mathcal{U} = \{U_i\}$ and then to chose local sections, $s_i : U_i \to P$. This gave $\{g_{ij} : U_{ij} \to G\}$ and $\{u_i : U_i \to Aut(G)\}$, so let us start off with these and see how much of P's structure we can retrieve.

Putting aside the u_is for the moment, we have a G-valued cocycle, $\{g_{ij}\}$, and we already have seen how to build a G-torsor from that information. Recall we take

$$P = \bigsqcup_i G(U_i)/\sim,$$

where $(g, i) \sim (gg_{ij}, j)$. (The basic relation is really that $(1_{U_i}, i) \sim (g_{ij}, j)$ with the left translation $G(U_{ij})$-action giving the more general form.) We thus have a lot of the structure already available. We are left to obtain a right G-action, which has to be 'independent' of the left action, *i.e.*, to commute with it as in the first definition of this section. (To avoid confusion between the two actions, we will pass to the (G,H)-bitorsor case so $u_i : U_i \to Isom(H,G)$, and will denote local elements that act on the right by h_i, whilst any acting on the left by g_i.)

In our 'reconstructed' P, there is clearly a natural choice for a local section over U_i, namely the equivalence class of the identity element $1_{U_i} \in G(U_i)$, or, more exactly of $(1_{U_i}, i)$, then we could define

$$[g,i].h := [g.u_i(h), i].$$

It is clear that this is a right action, since u_i is a homomorphism, and that it does not interfere with the left $G(U_i)$-action, which is $g'[g,i] = [g'g,i]$. Of course, we have to check compatibility with the equivalence relation, and that is exactly what is needed for checking that it works on adjacent patches / open sets of the cover. The key case is to work with a local section h of G over an open set, U, and examine what h does on patches U_i, U_j and their intersection. (Of course, this presupposes that we are intersecting U_i, etc., with U, *i.e.*, that we are effectively working with an open cover of U itself.)

We know how the U_i are related over the different patches, namely

$$u_i = i_{g_{ij}} u_j,$$

which on our local element, h, gives

$$u_i(h) = g_{ij} u_j(h) g_{ij}^{-1}.$$

As h is defined on U, the restrictions to the various U_i form a compatible family, (*i.e.*, we do not need to worry about transitions for h in formulae), so

$$[g,i].h = [gu_i(h), i] = [g.u_i(h)g_{ij}, j],$$

on the one hand, and also

$$[g.g_{ij}, j].h = [gg_{ij}u_j(h), j].$$

The earlier identity shows that

$$u_i(h)g_{ij} = g_{ij}u_j(h),$$

so these are the same local element of P over U_{ij}.

The u_i were introduced as the way to link local right and left actions,

$$u_i(h).s_i = s_i.h.$$

They also have an interpretation if we seek to study when a given left G-torsor, P, has an additional G-bitorsor, or more generally, a (G,H)-bitorsor structure. The cocycle rules linking the u_i with the g_{ij} involve the group homomorphism $i : G \to Aut(G)$. The g_{ij} part of the cocycle family only uses the left G-torsor structure on P. It is perhaps only because of 'natural curiosity', but it does seem natural to look at the $Aut(G)$-torsor, $i_*(P)$. Our earlier calculations show that suitable cocycles for this are given by $\{i(g_{ij})\} = \{i_{g_{ij}}\}$, but the u_i now look very like a coboundary! In fact that key equation, $u_i = i_{g_{ij}} u_j$, can obviously be rewritten as

$$i_{g_{ij}} = u_i u_j^{-1},$$

or

$$i_{g_{ij}} = u_i.1.u_j^{-1},$$

so the class of $\{i_{g_{ij}}\}$ is 'cohomologically null', *i.e.*, equivalent to 1 modulo coboundaries. In other words, $i_*(P) \cong T_{Aut(G)}$.

Conversely, if we have P and hence its cocycle representation, *and* a 0-cocycle trivialising $i_*(P)$, so $\{i_{g_{ij}}\}$ is a coboundary,

$$\{i_{g_{ij}}\} = \alpha_i \alpha_j^{-1},$$

then taking $u_i = \alpha_i$, we have a cocycle pair, (g_{ij}, u_i), giving P a G-bitorsor structure.

We clearly should look at this from the viewpoint of contracted products as they have a clearer *geometric* interpretation. The $Aut(G)$-torsor, $i_*(P)$, has a description as $Aut(G)_i \wedge^G P$, thus, by quotienting $Aut(G) \times P$ by the equivalence relation

$$(\alpha.g.p) \sim (\alpha \circ i(g), p).$$

The fact that $i_*(P)$ is locally trivial was given by the local sections induced by those $s_i : U_i \to P$ for P, namely

$$[(1, s_i)] : U_i \to Aut(G)_i \wedge^G P.$$

(Note this formulation is slightly different from that in Breen, [48], as he uses the opposite group $Aut^o(G)$ and i', but we can avoid that extra complication for our purposes here, since we really only need $\alpha = 1$ in the above.)

We can compare these local sections on overlaps U_{ij},

$$(1, s_i) \sim (1, g_{ij} s_j) \sim (i_{g_{ij}}, s_j) \sim (u_i u_j^{-1}),$$

but now our local sections $[(1, s_i)]$ are equivalent to others $t_i = [(u_i^{-1}, s_i)]$, which *agree* on overlaps

$$t_i = [(u_i^{-1}, s_i)] = [(u_i^{-1} u_i u_j^{-1}, s_i)] = t_j$$

over U_{ij}. These t_i thus form a global section for $i_*(P)$, which is hence the trivial torsor, up to isomorphism.

Reversing the argument, a global section of $i_*(P)$, together with the structural cocycle $\{g_{ij}\}$ for P gives a G-bitorsor structure on P. (We will return to this in more generality a bit later.)

We thus have that a G-bitorsor is a *relative* $\mathsf{Aut}(G)$*-torsor*, where $\mathsf{Aut}(G) = (G, Aut(G), \iota)$. It corresponds to a G-torsor, P, together with a trivialisation of $\iota_*(P)$. Using the fact that morphisms from the induced torsor $\iota_*(P)$ to $T_{Aut(G)}$ corresponds to morphisms over ι from P to $T_{Aut(G)}$, we get a second description, which is very useful for further generalisation.

7.5.5 Bitorsors, a simplicial view.

Pausing in our development, let us return to the simplicial viewpoint that we adopted earlier. The cover $\mathcal{U}$ gives a sheaf / bundle,

$$p : E = \sqcup \mathcal{U} \to B$$

and by repeated pullbacks, we get a simplicial sheaf / bundle,

$$N(\mathcal{U}) \to B.$$

The cocycle $\{(u_i, g_{ij})\}$ consists of a family $\{u_i\}$ giving a morphism,

$$\mathbf{g}_0 : N(\mathcal{U})_0 = \sqcup\mathcal{U} \to Aut(G),$$

together with a second family

$$\mathbf{g}_1 : N(\mathcal{U})_1 \to G \rtimes Aut(G).$$

This second piece of data is not quite as obvious as it might seem. The earlier model of the crossed view of group extensions used the crossed module, $\mathsf{Aut}(G) = (G, Aut(G), \iota)$ directly. Here we are using the cat^1-group / gr-groupoid / 2-group analogue, which can also be thought of simplicially as in our discussion of algebraic 2-types, page 97. Recall the face maps

$$d_i : G \rtimes Aut(G) \to Aut(G), \quad i = 0, 1,$$

are given by

$$\begin{aligned} d_1(g, \alpha) &= \alpha, \\ d_0(g, \alpha) &= i_g \circ \alpha \end{aligned}$$

and the degeneracy is

$$s_0(\alpha) = (1_G, \alpha).$$

The maps $\mathbf{g}_0$, $\mathbf{g}_1$ are to be hoped to be a part of a simplicial map from the simplicial sheaf $N(\mathcal{U})$ to the sheaf of simplicial groups, $K(\mathsf{Aut}(G))$, and to check that this is indeed the case, we need to recall that 'bundle-wise' the elements of $\sqcup\mathcal{U} = N(\mathcal{U})_0$ can usefully be thought of as pairs (x, U), where $U \in \mathcal{U}$ and $x \in U$. Of course, the projection maps p sends (x, U) to x itself. The 1-simplices of $N(\mathcal{U})$ therefore are given by triples (x, U_0, U_1) with $x \in U_0 \cap U_1$, so the corresponding face and degeneracy maps are

$$\begin{aligned} d_1(x, U_0, U_1) &= (x, U_0), \\ d_0(x, U_0, U_1) &= (x, U_1), \\ s_0(x, U) &= (x, U, U). \end{aligned}$$

We can thus see what this $\mathbf{g}$ must satisfy. We write $\mathbf{g}_1 = (g, \alpha)$ as before, and will try to identify what g and α must be. We have, then,

- $d_1\mathbf{g}_1 = \mathbf{g}_0 d_1$ means $\alpha = u_{|U_0} =: u_0$;
- $d_0\mathbf{g}_1 = \mathbf{g}_0 d_0$ means $i_g u_0 = u_{|U_1} =: u_1$;
- $s_o\mathbf{g}_0 = \mathbf{g}_1 s_0$ is a normalisation condition, which will make more sense when the first two conditions have been explored in more detail.

The obvious way to build $\mathbf{g}_1$, *i.e.*, g itself, is thus to take

$$\mathbf{g}(x, U_0, U_1) = (g_{10}(x), u_0(x)),$$

and to require that g_{ii} is 1_G restricted to $U_{ii} = U_i \cap U_i$ for the normalisation.

To continue our simplicial description, we should look at triple intersections, *i.e.*, $N(\mathcal{U})_2$, and the corresponding $K(\mathsf{Aut}(G))_2$. The points of $N(\mathcal{U})_2$ are, of course, represented by symbols such as (x, U_0, U_1, U_2), whilst those of $K(\mathsf{Aut}(G))_2$ above the point x, are of form $(g_2, g_1, \alpha)(x)$. The face maps of $N(\mathcal{U})$ are the obvious ones, $d_2(x, U_0, U_1, U_2) = (x, U_0, U_1)$, and so on, whilst

$$\begin{aligned} d_2(g_2, g_1, \alpha) &= (g_1, \alpha), \\ d_1(g_2, g_1, \alpha) &= (g_2 g_1, \alpha), \\ d_0(g_2, g_1, \alpha) &= (g_2, i_{g_1}\alpha), \end{aligned}$$

with the s_i inserting an identity in the appropriate place. (Of course, all these g_i, etc., are 'local elements', so are really local sections, and our formulae would have, over a given x, the values $g_2(x)$, etc., as above.)

We want $\mathbf{g}$ to be a simplicial morphism, so on 2-simplices we expect, for (x, U_0, U_1, U_2),

$$d_2\mathbf{g}_2 = \mathbf{g}_1 d_2,$$

etc., *i.e.*, if $\mathbf{g}_2(x, U_0, U_1, U_2) = (g_2, g_1, \alpha)(x)$, the d_2-face $(g_1, \alpha)(x) = (g_{10}(x), u_0(x))$, so $g_1, = g_{10}$, $\alpha = u_0$, and then the d_0 face gives $g_2 = g_{21}$. Finally the d_1-face gives

$$g_2 g_1 = g_{20},$$

so this gives us the cocycle condition

$$g_{21}g_{10} = g_{20}$$

over U_{012}.

The other simplicial morphism rules give compatibility with degeneracies, but using simplicial identities, these then give that $g_{01} = g_{10}^{-1}$, *i.e.*, again a normalisation condition.

We thus have

(i) the bundle of crossed modules $\mathsf{Aut}(G)$ given by $(G, Aut(G), \iota)$;

(ii) the corresponding bundle of simplicial groups, $K(\mathsf{Aut}(G))$;

(iii) the bundle / sheaf of simplicial sets, $N(\mathcal{U})$;
and

(iv) our local cocycle description of our bitorsor, P,

giving, it would seem, a simplicial map

$$\mathbf{g} : N(\mathcal{U}) \to K(\mathsf{Aut}(G)).$$

Conversely such a simplicial map gives a cocycle (for **you to check**).

(Here we are abusing notation slightly, since the domain of $\mathbf{g}$ is a bundle of simplicial sets, whilst the right hand side is the underlying simplicial set bundle of the simplicial group bundle, not that simplicial group bundle itself, however we have not shown that in the notation. It is, however, an important point to note.)

Continuing with this quite detailed look at the 'cocycles for bitorsors' context, we clearly have next to look at the 'change of local sections' from this simplicial viewpoint.

Suppose we change to local sections, $s'_i = g_i s_i$, so, as before, get

$$g'_{ij} = g_i g_{ij} g_j^{-1}$$

and

$$u'_i = i_{g_i} u_i.$$

If we are describing cocycles as simplicial maps, then fairly naturally, we might hope that the equivalence relation coming from coboundaries, as here, was something like homotopy of simplicial maps. We can see immediately that this looks to be not that stupid an idea, by looking at the base of the corresponding simplicial objects.

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & G^{(2)} \rtimes Aut(G) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G \rtimes Aut(G) \arrow[r, shift left=1] \arrow[r, shift right=1] & Aut(G) \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & N(\mathcal{U})_2 \arrow[u, shift left=1, "g_2"] \arrow[u, shift right=1, "g'_2"'] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & N(\mathcal{U})_1 \arrow[u, shift left=1, "g_1"] \arrow[u, shift right=1, "g'_1"'] \arrow[r, shift left=1] \arrow[r, shift right=1] & N(\mathcal{U})_0 \arrow[u, shift left=1, "g_0"] \arrow[u, shift right=1, "g'_0"']
\end{tikzcd}$$

then we would expect that a homotopy between $\mathbf{g}$ and $\mathbf{g}'$ would pick out, for each (x, U_0) in $N(\mathcal{U})_0$, an element $(g, \alpha) \in G \rtimes Aut(G)$ with $g = d_1(g, \alpha) = g_0$, $d_0(g, \alpha) = g'_0$, *i.e.*, $\alpha = u_0$ and $g'_0 = u'_0 = i_{g_0} \circ u_0$, exactly as needed. To see if this works in higher dimensions, we need to glance again at simplicial homotopies. We will take a fairly naïve view of them to start with. We have already met them in passing in our discussion of simplicial mapping spaces in Chapter 6.3, page 256.

Given $f, g : K \to L$, two morphisms of simplicial sets, a *simplicial homotopy* from f to g is, of course, a map

$$h : K \times \Delta[1] \to L$$

such that if $e_0 : \Delta[0] \to \Delta[1]$ is the 0-end of $\Delta[1]$, (so is actually represented by the d_1 face - beware of confusion) and $e_1 : \Delta[0] \to \Delta[1]$, gives the 1-end, then

$$f = h \circ (K \times e_0),$$

$$g = h \circ (K \times e_1).$$

(More on such cylinder based homotopies in abstract settings can be found in Kamps and Porter, [171]. In a general context, simplicial homotopy does *not* give an equivalence relation on the set of simplicial maps as, although it gives a reflexive, symmetry and transitivity depend on the existence of fillers in the simplicial set of morphisms.)

This is the neat geometric way of picturing simplicial homotopies. There is an alternative 'combinatorial' way that is also very useful (see [171], p.184-186, for a discussion - but not for the formulae which were left as an exercise!) This gives h being specified by a family of maps,

$$h^n_i : K_n \to L_{n+1},$$

indexed by $n = 0, 1, \ldots$, and i with $0 \leq i \leq n$, and satisfying some face and degeneracy relations that we will give later on. For the moment, we will only need to use these in low dimensions, so imagine the lowest dimension $h^0_0 : K_0 \to L_1$. For each vertex, k_0, we get an edge / 1-simplex in L_1 joining $f_0(k_0)$ and $g_0(k_0)$. Now if $k_1 \in K_1$, we expect a square in $K \times \Delta[1]$ looking like

$$\begin{tikzcd}
\cdot \arrow[r, "{(k_1, s_0 1)}"] \arrow[dr, phantom, "\tau_0", pos=0.3] & \cdot \\
\cdot \arrow[u, "{(s_0 d_1 k_1, \iota)}"] \arrow[r, "{(k_1, s_0 0)}"'] \arrow[ur] \arrow[ur, phantom, "\tau_1"', pos=0.7] & \cdot \arrow[u, "{(s_0 d_0 k_1, \iota)}"']
\end{tikzcd}$$

with $\iota \in \Delta[1]_1$, the unique non-degenerate 1-simplex, corresponding to $id : [1] \to [1]$. (Remember the product of simplicial sets, K and L, has $(K \times L)_q = K_q \times L_q$.) The homotopy h has to thus give two 2-simplices of L. These will be $h_0^1(k_1) := h(\tau_0)$ and $h_1^1(k_1) := h(\tau_1)$ respectively. We first note that $d_1\tau_0 = d_1\tau_1$, so

$$d_1 h_0^1 = d_1 h_1^1.$$

Likewise the geometric picture tells us that $d_2 h_1^1 = f_1$ and $d_0 h_0^1 = g_1$ and finally that $d_0 h_0^1 = h_0^0 d_0$, whilst $d_2 h_1^1 = h_0^0 d_1$.

In our special case of that general square, $k_1 = (x, U_0, U_1)$ with $d_0 k_1 = (x, U_1)$, $d_1 k_1 = (x, U_0)$, thus our earlier choices should mean the horizontal edges are mapped to

$$\begin{aligned} h((x, U_0, U_1), 0) &= (g_{10}(x), u_0(x)), \\ h((x, U_0, U_1), 1) &= (g'_{10}(x), u'_0(x)), \end{aligned}$$

and the vertical ones,

$$\begin{aligned} h((x, U_1), \iota) &= (g_1(x), u_1(x)), \\ h((x, U_0), \iota) &= (g_0(x), u_0(x)). \end{aligned}$$

They match up as required.

We need to work out h_0^1 and h_1^1. These will map (x, U_0, U_1) to 2-simplices of $K(\mathsf{Aut}(G))$, *i.e.*, to triples $(\gamma_2, \gamma_1, \alpha)$, with $\gamma_i \in G$ and $\alpha \in Aut(G)$. First we look at $h_0^1(x, U_0, U_1)$ and the faces we know of it.

Let $h_0^1(x, U_0, U_1) = (\gamma_2, \gamma_1, \alpha)$, then the two descriptions of $d_2 h_0^1$ give

$$(g_{10}(x), u_0(x)) = (\gamma_1, \alpha),$$

whilst for $d_0 h_0^1$, we have

$$(g_1(x), u_1(x)) = (\gamma_2, i_{\gamma_1} \circ \alpha).$$

We thus have $\gamma_1 = g_{10}(x)$, $\alpha = u_0(x)$ and $\gamma_2 = g_1(x)$ and we can check back that $i_{g_{10}} u_0 = u_1$ from earlier calculations. We have h_1^1 completely specified as

$$h_1^1(x, U_0, U_1) = (g_1(x), g_{10}(x), u_0(x)).$$

This gives $d^1 h_1^1(x, U_0, U_1) = (g_1(x) g_{10}(x), u_0(x))$, which we will need shortly.

We next turn to $h_0^1(x, U_0, U_1)$ and reset the meaning of γ_i and α, so this is $(\gamma_2, \gamma_1, \alpha)$. We do a similar calculation and this gives

$$h_0^1(x, U_0, U_1) = (g'_{10}(x), g_0(x), u_0(x)).$$

This 'feels' right, but we have to check it matches h_0^1 on the diagonal:

$$d_1 h_0^1(x, U_0, U_1) = (g'_{10}(x) g_0(x), u_0(x)),$$

but $g'_{10}(x) = g_1(x) g_{10}(x) g_0(x)^{-1}$, so this equals $(g_1(x) g_{10}(x), u_0(x))$, as hoped.

We have laboriously checked through the calculations of (h_0^1, h_1^1) to show how well behaved things really are. It is reasonably easy to extend the calculation to all dimensions. What needs to be retained is that h was completely specified by the coboundary and cocycle data and, conversely, if we were given any homotopy h between $\mathbf{g}$ and $\mathbf{g}'$, then $\mathbf{g}$ and $\mathbf{g}'$ will be equivalent. This suggests that the simplicial mapping sheaf or bundle $\underline{\mathcal{S}Sh_B}(N(\mathcal{U}), K(\mathsf{Aut}(G)))$, is what is really encoding the data in a neat way. (If you are hazy about simplicial mapping spaces, recall that if K and L are simplicial sets, $\underline{\mathcal{S}}(K, L)$ is the simplicial set of simplicial maps and (higher) homotopies, so

$$\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L).$$

Using the constant simplicial sheaves, $\Delta[n]_B$, to replace the use of the $\Delta[n]$ gives a similar simplicial enrichment for the category of simplicial sheaves / bundles on B, but this can be localised to make $\underline{\mathcal{S}Sh_B}(K, L)$, a simplicial sheaf as well.)

Earlier we omitted the detailed description of homotopies as families of maps. To complete our picture here, that description will now be useful. We first give it for simplicial sets, so in the very classical setting.

Let K and L be simplicial sets, and $f, g : K \to L$ two simplicial maps, then a homotopy

$$h : K \times I \to L$$

between f and g can be specified by a family of functions

$$h_, = h_i^n : K_n \to L_{n+1},$$

satisfying various relations. To understand how these arise, we use some simple notation extending that which we used above.

The non-degenerate $(n+1)$-simplices of $\Delta[n] \times \Delta[1]$ are of form $(s_j\iota_n, s_{\hat{j}}\iota_1)$, where $\iota_n \in \Delta[n]_n$ is the unique non-degenerate n-dimensional simplex corresponding to $id_{[n]} : [n] \to [n]$ in the description of $\Delta[n]$ as $\mathbf{\Delta}(-, [n])$, ι_1 being similarly specified for $n = 1$, and where $s_{\hat{j}}$ is the multiple degeneracy corresponding to $\hat{j} = (0, \ldots, \hat{j}, \ldots, n)$, *i.e.*, $s_n \ldots s_0$, but without s_j. (Any $(n+1)$ simplex of $\Delta[1]$ is given by an increasing map $[n+1] \to [1]$, so can be represented as a string $(0, \ldots, 0, 1, \ldots, 1)$, say with j zeroes. This will be $s_{\hat{j}}\iota_1$, since the first j degeneracies 'add in' 0s, whilst those after the $(j+1)^{st}$, that is, after the break, will add in 1s. The simplicial identities give $s_i s_j = s_j s_{i-1}$ if $i > j$, so $s_{\hat{j}}$ has a second useful description as $(s_{last})^{n-j}(s_0)^j$.)

For an n-simplex $k \in K$, we denote $(s_j k, s_{\hat{j}}\iota_1)$ by τ_j, or, more exactly, $\tau_j(k)$ if confusion might arise. We then encode our $h : K \times I \to L$ by $h_j^n(k) = h(\tau_j(k))$. The homotopy h is, of course, a simplicial map so, for any $0 \leq i \leq n+1$, we have $d_i h = h d_i$. These relations translate to give the following rules:

$$d_0 h_0 = g, \qquad\qquad d_{n+1} h_n = f,$$

$$\begin{array}{rcll} d_i h_j & = & h_{j-1} d_i & \text{for } i < j, \\ d_{j+1} h_{j+1} & = & d_{j+1} h_j, & \\ d_i h_j & = & h_j d_{i-1} & \text{for } i > j+1, \end{array}$$

and the corresponding degeneracy rules are

$$s_i h_j = h_{j+1} s_i, \qquad i \leq j,$$

$$s_i h_j = h_j s_{i-1}, \qquad i > j.$$

Of course, these h_js etc. are further indexed by a dimension h_j^n, so, for instance, $d_i h_j^n = h_{j-1}^{n-1} d_i$ is the full form of the second line of these.

Aside on Tensors and Cotensors: It is often the case, when considering simplicial objects in a category, $\mathcal{A}$, that one can form a 'tensor', $X \otimes I$, using a coproduct in each dimension, then one defines a homotopy to be a morphism

$$h : X \otimes I \to Y.$$

The construction of this 'tensor' is: given any simplicial set K, and a simplicial object X in $\mathcal{A}$, (where $\mathcal{A}$ has the coproducts that we will be using below),

$$(X \otimes K)_n = \bigsqcup_{k \in K_n} X_n(k) \text{ with each } X_n(k) = X_n,$$

i.e., a K_n-indexed copower of X_n. Using an element based notation, the usual way of denoting the copy of $x \in X_n$, in the k-indexed copy of X_n would be $x \otimes k$ and then face and degeneracy maps are given, in $X \otimes K$, by $d_i(x \otimes k) = d_i x \otimes d_i k$, etc., *i.e.*, 'component-wise'. In this setting again $h : X \otimes \Delta[1] \to Y$ can be decomposed to give a family $\{h^n_j : X_n \to Y_{n+1}\}$. The same description works if instead of a tensor, we have a cotensor.

The setting is that of $\mathcal{S}$-enriched categories having enough (finite) limits. Suppose now $\mathcal{C}$ is $\mathcal{S}$-enriched, so for objects $X, Y \in \mathcal{C}$, we can form a simplicial set $\underline{\mathcal{C}}(X,Y)$ of 'morphisms' from X to Y. A homotopy between $f, g \in \underline{\mathcal{C}}(X,Y)_0$ will, of course, be a 1-simplex $h \in \underline{\mathcal{C}}(X,Y)$ with $d_1 h = f$, $d_0 h = g$. If $\mathcal{C}$ is *cotensored* then, for any simplicial set K, there is a *cotensor*, $\overline{\mathcal{C}}(K,Y)$, for each Y in $\mathcal{C}$, such that

$$\mathcal{S}(K, \underline{\mathcal{C}}(X,Y)) \cong \mathcal{C}(X, \overline{\mathcal{C}}(K,Y)).$$

Of particular use is the case $K = \Delta[1]$, as a 1-simplex $h \in \underline{\mathcal{C}}(X,Y)$ can be represented by an element in $\mathcal{S}(\Delta[1], \underline{\mathcal{C}}(X,Y))$ and thus by an element of $\mathcal{C}(X, \overline{\mathcal{C}}(\Delta[1],Y))$. In other words, a homotopy is a morphism,

$$h : X \to \overline{\mathcal{C}}(\Delta[1],Y),$$

so $\overline{\mathcal{C}}(\Delta[1],Y)$ behaves like a path-space object or cocylinder on Y. The construction of $\overline{\mathcal{C}}(K,Y))$ uses limits and can be 'deconstructed' to give a family based description of homotopies, just as before. The nice thing about that description is, however, that it makes sense whatever category $\mathcal{A}$ is as it is merely governed by some small list of identities between composite maps. (For any $\mathcal{A}$, $Simp(\mathcal{A})$ is $\mathcal{S}$-enriched, so can be taken to be the $\mathcal{C}$ above; see Kamps and Porter, [171] for a discussion of some of these ideas, in particular on cylinders and cocylinders as a basis for 'doing' homotopy theory in some seemingly unlikely places! We will examine simplicially enriched categories more fully later on, starting on page 495.) A word of caution, however, is in order. As we mentioned earlier, homotopies are not always composable, nor reversible. If we have a homotopy, in this abstract setting, between morphisms f_0 and f_1 and another between f_1 and f_2, then there may not be one directly from f_0 to f_2. This is annoying! It depends on Kan filling conditions in the simplicial hom-sets. Luckily in many of the cases that we need, the composition of homotopies does work, however once or twice we will have to be careful in the wording. Of course, we could generate the equivalence relation defined by 'direct' homotopy, but, whilst this is very useful, it does often require a chain or 'zig-zag' of explicit 'direct' homotopies if it is to be of maximal use. Conditions on $\mathcal{A}$ can be found that imply that homotopy in $Simp(\mathcal{A})$ is an equivalence relation, (but I do not know if optimal such conditions are known).

Remark: We are heading for a fairly simplicial description of cohomology. A very useful reference at this point is Jack Duskin's memoir, [107], although that emphasises the Abelian theory only, and also his outline of a higher dimensional descent theory, [109]. From this simplicially based theory, it is then a short journey to give a 'crossed' description of the bitorsor based, (and then gerbe based), non-Abelian cohomology.

Pause: At this point, it is a good idea to take stock of what we have shown. We have used local sections $\{s_i\}$ to get cocycles $\{(g_{ij}, u_i)\}$ and have constructed the beginnings of a simplicial morphism $\mathbf{g}$ from $N(\mathcal{U})$ to $K(\mathsf{Aut}(G))$. So far we have explicitly given $\mathbf{g}_n$ for $n \leq 2$ only, and so should check higher dimensions as well. (Intuitively it would be strange if something came adrift in higher dimensions, since $\mathsf{Aut}(G)$ '*is* a 2-type', but we should make certain!) We also have to check our interpretation of homotopies in higher dimensions.

Let us see what $\mathbf{g}_n : N(\mathcal{U}) \to K(\mathsf{Aut}(G))$ would have to satisfy. Let

$$\mathbf{g}_n(x, U_0, \ldots, U_n) = (g_n, \ldots, g_1, \alpha),$$

then

$$\begin{aligned} d_n\mathbf{g}_n(x, U_0, \ldots, U_n) &= (g_{n-1}, \ldots, g_1, \alpha), \\ d_0\mathbf{g}_n(x, U_0, \ldots, U_n) &= (g_n, \ldots, g_2, i_{g_1} \circ \alpha), \\ d_i\mathbf{g}_n(x, U_0, \ldots, U_n) &= (g_n, \ldots, g_{i+1}g_i, \ldots, g_1, \alpha), \end{aligned}$$

for $0 < i < n$, so we *can* thus read off $\mathbf{g}_n$ from a knowledge of its faces! In other words, our intuition was right and $\mathbf{g}_0$, $\mathbf{g}_1$ and $\mathbf{g}_2$ determined $\mathbf{g}_n$ in all dimensions.

A very similar calculation shows that $\mathbf{h} : N(\mathcal{U}) \times I \to K(\mathsf{Aut}(G))$ corresponds to the 1-cocycle $\{g_i\}$ and nothing more.

We thus have established a one-one correspondence between the set of isomorphism classes of G-bitorsors that trivialise over $\mathcal{U}$ and the set $[N(\mathcal{U}), K(\mathsf{Aut}(G))]$ of homotopy classes of simplicial sheaf maps from $N(\mathcal{U})$ to the underlying simplicial sheaf of the simplicial group, $K(\mathsf{Aut}(G))$.

We should continue our pause here and make some comments about the overall situation. This set can be interpreted as a type of zeroth non-Abelian hyper-cohomology of B relative to the cover $\mathcal{U}$. It is $H^0(N(\mathcal{U}), \mathsf{Aut}(G))$. But what is hyper-cohomology? We will have a look at its classical Abelian form below, but note that the coefficients, here, are in a sheaf of crossed modules, so will also need to look at that in more detail. We saw earlier a related situation (in section 6.1) where we replaces the crossed module $\mathsf{Aut}(G)$ by a general one $\mathsf{Q} = (K, Q, q)$, when discussing non-Abelian extensions of G by K 'of the type of Q'. We there obtained a cohomology set, there called $H^2(G, \mathsf{Q})$, identifiable as $[\mathsf{C}(G), \mathsf{Q}]$, and the correspondence was obtained by identifying the cocycles as maps of crossed complexes and, as $\mathsf{C}(G)$ is 'free', it sufficed to give them on the generating elements, in other words on the analogue of $N(\mathcal{U})$.

The reason given for introducing the notion of extension of type Q was to obtain functoriality in the coefficients. (Recall that if $\varphi : G \to H$ is a homomorphism of groups then it is not clear when there is a morphism of crossed modules from $\mathsf{Aut}(G)$ to $\mathsf{Aut}(H)$ which is φ on the 'top group'.) This also gave a good possibility of a finer classification of *all* extensions of G by H: some will be of the type of a particular Q, others not.

In our bitorsor situation, the functoriality is once again important, but the second aspect gains an additional geometric significance. A very important part of classical fibre bundle theory relates to the possibility of 'reducing the group'. For instance, suppose we have a n-dimensional real manifold, X, then its tangent bundle is a fibre bundle with each fibre a vector space of dimension n and with the transition functions taking their values in $G\ell(n,\mathbb{R})$, *i.e.*, a n-dimensional vector bundle. (Its associated $G\ell(n,\mathbb{R})_X$-torsor is, as we saw, the frame bundle.) If X is at all 'nice', we can put a Riemannian metric on it, (*i.e.*, additional structure of considerable geometric importance), and this corresponds to showing that our transition functions can be replaced by ones taking values in $O(n,\mathbb{R})$, the corresponding group of orthogonal matrices, as these are the ones that preserve the metric/inner product. Note that the tangent bundle naturally has an action by $Aut(F)$, that is the corresponding automorphism group of the fibre, F. (With our bitorsors, the corresponding acting object is a strict automorphism gr-groupoid, and we have used the corresponding crossed module, $\mathsf{Aut}(G)$.)

Other examples would correspond to other subgroups of general linear groups. Foliated structures, systems of partial differential equations, etc., correspond to sub-bundles of bundles of jets on X. These structures may be on X itself or on some given fibre bundle $E \to X$ over X. In each case, giving a G-structure on E, for a group, G, which is a subgroup of the natural group of automorphisms, corresponds to 'reducing' the $Aut(F)$-torsor to a G-torsor. Another type of structure corresponds to 'lifting' the transition functions from some given H to a G, where $\varphi : G \to H$ is a nice epimorphism. For instance, the special orthogonal group $SO(n,\mathbb{R})$ for $n \geq 2$, has a universal covering group, $Spin(n) \to SO(n,\mathbb{R})$, and extra structure of use for some applications, corresponds to *lifting* the $u_{ij} : U_{ij} \to SO(n,\mathbb{R})$ to take values in $Spin(n)$. Of course, this is not always possible. Obstructions may exist to doing it, depending in part on the topological structure of X.

All these examples were of Lie groups, *i.e.*, groups in the category of differential manifolds, but a similar intuition was central to discussions in the 1960s and 1970s of the relationship between smooth and piecewise linear structures on topological manifolds, in which various *simplicial* groups of automorphisms were related and the obstructions to lifting transition functions of certain natural simplicial bundles were the key to the problem. Again analogous situations exist in algebraic geometry involving group schemes and their 'subgroups'. Here, as a group scheme over a fixed base $Spec(K)$ is in many ways a bundle of groups, the more general theory of group bundles and change of group bundles, rather than merely change of groups, as such, is what is important here.

It would almost be fair to say that, from a historical perspective, this is one modern interpretation of Klein's original intuition of what geometry is, *i.e.*, the study of the automorphisms that preserve some 'structure'. What seems now to be emerging is the relationship between higher level 'automorphism gadgets' such as $\mathsf{Aut}(G)$ and classical invariants such as cohomology and consequently, some appreciation of higher level 'structure'. Many of the ingredients of the theory are still missing or are merely 'embryonic' in the crossed module / 2-group case as yet, but the plan of action is becoming clearer.

Returning to the detail, we therefore consider a sheaf or bundle of crossed modules, $\mathsf{M} = (C, P, \partial)$, and look at data of the form

$$\mathbf{g} : N(\mathcal{U}) \to K(\mathsf{M}),$$

so $g_0(x, U_i) = p_i(x)$ with $p_i : U_i \to P$, a local section of P over U_i and $g_1(x, U_i, U_j) = (c_{ji}(x), p_i(x))$, where $c_{ji} : U_{ji} \to C$ is a local section of C over the intersection U_{ji}. These local sections satisfy

$$\partial(c_{ji})p_i = p_j \text{ and } c_{kj}c_{ji} = c_{ki}$$

over the intersections. Corresponding to a change in local sections will be a coboundary rule of the form:

$$c'_{ij} = c_i c_{ij} c_j^{-1},$$

and

$$p'_i = \partial(c_i) p_i,$$

i.e., a homotopy between $\mathbf{g}$ and $\mathbf{g}'$. The equivalence classes will be in $H^0(N(\mathcal{U}), \mathsf{M})$ and, both in this general case and in the particular case of $\mathsf{M} = \mathsf{Aut}(G)$, it is natural to pass to the limit over covers (or if working in a more general Grothendieck topos, over hypercovers) to get the zeroth Čech hyper-cohomology set with values in M, denoted $\check{H}^0(B, \mathsf{M})$.

We have $H^0(N(\mathcal{U}), \mathsf{M}) = [N(\mathcal{U}), K(\mathsf{M})]$, and it is reasonably safe to think of $\check{H}^0(B, \mathsf{M})$ in these terms, but, in fact, one really needs to introduce the category $D(\mathcal{E}) = Ho(Simp(\mathcal{E}))$, obtained by taking the category of simplicial objects in the topos, $\mathcal{E}$, in our simplest case that of simplicial sheaves on B, and inverting the 'quasi-isomorphisms', *i.e.*, those simplicial maps that induce isomorphisms on all homotopy groups. There are several detailed treatments of this type of construction in the literature - not all completely equivalent - so we will not give another one here!

We could, and later on will, go further. We could replace the crossed module, M, by a crossed complex, or, in general, could use a simplicial group, H, instead of $K(\mathsf{M})$. We will definitely keep this in mind, but just because it could be done, does not mean it needs doing *now*. The problem is that we, as yet, have only an embryonic understanding of the algebraic and geometric properties of the situation with M a crossed module or bundle / sheaf of such things. Past experience shows that the generalisation and abstraction *will be* worth doing, but we may not yet have the auxiliary concepts and intuitions to interpret what that theory will tell us, nor what are the *significant* new questions to ask and problems to solve. As yet, there are few signposts in that new land!

7.5.6 Cleaning up 'Change of Base'

Although we have considered change of base several times, we have not had available enough machinery to handle it really adequately. In particular, we have left the question of homotopic maps inducing 'isomorphic torsors' up in the air. Now we can give a reasonable treatment of that results and at the same time treat change of base for bitorsors, (and in such a way as to handle change of base for relative M-torsors as well, and we have not formally defined *them* yet).

One conceptual difficulty left over from earlier was that if f and f' were homotopic maps from B to B', and P was a G-torsor on B', we want to be able to say that somehow $f^*(P)$ and $(f')^*(P)$ are isomorphic, yet they are 'over' different groups bundles. The first is a $f^*(G)$-torsor, the second a $(f')^*(G)$-torsor. This problem did not arise with principal G-bundles as there the 'coefficient group' was just that, a group, corresponding to a constant sheaf of groups, so the two coefficient 'groups', $f^*(G)$ and $(f')^*(G)$ were the same. Both were trivial. Our first task is thus to look at a simplicial treatment of change of base, and once that is done, a lot of things will simplify!

Suppose that $f : B \to B'$ is a continuous map and $\mathbf{g} : N(\mathcal{U}) \to K(\mathsf{M})$ represents either a G-torsor, or a G-bitorsor or, looking forward to the next section, a relative M-torsor, for M a sheaf or bundle of crossed modules on B' and we assume that that object trivialises over the open cover $\mathcal{U}$. The continuous function f pulls back that cover to $f^{-1}(\mathcal{U})$. This can either be viewed as the result of pulling back each open set to get a cover, or, equivalently but perhaps better, by forming the sheaf / étale space, $\bigsqcup \mathcal{U}$ over B' and then pulling back that sheaf to $f^*(\bigsqcup \mathcal{U})$. The result is the same. In fact we saw earlier that f^* preserved pullbacks and so $N(f^*(\bigsqcup \mathcal{U}))$ is isomorphic to $f^*(N(\mathcal{U}))$. This isomorphism is given by examining local sections of the two simplicial sheaves, so local sections of $f^*(\bigsqcup \mathcal{U})$ are induced by composition of f with a local section of $\bigsqcup \mathcal{U}$. (A detailed treatment is not quite that simple. The map can better be examined at the level of germs of local sections as we did in our discussion of f^*, page 304.)

Remark: In situations where hypercovers are needed to give an adequate cohomology theory, the functor f^* still works more or less as above. Of course, the detailed geometric nature of the construction is a bit different as ideas of germs of local sections, etc., have to be interpreted slightly differently, say, in a topos, however the intuition is much the same.

Viewed as a pullback construction, there is a canonical map from $f^*(N(\mathcal{U}))$ to $N(\mathcal{U})$, namely the projection, and this is 'over' f itself. At the level of elements, this sends $(x, f^{-1}U_0, \dots f^{-1}U_n)$ to $(x, U_0, \dots, U_n)$. Abusing notation we will call this f as well. The induced cocycle is then just the composite, $\mathbf{g}f : N(f^{-1}\mathcal{U}) \to K(\mathsf{M})$, and this gives the induced torsor, but that is a $f^*(G)$-torsor. Thus at the level of the simplicial description of the induced torsor, the work is done for us without too much pain! We just have composition with f, and that, of course, is what we expected.

The next thing to look at is the connection between the induced functors for homotopic maps. We will restrict to compact spaces to simplify the discussion. If $h : f \simeq f' : B \to B'$, and we are looking at a torsor on B' that trivialises over the open cover $\mathcal{U}$, then we can get an open cover $h^{-1}(\mathcal{U})$ on $B \times I$ and a torsor on that space just by thinking of h as a continuous map. Because of our simplifying assumption of compactness, it is possible to refine $h^{-1}(\mathcal{U})$ to a cover of the form, $\{U \times V \mid U \in \mathcal{U}', V \in \mathcal{V}\}$ for $\mathcal{U}'$ an open cover of B and $\mathcal{V}$ an open cover of the unit interval I. We will denote this cover by $\mathcal{U} \times \mathcal{V}$. We can assume that the nerve of $\mathcal{I}$ is a simplicial sheaf that is essentially a subdivision $\underline{Sd}(\Delta[1])$ of the constant simplicial sheaf on I with value $\Delta[1]$. (The cover, $\mathcal{V}$, may need further refinement to get it to be of this form, and you should look at this point, but we also are using that I is contractible to get that we have a trivial sheaf.) The nerve of a product cover is isomorphic to the product of the nerves as can be seen by inspection. We thus have that $N(\mathcal{U} \times \mathcal{V})$ can be replaced by $N(\mathcal{U}) \times \underline{Sd}(\Delta[1])$. The subdivided $\Delta[1]$ is a concatenation of a number of copies of $\Delta[1]$, end to end, so the map induced at the simplicial level from $N(\mathcal{U} \times \mathcal{V})$ to $K(\mathsf{M})$ gives us not only the two maps induced by f and f', but also a sequence of simplicial homotopies between intermediate maps. These can be composed to get a simplicial homotopy between the original induced maps. Notice none of this uses any information about the actual torsor involved except the initial assumption that it trivialises over $\mathcal{U}$. This does it! We have a description of isomorphism classes of torsors in terms of homotopic maps, we have homotopic maps so .... .

From this lots of good things flow. Homotopically equivalent spaces, say B and B', give equivalent categories of torsors over 'linked' sheaves of groups, and, in particular, if G is a constant sheaf of groups, or M a constant sheaf of crossed modules, then over the two spaces the induced sheaves are also constant, hence we can talk of G-torsors over B or over B' without fussing too much about the fact that we really mean $\underline{G}_B$- and $\underline{G}_{B'}$-torsors.

The situation for contractible spaces is then simple. All torsors over $\underline{G}_B$ are trivial, and as a consequence, if B is a space which has an open cover by contractible open sets, and such that all finite intersections of the open sets are also contractible, (*i.e.*, a Leray cover), then we automatically have lots of local sections over that cover. As manifolds are examples of spaces with this property, this comes in to be very useful in applications of the torsors to geometry.

7.6 Relative M-torsors

(The basic references are Breen's paper [49], (but our conventions are different and so some of the results also look different), and also the papers of Jurčo, in particular, [170].)

7.6.1 Relative M-torsors: what are they?

What are the objects corresponding to a $\mathbf{g} : N(\mathcal{U}) \to K(\mathsf{M})$? We saw that this consisted of some local sections

$$p_i : U_i \to P$$

and others

$$c_{ij} : U_{ij} \to C$$

satisfying some evident relations, one of which was the cocycle condition

$$c_{kj}c_{ji} = c_{ki}.$$

These c_{ji} will give us a left C-torsor, E, say. We can examine the induced P-torsor, $\partial_*(E)$, and - surprise, surprise - the p_i part of the cocycle pair, $\{(c_{ij}, p_i)\}$, provides a trivialising coboundary, since

$$p_i = \partial(c_{ij})p_j$$

yields

$$\partial(c_{ij}) = p_i p_j^{-1} = p_i.1.p_j^{-1}.$$

Conversely suppose we have a C-torsor, E, and we know that $\partial_*(E)$ is trivial, then we can find p_is satisfying the above equations and making E into an M-torsor. If we look back to our motivating case with $\mathsf{M} = \mathsf{Aut}(G)$, then we can adapt the argument given there (page 335) to get an explicit global section of $\partial_*(E) = P_\partial \wedge^C E$, namely, for local sections e_i of E, define $\mathbf{t} = \{t_i\} = \{[p_i^{-1}, e_i]\}$ to get a compatible family and hence a global section, t, of $\partial_*(E)$. This process can be reversed, so from t and a choice of e_i, one can obtain p_i. We will see a neat way of doing this shortly.

What happens if we choose different local sections e_i' of E? These e_i' will give some c_is such that $e_i' = c_i e_i$, and also $p_i' = \partial(c_i)p_i$, but then

$$[(p_i')^{-1}, e_i'] = [p_i^{-1}\partial(c_i)^{-1}, c_i e_i] = [p_i^{-1}, e_i],$$

so the global section does not change.

We saw earlier that contracted product gave the category of G-bitorsors the structure of a group-like monoidal category with inverses, a gr-groupoid. (If P and Q are in $Bitors(G)$, then $P \wedge^G Q$ gave the 'product', whilst P^o was 'inverse' to P. Of course, the trivial bitorsor, T_G, was the identity object.) There is an obvious category of M-torsors, which we will denote by $\mathsf{M}{-}Tors$, (so $\mathsf{Aut}(G){-}Tors = Bitors(G)$), does this in general have any similar structure?

Before we attempt to answer that, we should give formal definitions of M-torsors, etc, as a base reference:

Definition: Let $\mathsf{M} = (C, P, \partial)$ be a bundle or sheaf of crossed modules over a space B, (or more generally a crossed module in a topos $\mathcal{E}$). By a *(relative)* M*-torsor*, or M*-relative torsor* we mean a left C-torsor together with a global section t of $\partial_*(E)$.

A morphism of M-torsors, $f : (E, t) \to (E', t')$, is a C-torsor morphism, $f : E \to E'$, such that

$$\begin{tikzcd} \partial_*(E) \arrow[rr, "\partial_*(f)"] & & \partial_*(E') \\ & B \arrow[ul, "t"] \arrow[ur, "t'"'] & \end{tikzcd}$$

commutes.

We will denote the category of M-torsors by $\mathsf{M}{-}Tors$.

Remark on terminology: The idea of a relative M-torsor lies between that of torsors and global sections and in the long exact sequences, the $\pi_0(\mathsf{M}{-}Tors)$-term is the transition from global section terms, $P(B)$, etc. to true torsor terms, $\pi_0(Tors(C))$. It is a Janus, looking back and forward. Various names have been applied to this. Breen in [48], following Deligne, used something of the form (C, P)-torsor, but that does not use the boundary map, ∂ and clearly various different crossed modules having the same C and P, but perhaps different actions or boundary maps might give differently behaved (C, P)-torsors. Aldrovandi, in conversation, favours a terminology that said, what we might write as $\pi_0(\mathsf{M}{-}Tors)$ or $\check{H}^0(B, K(\mathsf{M}))$, was a $\check{H}^0$-term so was the group of global sections of M. That is very good terminological reasoning, but it neglects the fact that the objects *are* C-torsors plus extra structure. It looks back in the sequence and neglects the future! Using the terminology of M-torsor, which I originally favoured, fails to look back and also hits the problem that the corresponding gr-groupoid $\mathcal{M} = \mathsf{M} - \mathsf{tors}$ is used later on to build $\mathcal{M}$-torsors, which are stacks with a nice action of $\mathcal{M}$, and these live at the next 'janus step' of the exact sequence. There seems no really good choice here. We have used 'relative M-torsor' or 'M-relative torsor' in the definition, but will continue to use 'M-torsor' later on as 'relative M-torsor' is quite tedious to type!

At this point, we need to revisit an old intuition that we have used several times before, but without which 'life' will seem unduly complicated! That intuition is that a principal G-set is a copy of G with an 'identity crisis'. In more detail, in situations such as that of universal covering spaces, E over a space B, the fibre is a copy of $\pi_1(B)$, but without a definite element being chosen as the identity. The natural path lifting property of covering spaces gives that any loop γ at a chosen base-point b_0 in B will lift uniquely to a path in the covering space, *once a start point* e_0 *above* b_0 *has been chosen*. If you choose a different start point e_0', you, of course, get a different lifted path. The end point of the lifted path will give the image of e_0 under the action of the path class $[\gamma] \in \pi_1(B)$. Thus once e_0 is chosen $p^{-1}(b_0) = E_{b_0}$ can be mapped bijectively to $\pi_1(B)$. (Remember we did say E was a *universal* covering space.) Under this bijection, the identity element of $\pi_1(B)$ corresponds to e_0, but our alternative choice, e_0', will give a bijection with e_0' itself corresponding to $1_{\pi_1(B)}$. There is no canonical choice of start point in E_{b_0}, so no definitive identification of E_{b_0} with $\pi_1(B)$.

For a G-bitorsor, with a local section $e_i : U_i \to E$, we have essentially the same situation. The left and right G-actions are globally independent and yet are locally linked by the $u_i : G_{U_i} \to G_{U_i}$. To use these it *is* necessary to use the e_i to temporarily pick a 'start point' in each fibre of E. Thus the equation,

$$u_i(g).e_i = e_i.g,$$

interprets as both the definition of u_i given the right action *and* conversely, given the u_i, as a defining equation of a right action. This does need to be spelt out again: given any local element

x of E over U_i, it has the form $x = g'e_i$ for some local element g' of G. Suppose we now operate with g on the right of x, then we get

$$x.g = g'e_i.g = g'u_i(g)e_i.$$

(This is very analogous to defining a linear transformation between vector spaces by transforming the elements of a chosen basis and then 'extending linearly'. Here we extend G-equivariantly for the *left* action, having transformed the 'basic' element e_i to $e_i.g$.)

The key transition equation for the u_is was

$$u'_i = i_{g_i} \circ u_i,$$

which emphasises this viewpoint. We changed e_i to e'_i using g_i, so $e'_i = g_ie_i$, but then, for right action by g,

$$e'_ig = u'_i(g)e'_i = u'_i(g)g_ie_i,$$

whilst also

$$e'_ig = g_ie_ig = g_iu_i(g).e_i,$$

giving the transition equation in the form $g_iu_i(g) = u'_i(g)g_i$.

We now need to translate this into a tool that can be used for M-torsors. The plan of action is to show that any M-torsor, E, has a natural C-bitorsor structure and for this we have to use $t : B \to \partial_*(E)$ to obtain a right C-action on E. In Lemma 49, (page 314), we saw how to go from a global section of a torsor to an identification of it as an 'identity-less' copy of the group bundle. We thus have that t allows us to identify $\partial_*(E)$ with T_P, *i.e.*, with P itself (as left P-torsor). We can unpack the recipe in Lemma 49, (but beware the change of notation, P is here the basic group of our crossed module M, but was the torsor in that earlier discussion). Any local element of $\partial_*(E)$ over some U_i is of form $[p, e]$, with p a local section of P over U_i and e a local section of E, again over U_i. We can get from t an expression $[p, e] = p'.t$ for some p' defined over U_i. Using the structural map of $\partial_*(E)$ as a P-torsor, we get

$$\partial_*(E) \overset{(t\pi,id)}{\to} \partial_*(E) \times \partial_*(E) \overset{\cong}{\to} P \times_B \partial_*(E) \overset{proj}{\to} P,$$

which, from $[p, e]$ gives the p'. (Recalling that, given e_i, the unadjusted choice of local sections is $[1, e_i]$, then this process picks out the corresponding p_i, so that $t = [p_i^{-1}, e_i]$.) Thus from t, we get a map from $\partial_*(E)$ to P.

In this 'game', it pays to go back-and-fore between the different descriptions and to revisit the special case, $\mathsf{M} = \mathsf{Aut}(G)$, for guidance, and, hopefully, inspiration. Our key equation defining the u_i was $u_i(g)e_i = e_i.g$. In our general case of $\mathsf{M} = (C, P, \partial)$, the rôle of the u_i is taken by the local elements p_i, which act on C (since, recall, that action is part of the crossed module structure) and the corresponding equation would be

$${}^{p_i}c.e_i = e_i.c,$$

but $e_i.c$ is not defined, a least not yet! We will take this as its definition (and remember our earlier discussion of right actions, and what here would be the C-equivariant extension), then see if it works!

First let us underline what the equation actually says. An arbitrary local element of E_{U_i} has form $e = c_i.e_i$ and the expression for $e.c$ will be $c_i.{}^{p_i}c.e_i$ as the right action has to be left C-equivariant, now if $c_1, c_2 \in C_{U_i}$, then

$$(e_i.c_1).c_2 = {}^{p_i}c_1.e_i.c_2 = {}^{p_i}c_1.{}^{p_i}c_2.e_i = {}^{p_i}(c_1c_2).e_i = e_i.(c_1.c_2),$$

so it does define an action, at least locally. Next we have to check on intersections. Supposing that p_i on U_i and p_j on U_j satisfy $p_j = \partial(c_{ji})p_i$, where $e_j = c_{ji}e_i$, then over U_{ij},

$$e_j.c. = c_{ji}e_i.c = c_{ji}{}^{p_i}c.e_i = c_{ji}{}^{p_i}c_{ji}^{-1}.e_j$$

and also

$$e_j.c = {}^{p_j}c.e_j = {}^{\partial(c_{ji})p_i}c.e_j,$$

and the Peiffer rule for crossed modules gives

$${}^{\partial c}c' = cc'c^{-1},$$

so the two local actions patch together neatly. We thus have an action of C on the right of E. Is it giving us a right C-torsor structure on C? This amounts to asking if locally the equation $x = yc$ can be solved uniquely for c in (some) terms of x and y over U_i, but $x = c'.y$ for a unique c', since E is a left C-torsor. The obvious element to try out as our required solution, c, is ${}^{p_i^{-1}}c'$ - **try it!** It works. We have proved:

Lemma 56 *If (E,t) is a* M*-torsor, then E is a C-bitensor.* ∎

From another perspective, this is quite clearly due to the natural map from M to $\mathsf{Aut}(C)$, given by the identity on C and the action map

$$\begin{tikzcd} C \arrow[r, "="] \arrow[d] & C \arrow[d] \\ P \arrow[r, "\alpha"'] & Aut(C) \end{tikzcd}$$

We would expect an M-torsor to be mapped to a $\mathsf{Aut}(C)$-torsor, that is, a C-torsor, via this morphism of crossed modules, so from this viewpoint the lemma may not seem surprising.

A few pages ago, we set out to extend the contracted product to M-torsors. Now that we have this lemma, we can, at least, work with a contracted product of the associated C-bitorsors. In other words, if (E_1, t_1), (E_2, t_2) are M-torsors, then we might tentatively explore a definition of $(E_1, t_1) \wedge^{\mathsf{M}} (E_2, t_2)$ as being $(E_1 \wedge^C E_2, t)$ with t still to be described. Here is a suitable, almost heuristic, approach that tells us we are going in the right direction.

We have $\partial_*(E) = P_\partial \wedge^C E_1$, where P_∂ is the trivial (left) P-torsor with, in addition, a right C-action given by : if $x \in P_\partial$, $x = p.t$, where t is a global section (fixed for the duration of the calculation), then, for $c \in C$, $x.c = p.\partial(c).t$. Now if $\partial_*(E)$ is assumed to have a global section, it is easy to show that it is, itself, isomorphic to P_∂. Next look at (E_1, t_1), and (E_2, t_2) and let us examine $\partial_*(E_1 \wedge^C E_2)$. This is $P_\partial \wedge^C E_1 \wedge^C E_2 = (P_\partial \wedge^C E_1) \wedge^C E_2 \cong P_\partial \wedge^C E_2$ by the above calculation, using t_1 to trivialise $(P_\partial \wedge^C E_1)$, and finally this is trivial using t_2.

This argument, although valid, merely shows that t exists. It could be taken apart further to get an explicit formula, but we will, instead, approach that through cocycles. We pick local sections of E_1 and E_2 over the same open cover $\mathcal{U}$. These we will denote by $e_i^1 : U_i \to E_1$, $e_i^2 : U_i \to E_2$. Given t_1 and t_1, we get local elements of P, p_i^1 and p_i^2, so that

$$t_1 = [(p_i^1)^{-1}, e_i^1],$$

and similarly for t_2. These p_i^1s are those for the local cocycle description of E_1 as (c_{ij}^1, p_i^1), so are the parts of the contracting homotopy on $\partial_*(E_1)$, etc.

Now look at $E_1 \wedge^C E_2$. The obvious local sections of this would be $e_i = [e_i^1, e_i^2]$, and using these we want to work out the corresponding cocycle pair. We need to work out the relationship of e_i with $e_j = [e_j^1, e_j^2]$. We have $e_i^1 = c_{ij}^1 e_j^1$, $e_i^2 = c_{ij}^2 e_j^2$, so

$$\begin{aligned}
(e_i^1, e_i^2) &= (c_{ij}^1 e_j^1, c_{ij}^2 e_j^2) \equiv c_{ij}^1 (e_j^1, c_{ij}^2 e_j^2) \\
&= c_{ij}^1 ({}^{p_j^1} c_{ij}^2 . e_j^1, e_j^2) = c_{ij}^1 \, {}^{p_j^1} c_{ij}^2 (e_j^1, e_j^2),
\end{aligned}$$

and we have $e_i = c_{ij}^1 \, {}^{p_j^1} c_{ij}^2 . e_j$. This C-coefficient may look familiar (or not), but before we identify it, we should look for the p_is. The obvious ones to try are $p_i = p_i^1 p_i^2$, *i.e.*, the product within P of the two values. We have a $c_{ij} = c_{ij}^1 . {}^{p_j^1} c_{ij}^2$, so can see if this works for the equation $p_i = \partial(c_{ij}) p_j$:

$$\begin{aligned}
p_i = p_i^1 p_i^2 &= \partial(c_{ij}^1) p_j^1 . \partial(c_{ij}^2) p_j^2 \\
&= \partial(c_{ij}^1) p_j^1 . \partial(c_{ij}^2) (p_j^1)^{-1} p_j^1 p_j^2 = \partial(c_{ij}) p_j.
\end{aligned}$$

The simplicial interpretation of the cocycles gave a map from $N(\mathcal{U})$ to $K(\mathsf{M})$, and in dimension 1, $K(\mathsf{M})$ is $C \rtimes P$. The multiplication in this semidirect product is

$$(c_1, p_1).(c_2, p_2) = (c_1 \, {}^{p_1} c_2, p_1 p_2).$$

In other words, if (E_1, t_1) corresponds to a simplicial map $\mathbf{g}_1 : N(\mathcal{U}) \to K(\mathsf{M})$ and similarly $\mathbf{g}_2$ corresponding (E_2, t_2), then $(E_1, t_1) \wedge^{\mathsf{M}} (E_2, t_2)$ is associated to the product $\mathbf{g}_1 . \mathbf{g}_2$,

$$N(\mathcal{U}) \to K(\mathsf{M}) \times K(\mathsf{M}) \to K(\mathsf{M}),$$

using the multiplication map of the simplicial group $K(\mathsf{M})$ corresponding to the crossed module, M. Does this give us a gr-groupoid structure on $\mathsf{M}{-}Tors$? The above description of the multiplication as corresponding to contracted product tells us that we can use the inverse of that multiplication to construct an inverse for the contracted product. The detailed formula for the inverse of an M-torsor, (E, t), is **left as an exercise**.

Note that we have not checked certain necessary facts about the (c_{ij}, p_j), namely that $c_{ij} c_{jk} = c_{ik}$ and they transform correctly under change of local sections. The details of these are **left to the reader**. They use the crossed module axioms several times. We have proved the following:

Proposition 82 *Under the identification of $\pi_0(\mathsf{M}{-}Tors)$ and $\check{H}^0(B, \mathsf{M})$, the group structure on the first given by the contracted product coincides with that given on the second under the group structure of $K(\mathsf{M})$, the associated simplicial group bundle of the bundle of crossed modules, M.* ∎

7.6.2 An alternative look at Change of Groups and relative M-torsors

When we discussed change of groups, we saw a neat induced torsor construction. Recall we had

$$\varphi : G \to H,$$

a morphism of sheaves of groups and a torsor E over G, we obtained $\varphi_*(E)$ by first forming H_φ, *i.e.*, the (H, G)-object with right G-action given via φ and then $\varphi_*(E) = H_\varphi \wedge^G E$.

This construction has various universal properties that we have not yet made explicit nor exploited, yet which are very useful. We will need to recall that if P and Q are two G-torsors, a morphism $f : P \to Q$ is a map over B such that $f(g.p) = g.f(p)$ for all $g \in G$ and $p \in P$. In other words, it is a sheaf map $f : P \to Q$, which is G-equivariant. We can represent this by a diagram:

$$\begin{tikzcd} G \times_B P \arrow[r, "G\times f"] \arrow[d] & G \times_B Q \arrow[d] \\ P \arrow[r, "f"'] & Q \end{tikzcd}$$

in which the vertical maps give the actions, and which is required to commute.

There is a neat notion from the theory of group actions (on sets), which adapts well to the torsor context. Suppose that $\varphi : G \to H$ is a homomorphism of ordinary groups, and (X, a_X) and (Y, a_Y) are a G-set and an H-set respectively, with $a_X : G \times X \to X$ and $a_Y : H \times Y \to Y$ being the actions. A map $f : X \to Y$ is said to be *over* φ if for all $x \in X$ and $g \in G$, we have $f(g.x) = \varphi(g).f(x)$. This is, of course easily represented by a similar commutative diagram:

$$\begin{tikzcd} G \times X \arrow[r, "\varphi\times f"] \arrow[d, "a_X"'] & H \times Y \arrow[d, "a_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

It thus follows that a G-map between G-sets is a slightly degenerate form of this notion.

Before we return to the situation of torsors, it will pay to note that φ makes H into a *right* G-set and that $\varphi_*(X)$ as being $H_\varphi \wedge^G X$, makes sense here as well. suppose $f : X \to Y$ is over φ in the above sense, then we look at f and see if it induces an H-map from $\varphi_*(X)$ to T. The elements of $\varphi_*(X)$ will be equivalence classes of pairs (h, x), where $(h, g.x) \equiv (h\varphi(g), x)$. We write $[(h, x)]$ for the equivalence class and try to guess what form an map induced from f might take. The obvious form to try would seem to be to set $\tilde{f}[(h, x)] = h.f(x)$ and to see if this works. Even though this is easy, let us do it explicitly:

$$h.f(g.x) = h.\varphi(g)f(x),$$

since f is over φ, but $\tilde{f}[(h\varphi(g), x)] = h.\varphi(g)f(x)$ as well, so we are done. We note, however, that this is really the only sensible way to define such a $\tilde{f}$. This is thus well defined as an H-map from $\varphi(X)$ to Y. (The fact that it is an H-map **should be clear**.)

We now have $f : X \to Y$ and $\tilde{f} : \varphi_*(X) \to Y$, so is there a possible factorisation of f as a composite of some map $X \to \varphi_*(X)$ over φ followed by $\tilde{f}$? There is an obvious map from X to $\varphi_*(X)$ namely that which sends x to $[(1_H, x)]]$. This then sends $g.x$ to $[(1_H, g.x)]$, which is the same as $[(\varphi(g), x)]$, which is $\varphi(g)[(1_H, x)]$, by the definition of the left H-action on $H_\varphi \wedge^G X$. This is thus a map over φ as expected and does not depend on f itself.

Going back to $\tilde{f}$, we hinted that this might be unique in some sense. What sense? First let us give a name to the map that we have just examined, say $\varphi_\sharp : X \to \varphi_*(X)$. We noted that $f = \tilde{f}\varphi_\sharp$ - but did not **check it**. That done, suppose we had some 'other' H-map $f' : \varphi_*(X) \to Y$, so that $f = f'\varphi_\sharp$, then $f'[(1, x)] = f(x)$, but f' is assumed to be an H-map, so $f'[(h, x)] = f'(h.[(1, x)]) = h.f(x)$ and $f' = \tilde{f}$.

If we write $Maps_\varphi(X, Y)$ for the set of maps from X to Y over φ, we have shown it to be isomorphic to $H-Sets(\varphi_*(X), Y)$. As both are functorial in Y, and (**for you to check**), the isomorphism is natural, we have shown that $Maps_\varphi(X, -)$ is a representable functor with $\varphi_*(X)$ as a representing object. There are still more things to work through and question here. What happens if we change X, for instance? But these can be **left to the reader**.

We did the above in the easy case of $Sets$, now transport the idea across to $Sh(B)$, or better still, to an arbitrary topos, $\mathcal{E}$. We have our original situation of a morphism, $\varphi : G \to H$, of sheaves of groups. We suppose E is a G-torsor and E' an H-torsor.

Definition: A sheaf map $f : E \to E'$ is said to be a *morphism of torsors over* φ if the diagram:

$$\begin{tikzcd} G \times_E \arrow[r, "\varphi\times f"] \arrow[d, "a_E"'] & H \times E' \arrow[d, "a_{E'}"] \\ E \arrow[r, "f"'] & E' \end{tikzcd}$$

commutes, the vertical arrows representing the actions.

We can equally well state this in terms of 'local elements'. (The choice of the approach used is largely a question of taste and is left to you.It is advisable to be able to follow and use any of the different methods when handling such discussions - although you may prefer one, say the diagrammatic one, to some other.)

We will write $Sh(B)_\varphi(E, E')$ for the sheaf of morphisms over φ from E to E'. (This is sloppy as E and E' really have to have the actions included in their labeling, but this is fairly anodyne sloppiness.) It should now be easy to prove:

Proposition 83 *(i) For any E, E' as above, there is a natural isomorphism of sheaves*

$$Sh(B)_\varphi(E, E') \cong Tors(H)(\varphi_*(E), E').$$

(ii) The functor $Sh(B)_\varphi(E, -)$ is representable. ■

Although easy, there are quite a lot of things to **check** here!

We thus have a neat universal property for φ_* as a functor from $Tors(G)$ to $Tors(H)$. We can now apply it to the case of relative M, where $\mathsf{M} = (C, P, \partial)$ is a sheaf of crossed modules. We had a description of a relative M-torsors as a C-torsor, E, together with a specified trivialisation $t : \partial_*(E) \overset{\cong}{\to} T_P$.

Proposition 84 *Suppose E is a C-torsor and $t : E \to T_P$, a morphism over ∂, then $(E, \tilde{t})$ is an* M*-torsor. Conversely f $E, f)$ is a relative* M*-torsor, then E is a C-torsor and $f\partial_\sharp : E \to T_P$ is a morphism of torsors over ∂.*

Proof: this is mostly just a corollary of our earlier result. The one point is that $\tilde{t} : \partial_*(E) \to T_P$ is a morphism of H-torsors, and hence is an isomorphism, hence, also, $\tilde{t}^{-1}(1_P)$ is a global section of $\partial_*(E)$. ■

We can use this to get a separate description of the category of M-torsors, which incidentally justifies the choice of name 'relative M-torsors' as they are somehow 'relative to T_P in a controlled way. In this description a morphism of M-torsors is a C-torsor morphism, f, making

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "t"'] & & E' \arrow[dl, "t'"] \\ & T_P & \end{tikzcd}$$

commute. (Here f is a C-torsor map, but t and t' are maps over φ. This diagram thus 'lives' in the category of sheaves on B.)

We will categorify this description later replacing M by a lax gr-groupoid, and, in fact, in a particular case by $\mathsf{M} - Tors$ itself, but all that requires stacks for a thorough handling, so must wait.

7.6.3 Examples and special cases

Right at the start of our discussion of crossed modules, in section 2.1, we gave various different examples. One was the $(G, Aut(G), \partial)$ case, where ∂ sending g to the inner automorphism determined by g. Others were normal subgroups and P-modules. We based the definition of (relative) M-torsor on that of G-bitorsor and thus on the first of these. What about the others?

(i) To take an almost silly example, let $\mathsf{M} = (1, P, inc)$, that is, the case $C = 1$. If $\mathcal{C}$ is our open cover, then the cocycle description of M-torsors gives us a family of local sections of P, say, $u_i : U_i \to P$, satisfying $p_i = p_j$ on intersections, $U_i \cap U_j$, but that means that the family glues to a global section of P. Conversely any global section of P gives a morphism from $N(\mathcal{U})$ to M. (We leave to the reader the examination of how this corresponds to a 1-torsor that yields a trivial P-torsor on application of ∂_*.) Thus in this case, M-torsors are just global sections of P and $\check{H}^0(B, \mathsf{M}) \cong \check{H}^0(B, P)$. (There is no question of coboundaries or equivalent cocycles as there is nothing above dimension 0 in M.)

(ii) The other extreme case is when C is Abelian and P is trivial. (We will sometimes write this as $C[1] = (C \to 1)$. It is a 'suspended' or 'shifted' form of C.) Here we just have a C-torsor E, and, of course $\partial_*(E)$ is a 1-torsor! There is not much choice of trivialisation, so we just have that C-torsor. In this case, we have $\check{H}^0(B, \mathsf{M}) \cong \check{H}^1(B, C)$, that is, cohomology in the old sense of Abelian cohomology.

(iii) The next obvious case is 'inclusion crossed modules' or 'normal subgroup pairs'. In other words, suppose N is a normal subgroup of P and M is the corresponding crossed module. (We write ∂ for the inclusion of N into P.) We would expect that, writing G for P/N, an M-torsor would be more or less the same, up to equivalence perhaps, as a $(1 \to G)$-torsor, *i.e.*, to a global section of G. The conditions on the local sections p_i over some cover $\mathcal{U}$, and the corresponding n_{ij} are now

$$p_i = n_{ij} p_j,$$

as well as $n_{kj} n_{ji} = n_{ki}$.

Remark: There is a morphism of crossed modules with kernel $(N, N, =)$ giving a short exact sequence,

$$\begin{tikzcd} N \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 1 \arrow[d] \\ N \arrow[r] & P \arrow[r, "\varphi"] & G \end{tikzcd}$$

we know that this will give a short exact sequence of simplicial groups and that M-torsors correspond to maps from $N(\mathcal{U})$ to $K(\mathsf{M})$ if they trivialise over the open cover $\mathcal{U}$. Our observation that M-torsors might lead to global sections of G relates to composition with the quotient map φ from M to $(1, G, inc)$. (This raises the question of maps of crossed modules inducing functors between the corresponding categories of torsors, in general. We will return to this shortly.)

Looking in more detail, suppose we have a M-torsor specified by a cocycle pair (p_i, n_{ij}) over some open cover $\mathcal{U}$, and we write g_i for $\varphi(p_i)$, then the g_is *do* form a global section of G, since they are compatible over the intersections. Conversely, given a global section g of G, we know that φ is an epimorphism of sheaves, so would like to lift g to something in P. This situation is one we have encountered before and will do so again later. An epimorphism of sheaves need not be an epimorphism of the underlying presheaves. In our spatial context, it will be an epimorphism on stalks, however. We thus do not know if there is a global section p of P satisfying $\varphi(p) = g$, but, thinking about the idea of stalk, for any $b \in B$, and any open set U containing b, there is a representative (g_U, U) of the element $g_b = g(b)$, which is in the stalk over b. As φ is an epimorphism on stalks, we can choose U such that there is a $p_u \in P(U)$ with $\varphi_U(p_U) = g_U$. This gives us an open cover $\mathcal{U}$ of B and a family of local section of P over $\mathcal{U}$. Next look at the intersections, $U \cap V$, of sets from $\mathcal{U}$. There the restrictions of p_U and p_V need not agree, but as their images are the same under φ, there is a $n_{U,V}$ in N over $U \cap V$, which satisfies $p_U = n_{U,V} p_V$, and the family of these ns satisfy the cocycle condition, so from our global section of G, we have constructed a cocycle pair for an M-torsor. Different liftings of g give local sections that agree up to a coboundary, n_u, (possibly on a joint refinement of the covers), so M-torsors do give global sections of G, and *vice versa*.

(iv) The last case is $\mathsf{M} = (M, G, 0)$, *i.e.*, M is a sheaf of G-modules. Here we have that cocycle pairs, (g_i, m_{ij}), must satisfy

$$g_i = \partial(m_{ij}) g_j,$$

but ∂ is trivial, so the g_is give a global section, whilst the m_{ij} give a M-torsor in the usual sense.

This example is good because it links M-torsors in this case with M-torsors *and* global sections, *i.e.*, some sort of 'extension', $G(B) \to \mathsf{M}{-}Tors \to Tors(M)$, or perhaps in the other order? We have not analysed the effect of the action of G on M. Does this mean that we have some sort of 'G-equivariant' cohomology, or cohomology of the sheaf of groups G with coefficients in the G-module M, ... and what about the gr-category structure. The detailed examination of all the structures involved is interesting and useful to do, so is, once again, **left as an exercise**.

This class of examples is also very important as amongst the examples of this type are, of course, the G-bitorsors with G a sheaf of Abelian groups, since for such a G, we have that $\mathsf{Aut}(G)$ is of the form $(G, Aut(G), 0)$. The best known example is where G is $U(1)$ or, equivalently, $G\ell(1, \mathbb{C})$, the group of unit modulus complex numbers. We will return to this later.

The above discussion suggests some interesting areas to explore. Reaction of these M-torsors to 'change of M', short exact sequences of sheaves of crossed modules and their 'reflection' in the behaviour of the M-torsors, etc. One particular short exact sequence is

$$\begin{tikzcd} K \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & N \arrow[d] \\ 1 \arrow[r] & P \arrow[r] & P, \end{tikzcd}$$

where $K = Ker\,\partial$ and $N = Im\,\partial$. It suggests that $\mathsf{M} - Tors$ is an extension of $G(B)$ by a category of K-torsors for an Abelian group sheaf, K, somehow twisted by the G-action. After examining one or two related subjects, we will be able to give a bit more insight and precision about this idea.

7.6.4 Change of crossed module bundle for 'bitorsors'.

We now have a very thorough knowledge of G-bitorsors and the more general (relative) M-torsors, via the link with simplicial maps from $N(\mathcal{U})$ to $K(\mathsf{M})$, but, of course, that link makes change of 'coefficients' more or less obvious.

First it should be noted, once again that the identification of $\check{H}^0(B, \mathsf{Aut}(G))$ as a second non-Abelian cohomology group of B with coefficients in G, runs foul of non-functoriality in G, but that this is not due to some subtle deep property of non-Abelian cohomology, rather it is due to the banal failure of $Aut(G)$ to be functorial in G, in other words, to a low level group theoretic fact, low level but in fact fundamental. It is here group theoretic, but generally automorphism groups do not vary functorially - and that opens the way to crossed modules.

If $\varphi : G \to H$ is a morphism of group bundles, then there may, or may not, be a morphism $\varphi' : Aut(G) \to Aut(H)$ such that

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "i"'] & H \arrow[d, "i"] \\ Aut(G) \arrow[r, "\varphi'"'] & Aut(H) \end{tikzcd}$$

is a morphism of crossed modules.

There is an induced morphism on $\check{H}^0(B, \mathsf{Aut}(G))$ if such a φ' does exist, and, of course, in more generality, if we have that $\varphi : \mathsf{M} \to \mathsf{N}$ is a morphism of crossed modules, then there is an induced homomorphism of groups

$$\varphi_* : \check{H}^0(B, \mathsf{M}) \to \check{H}^0(B, \mathsf{N}).$$

(It could happen that two crossed modules of the form $\mathsf{Aut}(G)$ could be linked by a zig-zag of other crossed modules so that the morphisms in the reverse direction were weak equivalences / quasi-isomorphisms in our earlier sense, and then there would be an induced map between the two $\check{H}^0(B, \mathsf{Aut}(G))$ groups. We will explore this more fully later on, using the beautiful theory of 'butterflies' as developed by Noohi, [222, 223].)

Exploring the above at a gr-groupoid level, *i.e.*, on $\mathsf{M}-Tors$ with contracted product, rather than at connected component / cohomology level, we get an induced gr-functor between $\mathsf{M}-Tors$ and $\mathsf{N}-Tors$, since it uses the functor K from crossed modules to simplicial groups. Explicitly $\varphi : \mathsf{M} \to \mathsf{N}$ induces $K(\varphi) : K(\mathsf{M}) \to K(\mathsf{N})$, a morphism of simplicial groups, but then our identification of the contracted product structure on $\mathsf{M}-Tors$ as being induced from the simplicial group structure of $K(\mathsf{M})$ immediately implies that $K(\varphi)$ induces a functor from M–$Tors$ to N–$Tors$ compatibly with the gr-groupoid structures.

7.6.5 Representations of crossed modules.

In the classical group based case, the naturally occurring vector bundles such as the tangent and normal bundles had the general linear group of some dimension as the basic G over which one worked. Extra structure corresponded to restricting to a subgroup or lifting to some 'covering group'. We recalled earlier, e.g., page 278, that the fibres of the bundles were vector spaces with an action of the chosen group, *i.e.*, a matrix representation of that group. What is, or should be, the representation theory 'of crossed modules'? There are several tentative answers.

A representation of a (discrete) group G and thus an action of G on some object, can be thought of in different ways. For instance, as a group homomorphism $G \to H$, where H is some group of permutations or matrices in which we can use methods from outside group theory, perhaps combinatorics, perhaps linear algebra, to analyse more deeply the properties of the elements of G. We could also consider this as a functor from $G[1]$, the corresponding groupoid with one object, to *Sets* for the permutation representations, or to some category of vector spaces or modules in the linear case.

The generalisations are to 'categorify' this second description by taking $\mathscr{X}(\mathsf{M})$, the 2-groupoid with one object (*i.e.*, the 2-group) of M, and looking for a nice category of '2-vector spaces' or '2-modules'. (The permutation version has not been that well explored yet, but we will see some ideas later on.) Some doubt exists as to what is the 'best' category of '2-vector spaces' to use, in fact the discussion is really about what that term should mean. We mention two possibilities here, but there are others and we will look at them later. The first is due independently to Forrester-Barker, [130], and to Baez and Crans, [14]. The second is based on an idea of Kapranov and Voevodsky, [175], using more monoidal category theory than we have been assuming so far.

Here we will adopt the simpler version, more as an illustration then as a claim that this is the 'correct' version. The motivation for the definition, used by Forrester-Barker and by Baez and Crans, is that, as crossed modules are category objects in the category of groups, for a linear representation theory of such things, it is natural to try category objects in the category of vector spaces, but such objects are equivalent to short complexes of vector spaces. The idea is also that some of the potential applications of the structures that we have been studying use chain complexes as coefficients. (We will see this more clearly in the later discussion of hyper-cohomology.) Keeping things simple, we look at chain complexes of vector spaces (or more generally of modules) of length 1. (Warning: for us here 'length 1' means *one morphism*, $C_1 \to C_0$, not 'one group' so our objects are linear transformation between vector spaces and our morphisms are commutative squares.) These are highly Abelian versions of crossed modules, so we will use similar notation such as C, D, etc., for them.)

We recall that chain complexes have a natural 'internal hom' construction, well known from classical homological algebra. (We will see this again in our discussion of hyper-cohomology so will treat it in more detail there.) The chain complex, $Ch(\mathsf{C}, \mathsf{D})$, has graded maps of degree n in dimension n, so, for instance, has chain homotopies in dimension 1. Putting $\mathsf{D} = \mathsf{C}$ and looking at the invertible maps gives an automorphism group, $\mathsf{Aut}(\mathsf{C})$, which is also a chain complex of groups, *i.e.*, we get a crossed module. If we have a general (discrete) crossed module M, we can consider a morphism $\mathsf{M} \to \mathsf{Aut}(\mathsf{C})$ as a representation of M, and can talk of M acting on C by 'linear maps'. We will not explore this further here, but note that we are very near the idea of representing a simplicial group as a simplicial group of simplicial automorphisms, somewhat as in section 6.3. At present, the available discussions of 2-group representations of this form include the thesis, [130], and papers, [14]. A more extensive use of monoidal category theory would allow us to consider a variant that considers 2-vector spaces to mean the 2-categorical version of the monoidal category of vector spaces. We will return to this later.

Chapter 8

Hypercohomology and exact sequences

8.1 Hyper-cohomology

8.1.1 Classical Hyper-cohomology.

We have several times mentioned this subject and so should provide some slight introduction to the basic ideas. We will go right back to basics, even though we have already used some of the ideas previously, usually without comment. Most of this first part may be well known to you.

The basic idea is that of a graded, or more precisely $\mathbb{Z}$-graded, group and variants such as graded vector spaces, or graded modules, or sheaves of such on some space, B or in some topos $\mathcal{E}$.

Definition (First form): A $\mathbb{Z}$*-graded vector space* (gvs) is vector space together with a direct sum decomposition, $\mathsf{V} = \bigoplus_{p\in\mathbb{Z}} V_p$. The elements of V_p are said to be *homogeneous of degree* p. If $x \in V_p$, write $|x| = p$.

A graded vector space could equally well be defined as a family $\{V_i\}_{i\in\mathbb{Z}}$ of vector spaces, since we could then form their direct sum and obtain the first version.

Definition (Second form): A $\mathbb{Z}$*-graded vector space* (gvs) is a $\mathbb{Z}$-indexed family, $\{V_i\}_{i\in\mathbb{Z}}$, of vector spaces.

(The definitions are, pedantically, not completely equivalent as one can have a constant family with all V_i equal, but that is really a smokescreen and causes no problem.)

Both versions are useful. For example, if K is a simplicial set, we can define a graded vector space using the second version by taking V_n to be the vector space with basis indexed by the elements of K_n if $n \geq 0$ and to be the trivial vector space if $n < 0$. From our treatment of simplicial sets, it would be somewhat artificial to define $\mathsf{V} = \bigoplus_{i\in\mathbb{Z}} V_i$. For another example, the other description fits better. The polynomial ring, $\mathbb{R}[x]$, is a graded vector space with V_n having basis $\{x^n\}$, *i.e.*, V_n is the subspace of degree n monomials over $\mathbb{R}$. The whole space, $\mathbb{R}[x]$, is here by far the more natural object.

For graded groups, etc., just substitute 'group' etc. for 'vector space' and correspondingly, 'direct product' for 'direct sum'.

Definition: A morphism $f : \mathsf{V} \to \mathsf{W}$ of graded vector spaces is *homogeneous* if $f(V_p) \subseteq W_{p+q}$ for all p and some common q, called the *degree of f*. The set of such morphisms of given degree is $Hom(\mathsf{V}, \mathsf{W})_q = \prod_p Hom(V_p, W_{p+q})$.

An endomorphism, $d : \mathsf{V} \to \mathsf{V}$, of degree -1 is called a *differential* or *boundary* (which depending largely on the context) if $d \circ d = 0$.

A gvs with a differential is really just a chain complex, where $d_n : V_n \to V_{n-1}$ and $d_{n-1}d_n = 0$.

Definition: A graded vector space together with a differential is variously called a *differential graded vector space* (dgvs), or a *chain complex*. Some authors reserve that latter term for a positively graded differential vector space, or module, or .... . The elements of V_n are called *n-chains*, those of $Ker\, d_n$, *n-cycles*, and those of $Im\, d_{n+1}$, *n-boundaries*.

A graded vector space V is *positively graded* if $V_i = 0$ for all $i < 0$. It is, on the other hand, *negatively graded* if $V_i = 0$ for $i > 0$.

The classical convention is to write V^{-n} instead of V_n for all n in the negatively graded case. This, of course, has the effect that if (V, d) is a differential graded vector space which is negatively graded, then d has apparent degree + 1, $d^n : V^n \to V^{n+1}$. In the usual terminology that will give a *cochain complex*. For some purposes, it is usual to adapt the terminology somewhat, for instance to use chain complex as a synonym for dgvs without mention of positive or negative, but then also to use cochain complex for what is essentially the same type of object, but with 'upper index' notation, so $\mathsf{V} = (V^n, d^n)$ with $d^n : V^n \to V^{n+1}$. Terms such as 'bounded above', 'bounded below' or simply 'bounded' are also current where they correspond respectively to $V_n = 0$ for large positive n, or large negative n or both. We will make little use, if any, of these in the context of these notes, but it is a good thing to be aware of the existence of the various conventions and to check before assuming that a given source uses exactly the same one as that which you are used to!

For simplicity of exposition, we will initially concentrate our attention on general dgvs, which we will often call *chain complexes* and will attempt to be reasonably consistent - although that is virtually impossible! We will extend that terminology to dg-modules and dg-groups if and when needed.

- The elements of a chain complex are called *chains*. If $c \in C_n$, it is an *n-chain*. If $dc_n = 0$, it is called an *n-cycle* and, if $c \in Im\, d_{n+1}$, an *n-boundary*. If 'n' is not important, or is understood, it may be omitted.

- A *chain map* $f : \mathsf{V} \to \mathsf{W}$ of chain complexes is a graded map of degree 0, $\{f_n : V_n \to W_n\}$ compatible with the differentials, so, for all n,

$$d_n^W f_n = f_{n-1} d_n^V,$$

  and, of course, we will drop the V and W superfixes whenever possible. The category of differential vector spaces and chain maps will be variously denoted dgvs, or Ch_k with variants $\mathsf{dgk-mod}$, $\mathsf{dgk-mod}_{\geq 0}$, Ch_k^+ and so on, denoting the k-module version, a positively graded variant, and an alternative notation. (These, and other, notations are all used in the literature with the precise convention usually evident from the context. To some extent the choice, say of dgvs as against Ch is determined by the use intended, but this is not completely consistent.)

- A *chain homotopy* between two chain maps $f, g : \mathsf{V} \to \mathsf{W}$ is a graded map of degree 1, $s : \mathsf{V} \to \mathsf{W}$ such that
$$f_n - g_n = d_{n+1}s_n + s_{n-1}d_n.$$
- The *homology* of a chain complex, $\mathsf{V} = (V, d)$, is the graded object
$$H_n(\mathsf{V}) = \frac{Ker\, d_n}{Im\, d_{n+1}}.$$
If we are using upper indices, for whatever reason, the more usual term will be 'cohomology',
$$H^n(\mathsf{V}^*) = \frac{Ker(d^n : V^n \to V^{n+1})}{Im(d^{n-1} : V^{n-1} \to V^n)}.$$
This most often occurs in the situation where C is a chain complex and A is a vector space / module or similar, then we form $Hom(\mathsf{C}, A)$, by applying the functor $Hom(-, A)$ to C. Of course, $d_n : C_n \to C_{n-1}$ induces a differential
$$Hom(C_{n-1}, A) \to Hom(C_n, A)$$
and the elements of $Hom(C_n, A)$ are called *cochains*, with *cocycles*, and *coboundaries* as the corresponding elements of kernels and images. The notation $Hom(\mathsf{C}, A)^n$ is used for the object $Hom(C_{-n}, A)$, so this 'dual' has negative grading if C has positive grading, and is given upper indexing. The homology of $Hom(\mathsf{C}, A)$ is then called the *cohomology of* C *with coefficients in* A. (We will try to follow usual terminology as given in standard homological algebra texts, e.g. the classic [191].)
- More generally, if C and D are both chain complexes (of modules), then we can form the graded Abelian group, $Hom(\mathsf{C}, \mathsf{D})$, with $Hom(\mathsf{C}, \mathsf{D})_n$ being the Abelian group of graded maps of degree n from C to D. This means, of course,
$$Hom(\mathsf{C}, \mathsf{D})_n = \prod_{p=-\infty}^{\infty} Hom(C_p, D_{p+n}),$$
as before.

We make this into a chain complex by specifying, for $f \in Hom(\mathsf{C}, \mathsf{D})_n$, its 'boundary' ∂f by, if $c \in C_p$,
$$(\partial f)_p c = \partial^{\mathsf{D}}(f_p c) + (-1)^{n+1} f_{p-1}(\partial^{\mathsf{C}} c).$$
(In the event that you have not seen this before, check that (i) $\partial\partial = 0$, (ii) if f is of degree 0, then it is a chain map if and only if $\partial f = 0$ and (iii) a chain homotopy, s, between two chain maps, $f, g \in Hom(\mathsf{C}, \mathsf{D})_0$, is precisely an $s \in Hom(\mathsf{C}, \mathsf{D})_1$ with $\partial s = f - g$.)

The homology of $Hom(\mathsf{C}, \mathsf{D})$ is called the *hyper-cohomology of* C *with coefficients in* D. The case where $D_0 = A$ and $D_n = 0$ if $n \neq 0$ is the cohomology we saw earlier. In general, $H^0(Hom(\mathsf{C}, \mathsf{D}))$, *i.e.*, chain maps modulo coboundaries, is just the group of chain homotopy classes of chain maps by (ii) and (iii) above. As is usual in homological (and homotopical) algebra, we usually need good conditions on C and D to get really good invariants from this construction - typically C needs to be 'projective' or D 'injective', or C needs to be 'fibrant' or D 'cofibrant'. Our use of this will be somewhat hidden by the situations we will be considering.

8.1.2 Čech hyper-cohomology

The main type of application for us will be the 'hyper'-version of Čech cohomology. In this, or at least in its simplest form, we have a space, X, and we form the colimit over the open covers, $\mathcal{U}$, of X of the hyper-cohomology groups $H^n(C(\mathcal{U}), \mathsf{D})$. In more detail:

The classical Čech cohomology of X with coefficients in a sheaf of R-modules, A, is defined via open covers $\mathcal{U}$ of X. If $\mathcal{U}$ is an open cover of X, then we form the chain complex, $C(\mathcal{U})$, by taking $N(\mathcal{U})$, the nerve of $\mathcal{U}$, and letting $C(\mathcal{U})_n$ be the sheaf of free R-modules generated by $N(\mathcal{U})_n$ with $\partial = \sum_{k=0}^{n}(-1)^k d_k$ being the differential. This can either be thought of as a complex of (sheaves of) R-modules or in the straight forward module version. We take coefficients in another sheaf of R-modules, A, and form $H^n(C(\mathcal{U}), A)$.

If $\mathcal{V}$ is a finer cover than $\mathcal{U}$, there is a chain map from $C(\mathcal{V})$ to $C(\mathcal{U})$. Recall if $\mathcal{V} < \mathcal{U}$, for each $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ with $V \subseteq U$, and $(x, V_0, \ldots, V_n) \in N(\mathcal{V})_n$, we can map it to a corresponding $(x, U_0, \ldots, U_n) \in N(\mathcal{U})_n$ with each $V_i \subseteq U_i$. This is not well defined as several Us might work for a particular V, so the construction of the chain map involves a choice, however it does induce, firstly, a chain map from $C(\mathcal{V})$ to $C(\mathcal{U})$, which is determined up to (coherent) homotopy and thus a *well defined* map on cohomology, $H^*(C(\mathcal{U}), A) \to H^*(C(\mathcal{V}), A)$.

The Čech cohomology, $\check{H}^*(X, A) = colim_{\mathcal{U}} H^*(C(\mathcal{U}), A)$, was the first, historically, of the sheaf type cohomologies. Others apply to a topos rather than merely a space. The obvious hyper-variant of this replaces A by a sheaf of chain complexes (of whatever variety you like, provided they are 'Abelian'), so $H^n(C(\mathcal{U}), \mathsf{D}) = H^n(Hom(C(\mathcal{U}), \mathsf{D}))$ and then $\check{H}^*(X, \mathsf{D}) = colim_{\mathcal{U}} H^*(C(\mathcal{U}), \mathsf{D})$.

We should 'deconstruct' this a bit to see why it is relevant to us.

To simplify our lives no end, we will assume D is a presheaf of chain complexes of R-modules which is positive, ($D_n = 0$ if $n < 0$). By the method of construction of colimits of modules, etc., we can find for any element of $\check{H}^*(X, \mathsf{D})$, an open cover $\mathcal{U}$ of X and a representing element in $H^*(C(\mathcal{U}), \mathsf{D})$. We can thus, further, find a representing n-cocycle from $C(\mathcal{U})$ to D, *i.e.*, an element in $\prod_p Hom(C(\mathcal{U})_p, D_{n+p})$.

To simplify still further, we look at low values of n:

- for $n = 0$, we have some $\mathbf{f} = \{f_p : C(\mathcal{U})_p \to D_p\}$, which satisfies $\partial\mathbf{f} = 0$, so $\mathbf{f}$ forms a chain map. In some of our most interesting cases, D is usually very short, e.g. $D_n = 0$ if $n > 1$, so $\mathsf{D} = (D_1 \to D_0)$ with zeroes elsewhere in other dimensions. Then the only f_ps that contribute to $\mathbf{f}$ are f_0 and f_1. Over an open set, U_i, of the cover, f_0 will be a local section, $f_{0,i}$, of D_0, since 0-simplices of $N(\mathcal{U})$ have form (x, U_i) over $x \in U_i$. Similiarly 1-simplices are, of course, represented by (x, U_i, U_j) with $x \in U_{ij}$, so f_1 corresponds to local sections $f_{1,ij} : U_{ij} \to D_1$. The boundary in $C(\mathcal{U})$ of (x, U_i, U_j) is $(x, U_j) - (x, U_i)$, so

$$d^{\mathsf{D}} f_{1,ij} = f_{0,j}(x) - f_{0,i}(x),$$

or

$$f_{0,j}(x) = d^{\mathsf{D}} f_{1,ij} + f_{0,i}(x).$$

If we look at the non-Abelian analogue of this, it gives

$$f_{0,j}(x) = d^{\mathsf{D}} f_{1,ij}.f_{0,i}(x),$$

which 'is' the equation $p_j = \partial(c_{ij})p_i$. (You could explore the cases where D is slightly longer, or what about a non-Abelian version?)

- for $n = 1$, we expect to find a formula corresponding to the coboundaries that we met on 'changing the local sections' for M-torsors. If h, (yes, 'h' as in 'homotopy') is a degree 1 map in $Hom(C(\mathcal{U}), \mathsf{D})$ and D has length 1 as above, the only case that contributes is $h_0 : C(\mathcal{U})_0 \to D_1$ and hence $h_{0,i} : U_i \to D_1$. You are **left to check** that this does give (the Abelian version of) the coboundary / chain homotopy formula.

8.1.3 Non-Abelian Čech hyper-cohomology.

The idea should be fairly obvious in its general form. We replace our overall structural viewpoint of chain complexes or sheaves of such, by our favorite non-Abelian analogue. For instance, we could take D to be a sheaf of simplicial groups, or crossed complexes, or n-truncated simplicial groups or ... . These would really include sheaves of 2-crossed modules and clearly we might try sheaves of 2-crossed complexes, and so on. Some of these classes of coefficient are very likely to turn out to be useful in the future if recent developments in algebraic and differential geometry are any indication. We cannot consider all of them here. The first is the easiest to deal with and to some extent includes the others. It is not structurally the neatest, but ... .

If D is a sheaf of simplicial groups, then we might be tempted to replace $C(\mathcal{U})$ by the free simplicial group sheaf on $N(\mathcal{U})$. It is very important to note that this is NOT the same as $\mathcal{G}(N(\mathcal{U}))$ and we should pause to consider this point.

Let K be a simplicial set and G a simplicial group. The set of simplicial maps from K to the underlying simplicial set of G is isomorphic to $Simp.Grps(FK, G)$ by the standard adjunction between the free group functor, F, and the forgetful functor, U from $Grps$ to $Sets$. Complications might seem to arise if one tries to work with $\underline{\mathcal{S}}(K, UG)$ and $\underline{Simp.Grps}(FK, G)$, as initially it needs to be noted that $\underline{\mathcal{S}}(K, UG) = \mathcal{S}(K \times \Delta[n], UG)$ and one has to think of the relationship between $F(K \times \Delta[n])$ and $F(K) \otimes \Delta[n]$, the latter in the sense of our earlier discussion of tensoring in simplicially enriched categories, page 341. (This problem is, in fact, not really there, as although F does not preserve products, the product $K \times \Delta[n]$ is actually being thought of, and constructed, as a *colimit* and F, as a left adjoint, behaves nicely with respect to such.) We will not explore that further here and will, in fact, stick with $\underline{\mathcal{S}}(N(\mathcal{U}), \mathsf{D})$ rather than use F. (Note that by a useful result of Milnor, FK and $\mathcal{G}SK$ are isomorphic for a reduced simplicial set K, where S is the reduced suspension; see [97] and the paper, [203], which can be found in Adams, [3].) The relationship between $\underline{\mathcal{S}}(K, UG)$ and other related constructions such as $\underline{\mathcal{S}}(K, \overline{W}G) \cong \underline{\mathcal{S}-Grpd}(\mathcal{G}(K), G)$, is given by the induced fibration sequence,

$$\underline{\mathcal{S}}(K, UG) \to \underline{\mathcal{S}}(K, WG) \to \underline{\mathcal{S}}(K, \overline{W}G),$$

coming from the fibration,

$$UG \to WG \to \overline{W}G.$$

If we work within our favourite topos $\mathcal{E}$, or with bundles over B, this still holds true. It is also the case that WG is (naturally) contractible.

Back with hyper-cohomology, let D be a sheaf of simplicial groups and form $\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(\mathsf{D}))$. We put forward the homotopy groups of this simplicial group as being one analogue of $H^*(C(\mathcal{U}), \mathsf{D})$ in this context. (If D is Abelian, it will be KD for some sheaf of chain complexes, D, and the Dold-Kan theorem, plus the freeness of $C(\mathcal{U})$, give a correspondence between the elements in the two cases. Since we have $\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(\mathsf{D}))$ is a simplicial Abelian group in that case, its homotopy is its homology and the detailed correspondence passes down to homology without any pain. We thus do have a generalisation of the Abelian situation with our formula.)

We have $\pi_n(\mathcal{U}, \mathsf{D}) := \pi_n(\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(\mathsf{D}))$ is thus a candidate for a 'non-Abelian' Čech cohomology relative to $\mathcal{U}$ with coefficients in D. (If $n > 1$, it is an Abelian group, which makes it suspiciously well behaved - in fact too well behaved! We really need not these π_n, but rather the various algebraic models for the various k-types of the homotopy type $\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(\mathsf{D}))$, *i.e.*, we could do with examining $M(\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(\mathsf{D})), k)$, the crossed k-cube of that simplicial group. (For those of you who hanker for the simple life, it should be pointed out that when discussing extensions, we already had that there was a groupoid of extensions $\mathcal{E}xt(G, K)$, and although we could extract information from that groupoid to get cohomology groups, the natural invariant is really that groupoid, not the cohomology group as such. We can extract information from such an invariant, just as we can extract homotopy information from a homotopy type. To keep the information tractable we often truncate, or kill off, some of the structure to make the extraction process more amenable to calculation.)

We are, however, running before we can walk here! The case we have met earlier is for $n = 0$, *i.e.*, $[N(\mathcal{U}), \mathsf{D}]$, and we could pass to the colimit over covers to get $\check{H}^0(B, \mathsf{D})$. This is without restriction on the sheaf of simplicial groups, D. Our earlier example was with $D = K(\mathsf{M})$ for $\mathsf{M} = (C, P, \partial)$, a sheaf of crossed modules. (Breen in [48] calls this the *zeroth cohomology of the crossed module,* M*,* but as it varies covariantly in M perhaps his later terminology, [51], as the *zeroth Čech non-Abelian cohomology of B with coefficients in* M, is more appropriate.)

What about $\check{H}^1(B, \mathsf{M})$?

This will be $colim_{\mathcal{U}} H^1(N(\mathcal{U}), \mathsf{M})$, which is $colim_{\mathcal{U}}\pi_1(\underline{Simp.\mathcal{E}}(N(\mathcal{U}), K(\mathsf{M}))$. From the long exact fibration sequence, this will be isomorphic to $colim_{\mathcal{U}}[N(\mathcal{U}), \overline{W}K(\mathsf{M})]$ and so should classify some sort of simplicial $K(\mathsf{M})$-bundles on B. It does, but we need to wait until a later chapter for the details.

The set $[N(\mathcal{U}), \overline{W}K(\mathsf{M})]$ has elements which are homotopy classes of maps from $N(\mathcal{U})$ to $\overline{W}K(\mathsf{M})$ and by the properties of the loop groupoid construction, $\mathcal{G}$ of section 6.2.1, page 249, each such is adjoint to a morphism of sheaves of $\mathcal{S}$-groupoids from $\mathcal{G}(N(\mathcal{U}))$ to $K(\mathsf{M})$. The category of crossed modules is equivalent, via K and $M(-, 2)$, to a full reflective subcategory / variety of $\mathcal{S}{-}Grpd$, and this extends to sheaves, so the elements of $[N(\mathcal{U}), \overline{W}K(\mathsf{M})]$ correspond to homotopy classes of crossed module morphisms from $M(\mathcal{G}N(\mathcal{U}), 2)$ to M. In particular, for nice spaces, B, one would expect there to be 'nice' covers $\mathcal{U}$, such that $N(\mathcal{U})$ corresponded, via geometric realisation, to B itself, then taking $\mathsf{M} = M(\mathcal{G}N(\mathcal{U}), 2)$ itself, one would have a sort of universal element in $\check{H}^1(B, \mathsf{M})$, corresponding in this level, to a universal simplicial sheaf over B, extending in part the construction and properties of the universal covering space. This argument is one form of the 'evidence' for believing Grothendieck's intuition in '*En Poursuite des Champs / Pursuing Stacks*', [140]. There seems no good reason why, for any nice class of simplicial groups that form a variety, $\mathcal{V}$, with perhaps having some stability with respect to homotopy types, there should not be a 'universal $\mathcal{V}$-stack' over B. The above corresponds to the case of crossed modules, but crossed complexes and many of the other types of crossed objects that we have met earlier would seem to be relevant here. The main hole in our understanding of this is not really how to do it, rather it is how to interpret the theory once it is there. This form of crossed homotopical algebra would extend Galois theory to higher 'levels', but what do the invariants tell us algebraically?

That provides some overview of this general case, but in our earlier situation, with extensions of groups, we used a crossed resolution of a group, G, not a simplicial one. We have also mentioned once or twice that the category, Crs, of crossed complexes is monoidal closed. This would suggest (i) that given a topos $\mathcal{E}$, and, in particular, given a space B and $\mathcal{E} = Sh(B)$, the category of crossed complexes *in* $\mathcal{E}$, denoted $Crs_{\mathcal{E}}$, would be monoidal closed, (ii) there would be a free crossed complex on a cover / hypercover in $\mathcal{E}$, *i.e.*, if we have a simplicial object K in $\mathcal{E}$, we would get a crossed complex object, $\pi(K)$, and if $K \to 1$ is a 'weak equivalence' then there would be a local contracting homotopy on $\pi(K)$, *i.e.*, $\pi(K) \to 1$ would be a 'weak equivalence' of crossed complex bundles (recall 1 is the terminal object of $\mathcal{E}$, so in the case of $\mathcal{E} = Sh(B)$ is the singleton sheaf), then (iii) if $\mathrm{C{\small RS}}_{\mathcal{E}}$ denotes the internal 'hom' of crossed complex bundles, we would be looking at the model $\mathrm{C{\small RS}}_{\mathcal{E}}(\pi(K), \mathsf{D})$ for a crossed complex, D, in $\mathcal{E}$ and would want the homotopy colimit of these over (hyper-)covers, K, so as to get a well-structured model. Of course, if $\mathcal{E} = Sh(B)$ and we have 'nice' (hyper-)covers K, then we would expect the homotopy type of this to stabilise, up to homotopy, so $\mathrm{C{\small RS}}_{\mathcal{E}}(\pi(K), \mathsf{D})$ would be the same, up to homotopy, as that homotopy colimit. This plan almost certainly works, but has not been followed through as yet, at least, in all its gory detail. The first part looks very feasible given the construction of $\mathrm{C{\small RS}}(\mathsf{C}, \mathsf{D})$ for (set based) crossed complexes, C and D. (A source for this is Brown and Higgins, [61] and it is discussed with some detail in Kamps and Porter, [171], p. 222-227.) We will not give the details here. The other parts also look to work as the set based originals are given by explicit constructions, all of which generalise to $Sh(B)$. If that does all work then one has a Crs-based 'hyper-cohomology' crossed complex, $\check{\mathrm{C{\small RS}}}(B, \mathsf{D}) = hocolim_K \mathsf{Crs}(\pi(K), \mathsf{D})$, whose homotopy groups represent the analogue of hyper-cohomology.

If you are wary of not having a group or groupoid as an 'answer' for what is this 'hypercohomology', think of various analogous situations. For instance, for total derived functor theory, in homological and homotopical algebra, from a functor you get a complex, but it is the homotopy type of that complex which is used, not just its homotopy groups. In algebraic K-theory, it is quite usual to refer to the algebraic K-theory of a ring as being the (homotopy type of) a simplicial set or space. The algebraic K-groups are then the homotopy invariants of that simplicial set. In other words, in 'categorifying', one naturally ends up with an object whose homotopy type encapsulates the invariants that you are mostly used to, but that object is the thing to work with, not just the invariants themselves.

8.2 Mapping cocones and Puppe sequences

Exact sequences in cohomology can be constructed in various ways. One of these is related to the fibration and cofibration seqences of homotopy theory. If one has a fibration of spaces, then it leads to a long exact sequence of homotopy groups. Of course, not all maps are fibrations, but any map, $f : X \to Y$, can be replaced, up to homotopy, by a fibration, and its fibre Γ_f, then codes up homotopy information about f. This fibre is usually called the *homotopy fibre* of f and we have already met it in our list of common examples leading to crossed modules; see page 45. Later on we will need to use the construction to extend our simplicial interpretations of non-Abelian cohomology, but, by way of introduction, to start with both that construction (mapping cocylinders and mapping cocones/homotopy fibres) and the resulting homotopy exact sequences (Puppe sequences) will be looked at in a much simpler setting, namely that of chain complexes. Initially we will concentrate on the dual situation as that is slightly easier to understand geometrically.

(A very useful concise introduction to this theory can be found in May's book, [200], starting about page 55, and, for results on chain complexes, page 90.)

8.2.1 Mapping Cylinders, Mapping Cones, Homotopy Pushouts, Homotopy Cokernels, and their cousins!

We need various 'homotopy kernels', 'homotopy fibres' and more general 'homotopy limits' for our discussion. We have also already mentioned 'homotopy colimits' in passing several times, and so it seems a good idea to examine this general area from an elementary point of view.

We will work with a chain map $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ of chain complexes of modules over some ring R. We will use a *cylinder* $\mathsf{C} \otimes \mathsf{I}$. This is given by tensoring C with the chain complex, I,

$$0 \longrightarrow R \xrightarrow{\partial} R \oplus R \longrightarrow 0,$$

$$\partial(e_1^1) = e_1^0 - e_0^0.$$

There is one generator, e_1^1, in dimension 1, and two in dimension zero, corresponding to the interval $I = [0,1]$ or $\Delta[1]$ having one 1-cell and two 0-cells, e_1^0 and e_1^0, the superfix denoting the dimension of that generator . We should give a formal definition of a tensor product of chain complexes, even though you may have met this before.

Definition: If C and D are chain complexes, their tensor product $\mathsf{C} \otimes \mathsf{D}$ has

$$(\mathsf{C} \otimes \mathsf{D})_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and boundary / differential given on generators by

$$\partial(c \otimes d) = (\partial c) \otimes d + (-1)^{|c|} c \otimes (\partial' d),$$

where $|c|$ is the degree of c, (that is, $c \in C_{|c|}$).

We note the connection between $\otimes$ and Hom, namely that, given chain complexes, C, D, and E, there are natural isomorphisms

$$Hom(\mathsf{C} \otimes \mathsf{D}, \mathsf{E}) \cong Hom(\mathsf{C}, Hom(\mathsf{D}, \mathsf{E})),$$

so $- \otimes \mathsf{D}$ and $Hom(\mathsf{D}, -)$ are adjoint.

Example:

$$\begin{aligned} (\mathsf{C} \otimes \mathsf{I})_n &\cong C_n \otimes I_0 \oplus C_{n-1} \otimes I_1 \\ &\cong C_n \oplus C_n \oplus C_{n-1} \end{aligned}$$

(We will denote elements in this direct sum as column vectors, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, but will usually write $(x, y, z)^t$, or even (x, y, z) if we are being lazy!)

The isomorphism matches $c_n \otimes e_0^0$ with $(c_n, 0, 0)^t$, $c_n \otimes e_1^0$ with $(0, c, 0)^t$ and $c_{n-1} \otimes e_1^1$ with $(0, 0, c_{n-1})^t$. We can therefore calculate $\partial(x, y, z)^t$ explicitly for $(x, y, z)^t \in C_n \oplus C_n \oplus C_{n-1}$.

$$\begin{aligned}
\partial(x, 0, 0)^t &= (\partial x, 0, 0)^t \\
\partial(0, y, 0)^t &= (0, \partial y, 0,)^t
\end{aligned}$$

and, as $(0, 0, z)^t$ corresponds to a "$c_{n-1} \otimes e_1^1$", its boundary is

$$\begin{aligned}
\partial(c_{n-1} \otimes e_1^1) &= \partial(c_{n-1}) \otimes e_1^1 + (-1)^{n-1} c_{n-1} \otimes \partial(e_1^1) \\
&= \partial(c_{n-1}) \otimes e_1^1 + (-1)^{n-1} c_{n-1} \otimes e_1^0 + (-1)^n c_{n-1} \otimes e_0^0
\end{aligned}$$

i.e. $\partial(0, 0, z)^t = ((-1)^n z, (-1)^{n+1} z, \partial z)^t$. This allows us to use, if we want to, a matrix representation of the boundary in $\mathsf{C} \otimes \mathsf{I}$ as

$$\begin{pmatrix} \partial & 0 & (-1)^{n-1} \\ 0 & \partial & (-1)^n \\ 0 & 0 & \partial \end{pmatrix}$$

and thus would allow us to use such a description to *define* a cylinder $\mathsf{C} \otimes \mathsf{I}$ for C, a chain complex in a more abstract setting such as that of an arbitrary Abelian category.

There are obvious chain maps,

$$\mathsf{e}_i : \mathsf{C} \to \mathsf{C} \otimes \mathsf{I},$$

$i = 0, 1$, corresponding to the ends of the cylinder, and a projection,

$$\sigma : \mathsf{C} \otimes \mathsf{I} \to \mathsf{C},$$

corresponding to 'squashing' the cylinder onto the base.

This, of course, leads to a notion of homotopy between chain maps.

Definition: A *(cylindrical) homotopy*, h, between two chain maps, $\mathsf{f}, \mathsf{g} : \mathsf{C} \to \mathsf{D}$, is a chain map,

$$\mathsf{h} : \mathsf{C} \otimes \mathsf{I} \to \mathsf{D},$$

with $\mathsf{he}_0 = \mathsf{f}$ and $\mathsf{he}_1 = \mathsf{g}$.

This notion of a 'cylindrical' homotopy, h, between two chain maps is easy to analyse. We have $h_n : C_n \oplus C_n \oplus C_{n-1} \to D_n$ and the conditions $\mathsf{he}_0 = \mathsf{f}$ and $\mathsf{he}_1 = \mathsf{g}$ become, in terms of coordinates, $h_n(x, 0, 0) = f_n(x)$, and $h_n(0, y, 0) = g_n(y)$, thus the 'free' or 'unbound' information for h is contained in $h_n(0, 0, z)$. This map, h , restricted to the C_{n-1}-summand gives a degree one map $\mathsf{h}' = \{h'_{n-1} : C_{n-1} \to D_n\}$. We have assumed that h is a chain map, so with our convention for the boundary on $\mathsf{C} \otimes \mathsf{I}$, we get:

$$\begin{aligned}
\partial h'_{n-1}(z) = \partial h_n(0, 0, z) &= h\partial(0, 0, z) \\
&= h((-1)^{n-1} z, (-1)^n z, \partial z) \\
&= (-1)^{n-1}(f_{n-1}(z) - g_{n-1}(z)) + h'(\partial z).
\end{aligned}$$

We thus have that, if we put $s_n = (-1)^n h'_n$, we will get a chain homotopy $\mathsf{s} : \mathsf{C} \to \mathsf{D}$, from f to g. Conversely any chain homotopy will yield a cylindrical homotopy.

Notational comment: The convention on signs that we have adopted is not the only on $\mathsf{C} \otimes \mathsf{I}$ and, as you can easily check, this will determine a different boundary on the chain complex, although the individual terms of the complex are still isomorphic to $C_n \oplus C_n \oplus C_{n-1}$.

Later we will consider the suspension $\mathsf{C}[1]$ of C and this has $C[1]_n = C_{n-1}$. Different sources on differential graded objects may adopt different conventions as to the form of the boundary for $\mathsf{C}[1]$. Quite often the convention chosen is $\partial_n^{\mathsf{C}[1]} = (-1)^n \partial_{n-1}^{\mathsf{C}}$, as this absorption of the $(-1)^n$ makes certain graded maps that naturally occur into chain maps and thus greatly simplifies the formulae and to some extent the theory.

These sign conventions are extremely useful in the study of differential graded algebras as in rational homotopy theory, cf. [128]. We are using chain complexes here mainly as an illustrative example, so will not need to adopt those conventions here. The reader is, however, advised that if working with differential graded (dg) structures, attention to the compatibility between the simplicial and 'dg' conventions is essential if your calculations are not going to look wrong! There is no essential difference in the geometric intuitions between the approaches, but confusion can easily arise if this is not recognised early on in work at this interface.

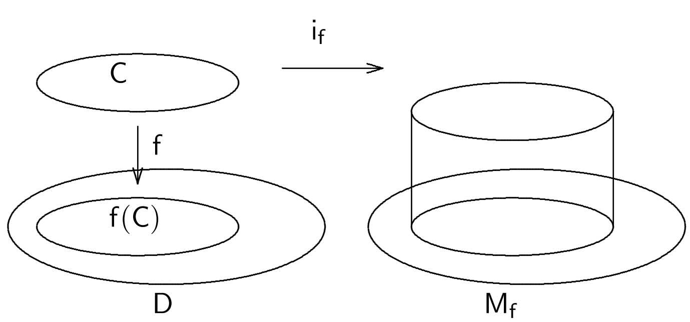
Given our chain map, $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, we can form a *mapping cylinder* on f by the pushout

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathsf{f}"] \arrow[d, "\mathsf{e}_0"'] & \mathsf{D} \arrow[d, "\mathsf{j}_\mathsf{f}"] \\
\mathsf{C} \otimes \mathsf{I} \arrow[r, "\pi_\mathsf{f}"'] & \mathsf{M}_\mathsf{f}
\end{tikzcd}$$

and we can set $\mathsf{i}_\mathsf{f} = \pi_\mathsf{f} \mathsf{e}_1$. The fact that the e_i are split by $\mathsf{s} : \mathsf{C} \otimes \mathsf{I} \to \mathsf{C}$ means that we can form a commutative square

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathsf{f}"] \arrow[d, "\mathsf{e}_0"'] & \mathsf{D} \arrow[d, "="] \\
\mathsf{C} \otimes \mathsf{I} \arrow[r, "\mathsf{fs}"'] & \mathsf{D}
\end{tikzcd}$$

and obtain an induced map $\mathsf{p}_\mathsf{f} : \mathsf{M}_\mathsf{f} \to \mathsf{D}$ satisfying $\mathsf{p}_\mathsf{f} \mathsf{j}_\mathsf{f} = id_\mathsf{D}$ and $\mathsf{p}_\mathsf{f} \pi_\mathsf{f} = \mathsf{fs}$. The second equation then gives $\mathsf{p}_\mathsf{f} \mathsf{i}_\mathsf{f} = \mathsf{f}$, as an easy consequence.



In addition, $\mathsf{j}_\mathsf{f} \mathsf{p}_\mathsf{f} : \mathsf{M}_\mathsf{f} \to \mathsf{M}_\mathsf{f}$ is homotopic to the identity by a homotopy that is constant on composition with j_f, *i.e.*, D is a strong deformation retract of M_f.

Note that we have not shown this last fact. That is **left for you to do**. We should also note that most of this does not use any specific properties of chain complexes nor of the cylinder that we have been using. The same arguments would work for any 'reasonable' cylinder functor on a category with pushouts. The construction of a homotopy from $\mathsf{j_f p_f}$ to the identity *does* use a few more properties. (**Try to investigate what is needed.** A quite detailed discussion of this from one point of view can be found in Kamps and Porter, [171], in a form fairly compatible with that used here.) We will need to use this mapping cylinder construction several times more in different contexts, so abstraction is useful.

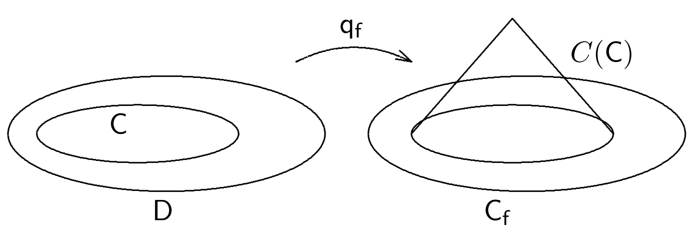
Aside: In [171], you will also find a proof that $\mathsf{i_f}$ satisfies a *homotopy extension property*, *i.e.*, it is a *cofibration*. The description above shows that any f can be factored as a cofibration composed with a strong deformation retraction.

Before we leave mapping cylinder-type constructions as such, we also need to comment on the dual situation, as that is really what we need for our immediate task. In many situation we can form a cocylinder, $\mathsf{D^I}$, either instead of, or as well as, a cylinder. For instance, in the setting of chain complexes, we can set $\mathsf{D^I} = Hom(\mathsf{I}, \mathsf{D})$ and then, as is easily checked, ${\mathsf{D^I}}_n \cong D_n \oplus D_n \oplus D_{n+1}$. The boundary is left to you to write down. The adjointness isomorphism gives the connection with the cylinder and also with chain homotopies. We can form a *mapping cocylinder* by a pullback:

$$\begin{tikzcd} \mathsf{M^f} \arrow[r, "\pi^{\mathsf{f}}"] \arrow[d, "\mathsf{j^f}"'] & \mathsf{D^I} \arrow[d, "\mathsf{e^0}"] \\ \mathsf{C} \arrow[r, "\mathsf{f}"'] & \mathsf{D}. \end{tikzcd}$$

There is a morphism $\mathsf{p^f} : \mathsf{C} \to \mathsf{M^f}$ splitting $\mathsf{j^f}$, so $\mathsf{j^f p^f} = id$, and also $\mathsf{p^f j^f} \simeq id$. Writing $\mathsf{i^f} = \mathsf{e_1}\pi^{\mathsf{f}}$, we have $\mathsf{i^f p^f} = \mathsf{f}$. This map $\mathsf{i^f}$ is a fibration, even in the abstract case under reasonable conditions on the context and the properties of the cocylinder functor, and we find, for instance in the topological setting, the method we used to replace an arbitrary map into a fibration, up to homotopy, (look back to page 45).

Returning now to mapping cylinders, we have $\mathsf{i_f} : \mathsf{C} \to \mathsf{M_f}$ inserting C as the 'top' of the cylinder part of $\mathsf{M_f}$. The *mapping cone*, $\mathsf{C_f}$, (or, sometimes, $C(\mathsf{f})$) of f is obtained by quotienting out by the image of $\mathsf{i_f}$. (This is usually visualised by imagining $\mathsf{C_f}$ as a copy of D together with a cone, $C(\mathsf{C})$ on C glued to it using f.)



We note that the map $\mathsf{j_f} : \mathsf{D} \to \mathsf{M_f}$ composed with the quotient $q : \mathsf{M_f} \to \mathsf{C_f}$ gives a map, $\mathsf{q_f} : \mathsf{D} \to \mathsf{C_f}$ and that the cone structure provides a homotopy between the composite, $\mathsf{C} \to \mathsf{D} \to \mathsf{C_f}$, and the trivial map, $\mathsf{C} \to \mathsf{C_f}$. We should look at this more closely.

If we compose the cylindrical homotopy given by the identity on $\mathsf{C} \otimes \mathsf{I}$ with π_{f}, we get a homotopy between $\pi_{\mathsf{f}}\mathsf{e}_0$ and $\pi_{\mathsf{f}}\mathsf{e}_1$, but $\pi_{\mathsf{f}}\mathsf{e}_0 = j_{\mathsf{f}}\mathsf{f}$ and $\pi_{\mathsf{f}}\mathsf{e}_1 = \mathsf{i}_{\mathsf{f}}$. Finally composing everything with $\mathsf{q} : \mathsf{M}_{\mathsf{f}} \to \mathsf{C}_{\mathsf{f}}$, we have a homotopy between $\mathsf{q}j_{\mathsf{f}}\mathsf{f} = \mathsf{q}_{\mathsf{f}}\mathsf{f}$ and $q\mathsf{i}_{\mathsf{f}}$, which latter map is trivial.

Dually we can get a *homotopy (mapping) cocone*: we take the homotopy cocylinder M^{f} and the map $\mathsf{i}^{\mathsf{f}} : \mathsf{M}^{\mathsf{f}} \to \mathsf{D}$ and form its fibre over the 'basepoint', that is the zero, of D. Of course that 'fibre' is just the kernel of i^{f} in our chain complex case study.

Aside on homotopy cokernels, etc.

In discussion on kernels and cokernels in Abelian and additive categories, it is quite often noted that the cokernel of a map, $\varphi : A \to B$, say in an Abelian category, gives a pushout

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & Coker\,\varphi \end{tikzcd}$$

and that the pushout square property is exactly the universal property defining cokernels. The construction of the mapping cone gives a similar square:

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathsf{f}"] \arrow[d] \arrow[dr, phantom, "\swarrow\!\!\!\!\!\!\swarrow"] & \mathsf{D} \arrow[d, "\mathsf{q}_{\mathsf{f}}"] \\ 0 \arrow[r] & C_{\mathsf{f}} \end{tikzcd}$$

but it is only homotopy commutative (or rather homotopy coherent as there is the natural *explicit* homotopy, $\mathsf{h}_{\mathsf{f}} : \mathsf{q}_{\mathsf{f}}\mathsf{f} \Rightarrow 0$). This homotopy coherent square has a universal property with respect to homotopy coherent squares based on $0 \leftarrow \mathsf{C} \xrightarrow{\mathsf{f}} \mathsf{D}$. This makes it reasonable to call the result a *homotopy pushout* and then to say that C_{f} is the *homotopy cokernel* or sometimes the *homotopy cofibre* of f. It is, of course, an example of a homotopy colimit, but note that it is necessary to give not only C_{f} plus q_{f} to get the full universal property (as would be the case for an ordinary colimit), but also h_{f}.

Exercise: The construction of the mapping cylinder is also a homotopy pushout. Try to formulate a good notion of homotopy pushout and identify that construction as an example of one such. The main idea is to start with two maps

$$B \xleftarrow{b} A \xrightarrow{c} C$$

with common domain and to form a homotopy coherent square

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "b"'] \arrow[dr, phantom, "\swarrow\!\!\!\!\!\!\swarrow"] & C \arrow[d, "b'"] \\ B \arrow[r, "c'"'] & D, \end{tikzcd}$$

where h is a homotopy $A \times I \to D$ between $b'c$ and $c'b$. For instance, use a repeated pushout operation on the diagram

$$\begin{tikzcd}
 & A \arrow[r, "c"] \arrow[d, "e_0"] & C \arrow[dd, dashed] \\
A \arrow[r, "e_1"] \arrow[d, "b"'] & A \times I & \\
B \arrow[rr, dashed] & & D
\end{tikzcd}$$

to construct its colimit, which will be a *double mapping cylinder*. The homotopy h is then clear. Specialise down to the case of b being the identity to complete. Note that homotopy pushouts are determined 'up to homotopy', not 'up to isomorphism', so you may not quite get what you expect and different construction may give different, but homotopic, models for it!

This discussion of homotopy cokernels is almost 'general'. It works, more or less, in any setting where there is a null object, corresponding to 0, having a nice cylinder that preserves pushouts, and, of course, enough pushouts. In our well behaved case study of chain complexes, we can track the construction in the direct sum decomposition if we so wish.

Homotopy commutative v. homotopy coherent: It is quite important to note a sort of theme that occurs both here and earlier in our discussion of bitorsors and M-torsors. An M-torsor was a C-torsor, E together with a definite choice of global section for $\partial_*(E)$. We did not just say the $\partial_*(E)$ is trivialisable, we specified a trivialisation as part of the structure.

Here with homotopy pushouts, we do not just have a homotopy commutative square, but specify a definite choice of homotopy linking the two composite maps around the square, *i.e.*, we give a 'homotopy coherent square'. This passage from 'there is a homotopy such that ...' to specifying one is of prime importance in interpreting non-Abelian cohomology.

We have concentrated, so far, on the case of chain complexes. We do need to caste a glance at the topological case. The above description in terms of homotopy cokernels goes through for pointed spaces.

Suppose $f : X \to Y$ is a map of pointed spaces, we can form M_f and factorise f as $p_f i_f = f$, where i_f is a cofibration and p_f is the retraction part of a strong deformation retraction, so in particular is a homotopy equivalence.

Using the cofibration $i_f : X \to M_f$, we divide out, identifying its image to a point, to get C_f as a quotient space, or directly as a homotopy pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & Y \arrow[d, "q_f"] \\
* \arrow[r] & C_f,
\end{tikzcd}$$

where $q_f = qj_f$ with $q : M_f \to C_f$ the quotient map.

8.2.2 Puppe exact sequences

The map q_f is a cofibration, under reasonable conditions on the spaces involved, and we can form the quotient of C_f by identifying the image of this map to a point: $SX \cong C_f/Y$, giving the (reduced) suspension, SX, on X. This can be defined directly as $(X \times I)/(X \times \{0,1\} \cup * \times I)$, where $*$ is the base point of X. It is also the homotopy pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\swArrow"] & * \arrow[d] \\ * \arrow[r] & SX, \end{tikzcd}$$

where the homotopy is the quotient map from $X \times I$ to SX.

This gives us a sequence of maps

$$X \xrightarrow{f} Y \to C_f \to SX \xrightarrow{Sf} SY \to SC_f \to S^2X \to \dots,$$

where we have extended the bit that we have actually constructed by applying S to it and grafting it to the old part. This sequence is known, variously, as the *long cofibre sequence* of f, the *Puppe sequence* of f or the *cofibre Puppe sequence*. It is 'homotopy exact' - what does that mean?

Recall that in an exact sequence, say, of Abelian groups, the kernel of one map is the image of the previous one, so in particular, the composition of pairs of maps in the sequence is always trivial. In the above sequence of pointed spaces, there is an *explicit* null-homotopy from each composition of pairs of adjacent maps to the corresponding trivial map that send the domain to the base point of the codomain. This is clear for the first composable pair $X \xrightarrow{f} Y \to C_f$ as that is exactly what C_f was designed to do! (Some treatments of these sequences in fact construct them by repeating that basic construction of C_f from f for subsequent maps starting with $Y \to C_f$, and then showing that the resulting terms match, up to homotopy, with those of the above sequence. We do not adopt that approach here, although it has some very good points to it.)

The next pair $Y \to C_f \to SX$ is trivial anyway. The checking that $C_f \to SX \xrightarrow{Sf} SY$ is homotopy exact is omitted. It can be found in the literature or you can attempt it yourself. This is thus the analogue of the composites being trivial in an exact sequence. The arguments used for these also show that an analogue of the other part of 'exactness' also holds. For this it seems advisable to indicate a more precise statement. (The temptation to use the words 'exact statement' here must be resisted!) That statement is the usual one here, and goes as follows. (It will need a certain amount of commentary, which will be given shortly.)

For any pointed space, Z, applying the functor $[-, Z]$ to the above sequence yields a long exact sequence of groups and pointed sets,

$$\dots \to [S^2X, Z] \to [SC_f, Z] \to [SY, Z] \to [SX, Z] \to [C_f, Z] \to [Y, Z] \to [X, Z].$$

We have already recalled the meaning of exactness for sequences of groups. The extension of that to pointed sets should be clear: we replace 'kernel' by 'preimage of the base point' whilst 'image' has the same meaning. If we examine the exactness at $[Y, Z]$, this says that if $g : Y \to Z$ is such that gf is null homotopic, (that is, there is some $h : gf \simeq *$), then there is some $\overline{g} : C_f \to Z$ such that $g = \overline{g}q_f$, and conversely. But that is just what the construction of C_f does, as the nullhomotopy extends the map on Y to the cone on the X part of C_f. In fact, of course, different nullhomotopies will extend to different maps on C_f and you are left to think about the way in which these different null homotopies are, or are not, 'observed' by the sequence. To start you thinking, if $h, h' : gf \simeq *$, then we have a self homotopy of $*$, intuitively, '$hh'^{(-1)}$'. The map

$hh'^{(-1)} : X \times I \to Z$ sends both ends of the cylinder to the basepoint and as it is constructed from pointed homotopies, it also sends $* \times I$ there. It thus induces a map from SX to Z, giving a possible link back to $[SX, Z]$. Again the theme of homotopy coherence v. homotopy commutativity is nearby as if we record the possible null homotopies then we get other information cropping up elsewhere in the sequence.

In this discussion of 'homotopy exact sequences', we have still to complete our discussion of the cofibre sequence of a chain map and also we will have need not so much of this cofibre form of the Puppe sequence, but rather the Puppe 'fibre' long exact sequence of a map. We start with the chain cofibre sequence.

So far we have

$$\mathsf{C} \to \mathsf{D} \to \mathsf{C_f}$$

and, in elementary terms,

$$(\mathsf{C_f})_n \cong D_n \oplus C_{n-1},$$

i.e., the pushout of D and a cone on C. (The differential / boundary is **left to you**.) There is an inclusion of D into $\mathsf{C_f}$, and, surprise surprise, the quotient is $\mathsf{C}[1]$, it has C_{n-1} in dimension n, so is the chain complex analogue of the suspension. (Here we must repeat the warning about sign conventions. The suspension is often considered to have boundary $(-1)^n\partial_n$, corresponding to the needs for the 'suspension map' to be a chain map. This is just due to a different convention on the boundary map of the cylinder. As we need this as a step to understanding the *simplicial* situation, our convention is slightly more appropriate.)

Of course, if E is another chain complex, then applying $[-, \mathsf{E}]$ should give us a long exact sequence. (All is not really as simple as that here as it is usually better to work in what is called the *derived category* of chain complexes rather than just dividing out by homotopy. Initially you should try this for chain complexes of free modules as you cannot always create the maps you want in more general contexts. This general situation *is* important and will be needed in certain aspects later on, but we will ignore the complication here. It is a very useful exercise to show the long exactness for chain complexes of free (or projective) modules, before trying to understand the complication if the freeness condition is removed.)

Now we turn to 'fibre Puppe sequences' in the topological case: we have our $f : X \to Y$ and form the *mapping cocylinder*, M^f, with $i^f : M^f \to Y$ being a fibration and $M^f \simeq X$ in a controlled way, (homotopy coherence again - and, yes, M^f *is* given by a homotopy pullback.) We form the fibre of i^f, and this is $C^f = F_h(f)$, the homotopy fibre of f that we have met before (cf. page 45). This is also a homotopy pullback:

$$\begin{tikzcd} C^f \arrow[r] \arrow[d, "f^f"'] \arrow[dr, phantom, "\swarrow"] & * \arrow[d] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

wher q^f is the composite $C^f \to M^f \to X$. We can realise this very neatly by first using the pullback

$$\begin{tikzcd} \Gamma Y \arrow[r] \arrow[d] & Y^I \arrow[d, "e_0"] \\ * \arrow[r] & Y \end{tikzcd}$$

giving the object of paths that start at $*$. This has a second map to Y induced by e_1, giving $\Gamma Y \to Y$, which is a fibration. This is the dual analogue of the cone on X in this dual context.

(The notation ΓY is 'traditional', but is also traditional for the set of global sections of a bundle! No confusion should arise!) This space ΓY is contractible in a geometrically pleasing way - the homotopy reduces the 'active' part of each path until it does nothing: if $\alpha : I \to Y$ with $\alpha(0) = *$, then $\alpha_t(s) = *$ if $s \leq t$ and is $\alpha(s-t)$ if $t \leq s \leq 1$. The α_t form a homotopy, essentially a path, from α to the constant path at $*$. We can realise C^f as the pullback:

$$\begin{tikzcd} C^f \arrow[r] \arrow[d] & \Gamma Y^I \arrow[d] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

(A useful observation here is that this pullback absorbs the homotopy of the homotopy pullback by replacing the $*$ by a contractible space. That *is* an example of a general process, a 'rectification' or 'rigidification' process, but this will not be explored until much later in these notes.)

Example 1: The neat example that illustrates the importance of this homotopy fibre construction is to take Y to be an arcwise connected space, X a proper subspace (so the inclusion f is very far from being a fibration). The fibre of f over a point $y \in Y$ is either a single point, if $y \in X$, or empty, if it is not. We think of y as being a map $y : * \to Y$, picking out that element, and change y along a path y_t, from being in X, say y_0, to not being in X, at y_1. That path is a homotopy between the maps y_0 and y_1, so although y_0 and y_1 are homotopic maps, the fibre over y_t changes homotopy type as t varies. On the other hand, the homotopy fibre has the same homotopy type along the whole of y_t. (We saw earlier (page 45) that the fundamental group of $F_h(f)$ was $\pi_2(Y, X)$ and does not change, up to specified isomorphisms, as one varies t between 0 and 1.)

Example 2: This first example was with f far from being a fibration. What if f *is* a fibration? (We, as usual, want to concentrate on the intuitions behind the facts here so will not explore this in depth, but it will be useful to have some picture of what is happening, leaving details either **to the reader** to provide or to find, as the results are fairly easy to find in the literature.)

First note the obvious

$$f^{-1}(*) = \{x \mid f(x) = *\},$$

whilst

$$C^f = \{(x, \lambda) \mid \lambda \in \Gamma Y, \lambda(0) = *, \lambda(1) = f(x)\},$$

so, in particular, there is a map from $f^{-1}(*)$ to C^f, mapping x to (x, c), where c is the constant path at $*$. We would like to see when this map is a homotopy equivalence. We have that underlying it, in some sense, is the map sending $*$ to $c \in \Gamma Y$, which is a homotopy equivalence, in fact a strong deformation retraction. If you try to see if this will induce in some way a retraction from C^f to $f^{-1}(*)$, then you hit the problem of what path an element (x, λ) should trace out in order to get to some $(x', c) \in f^{-1}(*)$. This would have to project down onto a path in X and in general there will not be one. If we assume that f is a fibration however, we can see more clearly what to do. (Recall that a fibration has a homotopy lifting property and it is that we will use.)

Examine the following diagram:

$$\begin{tikzcd} C^f \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ C^f \times I \arrow[r] & Y. \end{tikzcd}$$

The bottom horizontal map here is the composite $C^f \times I \to \Gamma Y \to Y$. The first of these is the inclusion, then the second is the homotopy retracting ΓY to a point, composed with the projection onto Y. The top horizontal map is q^f, so the diagram commutes. As f is assumed to be a fibration, there is a lift of the bottom map to a homotopy $C^f \times I \to X$, extending q^f on its 'zero' end. Its other end gives a map which has image in the fibre of f, so we have what we want - except for **checking details**!

This is very useful as it says: if f is a fibration, we do not need to turn it into one before taking its fibre! Why is that useful? Look at the fibre Puppe sequence so far

$$C^f \to X \to Y.$$

We said that ΓY is a fibration, so $q^f : C^f \to X$ is also a fibration. We can take its homotopy fibre, which will look messy to say the least, or its fibre, which is a lot easier to calculate!

$$\begin{aligned}(q^f)^{-1}(*_X) &= \{(\lambda, x) \mid \lambda(0) = *_Y, \lambda(1) = f(x), x = *_X\} \\ &= \{\lambda \mid \lambda(0) = \lambda(1) = *_Y\},\end{aligned}$$

so $(q^f)^{-1}(*_X) \cong \Omega Y$, the space of loops, at the base point,of Y. (This is neat, of course, as Ω is a functor, which is adjoint to S, the reduced suspension. Whether it is **right or left adjoint is left to you**! Thus we have a linkage between the right and left Puppe sequence constructions.) That fact gives us the tool to open up the whole of the sequence. It goes

$$\ldots \to \Omega^2 Y \to \Omega C^f \to \Omega X \stackrel{\Omega f}{\to} \Omega Y \to C^f \to X \stackrel{f}{\to} Y.$$

Given a pointed space Z, we can apply $[Z, -]$ to this sequence to get our long exact sequence

$$\ldots \to [Z, \Omega^2 Y] \to [Z, \Omega C^f] \to [Z, \Omega X] \stackrel{[Z,\Omega f]}{\to} [Z, \Omega Y] \to [Z, C^f] \to [Z, X] \stackrel{[Z,f]}{\to} [Z, Y],$$

(and once **you have sorted out** right or left adjunctions, you will find many terms you recognise from the other type of Puppe sequence).

Our treatment here has been deliberately informal. The importance of these sequences for cohomology cannot be over emphasised and we **suggest that you look** at some formal treatments, both for the algebraic case (via derived and triangulated categories, e.g. Neeman, [220]) and via the topological case consulting, say, May, [200] in the first instance before looking into the theory in other sources. There are abstract versions in homotopical algebra, see, for instance, in Hovey, [158], and a neat categorical treatment in Gabriel and Zisman, [132].

One final point before passing from descriptions of Puppe sequences to using them is the interpretation of exactness at the various points in the sequence. For instance, at $[Z, C^f]$, an element is represented by a map, g say, to C^f, and as C^f is given by a pullback, g decomposes via the two projections into a pair (g_X, g_Γ) with $g_X : Z \to X$ and $g_\Gamma : Z \to \Gamma Y$ such that $fg_X = e_1 g_\Gamma$. Going one step further, $\Gamma Y \subset Y^I$, so g_Γ gives a homotopy between $*$, the constant map to the basepoint, and fg_X. Now suppose $[Z, f] : [Z, X] \to [Z, Y]$ sends a homotopy class $[k]$ to the basepoint, then fk is homotopic to $*$ and we can build a $g : Z \to C^f$ from k and that homotopy. The more difficult part of the exactness at $[Z, X]$ follows. Back to $[Z, C^f]$, suppose our $g = (g_X, g_\Gamma)$ gets sent to the 'point' of $[Z, X]$, then $q^f g_X$ must be null homotopic. Pick such a null homootpy $h : Z \times I \to X$ and use the fact that q^f is a fibration to lift h to $\overline{h} : Z \times I \to C^f$. The 'other end ' of $\overline{h}$, *i.e.*, $\overline{h}e_1$ is such that $q^f \overline{h} e_1$ is $*$, so $\overline{h}e_1$ is into the fibre of q^f, but that is ΩY. It remains to put the various pieces together. The details can be found in many sources, but what is important to retain is the way of constructing a corresponding element in the previous stage. A trivialisation of an element yields a class in another stage. This should remind you of M-torsors, of categorisation and of homotopy cohenrence.

8.3 Puppe sequences and classifying spaces

8.3.1 Fibrations and classifying spaces

In his discussion of bitorsors, etc., in [48], Breen makes use of Puppe sequences of maps between classifying spaces. Suppose $v : H \to G$ is a morphism of simplicial groups, then we get an induced map of classifying spaces $Bv : BH \to BG$. We can take BG to be $\overline{W}G$ as being the neatest construction from our simplicial viewpoint. (Detailed calculations with $\overline{W}G$, etc., are quite easy in the simple case that we will need, but do get complicated if G has lots of non-trivial terms in its Moore complex. Another point worth making is that the detailed formulae for $\overline{W}G$ given earlier, page 254, use the algebraic composition order and therefore sometimes seem to reflect 'right actions'. This can be got around in either of two ways. The formulae for both $\overline{W}$ and G, the Dwyer-Kan $\mathcal{S}$-groupoid functor, can easily be reversed to get equivalent ones using the other composition order. This may be needed later when considering cocycles, etc., however the second argument uses that $\overline{W}G$ determines a Kan complex that is determined up to homotopy type - so either method will lead to the same $[-, \overline{W}G]$ and thus *most* of the time we can ignore the composition order. To ignore it, or forget it, completely is not a good idea, but we can face the problem, if and when it is needed.)

We thus are looking at $Bv : BH \to BG$. If v is not surjective, then we can use the mapping cocylinder construction, suitably adapted, to replace it by a fibration and fibrations of simplicial groups are exactly the surjective morphisms. We can thus study, without loss of generality, the surjective case and, of course, that means using the exact sequence

$$K \stackrel{u}{\to} H \stackrel{v}{\to} G$$

of simplicial groups and studying the effect of the functor B on it.

We 'clearly' get a long Puppe sequence, ending with

$$\ldots \to \Omega BH \to \Omega BG \to C^{Bv} \to BH \to BG.$$

Such a Puppe sequence can be constructed from the 'obvious' cocylinder functor, $\mathcal{S}_*(\Delta[1], -)$, but only works really well if applied to Kan complexes. Luckily these simplicial sets *are* Kan, so we can proceed accordingly. We note that as v is a fibration of simplicial groups, Bv is a fibration of simplicial sets, so we can hope that C^{Bv} can be more easily calculated than would be the case in general.

To see why Bv is a fibration, imagine we have a $\underline{g} \in BG_n$ and thus $\underline{g}$ has the form $(g_{n-1}, \ldots, g_0)$ with $g_i \in G_i$. We can find $h'_i \in H_i$ such that $v(\overline{h}'_i) = g_i$, $i = 0, \ldots, n-1$. If we are given a (n,k)-horn, $\overline{h}$, in BH that maps down to the (n,k)-horn, $(d_n\underline{g}, \ldots, \widehat{d_k\underline{g}}, \ldots, d_0\underline{g})$, of $\underline{g}$ (using the traditional $\widehat{\ }$ notation for an omitted element), then $\underline{h}^{-1}.\overline{h}'$ gives a horn over the trivial (n,k)-horn of BG, that is, we can *translate* the filling problem to the identity, where it is essentially that of proving that $\overline{W}G$ is a Kan complex, which is easier to handle and we will do so in a moment. Note this argument uses a transversal in each dimension, although we did not explicitly label it as being one, namely $g_i \mapsto h'_i$, which is suggestive of other uses of transversals in these notes.

An indirect, but neat, proof that $\overline{W}$ preserves fibrations and weak equivalences is to be found on p. 303 of the book, [137], by Goerss and Jardine. They note that this implies that $\mathcal{G}$ preserves cofibrations and weak equivalences, which is also very useful.

Postponing the proof that classifying spaces are Kan for the moment, the last thing to identify is the fibre of Bv, but this is easy, since we have an explicit description of Bv. It sends $\underline{h} = (h_{n-1}, \ldots, h_0)$ to $(v(h_{n-1}), \ldots, v(h_0))$, so its fibre is exactly the image by Bu of BK. We can thus use that, for fibrations, the fibre and homotopy fibre coincide up to equivalence, to conclude $C^{Bv} \simeq BK$ and our Puppe sequence now looks like

$$\ldots \to \Omega BH \to \Omega BG \to BK \to BH \to BG.$$

8.3.2 $\overline{W}G$ is a Kan complex

We have left this aside because we want to examine it in some detail, and those details were not needed at that point in our discussion. As an example of what might be done, suppose that G satisfies some extra condition such as the vanishing of its Moore complex in certain dimensions or that it satisfies the thin filler condition above some dimension, then the constructive description of $\overline{W}G$ suggests that it might be feasible to analyse $\overline{W}G$ to see if it satisfies some similar condition.

We will give the verification for a simplicial group, however, in many of the applications, we will need the construction for a simplicial group object in a topos, $\mathcal{E}$. This will allow us to talk of the classifying space of a sheaf of simplicial groups without worrying about the context. All the structure, however, is specified in a constructive way, and so goes across without any pain to a general topos. It also goes across without difficulty to an $\mathcal{S}$-groupoid. (I learnt this via Phil Ehlers' MSc thesis, [117], in which he did all the calculations explicitly.)

For convenience, we repeat the formulae for $\overline{W}G$, from page 254, making small adjustments, since we will not be looking at the groupoid case here, so let G be a simplicial group.

The simplicial set, $\overline{W}G$, is described by

- $(\overline{W}G)_0$ is a single point, so $\overline{W}(G)$ is a reduced simplicial set;
- $(\overline{W}G)_n = G_{n-1} \times \ldots G_0$, as sets, for $n \geq 1$.

The face and degeneracy mappings between $\overline{W}(G)_1$ and $\overline{W}(G)_0$ are the source and target maps and the identity maps of G_0, respectively; whilst the face and degeneracy maps at higher levels are given as follows:

The face and degeneracy maps are given by

- $d_0(g_{n-1}, \ldots, g_0) = (g_{n-2}, \ldots, g_0)$;
- for $0 < i < n$, $d_i(g_{n-1}, \ldots, g_0) = (d_{i-1}g_{n-1}, d_{i-2}g_{n-2}, \ldots, d_0g_{n-i}g_{n-i-1}, g_{n-i-2}, \ldots, g_0)$;

and

- $d_n(g_{n-1}, \ldots, g_0) = (d_{n-1}g_{n-1}, d_{n-2}g_{n-2}, \ldots, d_1g_1)$,

whilst

- $s_0(g_{n-1},\ldots,g_0) = (1, g_{n-1},\ldots,g_0)$;

and,

- for $0 < i \leq n$, $s_i(g_{n-1},\ldots,g_0) = (s_{i-1}g_{n-1},\ldots,s_0g_{n-i},1,g_{n-i-1},\ldots,g_0)$.

Let us start in a low dimension to see what problems there may be. For $n = 2$, suppose we had a $(2,2)$ box in $\overline{W}G$, so we have a pair, (x_0, x_1), of elements of $\overline{W}G_1$, which fit together, so $d_0x_0 = d_0x_1$. (We think of this as $(x_0, x_1, -)$, a list of possible faces, with a gap in the d_2-position.) We want some $y \in \overline{W}G_2$ such that $d_0y = x_0$ and $d_1y = x_1$.

Expanding things (in fact this is purely formal here, but lays down notation for later), we thus have $x_0 = (x_{0,0})$, $x_1 = (x_{1,0})$. (The condition on the faces happens to be trivial here since $\overline{W}G_0$ is a single point.) These $x_{i,0}$ are in G_0, for $i = 0, 1$. Similarly y will be of form (y_1, y_0), and we can examine what the desired conditions imply

$$\begin{aligned} x_{0,0} &= d_0y = y_0 \\ x_{1,0} &= d_1y = d_0y_1.y_0. \end{aligned}$$

We thus already know y_0 and need to find a y_1 with $d_0y_1 = x_{1,0}x_{0,0}^{-1}$. Clearly, we can find one, for instance, $s_0(x_{1,0}x_{0,0}^{-1})$ will do and we can even find *all* such, since any other suitable y_1 will have form $ks_0(x_{1,0}x_{0,0}^{-1})$ for some $k \in Ker\, d_o$. In other words, we really do know a lot about the possible fillers for our horn, even being able to count them if G is a finite simplicial group!

Next in line, we suppose that we have $(x_0, -, x_2)$ and want y such that $d_0y = x_0$, $d_2y = x_2$. Expanding these, using the same notation as before, we have, once again, that $x_{0,0} = d_0y = y_0$, but now

$$x_{2,0} = d_2y = d_1y_1.$$

Again we have y_0 and can solve $d_1y_1 = x_{2,0}$, using $y_1 = s_0(x_{2,0})$, and, to get all fillers, $ks_0(x_{2,0})$ with $k \in Ker\, d_1$.

That was easy! What about $(2,0)$-horns? These *are* slightly harder, as the other types did give us d_0y and thus handed us y_0 'on a plate', but it is only '*slightly*'.

We have $(-, x_1, x_2)$, $x_i = (x_{i,0})$ and want $y = (y_1, y_0)$. We thus know

$$\begin{aligned} x_{1,0} &= d_1y = d_0y_1.y_0 \\ x_{2,0} &= d_2y = d_1y_1. \end{aligned}$$

We do not know y_0, but do know d_1y_1 and can solve to get $y_1 = ks_0(x_{2,0})$ with $k \in Ker\, d_1$ as before. We then have $y_0 = (d_0(k)x_{2,0})^{-1}x_{1,0}$ for the general filler.

Although that is simple, it is also easy to see that it can be extended, with modifications, to higher dimensions.

If we have a (n,n)-horn in $\overline{W}G$, then we have $(x_0,\ldots,x_{n-1},-)$ with $x_i = (x_{i,n-2},\ldots,x_{i,0}) \in \overline{W}G_{n-1}$. for $i = 0, 1, \ldots, n-1$. The compatibility condition is non-trivial here, so we note that

$$d_ix_j = d_{j-1}x_i$$

if $i < j$.

We need to find all $y = (y_{n-1}, \ldots, y_0)$ with $d_i y = x_i$ for all $i < n$. We thus have

$$x_0 = d_0 y = (y_{n-2}, \ldots, y_0),$$

but this means that we know all but the top dimensional element of the string that is y. Next

$$x_1 = d_1 y = (d_0 y_{n-1}.y_{n-2}, \ldots, y_0),$$

so we glean the information that

$$d_0 y_{n-2} = x_{1,n-2}.x_{0,n-2}^{-1}.$$

Continuing, we get, for $k > 1$ and in the range $k < n$, that

$$x_k = d_k y = (d_{k-1} y_{n-1}, d_{k-2} y_{n-2}, \ldots, d_0 y_{n-k}.y_{n-k-1}, \ldots, y_0),$$

and here the only new information is that which we get on $d_{k-1} y_{n-1}$, which can be read off as being $x_{k,n-2}$.

We should note that the compatibility condition tells us that there will be no inconsistencies in the rest of this string. For instance, we seem to have

$$x_{k,n-k-1} = d_0 y_{n-k}.y_{n-k-1}.$$

As we know y_{n-k-1} and y_{n-k}, we can check that we do not have a conflict:

$$\begin{array}{rcl} y_{n-k} & = & x_{0,n-k} \\ y_{n-k-1} & = & x_{0,n-k-1}, \end{array}$$

but then $x_{k,n-k-1}$ needs to be $d_0 x_{0,n-k}.x_{0,n-k-1}$, which is the $(n-k-1)$-component of $d_k x_0$. The compatibility condition tells us

$$d_0 x_k = d_{k-1} x_0,$$

and we leave the reader to check that the $(n-k-1)$-component of this equation is exactly

$$x_{k,n-k-1} = d_0 x_{0,n-k}.x_{0,n-k-1},$$

as hoped for.

Collecting things up, we know $d_\ell y_{m-1}$ for $\ell = 0, \ldots, n-2$, *i.e.*, we have a $(n-1, n-1)$-horn in G itself. We know not only that G is a Kan complex, but how to fill horns algorithmically, so can find a suitable y_{n-1} and hence a filler, y, for the original (n,n)-horn in $\overline{W}G$.

The intermediate cases of (n,i)-horns in $\overline{W}G$ for $0 < i < n$ are very similar and are **left to you**. In each case, as we have $d_0 y = x_0$, we just have to work on the first element, y_{n-1} in the string giving us y. The other parts give us a horn in G, which encodes the available information on the faces of y_{n-1}. We fill this horn to get y_{n-1}, and hence to fill the original horn in $\overline{W}G$. In each case, we can fill because we know that the underlying simplicial set of G is a Kan complex. We have the algorithm for fillers and so can analyse the set of fillers for a given horn, the algorithm giving a definite coset representative. For instance, in the (n,n)-horn, above, we found y exactly except in the first, highest dimensional position, y_{n-1}. We use the algorithm to find *one* filler / solution for y_{n-1}, then know any other will differ from it by an element of $\bigcap_{i=0}^{n-2} Ker\, d_i$. This latter group is essentially a 'translate' of NG_{n-1} using the argument that Carrasco used to simplify Ashley's group T-complex condition (see the comment in the discussion of group T-complexes, page 40).

We still have to handle the $(n, 0)$-horn case, so should not be too pleased with ourselves yet! That was the slightly awkward case for the $n = 2$ situation that we studied earlier, as we do not have y_{n-2} given us initially.

Suppose $(-, x_1, \ldots, x_n)$ is the horn and we have to find a $y \in \overline{W}G_n$ satisfying $d_i y = x_i$ for $i = 1, \ldots, n$. Using the same notation, we have

$$x_1 = d_1 y = (d_0 y_{n-1}.y_{n-2}.y_{n-3}, \ldots, y_0)$$

and we get all the y_i except y_{n-1} and y_{n-2}. We then have

$$x_i = d_i y = (d_{i-1} y_{n-1}, \ldots, y_0)$$

and so get all the faces of y_{n-1}, except that zeroth one. We can thus fill the resulting $(n-1, 0)$-box in G (using the algorithm) to find a suitable y_{n-1}. We still do not have y_{n-2}, but as we now have y_{n-1}, we can read off $d_0 y_{n-1}$ from our solution to get

$$y_{n-2} = (d_0 y_{n-1})^{-1}.x_{1,n-1}.$$

We thus do get a filler for our $(n, 0)$-horn *and* can analyse the set of fillers / solutions if we need to.

Theorem 23 *For any simplicial group, G, the classifying space, $\overline{W}G$, is a Kan complex* ∎

Perhaps it occurs to you that it should be possible to adapt this constructive proof to give a proof that, if $f : G \to H$ is a surjection of simplicial groups, and thus a fibration, then $\overline{W}f$ will be a Kan fibration. We know already that $\overline{W}f$ *is* a fibration, as we saw this earlier, quoting some results in Goerss and Jardine, [137], but it should not be too difficult to construct a proof which took transversals in the necessary dimensions and *found* lifts for horns accordingly. This is left as a bit of a **challenge to the reader**. It is not just an exercise for amusement, however, as the analysis of fillers could give some interesting results in some cases.

We mentioned that most of this went across 'without pain' to the case of simplicial objects in a topos, $\mathcal{E}$, and hence to simplicial sheaves on a space. Perhaps a few words are needed, however, to show how this can be done. We start by thinking about how to talk about the Kan fibrations in $\mathcal{E}$, or more generally in any category with finite limits. For any object K in $Simp(\mathcal{E})$, we can form an object of $\mathcal{E}$ corresponding to the 'set of (n, k)-horns' in K. To see how to think about this, we look at $(2, 1)$-horns. These correspond, in the set based case, to pairs of 1-simplices, (x_0, x_2), with $d_0 x_2 = d_1 x_0$, so are elements of the pull back:

$$\begin{tikzcd} \Lambda^0[2](K) \arrow[r] \arrow[d] & K_1 \arrow[d, "d_1"] \\ K_1 \arrow[r, "d_0"'] & K_0 \end{tikzcd}$$

More generally, for a simplicial set K, $\Lambda^k[n](K)$, the set of (n,k)-horns in K is given by an iterated pullback or limit of a diagram. (If you have not seen this before, or ever handled it yourself, do try to formulate the diagram in as neat a way as possible - 'neat' is a question of taste! It is technically quite easy, but gives good practice in converting concepts across to diagrams and hence to finite limit categories.)

We thus can mimic this to get an object, $\Lambda^k[n](K)$, and an induced map, $K_n \to \Lambda^k[n](K)$, which maps an n-simplex to the (n,k)-horn of its faces other than the k^{th} one.

Definition: If $\mathcal{E}$ is a finite limit category, a morphism, $p : E \to B$, in $Simp(\mathcal{E})$ is a *Kan fibration* if the natural maps $E_n \to \Lambda^k[n](E) \times_{\Lambda^k[n](B)} B_n$ are all epimorphisms in $\mathcal{E}$.

We can equally obtain the meaning of a *Kan object* in $Simp(\mathcal{E})$.

Beke, [29], uses the term *local Kan fibration* for what has been called a Kan fibration in $\mathcal{E}$ above. That 'local' terminology is especially good when talking about the topos case, but with, later on in these notes, a use of 'locally Kan' enriched category, it did seem a bit risky to over use 'local Kan'!

We now return to the case of simplicial groups in the usual sense.

Corollary 13 *Suppose that $NG_{n-1} = 1$, then, for any i, with $0 \leq i \leq n$, any (n,i)-horn in $\overline{W}G$ has a unique filler.*

Proof: We noted that different fillers for an (n,i)-horn differed by elements of NG_{n-1}, or its translates, thus if that group is trivial, ... ■

Of course, we expect $\overline{W}G$ to have the same homotopy groups as G, displaced by one dimension, since there is the fibration sequence

$$G \to WG \to \overline{W}G$$

with WG contractible, so this corollary comes as no surprise. What is interesting is the detail that it gives us. If $NG_k = 1$, then clearly $\pi_k(G) = 1$ and hence $\pi_{k+1}(\overline{W}G)$ is trivial as well, but that there are unique fillers in the structure is perhaps a bit surprising, at least until one sees why.

Suppose that, as usual, G is a simplicial group and $D = (D_n)_{n\geq 1}$ is the graded subgroup of products of degeneracies. Within $\overline{W}G_n$, let

$$T_n = D_{n-1} \times G_{n-2} \times \ldots \times G_0,$$

be the subset of those elements whose first component is a product of degenerate elements, yielding a graded subset of $\overline{W}G$.

Corollary 14 *If G is a group T-complex, then $(\overline{W}G, T)$ is a simplicial T-complex.*

Proof: There is not that much to check. We know, by the proof of the theorem, that every horn has a filler in T. Uniqueness follows from the fact that G is a group T-complex. The other conditions are as easy to check as well, so are **left to you**. ■

Corollary 15 *If G is thin in dimensions greater than n, then $\overline{W}G$ has a unique T-filler for all horns above dimension $n+1$.* ■

The property of being a T-complex involves all dimensions and here we are meeting some sort of weaker 'filtered' condition. This condition was studied extensively by Duskin, and used in various forms in [107, 108] and in later work. It was also used by his students Glenn, [136], and Nan Tie, [218, 219], who looked at some of the links with T-complexes. They are also used, more recently, by Beke, [29], and we, in fact, studied his approach earlier when discussing the coskeleton functors, (in particular, in our brief discussion of exact n-types and n-hypergroupoids, cf. page 203).

8.3.3 Loop spaces and loop groups

We now turn to ΩBG. Although not strictly necessary, it will help to shift our perspective slightly and talk a bit more on some generalities. Let S^0 be the pointed simplicial set with two vertices and only degenerate simplices in dimensions higher than 1. In other words, it is the 0-sphere. The reduced suspension SS^0 is S^1, the circle, which can also be realised as $\Delta[1]/\partial\Delta[1]$, the circle realised as the interval with the ends identified to a single point. The loop space, ΩK, on a pointed connected simplicial set, K, is then $\underline{S}_*(S^1, K)$, or more briefly, K^{S^1}, the simplicial set of pointed maps from S^1 to K. (It will be a Kan complex if K is one.) As in the topological case, ΩK has the structure of an 'H-space'. This refers to a compositional structure *up to homotopy*, so we have

$$\mu : \Omega K \times \Omega K \to \Omega K,$$

given by composition of loops. Topologically this is just that: first do one loop, then the other, then rescale to get a map from $[0, 1]$ again. The rescaling means that this μ is not associative, but is associative up to a homotopy. There are also 'reverses', which are inverses up to homotopy, and it all fits together to make ΩK a 'group up to homotopy'. (Again the homotopies can be linked together to make a homotopy coherent version of a group.) The same can be done in the simplicial case provided that K is Kan. (This is a **good exercise to attempt**, to see once more the use of 'fillers' as a form of algebraic structure.)

If K is not reduced, we can replace it by a homotopy equivalent reduced simplicial set. (In fact we want $K = \overline{W}G$ and that *is* reduced.) For such a K, the simplicial group GK is often called the *loop group* of K. (Look back to page 249, if you need to review the construction of GK.) What is the connection between ΩK and GK?

It is clear there should be one as the free group construction involved in the definition of GK uses concatenation of strings of simplices and that is the algebraic analogue of composition of paths, however it is associative, has inverses, etc., as it gives a group. It looks like an abstract algebraic model of ΩK, which replaces the homotopy coherent multiplication by an algebraic one, but, as a result, gets a much bigger structure. Even in dimension 0, $\Omega K_0 \cong K_1$, whilst GK_0 is the free group on K_1. (This is again a **useful place** to see what the two structures look like, in low dimensions, and to see if there is a 'natural' map between them.) If we could replace Ω by G, our life would simplify as G is left adjoint to $\overline{W}$ and so, for any simplicial group, H, there is a natural map

$$G\overline{W}H \to H,$$

which is a weak equivalence, *i.e.*, it induces isomorphisms on all homotopy groups, then we would be able to identify three more terms of the Puppe sequence. In fact for any reduced K, GK and ΩK *are* weakly equivalent. We will not give the proof, referring instead to the discussion in Goerss and Jardine, [137], in particular the proof on p. 285. (This is very neat for us as it uses both ΓK, there called PK, and induced fibrations in a very similar way to our earlier treatment of the Puppe sequence.) If G is more interesting and is not reduced, then GK is equivalent to a disjoint union, indexed by $\pi_0(G)$, of simplicial sets that 'look like' copies of ΩG, namely loops, not at the identity element, but at some representative of a connected component of G. This will shortly be linked up with the décalage construction.

Putting all this together, we get that if

$$K \xrightarrow{u} H \xrightarrow{v} G$$

is a short exact sequence of simplicial groups, then the Puppe sequence of Bv ends:

$$\Omega G \to K \xrightarrow{u} H \xrightarrow{v} G \to BK \xrightarrow{Bu} BH \xrightarrow{Bv} BG.$$

We need to add what might be considered a cautionary note. To emphasise the *ideas* behind this sequence, we have handled the case of simplicial groups. For many of the applications, we have to work with sheaves of simplicial groups or, more generally, simplicial group objects in some topos, $\mathcal{E}$. In those cases the meaning of such terms as 'fibration' or 'weak equivalence' needs refining, much as the notion of 'equivalence' between categories needs adjusting before it can be used to its full potential with the 'stacks' that we will meet in the next chapter. The category in which one 'does' one's homotopy is then naturally to be considered with a Quillen model category structure and $[-,-]$ is replaced by $Ho(Simp(\mathcal{E}))(-,-)$, the 'hom-set' in the category obtained from that of simplicial objects in $\mathcal{E}$ by inverting the weak equivalences. These technicalities *do* complicate things to quite a large amount and are very non-trivial to describe in detail, however the idea is the same and the technicalities are there just to bring that idea to its most rigorous form. We have left out these technicalities to concentrate on the intuition, but they cannot be completely ignored. (Some idea of the possible detailed approaches to this can be found in Illusie's thesis, [163, 164], Jardine's paper, [166] and various more recent works on simplicial sheaves.)

8.3.4 Applications: Extensions of groups

Suppose we have our old situation, namely an extension of groups, or rather of sheaves of groups,

$$1 \to L \xrightarrow{u} M \xrightarrow{v} N \to 1$$

(as in section 7.4.6). We can replace each by a constant simplicial group, L by $K(L,0)$, etc. (To simplify notation we will, in fact, abbreviate $K(L,0)$ back to L, whenever this is feasible.) We now apply the classifying space construction and take the corresponding Puppe sequence. The result will be

$$1 \to L \xrightarrow{u} M \xrightarrow{v} N \to BL \to BM \to BN.$$

(Here we are abusing notation even more, as the first three terms are the underlying simplicial sheaves of the corresponding sheaves of simplicial groups, which are, ... and so on, but writing $U(K(L,0))$ seems silly and it would get worse, so ... .)

Note that in this sequence, we have that $\Omega^2 BN$ is equivalent to ΩN, which is contractible, which explains the 1 on the left hand end.. The classifying spaces are the nerves of the corresponding groupoids, $BL = Ner(L[1])$, etc.

All this is happening in $Sh(B)$ (or, more generally, in a topos, $\mathcal{E}$). Given an open cover $\mathcal{U}$ of B, with nerve $N(\mathcal{U})$, we get a long exact sequence of groups and pointed sets:

$$1 \to [N(\mathcal{U}), L] \to [N(\mathcal{U}), M] \to [N(\mathcal{U}), N] \to [N(\mathcal{U}), BL] \to [N(\mathcal{U}), BM] \to [N(\mathcal{U}), BN],$$

and passing to the colimit over coverings, this gives

$$1 \to L(B) \to M(B) \to N(B) \to \check{H}^1(B, L) \to \check{H}^1(B, M) \to \check{H}^1(B, N).$$

This is exactly the exact sequence that we discussed earlier, again in section 7.4.6. Note that we have not yet got our hands on any substitute for the $\check{H}^2(B, L)$, that exists in the Abelian case.

8.3.5 Applications: Crossed modules and bitorsors

Suppose $\mathsf{M} = (C, P, \partial)$ is a sheaf of crossed modules. It would be good to examine the simplicial view of relative M-torsors in a similar way. We have a sheaf of simplicial groups given by $K(\mathsf{M})$ and have identified $colim[N(\mathcal{U}), K(\mathsf{M})] = colim H^0(N(\mathcal{U}), \mathsf{M})$ with $\pi_0(\mathsf{M}{-}Tors)$, which is a group. We also showed that any M-torsor, (E, t), had that E is a C-torsor with t a trivialisation of $\partial_*(E)$. This suggests some sort of exact sequence:

$$\pi_0(\mathsf{M}{-}Tors) \to \pi_0(Tors(C)) \xrightarrow{\partial_*} \pi_0(Tors(P)),$$

i.e., anything in $Tors(C)$ that is sent to the base point (that is, the class of the trivial torsor) in $Tors(P)$, comes from an M-torsor. We can see this geometrically as we saw earlier. What is neat is that if (E, t) and (E', t') are M-torsors, with E and E' equivalent as C-torsors, then we can assume $E = E'$ and can use the trivialisations t and t' to obtain a global section, p, of P such that $t' = p.t$. The implication is that

$$P(B) \to \pi_0(\mathsf{M}{-}Tors) \to \pi_0(Tors(C))$$

is also exact. This can also be seen from the Puppe sequence.

First a very useful bit of the simplicial toolkit. We form the décalage of $K(\mathsf{M})$. (Recall $K(\mathsf{M})$ is the simplicial group associated to M, that is, it is formed as the internal nerve of the internal category corresponding to M, that it has P in dimension 0, $C \rtimes P$ in dimension 1, etc. It also has a Moore complex which is of length 1 and is exactly $C \xrightarrow{\partial} P$.)

What is the décalage?

Definition: The *décalage* of an arbitrary simplicial set, Y, is the simplicial set, $Dec Y$, defined by shifting every dimension down by one, 'forgetting' the last face and degeneracy of Y in each dimension. More precisely

- $(Dec Y)_n = Y_{n+1}$;
- $d_k^{n, Dec Y} = d_k^{n+1, Y}$;
- $s_k^{n, Dec Y} = s_k^{n+1, Y}$.

This comes with a natural projection, $d_{last} : Dec Y \to Y$, given by the 'left over' face map. (Check it is a simplicial map.) We will denote this by p, for 'projection'. Moreover this map gives a homotopy equivalence

$$Dec Y \simeq K(Y_0, 0),$$

between $Dec Y$ and the constant simplicial set on Y_0. The homotopy can be constructed from the 'left-over' degeneracy, s^Y_{last}. (A full discussion of the décalage can be found in Illusie's thesis, [163, 164] and Duskin's memoir, [107]. Be aware, however, some sources may use the alternative form of the construction that forgets the *zeroth* face rather than the *last* one. We will briefly discuss this later; see page 636. This works just as well. The translation between the two forms is quite easy, if sometimes a bit time consuming!)

Of course, this same construction works for simplicial objects in any category. We need it mainly for (sheaves of) simplicial groups and, in particular, as hinted at earlier, we need $Dec\, K(\mathsf{M})$. We list some properties of this simplicial group:

(i) $Dec\, K(\mathsf{M})_0 \cong C \rtimes P$, $Dec\, K(\mathsf{M})_1 \cong C \rtimes C \rtimes P$, and in general, $Dec\, K(\mathsf{M})_n \cong C^{(n+1)} \rtimes P$. The face maps are given by

$$\begin{array}{rcll} d_0(c_n, \ldots, c_0, p) & = & (c_n, \ldots, c_1, \partial c_0.p) & \\ d_i(c_n, \ldots, c_0, p) & = & (c_n, \ldots, c_i c_{i-1}, \ldots, c_0, p) & 0 < i < n \\ d_0(c_n, \ldots, c_0, p) & = & (c_n c_{n-1}, \ldots, c_0, p) & \end{array}$$

with degeneracies given by suitable insertions of identities.

(ii) $Dec\, K(\mathsf{M})$ has Moore complex isomorphic to one of the form

$$C \to C \rtimes P.$$

Here we clearly have $Ker\, d_1 = \{(c_1, c_0, p) \mid c_1 = c_0^{-1}, p = 1\} \cong C$. We also have a boundary, induced by d_0, so the boundary sends $(c^{-1}, c, 1)$ to $(c^{-1}, \partial c\}$. If this looks strange, **just check** that $(c^{-1}, c, 1)((c')^{-1}, c', 1) = ((cc')^{-1}, cc', 1)$. (Don't forget the Peiffer identity!)

(iii) The boundary is a monomorphism and its image is the kernel of the homomorphism from $C \rtimes P$ to P that sends (c, p) to $\partial c.p$. (That makes sense as that is the target / codomain map of the internal category or cat^1-group associated to M.)

(iv) $Dec\, K(\mathsf{M})$ is homotopy equivalent to the constant simplicial group on P. (This can be seen from the Moore complex, but also from the retraction of $Dec\, K(\mathsf{M})$ onto the subsimplicial group given by all $(1, \ldots, 1, p)$. That map is a deformation retraction with the 'extra degeneracy', $s_{\ell ast}$, of the décalage construction giving the homotopy, (**for you to check**). This is neat, because it is explicit and natural and thus can provide a more geometric picture than merely stating that there is a weak equivalence of simplicial groups between $Dec\, K(\mathsf{M})$ and $K(P, 0)$.)

(v) The morphism $\mathsf{p} : Dec\, K(\mathsf{M}) \to K(\mathsf{M})$ is an epimorphism, hence is a fibration. (It is, in fact, split at each level by the last degeneracy map of $K(\mathsf{M})$.) We can give p explicitly by $\mathsf{p}(c_n, \ldots, c_0, p) = (c_{n-1}, \ldots, c_0, p)$, hence:

(vi) The kernel of p is given by $Ker\, \mathsf{p} = \{(c, 1, \ldots, 1, 1) \mid c \in C\}$ with the face and degeneracy maps given by the restrictions of the above, so $Ker\, \mathsf{p}$ is isomorphic to $K(C, 0)$.

(vii) Within the context of our much earlier discussion of crossed modules as being given by fibrations (page 45), we had that if G is a simplicial group and $N \lhd G$ a normal simplicial subgroup, then applying π_0 to the inclusion of N into G gave us a crossed module. The proof that, up to isomorphism, all crossed modules arise in this way was left to the reader! Here it is:

If we take $G = Dec\, K(\mathsf{M})$, and $N = Ker\, \mathsf{p}$, then $\pi_0(N) \to \pi_0(G)$ is $\partial : C \to P$ and the actions agree, (all 'up to isomorphism', of course).

This is at the heart of the algebraic proof of Loday's theorem (see 5.5) that cat^n-groups / crossed n-cubes model all connected homotopy $(n+1)$-types. Its appearance here is not accidental.

We thus have an exact sequence of simplicial groups arising from M:

$$1 \to Ker\, \mathsf{p} \to Dec\, K(\mathsf{M}) \to K(\mathsf{M}) \to 1$$

corresponding to

$$K(C,0) \to K(P,0) \to K(\mathsf{M}),$$

(which is not exact!).

At a crossed module level, we get

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & C \arrow[d] \\ C \arrow[r] & P \arrow[r] & P \end{tikzcd}$$

is homotopy exact, or, more exactly (pun intended!) that

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C \arrow[d] \\ C \arrow[r] & C \rtimes P \arrow[r] & P \end{tikzcd}$$

is exact.

If we pass to the Puppe sequence, it will end

$$\Omega K(\mathsf{M}) \to C \to P \to K(\mathsf{M}) \to BC \to BP \to BK(\mathsf{M}).$$

Going through the usual process of applying $[N(\mathcal{U}), -]$ for an open cover $\mathcal{U}$ of the base space B, followed by the colimit over such $\mathcal{U}$s, we get

Proposition 85 *For any crossed module,* M*, there is an exact sequence*

$$1 \to \check{H}^{-1}(B,\mathsf{M}) \to C(B) \to P(B) \to \pi_0(\mathsf{M}{-}Tors) \to \pi_0(Tors(C)) \to \pi_0(Tors(P)) \to \check{H}^1(B,\mathsf{M}).$$

■

There are two 'mysterious' terms here. The second is the 1st Čech hypercohomology of B with coefficients in M. We have, sort of, met this earlier. It is

$$\check{H}^1(B,\mathsf{M}) = colim_{\mathcal{U}}[N(\mathcal{U}), BK(\mathsf{M})].$$

The treatment we have given it here, and the language we have available, is however not yet rich enough to yield a good geometric interpretation. For that we will need stacks and gerbes, and we will start on them in the next chapter!

The other strange term is $\check{H}^{-1}(B,\mathsf{M})$, which comes from the various $[N(\mathcal{U}), \Omega K(\mathsf{M})]$. We can calculate $\Omega K(\mathsf{M})$ explicitly using its description as the simplicial group of maps from S^1_* to $K(\mathsf{M})$.

Lemma 57 *(i) There are isomorphisms* $\Omega K(\mathsf{M}) \cong K(\pi_1(\mathsf{M}), 0)$*, the constant simplicial group on the kernel* $\pi_1(\mathsf{M}) = Ker(\partial : C \to P) \cong \pi_1(K(\mathsf{M}))$*.*

(ii) There are isomorphisms $\check{H}^{-1}(B,\mathsf{M}) = \check{H}^0(B, \pi_1(\mathsf{M})) \cong \pi_1(\mathsf{M})(B)$*, the group of global sections of* $\pi_1(\mathsf{M})$*.*

Proof: This is just a question of calculation so is left to you the reader. ■

8.3.6 Examples and special cases revisited

We can use the analyses of Puppe sequences and their applications to refine a bit more the information on relative M-torsors for the 'examples and special cases'. We first apply our exact sequence of the previous paragraph.

The first example is when $\mathsf{M} = (1, P, inc)$ and the exact sequence confirms the isomorphism between $P(B)$ and $\pi_0(\mathsf{M}{-}Tors)$. When M is $A[1] = (A \to 1)$ for Abelian A, the sequence gives, as expected, confirmation that $\pi_0(\mathsf{M}{-}Tors) \cong \pi_0(Tors(A))$ and that the latter has a group structure.

For an inclusion crossed module / normal subgroup pair, we can compare the exact sequence coming from $1 \to N \to P \to G \to 1$ with that from $\mathsf{M} = (N, P, \partial)$, with ∂ the inclusion. The induced maps give us a map of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & N(B) \arrow[r] \arrow[d, "="'] & P(B) \arrow[r] \arrow[d, "="'] & \pi_0(\mathsf{M}{-}Tors) \arrow[r] \arrow[d] & \pi_0(Tors(N)) \arrow[r] \arrow[d, "\cong"'] & \pi_0(Tors(P)) \arrow[r] \arrow[d, "\cong"'] & \check{H}^1(B, \mathsf{M}) \arrow[d, "?"'] \\
1 \arrow[r] & N(B) \arrow[r] & P(B) \arrow[r] & G(B) \arrow[r] & \check{H}^1(B, N) \arrow[r] & \check{H}^1(B, P) \arrow[r] & \check{H}^1(B, G)
\end{tikzcd}$$

which again gives $\pi_0(\mathsf{M}{-}Tors) \cong G(B)$, and suggests that the mysterious $\check{H}^1(B, \mathsf{M})$, in this special case, is our better known $\check{H}^1(B, G)$, *i.e.*, $\pi_0(Tors(G))$.

The last case we looked at was $\mathsf{M} = (M, G, 0)$. The long exact sequence has the induced map, ∂_*, trivial, so gives us

$$1 \to G(B) \to \pi_0(\mathsf{M}{-}Tors) \to \pi_0(Tors(M)) \to 1.$$

To examine the other situation considered on page 355, we need to apply our analysis of exact sequences of simplicial groups to another case.

8.3.7 Devissage: analysing $\mathsf{M}{-}Tors$

We saw that for any (sheaf of) crossed module(s) M, we had a short exact sequence

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & N \arrow[d] \\
1 \arrow[r] & P \arrow[r] & P,
\end{tikzcd}$$

or

$$\pi_1(\mathsf{M})[1] \to \mathsf{M} \to \pi_0(\mathsf{M})$$

if you prefer, (as $\pi_0(\mathsf{M}) = \pi_0(K(\mathsf{M})) = P/N$). (We only saw this for a crossed module, but clearly the argument goes through with only trivial changes in any topos, given suitable definitions!) Applying the associated simplicial group functor, K, this gives that

$$K(\pi_1(\mathsf{M}), 1) \to K(\mathsf{M}) \to K(\pi_0(\mathsf{M}), 0)$$

is an exact sequence of simplicial groups.

Theorem 24 *For any crossed module,* M*, there is an exact sequence*

$$1 \to \pi_0(Tors(\pi_1(\mathsf{M}))) \to \pi_0(\mathsf{M}{-}Tors)) \to \pi_0(\mathsf{M})(B) \to$$
$$\check{H}^2(B, \pi_1(\mathsf{M})) \to \check{H}^1(B, \mathsf{M}) \to \pi_0(Tors(\pi_0(\mathsf{M}))).$$

Proof: The proof merely is to identify the various terms from the Puppe sequence. Firstly the general form of such sequences, seen above, gives

$$\to \check{H}^{-1}(B, \pi_0(\mathsf{M})) \to \check{H}^0(B, \pi_1(\mathsf{M})[1]) \to \check{H}^0(B, K(\mathsf{M})) \to \check{H}^0(B, \pi_0(\mathsf{M})) \to \check{H}^1(B, \pi_1(\mathsf{M})[1]) \to \ldots$$

The first of these terms is trivial since for a general crossed module, $\Omega K(\mathsf{N})$ is $K(Ker\partial, 0)$, up to equivalence, so in our case in which $\mathsf{N} = (1 \to \pi_0(\mathsf{M}))$, it will be trivial. (Remember $\check{H}^{-1}(B, \mathsf{N}) = colim_{\mathcal{U}}[N(\mathcal{U}), \Omega K(\mathsf{N})]$.)

The next term $\check{H}^0(B, \pi_1(\mathsf{M})[1]) \cong \check{H}^1(B, \pi_1(\mathsf{M})) \cong \pi_0(Tors(\pi_1(\mathsf{M})))$, by our earlier calculations (case (ii) above). The next two terms are routine to handle, whilst that $\check{H}^1(B, \pi_1(\mathsf{M})[1])$ is isomorphic to $\check{H}^2(B, \pi_1(\mathsf{M}))$ is a classical result that is easy to check anyhow. Finally the remaining terms are standard. ■

Note that this gives some new information on $\mathsf{M}-Tors$, indicating the difference between this category for general M and for the particular special cases considered earlier.

Chapter 9

Non-Abelian Cohomology: Stacks

In passing from bundles and sheaves to 'higher categorified levels' and hence to higher cohomology, we need to apply some basic 'rules of thumb'. We should replace sets by (small) categories or groupoids, but as a (small) category, $\mathcal{C}$, will have 'hom-*sets*', $\mathcal{C}(x, y)$, etc., any category should be replaced by a 2-category, so that $\mathcal{C}(x, y)$ will itself be a category. We then need to replace functors by ... . At this point, we need to bring the other main 'rule of thumb' into play. In a set, equality of elements, $x = y$, seems a reasonable thing to work with, but already in a category, 'isomorphism' rather than 'equality' of objects is what is the natural idea and in a 2-category, 'equivalence of objects' replaces 'isomorphism'. The apparently natural notion of 'functor' (*i.e.*, '2-functor' between 2-categories) is thus not necessarily right for when we categorify things, rather a 'lax' or 'pseudo' functor of some form may be needed. In particular we had that 'sheaves' were special types of 'presheaves', quite typically $F : Open(B)^{op} \to Sets$, and corresponded to spaces over B with 'discrete fibres', but if we want or need more categorical structure in the fibres, what do we do? We will see that there are useful examples of 'fibred categories' corresponding to 'lax presheaves', and that there are objects analogous to sheaves, torsors, etc., in this categorified setting. Most importantly, these objects encode important algebraic and geometric information.

Sources and references: The important source for many of these ideas is SGA1, [141], so dates from 1960 or there abouts. Ideas relating to the more categorical aspects were then developed by Bénabou, for which see Streicher's notes, [254], whilst the link with non-Abelian cohomology was initially developed by Giraud, [135]. Some of the more recent geometric ideas can be conveniently found in Vistoli, [270].

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It is important to note that in fact we will be going back over ideas that we have already explored in other contexts. For instance, in Chapter 2, we looked at Cayley quivers and later on in section 4.5.14, we met the slice category as a tool for studying coverings of groupoids. Coverings are just the simplest cases of fibred categories, and sure enough we will be meeting slice / comma categories again. We will reprise some of the same material for convenience, so as not to disrupt the flow of ideas by requiring reference to another section in which slightly different motivations are at play. It is useful and important to know and to note that while we will be going over some of the same ground again, we will need to take that material and to categorify it, pushing things up from coverings to stacks and thence to higher stacks. Nearly the same ideas reappear time and time again, with slightly different emphasis, and then the earlier material comes in as crucial for helping one understand those various facets of the higher structured stuff.

9.1 Fibred Categories

9.1.1 The structure of $\mathsf{Sh}(B)$ and $\mathsf{Tors}(G)$

We will start with two 'case studies' based on ideas developed in the previous chapter.

We will look at $Sh(B)$, the category of sheaves on B, and how it relates to the $Sh(U)$ for open subsets, U, of B. After that we will do the analogous thing for $Tors(B, G)$, restricting that to open sets as well. These will form a sort of lax presheaf of categories. These are the two structures, $\mathsf{Sh}(B)$ and $\mathsf{Tors}(G)$, referred to in the title of this section. (Generally in this chapter, we will try to use a 'sans serif' font for such localised objects, with the more usual italic-style font for the mere category, rather than these 'fibred categories'.)

Suppose that G is a sheaf or bundle of groups on B (or in a topos, $\mathcal{E}$) and that U is an open set of B. We can restrict G to U to get a sheaf of groups, G_U, on U and hence a groupoid of G_U-torsors, $Tors(U; G)$. (We have abbreviated the notation $Tors(U; G_U)$ to $Tors(U; G)$ here as the extra mention of U seems unnecessary.)

Next look at $V \subset U$ and restrict the G_U-torsors to V. This gives a functor

$$res^U_V : Tors(U; G) \to Tors(V; G).$$

If $W \subset V \subset U$, then there is a natural isomorphism between res^U_W and the composite $res^V_W \circ res^U_V : Tors(U; G) \to Tors(W; G)$.

This looks very like a presheaf of categories (in fact, 'of groupoids', as each $Tors(U; G)$ is a groupoid as we have seen). Why is it not one? The point is how is res^U_V defined? The problem is most immediately seen in the related example of sheaves rather than torsors.

For each open set U in B, we have the category of sheaves on U, denoted $Sh(U)$, and we can represent the objects as étale spaces over U, so F corresponds to the sheaf of sections of some $E_F \to U$, say. If $V \subset U$, we can restrict the étale space to be over V, but how exactly is that done? *Pullback*.

$$\begin{tikzcd} i^*(E_F) \arrow[r] \arrow[d] & E_F \arrow[d] \\ V \arrow[r, "i"] & U \end{tikzcd}$$

with $i^*(E_F) = V \times_U E_F$. Now suppose $j : W \to V$, we have

$$j^*i^*(E_F) = W \times_V V \times_U E_F,$$

whilst

$$(ij)^*(E_F) = W \times_U E_F.$$

We are in a classic situation, very like that with a category with tensors, *i.e.*, a monoidal category. These objects are *not* equal, but *are* naturally isomorphic. (In fact you might ask what 'equality' really means, and it would be a good question!) A slightly more categorical way of viewing this is to say i^* is defined by pullback and pullbacks are only defined 'up to isomorphism', so we cannot guarantee 'equality' merely 'natural isomorphism'. The same is true for our torsors, res^U_V is really only specified up to isomorphism. (The first time you meet this it will seem strange since, surely, restriction is such a well behaved operation, but you have to think how it is done and then ... .)

(The notation is getting to be a bit heavy, so we will sometimes write $U_1 \xrightarrow{i} U_0$, and similar, to allow indexation, and will put indices rather than indexing by objects. We will then write i^* for $res^{U_1}_{U_0}$.)

There is a further property of these restriction functors. If we have

$$U_3 \xrightarrow{k} U_2 \xrightarrow{j} U_1 \xrightarrow{i} U_0$$

within $Open(B)$, then we have natural *isomorphisms*

$$\tau_{i,j} : (ij)^* \to j^*i^*$$

and similarly for the other possibilities. These give a diagram

$$\begin{tikzcd}
(ijk)^* \arrow[r, "\tau_{ij,k}"] \arrow[d, "\tau_{i,jk}"'] & k^*(ij)^* \arrow[d, "k^*.\tau_{i,j}"] \\
(jk)^*i^* \arrow[r, "\tau_{j,k}.i^*"'] & k^*j^*i^*
\end{tikzcd}$$

and, as usual in these situations, this commutes. (This is another form of cocycle condition as will become apparent later on.)

We return briefly to $i^* : Tors(G;U) \to Tors(G;V)$, and how it is formed. If P is a G_U-torsor on U, then we have to first form the *sheaf*, $i^*(P)$, over V, then look at the restricted sheaf, $i^*(G)$, of groups, then check that $i^*(P)$ is a $i^*(G)$-torsor.

It pays to verify this cocycle condition in several ways; for instance, using étale spaces and pullbacks to get explicit representatives for these objects and to use 'bare hands' calculations, but also look at the functorial properties of the functor $i^* : Sh(U) \to Sh(V)$ and check it for existence of adjoints. (Any standard text on sheaf theory will show you how.) With these categorical properties, you could give a description of $i^* : Tors(G;U) \to Tors(G;V)$ by showing that i^* on sheaves preserves torsors. This second neat method easily extends to the topos case, whilst the first argument can give a direct geometric 'hands-on' feel to what is happening.

9.1.2 Other examples

The situation that we noted for $Tors(G)$ and $Sh(B)$ also works for other situations such as for the category, $Vect(B)$, of vector bundles on B. We have a lot of locally defined categories, $Vect(U)$, for U open in B, fitting together neatly - clearly a descent situation. A similar situation occurs with the category of modules, not modules over a fixed ring, R, but modules. Here a module is a pair, (R, M), with R, an associative ring, and M, a left R-module, then a morphism of such objects is also a pair, (φ, f), where $\varphi : R \to S$ is a ring homomorphism and $f : M \to N$ is an Abelian group morphism such that for all $r \in R$, and $m \in M$, $f(r.m) = \varphi(r).f(m)$, in the obvious way, *i.e.*, it is a module morphism *over* φ. We have a forgetful functor $Mod \to Rings$ and also a 'functor' $F : Rings^{op} \to Cat$, given by $F(R) = R{-}Mod$, but it is not quite functorial as, given $R \xrightarrow{\varphi} S \xrightarrow{\theta} T$, the resulting triangle of categories and functors only commutes up to natural isomorphism,

$$F(\theta\varphi) \cong F(\varphi)F(\theta).$$

not 'on the nose' with an equality. We will not examine such 'pseudo' functors in full abstract generality yet, but would note that several of our crossed situations do give exactly this sort of structure.

9.1.3 Fibred Categories and pseudo-functors

For the moment, restricting our detailed attention to the spatial case, we abstract the structure of $Sh(B)$ and $Tors(G)$ to get the following:

Definition: (Pseudo-functor version) A *fibred category*, F, over B consists of

(i) a category, $F(U)$, for each open set U of B;

(ii) a functor, $i^* : F(U) \to F(V)$, for each inclusion $i : V \to U$ in $Open(B)$;

(iii) a natural isomorphism
$$\tau = \tau_{ij} : (ij)^* \to j^* i^*,$$
for each pair of inclusions $W \xrightarrow{j} V \xrightarrow{i} U$.

This data is to satisfy the 3-cocycle condition that, given inclusions

$$U_3 \xrightarrow{k} U_2 \xrightarrow{j} U_1 \xrightarrow{i} U_0,$$

the diagram

$$\begin{tikzcd}
(ijk)^* \arrow[r, "\tau_{ij,k}"] \arrow[d, "\tau_{i,jk}"'] & k^*(ij)^* \arrow[d, "k^*.\tau_{i,j}"] \\
(jk)^* i^* \arrow[r, "\tau_{j,k}.i^*"'] & k^* j^* i^*
\end{tikzcd}$$

commutes, where the arrows are induced from the τ-transformations.

Remark: A fibred category, in this sense, is 'exactly' a 'op-lax', pseudo functor from $Open(B)^{op}$ to Cat, the 'category' of categories, but note we are really using Cat as a 2-category, hence, we will try to use the notation, Cat, rather than simply Cat. (We will ignore difficulties of the size of the $F(U)$ here - they do not often cause any bother.) More generally we may also want to consider an 'op-lax pseudo'-functor, F, from a small category, $\mathcal{C}$, to Cat as there are aspects of the situation which are simpler to describe in this more general setting (due mostly to a cleaner notation).

We have hinted that 'lax' or 'op-lax' functors replace preservation of composition by preservation up to a 2-cell, *i.e.*, the codomain setting needs to be a 2-category or similar and then such a *lax-functor*, F, will replace the 'equality' in a composite $a \circ b = c$ by a 2-arrow $F(a) \circ F(b) \Rightarrow F(c)$. An *op-lax functor* has the 2-cell going in the opposite direction $F(c) \Rightarrow F(a) \circ F(b)$. (Which version is appropriate depends on the context and terminological conventions being employed.) When looked at in all generality, we also would have a 2-cell measuring the extent that F does / does not preserve the identity arrows.

By a 'pseudo-functor', we mean a lax or op-lax functor in which that 2-cell is always invertible, so its direction is not that important. We may say 'lax pseudo-functor' or 'op-lax pseudo-functor' meaning a pseudo-functor presented in its lax or op-lax form. It really is just a question of its 'presentation'. (Perhaps one should be saying the pseudo-functor is the data (F, τ, τ^{-1}), but that seems 'overkill'!)

If we have a lax pseudo-functor, then just replacing the structural 2-cells by their inverses and we will have an op-lax pseudo functor. We are often working with higher dimensional analogues of groupoids and there higher dimensional cells are invertible, so saying 'lax' or op-lax would have sufficed.

A good brief introduction to some aspects of lax 'pseudo' functors can be found in Borceux and Janelidze's book, [39]. We will look in some more details at lax and pseudo-functors later, (starting page 520), but, as this will only skate selectively over the surface of the theory, you may need to look up more details in 'the literature' in the mean-time. There is also a notion a pseudo-natural transformation, and once or twice in what follows, we will use the notion $Ps(\mathcal{C}, \mathsf{Cat})$ for the category of pseudo-functors and pseudo-natural transformation between them, having a category, $\mathcal{C}$, as domain and the category of categories as codomain. There is even a 2-categorical version which we will try to consistently denote $\mathsf{Ps}(\mathcal{C}, \mathsf{Cat})$

Examples of fibred categories: (i) Any presheaf of categories, $F : Open(B)^{op} \to \mathsf{Cat}$, gives a fibred category in which all the τ are identity transformations. The general case is thus a 'pseudo presheaf' of categories in a precise sense, or a 'presheaf up to isomorphisms'. This is a case of the fact that 'any functor is a pseudo-functor'.

(ii) The examples of sheaves and G-torsors give fibred categories that will be denoted $\mathsf{Sh}(B)$ and $\mathsf{Tors}(G)$, respectively.

(iii) When discussing non-Abelian group extensions, (Chapter 6.1, p. 246), from a general extension,

$$\mathcal{E} : \quad 1 \to K \xrightarrow{\iota} E \xrightarrow{p} G \to 1,$$

we saw that a choice of section, s, does not give an action of G on K, but does give a pseudo-functor from $G[1]$ to $Grps$. It will be useful to revisit this now. (First remember that $G[1]$ denote the group G 'thought of as a groupoid with a single object $*$'.)

Suppose given $s : G \to E$, a section of p, we try to define

$$F_s : G[1] \to Grp \hookrightarrow Grpd$$

by $F_s(*) = K$, the 'kernel' part of the extension

- for $g \in G$, $F_s(g) : K \to K$ is the automorphism of K given by

$$\iota(F_s(g)(k)) = s(g)\iota(k)s(g)^{-1},$$

but then we note that

$$\iota(F_s(g_2 g_1)(k)) = s(g_2 g_1)\iota(k)s(g_2 g_1)^{-1},$$

whilst

$$\iota(F_s(g_2)F_s(g_1)(k)) = s(g_2)s(g_1)\iota(k)s(g_1)^{-1}s(g_2)^{-1},$$

and these need not be equal. They are conjugate, however, and, if we define (cf. page 56), the factor set,

$$f : G \times G \to E$$

$$f(g_2, g_1) = s(g_2)s(g_1)s(g_2 g_1)^{-1},$$

then conjugating by $f(g_2, g_1)$ within E gives a 2-cell in the *groupoid*, $\mathsf{Grps}(K, K)$, from $F_s(g_2 g_1)$ to $F_s(g_2)F_s(g_1)$, *i.e.*, s gives a pseudo-functor from $G[1]$ to Grpd, here presented in its op-lax form.

We note that there was a neat construction, given F_s, of the centre term, E, of the extension (up to isomorphism), basically by taking as its underlying set the product *set*, $K \times G$, and defining a multiplication using both s and f.

By considering groups as groupoids, and thus as small categories, the extension thus gives a fibred category / pseudo-functor over $G[1]$, the group G considered as a groupoid. The use of techniques such as that of the crossed resolution, $\mathsf{C}(G)$, to encode the 'laxity' is typical of the process of resolving an object to handle choices 'up to isomorphism', or 'up to coherent homotopy', (see sections 11.2.3 and 11.5.2), and this shows the link with other cohomological tools.

9.2 The Grothendieck construction

This third example, together with the connection with presheaves, suggests that there should be a construction of an 'étale-space'-like category, $\mathcal{E}_{\mathsf{F}}$, with a functor $p : \mathcal{E}_{\mathsf{F}} \to \mathcal{C}$. (We treat the more general case with a general $\mathcal{C}$ not just in the case of $Open(B)$.) In fact, the term 'fibred category' would suggest such an interpretation anyway. How could one construct $\mathcal{E}_{\mathsf{F}} \xrightarrow{p} \mathcal{C}$ from $\mathsf{F} : \mathcal{C}^{op} \to \mathsf{Cat}$? There is an 'obvious' way. (It is known as the *Grothendieck construction*, but priority in the use of it is debatable as Ehresmann was using it about the same time that it was first used by Grothendieck, and both seem to have recognised it as being, to them, a mild generalisation of the construction of semi-direct products, or, more exactly, of the Schreier construction of an extension from a cocycle. It is also not that good a term since the 'Grothendieck construction' is also applied to the method of converting a semi-group into a group by adding inverses, as in Grothendieck's construction of the K-theory of vector bundles on a space.) We will now approach the problem without thinking too much about the group extension case, as it then can be seen to be very natural in general - it also more clearly relates to twisting a 'product bundle'.

9.2.1 The basic Grothendieck construction and its variants

If you look for the Grothendieck construction in the literature, initially, you will risk becoming, at very least, slightly confused. Sometimes the basic input is a functor $F : \mathcal{C} \to Cat$, sometimes $F : \mathcal{C}^{op} \to Cat$, but then F may be an op-lax or a lax functor, or more often a pseudo functor. The constructions given are clearly closely related, but they are not 'the same'. It therefore seems a good idea to set down a very basic version of the construction and then to look at variations on that. To add slightly to the confusion, we will sometimes have to convert from 'op-lax pseudo' to 'lax pseudo' or *vice versa* if we are handed a pseudo-functor in slightly the wrong format!

All that being said the basic construction may, or may not, be the one you will need and all of the possibilities are likely to be called *the Grothendieck construction*! We will give one form as basic, with three variants. The first of these variants is as 'basic' (and about as common) as the first one we will handle, so could equally well have been chosen as the basic form. Because of this, our 'basic' one may not be the basic one for someone else, just as semi-direct products are presented in several different ways.

The basic set up that we will choose will be that of a *normalised* op-lax functor $\mathsf{F} : \mathcal{C} \to \mathsf{Cat}$. We thus have $\mathsf{F} = (F, \tau)$, where, if $f : c \to c'$, and $g : c' \to c''$, $\tau_{f,g} : F(gf) \Rightarrow F(g)F(f)$ is a natural transformation, which satisfy a 3-cocycle condition, dual to that given on page 392, for composible triples of morphisms. (We will not assume that τ is necessarily a natural isomorphism.)

The category, $\mathcal{E}_{\mathsf{F}}$, will have

- as objects, pairs (x, c) with $c \in Ob(\mathcal{C})$ and $x \in Ob(F(c))$;
- as morphisms, pairs $(\alpha, f) : (x, c) \to (x', c')$, with $f : c \to c'$ (and thus $F(f) : F(c) \to F(c')$), and $\alpha : F(f)(x) \to x'$, a morphism in $F(c')$;
- as composition: in the situation
$$(x, c) \stackrel{(\alpha,f)}{\to} (x', c') \stackrel{(\beta,g)}{\to} (x'', c''),$$
the composite has $gf : c \to c''$ in its $\mathcal{C}$-component, and the composite
$$F(gf)(x) \stackrel{\tau_{f,g}(x)}{\to} F(g)F(f)(x) \stackrel{F(g)(\alpha)}{\to} F(g)(x') \stackrel{\beta}{\to} x'',$$
in the fibre over c'';
- as identities: given (x, c) in $\mathcal{E}_{\mathsf{F}}$, (id_x, id_c) is the identity at this object.

The verification of associativity uses the fact that τ satisfies a 3-cocycle condition, cf. page 392, and the identity works because F is assumed to be normalised.

We note that there is a projection, $p : \mathcal{E}_{\mathsf{F}} \to \mathcal{C}$, given by $p(x, c) = c$, $p(\alpha, f) = f$. We will look at this in some detail shortly, but will concentrate on one of the variants!

Remarks: (a) If $F : \mathcal{C} \to Cat$ is simply a functor, then each $\tau_{f,g}$ is the relevant identity transformation and the formulas simplify.

(b) If F is a pseudo-functor, (F, τ), but given in 'lax' form, so $\tau_{f,g} : F(g)F(f) \to F(gf)$, then we can replace τ by τ^{-1} to get F into op-lax form and use the above. It is this situation that occurs quite often.

(c) We could replace the codomain 2-category, Cat, by other similar 2-categories, such as Grpd with virtually no bother, but to go to a general 2-category (which would require a bit of extra structure to be made explicit, such as existence of colimits), we would need to use slightly more sophisticated tools, namely tensors / copowers and coends. We will see this in chapter 14, in discussing homotopy limits and colimits.

First variation: $\mathsf{F} : \mathcal{C}^{op} \to \mathsf{Cat}$ *is a lax functor* (so $\tau_{f,g} : F(f)F(g) \Rightarrow F(gf)$.)

We use a simple trick to see how this might be done. First note that $\mathsf{F}^{op} : \mathcal{C} \to \mathsf{Cat}^{op}$, and then, without agonising about the multiple types of duals / opposites that Cat has, try the basic formulas with reverse of the directions. If one does not work, reflect on the problem, check your working and ..., try another! The category $\mathcal{E}_{\mathsf{F}}$ should have for objects, pairs (x, c) with $c \in Ob(\mathcal{C})$ and $x \in Ob(F(c))$, as before, whilst a morphism
$$(\alpha, f) : (x, c) \to (x', c'),$$
will have $f : c \to c'$ (and so $f^{op} : c' \to c$), and then $\alpha : x \to F(f)(x')$ in $F(c)$. That looks feasible, so we now try composition:
$$(x, c) \stackrel{(\alpha,f)}{\to} (x', c') \stackrel{(\beta,g)}{\to} (x'', c'').$$

We have, clearly, $gf : c \to c''$ and need an arrow in $F(c)$ from x to $F(gf)$. We have

$$\alpha : x \to F(f)(x')$$

and

$$\beta : x' \to F(g)(x''),$$

so $F(f)(\beta) : F(f)(x') \to F(f)F(g)(x'')$ and we can use $\tau_{f,g}(x'')$ to get from $F(f)F(g)(x'')$ to $F(gf)(x'')$.

We again have to check associativity (which again follows from the cocycle condition of τ) and the existence of identities. We have a functor $p : \mathcal{E}_{\mathsf{F}} to \mathcal{C}$. (If we work with F^{op} more explicitly $\mathcal{E}_{\mathsf{F}^{op}}$ will come with a functor to $\mathcal{C}^{op}$, exactly as in the basic version, but then the construction of $\mathcal{E}_{\mathsf{F}}$ that we have given is, more or less, $(\mathcal{E}_{\mathsf{F}^{op}})^{op}$, so we get a functor to $\mathcal{C}$ itself.)

It is this version that is useful in many geometric situations, including that of stacks, as a presheaf of categories gives a functor $F : \mathcal{C}^{op} \to Cat$. In pratice, F is more often a pseudo-functor, so one uses either τ or τ^{-1}, (depending on the conventions in place!), to get the lax form of 'pseudo'.

The other two variants are of less immediate use for us, but we will sketch them anyhow.

2nd variation: $\mathsf{F} : \mathcal{C}^{op} \to \mathsf{Cat}$ *is an op-lax functor.*

(We can handle the 'pseudo' case of this using the first variant.)

As we have 'op-lax', we have $F(gf) \Rightarrow F(f)(F(g)$, and, imitating the other version, this suggests having morphisms $\alpha, f)$ with $\alpha : F(f)(x') \to x$. This thus takes a dual in the fibre. The details are **left to you**.

3rd variation: $\mathsf{F} : \mathcal{C} \to \mathsf{Cat}$ *is lax.*

Here we use morphisms $(\alpha, f) : (x, c) \to (x', c')$ with $f : c \to c'$ and $\alpha : x' \to F(f)(x)$, so again dualise in the fibre.

The most useful form for us is when it is assumed that we have a pseudo-functor, (F, τ), with $F : \mathcal{C}^{op} \to \mathsf{Cat}$, (presented in op-lax form, in agreement with the initial definition, although we will use τ^{-1} as well). We thus have, explicitly, a morphism in $\mathcal{E}_F$ from (x, c) to (y, d) is a pair, (α, f), where $f : c \to d$ in $\mathcal{C}$ and $\alpha : x \to F(f)(y)$ is a morphism in the 'fibre' over c, *i.e.*, in $F(c)$, and the composition of such morphisms,

$$(x, c) \xrightarrow{(\alpha,f)} (y, d) \xrightarrow{(\beta,g)} (z, e),$$

is

$$(\beta, g)\sharp_0(\alpha, f) = (\tau^{-1}_{(g,f)}(z) \circ F(f)(\beta) \circ \alpha, gf)$$

(It is useful to compare this with the formula in section 2.3 for the twisting of the multiplication in an extension using the 'factor set', $f(g_2, g_1)$.)

Remark: The various forms of the Grothendieck construction are 'homotopy colimits', (cf. [258]), so this relates to the type of construction described, in slightly vague terms, at the end of the previous chapter. We will revisit it later.

9.2.2 Fibred categories as Grothendieck fibrations

Fibred categories also arise as 'fibrations of categories'. From a pseudo-functor, $\mathsf{F} : \mathcal{C}^{op} \to \mathsf{Cat}$, we constructed a category, $\mathcal{E}_{\mathsf{F}}$, over $\mathcal{C}$. This is not just 'any old' functor, but has properties that resemble those of a fibration of spaces or simplicial sets. These properties correspond to a form of path lifting, but since a path in a category need not be reversible, and a path has two ends, the notion comes in two main flavours. We will give one of these. Many sources give the other, but it is easy to flip from one to the other. Approximately they correspond to the op-lax and lax forms of pseudo-functor, mixed with using the dual categories. They more or less coincide when handling pseudo-functors from $\mathcal{C}^{op}$ to the category of groupoids or 'fibrations with groupoid fibres' or 'categories fibred in groupoids' or . . . ; the terminology used is fairly transparent, but is quite varied! We will explore this without immediate reference to the preceding ideas, making the link later.

A motivating example: One motivating, and quite intuitively simple, example of a category over $\mathcal{C}$ with nice properties is when $\mathcal{C}$ has finite limits (so, in particular, pullbacks exist).

For each object c in $\mathcal{C}$, we have the category $\mathcal{C}/c$ of objects over c. (We saw this idea earlier, (for instance, page 274), with Top/B, in our initial discussion of bundle-like phenomena.) We here want to look at the pullback operation and its interaction with these 'objects over' categories and to do this in various different ways.

This category, $\mathcal{C}/c$, is the fibre over c of a functor defined on the category, $Arr(\mathcal{C}) := \mathcal{C}^{[1]}$, of arrows in $\mathcal{C}$. (The notation $\mathcal{C}^{[1]}$ refers to the identification of $Arr(\mathcal{C})$ as the category of functors from $[1]$, (yes, the small category corresponding to $0 \to 1$ or $0 < 1$) to $\mathcal{C}$). The objects are the arrows

$$c \to d$$

in $\mathcal{C}$, and these are 'the same as' functors from $[1]$ to $\mathcal{C}$, and the morphisms are the commutative squares: in other words,

$$(c \xrightarrow{\varepsilon} d) \xrightarrow{\varphi} (c' \xrightarrow{\varepsilon'} d')$$

is $\varphi = (\varphi_1, \varphi_0) : \varepsilon \to \varepsilon'$, and that is,

$$\begin{tikzcd} c \arrow[r, "\varphi_1"] \arrow[d, "\varepsilon"'] & c' \arrow[d, "\varepsilon'"] \\ d \arrow[r, "\varphi_0"] & d \end{tikzcd}$$

so, interpreted another way, they are 'natural transformations'.)

The assignment, $cod : Arr(\mathcal{C}) \to \mathcal{C}$, $cod(c \to d) = d$, is clearly a functor and the fibre $cod^{-1}(d)$ over d is precisely $\mathcal{C}/d$.

The 'game' is to identify the pullback squares in $Arr(\mathcal{C})$ by some neat universal property with regard to this functor, cod. Of course, a pullback square is just a particular type of morphism in $Arr(\mathcal{C})$. (There are two versions of the property - we will look at the stronger one first.)

Note on origin of terminology: An early used alternative name for 'pullback square' was *Cartesian square*; see, for instance, Gabriel and Zisman, [132].

Suppose we have a morphism, $\varphi : \varepsilon' \to \varepsilon$, in $Arr(\mathcal{C})$,

$$\begin{tikzcd} a' \arrow[r, "\varphi_1"] \arrow[d, "\varepsilon'"'] & a \arrow[d, "\varepsilon"] \\ b' \arrow[r, "\varphi_0"] & b \end{tikzcd}$$

We will say it is *Cartesian* if, for any other morphism $\psi : \varepsilon'' \to \varepsilon$ and $g : b'' \to b$, such that $\psi_0 = \varphi_0 g$

$$\begin{tikzcd} a'' \arrow[drr, bend left, "\psi_1"] \arrow[dd, bend right, "\varepsilon''"'] & & \\ & a' \arrow[r, "\varphi_1"] \arrow[d, "\varepsilon'"'] & a \arrow[d, "\varepsilon"] \\ b'' \arrow[r, "g"] \arrow[rr, bend right, "\psi_0"'] & b' \arrow[r, "\varphi_0"] & b \end{tikzcd}$$

there is a unique $\overline{g} : a'' \to a'$ such that $\gamma = (\overline{g}, g) : \varepsilon'' \to \varepsilon'$ in $Arr(\mathcal{C})$ and $\varphi \sharp_0 \gamma = \psi$.

If g was just the identity, this would be the ordinary pullback square property, and, of course, in this case, the more complex condition is a consequence of that property. We will see why this is useful later on.

We have:

Lemma 58 *For the functor, $cod : Arr(\mathcal{C}) \to \mathcal{C}$, the Cartesian morphisms are exactly the pullback squares.* ∎

The importance of such pullback situations in descent theory (of all flavours) led to the abstraction of the idea of a fibred category as a type of categorical fibration, (cf. Grothendieck, [139]).

The initial set up is a category, $\mathcal{B}$, as base. In addition we have another one, denoted $\mathcal{E}$, as the 'total' or 'top' space of the fibration, together with a functor, $p : \mathcal{E} \to \mathcal{B}$. We first look at a definition of Cartesian arrow, generalising and abstracting that above.

Definition: An arrow, $\varphi : e' \to e$, in $\mathcal{E}$ is said to be *Cartesian* if, given any other arrow, $\psi : e'' \to e$, in $\mathcal{E}$ with the same codomain and a factorisation of $p(\psi)$ through $p(e')$ and $p(\varphi)$,

$$\begin{tikzcd} p(e'') \arrow[dr, "g"'] \arrow[drr, "p(\psi)"] & & \\ & p(e') \arrow[r, "p(\varphi)"'] & p(e) \end{tikzcd}$$

then g *lifts* to a *unique* $\chi : e'' \to e'$ in $\mathcal{E}$ such that $\psi = \varphi \sharp_0 \chi$, and, of course, $g = p(\chi)$.

Remark: Thinking in terms of lifts in fibrations in a spatial or simplicial set context, the apparent extra complication here is due to the fact that the basic path from 0 to 1 in [1] is not reversible. The above idea thus reads: if you choose $p(e)$ as base point, and $p(\varphi)$ is the image of a path in the top category *ending* above $p(e)$, then you can lift factorisations down below to ones above in a unique way.

Example: Let $p : H \to G$ be an epimorphism of groups, then, of course, for the corresponding single object groupoids, $p[1] : H[1] \to G[1]$, is a functor. For the moment, we will **leave to the reader** the following result, but will return to this group epimorphism example in a lot more detail later on[1].

Lemma 59 *For this functor, any arrow in $H[1]$ is Cartesian.* ■

There is another weaker notion of Cartesian arrow, as follows:

Definition: An arrow, $\varphi : e' \to e$, in $\mathcal{E}$ is said to be *weakly Cartesian* if, given any other arrow, $\psi : e'' \to e$, in $\mathcal{E}$ with the same codomain such that $p(\psi) = p(\varphi)$, then there is a unique $\chi : e'' \to e'$ in $\mathcal{E}$ such that $\psi = \varphi\sharp_0\chi$ and $p(\chi) = id_{p(e')}$.

In our pullback example, this weaker property would seem to be nearer to the usual universal property of pullbacks, however, whilst the composite of Cartesian arrows will be Cartesian, (see the lemma below), the same is not necessarily true for the weaker form. Why is this important? The idea of Cartesian arrow is to capture that property of pullbacks for use in the many situations in which pullback-like constructions are needed (and especially in 'descent', where 'good' objects over an object are pulled and pushed around over subobjects, covers, and a mass of other variants).

Recall that if we have a diagram

$$\begin{tikzcd} A \arrow[r, "\alpha_1"] \arrow[d, "f"'] \arrow[dr, phantom, "①"] & B \arrow[r, "\beta_1"] \arrow[d, "g"'] \arrow[dr, phantom, "②"] & C \arrow[d, "h"] \\ A' \arrow[r, "\alpha_0"'] & B' \arrow[r, "\beta_0"'] & C' \end{tikzcd}$$

in an arbitrary category, and both the small squares ① and ② are pullbacks, then the big outer 'square', ③,

$$\begin{tikzcd} A \arrow[r, "\beta_1\alpha_1"] \arrow[d, "f"'] \arrow[dr, phantom, "③"] & C \arrow[d, "h"] \\ A' \arrow[r, "\beta_0\alpha_0"'] & C' \end{tikzcd}$$

is also one. You probably have seen this, but it will pay to recall the idea, so we will 'revise' it. You take a commutative square

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "a"'] & C \arrow[d, "h"] \\ A' \arrow[r, "\beta_0\alpha_0"'] & C' \end{tikzcd}$$

[1] and will give a the proof of this, although it is easy.

and use the universal property of square ② to get a unique morphism from X to B factoring c via β_1. You, then, check the resulting square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ A' \arrow[r, "\alpha_0"'] & B' \end{tikzcd}$$

commutes to get a factorisation via α_1 of the top arrow (using the universal property of ①). Finally you check that everything fits together as you hoped for.

Lemma 60 *Given a functor $p : \mathcal{E} \to \mathcal{B}$, if $\varphi_2 : e_2 \to e_1$ and $\varphi_1 : e_1 \to e$ are Cartesian arrows, so is $\varphi_1\varphi_2 : e_2 \to e$.* ■

No prizes for the proof! You just mimic the proof of 'pullbacks compose'. Note however that the proof uses the stronger Cartesian condition in a strong way. (You might with justice say that as the result was to be 'strong', you would expect to use 'strong' for the proof, but then a natural question is: does an analogous result hold for weak Cartesian arrows? There is a blockage. This is **worth investigating**.) We thus cannot assume that the composite of weak Cartesian arrows is weak Cartesian.

Returning to the strong case, we make a definition:

Definition: A functor, $p : \mathcal{E} \to \mathcal{B}$, is a *Grothendieck fibration* (usually abbreviated to *fibration*) if, for any object e in $\mathcal{E}$ and $f : b \to p(e)$ in $\mathcal{B}$, there is a Cartesian arrow, $\varphi : e' \to e$, in $\mathcal{E}$ with $p(\varphi) = f$. In this case, $\varphi : e' \to e$ may be called a *Cartesian lifting* of e along f.

A bit more terminology will be useful here.

If $p : \mathcal{E} \to \mathcal{B}$ is any functor[2], then a morphism $\varphi : e \to e'$ in $\mathcal{E}$ is said to be *vertical* if $p(\varphi)$ is the identity morphism on $p(e)$. For a given object b in $\mathcal{B}$, the subcategory of $\mathcal{E}$ consisting of those objects e of $\mathcal{E}$ 'above b' so satisfying $p(e) = b$, and the vertical morphisms between them is called the *fibre of $p : \mathcal{E} \to \mathcal{B}$ over b*. It will be denoted $\mathcal{F}_p(b)$ for the moment.

Remarks: (i) If a functor $p : \mathcal{E} \to \mathcal{B}$ is a fibration, then weak Cartesian arrows compose. Conversely, if the fibration condition holds with 'weak Cartesian' replacing 'Cartesian' and, if in addition, weak Cartesian arrows compose, then p will be a fibration. It was in this form that the definition of a fibred category as a fibration was given originally; see Grothendieck in [139, 141] and Giraud in [134, 135].

(ii) Notice that Cartesian liftings of an object, e, in the fibre over b are unique *up to vertical isomorphism*. The proof is a variant of the usual proof of 'uniqueness up to isomorphism' of an object given by a universal property.

Examples:

1. If $\theta : H \to G$ is a group epimorphism, then the corresponding functor, $\theta[1] : H[1] \to G[1]$, is a fibration.

[2] so not necessarily a fibration, but we will tend to use the term mostly in that case

2. If $\mathcal{C}$ has pullbacks, then $cod : Arr(\mathcal{C}) \to \mathcal{C}$ is a fibration, and conversely. This fibration is sometimes called the *fundamental fibration* of $\mathcal{C}$; see Streicher, [254], section 4.

The proofs of these two statements are **left to you**.

It is sometimes useful to use the following loose terminology:

If $\varphi : e' \to e$ is a Cartesian arrow of $p : \mathcal{E} \to \mathcal{B}$ and $p(\varphi) : p(e') \to p(e)$ is its image in $\mathcal{B}$, we may say that e' is a *pullback* of e over $p(\varphi)$.

In a fibration, $p : \mathcal{E} \to \mathcal{B}$, there are enough 'lifts' of arrows in $\mathcal{B}$. You specify an object e in $\mathcal{E}$ and an arrow ending at $p(e)$, then that arrow is the image of *at least one* Cartesian arrow back up in $\mathcal{E}$, ending at e, - so the solution set for the lifting problem is always non-empty.

Definition: Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration. A *cleavage* of p is a class, $\mathcal{K}$, of Cartesian arrows in $\mathcal{E}$ such that, for each e in $\mathcal{E}$ and $f : b \to p(e)$, there is a *unique* arrow, $\varphi : e' \to e$ in $\mathcal{K}$ satisfying $p(\varphi) = f$.

It is sometimes convenient to have a specific notation for this φ and we will follow [254] and introduce notation as follows: given e in $\mathcal{E}$ and $f : b \to p(e)$, the unique Cartesian lift over f in the cleavage will be denoted

$$Cart(f, e) : f^*(e) \to e.$$

This can be pictured as

$$\begin{tikzcd}
f^*(e) \arrow[r, dashed, "Cart(f,e)"] & e \\
b \arrow[r, "f"'] & p(e)
\end{tikzcd}$$

If needs be, we will use a subscript notation, $Cart_{\mathcal{K}}$, so as to identify which cleavage $\mathcal{K}$ is being used here. Of course, the use of $f^*(e)$ is intended to recall the example of pullback along f. This induces a functor $f^* : \mathcal{F}_p(p(e)) \to f^*(b)$, sometimes called the *reindexing functor*, since if $\alpha : e \to e'$ is a vertical morphism, thus in $\mathcal{F}_p(p(e))$, there will be a unique vertical morphism, $f^*(\alpha) : f^*(e) \to f^*(e')$, in the fibre over b. We will explore this briefly later, but also refer to Streicher's notes, [254], for more on this.

A way of thinking of a cleavage is that it is a categorification of a *transversal* in group theory. If we have a group epimorphism, $\theta : G \to H$, then a *transversal* for H in G can be variously defined as a section, $s : H \to G$, *i.e.*, a function / map on the underlying sets of the two groups, such that $\varphi s(h) = h$ for all $h \in H$. If you prefer to think of θ inducing an isomorphism $G/Ker\theta \cong H$, so elements of H 'are' cosets, the transversal is a set of coset representatives. Of course, s is not necessarily a splitting. It is not, in general, a homomorphism. Interpreting $\theta : G \to H$ as a functor, $\theta[1] : G[1] \to H[1]$, we have any cleavage, $\mathcal{K}$, corresponds exactly to a transversal.

Do they always exist? The axiom of choice tells one that

Proposition 86 *(If the Axiom of Choice holds in your context) every fibration has a cleavage.* ■

Remark: You may think that a strange way to state a proposition, so let us see why it is important. The axiom of choice states, in categorical language, that any epimorphism between sets if split, yet in many categories epimorphisms need not be split - that is the whole point of the notion of cleavage, in fact, from some points of view, it is the whole point about cohomology! Many of the ideas of this chapter so far, such as pseudo-functor, fibred category, fibration, work well (and usefully) for generalisations of the context. For instance, we might ask for fibrations of internal categories or of enriched categories. The ideas and intuitions make sense, although sometimes the definition may need reformulating to avoid too 'set biased' a language. The existence of a cleavage for a fibration, say, between *internal categories* will be dependent on where you are. (You may like the following simple case as (i) it uses ideas we do know well and (ii) it is relevant for later use. Consider a morphism, $p : E \to B$, between internal categories in the category of groups. When should it be considered to be a fibration? What should be the definition of a 'cleavage' of such a fibration? Remember you should be doing everything within the category of groups. Do they exist? When? Again, as we know well and will use again below, epimorphisms rarely split in the category of groups, ... This is **left to you to worry out**.)

Following on from this, there is another obvious definition.

Definition: A cleavage, $\mathcal{K}$, for a fibration, $p : \mathcal{E} \to \mathcal{B}$, is a *splitting* if it contains all identities and is closed under composition.

A fibration, $p : \mathcal{E} \to \mathcal{B}$, is a *splittable fibration* if it has a splitting. If that splitting is specified, the fibration together with the splitting is then called a *split fibration* or quite often a *cleaved fibration*.

Not every fibration is *isomorphic* to a splittable one. To see a simple example, let us return to the example of a group epimorphism, $\theta : H \to G$, as above, we get a fibration, $\theta[1] : H[1] \to G[1]$. It is fairly easy to check that if $\theta[1]$ is a splittable fibration, then θ would be a split epimorphism, with splitting being given by the cleavage. It is, then, easy to give an example of an epimorphism or groups that is not split, e.g., the natural quotient map from $C_\infty = \langle a \mid \emptyset \rangle$ to $C_2 = \langle a \mid a^2 \rangle$ would do. It should be clear that splittable fibrations of groups (as groupoids) need the top group to be a semi-direct product of the base and the 'fibre', *i.e.*, the kernel of the epimorphism.

9.2.3 From pseudo-functors to fibrations

We constructed a functor, $p : \mathcal{E}_\mathsf{F} \to \mathcal{B}$, from a pseudo-functor, $\mathsf{F} : \mathcal{B}^{op} \to \mathsf{Cat}$.

Proposition 87 *The functor, $p : \mathcal{E}_\mathsf{F} \to \mathcal{B}$, is a fibration with $\mathcal{K} = \{(id, f) \mid f : b' \to b \text{ in } \mathcal{B}\}$, being a cleavage of p.*

Proof: The easy way to check that a functor is a fibration is to give a cleavage, so here, as we are given a candidate cleavage, we just check that it is one.

First a bit more precision is needed. Given $f : b' \to b$ in $\mathcal{B}$ and an object, x, in $F(b)$, we have $(id_{F(f)(x)}, f) : (F(f)(x), b') \to (x, b)$ is in $\mathcal{K}$. We must check that this is a Cartesian arrow. (We bridge between the two notations and take $e = (x, b)$, $e' = (F(f)(x), b')$, $\varphi = (id_{F(f)(x)}, f)$.)

Suppose given $\psi : e'' \to e$ is in $\mathcal{E}_\mathsf{F}$, with $e'' = (x'', b'')$, and a factorisation $p(\psi) = fp(\varphi) = fg$, where $g : b'' \to x$. We thus have $\psi = (\beta, fg)$ for some $\beta : x'' \to F(fg)(x)$. We want $\chi : e'' \to e'$ with $\psi = \varphi\sharp_0\chi$ and $p(\chi) = g$. We thus know that χ has the form (γ, g) and so $\gamma : x'' \to F(g)(x')$, where $x' = F(f)(x)$. The condition that $\psi = \varphi\sharp_0\chi$ translates as the trivial expression, $fg = fg$, together with that β is the composite

$$x'' \xrightarrow{\gamma} F(g)F(f)(x) \xrightarrow{F(g)(id)} F(g)F(f)(x) \xrightarrow{\tau^{-1}_{(g,f)}} F(fg)(x).$$

We thus have $\beta = \tau^{-1}_{(g,f)}\gamma$, so can read off $\gamma = \tau_{(g,f)}\beta$ to find the unique χ satisfying the conditions. ∎

Corollary 16 *The cleaved fibration, $(\mathscr{E}_\mathsf{F}, p_\mathsf{F}, \mathscr{K})$, associated with a pseudo-functor, $\mathsf{F} = (F,\tau)$, is split if and only if F is a functor and τ is the identity.* ∎

This is just a question of checking that '$\mathscr{K}$ contains all the identities and is closed under composition' is equivalent to 'τ is trivial'. It is **left to you.**

9.2.4 ... and back

Suppose we have a fibration, $(\mathscr{E}, p)$, over $\mathscr{B}$ and choose a cleavage, $\mathscr{K}$. Is there an associated pseudo-functor, $\mathsf{F} = (F,\tau)$, and an isomorphism of fibrations between $(\mathscr{E}, p)$ and $(\mathscr{E}_\mathsf{F}, p_\mathsf{F})$? If there is then we will have two equivalent ways of looking at fibred categories, the fibrational one and that using pseudo-functors, and, of course, this *is* the case.

To make sense of this, the first thing to note is that we have not yet actually defined what is a morphism between fibrations. The definition is more or less obvious.

Definition (fibrational version): If $(\mathscr{E}, p)$ and $(\mathscr{E}', p')$ are two fibrations over $\mathscr{B}$, a *morphism of fibrations* from $(\mathscr{E}, p)$ to $(\mathscr{E}', p')$ is a functor, $F : \mathscr{E} \to \mathscr{E}'$, over $\mathscr{B}$ (so $p'F = p$) and such that F preserves Cartesian arrows.

We have an evident category, $Fib(\mathscr{B})$, of fibrations over $\mathscr{B}$, and an equally evident notion of isomorphism. To emphasise the defining properties of these morphisms of fibrations, it is often useful to stress the fact these preserve Cartesian arrows by using the terminology *Cartesian functor.*

In fact, $Fib(\mathscr{B})$ is better off being given the structure of a 2-category, so will turn aside for an instant to give that structure, although it will not be needed until later, becoming central to our discussion when we discuss change of base for fibred categories and later for stacks.

Definition: The 2-category, $\mathsf{Fib}(\mathscr{B})$, has fibrations over $\mathscr{B}$ as its objects, and if $p : \mathscr{E}_p \to \mathscr{B}$ and $q : \mathscr{E}_q \to \mathscr{B}$ are two such fibred categories, then $\mathsf{Fib}(\mathscr{B})((\mathscr{E}_p, p), (\mathscr{E}_q, q))$ often written more simply $\mathsf{Fib}(\mathscr{B})(p, q)$ is the category with objects the Cartesian functors from $(\mathscr{E}_p, p)$ to $(\mathscr{E}_q, q)$ and the *Cartesian natural transformations* between them as morphisms. Here, if $F, G : (\mathscr{E}_p, p) \to (\mathscr{E}_q, q)$ are Cartesian functors, a natural transformation, $\eta : F \Rightarrow G$ is said to be *Cartesian* if, for each object, e, of $\mathscr{E}_p$, $\eta_e : F(e) \to G(e)$ is a vertical morphism in $\mathscr{E}_q$.

Example: If $\mathscr{B}$ is the terminal category, then $\mathsf{Fib}(\mathscr{B})$ is isomorphic to the 2-category, Cat, of small categories, functors and natural transformations.

Notation and terminology: For much of the time we will be handling a general category, $\mathscr{B}$, as the 'base' for the fibred categories, however in a spatial context, we would have a *space*, B, and would take $\mathcal{B} = Open(B)$. In that case, we will often shorten 'fibred category over $Open(B)$' to 'fibred category over B', and use $\mathsf{Fib}(B)$ as a short form of $\mathsf{Fib}(Open(B))$. There is a very important point that for many geometric contexts, the natural base is *not* a space, but something nearer to a higher dimensional category. An instance of this is where we replace 'space' by 'orbifold' or 'moduli stack', and the detailed theory then needs adjusting. We will, however, continue with the simpler spatial context some of the time, as well as using the more general 'base category' idea.

Returning to the ' *... and back again*' theme:

Proposition 88 *A cleaved fibration, $(\mathcal{E}, p, \mathcal{K})$, over $\mathcal{B}$ defines a pseudo-functor, $(F : \mathcal{B}^{op} \to \mathsf{Cat}, \tau)$.*

Proof: (As you would suspect, the idea is that you 'unbuild' or 'deconstruct', the fibration, reversing the process given in previous sections.)

For b, an object of $\mathcal{B}$, let $\mathcal{F}(b)$ be the fibre of p above b, *i.e.*, the subcategory of $\mathcal{E}$, whose objects are the objects, e, of $\mathcal{E}$, which map down to b, so $p(e) = b$, and whose arrows, $\varphi : e \to e'$, are those of $\mathcal{E}$ satisfying $p(\varphi) = id_b$, so are the *vertical* arrows of $\mathcal{E}$.

Now suppose $f : b' \to b$ is an arrow in $\mathcal{B}$, we define $\mathcal{F}(f) : \mathcal{F}(b) \to \mathcal{F}(b')$ (and note the change in direction) by

- if $e \in \mathcal{F}(b)$, there is a unique Cartesian arrow, $\varphi : e' \to e$, *in the given cleavage*, $\mathcal{K}$, such that $p(\varphi) = f$, and we set $\mathcal{F}(f)(e) = e'$;
- if $\alpha : e_1 \to e$ is an arrow in $\mathcal{F}(b)$, then we have a unique Cartesian arrow, $\varphi_1 : e_1 \to e$, and $\mathcal{F}(f)(e_1) = e_1'$. We need $\mathcal{F}(f)(\alpha) : e_1' \to e'$, *i.e.*, $\mathcal{F}(f)(\alpha) : \mathcal{F}(f)(e_1) \to \mathcal{F}(f)(e)$. We have a diagram

$$\begin{tikzcd} e_1' \arrow[r, "\varphi_1"] & e_1 \arrow[d, "\alpha"] \\ e' \arrow[r, "\varphi"'] & e \end{tikzcd}$$

 with φ Cartesian, so have a unique $\chi : e_1' \to e'$ such that $\varphi\chi = \alpha\varphi_1$ and $p(\chi) = id_{b'}$. We set $\mathcal{F}(f)(\alpha) := \chi$.

We now check what happens if, in addition, we have $b : b'' \to b'$. We can work out $\mathcal{F}(fg)$ and $\mathcal{F}(g)\mathcal{F}(f)$ from $\mathcal{F}(b')$ to $\mathcal{F}(b'')$. For each object e in $\mathcal{F}(b)$, we have unique Cartesian arrows

$$\begin{aligned} \varphi : e' \to e &\quad \text{over} \quad f, \\ \gamma : e'' \to e' &\quad \text{over} \quad g, \\ \psi : e_1'' \to e &\quad \text{over} \quad fg, \end{aligned}$$

all in $\mathcal{K}$, however K was not assumed to be a splitting, so we do not know if $\varphi\gamma$ is in $\mathcal{K}$. It *will* be Cartesian however. We now need a 'useful lemma':

Lemma 61 *In a fibration, $(\mathcal{E}, p)$, over $\mathcal{B}$, if $\varphi : e' \to e$ and $\psi : e_1' \to e$ are both Cartesian arrows over $f : b' \to b$, then there is a unique isomorphism, $\chi : e_1' \to e'$, such that $\psi = \varphi\chi$ and $p(\chi) = id_{b'}$.*

Proof: This is fairly routine. You first find a unique χ using the Cartesian property for φ, then find a $\chi' : e' \to e_1'$ using the Cartesian property of ψ. Next look at $\chi\chi'$ and $\chi'\chi$ as lifts of the identity on b'_1 (and b' respectively), then use uniqueness once more and the fact that the identity arrows are Cartesian arrows to conclude that χ' is the inverse of χ. ■

Returning to the main proof, we have both ψ and $\varphi\gamma$ are Cartesian arrows over fg, so there is an arrow $\chi : e''_1 \to e''$, that is, from $\mathcal{F}(fg)(e)$ to $\mathcal{F}(g)\mathcal{F}(f)(e)$. We take this to be our $\tau_{(g,f)}(e)$ and check that it is an isomorphism (by the lemma) and is natural (by various uniqueness clauses). Finally we are left with the cocycle condition and that follows from another use of the uniqueness clause. (It is **worthwhile checking** this last point in a bit of detail.) We thus have that the allocation of the fibre of p defines a pseudo-functor as required, so we just take $\mathsf{F} = (\mathcal{F}, \tau)$ to complete the argument. ■

The following is now fairly obvious:

Corollary 17 *The pseudo-functor associated to a cleaved fibration is a functor if and only if the cleavage is a splitting.* ■

The proof is **left to you**.

Remark: It is instructive to view the diagram used above in the notation, which is nearer to that used by Streicher in [254]. It looks like

$$\begin{tikzcd} f^*(e_1) \arrow[r, "Cart(u,e_1)"] \arrow[d, dashed, "f^*(\alpha)"'] & e_1 \arrow[d, "\alpha"] \\ f^*(e) \arrow[r, "Cart(u,e)"'] & e \end{tikzcd}$$

with the dotted arrow given by the universal property of Cartesian morphisms. This emphasises the similarity of the argument to that encountered throughout more elementary parts of category theory.

Following those results we will **leave you to state and prove** hopefully now fairly obvious results linking $\mathsf{FibCat}(\mathcal{B})$ and $\mathsf{Fib}(\mathcal{B})$. A detailed account is given in Streicher's notes, [254]. The reader may be screaming that the 2-category structure on $\mathsf{FibCat}(\mathcal{B})$ has not been given and, of course, that is **part of the point of the challenge**. We will however need to give that structure shortly, and in detail, but do not want to spoil your fun!

Returning to split fibrations over $\mathcal{B}$, suppose we have two such split fibrations, $(\mathcal{E}, p, \mathcal{K})$ and $(\mathcal{E}', p', \mathcal{K}')$. Given a Cartesian functor, $F : (\mathcal{E}, p) \to (\mathcal{E}', p')$, we say F is *split Cartesian* if it preserves the splittings. Taking this apart, suppose e is in $\mathcal{E}$ and $f : b \to p(e)$, then $F(e) \in \mathcal{E}'$ and $p'(F(e)) = p(e)$. Of course, $Cart_{\mathcal{K}}(f, e) : f^*(e) \to e$ and the condition for preservation of the splittings is that

$$F(Cart_{\mathcal{K}}(f, e)) = Cart_{\mathcal{K}'}(f, F(e))$$

for all e, f, etc.

Taking a 2-arrow, $\eta : F \Rightarrow G$, between two split Cartesian functors, just to be a Cartesian natural transformation between then gives a 2-category, which will be denoted $\mathsf{Sp}(\mathcal{B})$. We will shortly look at the *Fibred Yoneda Lemma* making precise the relation between fibred categories and split fibrations over the same base, $\mathcal{B}$.

Finally for this section, you may also want to consider the following queries.

In the theory of group extensions, there are results comparing different sections of the 'right hand' epimorphism and linking that with the cohomological invariants of the target group. Are there analogues in the theory of fibrations using different cleavages for some fibration? What sort of theory might one hope for (i) in general, or (ii) in particular cases such as those we will study in section 9.2.5?

There are many interesting aspects of fibrations that we will not go into here. They are of considerable use in various logical situations, as well as in the cohomological and geometric ones that we will be considering here. For a development of this logical side, you can refer to Streicher's notes on Bénabou's ideas, in [254]. Those notes are also extremely useful for some of the basic theory of fibred categories, for instance the fibred Yoneda lemma, and we will, in section 9.3.4, give a version of that basic theory, avoiding some of the detailed explanations as being easily available in the Bénabou - Streicher notes, but adopting a fairly naive viewpoint to start with.

9.2.5 Two special cases and a generalisation

There are two special cases of fibred categories that are especially interesting for their links with other areas that we have met earlier. Of course, there will be other later on.

Categories fibred in groupoids

These can be specified most simply by a pseudo-functor, $F : \mathcal{B}^{op} \to \mathsf{Grpd}$, that is, one taking values in the full 2-subcategory of groupoids. Formally, and for ease of later use:

Definition: A fibred category / pseudo-functor, $\mathsf{F} = (F, \tau)$, over $\mathcal{B}$ is said to be *fibred in groupoids* if F is a pseudo-functor from $\mathcal{B}^{op}$ to Cat such that, for each object, c, $F(c)$ is a groupoid.

Suppose we view this from the fibration viewpoint. Clearly we will have a fibration, $p : \mathcal{E}_{\mathsf{F}} \to \mathcal{B}$, with each $p^{-1}(c)$, a groupoid, but there is a neater way of looking at this.

Proposition 89 *A functor, $p : \mathcal{E} \to \mathcal{B}$, corresponds to a category fibred in groupoids if, and only if, the following two conditions are satisfied:*

(i) every morphism is Cartesian;

and

(ii) given any object e of $\mathcal{E}$ and arrow $f : b' \to p(e)$, there is an arrow $\varphi : e' \to e$ in $\mathcal{E}$ with $p(\varphi) = f$.

Remark: In the particular case of fibrations of groupoids, that is when the base category, $\mathcal{B}$, is also a groupoid, the second condition is known as *costar surjectivity*. In work on groupoids[3], the *star* of an object, x, in the groupoid $\mathcal{G}$ is the set of g in $\mathcal{G}$ whose source is x, whilst the *costar* of x is the set of g whose target is x. (Both are particularly well behaved instances of comma categories.

[3] see Ronnie Brown's [57] and [59]

or more exactly, sets of generators for the comma categories at x.) A proof that star surjectivity is equivalent to a fibration condition is given in [171] on page 155, but notice that the version of fibration being used is more 'homotopically' based. We have already met the idea when handling coverings of 'scwols' in section 4.5.10.

Proof of proposition: If the two conditions hold for $(\mathcal{E}, p)$, then clearly this is a fibration. Now assume $\varphi : e' \to e$ is an arrow in some fibre, $p^{-1}p(e)$, then using (i), φ is Cartesian, so there is a $\psi : e \to e'$, also satisfying $p(\psi) = id_{f(e)}$ and $\varphi\psi = id_e$ by uniqueness. (We used this argument a short while ago.) Every arrow, thus, has a pre-inverse. The pre-inverse, ψ, also has a pre-inverse, and, by associativity, this will be φ (by the usual argument - think back to the beginning of most Group Theory courses!) We thus have that $p^{-1}(p(e))$ is a groupoid.

Now assume $(\mathcal{E}, p)$ is fibred in groupoids. Condition (ii) is immediate, so we just have to check (i). Given $\varphi : e' \to e$ in $\mathcal{E}$, and suppose $\psi : e'' \to e$ in $\mathcal{E}$, $g : p(e'') \to p(e)$. We know that, as $(\mathcal{E}, p)$ is a fibration, there is a Cartesian arrow $\varphi' : e''' \to e$ over $p(e') \to p(e)$, and a unique $\chi : e'' \to e'''$ factorising ψ (as $\psi = \varphi'\chi$), and over g. We also have a unique $\tau : e' \to e''$ factorising φ (as $\varphi = \varphi'\tau$) and over the identity on $p(e')$. The arrow $\tau^{-1}\chi : e'' \to e'$, then factorises ψ as $\varphi(\tau^{-1}\chi)$ and is over g; uniqueness is easy to check. We thus have that condition (i) holds: all morphisms in $\mathcal{E}$ are Cartesian. ■

Of course, this means that, if $(\mathcal{E}', p')$ is a fibration and $(\mathcal{E}, p)$ is a fibration fibred in groupoids, both over the base category $\mathcal{B}$, than for any functor $f : \mathcal{E}' \to \mathcal{E}$ over $\mathcal{B}$ (so $p'f = p$), will be a morphism of fibrations.

Discrete fibrations = Categories fibred in sets

A very particular (but also very important) case of the previous situation occurs if the pseudo-functor, $\mathsf{F} : \mathcal{B}^{op} \to \mathsf{Grpd}$, actually takes values in the subcategory of sets, that is, sets considered as groupoids only having identity arrows.

We have, in fact, looked in some detail at this before in the case in which $\mathcal{B}$ is a groupoid as back in section 4.5.14 we discussed the way that coverings of groupoids corresponded to local systems.

Definition: A fibration, $(\mathcal{E}, p)$, over $\mathcal{B}$ is said to be *discrete*, or *fibred in sets*, if, for any object b in $\mathcal{B}$, the only arrows in $p^{-1}(b)$ are identity arrows.

As you might expect, these special discrete fibrations have a special property.

Proposition 90 *Let $(\mathcal{E}, p)$ be a category over $\mathcal{B}$. It is a fibration fibred in sets if, and only if, for any object e of $\mathcal{E}$ and any arrow $f : b \to p(e)$, there is a unique arrow, $\varphi : e' \to e$, with $p(\varphi) = f$.*

Proof: For the 'forward implication', we know some φ exists, but if $\psi : e'' \to e$ was another such then there would be a χ *in the fibre over* b, factorising ψ through φ. The only such χ is an identity as those are the only arrows in the fibre, so uniqueness follows.

The converse is left to you to look at. ■

Remark: Thinking of the analogy with topological fibrations, this clearly has close links with the 'unique path lifting' type condition for covering spaces (cf. the discussion of covering spaces in section 7.1.2, and in many books on elementary homotopy theory.)

If we now look at such a category fibred in sets / discrete fibration, and we consider the associated pseudo-functor, $\mathsf{F} = (F, \tau) : \mathcal{B}^{op} \to Sets$, then what is τ? That is easy. It is a natural transformation between the pullback along a composite and the composite of the two pullbacks. Right, ... , how is that given? By a family of arrows in the fibre $F(b) = p^{-1}(b)$, where b is in the domain of the composite. However $F(c)$ is a set considered as a discrete category, so the only arrows there are identity arrows. We can thus derive this from the previous result:

Corollary 18 *Any fibration, $(\mathcal{E}, p)$, over $\mathcal{B}$, which is fibred in sets, corresponds to a functor, $F : \mathcal{B}^{op} \to Sets$,* i.e., *to a presheaf on $\mathcal{B}$.* ■

9.3 Fibred category theory: some elementary results

Let us spend some time exploring some of the concepts, and structures relating to fibred category theory. The added intuition this will provide will help later on when considering stacks, etc.

9.3.1 Fibred subcategories

The following is fairly obvious as a definition, but can be very useful as an idea.

Definition: Let $(\mathcal{E}, p)$ be a fibration / fibred category over $\mathcal{B}$ and suppose $\mathcal{D}$ is a subcategory of $\mathcal{E}$ such that, on writing i for the inclusion of $\mathcal{D}$ into $\mathcal{E}$,

(i) $(\mathcal{D}, pi)$ is a fibration over $\mathcal{B}$;

and

(ii) $i : (\mathcal{D}, pi) \to (\mathcal{E}, p)$ is a morphism of fibrations.

We say $(\mathcal{D}, pi)$ is a *fibred subcategory* or *subfibration* of $(\mathcal{E}, p)$.

Of course, we will loosely say '$\mathcal{D}$ is a fibred subcategory of $\mathcal{E}$' if no confusion is likely to arise.

Note the second condition of the definition implies that the Cartesian arrows for $(\mathcal{D}, pi)$ are also Cartesian for $(\mathcal{E}, p)$.

Suppose $p : \mathcal{E} \to \mathcal{B}$ is a fibration, and consider $\mathcal{D}$ a full subcategory of $\mathcal{E}$ with the property that if d is an object of $\mathcal{D}$ and $e \to d$ is a Cartesian arrow of $\mathcal{E}$, then e is also in $\mathcal{D}$, then $\mathcal{D}$ with the restriction, pi, of p to $\mathcal{D}$, is a sub-fibration of $(\mathcal{E}, p)$. The Cartesian arrows of $\mathcal{D}$ will be those of $\mathcal{E}$, whose codomains are in $\mathcal{D}$.

Definition: Let $(\mathcal{E}, p)$ be a fibration. The *fibred sub-category, $(\mathcal{E}_{Cart}, p)$, of Cartesian arrows* of $(\mathcal{E}, p)$ is specified by having the same objects as $\mathcal{E}$, but merely the Cartesian arrows (as in the discussion above).

From earlier work, we obtain:

Corollary 19 *For any fibration, $(\mathcal{E}, p)$, $(\mathcal{E}_{Cart}, p)$ is the largest sub-fibration of $(\mathcal{E}, p)$ that is fibred in groupoids.* ■

This is the fibred version of the obvious construction of the maximal groupoid of a category.

As we have hinted earlier, we have only scratched the surface of fibred category theory and have not touched on the applications. We need the theory for its input into the description of stacks, but before that we need to start to look at things simplicially and also to introduce an intermediate notion, prestack.

9.3.2 Fibred categories: a categorification of presheaves and a simplicial view

With the above discussion of fibred categories and fibrations, we can clearly see the 'categorification' aspect. With sheaves and étale spaces, the presheaf of sections gave the link between them. The fibres were sets. With fibred categories, the fibres are categories. For sheaves, the 2-cocycle condition was an equality, here it becomes an *isomorphism* and there is a 3-cocycle condition (page 392).

With regard to the 3-cocycle condition, the fact that this is a square is initially a bit confusing so let us explore that a bit more, but first draw the 2-cocycle rule as a triangle:

$$\begin{tikzcd} & 1 \arrow[dr, "F(g)"] & \\ 0 \arrow[ur, "F(f)"] \arrow[rr, "F(gf)"'] \arrow[rr, phantom, shift left=3, "\Uparrow\tau_{f,g}"] & & 2 \end{tikzcd}$$

now add another basic arrow giving a tetrahedron. We draw two views of this: from the basic

$$\begin{tikzcd} & 1 \arrow[ddr, "g" near end] \arrow[drr] & & \\ 0 \arrow[ur, "f"] \arrow[rrr] \arrow[dr] & & & 3 \\ & & 2 \arrow[ur, "h"'] & \end{tikzcd}$$

one gets the diagram of the d_0 and d_2 faces, (even faces),

- $$\begin{tikzcd} 1 \arrow[r, "F(g)"] \arrow[dr, "F(hg)"] & 2 \arrow[d, "F(h)"] \\ 0 \arrow[u, "F(f)"] \arrow[r, "F(hgf)"'] & 3 \end{tikzcd}$$

  plus 2-cells :

- $\tau_{gf,h} : F(hgf) \Rightarrow F(h)F(gf)$

- $\tau_{f,g} : F(gf) \Rightarrow F(g)F(f)$,

which give a composite 2-cell

$$(\tau_{g,h}.F(f))\tau_{f,gh} : F(hgf) \Rightarrow F(h)F(g)F(f);$$

then a diagram of odd faces (with d_1 and d_3)

-

$$\begin{tikzcd}
1 \arrow[r, "F(g)"] & 2 \arrow[d, "F(h)"] \\
0 \arrow[u, "F(f)"] \arrow[ur, "F(gf)"] \arrow[r, "F(hgf)"'] & 3
\end{tikzcd}$$

plus 2-cells :

- $\tau_{gf,h} : F(hgf) \Rightarrow F(h)F(gf)$
- $\tau_{f,g} : F(gf) \Rightarrow F(g)F(f)$

giving another composite 2-arrow:

$$(F(h).\tau_{f,g}.F(f))\tau_{gf,h} : F(hgf) \Rightarrow F(h)F(g)F(f).$$

The 3-cocycle condition says that these two composite 2-cells are equal, *i.e.*, the square diagram

$$\begin{tikzcd}
F(hgf) \arrow[r, Rightarrow, "\tau_{gf,h}"] \arrow[d, Rightarrow, "\tau_{f,hg}"'] & F(h)F(gf) \arrow[d, Rightarrow, "(F(h).\tau_{f,g})"] \\
F(hg)F(f) \arrow[r, Rightarrow, "(\tau_{g,h}.F(f))"'] & F(h)F(g)F(f)
\end{tikzcd}$$

commutes.

A neat quite 'geometric' intuition of 'why' it must commute is that, with fibred categories, one is using categories, functors (as 1-cells) and natural transformations (as 2-cells), with nothing corresponding to 3-cells as would be needed inside a tetrahedron as above, or, perhaps more exactly, only identities as 3-cells, so the 3-cell in the tetrahedron *must* be an identity, thus specifying equality, and the square *must* commute. This is basically the same point as when working with hyper-cohomology with coefficients in a short complex in the previous chapter. Degree n maps eventually become trivial as n increases. We have seen other similar things earlier in these notes as well.

We have met Cartesian morphisms of fibrations and then Cartesian natural transformations. Here are the corresponding ideas from the pseudo-functor viewpoint as promised, thus giving the 2-category structure on $\mathsf{FibCat}(\mathscr{B})$.

Definition (pseudo-functor version, first steps): Given two pseudo-functors / fibred categories, F and G, over B, a morphism, $\varphi : F \to G$, of fibred categories consists of:

- a functor, $\varphi_U = \varphi(U) : F(U) \to G(U)$, for each open U in B;
- for each morphism[4], $i : V \to U$, a natural isomorphism,

$$\alpha_i : \varphi_V i^* \overset{\cong}{\to} i^* \varphi_U,$$

which are to satisfy a compatibility condition with respect to the structural maps, τ, of F and G, namely: given $W \overset{j}{\to} V \overset{i}{\to} U$, the two composites

$$\varphi_W (ij)^* \overset{\alpha_{ij}}{\to} (ij)^* \varphi_U \overset{\tau\varphi_U}{\to} j^* i^* \varphi_U,$$

and

$$\varphi_W (ij)^* \overset{\varphi_W \tau}{\to} \varphi_W j^* i^* \overset{(\alpha_j) i^*}{\to} j^* \varphi_V i^* \overset{j^* \alpha_i}{\to} j^* i^* \varphi_U$$

are equal.

This condition also has a categorical / simplicial interpretation. First write $F(i)$ instead of $i^* : F(U) \to F(V)$, etc., then we have a square

$$\begin{tikzcd} F(U) \arrow[r, "\varphi_U"] \arrow[d, "F(i)"'] & G(U) \arrow[d, "G(i)"] \\ F(V) \arrow[r, "\varphi_V"'] & G(V). \end{tikzcd}$$

The first bit of extra structure corresponds to a 2-cell, α_i going 'up-right' across this square, *i.e.*, φ is *not* assumed to be a natural transformation, but *is* a 2-categorical analogue of one. (As the α_i are natural *isomorphisms*, this is a special type of 2-natural transformation. There is a wide range of terminology used in the 2-categorical literature for this. If we need to, we will continue to use the term 'pseudo-natural-transformation' for such a morphism between 'pseudo functors', F and G; again see Borceux and Janelidze, [39], for a discussion of pseudo-functors, etc. at about the level of these notes.)

Now with $W \overset{j}{\to} V \overset{i}{\to} U$, we can stack two of these squares, one on top of the other,

$$\begin{tikzcd} F(U) \arrow[r, "\varphi_U"] \arrow[d, "F(i)"'] & G(U) \arrow[d, "G(i)"] \\ F(V) \arrow[r, "\varphi_V"'] \arrow[d, "F(j)"'] & G(V) \arrow[d, "G(j)"] \\ F(W) \arrow[r, "\varphi_W"'] & G(W) \end{tikzcd}$$

plus 2-cells.

We can arrange this as a prism with base the α_{ij}-square and with two τ-triangles, one for F, one for G, on the ends. If we work out how these 2-cells paste together, we find (i) there are 5 faces to the prism, and (ii) we get two possible composites of 'whiskered' 2-cells, namely those in

[4]Of course, in *Open*(B) such a morphism is an inclusion of the open set, V, into U.

the compatibility condition. We are working within Cat, as a domain 2-category, for all our F, Gs, etc., so there are only identity 3-cells around and our prism must commute. (If we had started with $F : Open(B)^{op} \to 2{-}\mathsf{Cat}$, then we would ask for an invertible 3-cell within the prism as part of our data - a 3-cocycle type structure. Clearly this gives a simplicial appearance to the cocycle condition and a suggested intuition for higher dimensions.)

9.3.3 More structure: 2-cells, equivalences, etc.

There is clearly a category of fibred categories on B with the evident objects and morphisms, but as the morphisms are themselves (families of) functors, we can almost certainly go one stage further and get a 2-category of fibred categories on B, or, more generally, on any (small) category, $\mathcal{C}$. Let us try to see how this would go.

For two fibred categories, F and G, over B, a morphism, $\phi : F \to G$, had component functors, $\phi_U : F(U) \to G(U)$, together with for each $i : V \to U$, a natural isomorphism,

$$\alpha_i : \phi_V i^* \xrightarrow{\cong} i^*\phi_U,$$

satisfying a coherence condition on composites.

If we have $\phi, \psi : F \to G$, two such morphism then we could clearly look for families of natural transformations,

$$\omega_U : \phi_U \to \psi_U,$$

where clearly we should expect some compatibility with the α_i of ϕ and the corresponding β_i of ψ. The obvious sort of condition is the commutativity of

$$\begin{tikzcd}
\phi_V i^* \arrow[r, "\alpha_i"] \arrow[d, "\omega_V i^*"'] & i^*\phi_U \arrow[d, "i^*\omega_U"] \\
\psi_V i^* \arrow[r, "\beta_i"'] & i^*\psi_U
\end{tikzcd}$$

There is also the interaction between the ω and the two structural 2-cells, τ^F and τ^G. (**Draw a few diagrams to see what fits where.**) The key condition then looks to be simply

$$\omega_W \cdot \tau^F = \tau^G \cdot \omega_U$$

for a composition $W \to V \to U$, as before.

We thus have a candidate for an analogue of natural transformations between morphisms of fibred categories. We will simple refer to them as *2-arrows*. The following proposition is now fairly easy to prove:

Proposition 91 *Composition of the component natural transformation of 2-arrows is compatible with the side conditions and gives a 2-category,* $\mathsf{FibCat}(B)$, *of fibred categories on* B. ∎

The detailed checking is **left to you**. (Explicit lists of axioms for 2-categories can be found in very many places in the literature[5], and so we have not given them here.) We can either think of

[5] a useful one being Steve Lack' s companion, [182]

this as being instantiated by $\mathsf{Fib}(\mathcal{B})$ or by $\mathsf{FibCat}(\mathcal{B})$. We will tend to use $\mathsf{FibCat}(\mathcal{B})$ when talking generalities, but as they are equivalent it makes not much difference.

We will shortly need to use equivalences of fibred categories and there are two natural types that present themselves.

Definition: A morphism, $\varphi : F \to G$, (or, more precisely, (φ, α)), is called a *strong equivalence* if every φ_U is an equivalence of categories.

It is a *weak equivalence* if every φ_U is fully faithful and 'locally surjective on objects'.

By this latter condition, we mean that for every object $a \in G(U)$ and $x \in U$, there is a $V \xrightarrow{i} U$ with $x \in V$, and a $b \in F(V)$ such that $\varphi_U(b) \cong i^*(a)$ in $G(V)$. The morphism is, thus, 'essentially surjective on objects (eso) *after refinement*'.

9.3.4 The fibred Yoneda lemma

To start our investigation of Yoneda type results in a fibred setting, we will look a simple example that is an excellent one for giving intuition about splittings, etc. We will then use this example to explore general fibred categories. We take this route through the theory here, as we are using Streicher's notes on Bénabou's ideas, in [254] as one of the recommended sources. When we have finished our exploration of this fibred case, we may discuss the '2-Yoneda lemma' corresponding, in part, to the pseudo-functor approach to fibred categories, however both from a historical viewpoint (as Bénabou's lectures at Louvain-la-Neuve date from about 1980) and from the more geometric toolkit that the fibrational viewpoint gives one, we will start with this form[6].

We assume given a category, $\mathcal{B}$, as before, then for any object b in $\mathcal{B}$, we can form a special type of comma category[7] , $\mathcal{B} \downarrow b$, also thought of as the *over category* or *slice category*, $\mathcal{B}/b$. As we will need a bit of precision and notation here and have previously assumed some acquaintance with the idea, let us briefly recall what this looks like. The objects of $\mathcal{B} \downarrow b$ are the morphisms, $x : c \to b$ in $\mathcal{B}$, having the object b as their codomain, and a morphism, φ, from $x' : c' \to b$ to x is a morphism, $\varphi : c' \to c$, in $\mathcal{B}$ making

$$\begin{tikzcd} c \arrow[rr, "\varphi"] \arrow[dr, "x'"'] & & c \arrow[dl, "x"] \\ & b & \end{tikzcd}$$

commute, so that $x' = x\sharp_0\varphi$. (We use $\sharp_0$ to denote the composition in $\mathcal{B}$ in those instances in which in later work higher dimensional categories, together with their higher compositions, $\sharp_k$, may be needed.) There is a natural functor, $dom : \mathcal{B} \downarrow b \to \mathcal{B}$, given by $dom(x : c \to b) = c$, whilst $dom(\varphi : x' \to x) = (\varphi : c' \to c)$. (Note that the use of φ both for morphism in $\mathcal{B} \downarrow b$ and for the corresponding morphism in $\mathcal{B}$ *can be confusing*, so using the fuller form $\varphi : x' \to x$, etc, can be very useful.)

[6] We note that another useful source for this is in Vistoli's notes, [270].

[7] If you have not met the idea of 'comma category' in its full generality, it will be considered later on page 698, but we do not need that generality yet, although as it is an important concept in elementary category theory (and is not that hard to fathom), it is suggested that any reader who has not yet met this idea, search it out in the literature or on the internet.

Having outlined the construction of $\mathcal{B} \downarrow b$ and *dom*, we start our investigation of $dom : \mathcal{B} \downarrow b \to \mathcal{B}$ itself, by looking at its fibres. To simplify notation here, we will usually write p for *dom*, sometimes recording the b as well, so writing $p_b : \mathcal{B} \downarrow b \to \mathcal{B}$ if more than one p is being considered. If b' is an object of $\mathcal{B}$, then $p^{-1}(b')$ is the subcategory of $\mathcal{B} \downarrow b$ having as objects the morphisms from b' to b, and, if $x : b' \to b$, and $y : b' \to b$ are two such objects, a morphism $\varphi : x \to y$ will be in $p^{-1}(b')$ if, and only if, $p(\varphi)$ is the identity on b', but in that case $y = x$ and φ is the identity morphism on x. We thus have that $p^{-1}(b')$ is the discrete category on the set $\mathcal{B}(b', b)$. We will return to this shortly, but now ask if $dom : \mathcal{B} \downarrow b \to \mathcal{B}$ a fibration. If it is, then it will be a discrete fibration, of course.

What would be the Cartesian arrows? For that matter, what does the defining property of Cartesian arrows look like in this special situation. (We could jump straight to the fact that the fibres of p are discrete and argue from there, but will go 'the long way around' as that, hopefully, will help to see how simple things are in this setting.) We start with $\varphi : x' \to x$, so we picture it as

$$\begin{tikzcd} c' \arrow[rr, "\varphi"] \arrow[dr, "x'"'] & & c \arrow[dl, "x"] \\ & b & \end{tikzcd}$$

If φ is to be Cartesian, then for any other arrow $\psi : y \to x$, with $y : c'' \to b$ in $\mathcal{B} \downarrow b$, and a factorisation of $\psi : c'' \to c$ in the form

$$\begin{tikzcd} c'' \arrow[dr, "g"'] \arrow[drr, "\psi"] & & \\ & c' \arrow[r, "\varphi"'] & c \end{tikzcd}$$

we have to be able to lift g to a unique $\gamma : y \to x'$, satisfying $\psi = \varphi\sharp_0\gamma$, and given the specific situation here, this means that γ has to 'be' g, which must, thus, be over b, *i.e.*, $y = g\sharp_0 x'$.

We next suppose that we are given some $x : c \to b$ and $f : c' \to c$. We need some Cartesian arrow $\varphi : x' \to x$ with $p(\varphi) = f$. That means that φ must of the form

$$\begin{tikzcd} c' \arrow[rr, "\varphi"] \arrow[dr, "x'"'] & & c \arrow[dl, "x"] \\ & b & \end{tikzcd}$$

with that top arrow being f. The only thing left to find is x', but that has to be the composite $x\sharp_0 f$ as it has to make the triangle commute! Is this φ a Cartesian arrow? Yes, by Proposition 90, but if you would prefer to see this directly than assume given ψ and a factorisation in the same notation as above. From our earlier discussion, we have to check that $x'\sharp_0 g = y$, but we know $y = x\sharp_0\psi = x\sharp_0(f\sharp_0 g) = (x\sharp_0 f)\sharp_0 g = x'\sharp_0 g$ as required.

We thus have checked, in a lot of (simple) detail that:

Proposition 92 *The functor, $dom : \mathcal{B} \downarrow b \to \mathcal{B}$, is a discrete fibration.* ■

We will also denote this discrete fibration by $P_b := (\mathcal{B} \downarrow b, p_b)$.

Definition: A fibration or fibered category over $\mathcal{B}$ is said to be *representable* if it is equivalent to a fibred category of the form, P_b, for some b in $\mathcal{B}$.

In terms of other pieces of notation that we have used earlier: If $x : b' \to b$ and $f : c' \to b' = p(x)$, then

$$Cart(f, x) : f^*(x) \to x$$

is just

$$Cart(f, x) : (x\sharp_0 f : c' \to b) \to (x : b' \to b),$$

given by pre-composition with f. (This will be useful in the next stage of our exploration.)

Remark: Recall that the discussion that follows here is *very closely related* to parts of our treatment of groupoid coverings (starting in section 4.5.14) and slightly earlier to the idea of 'local systems', also on page 183, whilst discussing complexes of groups and related ideas.

As the above was a discrete fibration, it will correspond to a functor (not just a pseudo-functor) from $\mathcal{B}^{op}$ to $Sets$, that is, a presheaf on $\mathcal{B}$. Clearly we would expect that functor to be simple to describe given the simplicity of the situation we are looking at, and it is. As might be guessed from our calculation of the fibre $p^{-1}(b') = \mathcal{B}(b', b)$, the corresponding functor is the representable functor, $Y_{\mathcal{B}}(b) = \mathcal{B}(-, b) : \mathcal{B}^{op} \to Sets$. This is part of the Yoneda embedding set-up, where, of course, one also varies b to obtain $Y_{\mathcal{B}} = Yon_{\mathcal{B}} : \mathcal{B} \to [\mathcal{B}^{op}, Sets]$. That suggests that we should investigate varying the b in the comma category description as well.

Suppose $u : b \to c$ in $\mathcal{B}$, then we get an induced functor, $\mathcal{B} \downarrow u : \mathcal{B} \downarrow b \to \mathcal{B} \downarrow c$, given by 'post-composition', so, if $x : b' \to b$ is in $\mathcal{B} \downarrow b$, $u\sharp_0 x : b' \to c$ is in $\mathcal{B} \downarrow c$, and, as this just shifts the codomain along u, it is compatible with the two domain functors from $\mathcal{B} \downarrow b$ and $\mathcal{B} \downarrow c$ to $\mathcal{B}$, *i.e.*, , $\mathcal{B} \downarrow u$ is a functor *over* $\mathcal{B}$ from $p_b : \mathcal{B} \downarrow b \to \mathcal{B}$ to $p_c : \mathcal{B} \downarrow c \to \mathcal{B}$. Of course, we should see if $\mathcal{B} \downarrow u$ is a Cartesian functor. In fact, it is, and it is a split one, which is not hard to show as follows. (For later use, we will write P_u for this functor.)

As $Cart(f, x) : (x\sharp_0 f : c' \to b) \to (x : b' \to b)$ is 'pre-composition' with f,

$$\begin{tikzcd} c' \arrow[rr, "f"] \arrow[dr, "x\sharp_0 f"'] & & b' \arrow[dl, "x"] \\ & b & \end{tikzcd}$$

we have $P_u(Cart(f, x))$ is this 'triangle' post-composed with $u : b \to c$, thus giving

$$\begin{tikzcd} c' \arrow[rr, "f"] \arrow[dr, "u\sharp_0(x\sharp_0 f)"'] & & b' \arrow[dl, "u\sharp_0 x"] \\ & c & \end{tikzcd}$$

but this is exactly $Cart(f, P_u(x))$ by the associative law. We have

Proposition 93 *Given $u : b \to c$, $P_u : P_b \to P_c$ is a split Cartesian functor.* ■

We earlier noted that $\mathsf{FibCat}(\mathcal{B})$ was a generalisation of the 2-category Cat, as that was the case in which $\mathcal{B}$ was the terminal category having just one object, $*$, and the identity arrow on that, and nothing else. It is quite fun, but also, in fact, quite instructive, to see what $\mathcal{B} \downarrow b$ looks like in that case. Of course, the only object of $\mathcal{B} \downarrow b$ is $id_* : * \to *$, as b must be $*$. The only morphism is, again, the identity arrow from id_* to itself. You may think that it is silly to point this out as it is self-evident, - and it is - *but* at the same time, remember that in the proof of the standard Yoneda lemma, one examines $Nat(\mathcal{B}(-,b), F)$ and the proof, in part, hinges on the fact that, really, we know very little about $\mathcal{B}(-,b)$, but *we do know that* $\mathcal{B}(b,b)$ *is non-empty*, because id_b is in there! (That is almost all we do know in general.) We therefore use that to get for each natural transformation from $\mathcal{B}(-,b)$ to F a corresponding element of $F(b)$, and so on. In other words in such natural, and universal arguments the role of the identity arrow on an object is crucial. Here a closely related idea is apparent in section 4.5.14 where representable local systems and coverings of groupoids were discussed in some detail. We leave it **to the reader** to compare the groupoid case with the more general one being considered here.

In our fibred situation, we have not yet seen the identity arrows play any such crucial role, so we next look at P_b and the object $(id_b : b \to b)$. After all, it is the only object that we can count on to exist in $\mathcal{B} \downarrow b$. As $(id_b : b \to b)$ is the terminal object, sometimes denoted $\perp$, of $\mathcal{B} \downarrow b$, if we are given $x : b' \to b$, there is a unique morphism from x to $\perp$, and, of course, it is x in another guise:

$$\begin{tikzcd} b' \arrow[rr, "x"] \arrow[dr, "x"'] & & b \arrow[dl, "id_b"] \\ & b & \end{tikzcd}$$

and this is $Cart(x, id_b)$, so every arrow of $\mathcal{B}$ yields a Cartesian arrow of some P_b.

We will shortly look at the Cartesian functors from P_b to an arbitrary fibration, but first let us suppose we have P_b and P_c and $F : P_b \to P_c$. We saw above that any $u : b \to c$ would induce such a functor. This is less than half the story however. Recapping the Yoneda lemma, and its proof, as we know well, this states that, for a category $\mathcal{C}$, and object X of $\mathcal{C}$ and a functor, $F : \mathcal{C}^{op} \to Sets$, the set of natural transformations from $Y_{\mathcal{C}}(X) = \mathcal{C}(-,X)$ to F precisely corresponds to the set , $F(X)$. We recall that, in the proof, $\eta : \mathcal{C}(-,X) \to F$ is sent to $\eta_X(id_X)$, which is an element of $F(X)$. In particular, we have, for $F = \mathcal{C}(-,Y)$, that

$$Nat(\mathcal{C}(-,X), \mathcal{C}(-,Y)) \cong \mathcal{C}(X,Y).$$

Returning to the case of the discrete fibration, $P_b = (\mathcal{B}\downarrow, p_b)$, which is representable by $\mathcal{B}(-,b)$, we can read off that $\mathsf{Fib}(P_b, P_c)$ should be the discrete category on $\mathcal{B}(b,c)$ and hence what we saw above corresponded to the 'forward' part of a bijection. If $F : P_b \to P_c$ is a Cartesian functor, then, for any $x : b' \to b$ in $\mathcal{B} \downarrow b$, $F(x) : b' \to c$, since $p_c F = p_b$. In particular, $F(id_b)$ is an object in $\mathcal{B} \downarrow c$ is a morphism $u : b \to c$ (and, of course, this will be the morphism over in the other half of what we would expect to be a bijection!). We have a Cartesian arrow, $Cart(id_b, x) : x \to id_b$ and F preserves Cartesian arrows. (In fact, as the two fibrations are discrete, the Cartesian functor, F, is automatically 'split Cartesian', but we will go on not using that directly.) We thus have $F(Cart(id_b, x)) : F(x) \to F(id_b)$ is Cartesian, and remember that that codomain is u, so $F(x) = u\sharp_0 x$, as expected, and $F = P_u$. (We should check that F is the same as P_u on morphisms, but will leave this **for the reader**.) We record this as:

Proposition 94 *For b, c, objects in $\mathcal{B}$, the assignment $u \mapsto P_u$ gives a bijection between $\mathcal{B}(b,c)$ and $\mathsf{Fib}(P_b, P_c)$.* ■

Suppose now that we have an arbitrary fibration, $p : \mathcal{E} \to \mathcal{B}$, as always 'over $\mathcal{B}$', and $F : P_b \to (\mathcal{E}, p)$ is a Cartesian functor, then $F(id_b)$ is an object, e, say, in the fibre $\mathcal{F}_p(b)$. Suppose we have $f : b' \to b$ in $\mathcal{B}$, then, thought of as an object of $\mathcal{B} \downarrow b$, $F(f)$ is an object, e', in the fibre, $\mathcal{F}_p(b')$, over $b' = p(f)$. We have, therefore, a Cartesian arrow, $F(Cart(id_b, f)) : e' \to e$, which is a Cartesian lifting of e along f. Conversely, given any object, e, in $\mathcal{E}$, set $b = p(e)$, then for any $f : b' \to b$ in $\mathcal{B}$, any Cartesian lifting of e along f provides the data for a unique Cartesian functor, $F : P_b \to (\mathcal{E}, p)$. We have Cartesian liftings correspond to Cartesian functors with representable domains. There is an obvious functor,

$$E_{\mathcal{E},b} : \mathsf{Fib}(\mathcal{B})(P_b, \mathcal{E}) \to \mathcal{F}_p(b),$$

namely that sending F to $F(id_b)$, as, if $\tau : F \to G$ is in $\mathsf{Fib}(\mathcal{B})(P_b, \mathcal{E})$, then $\tau_{id_b} : F(id_b) \to G(id_b)$ is a morphism in the category $\mathcal{F}_p(b)$.

Proposition 95 *(Fibred Yoneda Lemma) The functor, $E_{\mathcal{E},b} : \mathsf{Fib}(\mathcal{B})(P_b, \mathcal{E}) \to \mathcal{F}_p(b)$, is an equivalence of categories.* ∎

We have given the basic ideas above so the details of the proof are, once again, **left to the reader**.

We note that that proof is essentially just a 2-categorical form of that of the Yoneda lemma and this aspect is at the fore of Vistoli's treatment in [270], which is worth consulting as well.

If we have $u : b \to c$ in $\mathcal{B}$, then we know $P_u : P_b \to P_c$ and hence

$$\mathsf{Fib}(\mathcal{B})(P_u, \mathcal{E}) : \mathsf{Fib}(\mathcal{B})(P_c, \mathcal{E}) \to \mathsf{Fib}(\mathcal{B})(P_b, \mathcal{E}),$$

so, apart from some questions on size (*i.e.*, do we need to use a bigger category than Cat?), we have a presheaf of categories,

$$\mathsf{Sp}(\mathcal{E}) : \mathcal{B}^{op} \to Cat,$$

defined by $\mathsf{Sp}(\mathcal{E})(b) = \mathsf{Fib}(\mathcal{B})(P_b, \mathcal{E})$ with $\mathsf{Sp}(\mathcal{E})(u) : \mathsf{Sp}(\mathcal{E})(c) \to \mathsf{Sp}(\mathcal{E})(b)$, the functor given by pre-composition with P_u, as above. This, of course, corresponds to a fibration, $\mathcal{S}p(\mathcal{E})$ over $\mathcal{B}$. This fibration, as it corresponds to a *functor*, is split.

The functors, $E_{\mathcal{E},b}$, combine to give a morphism, $E_{\mathcal{E}} : \mathcal{S}p(\mathcal{E}) \to \mathcal{E}$, and, as the components of $E_{\mathcal{E}}$ are equivalences of categories, $E_{\mathcal{E}}$ is an equivalence in the 2-category, $\mathsf{Fib}(\mathcal{B})$. We thus have that every fibred category is equivalent to a split one, and in fact the construction $\mathcal{S}p$ is clearly 2-functorial and forms a right 2-adjoint to the 2-functor from $\mathsf{Sp}(\mathcal{B})$ to $\mathsf{Fib}(\mathcal{B})$, that forgets the splitting. This aspect, and a statement and proof of a strong form of Fibred Yoneda Lemma is to be found as Theorem 3.1. of Streicher's Notes, [254], so will not be given here.

9.3.5 An interlude

We saw (page 402) that there were non-splittable fibrations, and the example we gave was of a non-split epimorphism, $\theta : H \to G$, of groups, considered as a fibration of groupoids, $\theta[1] : H[1] \to G[1]$. We now have that any fibration is equivalent to a split fibration. Largely because the calculation is quite fun, and hence revealing, although largely elementary, we will pause to look at what the theory that gave us the split fibration gives for a group epimorphism. This clearly cannot give us back a 'split' epimorphism of groups, but what does it give?

We will first work through the construction that we gave in the previous section with $\mathcal{B} = G[1]$, for a group, G. In such a $\mathcal{B}$, there is only one object, that we have denoted by $*$, and an object of

$\mathcal{B} \downarrow *$ is just an arrow, $* \xrightarrow{g} *$ in $\mathcal{B}$, ..., so is 'just' an element $g \in G$. An arrow, $g_1 \to g_2$, between two such objects is a commutative triangle,

$$\begin{tikzcd} * \arrow[rr, "g"] \arrow[dr, "g_1"'] & & * \arrow[dl, "g_2"] \\ & * & \end{tikzcd}$$

in $\mathcal{B}$, so 'is' an element, g, of G satisfying[8] $g_1 = g_2 g$, hence there is only one namely that corresponding to $g_2^{-1} g_1$. We note that, as we might expect, $\mathcal{B} \downarrow *$ is a groupoid. One extra detail is that the identity, $id_* = 1_G : * \to *$, gives a distinguished object if we need one.

We have seen this thing before. It is the codiscrete groupoid, $Codisc(X)$ on X, where, as before, we will sometimes write X for $U(G)$ in the context of the standard presentation of a group. Its underlying quiver is the Cayley quiver, $Cay(G, X)$, for that standard presentation; cf. page 53.

We know that $\mathcal{B} \downarrow *$ is only one part of the fibred category, P_*. We also have the domain functor

$$p = dom : (\mathcal{B} \downarrow *) \to \mathcal{B}.$$

Clearly $dom(* \xrightarrow{g} *) = *$, so it maps all objects in the only possible way, to the unique object of $\mathcal{B} = G[1]$, and $g_1 \xrightarrow{g} g_2$ goes to $* \xrightarrow{g} *$. We thus have that dom is, in fact, the groupoid morphism:

$$Codisc(X) = Cay(G, X) \to G[1],$$

that we met back on page 53.

The next step in showing the way the structures in our study of fibred categories generalise well known ideas from quite classical areas is to look at Cartesian functors. In our previous general scenario, we had the possibility of comparing the fibred categories, P_b, for different objects, b, but here we just have P_*, so will look at Cartesian functors, $F : P_* \to P_*$. Of course, by Proposition 94, we know that these correspond to elements in $\mathcal{B}$, and that is equal to G, with the composition being the multiplication in G. How does this work in terms of more classical structures?

Suppose that $g : * \to *$ in $\mathcal{B}$ (*i.e.*, g is an element of G). We get an induced functor, $\mathcal{B} \downarrow g : \mathcal{B} \downarrow * \to \mathcal{B} \downarrow *$, which sends $g_1 : * \to *$ to $gg_1 = g\sharp_0 g_1$, and remember we are using functional composition convention here. This is, thus, 'post-multiplication' or, with our convention, the left action of G on the set of elements of G.

We have that, if $x : * \to *$ is an object of $\mathcal{B} \downarrow *$ and $g : * \to *$ is a morphism in the base $\mathcal{B}$, ..., and, yes, because of the special situation we are looking at both 'are' elements of G, masquerading as different things! The Cartesian arrow, $Cart(g, x)$, is the triangle

$$\begin{tikzcd} * \arrow[rr, "g"] \arrow[dr, "xg"'] & & * \arrow[dl, "x"] \\ & * & \end{tikzcd}$$

so, as left actions commutes with right actions by associativity, P_g is a Cartesian functor.

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[8]Beware: as we are specialising from the general case of fibred *categories*, here we are using the functional composition convention continuing that use from the previous discussion. It is useful to keep that in mind, as some of the formulae we will obtain may look the wrong way around.

Conversely, suppose $F : P_* \to P_*$ is a Cartesian functor, then $F(1_G : * \to *)$ is some object, $u : * \to *$, of $\mathcal{B} \downarrow *$, and, as F is Cartesian, $F(x) = u\sharp_0 x$, (*i.e.*, ux), and F is the left action by u. In this situation, we thus get that the isomorphism

$$\mathsf{Fib}(P_*, P_*) \cong \mathcal{B}(*, *)$$

is related, not only to the Yoneda embedding and to the Cayley embedding[9] of G into the symmetric group, $Sym(X)$, for $X = U(G)$, but also to the identification of G as the symmetry group of $Cay(G, X)$, which is essentially the proof of Frucht's theorem that every (finite) group is the group of symmetries of a (finite) undirected graph[10].

Now consider the general situation of a group epimorphism, $\theta : H \to G$, (and the resulting functor $\theta[1] : H[1] \to G[1]$). We will be taking $\mathcal{E} = H[1]$, $\mathcal{B} = G[1]$ and $p = \theta[1]$ and plugging them into the discussion from the previous section, but before that it will help to take apart most of the ideas: *fibrations*, *cleavage*, *Cartesian arrows*, etc. and to see what they look like in this very special context[11].

To start with let us return to Lemma 59 and check that all arrows in $H[1]$ are Cartesian for this fibration[12], We look back to the definition on page 398. Suppose given $\varphi : e' \to e$ in $\mathcal{E}$, then this is just some $h : * \to *$ in $H[1]$. It will be Cartesian if, given any other $\psi : e'' \to e$, that is some $h' : * \to *$, and a factorisation of $\theta(h')$ through $\theta(h)$, (but according to the diagram in the definition this just says the we have some $g \in G$ such that $\theta(h') = \theta(h)g$, which, of course, means $g = \theta(h)^{-1}\theta(h')$, so there is only one factorisation possible and its existence is not in doubt), we have to find some $\chi : e'' \to e'$, that is, some $h'' : * \to *$ such that $\psi = \varphi\sharp_0\chi$, and $g = p(\chi)$, but that translates as $h' = h.h''$ and $\theta(h'') = g$. We then clearly must have that $h'' = h^{-1}h'$. We thus have that h was a Cartesian arrow.

We note that this does not depend on θ being an epimorphism. That condition is to ensure that any $g : * \to *$ *has* a lift to $H[1]$ and so that $\theta[1]$ is a fibration.

We next turn to 'cleavage'. Suppose that $\mathcal{K}$ is a cleavage for $p = \theta[1]$, then we have, for each object e in $\mathcal{E}$ (but there is only one, $*$), and $f : b \to p(e)$ (and so $f : * \to *$ in $G[1]$, *i.e.*, f is some $g \in G$), there is a unique Cartesian lift over f in $\mathcal{K}$, *i.e.*, for each $g \in G$, the cleavage is a choice of an $h \in H$ such that $\theta(h) = g$. We thus have that $\mathcal{K}$ is actually just the image of a section $\sigma : G \dashrightarrow H$ and specifying $\mathcal{K}$ is equivalent to specifying σ. (The cleavage is a splitting if $1_H \in \mathcal{K}$ and , if $h_1, h_2 \in \mathcal{K}$, then $h_1h_2 \in \mathcal{K}$, – but in the language of sections, this just says $\sigma(1_G) = 1_H$ and $\sigma(g_1g_2) = \sigma(g_1)\sigma(g_2)$, *i.e.*, that σ is a splitting for θ in the usual group theoretic sense.)

We now choose a cleavage, $\mathcal{K}$, (or a section σ), and take an elementary stroll through the 'cleaved fibrations give pseudo-functors' section (especially Proposition 88). We have set $\mathcal{F}(b)$ to be the fibre of p above an object, b, so, in our special setting, $\mathcal{F}(*)$ is the subgroupoids of $\mathcal{E}$ whose objects are the objects of $\mathcal{E}$ which map down to $*$ and whose arrows, $h : * \to *$, satisfy $\theta[1](h) = (1_G : * \to *)$. It is immediately clear that $\mathcal{F}(*)$ is just $K[1]$, where $K = Ker\,\theta$ is the usual group theoretic kernel of θ. Almost certainly you can predict how the next few parts of

9. Of course, it is well known that the Yoneda embedding is an extension of Cayley's result in general.

10. The finiteness assumption was removed in 1959 - 60, but was in the original version from 1939. The way in which one can replace a labelled directed graph / labelled quiver by an unlabelled undirected one is ingenious, so will be **left to you to dream up or find!** This also links back to covering spaces / covering groupoids, as the domain morphism *is* the labelling of the Cayley quiver, so 'deck transformations' of that preserve the labelling.

11. We have already mentioned some of this before, but will repeat some ideas here.

12. We will repeat several obvious facts, just to emphasise the way Grothendieck fibrations generalises this situation, as well as the geometric and the pullback ones.

the argument playout in this special case. If not, look back at our discussion of (non-Abelian) extensions, factor sets, cocycle conditions, etc. We have an extension,

$$1 \to K \xrightarrow{i} H \xrightarrow{\theta} G \to 1$$

corresponding to the fibration and the pseudo-functor. We can refer back to page 393 to complete the description[13]. We note that the only choice being made when getting a good description of the extensions / fibration is that of the section, (and we have already studied aspects of how that choice effects things, at least for extensions).

Almost finally, we look at a Cartesian functor, $F : P_* \to \mathcal{E}$. We note that for any $g \in G$, we have a Cartesian arrow, $Cart(g, 1_G)$, which is the triangle

$$\begin{tikzcd} * \arrow[rr, "g"] \arrow[dr, "g"'] & & * \arrow[dl, "1_G"] \\ & * & \end{tikzcd}$$

As F is 'over $\mathcal{B}$', $F(Cart(g, 1_G)) : * \to *$ is an arrow, $s(g) : * \to *$ of $H[1]$ with $\theta(s(g)) = g$, (as $\theta[1] \circ F = dom$, so F *is* a section – and thus the groupoid, $\mathcal{B} \downarrow *$ allows us to give a neat identification of a section as a particular type of morphism of groupoids over $G[1]$.

That handles the interpretation of a Cartesian functor, $F : P_* \to \mathcal{E}$, but we saw that there was a category (and here it will be a groupoid) $\mathsf{Fib}(G[1])(P_u, \mathcal{E})$ and so far we have only looked at its objects, namely those Cartesian functors. If g is in G, then we have $P_g : P_* \to P_*$, which we saw ws a Cartesian functor. We then get a presheaf of groupoids,

$$\mathsf{Sp}(\mathcal{E}) : G[1]^{op} \to Grpd,$$

given by $\mathsf{Sp}(\mathcal{E})(*) = \mathsf{Fib}(G[1])(P_u, \mathcal{E})$ with $\mathsf{Sp}(\mathcal{E})(g)$ being pre-composition with P_g. This then gives a category[14], $Sp(\mathcal{E})$, fibred over $G[1]$, of course, but the *groupoid*, $Sp(\mathcal{E})$, has an object for each section of θ, so is a groupoid, *not* 'a group'. The fibration, $Sp(\mathcal{E})$, is split, as it corresponds to a *functor* not just a pseudo-functor.

Examining this in a bit more detail, if we have two Cartesian functors, F_1 and $F_2 : P_* \to \mathcal{E}$, and $\eta : F_1 \to F_2$ is a natural transformation, then, naturally, for each $x : * \to *$ in $\mathcal{B} \downarrow *$, we have a morphism $\eta(x) : F_1(x) \to F_2(x)$, but both $F_1(x)$ and $F_2(x)$ are the single object, $*$, of $\mathcal{E}$, so this $\eta(x)$ is 'just' an element of $H[1]$, and η will be a Cartesian transformation if all the $\eta(x)$ are in $K[1]$. As η is to be natural, if we have a morphism, $g : x_1 \to x_2$ in $\mathcal{B} \downarrow *$, (so $x_1 = x_2 g$), then the diagram

$$\begin{tikzcd} F_1(x_1) \arrow[r, "\eta(x_1)"] \arrow[d, "F_1(g)"'] & F_2(x_1) \arrow[d, "F_2(g)"] \\ F_1(x_2) \arrow[r, "\eta(x_2)"'] & F_2(x_2) \end{tikzcd}$$

The generality of this setting makes things look more complicated than they really are, so let us try to simplify things without throwing away the essentials. We first note a useful if trivial consequence of $x_1 = x_2 g$ and that is that $g : x_1 \to x_2$ factorises as $x_1 \xrightarrow{x_1} 1_G \xrightarrow{x_2^{-1}} x_2$ as that

[13] but with the reversed notation giving a left action rather than $F : G[1]^{op} \to Cat$ as here.

[14] in fact a groupoid

composite is $x_1 \xrightarrow{x_2^{-1}x_1} x_2$, *i.e.*, $x_1 \xrightarrow{g} x_2$. This means that specifying a functor can just be done on morphisms of the form $x \xrightarrow{x} 1_G$, so in the above square we can restrict attention to the case in which $x_2 = 1_G$, and we may as well simplify notation and write x for that x_1. This now gives us $\eta(x) = F_2(x)^{-1}\eta(1_G)F_1(x)$, and we have that this is completely determined by $\eta(1_G)$. We have that if η is Cartesian then $\eta(1_G)$ will be in $K[1]$, and conversely, since $F_1(x)$ and $F_2(x)$ both project down to x, they differ by a vertical arrow, *i.e.*, an arrow of $K[1]$ and, as $K \lhd H$, the above formula for $\eta(x)$ shows this to be in $K[1]$.

We thus have that in this 'classical' situation of group extensions, the theory of fibred categories gives some interesting insights. Of course, the classical theory leads into non-Abelian cohomology, so we can expect a similar connection with non-Abelian cohomology in general, and, of course, that was one of the origins of the theory of stacks which we will be meeting shortly.

9.3.6 Change of base for fibrations / fibred categories? First thoughts.

We have seen that fibred categories have interpretations both as lax-, or pseudo-functors from some category $\mathcal{B}^{op}$ to Cat, but also as a particular type of 'fibration of categories', $\mathcal{E}_F \to \mathcal{B}$. We have seen that we can 'change base' along maps in various contexts, where fibrations, or bundles, or sheaves, occur, and it will be useful to be able to do this for 'fibred categories'. Perhaps a pullback type construction, or, as for sheaves, just composition would give that.

Let us start with $f : \mathcal{A} \to \mathcal{B}$, a functor, then there is a corresponding 'opposite', $f^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$ and so, if $F : \mathcal{B}^{op} \to \mathsf{Cat}$ is a pseudo-functor / fibred category, the composite $F \circ f^{op} : \mathcal{A}^{op} \to \mathsf{Cat}$ will likewise be a pseudo-functor / fibred category, this time over $\mathcal{A}$ rather than $\mathcal{B}$.

Remark: In the spatial context, a continuous map, $\varphi : B_1 \to B_2$, gives a functor in the opposite direction on the categories of open sets,

$$\varphi^t : Open(B_2) \to Open(B_1),$$

$$\varphi^t(V) = \varphi^{-1}(V),$$

in the usual sense of topology[15], so we get a way of building a fibred category over B_2 given one on B_1, completely analogous to the situation with (pre)sheaves[16]. Of course, that raises the question as to the 2-functoriality of the induced 'functor' between $\mathsf{FibCat}(B_1)$ and $\mathsf{FibCat}(B_2)$, and whether or not there are '2-adjoints'. There are, and this *will be* very useful. We will handle it in general before briefly returning to this spatial context.

We can look at this categorically and also in terms of fibrations. Suppose we have a general Grothendieck fibration, $P = (\mathcal{E}, p)$ over $\mathcal{B}$ and $f : \mathcal{A} \to \mathcal{B}$ is an ordinary functor, then we define $f^*(P)$ to be given by the pullback (in Cat):

$$\begin{tikzcd}
f^*(\mathcal{E}) \arrow[r, "k"] \arrow[d, "f^*(p)"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{A} \arrow[r, "f"'] & \mathcal{B}
\end{tikzcd}$$

[15] we use a superfix t to stand for transpose, to avoid an overuse of the $^{-1}$ notation and we use φ for the map to avoid over use of f!

[16] that we saw in section 7.3.8

This means that $f^*(\mathcal{E})$ has, as its objects, pairs, (a, e), with a an object of $\mathcal{A}$ and e, one of $\mathcal{E}$ and with $f(a) = p(e)$; similarly its arrows have form (u, φ) with $u : a \to a'$, $\varphi : e \to e'$ and $f(u) = p(\varphi)$. Of course, $f^*p(a, e) = a$ and $k(a, e) = e$, so the fibre $(f^*(p))^{-1}(a)$ is the subcategory of $f^*(\mathcal{E})$ having first 'variable' a and with $p(e) = id_{f(a)}$, etc. It is then not hard to show that, if $\mathcal{E} = \mathcal{E}_{\mathsf{F}}$ for a pseudo-functor, $\mathsf{F} : \mathcal{B}^{op} \to Cat$, then the pseudo-functor corresponding to $f^*(\mathcal{E}_{\mathsf{F}})$ will be $F \circ f^{op} : \mathcal{A}^{op} \to Cat$. Similarly, if we have an arrow (u, φ) in $f^*(\mathcal{E})$, then it would be Cartesian with respect to $f^*(p)$ if, and only if, φ is Cartesian with respect to p, *i.e.*, k preserves 'Cartesianness' of arrows. The situation is thus very similar to that for presheaves[17], so we will adopt a similar heading.

Changing the base induces a pair of 2-adjoint 2-functors.

For a functor, $f : \mathcal{A} \to \mathcal{B}$, suppose $f^* : \mathsf{Fib}(\mathcal{B}) \to \mathsf{Fib}(\mathcal{A})$ has a right 2-adjoint, f_*. (There are various interpretations of 'right 2-adjoint' with varying degrees of strictness and direction. Here we will mean that, if we have $\mathcal{X}$ in $\mathsf{Fib}(\mathcal{B})$ and $\mathcal{Y}$ in $\mathsf{Fib}(\mathcal{A})$, then there is a natural equivalence of categories:

$$\mathsf{Fib}(\mathcal{A})(f^*(\mathcal{X}), \mathcal{Y}) \simeq \mathsf{Fib}(\mathcal{B})(\mathcal{X}, f_*(\mathcal{Y})).$$

In fact, that could be taken apart further as the 2-naturality in $\mathcal{X}$ and $\mathcal{Y}$ could be spelt out more, but this level of detail will suffice for the limited use for which we will need this.)

If b is an object of $\mathcal{B}$, then we have the representable fibred category, $P_b = (\mathcal{B} \downarrow b, dom)$, and we can think of $\mathsf{Fib}(\mathcal{B})(P_b, f_*(\mathcal{Y}))$ as being the fibre of $f_*(\mathcal{Y})$ over b. Our assumption that f_* exists means that this is equivalent to $\mathsf{Fib}(\mathcal{A})(f^*(P_b), \mathcal{Y})$. Of course, this working 'up to equivalence' would need spelling out, but remember the values of an adjoint functor are determined by a universal property, so are determined only up to isomorphism, and, likewise, the values of a 2-adjoint 2-functor should be determined only up to equivalence. We thus know what $f_*(\mathcal{Y})$ must look like (up to equivalence) over each b in $\mathcal{B}$, and, by naturality, along each $u : b' \to b$ and thus can give a construction of $f_*(\mathcal{Y})$, initially as a pseudo-functor, but thus equivalently as a fibred category.

Remark: This right adjoint to f^* is sometimes written f_*, but also, as in Streicher, [254], as $\prod_f$ as it generalises constructions such as the product. (For the moment we will not use the left 2-adjoint of f^*, but that is written $\coprod_f$ as it generalises the coproduct, see Streicher, [254], section 14. We will see it shortly.)

Construction and definition: Given $f : \mathcal{A} \to \mathcal{B}$ and $\mathcal{Y}$ in $\mathsf{Fib}(\mathcal{A})$, the fibred category $\prod_f(\mathcal{Y})$, or $f_*(\mathcal{Y})$, is specified by: for $b \in \mathcal{B}$

$$\prod_f(\mathcal{Y})(b) = \mathsf{Fib}(\mathcal{A})(f^*(P_b), \mathcal{Y})$$

and, if $u : b' \to b$, inducing $P_u : P_{b'} \to P_b$, we set

$$\prod_f(\mathcal{Y})(u) = \mathsf{Fib}(\mathcal{A})(f^*(P_u), \mathcal{Y}).$$

The fibred category, $f_*(\mathcal{Y})$ or $\prod_f(\mathcal{Y})$, is said to be obtained by *change of base of $\mathcal{Y}$ along f.*

[17]back on page 302.

Examples: (i) If $\mathcal{B}$ is the terminal category, then, clearly, it has a single object, $*$ and just one morphism which is the identity on $*$. It is then clear that P_* is the identity functor on $\mathcal{B}$, so for $f : \mathcal{A} \to \mathcal{B}$, necessarily the unique such functor, $f^*(P_*)$ is the identity functor from $\mathcal{A}$ to itself considered as a fibration over $\mathcal{A}$. A Cartesian functor F from $f^*(P_*)$ to a fibred category $(\mathcal{E}, p)$ over $\mathcal{A}$ will give a *Cartesian section* of $(\mathcal{E}, p)$ as it fits into a triangle

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "id_{\mathcal{A}}"'] & & \mathcal{E} \arrow[dl, "p"] \\ & \mathcal{A} & \end{tikzcd}$$

This also works for morphisms, so $\prod_f(\mathcal{E})$ will be the category of Cartesian sections of $(\mathcal{E}, p)$. We will return to this shortly in the spatial setting.

(ii) If $f : \mathcal{A} \to \mathcal{B}$ has a right adjoint, $U : \mathcal{B} \to \mathcal{A}$, then $f^*(P_b)$ will be isomorphic to P_{Ub}, and hence $\prod_f \simeq U^*$.

(More to go here I think.)

9.4 Back to the Grothendieck construction ... and lax, op-lax and pseudo things

9.4.1 The Grothendieck construction as a (op-)lax colimit

(In this section, we will collect up some pieces from earlier discussions and look at them from a different perspective, in preparation for their reuse later one.)

The Grothendieck construction is often used to replaces the colimit in situation in which lax, op-lax or pseudo-functors are present. For instance, in the process of 'stackification' for pre-stacks, we cannot use an ordinary colimit as the 'functors' involved are not really functors, they are rather pseudo-functors and are usually definitely not 'strict'. (We will see later that even an 'op-lax' colimit does not quite do the trick and we will need a pseudo-colimit, which is slightly different. However that adapted version will be much easier to understand once the initial step from 'colimit' to 'lax colimit' has been made.)

The Grothendieck construction has the look of a categorified colimit in many ways, so that aspect of it needs some light shed on it. It is also a 'homotopy colimit' in a certain sense. The precise formulation is in Thomason's paper, [258]. We will need that aspect as well since it provides a means of further categorifying stacks and thus of more fully understanding what cohomology is about. That homotopy colimit aspect, though, will require other tools, so will be delayed until later. Here we will examine the Grothendieck construction as a laxified form of colimit. We will not always give fully motivated and formal definitions of ideas such as lax cone and cocone, or op-lax colimits, etc., until a slightly later chapter. Detailed treatments of these notions are better looked up in the literature devoted to the 2-categorical context although summaries will be given in these notes. The complications of lax and op-lax, and whether certain transformations should be invertible or not tends to obscure the basic ideas. We will, later on, often need the homotopy aspect slightly more than the lax one - so, from that point of view, the development here is a step on the way, rather that an end in itself.

First it will pay to look at the definition of the colimit of a functor, or diagram. We will assume that we have a functor, $F : \mathcal{B} \to Sets$, so as to keep things simple. We have, from standard texts on category theory, the idea of a cone and a cocone on a functor, F. As we are concentrating on colimits, we will look at a cocone. There are two equivalent ways of looking at a cocone. The slick way is to say:

Definition (categorical): A *cocone* on F with target, Y, is a natural transformation, η, from F to the constant functor, $cons_Y : \mathcal{B} \to Sets$.

The simplest example is probably to take $\mathcal{B}$ to be the ordered set, [1], *i.e.*, $0 \xrightarrow{(01)} 1$. The diagram, F, then looks like

$$\begin{tikzcd} F(0) \arrow[d, "F(01)"'] \\ F(1) \end{tikzcd}$$

and the cocone with target Y should look something like

$$\begin{tikzcd} F(0) \arrow[d, "F(01)"'] \arrow[dr, "\eta(0)"] & \\ & Y \\ F(1) \arrow[ur, "\eta(1)"'] & \end{tikzcd}$$

Later we will take this apart a bit more and will, as a result, check this is correct.

Definition: A *colimit* for F is a universal cocone for F.

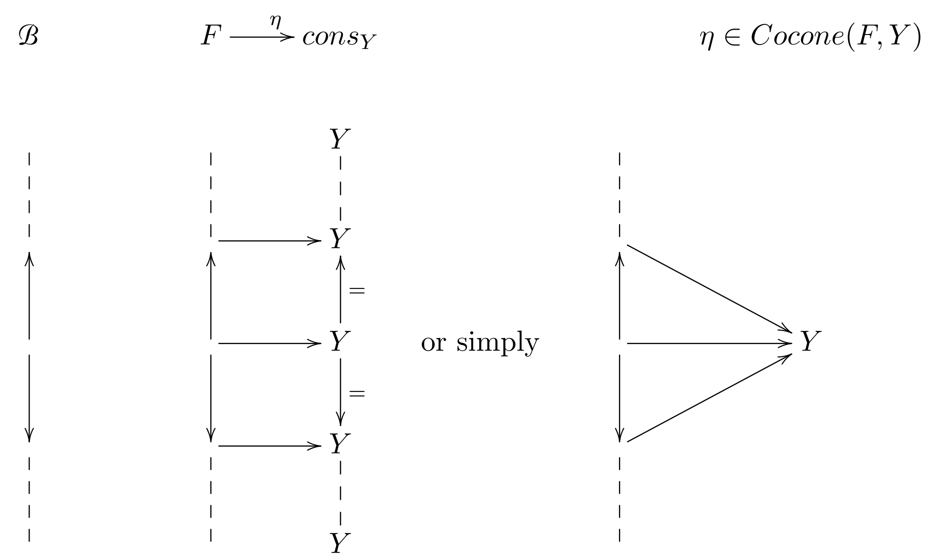
That has also to be taken apart. If $C = colim\, F$, it comes with a universal cocone, $\mu : F \to cons_C$, so that given any other cocone $\eta : F \to cons_Y$, there is a unique $\overline{\eta} : C \to Y$ in $Sets$ such that $\eta = cons_\eta \cdot \mu$.

It will be assumed that you are familiar with this idea, so, if you are not that used to colimits, spend a little time looking at a standard category theory text, concentrating on simple examples of colimits (coproducts, pushouts and coequalisers, in particular). You do not need to know much of the resulting theory in detail, but the *intuition* is very important.

We next will do the usual slightly more detailed deconstruction of 'cocones' as an idea[18]. (The functor, F, was deliberately given with codomain $Sets$ to allow categorification, but also to simplify exposition in certain places. We could, of course, replace it by any category we needed.) We think of functors from a small category as being 'diagrams' indexed by the category, in our case, $\mathcal{B}$. Natural transformations are then 'maps of diagrams', but, if a constant functor is the codomain of the natural transformation, the 'right hand' part of the resulting big diagram is really redundant

[18]This is, of course, standard material, and can be skimmed if you have met it before.

as all the maps in it are identities:



so it is natural to collapse that part of the big diagram to a point. This leads to a 'more elementary' form of the definition.

Definition (more 'elementary'): Given a functor, $F : \mathcal{B} \to Sets$, a cocone, $\eta : F \to Y$, on F is given by a family, $\{\eta(b) : F(b) \to Y \mid b \in Ob(\mathcal{B})\}$, of maps such that, if $f : b \to b'$ in $\mathcal{B}$, the diagram

$$\begin{tikzcd} F(b) \arrow[d, "F(f)"'] \arrow[dr, "\eta(b)"] & \\ F(b') \arrow[r, "\eta(b')"'] & Y \end{tikzcd}$$

commutes.

We will write $Cocone(F, Y)$ for the set of cocones on F with target, Y.

The definition of the colimit then requires there to exist a $\mu : F \to cons_C$ (and in which C will be the 'colimit') such that each $\eta \in Cocone(F, Y)$ corresponds *uniquely* to some $\overline{\eta} : C \to Y$ in $Sets$, *i.e.*, there is a (natural) isomorphism

$$Cocone(F, Y) \cong Sets(C, Y).$$

In other terminology, this requires that $Cocone(F, -) : \mathcal{B} \to Sets$ be a representable functor, represented by the colimit, C.

We can try to 'categorify' this, but must remember that we are not formalising this to any great extent. That can wait until we have a little more machinery available.

We can categorify the notion of cocone fairly easily, modelling the generalisation on simple intuitions. (Beware the intuitions that we will use will not necessarily be 'optimal', or general enough, so may need adjusting later[19]. The main point is to *start* building those intuitions, so as to see if they are adequate, or if they require more 'input'.) Remember in 'categorification', one of the things is to replace 'equality' by explicit 'isomorphism', or 'equivalence', or, at very least, an explicit natural transformation in one or other direction. Also we need to replace *Sets* by *Cat* or, better, by the corresponding 2-category, Cat. We thus expect, in the categorification process of the above ideas on cocones, etc., to have given a functor, $F : \mathcal{B} \to Cat$, or better, an op-lax one, $\mathsf{F} = (F, \tau) : \mathcal{B} \to \mathsf{Cat}$, and that is, of course, exactly the situation for the basic Grothendieck construction. We want an 'op-lax' cocone on F with target some small category, $\mathcal{Y}$. For this we would expect:

- for each $b \in Ob(\mathcal{B})$, a functor, $\eta(b) : F(b) \to \mathcal{Y}$, (and we can think of $\eta(b)$ as a '1-cell' or '1-arrow' if that is helpful);
- for each morphism, $f : b \to b'$, in $\mathcal{B}$, a natural transformation (2-arrow), $\theta_f : \eta(b) \Rightarrow \eta(b')F(f)$, replacing the old 'equality' in our classical cocone[20].

$$\begin{tikzcd}
F(b) \arrow[d, "F(f)"'] \arrow[dr, "\eta(b)"] \arrow[d, phantom, shift left=6ex, "\Downarrow\theta_f"] & \\
F(b') \arrow[r, "\eta(b')"'] & Y
\end{tikzcd}$$

and we will need a compatibility (cocycle!) condition for when we have two composible morphisms in $\mathcal{B}$:

$$b \xrightarrow{f} b' \xrightarrow{g} b''$$

gives us a triangle

$$\begin{tikzcd}
& F(b') \arrow[dr, "F(g)"] & \\
F(b) \arrow[ur, "F(f)"] \arrow[rr, "F(gf)"'] \arrow[rr, phantom, shift left=4ex, "\Uparrow\tau_{g,f}"] & & F(b'')
\end{tikzcd}$$

coming from the fact that F is an op-lax functor. We also have three θ-triangles, namely the one above together with one for g and one for gf. Looking at the corresponding part of the 'cocone', we have

$$\begin{tikzcd}
& F(b') \arrow[ddr] \arrow[drr, "\eta(b')"] & & \\
F(b) \arrow[ur] \arrow[rrr, "\eta(b)"] \arrow[drr] & & & \mathcal{Y} \\
& & F(b'') \arrow[ur, "\eta(b'')"'] &
\end{tikzcd}$$

[19] We will do this to some extent in a summary section looking at lax and pseudo-limits and colimits, (section 11.5.6, but fuller insights require more machinery together with the intuitions around that higher structure.

[20] These triangles will be referred to as being 'θ-triangles', below. This terminology is just for use here in this discussion, and will not be needed later on.

(plus corresponding 2-cells) which will need to commute. The missing 2-cells give

$$\begin{aligned}\tau_{g,f} : F(gf) &\Rightarrow F(g)F(f),\\ \theta_f : \eta(b) &\Rightarrow \eta(b')F(f),\\ \theta_g : \eta(b') &\Rightarrow \eta(b'')F(g),\\ \theta_{gf} : \eta(b) &\Rightarrow \eta(b'')F(gf),\end{aligned}$$

and to say the above tetrahedron 'commutes' is to say that the two possible composite 2-cells from $\eta(b)$ to $\eta(b'')F(g)F(f)$ have to be equal. In other words, the composites,

$$\eta(b) \Rightarrow \eta(b')F(f) \Rightarrow \eta(b'')F(g)F(f)$$

and

$$\eta(b) \Rightarrow \eta(b'')F(gf) \Rightarrow \eta(b'')F(g)F(f)$$

are equal (and **you are left to check** on what the 2-cells are).

For future developments, we note, once again, that 'have to be equal' is true because we are working with 2-categories and so there are no 3-cells or, if you prefer, 'only identity 3-cells'.

We now should expect $Cocone(\mathsf{F}, \mathcal{Y})$ to be a category if our perhaps naive view of 'categorification' is correct. The basic structure above involves the $\eta(b)$ as functors, so $\mathsf{Cocone}(\mathsf{F}, \mathcal{Y})$ should have natural transformations, $\mu(b) : \eta(b) \Rightarrow \eta'(b)$, somewhere around. They would need to be compatible with the θs ..., and **you are left to investigate in the usual way!** It **does** all fit together beautifully[21].

Following the categorification 'mantra', we have replaced sets by categories, so now need to define the op-lax colimit of F (if it exists) to be a 'representing object' for the 'functor' $\mathsf{Cocone}(\mathsf{F}, -)$. Of course, we really need to see how $\mathsf{Cocone}(\mathsf{F}, \mathcal{Y})$ varies with $\mathcal{Y}$. Is it functorial or merely op-lax functorial? We note that if this op-lax colimit exists then it will be determined, not up to isomorphism as with an ordinary colimit, but up to equivalence (but not just any old one). In fact the representing object, that we will call C, is to satisfy

$$\mathsf{Cocone}(\mathsf{F}, -) \simeq \mathsf{Cat}(C, -),$$

not $\cong$. It may be possible, and is often useful, to find a construction giving $\cong$, but that is not exactly what is required. Of course, if we find a C with a $\cong$ in the above, then it clearly also satisfies that with $\simeq$.

This leaves you with lots of **details to provide**. We will not give them as we will attack this later by a different route, firstly via a fuller treatment of lax and pseudo ideas, and then by indexed / weighted limits and colimits[22], but it *is* **worth your playing around** with the concepts, possibly looking up some of the details in the 2-categorical literature.

Remark: We fed 'op-laxness' into the definition of $\mathsf{Cocone}(\mathsf{F}, \mathcal{Y})$ by the direction of the 2-cells, θ_f. If we had had a lax functor, F, to start with, it would be more natural to use the 'lax' direction for these θ_f. In fact, for much of what we will need later the θ_f are usually invertible and F will

[21] We will give the neat 2-categorical approach to this later in section 11.5.6.

[22] using end, coends and their lax / pseudo analogues

be a pseudo-functor, so we would return to our earlier discussions about the direction of the 2-cell in the specifications of pseudo-functors. We will meet this several times more!

Starting now with an op-lax functor, $\mathsf{F} = (F, \tau)$, we can try to see if the basic $\mathcal{E}_{\mathsf{F}}$ has any of the characteristics of the op-lax colimit. For this, we will specify an op-lax cocone, $(\mathcal{Y}, \underline{\eta}, \underline{\theta})$, adopting the notation we have used above. We hope to be able to construct a functor, $\overline{\eta}$, from $\mathcal{E}_{\mathsf{F}}$ to $\mathcal{Y}$ using the cocone data. An object of $\mathcal{E}_{\mathsf{F}}$ is a pair, (x, b) with $b \in Ob(\mathcal{B})$ and $x \in Ob(F(b))$. The only 'obvious' way to define $\overline{\eta}(x, b)$ is $\eta(b)(x)$, since $\eta(b) : F(b) \to \mathcal{Y}$ is about the only thing available to us!

A morphism from (x, b) to (x', b') in this basic version of $\mathcal{E}_{\mathsf{F}}$ will be given by $f : x \to x'$, and $\alpha : F(f)x \to x'$ in $F(b')$, so we need $\overline{\eta}$ on such a morphism. This must be some morphism

$$\overline{\eta}(f, \alpha) : \overline{\eta}(x, b) \to \overline{\eta}(x', b')$$

in $\mathcal{Y}$. (Putting our 'jigsaw' pieces on the table, we should have to use $\theta_f : \eta(b) \Rightarrow \eta(b')F(f)$ as well as something derived from α.) Evaluating θ_f at the object x of $F(b)$, we get

$$\theta_f(x) : \eta(b)(x) \Rightarrow \eta(b')F(f)(x)$$

and now it should be clear. We compose this with $\eta(b')(\alpha)$ from $\eta(b')F(f)(x)$ to $\eta(b')(x')$. (**Pause:** $\eta(b') : F(b') \to \mathcal{Y}$ is a functor, so $\eta(b')(\alpha)$ makes sense and does what is claimed.) We thus take $\overline{\eta}(f, \alpha) = \eta(b')(\alpha)\sharp_0\theta_f(x)$. It is useful to check this is going to preserve composition. That will use the cocycle condition of the ηs together with the naturality of θ_g, the θ 2-cell corresponding to the second of the morphisms. Again this is routine, once you put the pieces together, so is **left to you**. That leaves preservation of identities by $\overline{\eta}$, ...

Everything works at this level, so now if $\mu : \eta \Rightarrow \eta'$ is a natural transformation between two such cocones, then we should get a natural transformation, $\overline{\mu} : \overline{\eta} \Rightarrow \overline{\eta'}$. As we left exploration of this aspect in $\mathsf{Cocone}(\mathsf{F}, \mathcal{Y})$ to you earlier, we leave this to you to check it all works.

This looks good. The constructions of $\overline{\eta}$ from η, etc., have exactly what we expect from universal constructions, that is *great naturality* in the non-technical sense as well as in the technical one. It is interesting to check that the resulting $\overline{\eta}$ is unique, ..., but we have not as yet given an op-lax cocone from F to $\mathcal{E}_{\mathsf{F}}$ itself, so should glance at that first. (This was left aside until we had some experience of handling op-lax cocones, but now ...)

We need a functor, that we will call $\eta(b)$, for lack of imagination, from $F(b)$ to $\mathcal{E}_{\mathsf{F}}$ for each b in $\mathcal{B}$, but sending x to (x, b) and $\alpha : x \to x'$ to (id, α) clearly gives one. What about behaviour with respect to an $f : b \to b'$? (This is fun!) We need a $\theta_f : \eta(b) \Rightarrow \eta(b')F(f)$, so will need it evaluated on $x \in F(b)$. It has to be a morphism from (x, b) to $x', F(f)(b))$ in $\mathcal{E}_{\mathsf{F}}$. No prizes for guessing which one!

You can now easily verify the uniqueness of the earlier assignment ..., over to you to finish things off.

That leaves the other variants of the Grothendieck construction to be looked at, but we will not do this as those are fairly routine for you to check on, and we will anyway, later on, be looking at the pseudo-colimit construction, which requires a bit more investigation. Before we do that we need to get back towards stacks.

9.4.2 Presenting the Grothendieck construction / op-lax colimit

Colimits are often constructed by taking a quotient. You start with a family of 'things' corresponding to the objects of your indexing category, then divide out by an equivalence relation or, more likely, a 'conguence[23]'. This gives a form of presentation akin to group presentations and, as we saw earlier, such information can be further analysed to gain a better understanding of the overall structures involved even, as was the case with higher syzygies, order relationships between the various 'elements' involved.

A similar process of 'presentation' can be done for the op-lax colimit. Here there is an intuition, which is very like that of colimits of groups if the individual groups are given with presentations. One of the simplest examples of this is in the classical form of van Kampen's theorem, as discussed in Brown, [59], Crowell and Fox, [96], or, for that matter, Gilbert and Porter, [133]. (The first of these discusses the groupoid form of the result.) The basic set-up is to get our hands on the pushout of a diagram

$$\begin{tikzcd} G_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & G_1 \arrow[d, dashed, "p_1"] \\ G_2 \arrow[r, dashed, "p_2"'] & G \end{tikzcd}$$

of groups. We assume that presentations, $\mathscr{P}_i = (X_i : R_i)$, are given for G_i, $i = 0, 1$ and 2, and that for each x in X_0, and $i = 1, 2$ a word in X_i representing $f_i(x)$ in G_i is specified. We will denote the chosen 'lift' of $f_i(x)$ by $\overline{f_i}(x)$. ('Lift' because this lives in $F(X_i)$, the free group on X_i, and there is, of course, an epimorphism

$$\varepsilon_i : F(X_i) \to G_i$$

for $i = 1, 2$. The meaning of 'lift' is thus that $\varepsilon(\overline{f_i}(x)) = f_i(x)$, as you probably guessed or knew!) The task is to use this data to give a presentation of G and descriptions of p_1 and p_2. The solution is well known, but we need to think of the proof as it will give insight into this 'presentations of op-lax colimits' problem.

The solution is that G has a presentation with set of generators, $X = X_1 \sqcup X_2$, the disjoint union of X_1 and X_2, and with relations of two different forms forming R. The two forms are:

1. If $r_i \in R_i$, we have a relation r_i in R (We use the inclusion $inc_i : X_i \to X$ to induce an 'inclusion'

   $$F(inc_i) : F(X_i) \to F(X),$$

   then $r_i \in F(X_i)$ and we really have $F(inc_i)(r_i) \in F(X)$, but we usually relabel it r_i as otherwise the notation can get 'impossible'.) We thus have a subset of our relations 'equal' to $R_1 \sqcup R_2$. (Notice that in this R_0 does not apparently play any role.)

2. For each $x \in X_0$, we have a relation

   $$\overline{f_1}(x)(\overline{f_2}(x))^{-1}$$

   in R. (Again, this is more accurately the word

   $$(F(inc_1)(\overline{f_1}(x)))(F(inc_2)(\overline{f_2}(x)))^{-1},$$

   but the simplified notation is much easier to work with an no confusion should occur.)

[23]that is an equivalence relation internal to 'things'

The two morphisms, p_1 and p_2, are induced by the two inclusions of generators. They can be factored via the 'free product' of G_1 and G_2, that is their coproduct, $G_1 \star G_2$, corresponding to the case where X_0 is empty, followed by the evident quotient of $G_1 \star G_2$ to G (by adding the *second* type of generator. (Note that $G_1 \star G_2$ is the coproduct in the category of groups, and if we think of the groups as groupoids this is not the coproduct in that category, which is just disjoint union. This is useful to note for what follows.)

Remark: (i) What is quite fun is to ask: what are the relations R_0 doing? They look to be not needed. They do influence the fact that f_1 and f_2 are homomorphisms, but that seems relatively minor. They do however influence things and we can see how if we try to work with identities amongst the relations.

What is neat is to calculate the identities for the presentation of G. If you do this, you realise that you will naturally get any identities among the relations for the presentations of G_1 and G_2 together with, you guessed, new identities coming from R_0. This behaviour was examined by Holz in his thesis, [157], and see also Abels and Holz, [2]. An approach using crossed resolutions was initiated by Moore, [209], and further informations and examples can be found in Brown, Moore, Porter and Wensley, [68]. These approaches use homotopy colimits, and that is suggestive given the links between homotopy colimits and op-lax colimits!

(ii) The direct proof that this *is* a presentation of G is not hard. You start with a commutative square

$$\begin{tikzcd} G_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & G_1 \arrow[d, "p_1"] \\ G_2 \arrow[r, "p_2"'] & G \end{tikzcd}$$

of groups and its data to construct a homomorphism from the group presented by $(X : R)$, first by defining it on the free group on X and then checking that the relations will be sent to the identity of H.

This presentation of G is nice, but it is not quite what we want as X_0 has taken a very different rôle to those played by X_1 and X_2. This is fine for a pushout, but it hides what would happen if we had a colimit over a more complicated diagram. There is, however, a variant of this presentation that makes a lot of sense, both algebraically and 'geometrically', and in which X_0 has a role which is easier to generalise to other situations. This second presentation is the following:

- set of generators, $X = \coprod X_i$, the disjoint union of the generating sets of the various groups;
- set of relations = union of two types of relation:

  (i) images in $F(X)$ of any $r_i \in R_i$, $i = 0, 1, 2$,

  (ii) for each $x \in X_0$ and $i = 1, 2$, $\overline{f_i}(x)x^{-1}$, so a relation that identifies x with its image in $F(X_i)$, but all happening in $F(X)$.

If you know about Tietze transformations, you can very quickly check that this second presentation is equivalent to the first. If you do not know them, they are four rules that allow transformations of presentations without changing the group being presented. They are two obvious substitution rules plus rules on insertion or deletion of redundant relations. Here we can informally manipulate the presentation: using $x \equiv \overline{f_1}(x)$, we substitute into all other relations that contain x. In particular we get, on substituting $\overline{f_1}(x)$ for x in $x \equiv \overline{f_2}(x)$, that $\overline{f_1}(x) \equiv \overline{f_2}(x)$, *i.e.*, that the relation $\overline{f_1}(x)(\overline{f_2}(x))^{-1}$ is a consequence of the presentation. A bit more subtle is the proof that the relations in R_0 all become redundant in the process. To prove that you need to use a bit more group presentation theory than we have assumed, but the results needed can either be found in texts on group presentations (such as Johnson, either [168] or the earlier, [167]) and, in any case, just use some fairly elementary group theory in their proofs.

The new presentation is now in a much better form that should, with care, generalise first to arbitrary colimits of groups, and then to op-lax colimits of op-lax functors, $\mathsf{F} = (F, \tau) : \mathcal{B} \to \mathsf{Cat}$. A sneaky way of looking at our pushout example is then as an op-lax pushout of categories. The only difference is that the result will be a *category* on three objects and, instead of the 'free product' of the G_is being an intermediate step, it will be the coproduct of the G_is *as categories* (or, if you do things carefully, as groupoids), instead of their coproduct *as groups*. This is still a bit vague, so let us proceed directly to a more detailed treatment.

We are given $\mathsf{F} = (F, \tau)$, as above, and first form a directed graph with set of vertices, O, and set of arrows, A, where

- $O = \coprod\{ObF(b) : b \in Ob(\mathcal{B})\}$ is the set of all the objects in all the categories $F(b)$. We denote an element of O by a pair, (x, b), with, as before, $x \in ObF(b)$;

- $A = (\coprod\{Arr(F(b)) : b \in Ob(\mathcal{B})\}) \sqcup \{h_{((x,b),f)} : ((x, b), f) \in O \times Arr(\mathcal{B})$ such that $dom(f) = b\}$, thus A consists of two types of arrow. The first is simply an arrow in some $F(b)$, with 'domain' and 'codomain' given by the obvious formulae, so, if $a : x \to x'$ within $F(b)$, then, within A, there is a corresponding a with $dom(a) = (x, b)$, $codom(a) = (x', b)$. The second type of arrow is here represented as an abstract label, $h_{((x,b),f)}$, with (x, b) an 'object' in O, whilst $f : b \to b'$ is a morphism in $\mathcal{B}$, starting at the object b. This arrow, $h_{((x,b),f)}$, will have domain (x, b) and codomain $(F(f)(x), b')$.

We now form the free category, W, on this directed graph, writing $\sharp_W$ for the composition in W.

The other part of the presentation will be a set, R, of relations. (As we are working within a category, not a group or groupoid, we write $a \equiv_R b$ instead of ab^{-1}, which need not make sense.) The final step will be to form the quotient of W by the smallest congruence containing all the relations in R. As you would expect, the relations in R come in various forms:

- (internally in the fibres) if a, a' are in $F(b)$ and $a'a$ is defined, then

$$a'\sharp_W a \equiv_R a'a.$$

  This relation thus ensures each of the $F(b)$ is copied into the quotient.

- (induced morphisms between fibres) suppose

$$b \xrightarrow{f} b' \xrightarrow{g} b'',$$

  and $x \in F(b)$, then we have arrows

$$h_{((x,b),f)} : (x, b) \to (F(f)(x), b'),$$

$$h_{((F(f)(x),b'),g)} : (F(f)(x), b') \to (F(g)F(f)(x), b''),$$

and also

$$h_{((x,b),gf)} : (x, b) \to (F(gf)(x), b'').$$

We also have

$$\tau_{(f,g)}(x) : F(gf)(x) \to F(g)F(f)(x)$$

in $F(b'')$. The relation is

$$\tau_{(f,g)}(x)\sharp_W h_{((x,b),gf)} \equiv_R h_{((F(f)(x),b'),g)}\sharp_W h_{((x,b),f)}.$$

- If $x \in F(b)$, then there is an identity, $id_x \in Arr(F(b)) \subseteq A$, but also we have the 'formal' identity on (x, b) within W, namely the empty string from (x, b) to itself:

$$id_x \equiv_R id^W_{(x,b)}.$$

- If $f : b \to b'$ in $\mathcal{B}$, and $a : x_0 \to x_1$ in $F(b)$, there is a morphism, $F(f)(a) : F(f)(x_0) \to F(f)(x_1)$, in $F(b')$ and a subsequent relation:

$$F(f)(a)\sharp_W h_{((x_0,b),f)} \equiv_R h_{((x_1,b),f)}\sharp_W a.$$

- For $b \in Ob(\mathcal{B})$ and $x \in Ob(F(b))$, we have

$$h_{((x,b),id_b)} \equiv_R id^W_{(x,b)}.$$

We **leave you to worry out** the proof that this gives us $\mathcal{E}_{\mathsf{F}}$, (up to equivalence).

We will see this sort of presentation again shortly when discussing pseudo-colimits. (The treatment here has been based on that in Fiore's A. M. S. Memoirs, [129], p. 21 - 22. That is actually given for pseudo-colimits, which is, in fact, the context in which we will need it mostly.)

9.5 Prestacks: sheaves of local morphisms

Let $F : \mathcal{C}^{op} \to \mathsf{Cat}$ be a fibred category in the wider sense and let $C \in Ob(\mathcal{C})$. Suppose $a, b \in Ob(F(C))$, then for any $f : D \to C$ in $\mathcal{C}$, we have $F(D)(f^*(a), f^*(b))$, the set of morphisms in $F(D)$ between the restrictions of a and b along f.

Now suppose we think of $f : D \to C$ as an object in $\mathcal{C}/C$ and consider a morphism of such:

$$\begin{tikzcd} D' \arrow[rr, "g"] \arrow[dr, "f'"'] & & D \arrow[dl, "f"] \\ & C, & \end{tikzcd}$$

so $f' = fg$. We get a composite

$$F(D)(f^*a, f^*b) \xrightarrow{g^*} F(D')(g^*f^*a, g^*f^*b) \xrightarrow{\tau^*} F(D')((fg)^*a, (fg)^*b),$$

where, given $\gamma : g^*f^*a \to g^*f^*b$, $\tau^*(\gamma) = (\tau_{f,g})_b^{-1}\gamma(\tau_{f,g})_a^{-1}$, *i.e.*, 'conjugation by τ':

$$\begin{tikzcd}
(fg)^*a \arrow[r, "\tau^*(\gamma)"] \arrow[d, "(\tau_{f,g})_a"'] & (fg)^*b \arrow[d, "(\tau_{f,g})_b"] \\
(g^*f^*)a \arrow[r, "\gamma"'] & (g^*f^*)b
\end{tikzcd}$$

commutes by definition of $\tau^*(\gamma)$.

Lemma 62 *Given F, C, a and b, the above defines a presheaf,*

$$Hom_F(a,b) : (\mathcal{C}/C)^{op} \to Sets.$$

Proof: This is **left to you** as it is quite straightforward. ■

Moreover any $\varphi : F \to G$ induces a morphism of presheaves on $(\mathcal{C}/C)$,

$$\varphi_{a,b} : Hom_F(a,b) \to Hom_G(\varphi_C(a), \varphi_C(b)).$$

Back to our case studies:

9.5.1 $\mathsf{Sh}(B)$

To get back to a more concrete example, let us examine this result in the simple case of $\mathsf{Sh}(B)$, *i.e.*, sheaves on B considered as a fibred category. (We will be working with several 'layers' of presheaves on various objects, so need to pay attention to terminology, etc.!)

Translating the above to this case

- $\mathcal{C} = Open(B)$;
- $C \in Ob(Open(B))$, so is an open set of B, and we will replace it notationally by U, as being our usual notation;
- $\mathcal{C}/U$ is the category of morphisms in $Open(B)$ with codomain U, so is precisely $Open(U)$.

Suppose now $F = \mathsf{Sh}(B)$, the fibred category of sheaves on B, and a and b are sheaves on U. For any $f : V \to U$, we have $Sh(V)(f^*(a), f^*(b))$, *i.e.*, the set of morphisms in $Sh(V)$ between the restricted sheaves, $f^*(a)$ and $f^*(b)$. If, further, $W \subset V$, and $g : W \to V$ is the inclusion, we *really* have $W \to V \to U$, but can picture it also as

$$\begin{tikzcd}
W \arrow[r, "g"] \arrow[dr, "(fg)"'] & V \arrow[d, "f"] \\
& U
\end{tikzcd}$$

that is, in $\mathcal{C}/U$. The obvious type of presheaf on U that we have here is

$$Hom_F(a,b)(V) = Sh(V)(f^*(a), f^*(b))$$

and, if $W \to V$, the corresponding function,

$$Hom_F(a,b)(V) \to Hom_F(a,b)(W),$$

is induced by restriction (but with the subtle point that it is better to assume $g^*f^*(a) \cong (fg)^*(b)$, usually not '=', especially in situations such as a more general Grothendieck topos).

There is an obvious question: when is this presheaf a sheaf?

As a start, we had better sort out what we are given or know, and what exactly we need to investigate further:

- We have an open set U of B and an open cover, $\mathcal{U} = \{U_i\}$ of U;
- We have inclusion maps: $\alpha^i : U_i \to U$, $\alpha_i^{ij} : U_{ij} \to U_i$, so $\alpha^i\alpha_i^{ij} = \alpha^j\alpha_j^{ij} = \alpha^{ij} : U_{ij} \to U$. We have a pair of sheaves, a, b, on U and hence their restrictions $(\alpha^i)^*(a)$, etc. Further restriction to U_{ij} gives the natural isomorphisms,

  $$(\alpha_i^{ij})^*(\alpha^i)^*(a) \to (\alpha^i\alpha_i^{ij})^*(a) = (\alpha^{ij})^*(a),$$

  which will be denoted $(\tau_i^{ij})_a$.
- We have a compatible family indexed by the cover,

  $$\varphi_i : (\alpha^i)^*a \to (\alpha^i)^*b.$$

  The restriction of this to U_{ij} is obtained by first applying the functor $(\alpha_i^{ij})*$ to get

  $$(\alpha_i^{ij}) * \varphi_i : (\alpha_i^{ij}) * (\alpha^i)^*a \to (\alpha_i^{ij}) * (\alpha^i)^*b,$$

  then applying τ, *i.e.*, 'conjugating' this with $(\tau_i^{ij})_a$ and $(\tau_i^{ij})_b$, to get

  $$(\tau_i^{ij})_b \circ (\alpha_i^{ij}) * \varphi_i \circ (\tau_i^{ij})_a^{-1} : (\alpha^{ij})^*(a) \to (\alpha^{ij})^*(b),$$

  which morphism we will denote φ_i^{ij}. We thus have, for compatibility, that

  $$\varphi_i^{ij} = \varphi_j^{ji} : (\alpha^{ij})^*(a) \to (\alpha^{ij})^*(b).$$

(It would be feasible to suppress some of this notation in this fairly elementary case, *but* taking care in simple cases often proves to be worth while in the more complex cases, so ... .)

We have to prove that such a compatible family glues to give a morphism, $\varphi : a \to b$. (We will actually check less than that as we will assume $x \in a(U)$ and will define $\varphi(x) \in b(U)$. The rest of the proof is similar, so is **left for the reader to think about**.) Given $x \in a(U)$, we restrict to U_i to get $x_i = (\alpha^i)^*(x) \in (\alpha^i)^*a(U_i)$ (which is really $a(U_i)$). As $\varphi_i : (\alpha^i)^*a \to (\alpha^i)^*b$, we have $\varphi_i(x) := \varphi_i((\alpha^i)^*(x)) \in (\alpha^i)^*b(U_i)$, with a little sensible (ab)use of notation).

Claim:

The family $(\varphi_i(x))$ is a compatible family in the sheaf, b, so defines a unique element in $b(U)$, which we denote $\varphi(x)$.

To prove compatibility, we need to compare

$$\alpha_i^{ij*}\varphi_i(x) = \alpha_i^{ij*}\varphi_i(x_i) = \alpha_i^{ij*}\varphi_i\alpha^{i*}(x)$$

with the corresponding element with the roles of i and j interchanged. That is not quite correct as this element is in $\alpha_i^{ij*}\alpha^{i*}(b)(U_{ij})$, not $\alpha^{ij*}b(U_{ij})$ for which we have to use the τ_i^{ij}s. We thus actually look at $(\tau_i^{ij})_b\alpha_i^{ij*}\varphi_i(x)$. We have a commutative square

$$\begin{tikzcd}
\alpha_i^{ij*}\alpha^{i*}(a) \arrow[r, "\alpha_i^{ij*}\varphi_i"] \arrow[d, "(\tau_i^{ij})_a"'] & \alpha_i^{ij*}\alpha^{i*}(b) \arrow[d, "(\tau_i^{ij})_b"] \\
\alpha^{ij*}a \arrow[r, "\varphi_i^{ij}"'] & \alpha^{ij*}b
\end{tikzcd}$$

and the restriction of φ_i to U_{ij} is φ_i^{ij} as defined. We can now complete the calculation:

$$\begin{aligned}
(\tau_i^{ij})_b\alpha_i^{ij*}\varphi_i(x) &= \varphi_i^{ij}(\tau_i^{ij})_a(\alpha_i^{ij})^*(x^i) \\
&= \varphi_i^{ij}(x^{ij}) \\
&= \varphi_j^{ij}(x^{ji})
\end{aligned}$$

by compatibility of the family, $\{\varphi_i\}$, and then unroll the argument going the other way to get this is equal to $(\tau_j^{ij})_b\alpha_j^{ij*}\varphi_j(x^j)$ as required. These thus glue to give us our required $\varphi(x)$. ∎

We have taken a lot of trouble to include 'detail', even when perhaps it would have been easy to cut corners, but, for instance, the role of the τs is crucial and can be obscured unless it is made explicit.

The situation here warrants a name!

Definition: A fibred category, F, over B is called a *prestack* if, for any open set, U, and objects $a, b \in F(U)$, the presheaf, $Hom_F(a, b)$, is a sheaf.

Terminology: Some authors use the term 'prestack' to mean any pseudo-functor,

$$F : Open(B)^{op} \to Cat$$

(or by extension, to any suitable 2-category). In that tradition, the above definition would be of a 'separated stack'.

In the 2-category of fibred categories on a space, B, we thus have the full 2-subcategory, $\mathsf{PreStacks}(B)$, determined by the prestacks and the morphisms between prestacks are just the morphisms of the corresponding fibred categories, similarly for the 2-arrows.

Summarising the above, we have

Proposition 96 *The fibred category,* $\mathsf{Sh}(B)$, *of sheaves on* B *is a prestack.* ∎

We can look at special sub-fibred categories of $\mathsf{Sh}(B)$ equally easily. For instance, consider sheaves of groups on B. This gives a fibred category $\mathsf{ShGrp}(B)$.

Proposition 97 *The fibred category,* $\mathsf{ShGrp}(B)$*, of sheaves of groups on* B *is a prestack.* ■

Within $\mathsf{Sh}(B)$, we can look at the full sub-fibred-category, $\mathsf{LCSh}(B)$, determined by the locally constant sheaves on B. The obvious adaptation of the above gives us:

Proposition 98 *The fibred category,* $\mathsf{LCSh}(B)$*, of locally constant sheaves on* B *is a prestack.* ■

9.5.2 $\mathsf{Tor}(B;G)$

If we turn our attention to our other case study, we can reuse most of our work on $\mathsf{Sh}(B)$, and then adapt and add the necessary to prove:

Proposition 99 *The fibred category,* $\mathsf{Tor}(B;G)$*, for a sheaf of groups,* G*, on* B *is a prestack.*

Proof: From the φ_i, which will now be G-torsor maps, we can certainly construct a sheaf map, φ, by the previous argument. We need to verify that φ is a torsor map, *i.e.*, that φ commutes with the action. For this, one compares $\varphi(g.x)$ and $g.\varphi(x)$, both of which 'glue' the $\varphi_i(g_i.x_i)$, so then uniqueness of 'gluing' gives the result. ■

We thus have two families of good examples of prestacks. In fact we have a lot more. Any set gives a category with only identity morphisms, so any presheaf, F, of sets yields a fibred category. If that fibred category is a prestack, then F itself would be a separated presheaf and conversely.

As one can 'sheafify' a presheaf, can one 'prestackify' a fibred category? Yes.

9.5.3 Prestackification!

In fact this is straightforward.

Proposition 100 *For any space* B*, the forgetful functor from the 2-category,* $\mathsf{PreStacks}(B)$*, of prestacks on* B *to that of fibred categories on* B *has a left 2-adjoint.*

Proof: The proof just takes each presheaf, $Hom_F(a,b)$, makes it into a sheaf, then checks that the result works. ■

An **interesting challenge is to investigate** what happens to 2-arrows during prestackification.

9.5.4 Change of base for prestacks

In section 7.3.8, we saw how a map, $f : X \to Y$, induced some functors between the categories of sheaves on the two spaces, X and Y. We had

$$f_* : Sh(X) \to Sh(Y),$$

and

$$f^* : Sh(Y) \to Sh(X),$$

and f^* is left adjoint to f_*. A natural question is to see what happens on prestacks[24] that is, does f induce 2-functors

$$f_* : \mathsf{PreStacks}(X) \to \mathsf{PreStacks}(Y),$$

and

$$f^* : \mathsf{PreStacks}(Y) \to \mathsf{PreStacks}(X)?$$

9.6 From prestacks to stacks

We thus have that in our examples, $\mathsf{Sh}(B)$ and $\mathsf{Tor}(B;G)$, the presheaf of 'local morphisms' between 'local objects' was a sheaf. We note, however, that the proof did use the adjustment transformations, τ, so was not, perhaps, quite so 'naively' constructed as one might pretend. Thus 'morphisms' glue. What about objects? Here we need to think again about the 'categorification' process.

You will recall that, at certain points, it has been useful to think of 'going up the dimensions' as corresponding to replacing sets by categories, categories by 2-categories, or similar, and as a consequence to replace 'equality' by 'isomorphism', or better 'equivalence', which is usually 'isomorphism up to an (invertible) 2-cell', thus 'fibred category' = 'pseudo-presheaf of categories' and we naturally involved the τ-transformations in the structure. We have asked 'do compatible families of objects glue?' in the prestacks $\mathsf{Sh}(B)$ and $\mathsf{Tor}(B;G)$. We first need to see what should replace 'compatible family' under categorification! 'Compatible families' are part of the descent data picture, so we introduce, for a fibred category F, a category of descent data relative to an open cover, $\mathcal{U}$, of an open set U. This fits well with the categorification yoga. We had a *set* of compatible families, and a fairly simply defined category of descent data back in section 7.1.1, but here we need a category of descent data with considerably more structure.

9.6.1 The descent category, $\mathsf{Des}(\mathcal{U}, F)$

Definition: Let F be a fibred category over B and let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of an open set, U, of B. The category, $\mathsf{Des}(\mathcal{U}, F)$, has

- **Objects:** systems, $(\underline{a}, \underline{\theta})$, where $\underline{a} = \{a_i : i \in I\}$, each a_i an object of $F(U_i)$ and $\underline{\theta} = \{\theta_{ij} : i, j \in I\}$ with $\theta_{ij} : \alpha_j^{ij*}(a_j) \overset{\cong}{\to} \alpha_i^{ij*}(a_i)$, an isomorphism in $F(U_{ij})$, these isomorphisms being required to satisfy the cocycle conditions:

$$\begin{aligned} \theta_{ii} &= 1 \\ \theta_{ij} \circ \theta_{jk} &= \theta_{ik} \end{aligned}$$

  in $F(U_{ijk})$;

- **Arrows:** $f : (\underline{a}, \underline{\theta}) \to (\underline{b}, \underline{\rho})$ is given by a family of arrows, $\{f_i : a_i \to b_i \in F(U_i)\}$, for which the diagrams

$$\begin{tikzcd} \alpha_j^{ij*}(a_j) \arrow[r, "\alpha_j^{ij*} f_j"] \arrow[d, "\theta_{ij}"'] & \alpha_j^{ij*}(b_j) \arrow[d, "\rho_{ij}"] \\ \alpha_i^{ij*}(a_i) \arrow[r, "\alpha_i^{ij*} f_i"'] & \alpha_i^{ij*}(b_i) \end{tikzcd}$$

[24] and later on stacks

commute.

The cocycle condition written

$$\theta_{ij} \circ \theta_{jk} = \theta_{ik}$$

is shorthand for a more complicated expression, as each term is restricted to U_{ijk}. If we write, for instance, $\theta_{ij}|_{U_{ijk}} = (\alpha_{ij}^{ijk})^*(\theta_{ij})$, then similarly for the others, the condition is

$$\theta_{ij}|_{U_{ijk}} \circ \theta_{jk}|_{U_{ijk}} = \theta_{ik}|_{U_{ijk}}.$$

How does $\mathsf{Des}(\mathcal{U}, F)$ vary with $\mathcal{U}$?

What happens if we change the cover? Recall that a morphism, $\alpha : \mathcal{V} \to \mathcal{U}$, between open covers of B is a map of the indexing sets, $\alpha : I(\mathcal{V}) \to I(\mathcal{U})$, such that $V \subseteq \alpha(V)$ for all $V \in \mathcal{V}$. (It induces a map of the simplicial sheaf nerves of the two covers, $N(\alpha) : N(\mathcal{V}) \to N(\mathcal{U})$, and we could work with that directly, but we do not yet have a "coordinate free" or "chart free" description of $\mathsf{Des}(\mathcal{U}, F)$, so will use the slightly stricter notion for the moment.) We would expect α to induce a function, α^*, from $\mathsf{Des}(\mathcal{U}, F)$ to $\mathsf{Des}(\mathcal{V}, F)$. (If you asked 'why in that direction?', think back to sheaves. There the compatible families of local sections over $\mathcal{U}$ *restrict* to ones over $\mathcal{V}$. We would not expect a map in the other direction, which would be *extending* the families.)

Suppose we have an object, $(\underline{a}, \underline{\theta})$, of $\mathsf{Des}(\mathcal{U}, F)$, then we have, for each $U \in \mathcal{U}$, $a_U \in Ob(F(U))$, etc., and we need an object $\alpha^*(\underline{a}, \underline{\theta})$, consisting of a family $\alpha^*(\underline{a})$ of objects in $F(V)$, $V \in \mathcal{V}$, but $V \subseteq \alpha(V) \in \mathcal{U}$, so we *can* restrict $a_{\alpha(V)}$ to V, to get the necessary objects. Similarly, we can restrict the isomorphisms, $\theta_{\alpha(V_i),\alpha(V_j)}$, to $V_{ij} := V_i \cap V_j$ and the normalisation and cocycle conditions will 'check-out' automatically.

This construction on objects easily extends to arrows, $f : (\underline{a}, \underline{\theta}) \to (\underline{b}, \underline{\rho})$, as it is just restriction, so α induces a functor,

$$\alpha^* : \mathsf{Des}(\mathcal{U}, F) \to \mathsf{Des}(\mathcal{V}, F),$$

and hence, ..., we get a 2-functor from $Cov(B)$, the category of covers, to Cat. **No.** What goes wrong is that restriction is specified up to isomorphism, so if $\alpha : \mathcal{V} \to \mathcal{U}$ and $\beta : \mathcal{W} \to \mathcal{V}$ are morphisms of covers, $(\alpha\beta)^*$ need not be the same as $\beta^*\alpha^*$. In $(\alpha\beta)^*$, we restrict $(\underline{a}, \underline{\theta})$ to a W via the inclusion of W into $\alpha\beta(W)$, but in $\beta^*\alpha^*$, this is done via the chain of inclusions $W \to \beta(W) \to \alpha\beta(W)$ and so in two stages. The data for the pseudo-functor F (corresponding to a specification / presentation of the fibred category) is easy to use to get the following:

Proposition 101 *Given a fibred category, F, over B, there is a pseudo-functor*

$$\mathsf{Des}(-, F) : Cov(B)^{op} \to \mathsf{Cat}$$

taking the value $\mathsf{Des}(\mathcal{U}, F)$ on an open cover, $\mathcal{U}$. ■

9.6.2 Simplicial interpretations of $\mathsf{Des}(\mathcal{U}, F)$: first steps

It will often be useful to have another view of the objects, etc., of $\mathsf{Des}(\mathcal{U}, F)$. We will formalise this later when looking at descent in much more generality and from a simplicial viewpoint, but it seems a good idea to start this process now.

The objects of $\mathsf{Des}(\mathcal{U}, F)$ are 'systems', $(\underline{a}, \underline{\theta})$. What are these 'simplicially'?

Recalling, (page 300), that an open cover, $\mathcal{U}$ gives us a simplicial sheaf, $N(\mathcal{U})$, on B, (you guessed!), we can interpret an object $(\underline{a}, \underline{\theta})$ in terms of this sheaf. (We have seen this sort of thing before, for instance, with the simplicial description of torsors, in section 7.4.5.) The basic sheaf / étale space is $\bigsqcup \mathcal{U} \to B$, but, as that is a bit awkward to write, we will just write $p : Y \to B$ for use during this brief snapshot of where this is going. We have a picture of $N(\mathcal{U})$ as the simplicial object:

$$N(\mathcal{U}): \quad \ldots \;\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}\; Y \times_B \cdots \times_B Y \;\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}\; \cdots \;\overset{d_0}{\underset{d_2}{\Rrightarrow}}\; Y \times_B Y \;\overset{d_0}{\underset{d_1}{\rightrightarrows}}\; Y \overset{p}{\dashrightarrow} B.$$

The pseudo-functor F gives categories $F(Y)$, $F(Y \times_B Y)$, etc, and induced coface and codegeneracy functors, d_i^*, and s_i^*, between them. Remembering what Y, $Y \times_B Y$, etc., are in terms of the open sets U_i of $\mathcal{U}$, we can interpret an $(\underline{a}, \underline{\theta})$ as consisting of an object, $\underline{a}$, of $F(Y)$ and a morphism, $\underline{\theta} : d_1^*(\underline{a}) \to d_0^*(\underline{a})$, in $F(Y \times_B Y)$. (Hold on, you should say: the pseudo-functor F is only defined on open sets of B and Y is a disjoint union of such, so is not defined as such. That is correct, so we have to extend F by defining $F(Y) := F(\bigsqcup \mathcal{U}) = \prod\{F(U) : U \in \mathcal{U}\}$ and similarly, as $Y \times_B Y$ can be identified to be the cover by intersections of sets from $\mathcal{U}$, we have $F(Y \times_B Y) := \prod_{i,j} F(U_{ij})$ and so on.) The cocycle condition will correspond to there being no non-trivial '2-cells' in $F(Y \times_B Y \times_B Y)$.

As usual, we can think of $F(U)$ not only as a (small) category, but also as that category's nerve. We seem then to be looking at $F(N(\mathcal{U}))$ as some cosimplicial simplicial set, or, more exactly perhaps, as a 'pseudo' version of such.

What about the morphisms / arrows in $\mathsf{Des}(\mathcal{U}, F)$? We had $f : (\underline{a}, \underline{\theta}) \to (\underline{b}, \underline{\rho})$ was a family of arrows, $f_i \in F(U_i)$, so is in $F(Y)_1$, (using a simplicial / nerve notation), with the commutativity condition being in $F(Y \times_B Y)_2$, *i.e.*, the square lives in there and commutes since all the 2-simplices there are degenerate.

Remarks: (i) It is sometimes useful to replace the notation $\mathsf{Des}(\mathcal{U}, F)$ by one emphasising the $p : Y \to B$ sheaf instead of the cover that gave it. In this case we will write $\mathsf{Des}(Y \to B, F)$. This notation also has the advantage of being transferrable to the situation found in a topos other than one of the form $Sh(B)$.

(ii) Quite a useful **exercise** here is to start with an even simpler situation. Take F to be a presheaf of sets on B, so just a functor $F : Open(B)^{op} \to Sets$. Look at the above description of $\mathsf{Des}(\mathcal{U}, F)$, considering each $F(U)$ as a discrete category. What sort of structure does $\mathsf{Des}(\mathcal{U}, F)$ have?

As we said, we will return to this simplicial description again later, to put more flesh on these 'bare bones'.

9.6.3 Stacks - at last

With sheaves, if F was a presheaf then each $x \in F(U)$ gave a compatible family of local sections over any open cover, $\mathcal{U}$, of U simply by restricting, $x_i := res^U_{U_i}(x)$. This gave a natural function, des, from $F(U)$ to the *set* of compatible families of local sections of F over $\mathcal{U}$ and F was a sheaf exactly when that function was a bijection. Similarly, given a fibred category, F, together with an open cover $\mathcal{U}$ of U, there is a natural *descent functor*,

$$des = des(\mathcal{U}, F) : F(U) \to \mathsf{Des}(\mathcal{U}, F),$$

so what is the obvious analogue of the sheaf condition?

Definition: The fibred category, F, is said to be a *stack* if each descent functor

$$des : F(U) \to \mathsf{Des}(\mathcal{U}, F)$$

is an equivalence of categories.

The idea of a stack is thus of a 2-sheaf, the analogue of a sheaf of categories, obtained by 'categorifying' the definition[25] of sheaf.

It will be important to 'deconstruct' the above definition. We first revisit the notion of equivalence of categories:

$F : \mathcal{C} \to \mathcal{D}$ is an equivalence if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms, $\eta : FG \stackrel{\cong}{\to} 1_{\mathcal{D}}$ and $\varepsilon : 1_{\mathcal{C}} \stackrel{\cong}{\to} GF$.

It is often easier (but with attendant disadvantages) to use an alternative formulation in which G, η and ε are not specified. Strictly this alternative is *not* completely equivalent, since it depends on the axiom of choice to rebuild a suitable G, η and ε from the specification and so depends on the set theory you are using. It thus is perhaps more a useful 'test' of equivalence rather than a completely equivalent formulation.

If F is an equivalence of categories, then F is full, faithful and essentially surjective on objects (eso).

'Deconstruction' is again in order:

- 'F is full' means that, for all x, y, objects of $\mathcal{C}$, the induced mapping,

  $$F_{x,y} : \mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$$

  is *surjective*;

- 'F is faithful' means that, for all x, y, $F_{x,y}$ is *injective*;

and

- 'F is essentially surjective on objects' (often abbreviated to 'eso') means that, if d is an object of D, then there is some $c \in \mathcal{C}$ such that $F(c) \cong d$.

Comment: The problem with taking this as a definition of equivalence is that essential surjectivity says there is a c, but does not construct one for us! Where possible, it is a good idea, given the d, to try to *construct* the c functorially, so that allows one to put $Gd = c$ and the rest usually falls into place. If one has to 'choose' a c, then the lack of naturality of the choice may be a problem, or rather a bothersome complication.

From this 'deconstruction', we can see that

(i) if F is a stack, then it is a prestack, since that corresponds to 'full and faithful' and also

(ii) that a stack is a prestack which satisfies: *for every cover, $\mathcal{U}$, of an open set U, any object, $(\underline{a}, \underline{\theta})$, of $\mathsf{Des}(\mathcal{U}, F)$ is isomorphic to an object of the form $des(x)$, for some $x \in F(U)$.*

[25] see page 294

It is worth noting, and will be important later, that if $i : V \subset U$ and $\mathcal{V} = \{U_i \cap V\}$, then there is a canonical functor

$$i^* : \mathsf{Des}(\mathcal{U}, F) \to \mathsf{Des}(\mathcal{V}, F),$$

and the diagram

$$\begin{tikzcd} F(U) \arrow[r, "des"] \arrow[d, "i^*"'] & \mathsf{Des}(\mathcal{U}, F) \arrow[d, "i^*"] \\ F(V) \arrow[r, "des"] & \mathsf{Des}(\mathcal{V}, F) \end{tikzcd}$$

commutes.

As stacks on B are just special fibred categories, there is no more obvious definition of morphisms of stacks than morphisms of the basic 'underlying' fibred category. The extra 'descent' structure both at the prestack and the stack level is not a question of extra operations merely extra conditions on existing structure. We thus have a 2-category[26], $\mathsf{Stacks}(B)$, of stacks on B, defined as a full 2-subcategory of $\mathsf{FibCat}(B)$.

The morphisms above are 'internal' to the context of a particular B. If one needs to compare stacks over different bases, then there is a notion of morphism in that case as well, but as with sheaves, the existence of a change of base construction allows one to push the stacks around moving them via the induction functors along continuous maps (or their analogues for sites and toposes). We will need this in certain cases slightly later so will briefly discuss that construction. In fact the cases we need *are* usually special so one can side-step the generalities if desired.

Let P be a stack on B and $f : A \to B$ be a continuous map, then we can build a new stack $f^*(\mathsf{P})$ on A in quite an obvious way. We know, from our previous discussions, how to go from $\mathsf{Stacks}(A)$ to $\mathsf{Stacks}(B)$ just by looking at $f_*(\mathsf{F})(U) = \mathsf{F}(f^{-1}(U))$, and as stacks are 'categorified sheaves', we may look at the definition of $f^*(F)$ for a sheaf F on B, for 'inspiration'. In that case, for U open in A,

$$f^*(F)(U) = colim\{F(V) \mid V \text{ open in } B, U \subseteq f^{-1}(V)\}.$$

The categorified version would then replace 'colim' by some lax or pseudo-colimit. This works well as the diagram one gets, consisting of the $F(V)$, is not commutative - remember, if $W \to V \to U$, we got a τ and F is a pseudo-functor, not a functor - but the lax or pseudo-colimit can handle that. If F is a stack of groupoids, then the colimit needs to be constructed with that in mind. We refer to various categorical papers for more details. In fact, the situation that we need is rather particular, so we can sidestep the more tricky generalities, for the moment at least!

Suppose f is the inclusion of an open subspace, A, then the index category for the colimit has an initial object, *i.e.*, A itself. We thus have that if F is a stack on B, $f^*(\mathsf{F})$ will be the stack F_A on A defined by: if $U \in Open(A)$, then $\mathsf{F}_A(U) = \mathsf{F}(U)$, which makes sense, since U will also be open in B. The other structure restricts in the same way. (Note that the universal property of the pseudo-colimit construction gives an equivalence between $f^*(\mathsf{F})$ and F_A, *not an isomorphism* and $f^*(\mathsf{F})$ will, in general, 'look bigger'.)

It is worth noting for later use that this assignment of $\mathsf{Stacks}(U)$ to U, for $U \in Open(B)$, yields a 'pseudo-2-functor', in some sense, from $Open(B)^{op}$ to $2-\mathsf{Cat}$, the category of 2-categories.

[26] This is a *strict* 2-category.

9.6.4 Back to $\mathsf{Sh}(B)$

In our case study of the properties of $\mathsf{Sh}(B)$, we have asked, essentially: Is $\mathsf{Sh}(B)$ a stack? We have that it is a prestack so now we have merely to prove that, when $F = \mathsf{Sh}(B)$, des is locally eso., *i.e.*, suppose that $(\underline{a}, \underline{\theta})$ is in $\mathsf{Des}(\mathcal{U}, \mathsf{Sh}(B))$, then there is a sheaf x on $U = \bigcup \mathcal{U}$ such that $des(x) \cong (\underline{a}, \underline{\theta})$.

We will start with an 'easy' case, namely we will assume all the θ_{ij} are identity morphisms. This *is* artificial, but gives some idea of how to proceed in general. We thus have $\underline{a} = \{a_i\}$, where a_i is a sheaf on U_i and where the restricted sheaves on intersections, U_{ij}, are *equal*. Since $\alpha_i^{ij} : U_{ij} \to U_i$ is the inclusion map, this says $\alpha_i^{ij*}(a_i) = \alpha_i^{ij*}(a_j)$.

We apparently have a diagram,

$$\textstyle\prod_i a_i \rightrightarrows \prod_{ij} \alpha_i^{ij*}(a_i) \ ,$$

and would like to take the equaliser of these two maps as this would encode compatibility, cf. section 7.3.1. Unfortunately, we have written down a diagram, but have not asked where it is living! Each object a_i is in the corresponding $Sh(U_i)$, so the left hand 'object' is not a valid one. As we need an object in $Sh(U)$, it would be a good idea to work only in that category. The inclusion $\alpha^i : U_i \to U$ induces the functor, $\alpha^{i*} : Sh(U) \to Sh(U_i)$, given by restriction, but also $\alpha^i_* : Sh(U_i) \to Sh(U)$, and α^i_* is right adjoint to α^{i*}. The first thing we might do, therefore, is to use $\prod_i \alpha^i_*(a_i)$ for the domain of our two morphisms. For the codomain, we can try the same trick: $\alpha^{ij} : U_{ij} \to U$ is equal to $\alpha^i \alpha^{ij}_i$, so $\alpha^i_* \alpha^{ij}_{i*}(\alpha_i^{ij*}(a_i))$ lives in the right place, *i.e.*, in $Sh(U)$. There is a natural morphism,

$$a_i \to \alpha^{ij}_{i*}(\alpha_i^{ij*}(a_i)),$$

coming from the fact that α^{ij}_{i*} is right adjoint to α_i^{ij*}, and, applying α^i_* to it, gives an ij-component of one of the two morphisms. We thus can drag our 'fictional' diagram into $Sh(U)$ and then form the equaliser of the two morphisms.

We *explicitly* used the adjointness of the two 'restriction' functors in order to show how this may be done in other, non-spatial, general situations. We can again 'deconstruct the construction'. We want to construct our sheaf x, so suppose $V \subset U$ and will try to construct $x(V)$. First look at the result of $\alpha^i_*(a_i)(V)$. The lower star version of the induced functor, f_*, is given by $f_*(F)(V) = F(f^{-1}(V))$, so here

$$\alpha^i_*(a_i)(V) = a_i(U_i \cap V),$$

and, if $U_i \cap V$ is empty, this will be a singleton set.

We know the restrictions to U_{ij} of a_i and a_j are equal, so over V, we have a diagram of sets,

$$\textstyle\prod_i a_i(U_i \cap V) \rightrightarrows \prod_{ij} a_i(U_{ij} \cap V)$$

with the two maps given by restriction. The natural thing to try is to define $x(V)$ to be the equaliser of these two maps.. We thus have $x(V)$ 'is' the set of compatible local sections of the a_is over V, which is a 'sensible' construction to make! We should still check x is a sheaf - but that is **left as an exercise**.

If we reinstate the θ_{ij}s, all we need to do is to change one of the maps. We keep

$$a_i \to \alpha^{ij}_{i*}(\alpha_i^{ij*}(a_i)),$$

but there is also

$$a_i \to \alpha^{ij}_{i*}(\alpha^{ij*}_j(a_j))$$

obtained by composing with $\alpha^{ij}_{i*}(\theta_{ij})$. Any apparent preference given to i over j here is an illusion since elsewhere in the product we have j then i.

We have sketched:

Theorem 25 *The fibred category,* $\mathsf{Sh}(B)$*, is a stack.* ■

We leave to the reader the extension of the above proof that is needed to show:

Theorem 26 *The fibred category,* $\mathsf{ShGrp}(B)$*, of sheaves of groups on* B *is a stack.* ■

We will also need to note that there is a 'full substack[27]' of $\mathsf{Sh}(B)$, determined by the locally constant sheaves of sets on B. This will be denoted $\mathsf{LCSh}(B)$. We will return to this later on, in section 9.8.

9.6.5 Stacks of Torsors

There are other fairly obvious examples of stacks. If we denote by $Vect(U)$, the category of vector bundles on U, for U an open set in B, then $F(U) = Vect(U)$ is part of the specification of a fibred category, $\mathsf{Vect}(B)$, on B, and, of course, it is a stack. More interestingly for us, if G is a sheaf or bundle of groups, we have:

Theorem 27 *The fibred category of* G*-torsors,* $\mathsf{Tors}(B)$*, on a space* B *is a stack.*

Proof: The earlier calculations that we did showed it was prestack, so we only have to check 'collatability', (cf. Mac Lane and Moerdijk, [194], for this term in their discussion of gluing of sheaves).

Suppose $F(U) = Tors(U, G)$ is the category of G_U-torsors on U. We can form $\mathsf{Des}(\mathcal{U}, F)$ for any open cover, $\mathcal{U}$, of U, and as G-torsors are sheaves, we can build a *sheaf*, x, from any 'descent data', $(\underline{a}, \underline{\theta})$, *i.e.*, forgetting the G-torsor structure, recording only the underlying sheaf. It thus remains to check that x is a G-torsor. To do this we can work locally - but then it is almost given to us on a plate! Each a_i is a G_{U_i}-torsor, so we get local sections for free for x, whilst the local actions of the G_U (on the various 'local' torsors) glue to give the structure of a G-torsor on x. (There *are* **details to check**, but they are not that hard.) ■

9.6.6 Strong and weak equivalences: stacks and prestacks

We can now make several observations about strong and weak equivalences of fibred categories, when applied to stacks and prestacks. Recall $\varphi : F \to G$, a morphism of fibred categories was called a *strong* equivalence if for each open set, U, of B, φ_U was an equivalence of categories, whilst it was a *weak* equivalence if each φ_U was full, faithful and locally essentially surjective on objects. This last condition was, intuitively, that one might have to refine before finding an object locally isomorphic to the given one. If, however, we can glue objects up to isomorphism, then if we have φ is a weak equivalence, the glued object will be isomorphic to the given one, so φ_U will actually be 'eso', *i.e.*, will be an equivalence. We thus have:

[27] in an obvious sense

Lemma 63 *If $\varphi : F \to G$ is a weak equivalence of prestacks over B and F is a stack, then φ is also a strong equivalence.* ∎

We can use gluing of objects to obtain other simple consequences.

Lemma 64 *Suppose given prestacks, F, G and H, morphisms $\varphi : F \to G$ and $\psi : F \to H$, and suppose (i) φ is a weak equivalence and (ii) H is a stack, then there is a morphism, $\tilde{\psi} : G \to H$, such that $\tilde{\psi}\varphi \cong \psi$ and $\tilde{\psi}$ is unique up to fibred isomorphism among such extensions.*

Proof: (Intuition only here - details **left to you**!) We have to define $\tilde{\psi}$ on some a, so use weak equivalence to find, locally, objects back in F, which 'almost' map to a. This gives descent data which we send to H via ψ, and which we reassemble there, using gluing, to get the object that we will take for $\tilde{\psi}(a)$. Now see what happens with morphisms. ∎

Weak equivalence together with '(pre-)stackness' thus behaves well. If $\varphi : F \to G$ is a strong equivalence, however, then F is a (pre)stack if and only if G is, so the 'object-gluing-up-to-isomorphism' condition will be preserved under strong equivalence, but clearly may not be under weak equivalence.

These last few comments indicate that when trying to stackify, weak equivalences are still very useful. We want the stack version of 'associated sheaf', so we try the following definition:

Definition: Let F be a prestack. An *associated stack* for F consists of a stack, $\tilde{F}$, and a weak equivalence, $\varphi : F \to \tilde{F}$.

We, of course, do not yet know if such things exist, but we do know:

Proposition 102 *Given a prestack F, if an associated stack, $(\tilde{F}, \varphi)$, exists for F,*
(i) it is unique up to strong equivalence,
and
(ii) if $\psi : F \to H$ is any other morphism into a stack, it factors through ϕ, $\psi \cong \tilde{\psi}.\varphi$, i.e., $(\tilde{F}, \varphi)$ *has a 'universal property up to isomorphism'.*

Proof: These are consequences of the lemmas. Suppose $\theta : F \to G$ is a weak equivalence into a stack G, then an extension, $\tilde{\theta}$, of θ over $\tilde{F}$ exists and is a weak equivalence, but then the earlier lemma shows it to be a strong equivalence as well. The second statement is attacked similarly. ∎

This result suggests that stacks satisfy a 2-universal property, in a fairly clear sense, and that the inclusion of $\mathsf{Stacks}(B)$ into $\mathsf{PreStacks}(B)$ should have a '2-adjoint'. We will explore this a bit slightly later after we have looked at constructing stacks from prestacks, which will give us a candidate for that '2-adjoint'.

9.6.7 'Stack completion' aka 'stackification'

This would, however, all be in vain if associated stacks did not exist! Luckily they do.

It pays, yet again, to step back and look at the sheaf case. The associated sheaf of a presheaf is made up of a colimit of compatible families of local sections. We could attempt a similar approach here. We could take a 'colimit' of the descent categories, $\mathsf{Des}(\mathcal{U}, F)$, as $\mathcal{U}$ varies over open covers of U, *i.e.*, something along the lines of

$$\hat{F}(U) = colim_{\mathcal{U}}\mathsf{Des}(\mathcal{U}, F).$$

The only problem is that an ordinary colimit of categories is not going to do the job, rather we need the 'pseudo-colimit' of these categories, or alternatively some 'homotopy colimit' of them in some sense. That way, we will record more of the data on the interrelationships between the $\mathsf{Des}(\mathcal{U}, F)$ as $\mathcal{U}$ varies.

We will tend to use the sheaf / topos theoretic notation, $Y \to U$, etc. for covers here. (It is less encumbered by indices and has the advantage that is only needs a little more work to make the transition from this to 'sites' and 'Grothendieck toposes'.

Discussion of how to build $\hat{F}$: Suppose F is a prestack on B and U is an open set of B. (The prestack condition will be needed in an essential way later on.)

Define a category, $\hat{F}(U)$, as follows:

• An object of $\hat{F}(U)$ consists of data, $(\pi, (\underline{a}, \underline{\theta}))$, where $\pi : Y \to U$ is a cover, and $(\underline{a}, \underline{\theta})$ is an object in $\mathsf{Des}(\pi : Y \to U, F)$. (We may sometimes write $(Y, (\underline{a}, \underline{\theta}))$ instead of the better $(\pi, (\underline{a}, \underline{\theta}))$.)

• A morphism from $(\pi, (\underline{a}, \underline{\theta}))$ to $(\pi', (\underline{a}', \underline{\theta}'))$ will be an equivalence class of locally defined morphisms over finer covers. In detail, given $\pi : Y \to U$ and $\pi' : Y' \to U$, there are covers $\rho : Z \to U$ finer than both, for instance, any cover finer than the pullback cover $Y \times_U Y' \to U$. (If you prefer to think in terms of open covers, $\mathcal{U}$ and $\mathcal{U}'$, this pullback cover is $\{U \cap U' \mid U \in \mathcal{U}, U' \in \mathcal{U}'\}$.)

We have maps of covers, $s : Z \to Y$, and $s' : Z \to Y'$, (or better $s : \rho \to \pi$ and $s' : \rho \to \pi'$), and so objects, $s^*(\underline{a}, \underline{\theta})$, and $s'(\underline{a}', \underline{\theta}') \in \mathsf{Des}(\rho, F)$. A *local morphism* from $(\pi, (\underline{a}, \underline{\theta}))$ to $(\pi', (\underline{a}', \underline{\theta}'))$ will be data (ρ, f), where ρ and f are as above. (Actually this is not quite right because we may need to register the relationship between ρ and π and π', *i.e.*, s and s', as we need to recall that refinement maps need not be unique between two covers. We will see an occasion later on when we will find it useful to record the extra data, but the notation will do fine for the present.)

There are at least two difficulties here. Certainly this corresponds to a reasonably good notion of locally defined morphism between the two objects, but it is very dependent on the *choice* of $\rho : Z \to U$. In our topological situation, we might be tempted to fix $Z = Y \times_U Y'$ and try to work with that, but that seems slightly odd as we might deny ourselves some morphisms which are *more* locally defined, that is, on finer covers, so that should be avoided. If we pass to finer covers than on our defining ρ, then we will get restrictions of any morphisms that we have already found and hence get, sort of 'in the limit', 'germs' of locally defined morphisms. In other words, we should consider some equivalence classes of locally defined morphisms under refinement rather than the basic morphisms themselves. That seems 'right' as it is a similar intuition to the idea in 'sheafification', where local sections are replaced by germs of local sections, and categorically, that is a colimit. This extra abstraction means that we can handle it in other situations than just $Sh(B)$, e.g., in the toposes that arise in non-topological contexts.

In fact, we assumed that F was a *prestack*, so the presheaf of local morphisms between objects *is a sheaf*, which means that our geometrically inspired idea above is very firmly based.

That is the first difficulty overcome. The second difficulty looks, initially, more serious - but, in fact, vanishes when we examine it closely as it is handled by the passage to 'germs' that we have already mentioned, and thus by the fact that F is assumed to be a prestack. The query is: "how are we to define composition of morphisms? In other words, we claimed that $\hat{F}(U)$ was a category, so we need to define its structure and we have not yet done that! We first need to set up the situation in a bit more detail.

We have three objects: $(\pi_i, (\underline{a}_i, \underline{\theta}_i))$, $i = 0, 1, 2$, and morphisms: $(\rho_{ij}, f_{ij}) : (\pi_i, (\underline{a}_i, \underline{\theta}_i)) \to (\pi_j, (\underline{a}_j, \underline{\theta}_j))$ for $(i, j) = (0, 1)$ and $(1, 2)$. We need to 'compose' f_{01} and f_{12}, but (oh dear!), they are in different categories:

- f_{01} is in $\mathsf{Des}(\rho_{01}, F)$;
- f_{12} is in $\mathsf{Des}(\rho_{12}, F)$,

but this viewpoint does not take account of the more geometric 'vision' of these locally defined morphisms as equivalence classes or 'germs', thus each morphism really contains not only the information that we see 'on the surface' notation, but also all its restrictions to finer covers.

We can find some Z_{012} finer than $Z_{12} \times_U Z_{01}$ giving a $\rho_{012} : Z_{012} \to U$, and can restrict f_{01} and f_{12} to Z_{012}, along those refinements. We thus have representatives, f'_{01} and f'_{12}, of the corresponding morphisms within $\mathsf{Des}(\rho_{012}, F)$, and, as is easily checked, *can* compose them. If we replace ρ_{012} by some finer cover, everything still works and is compatible with the restriction maps, (**left to you to check**), so we *do* have a well defined composition.

To summarise, the objects of $\hat{F}(U)$ are locally defined objects, whilst the morphisms are 'germs' of locally defined morphisms.

If $V \subset U$, then restriction all round yields a functor from $\hat{F}(U)$ to $\hat{F}(V)$, and, not surprisingly, if we have $W \subset V \subset U$, then we get natural transformations between the various functors yielding a pseudo-functor, $\hat{\mathsf{F}} : Open(B)^{op} \to \mathsf{Cat}$, *i.e.*, a fibred category $\hat{\mathsf{F}}$. There is clearly a morphism,

$$\omega : \mathsf{F} \to \hat{\mathsf{F}},$$

and the construction of $\hat{\mathsf{F}}$ makes it clear that locally defined objects glue "up to isomorphism", so $\hat{\mathsf{F}}$ is a stack, (but again **detailed checking is for you to follow up**).

We thus have:

Theorem 28 *For every prestack* F*, the above constructed* $\hat{F}$ *is an* associated stack*, (or* stack completion *or even* stackification*) of* F*.* ■

If F is an arbitrary fibred category, then we first take its prestackification, as explained earlier, then stack complete that prestack to get the *stack completion* of the original fibred category.

Summarising from our discussion here, we extract the following 'idea of a proposition':

Proposition 103 *The forgetful 2-functor,*

$$\mathsf{For} : \mathsf{Stacks}(B) \to \mathsf{PreStacks}(B),$$

has a left 2-adjoint 2-functor given by stackification. ■

We will not collect things up to give anything like a 'formal' proof, as this can be found in various forms in the literature, but we do need to put a little 2-categorical flesh on this to clarify the meaning[28]. It may help to look as well in Steve Lack's Companion, [182].

[28] although it should be fairly intuitively clear what the various terms mean,

We will, for simplicity, assume our 2-categories are strict for the moment.

Let $\mathcal{C}$ be a 2-category and $f : x \to y$ be a 1-morphism in $\mathcal{C}$. Thinking of the typical example of $\mathcal{C} = Cat$, it is clear what it should mean to say that f has a right adjoint.

Definition: A *right adjoint* to f will be a triple, (g, η, ε), with $g : y \to x \in \mathcal{C}(y, x)$, a 1-morphism, called 'the' right adjoint of f, and $\eta : 1_x \to gf$, $\varepsilon : fg \to 1_y$ are 2-morphisms / 2-cells, which satisfy the 'triangle equations', which can be found in any source on adjoint functors.

Similarly, we say f is an *adjoint equivalence* if there is structure (g, η, ε) as above, but with, in addition, η and ε invertible 2-morphisms.

This abstracts 'adjointness', but we still have to 'categorify' it.

Suppose now $\mathcal{C}$ and $\mathcal{D}$ are 2-categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are (strict) 2-functors, then there are two 2-functors, $\mathcal{D}(F-, -)$ and $\mathcal{C}(-, G-)$, from $\mathcal{C}^{op} \times \mathcal{D} \to Cat$.

Definition: A *2-adjunction* from $\mathcal{C}$ to $\mathcal{D}$ is given by a pair of 2-functors, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, together with a '2-equivalence[29] between $\mathcal{D}(F-, -)$ and $\mathcal{C}(-, G-)$.

The other terminology then follows the expected path.

Comments: All the above also exists in lax / op-lax / pseudo versions, which are not hard to find in the literature. For the moment we actually only seem to need this strict version.

9.6.8 Stackification and Pseudo-Colimits

The sheafification of a presheaf can be done using a colimit construction, something like

$$\tilde{F}(U) = colim_{\mathcal{U}} \mathsf{Des}(\mathcal{U}, F),$$

that is, a colimit of families of local sections, yielding 'germs' of local sections, in some sense.

Earlier we suggested (i) that $\hat{\mathsf{F}}$ should be given similarly by some formula such as

$$\hat{\mathsf{F}}(U) = ps{-}colim_{\mathcal{U}} \mathsf{Des}(\mathcal{U}, \mathsf{F}),$$

that is, a pseudo-coimit of the descent categories, $\mathsf{Des}(\mathcal{U}, \mathsf{F})$, over $Cov(U)$, the category of covers of the open set, U, and (ii) pseudo-colimits are a sort of 'homotopy colimit', (to be investigated later) and are given, up to equivalence, by a modification of the Grothendieck construction. We have examined that construction in quite a lot of detail above, so it seems a good idea to see how the description as a pseudo-colimit of $\mathsf{Des}(-, \mathsf{F})$ tallies with the construction we have given above. To start with we will work with the Grothendieck construction, which is *not quite the right one to use, and will need to be modified.* (This is on the principle that it is a good idea to start where you are, and not where you would like to be!) The Grothendieck construction is more exactly an op-lax colimit as we saw (section 9.4.1). The difference between this and the pseudo-colimit is that there are certain 2-cells that we would like to be invertible, but are not!

[29] that is, an equivalence in the 2-category $2Fun(\mathcal{C}^{op} \times \mathcal{D}, Cat)$, of 2-functors from $\mathcal{C}^{op} \times \mathcal{D}$ to Cat,

We have our prestack, F, and thus our pseudo-functor, $\mathsf{Des}(-,\mathsf{F}) : Cov(U)^{op} \to \mathsf{Cat}$. For brevity, let us call this pseudo-functor $\mathsf{X} : Cov(U)^{op} \to \mathsf{Cat}$, and, for the sake of comparison, keep to the sheaf theoretic view of covers as morphisms, $\pi : Y \to U$, with some nice properties such as stability under pullbacks. (Of course, this is *really* specifying the Grothendieck topology on $Sh(B)$, and, as was pointed out earlier, has the additional advantage of being much nearer the notation and terminology needed to make the transition from $Sh(B)$ to a general topos.)

We need to look at the category that we would have been calling, $\mathcal{E}_{\mathsf{X}}$, in our section on fibrations. We list the structure, transcribing from that earlier description:

- An object of $\mathcal{E}_{\mathsf{X}}$ is a pair $((\underline{a},\underline{\theta}),\pi)$, where $\pi : Y \to U$, and $(\underline{a},\underline{\theta})$ is an object of $\mathsf{Des}(\pi,\mathsf{F})$. ('So far so good', it has the same objects as $\hat{\mathsf{F}}(U)$, except for a different convention in the order of the pair, which should not disturb us unduly.)

- A morphism from $((\underline{a},\underline{\theta}),\pi)$ to $((\underline{a}',\underline{\theta}'),\pi')$ is a pair, (f,s), where $s : \pi \to \pi'$ in $Cov(U)$ and $f : (\underline{a},\underline{\theta}) \to (\underline{a}',\underline{\theta}')$ is in $\mathsf{X}(\pi)$.

This description of morphisms somehow looks completely different from that in $\hat{\mathsf{F}}(U)$, so what is going on here. We should examine the morphisms a bit more closely. (Once we have done that, the relationship is almost self evident and the differences will, it is hoped, look less stark. The analysis of the morphisms is also of use later on, so is not, in any case, a waste of effort.)

There are two obvious special types of morphism. The first has s the identity and so (f,s) is a morphism in the fibre, $\mathsf{X}(\pi)$. The second is, sort of, a morphism induced by an $s : \pi \to \pi'$, so there is some $(\underline{a}',\underline{\theta}')$ in $\mathsf{X}(\pi')$, and hence $s^*(\underline{a}',\underline{\theta}')$ in $\mathsf{X}(\pi)$. Of course, we therefore have (id_x, s) is a morphism, where $x = s^*((\underline{a}',\underline{\theta}')$. (We will usually just write (id,s) for this. We also note it is Cartesian.)

We look at a composite of the two types of morphism and note that $(f,s) = (id,s)\sharp_0(f,id)$, so any morphism in $\mathcal{E}_{\mathsf{X}}$ can be factorised in this way:

$$\begin{tikzcd} {} \arrow[r, "{(id,s)}"] & {} \\ {} \arrow[u, "{(f,id)}"] \arrow[ur, "{(f,s)}"'] & \end{tikzcd}$$

– – – – – – – – – – – –

$$\pi \xrightarrow{s} \pi'$$

It is quite interesting to see the composite of the other sort, *i.e.*, first an induced map and then one in the fibre. We will just give the answer, leaving you to **check it** using the formula for composition from earlier:

$$(g,id)\sharp_0(id,s) = (s^*(g),s)$$

This is reminiscent of the semi-direct product formula which is not that surprising.

We now look at the situation of morphisms in $\hat{\mathsf{F}}(U)$. We have covers $\pi : Y \to U$ and $\pi' : Y' \to U$, much as before but neither needs to be a refinement of the other. Instead, we have $\rho : Z \to U$, a joint refinement, so have $s : \rho \to \pi$ and $s' : \rho \to \pi'$. Pausing for a moment, that gives us, in $\mathcal{E}_{\mathsf{X}}$, some morphisms:

$$(id, s) : (s^*(\underline{a}, \underline{\theta}), \rho) \to ((\underline{a}, \underline{\theta}), \pi)$$

and

$$(id, s') : (s^*(\underline{a}', \underline{\theta}'), \rho) \to ((\underline{a}, \underline{\theta}), \pi).$$

We also have, in the fibre, $\mathsf{X}(\rho)$, a morphism

$$f : s^*(\underline{a}, \underline{\theta}) \to s^*(\underline{a}', \underline{\theta}'),$$

and so, again in $\mathcal{E}_{\mathsf{X}}$,

$$(f, id) : (s^*(\underline{a}, \underline{\theta}), \rho) \to (s^*(\underline{a}', \underline{\theta}'), \rho).$$

This makes it clear that in $\hat{\mathsf{F}}(U)$, our typical morphism would be thought of as a composite

$$(id, s')\sharp_0(f, id)\sharp_0(id, s)^{-1}.$$

The only problem is ... (id, s) is not invertible, as s is not invertible (except in exceptional cases).

As this does not seem to give what might be expected, let us go about it the other way around. Instead of writing morphisms in $\hat{\mathsf{F}}(U)$ in the language of $\mathcal{E}_{\mathsf{X}}$, look at the morphisms of $\mathcal{E}_{\mathsf{X}}$ and see if they interpret well in terms of $\hat{\mathsf{F}}(U)$. (We will have to adjust notation back again, so will have to be careful!)

Suppose we have (f, s) in $\mathcal{E}_{\mathsf{X}}$ with $s : \pi \to \pi'$ in $Cov(U)$ and $f : (\underline{a}, \underline{\theta}) \to s^*(\underline{a}', \underline{\theta}')$ in $\mathsf{X}(\pi)$. We do not here need to refine both our covers. Instead of

$$\begin{tikzcd}
 & \pi \\
\rho \arrow[ur, "s"] \arrow[dr, "s'"'] & \\
 & \pi'
\end{tikzcd}$$

in $Cov(U)$, we have $\rho = \pi$ and the s' will be our new s. This means that our basic morphism (f, s) becomes (π, f) in '$\hat{\mathsf{F}}(U)$-speak', and it looks as if there is an 'inclusion' or 'injection' of $\mathcal{E}_{\mathsf{X}}(((\underline{a}, \underline{\theta}), \pi), ((\underline{a}', \underline{\theta}'), \pi'))$ into $\hat{\mathsf{F}}(U)((\pi, (\underline{a}, \underline{\theta})), (\pi', (\underline{a}', \underline{\theta}')))$.

We should look at these more closely perhaps, but this is **for you to follow up.**

9.6.9 Stacks and sheaves

How different is a stack from a sheaf? The answer is 'very different'. To illustrate this, we will look at a sheaf of groups, G, on B. We can think of groups as single object groupoids to get a sheaf of categories, $G[1]$. Assume U is an open set of B and $\mathcal{U}$ is an open cover of U. What does $\mathsf{Des}(\mathcal{U}, G[1])$ look like? As each $G[1](U)$ has a single object, we have not that much choice for the $\underline{a}$ part of an object $(\underline{a}, \underline{\theta})$, but θ_{ij} is an arrow in $G[1](U_{ij})$, *i.e.*, an element of $G(U_{ij})$ and the cocycle conditions imply that $\underline{\theta}$ is a cocycle, determining a G-torsor on U, trivialised over $\mathcal{U}$. (This should make us expect that the morphisms in $\mathsf{Des}(\mathcal{U}, G[1])$ will be given by coboundaries!) Suppose $\underline{\theta}$ and $\underline{\rho}$ are two objects of this category $\mathsf{Des}(\mathcal{U}, G[1])$, then a morphism, $f : \underline{\theta} \to \underline{\rho}$, is given, yes, by a family, $\{f_i\}$, of arrows with f_i, an arrow in $G[1](U_i)$, hence 'really' by an element in $G(U_i)$ and the condition on these is that

$$\rho_{ij}.\alpha_j^{ij}(f_j) = \alpha_i^{ij}(f_i)\theta_{ij}.$$

The notation for the general case that we have used here is perhaps getting in the way a bit. If we write $g_{ij} = \theta_{ij}$, $g'_{ij} = \rho_{ij}$, $g_i = \alpha_i^{ij}(f_i)$, etc., then this is just

$$g'_{ij} = g_i g_{ij} g_j^{-1}$$

over U_{ij}, *i.e.*, it is *exactly* the coboundary relation. We thus have $\mathsf{Des}(\mathcal{U}, G[1])$ yields precisely the part of $Tor(U; G)$ of those G-torsors trivialised by $\mathcal{U}$ and, on forming the corresponding pseudo-colimit, we get the whole of $Tor(U; G)$. In other words, not only is $G[1]$ nowhere near being a stack, we have identified its 'stackification':

Theorem 29 *For a sheaf of groups, G, on B, the associated stack of* $G[1]$ *is* $\mathsf{Tors}(B; G)$ ∎

To help with the deciphering of the general situation, it is worth noting that the natural morphism

$$G[1] \to \mathsf{Tors}(B; G)$$

sends the single (global) object of $G[1]$ to the trivial G-torsor and similarly over any open set U. The local triviality condition on torsors then translates to saying that this morphism is a weak equivalence of fibred categories.

This example leads to the observation that for any prestack, F, on B, the associated stack $\hat{F}$ is characterised by the property that every object of $\hat{F}$ is locally contained in the essential image of F, *i.e.*, is locally isomorphic to an object of F.

9.6.10 What about stacks of bitorsors?

There is a certain implacable logic in the development of non-Abelian cohomology. Certain structures keep on coming up and then varying along the categorification process. Certain questions recur, usually in evolving form.

We have seen $Tors(B; G)$ gave $\mathsf{Tors}(B; G)$, the corresponding stack. Moreover this was the associated stack of the sheaf or bundle of groups, G, itself. Earlier we met bitorsors and relative M-torsors. It is natural to wonder if (G, H)-bitorsors on B form a stack and, of course they do as we have seen that left G-torsors form a stack, thus forgetting the right H-action, we can glue locally defined (G, H)-bitorsors up to isomorphism, then reinstate the H-action. That gives an idea of how to proceed with the proof of the last part of the verification. That it forms a prestack is also straightforward. We thus have, for G, H, two sheaves of groups on B, fibred categories, $\mathsf{Bitors}(G, H)$ and $\mathsf{Bitors}(G)$. What is more, the pairing structure given by the contracted product give morphisms of these categories. We have:

Theorem 30 *(i) Given sheaves of groups, G, and H,* $\mathsf{Bitors}(G, H)$ *is a stack.*

(ii) Given sheaves of groups, G, H and K, there is a morphism of fibred categories

$$\mathsf{Bitors}(G, H) \times \mathsf{Bitors}(H, K) \to \mathsf{Bitors}(G, K)$$

induced by contracted product.

(iii) For G a sheaf of groups, $\mathsf{Bitors}(G)$ *is a* gr-stack, i.e., *each of the fibres, $Bitors(G_U)$, is a gr-category,* i.e., *a group-like monoidal category, with the restrictions respecting the structure.* ∎

(The second part requires the definition of product of fibred categories, but that is given by fibrewise product **so should cause no technical 'difficulties to the reader'**.)

We take the obvious next step, that is to examine the fibred category, $\mathsf{M}-\mathsf{Tors}$ (or '$\mathsf{M}-\mathsf{Tors}(B)$', if need be). First we note that if $\varphi : G \to H$ is a morphism of sheaves of groups, then the induced functor, φ_* from $Bitors(G)$ to $Bitors(H)$, 'localises' so as to give a morphism of fibred categories, which is given by $\varphi_*(E) = H_\varphi \wedge E$.

If $\mathsf{M} = (C, P, \partial)$ is a sheaf of crossed modules, then any relative M-torsor is a C-torsor, E, together with a global section, t, of $\partial_*(E)$. Restriction and contracted product work well together. Contracted product is given by a coequaliser of a pair of morphisms of sheaves, so restricts without problem from an open set U to an open subset, V, of U, or to an open cover, $\mathcal{U}$, for that matter. The prestack condition is thus reasonably easy to check - local compatibility with a given global section, t, transfers to any glued morphism. More precisely, if a, b are M-torsors over U, $a = (E, s)$, $b = (E', t)$, then the restricted bitorsors $f^*(a)$, $f^*(b)$ for $f : V \to U$ have the form $(f^*(E), t|_V)$ so, given a family of bitorsor morphisms, $\varphi^i : (\alpha^i)^*(a) \to (\alpha^i)^*(b)$, over an open cover $\{U_i\}$, the resulting glued morphism from a to b is compatible with s and t, since locally these φ^i were. We thus have that $\mathsf{M}-\mathsf{Tors}$ is at least a prestack.

Now assume we glue together any descent data, $(\underline{a}, \underline{\theta})$, for M-torsors, considering them as C-torsors, to get, at very least, a C-torsor, E, locally isomorphic to the C_{U_i}-torsor, E_i, over the set U_i of the open cover $\mathcal{U}$. We then get a P-torsor, $\partial_*(E)$, and a family of local isomorphisms, $\sigma_i : E_i \cong E|_{U_i}$, and thus

$$\partial_*(\sigma_i) : \partial_*(E_i) \cong \partial_*(E)|_{U_i} = \alpha^{i*}\partial_*(E) = \partial_*\alpha^{i*}(E).$$

Now $a_i = (E_i, t_i)$, where t_i is a global section of $\partial_*(E_i)$ over U_i. We use this local section, t_i, to obtain a local section, $t'_i = \partial_*(\sigma_i)t_i$, of $\partial_*(E)|_{U_i}$ over U_i. Over U_{ij}, we have an isomorphism of M-torsors

$$\theta_{ij} : \alpha_j^{ij*}(a_j) \to \alpha_i^{ij*}(a_i),$$

so

$$\begin{tikzcd} \partial_*\alpha_j^{ij*}(a_j) \arrow[rr, "\partial_*\theta_{ij}"] & & \partial_*\alpha_i^{ij*}(a_i) \\ & U_{ij} \arrow[ul, "t_j"] \arrow[ur, "t_i"'] & \end{tikzcd}$$

commutes and the σ_is are compatible, as sheaf isomorphisms, with these θ_{ij}s. This implies that the t'_i form a compatible family of local sections of $\partial_*(E)$, which glue to form a global section t of $\partial_*(E)$, *i.e.*, $(E, t) \in \mathsf{M}-Tors(U)$. We have thus checked:

Theorem 31 *If $\mathsf{M} = (C, P, \partial)$ is a sheaf of crossed modules over B, then the fibred category $\mathsf{M}-\mathsf{Tors}(B)$ is a stack, in fact, a gr-stack.* ■

The final comment follows from the structure of a gr-groupoid on each 'fibre', compatibly with restriction.

As M defined a sheaf of gr-groupoids, we are led to another query. A single sheaf of groups, G, led after stackification to a stack which was equivalent to the stack of G-torsors. If we replace G by

M, and think of it as a sheaf of gr-groupoids, we must surely get $\mathsf{M}-\mathsf{Tors}(B)$ after stackification, mustn't we?

To investigate this, learning from the case of G-torsors, we take a direct approach. Let $\mathcal{X}(\mathsf{M})$ denote the sheaf of gr-groupoids associated to M, so $\mathcal{X}(\mathsf{M})$ has for its sheaf of objects the sheaf P and for its sheaf of arrows, $C \rtimes P$. This $\mathcal{X}(\mathsf{M})$ will be our F for this example. We explore what $\mathsf{Des}(\mathcal{U}, F)$ looks like for this F and an open cover, $\mathcal{U}$, of an open set U of B. We translate the definition of $\mathsf{Des}(\mathcal{U}, F)$ to this context. It gives:

- **Objects:** $(\underline{a}, \underline{\theta})$, where $\underline{a} = \{a_i\}$ with $a_i \in \mathcal{X}(\mathsf{M})(U_i) = P_i = P(U_i)$ and $\underline{\theta} = \{\theta_{ij}\}$, where

$$\theta_{ij} : \alpha_j^{ij*}(a_j) \overset{\cong}{\to} \alpha_i^{ij*}(a_i).$$

Thus $a_i \in P_i$ and, to make our lives more interesting, we will write p_i instead of a_i. The θ_{ij} are arrows, which are naturally invertible in this context, from $p_j|_{U_{ij}}$ to $p_i|_{U_{ij}}$. As such they will be of the form $(c_{ij}, p_j) \in (C \rtimes P)(U_{ij})$. (For obvious reasons we will, for the moment, throw away the α_i^{ij*}-notation, reverting to our earlier notation of writing 'p_j^i over U_{ij}' or saying that an equation holds 'over U_{ij}', as here it has no risk attached, unlike in some other contexts.) As (c_{ij}, p_j) has target $\partial(c_{ij})p_j$, this gives us $p_i = \partial(c_{ij})p_j$ over U_{ij}. (We have seen that before!)

- **Arrows:** $f : (\underline{a}, \underline{\theta}) \to (\underline{b}, \underline{\rho})$, or, changing notation, $f : (\underline{p}, \underline{c}) \to (\underline{p'}, \underline{c'})$, will be a family of arrows, $f_i : p_i \to p'_i$, in $\mathcal{X}(\mathsf{M})(U_i)$, but that gives a family $\{c_i\}$ with $c_i \in C_i$ such that $p'_i = \partial c_i.p_i$. These f_i have to satisfy the compatibility condition with regard to the θ_{ij} part of the objects - and, yes, you guessed, this translates to

$$c'_{ij} = c_i c_{ij} c_j^{-1}.$$

In other words we have exactly the objects and arrows we need to get:

Theorem 32 *Given a sheaf of crossed modules,* M*, the associated stack of the sheaf of gr-groupoids,* $\mathcal{X}(\mathsf{M})$*, is the gr-stack,* $\mathsf{M}-\mathsf{Tors}(B)$*.*

Of course, there is a lot still to check, e.g., that this local description of $\mathsf{Des}(\mathcal{U}, F)$ does pass to the colimit, that everything is compatible with the gr-groupoid / contracted product structure, etc. but **this can all be safely left 'to the reader'**.

As a corollary we get

Corollary 20 *The gr-stack,* $\mathsf{Bitors}(G)$*, is the associated gr-stack of the sheaf of crossed modules,* $G \to Aut(G)$*,* i.e.*, of* $\mathsf{Aut}(G)$*.*

(We should note that the use of the 'the' in 'the associated stack' in these results is not quite right, as associated stacks are only defined 'up to equivalence'.)

We have seen that $\mathsf{M}-Tors$ is a gr-category and that the corresponding stack, $\mathsf{M}-\mathsf{Tors}$, is a gr-stack and thus that this is true, in particular, for $\mathsf{M} = \mathsf{Aut}(G)$. An important case of this is when G is a sheaf of Abelian groups, then $\mathsf{Aut}(G) = (G, Aut(G), 0)$, since G will have no non-trivial inner automorphisms. This has several implications. Most of these apply in more generality so we will look at a general crossed module of form $\mathsf{M} = (C, P, 0)$, so C is a P-module and the 'boundary map', ∂ is the trivial homomorphism. This assumption means that any representing map $\underline{\mathbf{g}} : N(\mathcal{U}) \to K(\mathsf{M})$ reduces to an assignment of elements c_{ij} to U_{ij} and p_i to U_i such that

$c_{ij}c_{jk} = c_{ik}$ and $p_j = p_i$ over U_{ij}. We thus have a C-torsor, E, on which P acts and a global section of P.

For the gr-stack structure, the C-torsor, E, that one gets is both a right and left C-torsor, as we have seen. The right action need not be the obvious one from symmetry as we have a formula for it as $e_i.c = {}^{p_i}c_i.e_i$ (see page 348 and the discussion there). This has to be interpreted with care: $e_i.c$ is the result of acting with c on the right of the local section e_i. It is not obtained by multiplication. The contracted product is symmetric as again we saw earlier (**if you did the exercise**!) and so, of course, $\pi_0(\mathsf{M} - \mathsf{Tors})$ is an Abelian sheaf.

9.6.11 Stacks of equivalences

What we next look at could have been discussed at any point almost from the first chapter onwards. We saw there that a group, G, can be considered as a groupoid with one object, for which we have often written $G[1]$, indicating a suspended or categorified version of G. Also very early on, we met the crossed module, $\mathsf{Aut}(G) = (G, Aut(G), \iota)$, and have used it many times in later chapters. There is a neat link between them.

Looking at two groups, G, H, we can examine the interpretation of categorical notions and constructions such as functor, natural transformation, equivalence of categories, etc., for $G[1]$ and $H[1]$. For instance, a functor from $G[1]$ to $H[1]$ is clearly just a homomorphism from G to H. A natural transformation is a little bit more subtle. A natural transformation $\eta : f \Rightarrow g$ between two such functors picks out an arrow, $\eta(a) : f(a) \to g(a)$, in $H[1]$ for each object a of $G[1]$, but there is but one such object and as arrows in $H[1]$ are just elements of H, η 'is' an element h of H such that $h.f(x) = g(x).h$ for all $x \in G$, *i.e.*, as we saw earlier, $g = h.f.h^{-1} = i_h \circ f$.

If we ask for conditions on $f : G \to H$, so that $f[1]$ is an equivalence of groupoids, we will get a $f' : H \to G$ and natural isomorphisms, $\eta : ff' \Rightarrow Id_H$, $\varepsilon : Id_G \Rightarrow f'f$, so there are elements $g \in G$ and $h \in H$ such that for all $y \in H$, $ff'(y) = hyh^{-1}$ and for all $x \in G$, $gxg^{-1} = f'f(x)$. We thus have that f almost looks like an isomorphism - is it in fact one? We can try to prove that it is and see what happens. For instance, f is easily seen to be a monomorphism since if $f(x) = 1$, then $gxg^{-1} = 1$, *i.e.*, $x = 1$. Is it an epimorphism? If we have $y \in H$, set $y' = h^{-1}yh$ to find $f(f'(y')) = y$, so it is. An equivalence is an isomorphism therefore. (Another amusing way to prove that fact is to find an inverse isomorphism by manipulating f' - this is **left to you**!)

The most immediately important example of this type is the case when $G = H$ and one is looking at self equivalences of $G[1]$. As $G[1]$ is a groupoid, we can form the category of functors from $G[1]$ to itself. Of course, this also has a monoid multiplication,

$$Gpd(G[1], G[1]) \times Gpd(G[1], G[1]) \to Gpd(G[1], G[1]),$$

given by composition in the 2-category of groupoids (so this multiplication is a functor). We restrict to the subcategory $Aut(G[1]) \subseteq Gpd(G[1], G[1])$, where, of course, this stands for the automorphism 2-category of $G[1]$; again, of course, this is both a group object and a groupoid, *i.e.*, after a tiny bit of checking, it is an internal group in *Gpd* or an internal groupoid in *Groups* or a strict gr-groupoid. This means we should be able to identify the associated crossed module - the group of objects is just the automorphism group of G and as 'natural transformation are conjugations', the top group is isomorphic to G itself with $\partial = i : G \to Aut(G)$, so the associated crossed module of $Aut(G[1])$ is $\mathsf{Aut}(G)$.

Breen, in [50], notes a neat way of looking at $Aut(G[1])$. We will adopt his notation for the discussion, writing BG for $Ner(G[1])$, as we did in our discussion of Puppe sequences. (The geometric realisation of this is the classifying *space* of G, which is what is the thing more normally denoted BG.) Extending this to a sheaf of groups, G, we get a prestack of (local) self equivalences of BG, denoted $Eq(BG)$. Of course, BG is a simplicial sheaf and the equivalences are equivalences of the *sheaf* so *do* restrict in a reasonable way. An equivalence of BG is just an automorphism of G, and, again of course, the natural transformations are given by conjugation. It is easy to check that this identifies $Eq(BG)$ with $Aut(G[1])$, the gr-prestack that we have considered earlier. One can trace this phenomenon, that the equivalences of BG are just the automorphisms, as being due to the fact that the nerve functor from groupoids (or more generally small categories) into $\mathcal{S}$ is a full embedding. Of interest also is to calculate $\mathsf{aut}(BG)$ in the sense of section 6.3. Recall that if Y is a simplicial set, $\mathsf{aut}(Y)_n$ consisted of morphisms $\xi_n : Y \times \Delta[n] \to Y$, that when we form $(\xi_n, p) : Y \times \Delta[n] \to Y \times \Delta[n]$, we have an automorphism over $\Delta[n]$, *i.e.*, the diagram

$$\begin{tikzcd} Y \times \Delta[n] \arrow[rr, "(\xi_n,p)"] \arrow[dr, "p"'] & & Y \times \Delta[n] \arrow[dl, "p"] \\ & \Delta[n] & \end{tikzcd}$$

is commutative, where the two slanting arrow are the obvious projection, p, to $\Delta[n]$,. The face and degeneracy maps are induced in the obvious way. Examination of this when $Y = BG$ shows that such a ξ is determined by a sequence, $h_1, \ldots, h_n$, of elements of G together with a starting automorphism. If that looks familiar, **check up** on face and degeneracy maps and you will get an isomorphism

$$\mathsf{aut}(BG) \cong \mathsf{K}(\mathsf{Aut}(G)),$$

the nerve of the associated groupoid structure of the crossed module $\mathsf{Aut}(G)$. (The only annoyance is with the order of composition that must be handled carefully!) This is natural with respect to the sheaf structure induced from that on G.

If we replace auto-equivalences by equivalences between G and a second sheaf of groups H, the same analysis works almost word for word, except of course that $Eq(BG, BH)$ does not have a compositional monoid structure. What replaces that is an action of $Eq(BG)$ by *precomposition* and one by $Eq(BH)$ by *postcomposition*[30].

We thus have

Lemma 65 *(i) For G, a sheaf of groups on B, the gr-prestack, $Eq(BG)$, is determined by the sheaf of crossed modules, $\mathsf{Aut}(G)$.*

(ii) For G, H, sheaves of groups on B, the pre-stack, $Eq(BH, BG)$, of equivalences is the prestack defined by $Isom(H, G)$.

(iii) The action of $\mathsf{Aut}(G)$ on $Isom(H, G)$ extends to one of $Eq(BG)$ on $Eq(BH, BG)$. ■

The constructions being natural, we can stack-complete, noting that the process of localising over B changes nothing of the structure. The self equivalences of BG then give us self-equivalences of $\mathsf{Tors}(G)$ and the lemma transforms to give:

Proposition 104 *(i) The gr-stack, $Eq(\mathsf{Tors}(G))$, of self-equivalences of the stack, $\mathsf{Tors}(G)$, is the stack associated to the gr-prestack, $\mathsf{Aut}(G[1])$,* i.e., *the stack, $\mathsf{Bitors}(G)$, of G-bitorsors on B.*

[30] These terms 'pre-' and 'post-composition' are neutral with respect to conventions of notation. Functional order makes $Eq(BG, BH)$ a left $Eq(BH)$-object, but algebraic order would change 'left' for 'right'. Care does need to be taken here.

(ii) The stack, $Eq(\mathsf{Tors}(H), \mathsf{Tors}(G))$, of equivalences between the stacks $\mathsf{Tors}(H)$ and $\mathsf{Tors}(G)$ is the stack associated to the prestack, $Isom(H, G)$, of isomorphisms from H to G, and the action of $Eq(\mathsf{Tors}(G))$ on this stack, by post-composition, is that induced from the action of the gr-prestack $\mathsf{Aut}(G)$. The stack, $Eq(\mathsf{Tors}(H), \mathsf{Tors}(G))$, is equivalent to that of (G, H)-bitorsors.

Proof: The argument given earlier, although valid, requires a certain amount of calculation / verification to be completely 'water tight'. Here, therefore, is a separate argument.

Let U be an open set of B and $u : U \to Aut(G[1])$ be a local section over U, then u is a (local) automorphism of $G_U[1]$ and we associated to this a G-bitorsor with trivial underlying left G-torsor structure and, of course, right G-action given via u. More precisely, let $\Lambda(U)$ be T_{G_U} with trivial section, $s : U \to T_{G_U}$, and where

$$s.g := u(g).s.$$

(Writing h for another local element of G, remember $(h.s).g = hu(g).s$, so the actions are independent. We saw this before, of course, when discussing cocycles for bitorsors.) This gives a morphism Λ of fibred categories

$$\Lambda : \mathsf{Aut}(G[1]) \to \mathsf{Bitors}(G)$$

and, of course, it needs to be checked that it works at the level of morphisms - which is '**left to the reader**' as it is a repetition of arguments already rehearsed! It is also easily seen that it is full, faithful and locally 'eso', the latter being by using local triviality of bitorsors, so Λ is the 'stackification' morphism. ■

This whole discussion 'should' be reminding you of our brief excursion into Morita theory. It is time for a revisit, but before we do that note that it is very easy to get various group structures and group-like structures reversed when discussing bitorsors, etc. Two different sources can adopt different conventions leading to confusion. (The author of these notes knows this to his cost *and* does not guarantee to have always resolved the notational problems consistently! For instance, a slight change in convention and notation results in there being an opposite group structure in Breen's [48]. I think that I have an internally consistent convention, but suggest that the reader *always* work with the convention that suits their application and again should *always* be aware that different motivations and different intuitions can lead to different sensible conventions - so *always* check!)

9.6.12 Morita theory revisited

We saw earlier, in Proposition 80, page 331, that any (G, H)-bitorsor, Q, on B gave an equivalence of categories

$$\Phi_Q : Tors(H) \to Tors(G)$$

given by $\Phi_Q(M) = Q \wedge^H M$. This was very well behaved, since we could easily check, from associativity, up to isomorphism, of the contracted product, that Φ_{Q^o} given by Q^o, the (H, G)-bitorsor obtained by reversing the two actions, was an inverse for Φ_Q.

Clearly we can obtain a localised stack version of this very easily. In other words, restricting to open sets U in B, Q_U determines an equivalence between $Tors(U; H)$ and $Tors(U; G)$ and this is compatible with restriction (up to isomorphism) inducing a strong equivalence of stacks between $\mathsf{Tors}(H)$ and $\mathsf{Tors}(G)$.

Conversely, given any equivalence

$$\Phi : \mathsf{Tors}(H) \to \mathsf{Tors}(G)$$

either just at the category level, or of fibred categories, we can find a (G, H)-bitorsor, Q, since $\Phi(T_H)$, the image of the trivial H-torsor gives us one. For simplicity, we will assume Φ is an equivalence of fibred categories, then Q trivialises over some open cover $\mathcal{U}$ giving $Q_U \cong T_{G,U}$ over each U in the cover. We then can use the reverse equivalence, Ψ_U, of Φ_U to obtain a (H_U, G_U)-bitorsor getting us back to $T_{H,U}$. Checking over intersections gives that this identifies Φ_Q as being Φ itself, up to isomorphism. We thus have, again, that $Eq(\mathsf{Tors}(H), \mathsf{Tors}(G))$ is equivalent to $\mathsf{Bitors}(G, H)$, by a more geometric argument.

9.7 Base change for stacks

Continuing our discussion of base change, we have seen how a map, $f : X \to Y$, induced some functors between the categories of sheaves on the two spaces, X and Y:

$$f_* : Sh(X) \to Sh(Y),$$

and

$$f^* : Sh(Y) \to Sh(X),$$

with f^* left adjoint to f_*.

9.8 Locally constant stacks

Chapter 10

Non-Abelian Cohomology: Gerbes

Stacks and gerbes are very closely related. Stacks are the categorified analogues of sheaves of groupoids. Gerbes are stacks with some side conditions. Because of their importance for non-Abelian cohomology, however, they deserve a separate chapter, but to some extent, what goes into a chapter on gerbes could equally well be in one on stacks!

10.1 Gerbes

Before launching into the subject of gerbes, we need first to revisit the relationship between groups and groupoids. We have used many times the fact that if G is a group, it can be thought of as a single object groupoid, usually denoted $G[1]$. We have discussed at various points the role of homomorphisms between groups yielding functors of the corresponding categories / groupoids and conjugations yielding natural transformations. This culminated in our discussion of equivalences at the end of the last section.

All this traffic of ideas may seem one way, from groups to groupoids, but can we see what happens in the opposite direction? Another closely related point for consideration is 'what are the differences?'

Firstly a difference, we may quite often say 'a groupoid is a group with many objects' as a means of expressing the intuition of the relationship, but a groupoid need not have many objects, ... it need not have any objects! An equivalence relation always yields a groupoid as we saw early on. In particular, the empty equivalence relation on the empty set yields, yes, the empty groupoid. This is allowed since the axioms of a (small) category specify a set of objects and a set of arrows, that for each object there is an identity arrow, etc., but if the set of objects is empty, ... ! A group, in the usual definition, cannot be empty as there is an unconditional existence statement for the identity element. Even considering a group as a groupoid, one says it is a groupoid with one object, so is not empty.

If we take two groups, G and H, say, then their coproduct, $G * H$ within the category of groups is what is often called the free product, obtained by freely forming words which alternate between elements of G and those of H. Composition is by concatenation followed by reduction to that alternating form. Take now the groupoids $G[1]$ and $H[1]$ and form their coproduct. This is given by disjoint union, so has 2 objects. It is clearly not $(G * H)[1]$, thus the process of categorification does not preserve coproducts, $G[1] \sqcup H[1]$ is not even a connected groupoid, i.e., its π_0 is not a singleton. This distinction between connected and non-connected groupoids is important. If now

G denotes a groupoid and it is connected, this means explicitly that for any two objects x, y of G, the set $G(x,y)$ of arrows from x to y is non-empty. If we pick an object x_0 in a connected groupoid then for each other object y, we can pick an arrow e_y from x_0 to y. Consider the inclusion of $G(x_0)$ into G or pedantically of $G(x_0)[1]$ into G. This is an equivalence of categories, or, in its homotopy theoretic form, a homotopy equivalence, even a strong deformation retraction. What is the retraction? If $g \in G(y,z)$, then send g to the composite $x_0 \xrightarrow{e_y} y \xrightarrow{g} z \xrightarrow{e_z^{-1}} x_0$, which is in $G(x_0)[1]$. (That this is an equivalence is **left as an exercise**.) (Good references for these sorts of argument in groupoids can be found in Brown's book, [59] or Higgins, [151].) We thus have:

any non-empty connected groupoid is homotopy equivalent to any of its vertex groups.

It is useful to note that the actual equivalence depends on the choice of base point, x_0 and also on that of the chosen edges, e_y.

10.1.1 Definition and elementary properties of Gerbes

(Throughout the sections on gerbes, as such, we will follow and expand on Breen's exposition from [51])

The term 'gerbe' refers to a special sort of stack of groupoids. A gerbe is to a general stack what, up to equivalence, a group is to a general groupoid. (Because of the importance of certain very special types of gerbe in applications, some authors restrict the term to that subclass, but here we will adopt the general terminology as originally used by Giraud and Grothendieck. Another very particular 'misuse' of terminology in some sources is to consider only Abelian gerbes, but to use the term 'gerbe' for all the objects. This can be very confusing to the beginning 'gerbologist', so be warned, always check which definition is being used when using an article on gerbes. Some authors state the assumptions clearly and 'up front', others, unfortunately, not so clearly.)

Definition: (i) A stack of groupoids, F, on B is *locally non-empty* if there is an open cover, $\mathcal{U}$, of B for which each groupoid $F(U)$ is non-empty, for $U \in \mathcal{U}$.

(ii) A stack of groupoids, F, on B is said to be *locally connected* if there is an open cover, $\mathcal{U}$, of B for which each groupoid $F(U)$ is connected, for $U \in \mathcal{U}$.

(iii) A *gerbe* F on B is a locally non-empty, locally connected stack of groupoids on B.

Local connectedness can be well stated by saying that for the various U, if x and y are local objects defined over U, the set $F(U)(x,y)$ is not empty.

Example: Let G be a sheaf or bundle of groups on B and $\mathsf{Tors}(G)$, the stack of G-torsors. If U is any open set in B, then as $\mathsf{Tors}(G)(U) = Tors(U;G)$, the category of G_U-torsors over U, it has at least the trivial G_U-torsor amongst its objects, so $\mathsf{Tors}(G)$ is locally non-empty.

Next look at $\mathsf{Tors}(G)(U)$ again. Any two G_U-torsors are locally isomorphic to each other, since they are both locally isomorphic to the trivial G-torsor, so, if F and F' are two G_U-torsors, these is an open cover such that over that cover F and F' are isomorphic, hence $\mathsf{Tors}(G)$ is locally connected. We thus have that $\mathsf{Tors}(G)$ is a gerbe.

The point about the example is that $\mathsf{Tors}(G)$ has a global object. Given G, we have T_G, the trivial G-torsor over B, i.e., $\mathsf{Tors}(G)(B)$ is non-empty. The automorphism group of T_G is G itself. (This requires a bit of thought perhaps. The automorphisms of T_G include those that are locally defined, i.e., that are in $Aut(T_G)(U)$ for some open set U of B. As we have noted before, $Aut(T_G)$

is a sheaf and it is easy to see that an automorphism sends the trivial section to ... something, and that something is in G and determines the automorphism. We have seen this argument before in several guises, so details should be '**left to the reader**'.)

We also have looked at the 'homs', $\mathsf{Tors}(G)(Y, T_G)$. This is again a sheaf and it has a left action by $Aut(T_G)$, by composition, and, yes, it *is* a $Aut(T_G)$-torsor as is easily checked. Identifying $Aut(T_G)$ with G, identifies $\mathsf{Tors}(Aut(T_G))$ and $\mathsf{Tors}(G)$, and the correspondence is an equivalence of stacks. In other words, we have retrieved $\mathsf{Tors}(G)$ from its internal structure.

We can apply this idea to gerbes in general as follows:

Definition: We say a gerbe, P, is a *neutral gerbe* or is *trivial* if $\mathsf{P}(B)$ is non-empty.

Proposition 105 *If* P *is trivial and* x *is an object of* $\mathsf{P}(B)$*, then defining* $G = Aut_{\mathsf{P}}(x)$ *to be the automorphism sheaf at* x *in* P*, there is an equivalence of gerbes between* P *and* $\mathsf{Tors}(G)$*.*

Proof: First note that G is a sheaf of groups. Using, it is hoped, an obvious notation, for U an open set of B, $G(U) = Aut_{\mathsf{P}(U)}(x_U)$, that is, the vertex group of the object x_U in $\mathsf{P}(U)$, also denoted $\mathsf{P}(U)(x_U)$. This is a sheaf by virtue of the second axiom of stacks, i.e., morphisms glue.

The rest of the proof follows the discussion above for $\mathsf{Tor}(G)$ itself. We note for an object y of $\mathsf{P}(U)$, that $\mathsf{P}(U)(y, x)$ is a left $G(U)$-set, compatibly with the restriction maps to smaller open sets. The action is just composition: writing $\mathsf{P}(y, x)$ instead of $\mathsf{P}(U)(y_U, x_U)$ for convenience, we have

$$\mathsf{P}(x, x) \times \mathsf{P}(y, x) \to \mathsf{P}(y, x)$$

$$(g, h) \mapsto g \circ h$$

in the functional order. This makes $\mathsf{P}(y, x)$ into a G-torsor and the assignment to y of this torsor defines a morphism of stacks from P to $\mathsf{Tor}(G)$. We claim this is an equivalence of stacks.

As P is a stack, we have only to check for each U in B, that the corresponding functor, over U, is full, faithful and locally eso.

For U, $\mathsf{P}(U)$ is a groupoid as is $Tors(G_U)$. The functor sends y, which is in $\mathsf{P}(U)$, to $\mathsf{P}(U)(y, x)$. It sends a morphism $k : z \to y$ to the morphism

$$\mathsf{P}(k^{-1}, x) : \mathsf{P}(z, x) \to \mathsf{P}(y, x),$$

$$h \mapsto hk^{-1}.$$

If we consider a morphism, $\alpha : \mathsf{P}(z, x) \to \mathsf{P}(y, x)$, of $\mathsf{P}(x, x)$-sets, then we have, for each $h : z \to x$, $\alpha(h)^{-1}h \in \mathsf{P}(z, y)$. We claim this is independent of the choice of h. To see why, consider another $h_1 : z \to x$, then $h_1 = h_1 h^{-1} h$, of course, but $h_1 h^{-1} \in \mathsf{P}(x, x)$, so h and h_1 differ only by the action of $\mathsf{P}(x, x)$. The morphism α preserves the action, so $\alpha(h)^{-1}h$ is the same as $\alpha(h_1)^{-1}h_1 = k$, say. Now $k \in \mathsf{P}(z, y)$, and we calculate

$$hk^{-1} = hh^{-1}\alpha(h) = \alpha(h).$$

Thus we have that our functor is full and faithful. It remains to show that it is locally eso., but as P is locally connected, this is almost immediate, since although $\mathsf{P}(U)$ may not be connected, there is an open cover of U such that over each V of that covering $\mathsf{P}(V)$ is connected. Suppose Q is a G_U-torsor, then there is some open cover $\mathcal{V}$ of U, which we can assume finer than $\mathcal{U}$, and isomorphisms $Q_V \cong T_{G_V}$ for $V \in \mathcal{V}$. Over intersections, $V_1 \cap V_2$, of sets of $\mathcal{V}$, we have elements of $G_{V_1 \cap V_2}$, which link the restrictions of the chosen isomorphisms. (We will not give labels and will do everything informally **for the reader to formalise**!) Also $T_{G_V} \cong \mathsf{P}(x_V)$ as a G_V-set. We form descent data relative to $\mathcal{V}$ by picking x_V over V, 'gluing' via the isomorphisms over the intersections. As P is a stack, this is going to give an object y over U, which (i) is isomorphic to x_U, since the locally defined isomorphisms glue to give an arrow in $\mathsf{P}(U)$, and (ii) its image in $Tors(G_U)$ is isomorphic to Q. We thus have that the functor from P to $\mathsf{Tors}(G_U)$ is locally eso. and hence is an equivalence. ■

There are various points to make here. We started with x, a global object and constructed an equivalence beteen P and a gerbe of G-torsors. If we change x, we change the equivalence. We thus may have P equivalent to many different gerbes of G-torsors, for different G. At this point we need to look back at the Morita theory from the last section and subject the lessons of that theory to scrutiny from the perspective here. (We **leave this to you** to do.)

A second point is to note the conceptual similarity between this result and the earlier one which stated that a torsor with a global section is isomorphic to the trivial torsor. Even the proof is conceptually similar. It is a categorification of the earlier one. There are differences as well as we do not have as much structure on $\mathsf{Tors}(G)$ as on an individual torsor. It does not, for example, have the analogue of a multiplication, even though it has a sort of identity object, namely the trivial torsor. The stack of G-bitorsors does have a 'categorified multiplication', but as we will see, is not a gerbe in general.

The third point is that an arbitrary gerbe, P, has a trivialising cover, i.e., there is an open cover $\mathcal{U}$ such that each $\mathsf{P}(U)$ is non-empty, hence P_U, the restriction of P to U, is equivalent to $\mathsf{Tors}(G)$ for some sheaf of groups G on U. Beware, however, even though P_U and P_V for $U, V \in \mathcal{U}$, can both be identified with gerbes of torsors, the corresponding sheaves of groups on U and V are difficult to link up over the intersection. (By now you should be able to guess the sort of construction needed. Over $U \cap V$, there will be two descriptions of $\mathsf{P}(U \cap V)$, linked to two restricted gerbes of torsors, so these restricted gerbes are equivalent, hence we use Morita theory to get a bitorsor over the intersection.) We will return to this later.

We have described gerbes relative to open covers of a space B. We could equally well describe them for a general topos, $\mathcal{E}$, using hypercoverings. There are also intermediate positions that are very useful and that we will visit shortly.

There is the problem that, over two open sets of the cover $\mathcal{U}$, we may get only loosely related sheaves of groups means that these sheaves of groups may not glue together to form a single globally defined G that can be restricted to the U and V to give sheaves G_U and G_V such that $\mathsf{P}_U \simeq \mathsf{Tors}(G_U)$ and $\mathsf{P}_V \simeq \mathsf{Tors}(G_V)$. This problem is one of 'strictification' of the data. To simplify matters, it is useful to assume that there is a global sheaf of groups which does work. Although a restriction on the generality, this does allow much greater progress in the development of the theory to be made, setting up some intuition that can be used if this more general situation is required. This 'global G' situation *is* less general, but as we will see it still includes some very interesting examples.

10.1.2 G-gerbes and the semi-local description of a gerbe

We examine that point in more detail next.

Definition: Let G be a sheaf of groups on B and P a gerbe. We say P is a *G-gerbe* if there is an open cover $\mathcal{U} = \{U_i : i \in I\}$ of B, objects x_i in $\mathsf{P}(U_i)$ and isomorphisms, over each U_i, $G|_{U_i} \cong Aut_{\mathsf{P}_{U_i}}(x_i)$.

If P is a G-gerbe on B and we have chosen local objects x_i over U_i, an open set in the given cover $\mathcal{U}$, then there will be a nice local description of P, namely, there are equivalences

$$\Phi_i : \mathsf{P}_{U_i} \to \mathsf{Tors}(G)|_{U_i}$$

over U_i. If we choose 'quasi-inverses' for these Φ_i, then we can get, over U_{ij}, self-equivalences

$$\Phi_{ij} := \Phi_i|_{U_{ij}} \circ \Phi_j|_{U_{ij}}^{-1} : \mathsf{Tors}(G)|_{U_{ij}} \to \mathsf{Tors}(G)|_{U_{ij}},$$

but thus we get a family, $\{P_{ij}\}$, of G-bitorsors over U_{ij}. These glue on local intersections, U_{ijk}, since there are natural transformations

$$\Psi_{ijk} : \Phi_{ij}\Phi_{jk} \Rightarrow \Phi_{ik},$$

which satisfy a cocycle condition over 4-fold intersections.

At the G-bitorsor level, the Ψ_{ij} define isomorphisms of G-bitorsors

$$\psi_{ijk} : P_{ij} \wedge^G P_{jk} \to P_{ik},$$

above U_{ijk} and above $U_{ijk\ell}$, we have that

$$\begin{tikzcd}
P_{ij} \wedge^G P_{jk} \wedge^G P_{k\ell} \arrow[r] \arrow[d] & P_{ik} \wedge^G P_{k\ell} \arrow[d] \\
P_{ij} \wedge^G P_{j\ell} \arrow[r] & P_{i\ell}
\end{tikzcd}$$

commutes, each arrow being the evident one.

This is called the 'semi-local description' of P by Breen, [51].

Any reader who is used to fibre and vector bundles may be feeling that G-gerbes are 'locally trivial' in a very analogous way to, say, a bundle with isomorphisms $E|_{U_i} \cong U_i \times F$, thus allowing the use of local coordinates, etc. We have that the 'fibre' here is a groupoid, $Tors(G)$, and possible variation in G over different parts of the cover, (as it need not be a constant sheaf of groups), make this intuition a good, and very rich, one to explore. (The case of a constant sheaf of groups, G, is still an important one, although the more natural general case is not that much more work!)

10.1.3 Some examples and non-examples of gerbes

Of course, for any sheaf of groups, G, the stack $\mathsf{Tors}(B; G)$ of G-torsors on our space, B, is a gerbe, but few of our other examples of stacks are gerbes, without some side conditions. Of course, from our decomposition results just discussed we could construct examples, but that does seem the wrong way around. We will later produce some good examples of gerbes other than just $\mathsf{Tors}(B; G)$, but it is instructive to examine our other stacks for 'gerbeness' as well.

Clearly the stack, $\mathsf{Sh}(B)$, of sheaves on B does not even give us a stack of groupoids, so cannot be a gerbe. Our other prime example of stacks were the stacks of (G, H)-bitorsors, G-bitorsors, and, most generally, M-torsors for M a sheaf of crossed modules. For the sake of clarity, let $\mathsf{M} = (C, P, \partial)$ as before, then any M-torsor is a C-torsor (with conditions as we have seen), so every morphism of M-torsors is a C-torsor morphism and is thus invertible as a C-torsor morphism. It is then easy to see that the inverse of any such must also be an M-torsor morphism and thus to conclude that $\mathsf{M}-\mathsf{Tors}$ is a stack of groupoids. That raises the question as to the local conditions: non-emptiness (yes) and local connectedness (sometimes).

Proposition 106 *The stack* $\mathsf{M}-\mathsf{Tors}$ *is a gerbe if and only if* $\pi_0(\mathsf{M})$ *is a singleton sheaf.*

Proof: We need only check local connectedness, so suppose given two M-torsors, (E, t) and (E', t'). (Recall that E here is a C-torsor and t is a given trivialisation of $\partial_*(E) = P_\partial \wedge^C E$.) We can suppose both E and E' are trivialised over some open cover $\mathcal{U}$, and examine $\mathsf{M}-\mathsf{Tors}$ for when it is (locally) connected. We investigate the conditions for there to be a morphism $\varphi : (E, t) \to (E', t')$ (over some U in $\mathcal{U}$).

Picking local sections s (resp. s') of E (resp. E'), $\varphi(s) = c.s'$ for some c in C, of course, over U.

This uses the fact that φ will be a map of C-torsors. We need to check for compatibility with t and t'. The torsor $\partial_*(E)$ has a local section, induced by s, namely $[1, s]$, (cf. the discussion on page 324). Recall that elements of $P_\partial \wedge^C E$ are equivalence classes of pairs (p, e), where $(p, c.e) \equiv (p.\partial c, e)$. The global section / trivialisation, t, can be specified by $t = [p, s]$ and similarly $t' = [p', s']$. The morphism $\partial_*(\varphi)$ is given by

$$\partial_*(\varphi)[p, s] = [p, c.s],$$

and this is, of course, $[p.\partial c, s']$. As $\partial_*(t) = t'$ for compatibility, we must have the corresponding local sections of P linked by $p' = p.\partial c$, i.e., p and p' determine the same element of $\pi_0(\mathsf{M})$.

Conversely, if (E, t) and (E', t') are given and $p' = p.\partial c$ for some $c \in C$, then define $\varphi : E \to E'$ by $\varphi(s) = c.s'$ to get a morphism of C-torsors over U. (Investigation of descent conditions is **left to you**.) This gives a locally defined automorphism φ. ■

The stack of M-torsors is thus not a gerbe unless M is really a central extension

$$1 \to \pi_1(\mathsf{M}) \to C \xrightarrow{\partial} P \to 1.$$

If we look at the local sheaves of groups determined by $\mathsf{M}-\mathsf{Tors}$ in this situation, we have for any object $x = (E, t)$ over an open set U of B, that $Aut(x)$ can be calculated using a similar analysis to that above. Picking local sections s_V of E over some trivialising open cover $\mathcal{V}$ of U, we get that any automorphism φ of (E, t) determines a family $\{c_V\}$ of local elements of C given by $\varphi(s_V) = c_V.s_V$, as φ is an automorphism of the C-torsor structure of E. The compatibility with the trivialisation t of $\partial_*(E)$ now translates as: $t = [p_V, s_V]$ and $p_V = p_V.\partial c_V$, so $c_V \in Ker\,\partial$ (over the open set V).

We need to see the dependency of the c_V on the choice of local sections $\{s_V\}$. We **leave you to check** that this results in a conjugation of c_V, and hence that the actual isomorphism between $Aut(x)_V$ and $Ker\,\partial|_V$ *is* dependent on the choices made. (It may help to recall that, in a connected groupoid, all the vertex groups are isomorphic, but the actual isomorphisms involved depend, up to conjugation, on the choice of a maximal tree within the groupoid. We should also recall that conjugation is the groupoid form of homotopy.)

This is an important point. A G-gerbe specification (as given above) is an existence condition: there *is* a G *and* an open cover $\mathcal{U} = \{U_i\}$ *and* a family of objects x_i *and* a family of isomorphisms $\varphi_i : G|_{U_i} \cong Aut(x_i)$. There is no statement of uniqueness at any stage and no analysis of the dependence of this data on choices.

Corollary 21 *If* M *is a connected sheaf of crossed modules, so* $\pi_0(\mathsf{M})$ *is the terminal singleton sheaf, then* $\mathsf{M}-\mathsf{Tors}$ *is a* $\pi_1(\mathsf{M})$*-gerbe.* ∎

Note that for arbitrary M, the above argument shows that the stack $\mathsf{M}-\mathsf{Tors}$ of relative M-torsors is such that the local automorphisms of local objects give a family of groups isomorphic to $\pi_1(\mathsf{M})$.

Corollary 22 *The stack of G-bitorsors is a gerbe if and only if all automorphisms of G are inner, i.e., the outer automorphism sheaf of G is trivial. When this occurs,* $\mathsf{Bitors(G)}$ *is a* $Z(G)$*-gerbe, where* $Z(G)$ *is the centre of* G*.* ∎

We thus have an important class of stacks that are not gerbes, however there are still many other important instances of stacks that are gerbes.

Another link with G-bitorsors needs commenting on. We saw earlier, Theorem 30, page 450, that for G a sheaf of groups, the stack of G-bitorsors has a monoidal structure given by contracted product, and that this is 'group-like', i.e., $\mathsf{Bitors(G)}$ is a gr-stack. This means that it is very like a sheaf of 'gr-groupoids', and, of course, we saw that it was the stack completion of the (pre-)sheaf of the internal categories associated to the sheaf, $\mathsf{Aut}(G) = (G, Aut(G), \iota)$, of crossed modules. If we change our viewpoint from that of $\mathsf{Bitors(G)}$ being a stack, that is telling us about objects defined by G, i.e., a 'large' object containing the various 'small' objects of interest to us, to one where it is an algebraic object *derived* from our original object G, then we can view the isomorphisms, Ψ_{ijk}, above as defining a 1-cocycle on B with values in this monoidal stack. (Breen, [51], suggests the term 'bitorsor cocycle' for such a family.) This is a useful change to make and is thoroughly in line with the categorification. We could replace $\mathsf{Aut}(G)$ by an arbitrary sheaf of crossed modules, M, then stack completing it, could define a notion of M-gerbe. Of course, that would end up with the Ψ_{ijk}s being isomorphisms of M-torsors. This may look like generalisation for the sake of it, but recall our intuition that structure on a space is given by reduction of the group of transitions to a subgroup or by lifting them to a 'supergroup'. This again relates to extensions of non-Abelian cohomology to higher dimensions. Both directions will be explored more thoroughly later.

Yet another intuition is that these 1-cocycles, thought of as G-bitorsors over the intersections, U_{ijk}, are cocycles with values themselves determined by cocycles, since the G-bitorsors are themselves given by cocycle pairs (g_{ij}, u_i) with values in $\iota : G \to Aut(G)$. Is it feasible to work with some sort of double cocycle? Again we will investigate later.

In the case of a general gerbe, P, there may not be a single G making P into a G-gerbe, but the 'semi-local' description adapts quite well. We have an open cover $\mathcal{U}$ of B such that for each U_i of the cover, there is an equivalence

$$\Phi_i : \mathsf{P}_{U_i} \to \mathsf{Tors}(G_i),$$

where G_i is a sheaf of groups on U_i, namely $Aut(x_i)$ for some chosen object x_i in $\mathsf{P}(U_i)$. These groups need not form part of a single sheaf of groups on B, but choosing a 'quasi-inverse' for each equivalence, we get

$$\Phi_{ij} : \mathsf{Tors}(G_j)_{U_{ij}} \to \mathsf{Tors}(G_i)_{U_{ij}},$$

given by $\Phi_i \circ \Phi_j^{-1}$, and thus natural transformations

$$\Psi_{ijk} : \Phi_{ij} \circ \Phi_{jk} \Rightarrow \Phi_{ik},$$

induced by the cancellation transformation : $\Phi_j^{-1} \circ \Phi_j \Rightarrow Id$. These Φ_{ij} correspond to (G_j, G_i)-bitorsors, P_{ij}, rather than just to G-bitorsors, and these P_{ij} come with natural isomorphisms,

$$\psi_{ijk} : P_{ij} \wedge^{G_j} P_{jk} \to P_{ik},$$

over U_{ijk} and a corresponding coherence square over any $U_{ijk\ell}$.

Remark: It is important to note that the second gerbe axiom (local connectedness) will only tell us that different choices of the local objects x_i will be *locally* isomorphic over U_{ij}, i.e., there will be an open cover of U_{ij} over which $x_i|_{U_{ij}}$ and $x_i'|_{U_{ij}}$ will be isomorphic. This is again that question of coverings versus hypercoverings that we have briefly mentioned earlier. It is usual, and very useful, to simplify the discussion of gerbes in a first treatment of their properties by assuming that coverings suffice. In the more usual topological situations, this is completely adequate as if, for instance, B is paracompact, Čech cohomology and the cohomology defined via hypercoverings coincide, cf. Spanier, [250] p. 342. In the algebraic geometry context, if B is a scheme which is quasi-projective over a ring and we use the étale topology, then, by a theorem of M. Artin, [8], again Čech covers are cofinal amongst the hypercoverings of B, so we can always refine a cover to avoid the necessity of using hypercoverings.

10.2 Geometric examples of gerbes

Our earlier discussion of examples only turned up one type of example of gerbes, namely $\mathsf{Tors}(G)$, yet we have then called this example trivial! None of the other examples of stacks gave us an example without at least some additional assumption. We therefore could do with some examples that are non-trivial, otherwise the theory would not be worth studying! Earlier when discussing torsors, both geometry / topology and algebra gave examples. A similar thing happens here. We will start with some background ideas before turning to several special types of gerbe that occur in areas of geometry and topology.

A word of warning may be in order here. In this geometric setting, gerbes have often been considered as generalisations (actually 'categorifications) of line bundles and as such are thought of as merely a geometric realisation of an integral cohomology class in $H^3(B, \mathbb{Z})$. This gives a very important class of gerbe, but the prevalence of this class in applications leads to some confusion and to an enormous constriction in the terminology. For us here, as for the original motivation in the work of Giraud, Grothendieck, etc., gerbes are geometric objects in their own right. They may be classified by cohomology classes and thus give a representation of the elements of some cohomology group, but that is not their only *raison d'être*. The restricted focus of looking just at $H^3(B, \mathbb{Z})$ seems very like saying that, as general real vector spaces are sums of copies of $\mathbb{R}$, we need only consider one dimensional vector spaces. That there is a beautiful theory for those gerbes is without doubt (see, for instance, the brief description in Hitchin's 'What is' article, [156] or his longer article, [155]), but to ignore the other gerbes does seem a very silly restriction. From a practical point of view, especially for the beginner, this occasional restriction in terminology means that **it is essential to check when consulting an article if the general form or some restricted form of gerbe is being considered**.

10.2.1 Line bundles

Let us start by examining the sequence of ideas that lead from ordinary cohomology to that class of gerbes that are thought of as the 'categorification' of line bundles. The classical topological cohomology of a space, X, is given either by a singular or Čech type cochain complex and the two approaches coincide for 'nice' spaces such as manifolds. The cochain complex is given by $Ch(C_*(K), \mathbb{Z})$, where K is a simplicial set which hopefully approximates X well, e.g. $K = Sing(X)$ or $N(\mathcal{U})$, for $\mathcal{U}$ a 'good' open cover of X. In the latter case, we would need to pass to the limit over refinements of $\mathcal{U}$ unless X is a space such as a manifold, where local 'niceness' conditions will ensure that $|N(\mathcal{U})| \simeq X$ for fine enough covers. The cohomology is then the sequence of groups $H^n(X, \mathbb{Z})$. The idea is to represent the cohomology classes as more *geometric* objects than the cocycles, $f : N(\mathcal{U}) \to \mathbb{Z}$.

We know that exact sequences of coefficients for cohomology yield long exact sequences of cohomology groups. The basic short exact sequence that we will be needing is

$$0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 1,$$

where $U(1)$ is the unitary group of 1×1 unitary (complex) matrices, and so is just the group of unit moduli complex numbers. In other words, it is the circle group, S^1. There are various viewpoints that are potentially interacting here. This is a Lie group, but is also the common or garden circle and the sequence is the fibration sequence coming from the universal cover of S^1, as the map from $\mathbb{R}$ to $U(1)$ is the usual exponential map, $exp(t) = e^{2\pi it}$. Of course, this Lie group, $U(1)$, is the start of a family of unitary groups, $U(n)$, where $U(n)$ is the group of unitary $n \times n$ complex matrices. (There is even an infinite dimensional relative, $U(\mathscr{H})$, where $\mathscr{H}$ is an infinite dimensional separable Hilbert space, and the elements of the group are the unitary operators on it.)

From any such exact sequence, given any space, we can get an exact sequence of Lie group bundles on X: if G is a Lie group, we will write $\underline{G}_X := (G \times X \to X)$ as a Lie group bundle. From this, assuming that X is a smooth manifold, we get an exact sequence of sheaves of groups by taking sheaves of (smooth) local sections, $\underset{\sim}{G}_X := \Gamma_X(\underline{G}_X)$ to get, in our example,

$$0 \to \underset{\sim}{\mathbb{Z}}_X \to \underset{\sim}{\mathbb{R}}_X \to \underset{\sim}{U(1)}_X \to 1.$$

As $\mathbb{Z}$ is a discrete group, $\underset{\sim}{\mathbb{Z}}_X$ is the sheaf of locally constant integer valued functions on X; $\underset{\sim}{\mathbb{R}}_X$ is isomorphic to $C^\infty_X(\mathbb{R})$, the sheaf of smooth real valued functions on X and, similarly, $\underset{\sim}{U(1)}_X$ is the sheaf of unit moduli complex (local) functions, $\sigma : U \to \mathbb{C}$, $|\sigma(x)| = 1$ for all $x \in U$, an open set of X. We note that the sheaf cohomology, $H^n(X, \underset{\sim}{\mathbb{R}}_X)$, is trivial in positive dimensions as $\underset{\sim}{\mathbb{R}}_X$ is what is called a *fine* sheaf. (Here is not the place to handle this in detail, see Spanier, [250], Chapter 6, section 8, or Wikipedia.) Applying this observation to the long exact sequence in cohomology, we get that $H^n(X, \mathbb{Z}) \cong H^{n-1}(X, \underset{\sim}{U(1)}_X)$, and, in particular, $H^2(X, \mathbb{Z}) \cong H^1(X, \underset{\sim}{U(1)}_X)$.

Next let us return to our description of n-dimensional vector bundles on X. (Here we will assume that they are *complex* vector bundles, so locally are isomorphic to $U \times \mathbb{C}^n$ for some n.) We thus have an open cover, $\mathcal{U}$, over which our vector bundle, $E \to X$, trivialises and thus gives a family of transition functions, $g_{ij} : U_{ij} \to G\ell_n(\mathbb{C})$, which on triple intersections satisfy a cocycle condition,

$$g_{ij}g_{jk}g_{ki} = I_n,$$

the identity $n \times n$ matrix. (Note that this is another form of the cocycle conditions that we have seen so often now, as the transition functions can be thought of as forming a map from the simplicial sheaf, $N(\mathcal{U})$, to the simplicial sheaf, $BG\ell_n(\mathbb{C})_X$ by our earlier discussion of simplicial descriptions of torsors, etc.) As $U(1)$ is Abelian, in the case $n = 1$, we can and will sometimes write the cocycle condition in that case as

$$g_{ij} + g_{jk} - g_{ik} = 0.$$

A cohomology class, $\gamma \in H^1(X, \underset{\sim}{U(1)}_X)$, will be given by a family of cocycles, $g_{ij} : U_{ij} \to U(1)$. Using the canonical action of $U(1)$ on $\mathbb{C}$, we get a *line bundle* on X, i.e., a 1-dimensional vector bundle. We thus have that a cohomology class in $H^2(X, \mathbb{Z})$ can be represented by an isomorphism class of line bundles (here there are **details for you to check about why 'isomorphism classes'**) and, in fact, *vice versa*. There is no real difference between line bundles, which have 'gauge' group $G\ell_1(\mathbb{C}) \cong \mathbb{C}^\times$, the multiplicative group of non-zero complex numbers, and $U(1)$-bundles. To get from an ordinary line bundle to a $U(1)$-bundle, i.e., to reduce the group from $G\ell_1(\mathbb{C})$ to $U(1)$, one chooses an inner product on the fibres so as to get a Hermitian line bundle. (That any vector bundle over a *paracompact* space has a metric / inner product and thus that its structure group restricts to the group of unit norm matrices is a classical result to be found, for instance, in Husemoller, [160], Chapter 5, section 7.) Now, in a given Hermitian line bundle take the subspace of unit norm vectors to get a principal $U(1)$-bundle / $U(1)$-torsor.

Remark: If X is a complex manifold then, as above, the sheaf of holomorphic functions on it is essentially the same as that of holomorphic sections of the bundle, $\mathbb{C} \times X$, over X. It is the *structure sheaf*, $\mathcal{O}_X$, of the manifold, when that manifold is viewed from the point of view of complex algebraic geometry. This sheaf, $\mathcal{O}_X$, is a sheaf of rings and the sheaf, $\underset{\sim}{(\mathbb{C}^\times)}_X$, is isomorphic to $\mathcal{O}^*_X$, the sheaf of units of $\mathcal{O}_X$. The analogue of a line bundle in an algebraic geometric context is thus represented by a cohomology class in $H^1(X, \mathcal{O}^*_X)$, where now X may be a scheme or some other ringed space (= space with a given sheaf of rings on it). In this context the sheaf of (structure preserving, i.e., smooth, holomorphic or whatever) sections of a vector bundle on X becomes a module over the sheaf of rings, so in general in the algebraic geometry context vector bundles are replaced by (certain types of) modules over the ringed space.

For any two *vector* bundles, E_1 and E_2, over X, we can form their (fibrewise) tensor product $E_1 \otimes E_2$. If E_i has dimension n_i, then $E_1 \otimes E_2$ has dimension $n_1.n_2$, so if both E_1 and E_2 are line bundles, so is $E_1 \otimes E_2$. If we choose an open cover over which both E_1 and E_2 trivialise, then there are transition functions, g^1_{ij} and g^2_{ij}, defined on the intersections U_{ij}, taking values in $U(1)$. There are isomorphisms,$E_1|_{U_i} \cong U_i \times \mathbb{C}$, etc., and, together with the canonical isomorphism, $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$, these give that $E_1 \otimes E_2$ has transition functions given by the products, $g^1_{ij}.g^2_{ij}$. This implies, after checking of 'well definition' of everything, compatibility with coboundaries, etc., that if E is a line bundle, then there is another line bundle, E^{-1}, (whose transition functions are the inverses of those for E) such that $E \otimes E^{-1}$ is a trivial line bundle, i.e., is $X \times \mathbb{C}$. (We have essentially seen this argument before, in fact, in a more general case, namely that of bitorsors. Look back at the discussion on page 330 as well as later material on this idea. It is **left to you** to ask what questions arise via this linkage.)

We could equally well look at the sheaves, L, of local sections of these line bundles. The sheaf in these cases is, as we said just now, a module over $\mathcal{O}_X$, provided the structure mentioned earlier is present, and the notion of *invertible sheaf* is used, since $L \otimes L^{-1} \cong \mathcal{O}_X$. We thus have that isomorphism classes of line bundles, or invertible sheaves, or ... form a group. This is called the *Picard group* of $(X, \mathcal{O}_X)$, and we note that it does depend on what sheaf of rings is being thought of as the structure sheaf of the context. This applies also in algebraic geometry, where $H^1(X, \mathcal{O}_X^*)$, the cohomology group of a 'scheme' X with coefficients in the sheaf of units of the structure sheaf, forms exactly the Picard group of this ringed space. (Again to explore this thoroughly would lead too far away, however see the Wikipedia entry for 'Picard group' as a start and do not forget the link with bitorsors that we hinted at slightly earlier.)

Returning to cohomology, of course the isomorphism, $H^1(X, U(1)) \cong H^2(X, \mathbb{Z})$, is just the 'tip of the iceberg'. There is an infinite family of such isomorphisms, $H^n(X, U(1)) \cong H^{n+1}(X, \mathbb{Z})$. The next case to examine is $n = 2$, of course. Here a cohomology class in $H^3(X, \mathbb{Z})$ can be thought of as being one in $H^2(X, U(1))$. (As $U(1)$ is an Abelian group, the sheaf cohomology here can be handled using a slightly simpler set of machinery than in the non-Abelian situation, i.e., using chain complexes as well as simplicial things, and using additive notation if it eases the calculations.)

A cohomology class in $H^2(X, \underset{\sim}{U}(1)_X)$, (so reverting to the sheaf notation), can be given in terms of Čech 2-cocycles over some cover, $\mathcal{U}$. The simplicial sheaf, $N(\mathcal{U})$, then interprets as intersections of the open sets, and so a 2-cocycle will be given by a family of functions,

$$g_{ijk} : U_{ijk} \to U(1),$$

defined on the triple intersections with values in this group and satisfying a cocycle condition over 4-fold intersections. If we write things additively, this would be that, on $U_{ijk\ell}$,

$$g_{jk\ell} - g_{ik\ell} + g_{ij\ell} - g_{ijk} = 0,$$

i.e., thinking of $\mathbf{g}$ as a morphism of simplicial sheaves, for any $\sigma \in N(\mathcal{U})_3$,

$$\sum (-1)^i \mathbf{g} d_i \sigma = 0.$$

Remarks: (i) First a 'warning', this formula is evidently written additively and $U(1)$ is Abelian, so the order of terms clearly does not matter here, but when working with higher $U(n)$, which are not Abelian, order would matter and the considerations needed for handling that non-Abelian case give us the link to our earlier discussions of models for homotopy n-types, crossed complexes, etc., as this formula is a form of the homotopy addition lemma.

(ii) Thinking of $\mathbf{g}$ as a simplicial map, this cocycle condition has some nice consequences, even at the elementary level. What is the codomain of $\mathbf{g}$? We can think of $U(1)$ as a groupoid (we will avoid here the formal notation in which a group, G, corresponds to a groupoid with one object, $G[1]$, as that gives us $U(1)[1]$, which is a bit much notationally!). The codomain of $\mathbf{g}$, then, is the nerve of this groupoid, i.e., $BU(1)$. It is then easy to see that if we look at, say, U_{iik}, this is degenerate in $N(\mathcal{U})$, so the corresponding g_{iik} will be trivial, similarly g_{ijk} is trivial if any two of i, j and k are the same. As a result $g_{ijk} = -g_{jik}$, since we need only look at the cocycle condition in the case of the index $ijik$. Of course, permuting the indices of g_{ijk} in any way leaves it fixed if the permutation is even and multiplies it by -1 if it is odd.

The cocycle $\mathbf{g}$ will be a coboundary and thus cohomologically trivial if there is a morphism, $\mathbf{f}$, from $N(\mathcal{U})$ to $U(1)$, concentrated in dimension 1, (so $\mathbf{f}$ is determined by the family, $f_{ij} : U_{ij} \to U(1)$, on the 2-fold intersections) such that $\mathbf{g} = \mathbf{f}\partial$, i.e. $g_{ijk} = f_{ij} + f_{jk} - f_{ik}$. In this case we also say $\mathbf{f}$ is a *trivialisation* of $\mathbf{g}$.

(Again as this is an Abelian situation, the question of functional as against algebraic compositional order is avoided. Our usual 'conventional' diagram would be

$$\begin{tikzcd} & j \arrow[dr, "f_{ij}"] & \\ k \arrow[ur, "f_{jk}"] \arrow[rr, "f_{ik}"'] & g_{ijk} & i \end{tikzcd}$$

in functional order.)

Now, if f_{ij} and f'_{ij} are two trivialisations, then $f_{ij} - f'_{ij} =: h_{ij}$ satisfies

$$h_{ij} + h_{jk} - h_{ik} = 0.$$

If we write this multiplicatively, this gives $h_{ij}h_{jk}h_{ik}^{-1} = 1$, and the family, $\mathbf{h}$, determines a line bundle, or so it seems, but on what space?

Pick some U_0 in $\mathcal{U}$, then $\mathbf{g}$ trivialises over the cover $\mathcal{U}|_{U_0}$ obtained by intersecting U_0 with the *other* open sets of $\mathcal{U}$. If, for the moment, we set, for $i, j \neq 0$, $f^0_{ij} = g_{0ij}$, then the cocycle condition for $\mathbf{g}$ gives: for index $0ijk$, and thus over $U_0 \cap U_{ijk}$,

$$g_{ijk} = g_{0jk} - g_{0ik} + g_{0ij} = f^0_{jk} - f^0_{ik} = f^0_{ij}.$$

(Again note that our 'sloppy' writing of the cocycle condition earlier means that the order here is not what we had before. Of course, it does not matter as $U(1)$ is Abelian, but reminds us that order of composition is more likely to be delicate in non-Abelian contexts.) In any case, this shows that (f_{ij}) forms a trivialisation of $\mathbf{g}|U_0$ over this cover $\mathcal{U}|_{U_0}$. We repeat this for all open sets, U, in $\mathcal{U}$.

We note that f^0_{ij} was studied above with the condition that $i, j \neq 0$. If we, however, fed the formula with $i = 0$, say, we would get $f^0_{0j} = 0$ (or 1 depending on additive or multiplicative notation). It is thus convenient to extend the definition and to put $f^0_{0j} = 0$ for all j and similarly for $f^0_{i0} = 0$ for all i, and this then specifies a trivialisation localised on U_0 and, more generally, the method will lead to trivialisations localised on each $U_\alpha \in \mathcal{U}$.

If we look on $U_{ij} = U_i \cap U_j$, we now have two trivialisations, $\mathbf{f}^i$ and $\mathbf{f}^j$, (both restricted to U_{ij}). By our previous discussion, we have a family, $\mathbf{h}^{ij}$, given by

$$h^{ij}_{k\ell} = f^i_{k\ell} - f^j_{k\ell}$$

and this family determines a line bundle, L_{ij}, over U_{ij}. We note that $h^{ij}_{k\ell} = g_{ik\ell} - g_{jk\ell}$, by definition, so $L_{ij} \cong L^{-1}_{ji}$.

Lemma 66

$$L_{ij}L_{jk}L^{-1}_{ik} \cong 1,$$

the trivial line bundle on U_{ijk}.

Proof: We have to calculate the sum

$$\mathbf{A} = \mathbf{h}^{ij} + \mathbf{h}^{jk} - \mathbf{h}^{ik}$$

over some $U_{\alpha\beta} \cap U_{ijk}$. This gives

$$A_{\alpha\beta} = g_{\alpha ij} - g_{\beta jk} + g_{\alpha jk} - g_{\beta ij} - g_{\alpha ik} + g_{\beta ik},$$

but, on $U_{\alpha\beta} \cap U_{ijk}$, we have a local section, $g_{\alpha\beta ij}$, and $\partial g_{\alpha\beta ij} = g_{\beta ij} - g_{\alpha ij} + g_{\alpha\beta j} - g_{\alpha\beta i}$, so $g_{\alpha ij} - g_{\beta ij} = g_{\alpha\beta j} - g_{\alpha\beta i} + \partial g_{\alpha\beta ij}$ and consequently $A_{\alpha\beta} = \partial(g_{\alpha\beta ij} + g_{\alpha\beta jk} - g_{\alpha\beta ik})$. We thus have that $\mathbf{A}$ is a boundary everywhere on U_{ijk}, so the corresponding product line bundle, $L_{ij}L_{jk}L_{ik}^{-1}$, is trivial as claimed. ■

Note that not only does show that $L_{ij}L_{jk}L_{ik}^{-1}$ is trivial, but, starting with $\mathbf{g}$, it gives a specific trivialisation of that bundle, determined *explicitly* by the simplicial map, $\mathbf{g}$, corresponding to the original cocycle, or, if you are not yet needing the non-Abelian (and thus simplicial) viewpoint, the map of chain complexes from $C(\mathcal{U})$ to $U(1)$, with an adjustment of dimensions to get the grading right. (The argument is, however, essentially simplicial even in the Abelian case, and that viewpoint is *very useful*.)

We write $\theta_{ijk}^{\alpha\beta} = g_{\alpha\beta ij} + g_{\alpha\beta jk} - g_{\alpha\beta ik}$.

We thus have that a cohomology class in $H^2(X, \underset{\sim}{U(1)}_X)$, which is defined over a cover, $\mathcal{U}$, determines

- a line bundle, L_{ij}, over each $U_i \cap U_j$ such that
- $L_{ij} \cong L_{ji}^{-1}$,

together with

- a trivialisation, θ_{ijk}, of $L_{ij}L_{jk}L_{ik}^{-1}$, where $\theta_{ijk} : U_{ijk} \to U(1)$ is a 2-cocycle.

(We **leave the checking** that θ_{ijk} is a 2-cocycle on U_{ijk} to you. It is just a simple verification of the equations.)

Given our earlier simplicial descriptions of torsors, etc., it is perhaps quite natural to rework the above, replacing the open cover, $\mathcal{U}$, by the corresponding sheaf / étale space over X, $U \to X$, where $U = \bigsqcup \mathcal{U}$. (We can also think of this as $\mathsf{U} \to 1$ in $Sh(X)$, as the identity function on X considered as 'X over itself', is the terminal object, 1, of $Sh(X)$.) The intersections, as you will probably recall, correspond to $U \times_X U$, which we will denote by $U^{[2]}$, (and will extend the notation in the obvious way), so the above data corresponds to a line bundle L over $U^{[2]}$ with a trivialisation (= global section) over $U^{[3]}$ of $d_0^*(L)d_2^*(L)d_1^*(L)^{-1}$. (Here we are, of course, using the simplicial structure of $N(\mathcal{U})$, see page 300.) This, in part, gives a geometric candidate for a type of object representing the cohomology classes in $H^3(X, \mathbb{Z})$, thus generalising line bundles. In [211], Murray put forward a generalisation of this, which gives an even more geometric flavour to the objects.

10.2.2 Line bundle gerbes

There are various generalisations of the above situation. Just as, for a differential geometric context, bundles of groups can be more useful than sheaves of groups as the concept more easily allows non-trivial topologies on the groups (e.g. with bundles of Lie groups), so gerbes as defined above sometimes need a more 'bundle-like' version. This leads to various forms of 'bundle gerbe', a concept developed by Murray, [211]. There are various extensions of his initial definition which we will look at later. Bundle gerbes generalise line bundles to the next dimension using some neat extensions of the ideas we have just seen. The simplest of these is to replace $U \to X$ by a suitable locally split map. This allows one to introduce more structured fibres for the covering (and, to some extent, looks at a Grothendieck topology as well as the standard topology). This can also be thought of as a step in the direction of hypercoverings.

Definition: A continuous (or smooth or ...) map $\pi : Y \to X$ is said to be *locally split* if it admits enough local sections, more precisely, for each $x \in X$, there is an open neighbourhood U of x and a section $s : U \to Y$, so $\pi s = id_U$.

Examples: Locally trivial fibrations are locally split and étale maps arising from an open cover $\mathcal{U}$ of X (as above), or from an étale space corresponding to a sheaf, are as well. If we are considering smooth manifolds and smooth maps between them any surjective submersion, $f : N \to M$, is locally split. (Recall that a *submersion* is a smooth map for which the induced map on tangent spaces $Df_p : TN_p \to TM_{f(p)}$ is a surjective linear map for all $p \in N$. In local coordinates such a submersion looks like the standard projection of $\mathbb{R}^n$ onto $\mathbb{R}^m$.)

The following is, essentially, taken from [43] with minor modifications.

Definition: A *(Hermitian) line bundle gerbe* on a space X is a pair, $(L, \pi : Y \to X)$, where $\pi : Y \to X$ is a locally split map (sometimes called the 'fibering' of the line bundle gerbe) and L is a (Hermitian) line bundle $L \to Y^{[2]}$, on the pullback, $Y^{[2]} = Y \times_X Y$, together with an associative 'product'

$$L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \xrightarrow{\cong} L_{(y_1,y_3)}$$

for every (y_1, y_2) and (y_2, y_3) in $Y^{[2]}$, which is an isomorphism.

We think of $L_{(y_1,y_2)}$ as the 'space' of arrows *from y_2 to y_1* in accordance with our functional composition convention. The multiplication is the composition of a category structure as we will see.

If X is a smooth manifold, then we require the product to be smooth in y_1, y_2 and y_3, and in any case, the product is to be a (Hermitian) isomorphism. (Recall all these fibres, $L_{(y_1,y_2)}$, etc., are copies of $\mathbb{C}$.) The product is associative whenever the triple product is defined. (Note that the coherence of the isomorphisms giving the monoidal category structure on complex vector spaces is underlying this to a small extent, but no more than if we were asking that a vector space, A, have an algebra structure. In both cases we need to be conscious of the slight difficulties that arise from the lack of associativity of the tensor product construction, but we know it causes no lasting problem so can safely set it aside.)

Proposition 107 *(i) For any $y \in Y$, $L_{(y,y)} \cong \mathbb{C}$, so that $1 \in \mathbb{C}$ corresponds to an identity for the composition.*

(ii) For any $(y_1, y_2) \in Y^{[2]}$, there is a natural isomorphism

$$L_{(y_1,y_2)} \cong L^*_{(y_2,y_1)},$$

where * indicates the dual space.

Proof: (The ideas are ones that we have seen many times now and are worth looking at from our general perspective. They are simple, but are worth giving explicitly for that reason.) It is simpler to work with $\mathbb{C}^\times$ bundles rather than the corresponding line bundles.

There is the multiplication isomorphism,

$$L_{(y_1,y_2)} \otimes L_{(y_2,y_2)} \stackrel{\cong}{\to} L_{(y_1,y_2)},$$

which we will write as concatenation, with a dot when it is needed to avoid ambiguity, and our assumption that we are working with the principal $\mathbb{C}^\times$ bundles means that if $p \in L_{(y_1,y_2)}$ and $q \in L_{(y_2,y_2)}$, there is some non-zero complex number z such that $p.q = zp$. Set $e = z^{-1}q$, so $p.e = p$. (Where necessary we will write e_{y_2} for this element of $L_{(y_2,y_2)}$.)

Suppose we had used wp instead of p in the above for some $w \in \mathbb{C}^\times$, then $(wp).e = w(p.e) = wp$, and we have that $p'.e = p'$ for all $p' \in L_{(y_1,y_2)}$, so e is an identity for pre-composition.

If now we take $r \in L_{(y_2,y_3)}$ for some $(y_2, y_3) \in Y^{[2]}$, then we consider $e.r \in L_{(y_2,y_3)}$. This is rz for some probably different $z \in \mathbb{C}^\times$. The composition is associative, so

$$(p.r)z = p.(rz) = p.(e.r) = (p.e).r = p.r.$$

(We must keep awake here! p and r are elements in fibres of L, z is not. It is just a non-zero complex number.) As the action is effective, we must have $z = 1$, i.e., $e.r = r$ and e is also an identity for post-composition. We thus have an identity e_y at each diagonal element (y, y) in $Y^{[2]}$. (**Check it is unique!**)

Finally, in this preparatory phrase, as composition is an isomorphism, we can note that there is a unique $p^{-1} \in L_{(y_2,y_1)}$ such that $p^{-1}.p = e_{y_2}$ and **will leave you to give** the obvious argument that $pp^{-1} = e_{y_1}$.

Turning to (i) $L_{(y,y)} \cong \mathbb{C}$, since if $q \in L_{(y,y)}$, there is a unique $z \in \mathbb{C}$ such that $q = ze_y$, compatibly with the action of $\mathbb{C}$ on both sides of the isomorphism. (Details again **left to you**.)

For (ii), we have that composition gives a linear isomorphism (using (i))

$$L_{(y_1,y_2)} \otimes L_{(y_2,y_1)} \stackrel{\cong}{\to} L_{(y_1,y_1)} \cong \mathbb{C},$$

and flipping this through the adjointness isomorphism for tensors and 'homs', we get

$$L_{(y_1,y_2)} \cong Hom(L_{(y_2,y_1)}, \mathbb{C}) = L^*_{(y_2,y_1)}.$$

(If you prefer, for $p \in L_{(y_1,y_2)}$, define the linear form, $p^* : L_{(y_2,y_1)} \to \mathbb{C}$, by

$$p^*(q)e_{y_1} = p.q,$$

and check that $p \mapsto p^*$ is an isomorphism.) ■

A bundle gerbe thus consists of a map $\pi : Y \to X$ and a line bundle on $Y^{[2]}$, so it is clear what a morphism of bundle gerbes should be. It should have three interconnected parts: a (smooth) map $\gamma : X \to X'$ of the bases, a map $\beta : Y \to Y'$ for the top spaces and a map α of the line bundles. Of course, we want β to be a map 'over γ', that is, that $\gamma\pi = \pi'\beta$. This will imply that there will be an induced map on the pullbacks $\beta^{[2]} : Y^{[2]} \to (Y')^{[2]}$. The final condition will be that α is a map of line bundles covering $\beta^{[2]}$, and that it preserves the product (and hence both the identities and the inverses).

We can analyse this data somewhat differently, reducing its complexity a bit. The simplest case would be with β and γ both the identity maps, and α a line bundle morphism preserving the composition. The next level would be with just γ the identity, so one would have morphism of bundle gerbes 'over X'. The third level is the general one. Going through this backwards, given $\gamma : X \to X'$ and $\pi' : Y' \to X'$, we can pull back to get $\gamma^*(Y') \to X$. This is locally split (**for you to check**) and β will induce a map $\beta' : Y \to \gamma^*(Y')$ over X. Since the pullback $\gamma^*(Y')^{[2]}$ is a multiple pullback / limit of a particular diagram, it is easy to check that it is, up to isomorphism, the same as $\gamma^*((Y')^{[2]})$, (if in doubt try to draw the diagrams involved). The map $\beta^{[2]} : Y^{[2]} \to (Y')^{[2]}$ allows us to pull back L' to a line bundle over $Y^{[2]}$ and so to factor α into a composite in which one part is over $Y^{[2]}$ and the other part is independent of α, being from $(\beta^{[2]})^*(L') \to L'$. We thus can decompose the original map into a composite of maps of the special types.

These same ideas and constructions allow us to pullback a bundle gerbe on X', say, along any smooth map, $f : X \to X'$, giving an induced bundle gerbe over X and a morphism of bundle gerbes from the induced one to the original one. Similarly we can form a 'product' of two bundle gerbes over X. The product of $(L, \pi : Y \to X)$ and $(L', \pi' : Y' \to X)$ would be denoted $(L \otimes L', Y \times_X Y' \to X)$. To see what this must be note first that $Y \times_X Y'$ consists of pairs, (y, y'), such that $\pi(y) = \pi'(y')$, so we have to describe the line bundle fibre over some (y_1, y_1'), $(y_2, y_2') \in (Y \times_X Y')^{[2]}$. Of course (again!), the only candidate staring us in the face is $L_{(y_1,y_2)} \otimes L_{(y_1',y_2')}$, which is what $L \otimes L'$ is. (The details are now **quite easy to check**.)

There is thus quite a lot of interesting structure on the category of bundle gerbes on a fixed X and given a bundle gerbe, we can generate others - but so far we have not produced a single one!

Examples: (i) First a 'trivial' example. Let $Q \to Y$ be a $\mathbb{C}^\times$-torsor on Y, then set

$$P_{(y_1,y_2)} = Aut_{\mathbb{C}^\times}(Q_{y_1}, Q_{y_2}) \cong Q^*_{y_1} \otimes Q_{y_2}.$$

The multiplication is given by

$$P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \cong Q^*_{y_1} \otimes Q_{y_2} \otimes Q^*_{y_2} \otimes Q_{y_3} \to Q^*_{y_1} \otimes \mathbb{C} \otimes Q_{y_3} \overset{can}{\cong} Q^*_{y_1} \otimes Q_{y_3},$$

induced by the canonical pairing between Q_{y_2} and its dual. We can also write

$$\begin{aligned} P &= Aut_{\mathbb{C}^\times}(\pi_1^{-1}Q, \pi_2^{-1}Q) \\ &\cong \pi_1^{-1}Q^* \otimes \pi_2^{-1}Q, \end{aligned}$$

where π_1 and π_2 are the two projections from $Y \times_X Y$ to Y.

(ii) We next consider an 'obstruction problem', i.e., determining if a change of structure group is possible for torsors, in this case a lifting to an 'overgroup'.

Consider a Lie group, G, and a central extension

$$\mathbb{C}^\times \xrightarrow{\iota} \tilde{G} \xrightarrow{p} G.$$

We have seen that there is an induced sequence,

$$\ldots \to Tors(\mathbb{C}^\times) \to Tors(\tilde{G}) \to Tors(G),$$

and as $\mathbb{C}^\times$ is Abelian, we would expect to be able to continue this with $H^2(X, \mathbb{C}^\times)$, Of course, from our build up, we expect elements of $H^2(X, \mathbb{C}^\times)$ to 'be' isomorphism classes of line bundles. The group makes sense anyway as $\mathbb{C}^\times$ *is* Abelian and we know it is isomorphic to $H^3(X,\mathbb{Z})$ as we saw earlier.

Suppose we have a G-torsor, $Y \to X$, and we want to ask if it is induced from some $\tilde{G}$-torsor, $E \to X$, i.e., is $p_*(E) \cong Y$. (We recall that $p_*(E) = G \wedge^{\tilde{G}} E$, and the cocycles for E give those for $p^*(E)$ on composition with p.) We have already used this sort of construction before. Let $\mathcal{U}$ be a trivialising cover for Y and $g_{ij} : U_{ij} \to G$ be a family of transition functions / cocycles for Y. Pick $\tilde{g}_{ij} : U_{ij} \to \tilde{G}$, (although we may need to refine $\mathcal{U}$ before this works), so that $p(\tilde{g}_{ij}) = g_{ij}$. Of course, $\tilde{g}_{ij}$ may not satisfy the cocycle condition, although

$$p(\tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki}) = 1$$

since (g_{ij}) *is* a cocycle. Hence $\tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} = \iota(c_{ijk})$ for some $c_{ijk} : U_{ijk} \to \mathbb{C}^\times$. Is c_{ijk} a 2-cocycle?

We have more or less seen this situation before when examining the 'long exact' sequences and Puppe sequences of a short exact sequence of simplicial groups in a previous chapter. Here we have less generality as we are in an Abelian setting, so will 'cheat' and make life easier for ourselves by writing things additively. We will look at non-Abelian analogues later so cannot escape!

Additively we have

$$\iota(c_{ijk}) = \tilde{g}_{ij} + \tilde{g}_{jk} - \tilde{g}_{ik},$$

since we can assume $\tilde{g}_{ki} = -\tilde{g}_{ik}$, and

$$\iota(c_{jk\ell} - c_{ik\ell} + c_{ij\ell} - c_{ijk}) = 0,$$

as on expanding out, the terms cancel in pairs. (Note this uses that $\mathcal{C}^\times$ is *Abelian*, not that the extension is central. In fact with more care even Abelianess is not needed.) Now we invoke that ι is a monomorphism so the c_{ijk} satisfy the cocycle condition.

The (c_{ijk}) thus define a cohomology class in $H^2(X,\mathcal{C}^\times)$ and thus one in $H^3(X,\mathbb{Z})$. If the G-torsor, Y, is an image of a $\tilde{G}$-torsor, E, then the transition functions for E give, after composition with p, equivalent ones for Y, so we can pick the cover $\mathcal{U}$ and g_{ij} such that there is some lift, $\tilde{g}_{ij}$, which is a cocycle (just take the ones for E!), and thus the c_{ijk} will be trivial in this case.

There is quite a lot of (useful) checking **to be done here**. What happens to (c_{ijk}) if we change g_{ij} by a coboundary? If (c_{ijk}) is itself a coboundary, what are the implications? We would expect that the formula for c_{ijk} as coboundary would give some elements that would allow us to deform our choices $\tilde{g}_{ij}$, so that they themselves give a cocycle and thus a $\tilde{G}$-torsor. Does this happen? This is left '**to you the reader**'. Of course, $\mathbf{c} = (c_{ijk})$ is a coboundary exactly when Y is isomorphic to an image, $p^*(E)$, of a $\tilde{G}$-torsor, i.e., $[\mathbf{c}] \in H^2(X,\mathbb{C}^\times)$ is the *obstruction* to lifting the G-torsor structure.

This calculation is very instructive and can be 'geometrised' to give a bundle gerbe as follows. We start as before with a G-torsor, $\pi : Y \to X$, and use this as the 'fibering' / covering for a bundle gerbe. Let $(y_1, y_2) \in Y^{[2]}$ and set

$$P_{(y_1,y_2)} = \{\tilde{g} \in \tilde{G} \mid p(\tilde{g})y_2 = y_1\}.$$

This is the set of lifts of the g, which gives $g.y_2 = y_1$. (Warning: we are writing this in left torsor, functional order notation, not as in Murray's paper, [211], which has right torsors and uses the algebraic / concatenation composition.)

Given y_1, y_2, y_3, in the same fibre of π and element $\tilde{g}_{12}$, $\tilde{g}_{23}$ such that

$$p(\tilde{g}_{12}).y_2 = y_1$$

and

$$p(\tilde{g}_{23}).y_3 = y_2,$$

then $\tilde{g}_{23}.\tilde{g}_{12} \in P_{(y_1,y_3)}$, so multiplication in $\tilde{G}$ provides the multiplication / composition within the line bundle. 'Line bundle'? Yes, as $\pi : Y \to X$ is a G-torsor, there is a unique g_{12} such that $g_{12}.y_2 = y_1$, hence $P_{(y_1,y_2)}$ *is* a copy of $\mathbb{C}^\times$. Of course, the composition is associative, since it is got from the multiplication of $\tilde{G}$. This simply defined bundle gives a cohomology class in $H^2(X, \mathbb{C}^\times)$, which should be the obstruction class. To see that it is, we look at when it vanishes.

Suppose $E \to X$ is a $\tilde{G}$-torsor on X that maps to $Y \to X$, i.e., $p_*(E) \cong Y$, then there is a projection map, q from E to Y (over X) corresponding to the epimorphism $p : \tilde{G} \to G$. We identify Y with $p_*(E)$ to make the discussion easier. To see what we must do, pick an element e in E (really a local element or local section of E, but think of it as an element), so in the fibre over $\pi'(e)$, and a $y \in \pi^{-1}(\pi'(e))$, and define $q(e) = y$. This extends to a map on fibres using p, so if e' is another element of that fibre in E, then $e' = \tilde{g}.e$ for some $\tilde{g}$ and we set $q(e') = p(\tilde{g})y$. Now we take a bit more care and, choose a local section e of E and a local section y of E over the same open set U, and define q via local sections. As we are considering locally split maps, and G-bundles, etc., are such, this works well, but the details **do need chcking up on**; they are often neglected in treatments of this! What conditions are needed for q to be continuous? ... smooth? Does q depend on the choices made and if it so, does it matter to the end result? and so on.

We will see other examples later.

We are still giving a development of these ideas that is largely independent or our early sections, so to start the process of comparison, we will describe the cohomology class corresponding to a line bundle gerbe. This is very near to the semi-local description of a gerbe as a stack with special properties. We will then take this one step nearer to gerbes but the more detailed actual comparison will come slightly later. This also introduces the important idea of the characteristic class of a bundle gerbe.

The characteristic class of a line bundle, L, is the cohomology class in $H^2(X, \mathbb{Z})$ which it determines. In other words it is central to the classification of line bundles, or inversely is at the heart of the representation theory of cohomology classes by line bundles. The theory of characteristic classes in general, and how they relate to differential forms and the geometry of the manifold, is enormous, so cannot be handled here.

Definition: The *characteristic class* of a line bundle gerbe, (L, π), is the class in $H^3(X, \mathbb{Z})$ that it determines. (This is called the *Dixmier-Douady class* of (L, π).)

How should we think of this?

We start with a line bundle gerbe, $(L, \pi : Y \to X)$ and so $L \to Y^{[2]}$ is the line bundle part of it. As π is locally split, we can choose an open cover $\mathcal{U}$ of X such that there are sections, $s_i : U_i \to Y$, over each $U_i \in \mathcal{U}$. On double intersections, we have

$$(s_i, s_j) : U_{ij} \to Y^{[2]},$$

defined by $(s_i, s_j)(x) = (s_i(x), s_j(x))$, and we pull L back along this map to get a line bundle L_{ij} on U_{ij}. (We note that although U_{ij} and U_{ji} are the same open set, here they are considered twice, corresponding to the construction of the simplicial sheaf from the open cover that we used earlier. We think of U_{ij} as 'going from i to j' and U_{ji} going in the other direction.) We have that $L_{ij} \otimes L_{jk} \cong L_{ik}$ over U_{ijk} by the composition isomorphism.

If we assume that $\mathcal{U}$ is a Leray cover, (so all the U_i and all the finite intersections, U_α, are contractible), then L_{ij} will have a (non-zero) section σ_{ij}, as it will be isomorphic to a product line bundle, $U_{ij} \times \mathbb{C}$. Moreover we can define a $\mathbb{C}^\times$ valued function,

$$g_{ijk} : U_{ijk} \to \mathbb{C}^\times,$$

which measures the failure of the σ_{ij} to define a 1-cocycle, i.e. ,

$$g_{ijk} = \sigma_{ij}\sigma_{jk}\sigma_{ik}^{-1},$$

and, as we have already calculated above, it is clear that the g_{ijk} satisfy the 2-cocycle condition over $U_{ijk\ell}$ and so gives a class in $H^2(X, \mathbb{C}^\times)$. Using the isomorphism between this group and $H^3(X, \mathbb{Z})$ gives the Dixmier-Douady class, $d(L, \pi)$, of the bundle gerbe.

(You will, no doubt, have noticed the number of choices of sections, etc., involved here, so will **need to see what happens** when these choices are changed.)

Proposition 108 *A line bundle gerbe $(L, \pi : Y \to X)$ has zero Dixmier-Douady class precisely when it is trivial.*

Proof: Suppose $Q \to Y$ is a line bundle on Y and we write $P = \delta(Q) := \pi_1^{-1}Q^* \otimes \pi_2^{-1}Q$ and will examine its Dixmier-Douady class. We use the $s_i : U_i \to Y$ that locally split $\pi : Y \to X$ and set $Q_i = s_i^*(Q)$, the pullback of Q over U_i. There are natural isomorphisms

$$P_{ij} \cong Q_i^* \otimes Q_j.$$

Each U_i is contractible, as the open cover can be assumed to be a Leray cover, so there is a section $q_i : U_i \to Q_i$, (non-zero), and we can choose $\sigma_{ij} = q_i^{-1} \otimes q_j$ over U_{ij}, since the transition functions of Q_i^* can be chosen to be the inverses of those for Q_i. (Remember, $Q^* = Q^{-1}$ as a line bundle!) Now working out g_{ijk}, we find

$$g_{ijk} = q_i^{-1} \otimes q_j \otimes q_j^{-1} \otimes q_k \otimes q_k^{-1} \otimes q_i,$$

so it is trivial when composed, giving the trivial element in $H^2(X, \mathbb{C}^\times)$. The Dixmier-Douady class is thus trivial.

Now suppose that we are given (L, π) such that $d(L, \pi) = 0$. We pick a Leray open cover, etc., of X as before, and get our cocycle g_{ijk}. This is assumed to be a trivial cocycle, so must be a coboundary. It itself is a family, $\mathbf{g}$, of maps, $g_{ijk} : U_{ijk} \to \mathbb{C}^\times$ (or into $U(1)$ if you prefer as it makes no difference), and to say that it is a coboundary is to say that there is a family of functions, $\mathbf{f} = \{f_{ij} : U_{ij} \to \mathbb{C}^\times\}$ such that $\mathbf{g} = \mathbf{f}\partial$. In other words, $g_{ijk} = f_{ij}f_{jk}f_{ik}^{-1}$. We can adjust the σ_{ij}, multiplying each by the corresponding f_{ij}^{-1} yet not changing the line bundle, so we can assume that g_{ijk} is always equal to 1.

Restrict Y to U_i, writing $Y_i = \pi^{-1}(U_i)$ and define Q_i over Y_i by setting its fibre over y to be

$$(Q_i)_y = P_{(y, s_i(\pi(y)))}.$$

The σ_{ij} live in

$$\begin{aligned} P_{(s_i(\pi(y)), s_j(\pi(y))} &\cong P^*_{(s_i(\pi(y)), y)} \otimes P^*_{(s_j(\pi(y)), y)} \\ &= (Q_i^*)_y \otimes (Q_j)_y. \end{aligned}$$

In other words, you compare the fibres by referring always to the chosen y. (We have given this as 'in the fibre', but it can also be done, more correctly, using local sections.) This is easily seen to give a line bundle over Y_i - but **do check that it is**.

It is clear that the σ_{ij} thus define isomorphisms between Q_i and Q_j over $Y_i \cap Y_j$ and so, by 'descent', give a line bundle Q over Y itself. By construction, $P \cong \delta(Q)$, as hoped for. ■

Several remarks are in order here. We have deliberately confused line bundles, $\mathbb{C}^{\times}$-torsors and to some extent $U(1)$-torsors. Geometrically it seems that line bundles 'feel' nicest as they seem least abstract! Any line bundle has a trivial zero section, however, so if one sticks with them one really needs to be tagging all sections with the label 'non-zero', i.e., corresponding to a section of the corresponding $\mathbb{C}^{\times}$-torsor. This gets annoying! It is thus useful to refer to line bundles, but to think $\mathbb{C}^{\times}$-bundles or $\mathbb{C}^{\times}$-torsors!

The Dixmier-Douady class behaves naturally with respect to the operations of inversion, pull-back, tensor product, etc. (This should remind you of the way in which natural constructions on bitorsors (contracted product, etc.) corresponded to multiplication, inversions, etc., in the cohomology group.)

Proposition 109 *(i) Suppose given a map, $\phi = (\phi_0, \phi_1)$, of 'fibre maps'*

$$\begin{tikzcd} Y' \arrow[r, "\phi_1"] \arrow[d, "\pi'"'] & Y \arrow[d, "\pi"] \\ X' \arrow[r, "\phi_0"'] & X \end{tikzcd}$$

and (L, π) a bundle gerbe on X. The induced homomorphism satisfies

$$d(\phi_1^*(L), \pi') = \phi_0^* d(L, \pi).$$

(ii) If (L, π) is a bundle gerbe on X, then so is (L^*, π) and

$$d(L^*, \pi) = -d(L, \pi).$$

(iii) If (L, π) and (L', π') are bundle gerbes on X, then, writing $\pi'' : Y \times_X Y' \to X$ for the natural diagonal composite map in the pullback square,

$$d(L \otimes L', \pi'') = d(L, \pi) + d(L', \pi').$$

Proof: These are proved using the cocycle description and are **left as an exercise**. (There may be some intermediate results that will be needed - the proof is not a 'one-liner'!) ■

A particular case of part (i) of this result is very useful. If the map on the bases. $\phi : X' \to X$ is the identity map on X, then ϕ_* is, of course, the identity on $H^3(X, \mathbb{Z})$. Part (i) then gives:

Corollary 23 *If $\phi_1 : (Y', \pi') \to (Y, \pi)$ is a map of locally split fiberings over X, then for any bundle gerbe, (L, π) on X,*

$$d(\phi_1^*(L), \pi') = d(L, \pi).$$

■

This means that d cannot tell the difference between (L, π) and its pullback to Y'. (We can think of (Y', π') as perhaps being a 'refinement' of (Y, π) - thinking of 'hypercoverings' which are not that far away here - and this then says that if we have a representative of a class in $H^2(X, U(1))$ 'defined over' (Y, π), then it is defined over any finer (Y', π').) What it tells us is that there are potentially many representatives of a given class in $H^3(X, \mathbb{Z})$ (or $H^2(X, U(1))$) amongst the line bundle gerbes and they need not be 'isomorphic' as ϕ_1 may not be a homeomorphism over X.

We can squeeze a bit more out of this result and its corollary:

Proposition 110 *If (L, π) and (L', π') are two line bundle gerbes, having the same Dixmier-Douady class in $H^3(X, \mathbb{Z})$, then $(L^* \otimes L')$ is trivial, and conversely.*

Proof: Calculate $d(L^* \otimes L')$. It is $d(L^*) + d(L')$ by (iii) of the previous result. This is $-d(L) + d(L')$ by (ii) and this is zero if the two classes coincide. Thus, if $d(L) = d(L')$, then $(L^* \otimes L')$ is a trivial bundle gerbe. For the converse, ... run the argument backwards! ∎

Definition: Two line bundle gerbes, (L, π) and (L', π'), are said to be *stably isomorphic* if $(L^* \otimes L')$ is trivial. In this case a trivialisation of $(L^* \otimes L')$ is called a *stable isomorphism* from (L, π) to (L', π').

If two line bundles $p : L \to X$ and $p' : L' \to X$ are isomorphic, then there is a global section of $Iso_X(p, p')$, the sheaf of local isomorphisms of the two bundles, and hence a global section of the bundle, $L^* \otimes L' \to X$, and that gives a trivialisation of that line bundle, thus stable isomorphism as above seems a neat generalisation of the lower dimensional case. (It would be useful here to look back at the material on automorphisms of G-torsors, contracted product etc. from the early parts of the previous chapter. Contracted product is the analogue for G-bitorsors of the tensor products used here for line bundles.)

The notion of stable isomorphism was introduced by Murray and Stevenson, [212], but is clearly also the bundle analogue of ideas on gerbes, in general, that date back further. We should make this more transparent by solidifying the connections between these bundle gerbes and gerbes *per se*.

10.2.3 From bundles gerbes to gerbes

Let us start with a line bundle gerbe, $(L \to Y^{[2]}, \pi : Y \to X)$, on X and with composition isomorphisms

$$L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \xrightarrow{\cong} L_{(y_1,y_3)}.$$

We have already looked at such an object locally, so let us briefly rerun the analysis. We know that π is locally split, so can find a cover $\mathcal{U}$ of X such that over each U_i, there is a section of π, thus on the overlap U_{ij}, there are sections

$$(s_i, s_j) : U_{ij} \to Y^{[2]},$$

and we set L_{ij} to be the pullback of L over U_{ij} along this section. The composition gives an isomorphism

$$L_{ij} \otimes L_{jk} \xrightarrow{\cong} L_{ik}$$

over U_{ijk}. When defining $d(L, \pi)$, we looked at (local) sections σ_{ij} of L_{ij} and found a 2-cocycle, $g_{ijk} = \sigma_{ij}\sigma_{jk}\sigma_{ik}^{-1} : U_{ijk} \to \mathbb{C}^\times$.

To get the σ_{ij}, we may need to refine the cover to ensure that global sections exist over U_{ij}. (We know that local sections exist since L_{ij} is a line bundle on U_{ij}, so we take a cover $\mathcal{U}'$ finer than $\mathcal{U}$, if necessary, to ensure that, for the corresponding $U'_{\alpha\beta}$, global sections exist. This type of argument needs examining *in detail* as it is at the heart of the matter - but that is **left to you to do**.) We assume therefore that $\mathcal{U}$ is fine enough for the σ_{ij} and thus that the g_{ijk} exist. Now with this line bundle gerbe, (L, π), we define a sheaf of groupoids, $\mathsf{G} = \mathsf{G}_{(L,\pi)}$, on X as follows:

- The sheaf of objects G_0 is the sheaf of sections of $\pi : Y \to X$;
- The sheaf of arrows G_1 is defined by:

  if $a, b : U \to Y$ are local sections of π over an open set U in X, then an arrow $g : a \to b$ is a section of the pullback of $L \to Y^{[2]}$ along $(a, b) : U \to Y^{[2]}$.

As π is locally split, the stalk of G at any $x \in X$ is non-empty, and we have seen that $L \to Y^{[2]}$ is locally split as well, so G is locally connected. It follows that the associate stack of G is a gerbe. Of course, our transition from line-bundle gerbes to gerbes is functorial.

Later we will see more fully how certain gerbes give line bundle gerbes, but before that we should note that as Y was not necessarily the étale space of a sheaf on X, it seems highly unlikely that, in general, we could start with a gerbe and retrieve some (L, π). The fibres of étale spaces are discrete but in general the fibres of locally split maps need not be.

Before we continue this investigation we will look at other aspects of what we have done so far.

We would expect that the gerbe given by the above process would be a "$\mathbb{C}^\times$-gerbe", i.e., its sheaf of local automorphisms should be the constant sheaf on X with "value" $\mathbb{C}^\times$ or, if looking at the Hermitian flavoured case, $U(1)$. How can we verify this? We can find an open cover $\mathcal{U}$ given by those U over which local sections of π exist, thus over such a U there is a global section, a, of G_0 and hence of the object part of the stack completion of G. The automorphism sheaf of a is the group of sections of the pullback of L along $(a, a) : U \to Y^{[2]}$, but that is $\mathbb{C}^\times$ or $U(1)$, depending on the viewpoint taken. This is a constant sheaf and so G is a $\mathbb{C}^\times$-gerbe.

10.2.4 Bundle gerbes and groupoids

As we saw at the beginning of these notes, an equivalence relation, R, on a set, Y, gives a groupoid. As any (surjective) function, $\pi : Y \to X$, yields the standard equivalence relation: $y_1 R y_2$ if and only if $\pi(y_1) = \pi(y_2)$, for which X can be identified with the set of equivalence classes (which is why we added 'surjective' above), any such function yields a groupoid, and of course, viewed as a small category, this is

$$Y^{[2]} \underset{t}{\overset{s}{\rightrightarrows}} Y \ ,$$

where $Y^{[2]}$ is, as before, the pullback, $Y \times_X Y$, and s and t are the projections. The map, i, which picks out 'identity arrows' for each object, is the diagonal, of course, the composition is

$$((y_1, y_2), (y_2, y_3)) \mapsto (y_1, y_3),$$

in algebraic order.

We have used this many times now and have also met it in other contexts, such as internally to some category such as groups. In our current context of line bundle gerbes, we have used this structure in *indexing* the multiplication of L

$$L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \overset{\cong}{\to} L_{(y_1,y_3)}.$$

The line bundle $L \to Y^{[2]}$ can also be interpreted as

$$L \rightrightarrows Y^{[2]}$$

by composing the projection of the line bundle with the two projections and the subsequent interpretation of our earlier results (page 470) is that this is a groupoid as well.

If we are working with smooth manifolds and maps then not only is

$$Y^{[2]} \underset{t}{\overset{s}{\rightrightarrows}} Y$$

a topological groupoid (i.e., there is a space of objects and a space of arrows, and all the structure maps are continuous), but, under reasonable extra conditions, it is a *Lie groupoid* as all the structure maps s, t, i and the composition and inversion maps are all smooth. The one problem that can occur is that $Y \times_X Y$ is not in general a smooth manifold. It is, however, if $\pi : Y \to X$ is a submersion and that is why all through the discussion of smooth line bundle gerbes, the fibering π was required to be a submersion. This, for instance, occurs if $Y = \sqcup \mathcal{U}$, the étale space associated to an open cover of X.

A Lie groupoid is, of course, the multi-object analogue of a Lie group. Another example of a Lie groupoid on X comes from any Lie group, G. We then get a *bundle of Lie groups*, $\underline{G} = G \times X \to X$. The source and target maps are both the projection onto X.

Now assume we have that (L, π) is a line bundle gerbe, then we have a smooth surjective morphism of Lie groupoids

$$L \to Y^{[2]}$$

and hence, intuitively, an extension

$$? \to L \to Y^{[2]}$$

of such objects. Thinking of L as a $\mathbb{C}^\times$-bundle on $Y^{[2]}$ or, in the Hermitian flavoured version, a principal $U(1)$-bundle / $U(1)$-torsor on $Y^{[2]}$, we get that the left hand term is $\underline{\mathbb{C}^\times}$ or $\underline{U(1)}$.

This gives an equivalent definition of a line bundle gerbe that can be found, for instance, in Moerdijk's notes, [206]. In fact, as is pointed out there, it gives a neat way to generalise line bundle gerbes.

First we define:

Definition: An *extension of Lie groupoids* over Y is a sequence of Lie groupoids over Y

$$K \xrightarrow{j} G \xrightarrow{\varphi} H,$$

where φ is a surjective submersion and j is an embedding onto a submanifold, $Ker\,\varphi = \{g \in G \mid \varphi(g) \text{ is a unit of } H\}$.

We note that maps of 'groupoids over Y' means that both j and φ are the identity map on objects, so K satisfies $sj(k) = tj(k)$ and so K is a bundle of groups.

Now let G be a fixed Lie group.

Definition: A *G-bundle gerbe* over a manifold X is a pair (β, π), where $\pi : Y \to X$ is a surjective submersion and β is an extension

$$\beta = (\underline{G} \to L \stackrel{\varphi}{\to} Y^{[2]})$$

of Lie groupoids.

Our previous discussion implies the following result:

Proposition 111 *(i) A line bundle gerbe (L, π) is equivalent to a $\mathbb{C}^\times$-bundle gerbe, (β, π), with extension,*

$$\beta = (\underline{\mathbb{C}^\times} \to L \to Y^{[2]}),$$

(in the same notation as before).

(ii) A Hermitian line bundle gerbe (L, π) is equivalent to a $U(1)$-bundle gerbe, (β, π), with extension

$$\beta = (\underline{U(1)} \to L \to Y^{[2]}).$$

■

Suppose we have a Lie groupoid $G \underset{t}{\overset{s}{\rightrightarrows}} M$ together with a submersion $\pi : M \to Y$ for which $\pi s = \pi t$, then we will call this a *family of groupoids on X*. For each such family and each point $x \in X$, the fibre $G_x \rightrightarrows M_x$ is a Lie groupoid.

A family of groupoids on X is almost the same as a sheaf of groupoids on X except that, for the latter, one would have that the maps and the composites πs $(= \pi t)$ would be étale (in this case, local diffeomorphisms, cf. page 295, as the map is smooth). This condition would then imply that s and t, i and the composition and inversion maps were all étale maps as well, so the basic Lie groupoid $(G, M, s, t, i, \ldots)$ would be an *étale groupoid.*

Proposition 112 *Suppose that G is a Lie group and (β, π) is a G-bundle gerbe, in the above sense, then $L \underset{t}{\overset{s}{\rightrightarrows}} Y$ is a family of groupoids on X, where $s = \pi_1\varphi$, $t = \pi_2\varphi$ for $\pi_i : Y^{[2]} \to Y$, the two projections. Moreover*

(i) $\varphi = (s, t) : L \to Y^{[2]}$ is a surjective submersion,

and

(ii) there is an isomorphism of Lie groupoids

$$j_m : G \to Aut_L(m),$$

which identifies each vertex group in L with G, this isomorphism varying smoothly in the local object, m. Conversely given a family of groupoids satisfying these conditions, (β, π) is a G-bundle gerbe. ■

The proof is just: reformulate the definition and check! As Moerdijk comments in [206], the first condition states strongly that each fibre is non-empty, whilst it also says that that fibre groupoid is connected. This reformulation shows very neatly the way that G-bundle gerbes are a neat extension of the idea of gerbe, which allows non-trivial topology in the fibres, just like bundles of groups generalise sheaves of groups.

In the above theory, the generalisation from the groupoid corresponding to an open cover $\mathcal{U}$ of X to a submersion $\pi : Y \to X$ and the groupoid

$$Y^{[2]} \rightrightarrows Y$$

was important. Further generalisations are possible and important. We can replace the manifold X by an orbifold. (As usual Wikipedia is a good place to start for these.) An orbifold is approximately the quotient of an n-manifold, M, by the action of a finite group, or more exactly a space which has local patches given by quotienting $\mathbb{R}^n$ by the action of a finite group. There has to be compatibility conditions on double overlaps and, surprise, a cocycle condition on triple ones. It is not surprising that as a group action gives rise to a groupoid (as in our very first section), so an orbifold gives rise to an étale Lie groupoid by putting together the action groupoids of the 'local actions'. The notion of bundle gerbes over manifolds then gives a rich theory for describing the geometry of classes in $H^3(X)$, and more generally. The basic reference for this is the paper by Lupercio and Uribe, [189]. We will not describe more of that theory here as it would take us too far away from the development of our main themes.

10.3 Cocycle description of gerbes

For the moment we will leave aside the bundle version of gerbes and also the geometric constructions related to bundle gerbes. We will revisit these later.

When we last looked at gerbes as such, we had the semi-local description of a gerbe, P; see page 461. We assume, for simplicity as there, that P is a G-gerbe for some sheaf of groups, G. With the insights of the bundle gerbe theory, at least its elementary parts, we can glance at that from the 'semi-local' perspective.

For the semi-local description, we had an open cover $\mathcal{U}$ and over each U, an equivalence

$$\mathsf{P}(U) \overset{\simeq}{\to} \mathsf{Tor}(G, U)$$

obtained by choosing an object in $\mathsf{P}(U)$. We thus had G-bitorsors, P_{ij}, over U_{ij}, which gave the transition from $\mathsf{Tor}(G, U_j)$ to $\mathsf{Tor}(G, U_i)$ over the intersection. Now assume we have a Grothendieck topology of some sort, and replace $\mathcal{U}$ by a single covering morphism $Y \to X$. We can rerun the description with $\bigsqcup \mathcal{U}$ replaced by Y. An object x in P_Y gives a sheaf of groups, $G = Aut_{\mathsf{P}_Y}(x)$ over Y together with a (p_2^*G, p_1^*G)-bitorsor on $Y^{[2]}$ satisfying a coherence condition on $Y^{[3]}$. Of course, if the Y is really the Y_0 of a hypercovering then in the above we should replace $Y^{n]}$ by Y_{n-1}. In other words, although initially bundle gerbes look very different to standard gerbes, they are, in fact, very closely related.

We will see the usefulness of the covering idea again shortly. The point that is important is that a gerbe is locally non-empty and locally connected. The first condition gives an open cover, $\mathcal{U}$, or covering family if working with a topos, such that the gerbe pulled back that cover is non-empty, but it is still only locally connected, i.e., we may still have to find a finer cover than $\mathcal{U}$ before getting to a connected situation. After the first step, we have $\{U_i\}$, after the second $\{U_{i,\alpha}\}$. If handling a topological situation, i.e., working with $Sh(B)$, and provided B is paracompact, we can assume that we can refine the first situation so that each groupoid $\mathsf{P}(U)$ is both non-empty *and* connected. In other word, repeating what has been mentioned before, if B is paracompact then we can use coverings rather than hypercoverings. This avoids multiple indices! Once we understand the situation simplicially, then we can replace $N(\mathcal{U})$ by a hypercovering without added pain! There is a downside, however, as there is some loss or mutation of the geometric intuition, which can be awkward to the beginner in the subject. Because of this we will usually work with coverings.

10.3.1 The local description

(We will continue to follow and to expand on Breen's exposition from [51].)

Let P be a G-gerbe, then there is an open cover $\mathcal{U}$ for which each P_U is non-empty. Pick an object x_i in P_{U_i}. On U_{ij}, we will assume $\mathsf{P}_{U_{ij}}$ is connected. (In general we might have to cover U_{ij} more finely before getting connectedness.) We pick an arrow

$$\phi_{ij} : x_j \to x_i$$

in $\mathsf{P}_{U_{ij}}$. (Note the abuse of notation, writing x_j for $x_j|_{U_{ij}}$.) We have, as in the semilocal description (page 461), an identification of $G_i := G|_{U_i}$ with $Aut_{\mathsf{P}}(x_i)$ and over U_{ij}, the arrow φ_{ij} induces an isomorphism

$$\lambda_{ij} : G_j|_{U_{ij}} \to: G_i|_{U_{ij}}$$

given by conjugation: $\lambda_{ij}(\gamma) = \varphi_{ij}\gamma\varphi_{ij}^{-1}$ within the groupoid $G|_{U_{ij}}$.

$$\begin{tikzcd} x_j \arrow[r, "\gamma"] \arrow[d, "\varphi_{ij}"'] & x_j \arrow[d, "\varphi_{ij}"] \\ x_i \arrow[r, "\lambda_{ij}(\gamma)"'] & x_i \end{tikzcd}$$

Remark: The point is here that λ_{ij} induces the equivalence

$$\Phi_{ij} : \mathsf{Tors}(G)|_{U_{ij}} \to \mathsf{Tors}(G)|_{U_{ij}},$$

of the semi-local description

$$\Phi_{ij} = \lambda_{ij*},$$

and the (G_j, G_i)-bitorsor, P_{ij} is $(T_{G_i})_{\lambda_{ij}}$, that is, the 'group' G_i considered as a trivial left G_i-torsor with right G_j-action induced by λ_{ij}:

$$g_i.g_j := g_i\lambda_{ij}(g_j),$$

all of this happening over U_{ij}. It is also worth noting that, although we have assumed that P is a G-gerbe, to examine the above point it becomes clearer if the various G_i are kept notationally apart! The group G_j has to act on the right of G_i via λ_{ij}.

The description of the isomorphisms, λ_{ij}, relates well to behaviour over triple intersections U_{ijk}. There we have three locally chosen objects x_i, x_j, x_k and a diagram

$$\begin{tikzcd} x_k \arrow[r, "\varphi_{jk}"] \arrow[d, "\varphi_{ik}"'] & x_j \arrow[d, "\varphi_{ij}"] \\ x_i \arrow[r, dashed, "?"] & x_i \end{tikzcd}$$

As the φs were merely 'chosen', we do not know that they satisfy any nice cocycle condition, but we will have a $g_{ijk} \in G_i|_{U_{ijk}}$ completing the square. We combine the two types of square as follows:

$$\begin{tikzcd}
x_i \arrow[rrr, "\lambda_{ij}(\gamma)"] \arrow[ddd, "g_{ijk}"'] & & & x_i \arrow[ddd, "g_{ijk}"] \\
& x_k \arrow[ul, "\varphi_{ik}"'] \arrow[r, "\gamma"] \arrow[d, "\varphi_{jk}"'] & x_k \arrow[ur, "\varphi_{ik}"] \arrow[d, "\varphi_{jk}"] & \\
& x_j \arrow[dl, "\varphi_{ij}"] \arrow[r, "\lambda_{jk}(\gamma)"] & x_j \arrow[dr, "\varphi_{ij}"'] & \\
x_i \arrow[rrr, "\lambda_{ij}\lambda_{jk}(\gamma)"'] & & & x_i
\end{tikzcd}$$

i.e., $\lambda_{ij}\lambda_{jk}(\gamma) = g_{ijk}\lambda_{ik}(\gamma)g_{ijk}^{-1}$ within $G_{U_{ijk}}$, but this means that

$$\lambda_{ij}\lambda_{jk} = \iota_{g_{ijk}}\lambda_{ik},$$

where ι is, here as usual, the natural (left) conjugation morphism from $G_{U_{ijk}}$ to $Aut(G_{U_{ijk}})$.

For comparison, both forwards and backwards in this discussion, it may help to think of the square that defines g_{ijk} as a 2-simplex

$$\begin{tikzcd}
& {} \arrow[dr, "\varphi_{ij}"] & \\
{} \arrow[ur, "\varphi_{jk}"] \arrow[rr, "\varphi_{ik}"'] \arrow[rr, phantom, shift left=3ex, "g_{ijk}"] & & {}
\end{tikzcd}$$

with the g_{ijk} the obstruction to the cocycle condition being satisfied, but even more striking is the corresponding diagram coming from the λ_{ij}s,

$$\begin{tikzcd}
& j \arrow[dr, "\lambda_{ij}"] & \\
k \arrow[ur, "\lambda_{jk}"] \arrow[rr, "\lambda_{ik}"'] \arrow[rr, phantom, shift left=3ex, "g_{ijk}"] & & i
\end{tikzcd}$$

which is reminiscent of the diagrams for maps from $N(\mathcal{U})$ into $K(\mathsf{Aut}(G))$. Keeping that in mind, we look at a 4-fold intersection, $U_{ijk\ell}$. We have a tetrahedron:

$$\begin{tikzcd}
& k \arrow[dr, "\lambda_{ik}"] \arrow[dd, "\lambda_{jk}"] & \\
\ell \arrow[ur, "\lambda_{k\ell}"] \arrow[rr, dash] \arrow[dr, "\lambda_{j\ell}"'] & & i \\
& j \arrow[ur, "\lambda_{ij}"'] &
\end{tikzcd}$$

with $\lambda_{i\ell}$ on the level map at the back, and with the corresponding g_{ijk} etc. in the faces. The faces fit together giving a square

$$\begin{tikzcd}
x_i \arrow[r, "g_{ij\ell}"] \arrow[d, "g_{ik\ell}"'] & x_i \arrow[d, "\lambda_{ij}(g_{jk\ell})"] \\
x_i \arrow[r, "g_{ijk}"'] & x_i
\end{tikzcd}$$

and so we will get an equation

$$g_{ijk}g_{ik\ell} = \lambda_{ij}(g_{jk\ell})g_{ij\ell}.$$

The only mysterious thing here is the $\lambda_{ij}(g_{jk\ell})$ term. Why is it there? The three other faces correspond to d_0, d_1 and d_2 of the tetrahedron and so end up at i. This term tries to end up at j, so we drag it through to x_i using λ_{ij}. That hopefully gives some intuition as to what it does, but to see why it has exactly the form it has, we need to go back from the λ_{ij}s to the φ_{ij}.

The g_{ijk}s, etc., all came from filling a square and so we try to fit these squares together into a cube. We have

$$\begin{tikzcd}
& x_\ell \arrow[rr, "\varphi_{j\ell}"] \arrow[dl, "\varphi_{i\ell}"'] \arrow[dd, "\varphi_{k\ell}"] & & x_j \arrow[dl, "\varphi_{ij}"] \arrow[dd, "g_{jk\ell}"] \\
x_i \arrow[rr, "g_{ij\ell}"] \arrow[dd, "g_{ik\ell}"'] & & x_i \arrow[dd] & \\
& x_k \arrow[rr] \arrow[dl, "\varphi_{ik}"'] & & x_j \arrow[dl, "\varphi_{ij}"] \\
x_i \arrow[rr, "g_{ijk}"'] & & x_i &
\end{tikzcd}$$

All but the right side face and the front face have the form defining a 'g-term'. The right face (if we rotate it anticlockwise) looks like

$$\begin{tikzcd}
x_j \arrow[r, "g_{jk\ell}"] \arrow[d, "\varphi_{ij}"'] & x_j \arrow[d, "\varphi_{ij}"] \\
x_i \arrow[r, "?"'] & x_i
\end{tikzcd}$$

and so the missing edge will be $\lambda_{ij}(g_{jk\ell})$. As each face so far considered is commutative *and* all the arrows are invertible, the final face, i.e., the front one, is also commutative, so we get

Lemma 67 *The elements λ_{ij} and g_{ijk} satisfy the equations $\lambda_{ij}\lambda_{jk} = i_{g_{ijk}}\lambda_{ik}$, on the U_{ijk} and $g_{ijk}g_{ik\ell} = \lambda_{ij}(g_{jk\ell})g_{ij\ell}$ on $U_{ijk\ell}$.* ∎

We clearly have here the beginnings of a simplicial description of G-gerbes. Not only does it involve several 'simplicial' diagrams, but the interpretation is clearly related to our earlier simplicial descriptions. We earlier had the end term of our exact sequence, left without a neat interpretation. The above looks as if it might be the start of such an interpretation, but it is just a start and we need to look at coboundaries, choices, etc., before being sure.

Our initial step was to pick the x_is then the $\phi_{ij} : x_j \to x_i$, so we should examine what happens if we pick other objects and / or arrows.

If $\{x'_i\}$ is another family of objects in the $\mathsf{P}_{U_{ij}}$ relative to an open cover $\mathcal{U}$, then, refining the cover if necessary, we can find arrows

$$\chi_i : x_i \to x'_i$$

in these groupoids, linking the old and new choices. Likewise we choose

$$\phi'_{ij} : x'_j \to x'_i,$$

(although this may again require further refinement of the cover). We have a diagram

$$\begin{tikzcd}
x_j \arrow[r, "\varphi_{ij}"] \arrow[dd, "\chi_j"'] & x_i \arrow[d, "\chi_i"] \\
 & x_i' \arrow[d, dashed] \\
x_j' \arrow[r, "\varphi_{ij}'"'] & x_i'
\end{tikzcd}$$

and we obtain an arrow $\theta_{ij} : x_i' \to x_i'$ in $\mathsf{P}_{U_{ij}}$ that measures the lack of coherence of the χ_i with respect to previous choices. We have

$$\theta_{ij} = \varphi_{ij}'\chi_j\varphi_{ij}^{-1}\chi_i^{-1}$$

(An important special case of this is when the x_is are left as they were but another $\varphi_{ij}' : x_j \to x_i$ is chosen. In that case θ_{ij} is just $\varphi_{ij}'\varphi_{ij}^{-1}$.)

If we have G-gerbes, each object x_i or x_i' determines a copy of G_{U_i}, but we need to keep track of 'which copy' so will write G_i for $Aut_{\mathsf{P}}(x_i)$ and similarly G_i' for $Aut_{\mathsf{P}}(x_i')$. These two sheaves of groups are isomorphic since the objects x_i and x_i' are linked via χ_i. We denote by $r_i : G_i \to G_i'$, the isomorphism that results by conjugation. If $u : x_i \to x_i$ is in G_i, then we have a diagram

$$\begin{tikzcd}
x_i \arrow[r, "u"] \arrow[d, "\chi_i"'] & x_i \arrow[d, "\chi_i"] \\
x_i' \arrow[r, "r_i(u)"'] & x_i'
\end{tikzcd}$$

How does this change, of the local objects and the morphisms between them, change the λ_{ij}s? As the r_i are isomorphisms, the easier thing is to calculate $\lambda_{ij}'(r_j(\gamma))$ for γ, as before, from x_j to itself. An easy calculation shows that the corresponding diagram to the above one defining θ_{ij} is

$$\begin{tikzcd}
G_j \arrow[r, "\lambda_{ij}"] \arrow[dd, "r_j"'] & G_i \arrow[d, "r_i"] \\
 & G_i' \arrow[d, "\iota_{\theta_{ij}}"] \\
G_j' \arrow[r, "\lambda_{ij}'"'] & G_i'
\end{tikzcd}$$

(This is **a good point to check**. The necessary diagram *is* quite easy to construct. Start with γ and transform it in the two ways given by the two paths in the above. Each arrow in this diagram will give you a square in the required diagram. This shows immediately the links between an edge in the groupoid and the conjugation that it gives, but is best done by the reader!) We thus have $\lambda_{ij}'r_j = \iota_{\theta_{ij}}r_i\lambda_{ij}$. We have seen that conjugation in the context of groupoids is closely linked to homotopies of groupoid morphisms and one way to express this was simplicially. Here a simplical view looks very neat. It gives

$$\begin{tikzcd}
G_j \arrow[r, "\lambda_{ij}"] \arrow[d, "r_j"'] \arrow[dr, "\theta_{ij}"] \arrow[d, phantom, shift left=3ex, "="] & G_i \arrow[d, "r_i"] \\
G_j' \arrow[r, "\lambda_{ij}'"'] & G_i'
\end{tikzcd}$$

In other words, a homotopy from λ_{ij} to λ'_{ij}. The θ_{ij} labelling the top right 2-simplex has boundary given in the diagram with the diagonal being the morphism $\iota_{\theta_{ij}} r_i \lambda_{ij}$. This begs for some simplification, and this will be done shortly. It clearly also could be simplified somewhat by replacing $\iota : G \to Aut(G)$ by an arbitrary sheaf of crossed modules, but we must wait to do this until we have looked at the effect of the changes of the choices of objects, etc., on other parts of the structure.

The cocycle pair describing P consists of two families, one $\{\lambda_{ij}\}$ of pairwise transitions, the other the family $\{g_{ijk}\}$ of local sections of G over triple intersections. These satisfy a linking cocycle condition on the triple intersections and a cocycle condition on the 4-fold intersections as in Lemma 67. Our recent discussion suggested a boundary relation for the λ_{ij}s namely the existence of θ_{ij}s and r_is satisfying

$$\lambda'_{ij} r_j = \iota_{\theta_{ij}} r_i \lambda_{ij},$$

or, alternatively, $\lambda'_{ij} = \iota_{\theta_{ij}} r_i \lambda_{ij} r_j^{-1}$, and this does *look* right as it does seem to correspond to some sort of simplicial homotopy relation, but we would expect a second compatibility condition involving the g_{ijk}s and the corresponding g'_{ijk}s. (If you want to see why such a second condition should be there, look at the Abelian case and ideas of classical hypercohomology, and there replace the crossed module $\iota : G \to Aut(G)$ by a two term chain complex concentrated in dimensions 1 and 2. The cocycle pairs give chain maps from $N(\mathcal{U})$ to the coefficients and the relation above describes the first part of a chain homotopy condition, but we also need a map from $N(\mathcal{U})_2$ to dimension 3 of the coefficients. What that map is is no problem as that position is trivial, but this does impose a condition on the two level two maps which are given by the g_{ijk}s.)

The g'_{ijk}s are, of course, defined by the analogous square to those used for the g_{ijk}s, with φ's replacing φs, so $g'_{ijk} = \varphi'_{ij}\varphi'_{jk}\varphi'^{-1}_{ik}$. We draw a cube with this at the base and with expressions for φ'_{ij}, etc. giving the three corresponding vertical faces with the g_{ijk} square at the top. (There is one subtlety. The neat diagrammatic representation we will give is due to Breen in his notes, [51].)

$$\begin{tikzcd}
 & x_k \arrow[dl, "\varphi_{jk}"'] \arrow[rr, "\varphi_{ik}"] \arrow[dddd, "\chi_k"] & & x_i \arrow[dl, "g_{ijk}"'] \arrow[d, "\chi_i"] \\
x_j \arrow[rr, "\varphi_{ij}"] \arrow[dd, "\chi_j"'] & & x_i \arrow[d, "\chi_i"'] & x'_i \arrow[dl, "r_i(g_{ijk})"] \arrow[ddd, "\theta_{jk}"] \\
 & & x'_i \arrow[d, "\theta_{ij}"] & \\
x'_j \arrow[rr, "\varphi'_{ij}"] \arrow[dd, "\theta_{jk}"'] & & x'_i \arrow[dd, "\lambda'_{ij}(\theta_{jk})"'] & \\
 & x'_i \arrow[dl, "\varphi'_{jk}"'] \arrow[rr] & & x'_i \arrow[dl, "g'_{ijk}"] \\
x'_j \arrow[rr, "\varphi'_{ij}"'] & & x'_i &
\end{tikzcd}$$

The subtlety is the need for the 'induction square' giving $\lambda'_{ij}(\theta_{jk})$. (If you draw the g_{ijk} and g'_{ijk} as filling the 2-simplices coming from the λs, then it is easy to construct a prism with these on the ends, the r_is as the joining edges and with θs filling the faces. This prism then shows the induced term once again, as it has two terms ending at the i-vertex whilst the θ_{jk} does not fit there unless dragged there by λ'_{ij}. A filling scheme in the simplicial set $\overline{W}K(\mathsf{Aut}(G))$ will give another derivation of this result.)

We can read off from the above diagram that

$$g'_{ijk} = \lambda'_{ij}(\theta_{jk})\theta_{ij}r_i(g_{ijk})\theta_{ik}^{-1},$$

which is the second coboundary relation that we were seeking.

Definition: Two cocycle pairs, (λ_{ij}, g_{ijk}) and $(\lambda'_{ij}, g'_{ijk})$, are *cohomologous* if there are isomorphisms $r_i \in Isom(G_i, G'_i)$ and sections $\theta_{ij} \in G'_i|_{U_{ij}}$ satisfying

$$\begin{cases} \lambda'_{ij} &= i_{\theta_{ij}} r_i \lambda_{ij} r_j^{-1}, \\ g'_{ijk} &= \lambda'_{ij}(\theta_{jk})\theta_{ij}r_i(g_{ijk})\theta_{ik}^{-1}. \end{cases}$$

This is valid even for a general gerbe, but when we assume that P is a G-gerbe, then we get $r_i \in Aut(G)$, and the θ_{ij} are in $G|_{U_{ij}}$.

We take the set of equivalence classes of cocycle pairs modulo this relation of 'cohomologous' to be the definition of $H^1(\mathcal{U}, \mathsf{Aut}(G))$. Clearly it needs to be generalised to take coefficients in a general sheaf of crossed modules, $\mathsf{M} = (C, P, \partial)$, and to then pass to the colimit over refinements of the covers. To spell this out a bit more, we take

- a cocycle pair, (p_{ij}, c_{ijk}), over $\mathcal{U}$ with values in M, to consist of a family of local sections $p_{ij} \in P(U_{ij})$, and a family $c_{ijk} \in C(U_{ijk})$ such that for all i, j, k, (as usual),

- $p_{ij}p_{jk} = \partial c_{ijk}.p_{ik}$,

and

- $c_{ijk}c_{ik\ell} = {}^{p_{ij}}c_{jk\ell}.c_{ij\ell}$.

The 'picture' is

$$\begin{tikzcd} & \cdot \arrow[dr, "p_{ij}"] & \\ \cdot \arrow[ur, "p_{jk}"] \arrow[rr, "p_{ik}"'] \arrow[rr, phantom, shift left=4ex, "c_{ijk}"] & & \cdot \end{tikzcd}$$

for the first of these, with an obvious tetrahedron for the second.

This is a good place to refer back to two earlier discussions. On page 252, we looked at the formulae and diagrams for the classifying space of a crossed complex. At that point, we were still using the algebraic composition convention, so the reader will need to take some care and work through with that in mind, but the diagrams indicate the close connection with what we have here. The other discussion is in section 7.6 on M-torsors. The cocycle pairs there gave p_is and c_{ij}s and corresponded to simplicial maps $\mathbf{g} : N(\mathcal{U}) \to K(\mathsf{M})$. Here we have a cocycle pair that corresponds to a simplicial map from $N(\mathcal{U})$ to $BK(\mathsf{M})$, i.e., to $\overline{W}K(\mathsf{M})$ - and we are approaching an interpretation of our mysterious $\check{H}^1(B, \mathsf{M})$ term from the last chapter, (pages 386 and 387).

Back to our definition, two cocycle pairs, (p_{ij}, c_{ijk}) and (p'_{ij}, c'_{ijk}), are *cohomologous* if there are families of local sections, $r_i \in P(U_i)$ and $t_{ij} \in C(U_{ij})$, satisfying, for all i, j, k,

- $p'_{ij} = \partial(t_{ij})r_i p_{ij} r_j^{-1}$ over U_{ij};
- $c'_{ijk} = {}^{p_{ij}}t_{jk}.t_{ij}r_i(c_{ijk})t_{ik}^{-1}$ over U_{ijk}.

We saw that such a pair (r_i, t_{ij}), in fact, corresponds to a homotopy between the simplicial maps from $N(\mathcal{U})$ to $BK(\mathsf{M})$ given by the cocycle pairs. In fact, given the filling properties of $BK(\mathsf{M})$, which not only is a Kan complex, but in which the fillers can be algebraically derived as we have seen, the correspondence is reversible, so given an arbitrary homotopy between two maps $\mathbf{g}, \mathbf{g}' : N(\mathcal{U}) \to BK(\mathsf{M})$, corresponding to the cocycle pairs as above, we can solve to get the rs and ts as above. (This is 'fairly obvious' given our earlier discussion, but still needs some work. The rs correspond to the edges of the squares in the homotopy:

$$\begin{tikzcd}
\cdot \arrow[r, "p_{ij}"] \arrow[d, "r_j"'] \arrow[dr] \arrow[dr, phantom, "a", shift left=3] \arrow[dr, phantom, "b"', shift right=3] & \cdot \arrow[d, "r_i"] \\
\cdot \arrow[r, "p'_{ij}"'] & \cdot
\end{tikzcd}$$

but, in general, the bottom left corner, below the diagonal, will not be an identity. It is then necessary to replace the given homotopy by one in which these elements *are* identities. This is done by solving the relevant simplicial identities referring back to the structure of $BK(\mathsf{M})$, - and then the relevant equations need checking.)

Remark: The algebraic form of the cohomology relations here is beginning to be near the limit of what we can handle using cocycle type descriptions. The formulae also are beginning to be 'obscure' geometrically. Because of this, this cocycle description tends to be surplanted by the simplicial description in much of the work on this topic.

10.3.2 From local to semi-local

In the local description of a gerbe, we have cocycle pairs, (λ_{ij}, g_{ijk}), or, more generally, (p_{ij}, c_{ijk}), but in the semi-local description that linked so well with bundle gerbes, we had G-bitorsors over the intersections. We know bitorsors have themselves a cocycle description, so what is the translation between these different formulations?

The translation proceeds by a careful look at the two constructions involved:

- In both descriptions, we have an open cover $\mathcal{U}$ and over the open set U_i, we choose an object x_i in $\mathsf{P}(U_i)$. We have a sheaf of groups $G_i := Aut_{\mathsf{P}}(x_i)$. (We will be concentrating on G-gerbes, but for the 'book-keeping', it is advantageous to denote $G!_{U_i}$ by G_i, which means the extension to general gerbes is then easy.)
- For the semi-local description, we have a (G_i, G_j)-bitorsor, P_{ij} over U_{ij}, which gives the equivalence between $Tors(G_j)|_{U_{ij}}$ and $Tors(G_i)|_{U_{ij}}$;
- For the local description, we choose an arrow from x_j to x_i over the intersection, U_{ij} and, by conjugation, we get an isomorphism $\lambda_{ij} : (G_j)|_{U_{ij}} \to (G_i)|_{U_{ij}}$.

To get the bitorsor P_{ij}, one uses the equivalence $\Phi_{ij} = \Phi_i \circ \Phi_j^{-1}$ for some *choice* of quasi-inverse Φ_j^{-1} for $\Phi_j : \mathsf{P}_{U_j} \to \mathsf{Tors}(G_j)$, restricted to U_{ij}. The morphism λ_{ij} induces an equivalence and corresponds to such a choice of quasi-inverse. (Note that as Φ_j is an equivalence and not an isomorphism, there may be many such quasi-inverses.) The description of Φ_{ij} means that we have to run the construction in the proof of Proposition 104, page 454, that proves that Φ_{ij} is essentially surjective on objects, only now just with target torsor $Q = T_{G_j}$. In this case, referring back to that proof, Q is already trivialised over the cover and one just needs to pick the y isomorphic to x_j over U_j. On restricting to U_{ij}, this amounts to picking an object over U_{ij} and an isomorphism from the restriction of x_j to U_{ij} to that object. Of course, $x_i|_{U_{ij}}$ is such an object and ϕ_{ij} is such an isomorphism, inducing λ_{ij} on the vertex groups. This gives an explicit description of P_{ij} given a choice of ϕ_{ij}, namely as $(T_{G_i})_{\lambda_{ij}}$, the (G_i, G_j)-bitorsors that is the trivial left G_i-torsors with right G_j-action given by λ_{ij}, (see the discussion of contracted product and change of groups, in section 7.4.4, starting on page 320. We think of this as being defined via the natural global section, $!_i$, of T_{G_i}. Any 'local element' of T_{G_i} is of form $x = g.1_i$, so given g_j, a local element of G_j, $x.g_j = g.1_i.\lambda_{ij}(g_j) = g.\lambda_{ij}(g_j).1_i$, but this then hands us 'on a plate' the beginnings of the cocycle description of P_{ij}

The left G_i-torsor, P_{ij}, is already trivial on U_{ij} by the above, so there are no transition cocycles needed if we stick with the cover $\{U_{ij}\}$! (The reader may want to think about this again when we are in a context where hypercoverings are needed instead of covers. There is also some advantage in using the cover $\{U_{ijk}\}_k$, i.e., the triple intersections with ij fixed and k varying, but at least, for the moment, let us explore this ultra simple choice with a single open set in the cover!) So what are the isomorphisms u in this context. The description as $u_i(h)s_i = s_i.h$ from page 333, indicates clearly that it is λ_{ij}. (Beware P_{ij} is a bitorsor on U_{ij} with cocycle $(1, \lambda_{ij})$ - but i and j are fixed here, not variable as in the earlier discussion of bitorsors in general. For instance, the equation '$u_i = \iota_{g_{ij}} u_j$' is trivial here, since there is only one open set in the cover being considered.)

Our local description is governed by cocycle pairs, (λ_{ij}, g_{ijk}), and we have identified the meaning of the λ_{ij}s in the semi-local description. What are the g_{ijk} in this context? We had, in our semi-local description, that on triple intersections, there were isomorphisms,

$$\psi_{ijk} : P_{ij} \wedge^G P_{jk} \to P_{ik}.$$

The left hand side of this is the bitorsor corresponding to $\Phi_{ij}\Phi_{jk}$, or, from our discussion above, $(\lambda_{ij}\lambda_{jk})_*$, whilst the right hand one to Φ_{ik} and thus to λ_{ik*}. By Lemma 52, page 321, the natural transformation will be determined by a section of G_i and, of course, this is the g_{ijk} of the other description, and satisfies

$$\lambda_{ij}\lambda_{jk} = \iota_{g_{ijk}}\lambda_{ik}.$$

The ψ_{ijk} must satisfy an associativity coherence condition over 4-fold intersections and that translates to

$$g_{ijk}.g_{ik\ell} = {}^{\lambda_{ij}}g_{jk\ell}.g_{ij\ell}.$$

Thus the translation between semi-local and local descriptions is fairly straightforward once the indices are sorted out.

In the above, we were able to simplify the discussion no end since P_{ij} was trivial over U_{ij}. In the more general situation, one starts with an open cover $\mathcal{U} = \{U_i\}$ over which P is non-empty allowing the choice of local objects x_i, then over U_{ij}, we would need a cover over which $\mathsf{P}_{U_{ij}}$ was locally connected, so the λ_{ij}s are only *locally defined* and need additional indices, (see Breen's treatment of this in [50], section 2.4).

We have not actually shown that such a cocycle decomposition of a gerbe, in either form, can be reversed, i.e., given a collection (λ_{ij}, g_{ijk}) or more generally.(p_{ij}, c_{ijk}), satisfying the various cocycle conditions, one can construct a gerbe with that description. Breen discusses this in detail in the monograph [50], section 2.6, using 2-descent data and a 2-stack. As we have not met these so far in these notes, we cannot treat that yet, however we will start preparing the ground so as to discuss such ideas shortly.

Chapter 11

Homotopy Coherence and Enriched Categories.

We are getting to a point where we need some more powerful insights on homotopy coherence and descent, so in the next few chapters we will examine these topics in some detail. This will give us some useful tools for later use. (These chapters are quite long can be skimmed at first reading, but as the tools will be used later, the material is important for later sections.)

At several points in earlier chapters, we have had to replace colimits by 'pseudo' or 'lax' colimits. We have, especially when 'categorifying', had to replace equality or commutativity in some context, by 'equivalence' or 'coherence'. We have now some experience in handling such ideas and hopefully have built up some intuition, gaining a 'feel' for the general method. It is time now to devote some space to solidifying that intuition a bit further as we will be needing to go in more deeply in future sections.

We will not give a full treatment however as that would take up a lot of space and also would detract from the development of gerbes as such. We will discuss various aspects of the problem and various approaches. Some will involve homotopy theoretic viewpoints, others multiple category theoretic ones. The point is that each approach models certain aspects more transparently than others, so it helps to have a 'multiple model' view. There are possible 'unified models', but they tend to be better handled once the partial approaches - simplicial, homotopy theoretic, n-categorical ones - have been at least met and partially mastered.

11.1 Case study: examples of homotopy coherent diagrams

(Before we get into some examples, it is useful to introduce a bit of terminology that we will use from time to time. If we have a 'diagram' in a category $\mathscr{A}$, then we have, more exactly, some functor, $F : \mathscr{I} \to \mathscr{A}$. We will refer to $\mathscr{I}$ as the 'template' of the diagram, as it gives us the shape of the diagram, that is, what the diagram 'looks like'. We may sometimes give just a graph or more likely a directed graph as a 'template' in which case the corresponding free category on that directed graph will be the domain of the functor. We will also extend the use of 'template' to other similar situations in particular to homotopy coherent diagrams.)

The situation we will start with is a triangular diagram

$$\begin{tikzcd} & K_1 \arrow[dr, "k_{12}"] & \\ K_0 \arrow[ur, "k_{01}"] \arrow[rr, "k_{02}"'] & & K_2 \end{tikzcd}$$

of three spaces or, preferably, simplicial sets, and three maps such that, for the moment, $k_{12} \circ k_{01} = k_{02}$. We can, and will, consider this as a functor

$$K : [2] \to \mathcal{S},$$

where, as always, [2] is the ordinal $\{0 < 1 < 2\}$, considered as a small category. (It is the 'template' for this type of diagram.)

Suppose now that we want to change each K_i to a corresponding object, L_i, which is homotopy equivalent to it. This often occurs when, for instance, the K_is are $K(G,1)$s, and so have only their fundamental groups non-trivial amongst their homotopy groups. It may be thought useful to replace the K_is by smaller or simpler models that reflect the structure of the $\pi_1(K_i)$s. Suppose, therefore, that we have specified maps

$$\left.\begin{array}{l} f_i : K_i \to L_i \\ g_i : L_i \to K_i \end{array}\right\} \quad i = 0, 1, 2,$$

and homotopies

$$\left.\begin{array}{l} \mathbf{H}_i : Id_{K_i} \simeq g_i f_i \\ \mathbf{K}_i : Id_{L_i} \simeq f_i g_i \end{array}\right\} \quad i = 0, 1, 2.$$

We had a commutative diagram linking the K_is. Can we construct some similar diagram from the L_is? The answer is 'yes, but ...'.

We, of course, need some maps $\ell_{ij} : L_i \to L_j$, and there seems only one possible method of obtaining them in a sensible way, namely, use g to get back to K, go around the K-diagram and then pop back to L using f, *i.e.*, define $\ell_{ij} : L_i \to L_j$ by $\ell_{ij} := (L_i \stackrel{g_i}{\to} K_i \stackrel{k_{ij}}{\to} K_j \stackrel{f_j}{\to} L_j)$. This seems the only way - yet it will not work in general. Yes, these ℓ_{ij}s will exist, but

$$\begin{tikzcd} & L_1 \arrow[dr, "\ell_{12}"] & \\ L_0 \arrow[ur, "\ell_{01}"] \arrow[rr, "\ell_{02}"'] & & L_2 \end{tikzcd}$$

will not commute in general. In fact,

$$\ell_{12} \circ \ell_{01} = f_2 k_{12} g_1 f_1 k_{01} g_0.$$

whilst

$$\ell_{02} = f_2 k_{02} g_0 = f_2 k_{12} k_{01} g_0,$$

so we have g_1f_1 blocking the way! As $Id_{K_1} \simeq g_1f_1$, $\ell_{02} \simeq \ell_{12} \circ \ell_{01}$, the triangle *is* homotopy commutative, but it is more than that since we were told a homotopy $\mathbf{K}_1 : Id_{K_1} \simeq g_1f_1$, and so have a specific homotopy that does the job, namely $\mathbf{L}_{012} := f_2k_{12}\mathbf{K}_1(k_{01}g_0 \times I)$.

$$\begin{tikzcd}
& L_1 \arrow[dr, "\ell_{12}"] \arrow[d, phantom, "\Uparrow"] & \\
L_0 \arrow[ur, "\ell_{01}"] \arrow[rr, "\ell_{02}"'] & {} & L_2
\end{tikzcd}$$

Remark: The homotopies we used above went from the identity maps to the composites. We could equally well have written them around the other way. The only difference would be that the arrow in the above diagram would go down instead of up. The conventions here vary from source to source. The above is useful here because it will reflect the cocycle formulae that we have already used, but at other points in our discussion, it will not necessarily be the optimal choice. As homotopies are reversible, it essentially makes no difference here, but it can lead to different formulae and some confusion if this is forgotten.

Now we try to do a slightly harder example. The input this time will be

$$K : [3] \to \mathcal{S},$$

together with $f_i : K_i \to L_i$, $g_i : L_i \to K_i$, $\mathbf{H}_i$, and $\mathbf{K}_i$, for $i = 0, \ldots, 3$. We have maps ℓ_{ij} as before, but also homotopies $\mathbf{L}_{ijk} : \ell_{ik} \simeq \ell_{jk} \circ \ell_{ij}$ for $i < j < k$ within [3], given by $\mathbf{L}_{ijk} := f_kk_{jk}\mathbf{K}_j(k_{ij}g_i \times I)$.

(Any doubts as to why we are going on this excursion into homotopy coherence should be beginning to dissipate by now!) We thus have a tetrahedral diagram

$$\begin{tikzcd}
& L_1 \arrow[drr, "\ell_{13}"] \arrow[ddr, "\ell_{12}"] & & \\
L_0 \arrow[ur, "\ell_{01}"] \arrow[rrr, "\ell_{03}"] \arrow[drr, "\ell_{02}"'] & & & L_3 \\
& & L_2 \arrow[ur, "\ell_{23}"'] &
\end{tikzcd}$$

with homotopies, as above, in each face.

We saw this sort of diagram when we were discussing fibred categories and, in particular, the 3-cocycle condition which mysteriously came out to be written as a square (cf. page 409). Here also we can analyse our tetrahedral diagram as a square with vertices corresponding to paths through the diagram from L_0 to L_3 and with edges corresponding to the homotopies in the faces. Of course, for instance, $\mathbf{L}_{123} : \ell_{13} \simeq \ell_{23} \circ \ell_{12}$, so it contributes a 'whiskered homotopy' $\mathbf{L}_{123} \circ \ell_{01} : \ell_{13} \circ \ell_{01} \simeq \ell_{23} \circ \ell_{12} \circ \ell_{01}$. (Note we are here being lazy, using the convenient notation $\mathbf{L}_{123} \circ \ell_{01}$ instead of the more exact $\mathbf{L}_{123} \circ (\ell_{01} \times I)$, which, however, is sometimes essential!)

$$\begin{tikzcd}
\ell_{03} \arrow[r, Rightarrow, "\mathbf{L}_{023}"] \arrow[d, Rightarrow, "\mathbf{L}_{013}"'] & \ell_{23}\ell_{02} \arrow[d, Rightarrow, "\ell_{23} \circ \mathbf{L}_{012}"] \\
\ell_{13}\ell_{01} \arrow[r, Rightarrow, "\mathbf{L}_{123} \circ \ell_{01}"'] & \ell_{23}\ell_{12}\ell_{01}
\end{tikzcd}$$

We can compose these homotopies to get two, in general distinct, homotopies from ℓ_{03} to $\ell_{23}\ell_{12}\ell_{01}$, explicitly calculable in terms of $\mathbf{K}_1$ and $\mathbf{K}_2$. (A useful observation here is that the indices 1 and 2 are in the middle of all the homotopies' indices, never 0 or 3, as should be clear from the constructions, so our homotopies use $\mathbf{K}_1$ and $\mathbf{K}_2$, not the others.)

Remark: These can be viewed as defined from $L_0 \times I$ to L_3. This is most easily seen in the topological case as we have an obvious homeomorphism, $[0,1] \cong [0,\frac{1}{2}] \cup [\frac{1}{2},1]$, which allows a neat concatenation of homotopies. It also works well in the simplicial case provided we have the our objects satisfy the Kan condition, *i.e.*, are Kan complexes.

Simplicially the composition of homotopies is done via a choice of filler. We have two maps

$$L_0 \times \Delta[1] \to L_3$$

i.e., two 1-simplices in $\underline{\mathcal{S}}(L_0, L_3)$, which as we saw earlier (cf. page 256) is the simplicial set of maps of various 'degrees' from L_0 to L_3, given precisely by

$$\underline{\mathcal{S}}(K,L)_n = \mathcal{S}(K \times \Delta[n], L),$$

in general. From the two composable homotopies, we obtain a map

$$L_0 \times \Lambda^1[2] \to L_3$$

or equivalently a (2,1)-horn

$$\Lambda^1[2] \to \underline{\mathcal{S}}(L_0, L_3).$$

If L_3 is a Kan complex, then so is $\underline{\mathcal{S}}(L_0, L_3)$. (If you have not met the proof, it is worth looking up. You should find it in more or less any text with a section on simplicial homotopy theory.) From our $(2,1)$-horn, we will get a filler:

$$\Delta[2] \to \underline{\mathcal{S}}(L_0, L_3),$$

and the d_i-face of this is a composite homotopy.) Note it is *a*, not *the*, composite homotopy, as we obtained *a* filler by the Kan condition and could not demand it had any special properties such as 'uniqueness'. This point is also valid working with topological homotopies. We conveniently compose homotopies by gluing one copy of a cylinder $X \times I$ to a second one and rescaling. The usual formula looks like

$$H * K(t) = \begin{cases} H(2t) & 0 \leq t \leq \frac{1}{2} \\ K(2t-1) & \frac{1}{2} \leq t \leq 1, \end{cases},$$

but this is just one very convenient composite and we could have used many other conventions, for instance, $H(3t)$ for $0 \leq t \leq \frac{1}{3}$, and $K(3t-2)$ for $\frac{1}{3} \leq t \leq 1$. Any homeomorphism $h : [0,1] \to [0,2]$ such that $h(0) = 0$ and $h(1) = 2$ will give another composite homotopy.)

That being said, the really neat way to treat this square is ... as a square! We need to specify a 2-fold homotopy, so want a map $\theta : L_0 \times I^2 \to L_3$, which fills the square, *i.e.*, $\theta(x,s,t) \in L_3$ for $(s,t) \in I^2$ and for $x \in L_0$, with

$$\begin{aligned} \theta(x,s,0) &= \mathbf{L}_{023}(x,s), \\ \theta(x,s,1) &= \mathbf{L}_{123}(\ell_{01}(x),s), \\ \theta(x,0,t) &= \mathbf{L}_{013}(x,t), \\ \theta(x,1,t) &= \ell_{23}\mathbf{L}_{012}(x,t). \end{aligned}$$

In the topological case, such a θ would need, of course, to be continuous, but would then be a suitable level 2 homotopy, $\mathbf{L}_{0123}$, completing our solution. We have not said how to construct this θ, but you have all the necessary machinery to do so. It only uses the elements of the data that have already been given. Its construction is **quite useful to do yourselves**, as it shows you how the low dimensional homotopies combine quite simply to give the level 2 homotopy that is needed. It uses a bit of topology, but only in a minimal way.

If we need a simplicial analogue of this, then we would need $\mathbf{L}_{0123} \in \mathcal{S}(L_0 \times \Delta[1]^2, L_3)$. Our simplicial mapping space, $\underline{\mathcal{S}}(L_0, L_3)$, initially looks slightly wrong for this since we need two 2-simplices with one matching common d_1-face to get $\Delta[1]^2$ and all the simplices in $\underline{\mathcal{S}}(L_0, L_3)$ have form $L_0 \times \Delta[n] \to L_3$. In fact this is easy to get around. The category of simplicial sets is Cartesian closed with its internal mapping object given exactly by this $\underline{\mathcal{S}}(K, L)$ construction, so we have, for each triple, K, L, M, of simplicial sets;

$$\underline{\mathcal{S}}(K \times L, M) \cong \underline{\mathcal{S}}(K, \underline{\mathcal{S}}(L, M)).$$

(If you are not familiar with Cartesian closed categories, then do glance at a suitable survey article or category theory textbook, e.g. [38]. The Wikipedia article on the subject will also give you some basic facts and ideas about the concept. You should also consult the n-Lab.)

We can use this isomorphism to convert our desired level 2 homotopy into a simplicial map

$$\Delta[1]^2 \to \underline{\mathcal{S}}(L_0, L_3).$$

(For formalities sake, it may be better to think of $\mathbf{L}_{0123}$ as being

$$\Delta[1]^2 \times L_0 \to L_3$$

instead of as having domain $L_0 \times \Delta[1]^2$.)

This is using the simplicially enriched category structure of $\mathcal{S}$, and allows us to produce and interpret a similar construction in many other simplicially enriched contexts. To do this we will need some more elements of the notions of simplicially enriched categories, also called $\mathcal{S}$-categories. These are just one of the ways of encoding *homotopy coherence*, but they fit neatly into our general approach. Other related concepts would include dg-categories that is, differential graded categories, which are categories enriched over the category of chain complexes. We will have a look at these later.

11.2 Simplicially enriched categories

These are, intuitively, just categories with simplicial 'hom-sets'. We will also call them *$\mathcal{S}$-categories*.

11.2.1 Categories with simplicial 'hom-sets'

We assume we have a category, $\mathcal{A}$, whose objects will often be denoted by lower case letter, x, y, z, ..., at least in the generic case, and for each pair of such objects, (x, y), a simplicial set, $\mathcal{A}(x, y)$, is given. For each triple x, y, z of objects of $\mathcal{A}$, we have a simplicial map, called *composition*,

$$\mathcal{A}(x, y) \times \mathcal{A}(y, z) \longrightarrow \mathcal{A}(x, z);$$

and for each object x, a map,

$$\Delta[0] \to \mathcal{A}(x, x),$$

that 'names' or 'picks out' the 'identity arrow' in the set of 0-simplices of $\mathcal{A}(x,x)$. This data is to satisfy the obvious axioms, associativity and identity, suitably adapted to this situation.

Definition: Such a set-up, as detailed above, will be called a *simplicially enriched category* or, more simply, an *$\mathcal{S}$-category*.

Enriched category theory is a well established branch of category theory. It has many useful tools and not all of them have yet been fully explored for the particular case of $\mathcal{S}$-categories and its applications in homotopy theory.

Warning: As we have mentioned before, some authors use the term 'simplicial category' for what we have termed a simplicially enriched category. There is a close link with the notion of simplicial category that is consistent with usage in simplicial theory *per se*, since any (small) simplicially enriched category can be thought of as a simplicial object in the 'category of categories', but a simplicially enriched category is not just a simplicial object in the 'category of categories' and not all such simplicial objects correspond to such enriched categories. That being said, that usage need not cause problems provided you are aware of the usage in the paper to which reference is being made.

11.2.2 Examples of $\mathcal{S}$-categories

We have seen the first example several times before, but will repeat it for convenience:

(i) $\mathcal{S}$, the category of simplicial sets:
here
$$\underline{\mathcal{S}}(K,L)_n := \mathcal{S}(\Delta[n]\times K, L);$$
Composition : for $f\in\underline{\mathcal{S}}(K,L)_n$, $g\in\underline{\mathcal{S}}(L,M)_n$, so $f:\Delta[n]\times K\to L$, $g:\Delta[n]\times L\to M$,
$$g\circ f := (\Delta[n]\times K \overset{diag\times K}{\longrightarrow} \Delta[n]\times\Delta[n]\times K \overset{\Delta[n]\times f}{\longrightarrow} \Delta[n]\times L \overset{g}{\to} M);$$
Identity : $id_K : \Delta[0]\times K \overset{\cong}{\to} K$.

Notational remark: Perhaps a word on notation is needed here. Above we have used $\mathcal{S}(\Delta[n]\times K, L)$, but as the product is symmetric, we could equally well have used $\mathcal{S}(K\times\Delta[n], L)$, and although in writing these notes I have tried to be consistent for the first of these, there will certainly be instances of the second convention that have crept in as both are used in the source material that I have used! It makes no real difference to the theory, but can make a difference to the formulae. Similar notational conventions, and similar probable errors in the notation, apply to the other examples below.

(ii) Top, 'the' category of spaces (of course, there are numerous variants but you can almost pick whichever one you like as long as the constructions work):
$$\underline{Top}(X,Y)_n := Top(\Delta^n\times X, Y).$$
Composition and identities are defined more or less as in (i).

If our favourite category, Top, of topological spaces has mapping spaces, Y^X, so is itself Cartesian closed, then $\underline{Top}(X,Y)$ can be identified with $Sing(Y^X)$, and this is also true if Y^X exists in Top for some pair of spaces X and Y, even if not all such pairs may have this property.

(iii) For each X, $Y \in Cat$, the category of small categories, then we similarly get $\underline{Cat}(X,Y)$,

$$\underline{Cat}(X,Y)_n = Cat([n] \times X, Y).$$

We leave the other structure up to the reader.

Of course, Cat *is* Cartesian closed and $\underline{Cat}(X,Y) = Ner(Y^X)$, up to isomorphism.

(iv) Crs, the category of crossed complexes: see section 3.1, for the definitions and additional references, [171] for some introductory background, and Tonks, [264] for a more detailed treatment of the simplicially enriched category structure;

$$\mathsf{Crs}(A,B) := Crs(\pi(n) \otimes C, D).$$

Composition has to be defined using an approximation to the identity, again see [264]. (As mentioned before, the book by Brown, Higgins and Sivera, [64], contains a coherent exposition of most of the theory of crossed complexes.)

(v) Ch^+_K or, more expansively, $Ch^+(K\text{–}Mod)$, the category of positive chain complexes of modules over a (commutative) ring K. Details are left to **the reader**, or follow from the Dold-Kan theorem and example (vi) below. We will examine this in more detail later on, but will also look at a different enrichment for this category.

(vi) $Simp.K\text{-}Mod$, the category of simplicial K-modules. The structure uses tensor product with the free simplicial K-module on $\Delta[n]$ to define the 'hom' and the composition, so is very much like (i). It is better viewed as being enriched over itself and we will examine it from that viewpoint slightly later.

(vii) Any simplicial monoid is a simplicially enriched category, so also any simplicial group is one. Of course, they only have a single object. Conversely an $\mathcal{S}$-category that has a single object only is a simplicial monoid. The multiplication in the simplicial monoid is the composition in the category etc.

(viii) Any category, C, will give us a $\mathcal{S}$-category, namely the corresponding *trivially enriched* or *locally discrete* $\mathcal{S}$-category. This leads to:

Definition: A $\mathcal{S}$-category, $\mathcal{B}$, is *locally discrete* or, equivalently, *trivially enriched* if each $\mathcal{B}(x,y)$ is a discrete simplicial set.

(ix) Any 2-category, C, will give us an $\mathcal{S}$-category. In fact, a 2-category is precisely a Cat-enriched category, so each 'hom' is a small category. In more detail, suppose C is a 2-category and x, y and z are objects, then the composition

$$c_{x,y,z} : C(x,y) \times C(y,z) \to C(x,z)$$

is a functor. The obvious way to get a simplicial set from $C(x,y)$ is to apply the nerve functor. We let $C^\Delta(x,y) = Ner(C(x,y))$ and we use the fact that we have already noted, that the nerve functor preserves products, then we define the $\mathcal{S}$-category, C^Δ, by the above simplicial 'homs' with composition

$$C^\Delta(x,y) \times C^\Delta(y,z) \cong Ner(C(x,y) \times C(y,z)) \xrightarrow{Ner(c_{x,y,z})} Ner(C(x,z)) \cong C^\Delta(x,z).$$

The identities look after themselves; associativity and unit axioms are then easily checked. In fact, as the nerve functor *embeds Cat* as a subcategory of $\mathcal{S}$, the resulting $\mathcal{S}$-category is really just the original 2-category in disguise.

(x) We saw in section 6.2.1 how to construct a simplicially enriched groupoid, GK, from a simplicial set, K. The terminology *is* consistent. Recall that the set of objects of GK was the set of vertices of K itself and that there were two maps, source and target, given by iterated face maps to K_0, (cf. page 249). To rewrite GK as a simplicially enriched category, we just take, for objects, x and y of GK, $GK(x,y)_n$ to be the set of arrows in GK_n that start at x and have target y. The composition in GK_n works by construction and all this is compatible with face and degeneracy maps. (The details **should be looked at a bit** as it is very often useful to be able to swap between the two ways of viewing GK. Thinking of the Dwyer-Kan loop groupoid as a simplicially enriched category is akin to thinking of a group G as a small category, so this is central to the 'categorification' story.)

(xi) An important set of examples of nice small $\mathcal{S}$-categories is given by the simplicially enriched category versions of the simplices. These are built from the ordered sets $[n] = \{0 < 1 < \ldots < n\}$ and will be denoted $S[n]$. We will give two equivalent definitions of them, one simple one here, another shortly using a comonadic resolution. The latter is very useful for linking the construction with the cohomology of categories, but the first is very pretty and simple and is easier to understand.

First note that if i and j are in $[n]$, then there are no paths from i to j if $i > j$, but if $i \leq j$, there are 2^{j-i} such paths. (Experiment a bit with simple examples if you do not see this.) More precisely, a path in a category $\mathcal{C}$ from an object, x, to an object, y, is a sequence of arrows in $\mathcal{C}$ joining the two objects:

$$x = c_0 \xrightarrow{a_1} c_1 \xrightarrow{a_2} \ldots \xrightarrow{a_k} c_k = y.$$

It thus determines a functor $a : [k] \to \mathcal{C}$ and, at this stage incidently, a simplex of $Ner(\mathcal{C})$. As $[n]$ is a totally ordered set, each (non-degenerate) such path from i to j is specified just by the set of intermediate objects, (as there are unique arrows between them so there is no choice of the a_ms). It is now clear that there are $j - i - 1$ intermediate elements, between i and j, and so 2^{j-i-1} such paths including the direct path that corresponds to the empty set of intermediate objects. The combinatorial structure of the partially ordered set of such paths is clearly that of $\{0 < 1\}^{j-i-1}$, as each path corresponds to a subset of the intermediate objects of $[n]$. The nerve of this partially ordered set is $\Delta[1]^{j-i}$. If $i \leq j \leq k$, we can define a composition pairing

$$\Delta[1]^{j-i-1} \times \Delta[1]^{k-j-1} \to \Delta[1]^{k-i-1}$$

given by sending a pair consisting of a subset A of $\{i, \ldots, j\}$ and a subset B of $\{j, \ldots, k\}$ to $A \cup \{j\} \cup B$. Note the inclusion of $\{j\}$. It will always be there in that composite. (Here we are working in several contexts at once, paths, subsets of sets of intermediate elements, and simplicial mappings, so it may pay to pause and check details such as compatibility with face and degeneracy maps etc., just to make sure your intuition on what is happening here, and why it works, is up to speed.)

Definition: Let $S[n]$ be the $\mathcal{S}$-category having the same objects as the category $[n]$, with $S[n](i,j)$ empty if $j < i$ and isomorphic to $\Delta[1]^{j-i-1}$ if not, and with the above composition pairing as the composition. We will call $S[n]$ the *$\mathcal{S}$-categorical n-simplex.*

(xii) In general, any category of simplicial objects in a 'nice enough' category has a simplicial enrichment, although the general argument that gives the construction does not always make the structure as transparent as it might be.

Proposition 113 *If $\mathcal{A}$ is any category, $Simp(\mathcal{A}) = \mathcal{A}^{\Delta^{op}}$ is an $\mathcal{S}$-category.*

Proof: Let K to be any simplicial set, then Δ/K is the comma category with objects $([n], x)$ with $x \in K_n$ and morphisms $\mu : ([n], x) \to ([m], y)$ being those $\mu : [n] \to [m]$ in Δ such that $K(\mu)(y) = x$. There is a forgetful functor

$$\delta_K : \Delta/K \to \Delta, \qquad \delta_K([n], x) = x.$$

Now given $X, Y \in Simp(\mathcal{A})$, define

$$Simp(\mathcal{A})(X,Y)_n = NatTrans(X\delta^{op}_{\Delta[n]}, Y\delta^{op}_{\Delta[n]})$$

■

Several times above we have use a notational convention that can be very useful. If a category, $\mathcal{A}$, is to be regarded both as an ordinary category and a simplicially enriched one, there arises a problem of what notation to use for the two types of hom-object. One simple and quite effective solution is to use $\mathcal{A}(A,B)$ if thinking of the *set* of morphisms and an underlined version $\underline{\mathcal{A}}(A,B)$ if it is the simplicial set of morphisms that we mean. Then it is also natural to refer to the basic category as $\mathcal{A}$ and the $\mathcal{S}$-enriched version as $\underline{\mathcal{A}}$. We probably have not been consistent about this, but will try!

There is an evident notion of $\mathcal{S}$-enriched functor, so we get a category of 'small' $\mathcal{S}$-categories, denoted $\mathcal{S}$–Cat. Of course, some of the above examples are not 'small'. (With regard to 'smallness', although sometimes a smallness condition is essential, one can often ignore questions of smallness and, for instance, consider simplicial 'sets' where actually the collections of simplices are not truly 'sets' (depending on your choice of methods for handling such foundational questions).)

11.2.3 From simplicial resolutions to $\mathcal{S}$-categories

The construction of $S[n]$ from $[n]$ is an example of a general construction for any small category. One can approach it via paths as we did above or via a free category construction. This latter approach has the advantage that it emphasises the link between the constructions of the categorical approach to homotopy coherence and the constructions of categorical cohomology theory, as exemplified by the comonadic resolution construction that we used earlier in a particular case, cf. section 3.5.3, page 90. It is therefore useful to present both approaches.

The forgetful functor, $U : Cat \to DGrph_0$, has a left adjoint, F, as we mentioned back in section 2.2.3. Here $DGrph_0$ denotes the category of directed graphs with 'identity loops', so U forgets just the composition within each small category, but remembers that certain loops are special 'identity loops'. The free category functor here takes, between any two objects, all strings of composable *non-identity* arrows that start at the first object and end at the second. One can think of F identifying the old identity arrow at an object x with the empty string at x.

This adjoint pair gives a comonad on Cat in the usual way, and hence a functorial simplicial resolution, as we saw on page 90. Here we will use the alternative form of the construction. This takes the face and degeneracy maps in the opposite direction, but is otherwise more or less completely equivalent, yielding the conjugate simplicial object. We will denote this, for a small category $\mathbb{A}$, by $S(\mathbb{A}) \to \mathbb{A}$. In more detail, we write $L = FU$ for the functor part of the comonad, the unit of the adjunction, $\eta : Id_{DGrph_0} \to UF$, gives the comultiplication, $F\eta U : L \to L^2$, and the counit of the adjunction gives $\varepsilon : FU \to Id_{Cat}$, that is, $\varepsilon : L \to Id$. Now, for $\mathbb{A}$ a small category, set $S(\mathbb{A})_n = L^{n+1}(\mathbb{A})$ with face maps $d_i : L^{n+1}(\mathbb{A}) \to L^n(\mathbb{A})$ given by $d_i = L^i\varepsilon L^{n-i}$, and similarly for the degeneracies, which use the comultiplication in an analogous formula. (Note that the other convention would use $d_i = L^{n-i}\varepsilon L^i$. The only effect of such a change is to reverse the direction of certain 'arrows' in diagrams. The two simplicial structures are 'dual' to each other. The difference is exactly that which we noted when we first wrote the homotopy coherent triangle in our first example.)

This $S(\mathbb{A})$ is a simplicial object in Cat, $S(\mathbb{A}) : \mathbf{\Delta}^{op} \to Cat$, so does not immediately gives us a simplicially enriched category, however its simplicial set of objects is constant because U and F took note of the identity loops.

In more detail, let $ob : Cat \to Sets$ be the functor that picks out the set of objects of a small category, then $ob(S(\mathbb{A})) : \mathbf{\Delta}^{op} \to Sets$ is a constant functor with value the set $ob(\mathbb{A})$ of objects of $\mathbb{A}$. More exactly, it is a discrete simplicial set, since all its face and degeneracy maps are bijections. Using those bijections to identify the possible different sets of objects, yields a constant simplicial set where all the face and degeneracy maps are identity maps, *i.e.*, we do now have a *constant* simplicial set of objects.

Lemma 68 *Let $\mathcal{B} : \mathbf{\Delta}^{op} \to Cat$ be a simplicial object in Cat such that $ob(\mathcal{B})$ is a constant simplicial set with value B_0, say. For each pair $(x, y) \in B_0$, let*

$$\mathcal{B}(x, y)_n = \{\sigma \in \mathcal{B}_n \mid dom(\sigma) = x, codom(\sigma) = y\},$$

where, of course, dom *refers to the domain function in $\mathcal{B}_n$, similarly for* codom.

(i) The collection $\{\mathcal{B}(x, y)_n \mid n \in \mathbb{N}\}$ has the structure of a simplicial set, $\mathcal{B}(x, y)$, with face and degeneracies induced from those of $\mathcal{B}$.

(ii) The composition in each level of $\mathcal{B}$ induces

$$\mathcal{B}(x, y) \times \mathcal{B}(y, z) \to \mathcal{B}(x, z).$$

Similarly the identity map in $\mathcal{B}(x, x)$ is defined as id_x, the identity at x in the category $\mathcal{B}_0$.

(iii) The resulting structure is an $\mathcal{S}$-enriched category, that will also be denoted $\mathcal{B}$. ■

The proof is just a matter of checking formulae, and is **left to the reader**.

In particular, this shows that $S(\mathbb{A})$ is a simplicially enriched category. The augmentation of the comonadic resolution yields an $\mathcal{S}$-functor, denoted $d_0 = \eta := \eta_{\mathbb{A}} : S(\mathbb{A}) \to \mathbb{A}$, from $S(\mathbb{A})$ to the locally discrete $\mathcal{S}$-category corresponding to $\mathbb{A}$. (The d_0 notation is useful if considering the whole structure as enriched over *augmented simplicial sets*, .)

Definition: For a small category $\mathbb{A}$, the $\mathcal{S}$-category $S(\mathbb{A})$ is the *free $\mathcal{S}$-category resolving* $\mathbb{A}$ The $\mathcal{S}$-functor $\eta := \eta_{\mathbb{A}} : S(\mathbb{A}) \to \mathbb{A}$ is the *augmentation of this resolution.*

The description of the simplices in each dimension of $S(\mathbb{A})$ that start at a and end at b is intuitively quite simple. The arrows in the category, $L(\mathbb{A})$, correspond to strings of symbols representing non-identity arrows in $\mathbb{A}$ itself, those strings being 'composable' in as much as the domain of the i^{th} arrow must be the codomain of the $(i-1)^{th}$ one, and so on. Because of this we have:

- $S(\mathbb{A})_0$ consists exactly of such composable chains of maps in $\mathbb{A}$, none of which is the identity;
- $S(\mathbb{A})_1$ consists of such composable chains of maps in $\mathbb{A}$, none of which is the identity, together with a choice of bracketting;
- $S(\mathbb{A})_2$ consists of such composable chains of maps in $\mathbb{A}$, none of which is the identity, together with a choice of two levels of bracketting;
- ... and so on.

Face and degeneracy maps remove or insert brackets, but care must be taken when removing innermost brackets as the compositions that can then take place can result in chains with identities, which then need removing, see [83], that is why the comonadic description is so much simpler, as it manages all that itself.

To understand $S(\mathbb{A})$ in general, it pays to examine the simplest few cases. The key cases are when $\mathbb{A} = [n]$, the ordinal $\{0 < \ldots < n\}$ considered as a category as before. We gave these earlier from the other viewpoint, so how do they look from the comonadic one? This sheds light on the links between the two approaches.

The cases $n = 0$ and $n = 1$ give no surprises:

- $S[0]$ has one object 0 and $S[0](0,0)$ is isomorphic to $\Delta[0]$, as the only simplices are degenerate copies of the identity.
- $S[1]$ likewise has a trivial simplicial structure, being just the category $[1]$ considered as an $\mathcal{S}$-category.
- Things do get more interesting at $n = 2$. The key here is the identification of $S[2](0,2)$. There are two non-degenerate strings or paths that lead from 0 to 2, so $S[2](0,2)$ will have two vertices. The bracketted string $((01)(12))$ on removing inner brackets gives (02) and outer brackets, $(01)(12)$, so represents a 1-simplex,

$$(02) \xrightarrow{((01)(12))} (01)(12),$$

Other simplicial homs are all $\Delta[0]$ or empty. It thus is possible to visualise $S[2]$ as a copy of $[2]$ with a 2-cell going towards the top:

$$\begin{tikzcd} & 1 \arrow[dr, "(12)"] & \\ 0 \arrow[ur, "(01)"] \arrow[rr, "(02)"'] & \Uparrow & 2 \end{tikzcd}$$

- The next case $n = 3$ is even more interesting: $S[3](i, j)$ will be (i) empty if $j < i$, (ii) isomorphic to $\Delta[0]$ if $i = j$ or $i = j - 1$, (iii) isomorphic to $\Delta[1]$, by the same reasoning as we just used, for $j = i + 2$ and that leaves $S[3](0, 3)$. This is a square, $\Delta[1]^2$, as follows:

$$\begin{tikzcd} (03) \arrow[r, "((02)(23))"] \arrow[d, "((01)(13))"'] \arrow[dr, "diag"] \arrow[d, phantom, shift left=6ex, "a"] & (02)(23) \arrow[d, "((01)(12))((23))"] \arrow[d, phantom, shift right=6ex, "b"] \\ (01)(13) \arrow[r, "((01))((12)(23))"'] & (01)(12)(23) \end{tikzcd}$$

where the diagonal $diag = ((01)(12)(23))$, $a = (((01))((12)(23)))$ and $b = (((01)(12))((23)))$. (It is instructive to check that this is correct, firstly because I may have slipped up (!) as well as seeing the mechanism in action. Removing the outermost brackets is d_0, and so on.)

- The case of $S[4]$ is worth doing. (Yes, that means it is suggested as an **exercise**. As might be expected, $S[4](0, 4)$ is a cube.)

The simplicial resolution construction of $S(\mathbb{A})$ from $\mathbb{A}$ was cross referenced to our earlier use of comonadic simplicial resolutions for groups and the link of that with cohomology, see page 90. So as to investigate the link between the two instances of this that we have seen, it is useful to look at a special case of the $\mathcal{S}$-construction, namely when the given small category is a monoid and, in particular, when it is a group.

Let $\mathbb{A}$ be a monoid, thought of as a small category with a single object. The adjoint pair of functors,

$$U : Cat \rightleftarrows DGrph_0 : F \ ,$$

restricts to the category of monoids on the one hand and to that, $Sets_0$, of pointed sets on the other:

$$U : Mon \rightleftarrows Sets_0 : F \ .$$

The basic step in the construction is that of forming the free monoid on the set of the non-identity elements of a monoid, and so the bracketing terminology works well still in this particular situation.

We thus have that $S(\mathbb{A})$ is a simplicial monoid in the ordinary sense of the term. If $\mathbb{A}$ is actually a group rather than 'merely' a monoid, then $S(\mathbb{A})$ is still only a simplicial monoid, but for any $g \in \mathbb{A}$, there are 'generators' $\langle g\rangle$ and $\langle g^{-1}\rangle$ in $S(\mathbb{A})_0$ and a 1-simplex, $(\langle g\rangle, \langle g^{-1}\rangle)$ in $S(\mathbb{A})_1$. We can calculate the vertices on the two ends of this: as $d_0 = \varepsilon T$ and $d_1 = T\varepsilon$,

$$d_0(\langle g\rangle, \langle g^{-1}\rangle) = \langle g\rangle\langle g^{-1}\rangle,$$

and

$$d_1(\langle g\rangle, \langle g^{-1}\rangle) = 1,$$

since $\varepsilon(\langle g\rangle, \langle g^{-1}\rangle) = 1_{\mathbb{A}}$). The 1-simplex thus looks like

$$1 \to \langle g\rangle\langle g^{-1}\rangle.$$

Of course, there is another one from 1 to $\langle g\rangle\langle g^{-1}\rangle$. As $S(\mathbb{A})_0$ is a free monoid, we do not have elements such as $\langle g\rangle^{-1}$ around and so do not get a corresponding 1-simplex *ending* at 1.

Remark: The history of this S-construction is interesting. A variant of it, but with topologically enriched categories as the end result, is in the work of Boardman and Vogt, [36], and also in Vogt's paper, [271]. Segal's student Leitch used a similar construction to describe a homotopy commutative cube (actually a *homotopy coherent cube*), cf. [185], and this was used by Segal in his famous paper, [244], under the name of the 'explosion' of $\mathbb{A}$. All this was still in the topological framework and the link with the comonad resolution was still not in evidence.

Although it seems likely that Kan knew of this link between homotopy coherence and the comonadic resolutions by at least 1980, (cf. [111]), the construction does not seem to appear in his work with Dwyer as being linked with coherence until much later. Cordier made the link explicit in [83] and showed how Leitch and Segal's work fitted in to the pattern. His motivation was for the description of homotopy coherent diagrams of topological spaces. Other variants were also apparent in the early work of May on operads, and linked in with Stasheff's work on higher associativity and commutativity 'up to homotopy', and it would be possible to write a whole course on those and not to stray too far from our theme of non-abelian cohomology either.

Cordier and Porter, [84], used an analysis of a locally Kan simplicially enriched category involving this construction to prove a generalisation of Vogt's theorem on categories of homotopy coherent diagrams of a given type. (We will return to this aspect a bit later in these notes, but an elementary introduction to this theory can be found in [171].) Finally Bill Dwyer, Dan Kan and Jeffrey Smith, [114], introduced a similar construction for an $\mathbb{A}$ which is an $\mathcal{S}$-category to start with, and motivated it by saying that $\mathcal{S}$-functors with domain this $\mathcal{S}$-category corresponded to ∞*-homotopy commutative* $\mathbb{A}$*-diagrams*, yet they do not seem to be aware of the history of the construction, and do not really justify the claim that it does what they say. Their viewpoint is however very important as, basically, within the setting of Quillen model category structures, this provides a cofibrant replacement construction. We will look at cofibrant replacements in another context later on in this chapter. (If you want to check up on this idea now, a good source is Hovey's book, [158].) Of course, any other cofibrant replacement could be substituted for it and so would still allow for a description of homotopy coherent diagrams in that context. This important viewpoint can also be traced to Grothendieck's *Pursuing Stacks*, [140].

The extension of the construction in [114], although simple to do, is often useful and so will be outlined next.

If $\mathbb{A}$ is already a $\mathcal{S}$-category, think of it as a simplicial category, then applying the S-construction to each $\mathbb{A}_n$ will give a bisimplicial category, *i.e.*, a functor $S(\mathbb{A}) : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to Cat$. Of this we take the diagonal, so the collection of n-simplices is $S(\mathbb{A})_{n,n}$, and, by noticing that the result has a constant simplicial set of objects, then apply the lemma.

Before leaving this construction, let us just comment that if we had used the other formulae for the simplicial resolution, the only difference would be that the higher dimensional arrows would be reversed in direction, so that, for instance, in $S[2]$, we would have had the arrow going from the composite of the d_2 and the d_0 *to* the d_1-face, not the other way around.

11.3 Structure

As one can 'do' homotopy theory with simplicial sets, one can adapt that theory to give a basic homotopy theory in any $\mathcal{S}$-category. Of course, some of these homotopy theories will be richer than others.

11.3.1 The 'homotopy' category

If $\mathcal{C}$ is an $\mathcal{S}$-category, we can form a category $\pi_0\mathcal{C}$ with the same objects and having

$$(\pi_0\mathcal{C})(X,Y) = \pi_0(\mathcal{C}(X,Y)).$$

This is known as the *homotopy category of the $\mathcal{S}$-category*. For instance, if $\mathcal{C} = \mathcal{CW}$, the category of CW-complexes, then $\pi_0\mathcal{CW} = Ho(CW)$, the corresponding homotopy category. Similarly we could obtain a groupoid enriched category using the fundamental groupoid (cf. Gabriel and Zisman, [132]), that is, by applying the fundamental groupoid functor, Π_1, to each 'hom'

$$(\Pi_1\mathcal{C})(X,Y) = \Pi_1(\mathcal{C}(X,Y)).$$

This works because Π_1 preserves products. (We will see many similar results later, in which the type of enriched structure is transformed using a 'monoidal functor', *i.e.*, one that is compatible with the monoidal category structures being used. All will be revealed later, in Chapter 12.)

Remarks: (i) This notion of a groupoid enriched category has been called a *track category* by Baues; see [26], for instance, and also a *(2,1)-category*, a term we will see in more detail later on[1]. The terminology is not quite precise enough for our uses as we will have track n-categories to handle later on, so we will call this 2-dimensional version a *track 2-category*. Formally we have:

Definition: A 2-category, $\mathbf{C}$, is a *track 2-category* or a *groupoid enriched category* if each $\mathbf{C}(x,y)$ is a groupoid.

These track 2-categories / groupoid enriched categories have a reasonably rich 'abstract' homotopy theory, as is shown by the book by Gabriel and Zisman, [132], or the article by Fantham and Moore, [127]. More recently they have been used extensively by Baues, [26].

One can 'do' some elementary homotopy theory in *any* $\mathcal{S}$-category, $\mathcal{C}$, by saying that two maps $f_0, f_1 : X \to Y$ in $\mathcal{C}$ are *homotopic* if there is an $H \in \mathcal{C}(X,Y)_1$ with $d_0H = f_1$, $d_1H = f_0$.
This theory will not be very rich, however, unless at least some low dimensional Kan conditions are satisfied.

Definition: The $\mathcal{S}$-category, $\mathcal{C}$, is called *locally Kan* if each $\mathcal{C}(X,Y)$ is a Kan complex; *locally weakly Kan* if ..., etc.

[1] Relying on some intuitive vague idea of higher categories in general, i.e. beyond dimension 2, the idea is that an (n,r)-category is a higher category such that (i) all k-morphisms for $k > n$ are trivial, *i.e.*, identities of the appropriate dimension, whilst (ii) all k-morphisms for $k > r$ are 'reversible', so in a $(2,1)$ category, you have a 2-category in which the 2-cells are invertible, so it is groupoid enriched.

(If you have not met 'weak Kan complexes', you will soon meet them in earnest! We will define them properly before using them, so don't worry.)

The theory is 'geometrically' nicer to work with if $\mathcal{C}$ is *tensored* or *cotensored*.

11.3.2 Tensoring and Cotensoring

We have already met the idea of tensoring and cotensoring briefly when discussing simplicial homotopies, (page 341 in section 7.5.5). The notions of tensors and cotensors make sense in any enriched category setting, but for the moment we will just give the case of simplicially enriched category[2].

Definition: If for all $K \in \mathcal{S}$, $X, Y, \in \mathcal{C}$, there is an object $K\bar{\otimes}X$ in $\mathcal{C}$ such that

$$\mathcal{C}(K\bar{\otimes}X, Y) \cong \mathcal{S}(K, \mathcal{C}(X, Y)$$

naturally in K, X and Y, then $\mathcal{C}$ is said to be *tensored* over $\mathcal{S}$.

Definition: Dually, if we require objects $\bar{\mathcal{C}}(K, Y)$ such that

$$\mathcal{C}(X, \bar{\mathcal{C}}(K, Y)) \cong \mathcal{S}(K, \mathcal{C}(X, Y)$$

then we say $\mathcal{C}$ is *cotensored* over $\mathcal{S}$.

Remark on terminology: In many ways this terminology is not a good one. Usually 'tensors' are given by colimit type constructions, whilst cotensors are limit-type constructions. A cotensor is interpreted as if it was a function, or mapping 'space', and in the simple case of a *Set*-enriched setting, (*i.e.*, standard category theory) is a *power* operation. If X, Y are objects in a category, $\mathcal{C}$, and K is just a set, $\bar{\mathcal{C}}(K, Y)$ is Y^K, the K-fold power of Y, that is, the product of K-many copies of the object, Y. Dually $K\bar{\otimes}X$ will be the K-fold copower of X, that is, the coproduct of K-many copies of the object X. Because of this, an alternative terminology to the above has been suggested:

'standard'	alternative
tensored	copowered
cotensored	powered

(see the discussion of this in the nLab, [221].) (This terminology is probably still unstable but should stabilise soon.)

The example that we have seen most of this type of structure is in $\mathcal{S}$, where, for K in $\mathcal{S}$, and, this time, also X in $\mathcal{S}$, $K\bar{\otimes}X$ is just $K \times X$ and, dually, for Y in $\mathcal{S}$, $\bar{\mathcal{C}}(K, Y)$ is $\underline{S}(K, Y)$, the simplicial function space of maps from K to Y. To gain some intuitive feeling for these two concepts in general, we can think of $K\bar{\otimes}X$ as being 'K product with X', and $\bar{\mathcal{C}}(K, Y)$ as the object of functions from K to Y. These words do not, as such, make sense in all generality, but do tell one the sort of tasks these constructions will be set to do. They will not be much used explicitly here, however, their application to constructing homotopy limits and colimits will be looked at in detail later on.

[2]We note however that in the general case of a $\mathcal{V}$-enriched category that we will meet in section 12.1.1, it is easy to adapt the definitions, as we will do!

The following also gives an indication of other uses. Some of the terminology has not been explicitly explained, but the results do give an idea of the structure available.

Proposition 114 *(cf. Kamps and Porter, [171]) If $\mathcal{C}$ is a locally Kan $\mathcal{S}$-category tensored over $\mathcal{S}$, then, taking $I \times X := \Delta[1]\bar{\otimes}X$, we get a good cylinder functor such that for the cofibrations relative to I and weak equivalences taken to be homotopy equivalences, the category $\mathcal{C}$ has a cofibration category structure.* ■

A cofibration category structure is just one of many variants of the abstract homotopy theory structure introduced to be able to push through homotopy type arguments in particular settings. There are variants of this result, due to Kamps, see references in [171], where $\mathcal{C}$ is both tensored and cotensored over $\mathcal{S}$ and the conclusion is that $\mathcal{C}$ has a Quillen model category structure. The examples of locally Kan $\mathcal{S}$-categories include Top, and Kan, that is the full subcategory of $\mathcal{S}$ given by the Kan complexes, also $Grpd$ and Crs, but not Cat or $\mathcal{S}$ itself.

11.4 Nerves and Homotopy Coherent Nerves

Before we get going on this section, it will be a good idea to bring to the fore, as promised, the definitions of *weak Kan complex* (or *quasi-category*). We first recall and repeat from the first chapter, the notions of Kan fibration and Kan complex, as these are central to what follows and it is convenient not to have to be flipping back and fore to the earlier discussion.

11.4.1 Kan and weak Kan complexes

As usual, we set $\Delta[n] = \mathbf{\Delta}(-, [n]) \in \mathcal{S}$, then for each i, $0 \leq i \leq n$, we can form a subsimplicial set, $\Lambda^i[n]$, of $\Delta[n]$ by discarding the top dimensional n-simplex (given by the identity map on $[n]$) and its i^{th} face. We must also discard all the degeneracies of these simplices. This informal definition does not give a 'picture' of what we have, so we will list the various cases for $n = 2$.

$\Lambda^0[2] =$

$$\begin{tikzcd} & 1 \arrow[dr, dotted, dash] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

$\leftarrow 0^{th}$ face missing

$\Lambda^1[2] =$

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr, dotted, dash] & & 2 \end{tikzcd}$$

$\Lambda^2[2] =$

$$\begin{tikzcd} & 1 \arrow[dr] \arrow[dl, dotted, dash] & \\ 0 \arrow[rr] & & 2 \end{tikzcd}$$

A map $p : E \to B$ is a *Kan fibration* if given any n, i, as above, and any (n, i)-horn in E, *i.e.*, any map $f_1 : \Lambda^i[n] \to E$, and n-simplex, $f_0 : \Delta[n] \to B$, such that

$$\begin{tikzcd} \Lambda^i[n] \arrow[r, "f_1"] \arrow[d, "inc"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "f_0"'] & B \end{tikzcd}$$

commutes, then there is an $f : \Delta[n] \to E$ such that $pf = f_0$ and $f.inc = f_1$, *i.e.*, f lifts f_0 and extends f_1.

A simplicial set, K, is a *Kan complex* if the unique map $K \to \Delta[0]$ is a Kan fibration. This is equivalent to saying that every horn in K has a filler, *i.e.*, any $f_1 : \Lambda^i[n] \to K$ extends to an $f : \Delta[n] \to K$. This condition looks to be purely of a geometric nature but in fact has an important algebraic flavour; for instance, if $f_1 : \Lambda^1[2] \to K$ is a horn, it consists of a diagram

$$\begin{tikzcd} & {} \arrow[dr, "b"] & \\ {} \arrow[ur, "a"] & & {} \end{tikzcd}$$

of 'composable' arrows in K. If f is a filler, it looks like

$$\begin{tikzcd} & {} \arrow[dr, "b"] \arrow[d, phantom, "f"] & \\ {} \arrow[ur, "a"] \arrow[rr, "c"'] & {} & {} \end{tikzcd}$$

and one can think of the third face, c, as a composite of a and b. This 'composite', c, is not usually uniquely defined by a and b, but is determined 'up to homotopy'. If we write $c = ab$ as a shorthand then if $g_1 : \Lambda^0[2] \to K$ is a horn, we think of g_1 as being

$$\begin{tikzcd} & {} & \\ {} \arrow[ur, "d"] \arrow[rr, "e"'] & & {} \end{tikzcd}$$

and to find a filler is to find a diagram

$$\begin{tikzcd} & {} \arrow[dr, dashed, "x"] & \\ {} \arrow[ur, "d"] \arrow[rr, "e"'] & & {} \end{tikzcd}$$

and thus to 'solve' the equation $dx = e$ for x in terms of d and e. It thus requires, in general, some approximate inverse for d, in fact, taking e to be a degenerate 1-simplex puts one in exactly such a position. Thus Kan complexes have a very weak 'algebraic' structure. There is a sort of composition, *up to homotopy*, which is sort of associative, *up to homotopy*, and has sort of inverses, yes, you guessed, *up to homotopy*.

In many useful cases, we do not always have inverses and so want to discard any requirement that would imply they always exist. This leads to the weaker form of the Kan condition in which in each dimension no requirement is made for the existence of fillers on horns that miss out the zeroth or last faces. More exactly:

Definition: A simplicial set $\mathbf{K}$ is a *weak Kan complex* or *quasi-category* if for any n and $0 < k < n$, any (n, k)-horn in K has a filler.

Remarks: (i) Joyal, [169], uses the term *inner horn* for any (n, k)-horn in K with $0 < k < n$. The two remaining cases are then conveniently called *outer horns*.

(ii) Quasi-categories are an important model for a class of ∞-category. We will not give a definition of (weak) ∞-category for the moment, using the intuition that such should consist of

k-arrows for all k, which can be composed, up to coherence conditions. The class of ∞-categories given by quasi-categories is determined by the condition that in ∞-categories of this form, all the k-arrows are 'invertible up to higher arrows', for all $k > 1$. They are $(\infty, 1)$-categories. Similarly '∞-groupoids' are exactly '$(\infty, 0)$-categories', as all k-arrows are invertible up to higher arrows for all $k > 0$.

11.4.2 Categorical nerves

As we saw in section 1.3.1, the categorical analogue of the singular complex is the nerve: if C is a category, its *nerve*, $Ner(\mathsf{C})$, is the simplicial set with $Ner(\mathsf{C})_n = Cat([n], \mathsf{C})$, where $[n]$ is the category associated to the finite ordinal $[n] = \{0 < 1 < \ldots < n\}$. The face and degeneracy maps are the obvious ones using the composition and identities in C.

The following is well known and easy to prove (*i.e.*, **left to you**).

Lemma 69 *(i) $Ner(\mathsf{C})$ is always weakly Kan.*
(ii) $Ner(\mathsf{C})$ is Kan if and only if C is a groupoid. ■

Of course more is true. Not only does any inner horn in $Ner(\mathsf{C})$ have a filler, it has exactly one filler. To express this with maximum force, the following idea, often attributed to Graeme Segal or to Grothendieck, is very useful, [243]. (See the discussion in the n-Lab, [221], under 'Segal condition' for more details and the development of the idea in some directions that we will not be following below.)

Let $p > 0$, and consider the increasing maps, $e_i : [1] \to [p]$, given by $e_i(0) = i$ and $e_i(1) = i + 1$. For any simplicial set, A, considered as a functor $A : \mathbf{\Delta}^{op} \to Sets$, we can evaluate A on these e_i and, noting that $e_i(1) = e_{i+1}(0)$, we get a family of functions $A_p \to A_1$, which yield a cone diagram, for instance, for $p = 3$:

$$\begin{tikzcd}
A_3 \arrow[drrr, "A(e_1)"] \arrow[ddrr, "A(e_2)"] \arrow[dddr, "A(e_3)"'] & & & \\
& & & A_1 \arrow[d, "d_0"] \\
& & A_1 \arrow[r, "d_1"] \arrow[d, "d_0"] & A_0 \\
& A_1 \arrow[r, "d_1"'] & A_0 &
\end{tikzcd}$$

and in general, thus yield a map

$$\delta[p] : A_p \to A_1 \times_{A_0} A_1 \times_{A_0} \ldots \times_{A_0} A_1.$$

The maps, $\delta[p]$, have been called the *Segal maps*.

Lemma 70 *If $A = Ner(\mathsf{C})$ for some small category C, then for A, the Segal maps are bijections.*

Proof: A simplex $\sigma \in Ner(\mathsf{C})_p$ corresponds uniquely to a composable p-chain of arrows in C, and hence exactly to its image under the relevant Segal map. ■

Better than this is true:

Proposition 115 *If A is a simplicial set such that the Segal maps are bijections, then there is a category structure on the directed graph,*

$$A_1 \rightrightarrows A_0 \leftarrow ,$$

making it a category whose nerve is isomorphic to the given A.

Proof: To get composition you use

$$A_1 \times_{A_0} A_1 \stackrel{\cong}{\to} A_2 \stackrel{d_1}{\to} A_1.$$

Associativity is given by A_3. The other laws are easy, and illuminating, to check. ■

The condition 'Segal maps are bijections' is closely related to notions of 'thinness' as used by Brown and Higgins in the study of crossed complexes and their relationship to ω-groupoids, (see, for instance, [64], and here in our discussion of T-complexes, starting on page 38), and it also relates to Duskin's 'hypergroupoid' condition, [107][3].

Another result that is sometimes useful is a refinement of the 'groupoids give Kan complexes' lemma, Lemma 1 on page 36. The proof is 'the same' and is equally **left to the reader**.

Lemma 71 *Let $A = Ner(\mathsf{C})$, the nerve of a category C.*

(i) Any $(n, 0)$-horn

$$f : \Lambda^0[n] \to A$$

for which $f(01)$ is an isomorphism has a filler. Similarly any (n, n)-horn $g : \Lambda^n[n] \to A$ for which $g(n-1\ n)$ is an isomorphism, has a filler.

(ii) Suppose f is a morphism in C with the property that, for any n, any $(n, 0)$-horn $\varphi : \Lambda^0[n] \to A$ having f in the $(0, 1)$ position, has a filler, then f is an isomorphism. (Similarly with $(n, 0)$ replaced by (n, n) with the obvious changes.) ■

Again the proof is not hard and reveals some neat arguments, so ... .

Remark: Joyal in [169] suggested that the name 'weak Kan complex', as introduced by Boardman and Vogt, [36], could be changed to that of 'quasi-category', partially, to stress the analogy with categories *per se* as '*Most concepts and results of category theory can be extended to quasi-categories*', [169][4].

It would have been nice to have explored Joyal's work on quasi-categories more fully, e.g. [169], but that would take us too far from our central themes. The following few sections just skate the surface of that theory.

[3] see here the discussion centred on page 203.

[4] In fact, many of these ideas had been implicit in some of Vogt's work, and Cordier and Porter had explored it in about 1984, but this was not published as it was decided to concentrate on the $\mathcal{S}$-categorical model for homotopy coherence.

11.4.3 Quasi-categories

Categories yield quasi-categories via the nerve construction as we have seen. Quasi-categories yield categories by a 'fundamental category' construction that is left adjoint to nerve. This can be constructed using the free category generated by the 1-skeleton of A, and then factoring out by a congruence generated by the basic relations : $gf \equiv h$, one for each commuting 1-sphere (g,h,f) in A. By a *1-sphere* is meant a map $a : \partial\Delta[2] \to A$, thus giving three faces, (a_0, a_1, a_2), linked in the obvious way. The 1-sphere is said to be *commuting* if there is a 2-simplex, $b \in A_2$, such that $a_i = d_i b$ for $i = 0, 1, 2$.

Definition: The *fundamental category of a quasi-category*, A, is the category with presentation:
• generators = the 1-skeleton of A,
and
• relations $gf \equiv h$ as above.

This 'fundamental category' functor also has a very neat description due to Boardman and Vogt. (The treatment here is, again, adapted from [169].)

We assume given a quasi-category, A. Write $gf \sim h$ if (g,h,f) is a commuting 1-sphere. Let $x, y \in A_0$ and let $A_1(x,y) = \{f \in A_1 \mid x = d_1 f, y = d_0 f\}$. If $f, g \in A_1(x,y)$, then, suggestively writing $s_0 x = 1_x$,

Lemma 72 *The four relations $f1_x \sim g$, $g1_x \sim f$, $1_y f \sim g$ and $1_y g \sim f$ are equivalent.* ■

The proof is easy and is **left as an exercise**.

We will say $f \simeq g$ if any of these is satisfied and call $\simeq$, the *homotopy relation*. It is an equivalence relation on $A_1(x,y)$. Set $ho\, A_1(x,y) = A_1(x,y)/\simeq$.

If $f \in A_1(x,y)$, $g \in A_1(y,z)$ and $h \in A_1(x,z)$, then the relation $gf \sim h$ induces a map:

$$ho\, A_1(x,y) \times ho\, A_1(y,z) \to ho\, A_1(x,z).$$

Proposition 116 *The maps*

$$ho\, A_1(x,y) \times ho\, A_1(y,z) \to ho\, A_1(x,z)$$

give a composition law for a category, $ho\, A$, the homotopy category of A. ■

Definition: This category, $ho\, A$, is called the *homotopy category* of A.

Of course, $ho\, A$ is the fundamental category of A up to natural isomorphism. From previous comments we have:

Corollary 24 *A quasi-category A is a Kan complex if and only if $ho\, A$ is a groupoid.* ■

11.4.4 Homotopy coherent diagrams and homotopy coherent nerves

(The notion was explicitly introduced by Cordier, [83], adapting ideas from Boardman and Vogt, [36]. There is an overview of this theory in [228] and a thorough introduction in [171]. The construction of the homotopy coherent nerve is also used, extensively, by Lurie in [190], and by Hinich, [154].)

Before handling this topic, we quickly recall some of the intuition behind homotopy coherent (h. c.) diagrams, as we saw a few pages back.

A diagram indexed by the small category, [2],

$$\begin{tikzcd} & X(1) \arrow[dr, "X(12)"] & \\ X(0) \arrow[ur, "X(01)"] \arrow[rr, "X(02)"'] & & X(2) \end{tikzcd}$$

is h. c. if there is specified a homotopy

$$X(012) : I \times X(0) \to X(2),$$

$$X(012) : X(02) \simeq X(12)X(01).$$

For a diagram indexed by [3]: Draw a 3-simplex, marking the vertices $X(0), \ldots, X(3)$, the edges $X(ij)$, etc., the faces $X(ijk)$, etc. The homotopies $X(ijk)$ fit together to make the sides of a square

$$\begin{tikzcd} X(13)X(01) \arrow[r, "X(123)X(01)"] & X(23)X(12)X(01) \\ X(03) \arrow[u, "X(013)"] \arrow[r, "X(023)"'] & X(23)X(02) \arrow[u, "X(23)X(012)"'] \end{tikzcd}$$

and the diagram is made h. c. by specifying a second level homotopy

$$X(0123) : I^2 \times X(0) \to X(3)$$

filling this square.

These can be continued for larger $[n]$, as we have hinted.

We have seen that the 'same' diagrams occur when we draw what seems to be a reasonable example of the intuitive form of homotopy coherent diagram in Top and in the $\mathcal{S}$-categories, $S(\mathbb{A})$. This suggests the definition of a homotopy coherent diagram in an arbitrary $\mathcal{S}$-category. This form is due to Cordier, [83], extending the earlier work of Boardman and Vogt.

Definition: Let $\mathbb{A}$ be a small category and $\mathcal{B}$, an $\mathcal{S}$-category.

(i) A *homotopy coherent diagram of type* $\mathbb{A}$ *in* $\mathcal{B}$ is a $\mathcal{S}$-functor, $F : S(\mathbb{A}) \to \mathcal{B}$.

(ii) If $F_0, F_1 : S(\mathbb{A}) \to \mathcal{B}$ are two such diagrams, a *homotopy coherent map* between them is a diagram of type $\mathbb{A} \times [1]$ agreeing with F_0 on $\mathbb{A} \times \{0\}$ and with F_1 on $\mathbb{A} \times \{1\}$.

Of course, we refer to $\mathbb{A}$ as the *template* of the h. c. diagram, F.

We should pause to examine this notion of homotopy coherent map in more detail, via our low dimensional examples, *i.e.*, with $\mathbb{A} = [n]$ for small values of n.

For $n = 0$, this is unenlightening: $F_0, F_1 : S[0] \to \mathcal{B}$, so they are really just two objects of $\mathcal{B}$, and a h.c. map between them in then just a map between $F_0(0)$ and $F_1(0)$ in $\mathcal{B}$.

For $n = 1$, it is already a much richer picture. This time, F_0 and F_1 pick out two maps in $\mathcal{B}$, namely $F_i(0) \stackrel{F_i(01)}{\to} F_i(1)$ for $i = 0, 1$. A homotopy coherent map, $\eta : F_0 \to F_1$, is a h.c. diagram of type $[1] \times [1]$, so is a square of form

$$\begin{tikzcd}
F_1(0) \arrow[r] \arrow[dr, phantom, "\Nwarrow" near start] & F_1(1) \\
F_0(0) \arrow[u] \arrow[ur] \arrow[r] & F_0(1) \arrow[u] \arrow[ul, phantom, "\Searrow" near end]
\end{tikzcd}$$

and will specify $\eta(i) : F_0(i) \to F_1(i)$ for $i = 0, 1$, but also a diagonal map, which we will write $\eta_0^1 : F_0(0) \to F_1(1)$, then also we will have homotopies as shown from η_0^1 to $F_1(01)\eta(0)$ and to $\eta(1)F_0(01)$, respectively.

It is worthwhile pausing to note that, in this simplicial approach, there is an avoidance of questions of directions of 2-cells (and higher order ones). Often when looking at diagrams showing lax or pseudo morphisms between lax or pseudo functors, one or other of the directions is chosen, e.g., here it might typically be $\eta : \eta(1)F_0(01) \Rightarrow F_1(01)\eta(0)$. If we are in a 'pseudo' context, this choice, although arbitrary, is somewhat immaterial as η will be invertible, but this need not be the case for a lax morphism. Nothing dictates which direction is 'better' and both are present in this simplicial approach. If someone gives you $\eta : \eta(1)F_0(01) \Rightarrow F_1(01)\eta(0)$, you can take $\eta_0^1 = \eta(1)F_0(01)$ and set the bottom right homotopy to be the identity. Likewise if η is presented in the reverse direction, just set the top left cell to be the identity two cell and use the given η in the bottom right. Some people do not like this as they prefer one choice or other, usually for a good reason from the situation being handled, yet, simplicially, it is more or less required to have the diagonal and the two 2-cells.

For $n = 2$, we have a prism, $[2] \times [1]$, and you have to specify η on three tetrahedra in this, agreeing on the overlaps. Here is a possible notation and the beginnings of a detailed discussion which can be extended to higher dimensions. (The rest is not hard, but does **really involve reader participation**!)

$$\begin{tikzcd}
& (1,0) \arrow[rr] \arrow[dd] \arrow[ddrr] & & (1,1) \arrow[dd] \\
(0,0) \arrow[ur] \arrow[rr] \arrow[dr] & & (0,1) \arrow[ur] \arrow[dr] & \\
& (2,0) \arrow[rr] & & (2,1)
\end{tikzcd}$$

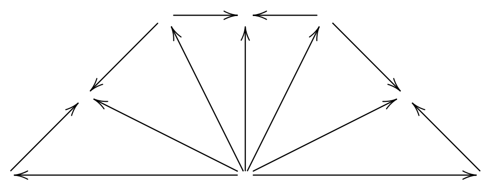
We suggest a matrix notation. For this the use of column 'vectors' is preferable to rows, so (1,0) becomes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as a vertex label; the edge from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is then clearly $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; the shown diagonal is $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$. (Two diagonals have been left out of the diagram.)

We mentioned three tetrahedra. These are

$$\sigma_0 = \left(\begin{smallmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right), \quad \sigma_1 = \left(\begin{smallmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right), \quad \sigma_2 = \left(\begin{smallmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{smallmatrix}\right).$$

The first and second share a d_2-face, namely $\left(\begin{smallmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, whilst the second and third share a d_1-face, *i.e.*, $\left(\begin{smallmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \end{smallmatrix}\right)$.

The comments above about 'orientation' or 'direction' are even more pertinent here. For each tetrahedron, we have a copy of $S[3]$, so in particular $S[3](0,3)$ is there 3 times. As $S[3](0,3)$ is a square, $\Delta[1]^2$, we have 6 two simplices in $S([2]\times[1])((0,0),(2,1))$. They fit together to give half a hexagon:



Each subdivided segment is a square in disguise! (You get half a hexagon because the prism is half of the cube $[1]^3$, and $S([1]^3)$ is a barycentrically subdivided hexagon.) Of the six 2-simplices, if you check their orientation, half go anticlockwise, half go clockwise. Later in our discussion of 2-dimensional descent data, we will have a prismical diagram. In each rectangular side face, we choose the convention as above, putting the 'active' face in one of the two 2-simplices. This means three of the boundary arrows in the above will be set to be equalities. The diagram we will use there is a commuting pentagon of 2-cells in a 2-category, and, of course, this can be derived from the above by noting that in 2-categories, there are no 3-cells, so $S([2]\times[1])((0,0),(2,1))$ will be mapped to a category, something like $\mathcal{B}(F_0(0), F_1(2))$, but that has no non-identity 2-cells, so the 2-simplices will be sent to identity homotopies. The other input is that $5 = 8-3$ (proof left to the reader - no calculators permitted - other than your fingers!!!) We will refer back to this when we are looking at 2-dimensional descent. It permits us to see the phenomena there as being very much akin to those with homotopy coherence.

This type of combinatorial analysis can be very useful when handling maps of homotopy coherent diagrams and relating them to other descriptions (lax, pseudo , etc.) of the same situations. It is not the only way of handling these ideas however, and the simplicial set of maps between two $\mathcal{S}$-functors, $F, G : S(\mathbb{A}) \to \mathcal{B}$, can be handled categorically as well. The basic intuition is, however, very much the same, and the resulting problems are there, whichever way you approach this. Use of more high powered categorical machinery, quasi-categories, etc. can make the theory much easier to apply, but also then you need to keep in sight the basic intuitions and to see how the combinatorics related to that is encoded in the machine you are using.

We mentioned 'problems' ... what are they?

In general, homotopy coherent maps, as defined here, need not compose, even when they might be expected to. The problem is analogous to that of composing homotopies between simplicial maps, that we met a short while ago. Unless the codomains are Kan complexes, there is no guarantee that such homotopies can be composed. Even when they compose, of course, there will, in general, be many composites. Those composites *will* be themselves homotopic and so on. Here with homotopy coherent maps, provided that the ambient category, $\mathcal{B}$, is *locally weakly Kan*, (*i.e.*, is 'quasi-category'-enriched), then they do compose, up to homotopy. The result is a sort of 'A_∞-category' structure, (see Batanin's paper, [24]), but also has a quasi-categorical description, which we will meet shortly. One can also use Verity's theory of complicial sets, [266] and their weakened form, [267–269]. These are closely related to the simplicial T-complexes we saw in section 1.3.6.

The theory of homotopy coherence was initially developed explicitly by Vogt, [271], following methods introduced with Boardman, [36], (see also the references in that source for other earlier papers on the area), then Cordier, [83], provided the simple $\mathcal{S}$-category theory way of working with h. c. diagrams and hence released an 'arsenal' of categorical tools for working with h. c. diagrams. Some of that is worked out in the papers, [85–88]. We illustrate this with some results taken from those sources.

Proposition 117 *Let $X : \mathbb{A} \to Top$ be a commutative diagram and suppose that each object, $X(a)$ is given to be homotopy equivalent to some $Y(a)$ with specified homotopy equivalence data:*

$$f(a) : X(a) \to Y(a), \quad g(a) : Y(a) \to X(a)$$

$$H(a) : g(a)f(a) \simeq Id, \quad K(a) : f(a)g(a) \simeq Id,$$

then there is a h. c. diagram Y based on the objects, $Y(a)$, and homotopy coherent maps

$$f : X \to Y, \quad g : Y \to X, \ \textit{etc.},$$

making X and Y homotopy equivalent as h. c. diagrams. ■

In other words, our earlier simple examples can be handled for any indexing category. More generally, if X is a h.c. diagram, and we are given the $Y(a)$s, etc., as above then we can construct a h.c. Y. (This is 'really' a result about quasi-categories, see [169].)

Theorem 33 *Vogt, [271]. If $\mathbb{A}$ is a small category, there is a category $Coh(\mathbb{A}, Top)$ of h. c. diagrams and homotopy classes of h. c. maps between them. Moreover there is an equivalence of categories*

$$Coh(\mathbb{A}, Top) \xrightarrow{\simeq} Ho(Top^{\mathbb{A}})$$

where $Ho(Top^{\mathbb{A}})$ is obtained from the category of functors from $\mathbb{A}$ to Top by inverting objectwise homotopy equivalences.

This was extended replacing Top by a general locally Kan simplicially enriched complete category, $\mathcal{B}$, in [84].

(iii) Cordier (1980), [83]. Given $\mathbb{A}$, a small category, then the $\mathcal{S}$-category $S(\mathbb{A})$ is such that a h. c. diagram of type $\mathbb{A}$ in Top is given precisely by an $\mathcal{S}$-functor

$$F : S(\mathbb{A}) \to Top$$

This suggested the extension of h. c. diagrams to other contexts such as a general locally Kan $\mathcal{S}$-category, $\mathcal{B}$, and suggests the definition of homotopy coherent diagram in a $\mathcal{S}$-category and thus a h. c. nerve of an $\mathcal{S}$-category.

Definition: (Cordier (1980), [83]) Given a simplicially enriched category $\mathcal{B}$, the *homotopy coherent nerve* of $\mathcal{B}$, denoted $Ner_{h.c.}(\mathcal{B})$, is the simplicial 'set' with

$$Ner_{h.c.}(\mathcal{B})_n = \mathcal{S}-Cat(S[n], \mathcal{B}),$$

and with the induced face and degeneracy maps.

Remark on terminology: Cordier, [83], initially used the term 'homotopy coherent nerve' for the above as he was primarily interested in its use in that area although in his subsequent work with Porter, [85–88], the quasi-categorical and ∞-categorical aspects were often a priority. Lurie, [190], has called this the *simplicial nerve functor* as his applications are not explicitly concerned with homotopy coherence.

To understand h. c. diagrams and thus $Ner_{h.c.}(\mathcal{B})$, we will unpack the definition of homotopy coherence. The first thing to note is that, as we saw, for any n and $0 \leq i < j \leq n$, $S[n](i,j) \cong \Delta[1]^{j-i-1}$, the $(j-i-1)$-cube given by the product of $j-i-1$ copies of $\Delta[1]$. Thus we can reduce the higher homotopy data to being just that, maps from higher dimensional cubes.

Next some notation:

Given simplicial maps

$$f_1 : K_1 \to \mathcal{B}(x, y),$$

$$f_2 : K_2 \to \mathcal{B}(y, z),$$

we will denote the composite

$$K_1 \times K_2 \to \mathcal{B}(x, y) \times \mathcal{B}(y, z) \xrightarrow{c} \mathcal{B}(x, z)$$

just by $f_2.f_1$ or $f_2 f_1$. (We already have seen this in the h. c. diagram above for $\mathbb{A} = [3]$. $X(123)X(01)$ is actually $X(123)(I \times X(01))$, whilst $X(23)X(012)$ is exactly what it states.)

Suppose now that we have the h. c. diagram, $F : S(\mathbb{A}) \to \mathcal{B}$. This is an $\mathcal{S}$-functor and so:

- to each object a of $\mathbb{A}$, it assigns an object $F(a)$ of $\mathcal{B}$;
- to each string of composable maps in $\mathbb{A}$,

  $$\sigma = (f_0, \ldots, f_n)$$

  starting at a and ending at b, it assigns a simplicial map

  $$F(\sigma) : S(\mathbb{A})(0, n+1) \to \mathcal{B}(F(a), F(b)),$$

  that is, a higher homotopy

  $$F(\sigma) : \Delta[1]^n \to \mathcal{B}(F(a), F(b)),$$

  such that
- if $f_0 = id$, $F(\sigma) = F(\partial_0 \sigma)(proj \times \Delta[1]^{n-1})$;

- if $f_i = id$, $0 < i < n$,
$$F(\sigma) = F(\partial_i \sigma).(I^i \times m \times I^{n-i}),$$
where $m : I^2 \to I$ is the multiplicative structure on $I = \Delta[1]$ induced by the 'max' function on $\{0, 1\}$;
- if $f_n = id$, $F(\sigma) = F(\partial_n \sigma)$;
- ${}_i\ F(\sigma)|(I^{i-1} \times \{0\} \times I^{n-i}) = F(\partial_i \sigma), 1 \leq i \leq n-1$;
- ${}_i\ F(\sigma)|(I^{i-1} \times \{1\} \times I^{n-i}) = F(\sigma'_i).F(\sigma_i)$, where $\sigma_i = (f_0, \ldots, f_{i-1})$ and $\sigma' = (f_i, \ldots, f_n)$.

We have used ∂_i here for the face operators in the nerve of $\mathbb{A}$.

The specification of such a homotopy coherent diagram can be split into two parts:

(a) specification of certain homotopy coherent *simplices*, *i.e.*, elements in $Ner_{h.c.}(\mathcal{B})$; and

(b) specification, via a simplicial mapping from $Ner(\mathbb{A})$ to $Ner_{h.c.}(\mathcal{B})$, of how these individual parts (from (a)) of the diagram are glued together.

The second part of this is easy as it amounts to a simplicial map $Ner(\mathbb{A}) \to Ner_{h.c.}(\mathcal{B})$, and so we are left with the first part. The following theorem was proved by Cordier and Porter, [84], but many of the ideas for the proof were already in Boardman and Vogt's lecture notes, like so much else!

Theorem 34 *([84]) If $\mathcal{B}$ is a locally Kan $\mathcal{S}$-category, then $Ner_{h.c.}(\mathcal{B})$ is a quasi-category.* ■

It is not clear what happens if $\mathcal{B}$ is only locally weakly Kan, is $Ner_{h.c.}(\mathcal{B})$ then a quasi-category? It is probably a known result, maybe even clear, but may not be in published form.

The proof of the theorem is in the paper, [84], and is not too complex. The essential feature is that the very definition (unpacked version) of homotopy coherent diagram makes it clear that parts of the data have to be composed together, (recall the composition of simplicial maps

$$f_1 : K_1 \to \mathcal{B}(x, y),$$

$$f_2 : K_2 \to \mathcal{B}(y, z),$$

above and how important that was in the unpacked definition).

We thus have that a homotopy coherent diagram 'is' a simplicial map, $F : Ner(\mathbb{A}) \to Ner_{h.c.}(\mathcal{B})$, and that $Ner_{h.c.}(\mathcal{B})$ is a quasi-category. Of course, the usual proof that, if X and Y are simplicial sets, and Y is Kan, then $\underline{\mathcal{S}}(X, Y)$ is Kan as well, extends to having Y a quasi-category and the result being a quasi-category. Earlier we referred to $Coh(\mathbb{A}, \mathcal{B})$ in connection with Vogt's theorem. The neat way of introducing this is as $ho\,\mathcal{S}(Ner(\mathbb{A}), Ner_{h.c.}(\mathcal{B}))$, the fundamental category of the function quasi-category. In fact, this is essentially the way that Vogt first described it.

If $\mathcal{A}$ and $\mathcal{B}$ are both $\mathcal{S}$-categories, and $F : \mathcal{A} \to \mathcal{B}$ is an $\mathcal{S}$-functor, then clearly F induces a simplicial map
$$Ner_{h.c.}(F) : Ner_{h.c.}(\mathcal{A}) \to Ner_{h.c.}(\mathcal{B}).$$

In other words $Ner_{h.c.}$ is a functor from $\mathcal{S}-Cat$ to $\mathcal{S}$, ignoring any problems due to 'size' of the categories involved. We will see later (Proposition 121 and the discussion around that result) that there may be simplicial maps between $Ner_{h.c.}(\mathcal{A})$ and $Ner_{h.c.}(\mathcal{B})$ that do not come from $\mathcal{S}$-functors.

As the category, $\mathcal{S}-Cat$, of (small) $\mathcal{S}$-categories and $\mathcal{S}$-functors between them is cocomplete, there is a left adjoint to this nerve functor in the usual way. The general picture of such adjoint pairs induced by some 'models' here looks like this: we have $S : \mathbf{\Delta} \to \mathcal{S}-Cat$ and $\Delta : \mathbf{\Delta} \to \mathcal{S}$, the Yoneda embedding, and these induce the nerve and 'realisation' adjoint pair. (If you replace $\mathcal{S}-Cat$ by Top, you get the singular complex / geometric realisation adjoint pair, that you have met earlier.) As the nerve functor has a left adjoint, it preserves limits and, in particular, products.

11.4.5 Simplicial coherence and models for homotopy types

Before we look at more direct applications of simplicially based homotopy coherence, there is a point that is worth noting for the links with algebraic and categorical models for homotopy types. The $\mathcal{S}$-categories, $S[n]$, contain a lot of the information needed for the construction of such models. A good example of this is the interchange law and its links with Gray categories and Gray groupoids.

Consider $S[4]$. The important information is in the simplicial set $S[4](0,4)$. This is a 3-cube, so is still reasonably easy to visualise. Here it is. The notation is not intended to be completely consistent with earlier uses, but is meant to be more or less self explanatory.

$$\begin{tikzcd}
& (01)(13)(34) \arrow[rr] & & (01)(12)(23)(34) \\
(01)(14) \arrow[ur] \arrow[rr] & & (01)(12)(24) \arrow[ur] & \\
& (03)(34) \arrow[uu] \arrow[rr] & & (02)(23)(34) \arrow[uu] \\
(04) \arrow[uu] \arrow[ur] \arrow[rr] & & (02)(24) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

This looks mysterious! A 4-simplex has 5 vertices, and hence 5 tetrahedral faces. Each of the 5 tetrahedral faces will contribute a square to the above diagram, yet a cube has 6 square faces! Where does the 'extra' face come from? (Things get 'worse' in $S[5](0,5)$, which is a 4-cube, so has 8 cubes as its faces, but $\Delta[5]$ has only 6 faces.) Back to the 'extra' face, this is

$$\begin{tikzcd}
(01)(12)(24) \arrow[r, "(01)(12)(234)"] & (01)(12)(23)(34) \\
(02)(24) \arrow[u, "(012)(24)"] \arrow[r, "(02)(234)"'] \arrow[ur, phantom, "(012)(234)"] & (02)(23)(34). \arrow[u, "(012)(23)(34)"']
\end{tikzcd}$$

The arrow $(012) : (02) \to (01)(12)$ will, in a homotopy coherent diagram, make its appearence as the homotopy,

$$X(012) : I \times X(0) \to X(2),$$

$$X(012) : X(02) \simeq X(12)X(01),$$

thus this square implies that the homotopies $X(012)$ and $X(234)$ interact minimally. Drawing homotopies as 2-cells in the usual way, the square we have above is the interchange square and the interchange law will hold in this system provided this square is, in some sense, commutative. In models for homotopy n-types for $n \geq 3$, these interchange squares give part of the pairing structure between different levels of the model. They are there in, say, the Conduché model (2-crossed modules, cf. Conduché, [81] and here, section 5.3.4) as the *Peiffer lifting*, and in the Loday model, (crossed squares, cf. [187]), as the h-map. In a general dimension, n, there will be pairings like this for any splitting of $\{0, 1, \ldots, n\}$ of the form $\{0.1.\ldots, k\}$ and $\{k, \ldots, n\}$. These are related to the Peiffer pairings that we have mentioned several times.

11.5 Useful examples

By the main title of this section, we intend to concentrate attention on the ways in which homotopy coherence techniques clarify what is going on at certain points of the development of cohomology and related areas. Mostly these are instances of more general results listed or mentioned earlier in this chapter.

11.5.1 G-spaces: discrete case

The first example concerns a G-space for G a discrete group. (For G a topological group, more subtle arguments are needed although, as we will see later, the basic idea is the same.) We therefore have a space, X, and an action

$$a : G \times X \to X, \qquad a(g, x) = g \cdot x,$$

being a continuous map from the product to X satisfying some rules. We have considered such a G-object in several different ways, and settings, not all of them 'spatial'. One was to consider the group, G, as a groupoid with a single object. This groupoid has usually been written $G[1]$, with the single object denoted by $*$ or similar. We then built a functor, $\mathbb{X} : G[1] \to Top$, as follows:

- $\mathbb{X}(*) = X$;
- if $g : * \to *$ in $G[1]$ and $x \in X$, then $\mathbb{X}(g) : \mathbb{X}(*) \to \mathbb{X}(*)$ is simply $\mathbb{X}(g)(x) = g \cdot x$, and, of course, $\mathbb{X}(g_1g_2) = \mathbb{X}(g_1)\mathbb{X}(g_2)$.

If we need another description of functors than merely elementwise, (which can be awkward for categorification), it may help to replace the second part of the above by

$$\mathbb{X} : G[1](*, *) \to Top(\mathbb{X}(*), \mathbb{X}(*)),$$

as being the analogue of the usual : if $F : \mathbb{A} \to \mathbb{B}$, then, for any objects a_1, a_2 in $\mathbb{A}$, there is a map,

$$F : \mathbb{A}(a_1, a_2) \to \mathbb{B}(F(a_1), F(a_2)),$$

which has to satisfy some composition preservation rules (and some tightening up on notation, since this F is really something like F_{a_1,a_2}, and so on).

The point of this second description is two fold. We have, once unpacked from its notation, just a function

$$G \to Top(X, X),$$

(and the codomain here is a monoid under composition of functions), which preserves multiplication and identity. The image of this function is thus within $Aut(X) \subseteq Top(X, X)$, the group of self homeomorphisms of X, and so we get back to the other description of an action as a homomorphism,

$$G \to Aut(X).$$

(If G is a topological group and Top is Cartesian closed, then $Aut(X)$ will be a topological group, and a continuous action will correspond to a *continuous* homomorphism of the same form. If G is a simplicial group and X is a simplicial set, we get back simplicial automorphisms and simplicial actions as we looked at earlier (in section 6.3, starting on page 256, and the section following that). Here, of course, $G[1]$ is a simplicially enriched groupoid and the action yields an $\mathcal{S}$-functor, $\mathbb{X} : G[1] \to \mathcal{S}$, and so on. (You should play around with the different types of contexts to see what works well and what less well.))

Each of these descriptions of G-actions is useful. Here we will take the description of a G-space as

$$\mathbb{X} : G[1] \to Top.$$

(From now on, we drop the 'blackboard' font, $\mathbb{X}$, for this and merely write X.) Now suppose that we replace our space X by a homotopy equivalent one, Y, (along a homotopy equivalence, $(f : X \to Y,\ f' : Y \to X,\ \mathbf{H}$ and $\mathbf{K})$), then we do not usually get a G-action on Y. (The situation is, of course, essentially that which we examined in section 11.1, and it is worthwhile to see what a 'bare hands' approach gives in this situation.)

The theoretical, general, results that we have quoted give us a homotopy coherent diagram

$$Y : S(G[1]) \to \underline{Top},$$

where $\underline{Top}$ is the simplicial enrichment of Top.

Of course, there is nothing magical about $\underline{Top}$ here and we could have equally well have used $\mathcal{S}$ or a general simplicially enriched category, $\mathcal{B}$. (Of course, for some purposes, we would need for $\mathcal{B}$ to be 'locally Kan' and / or for certain limits or colimits to exit, in order to get a neat theory here.)

The points to retain from this are that $S(G[1])$ is almost the 'free-group' comonadic simplicial resolution of G. It is a simplicial monoid, not a simplicial group however. We have *deformed* the group action to a homotopy coherent action and this is done by replacing G by a free simplicial resolution of G. (This is another instance of 'cofibrant replacement'.) The role of $Aut(X)$ is no longer viable. We cannot use $Aut(Y)$ in its place because, if we have $g \in G$, then we have a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "X(g)"'] & Y \arrow[d, dashed, "Y(\langle g\rangle)"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and $Y(\langle g\rangle) = fX(g)f'$, at least according to the recipe that we found in our earlier analysis. We cannot guarantee that $Y(\langle g\rangle)$ will be an 'automorphism' of Y. We do have $X(g^{-1}) : X \to X$, but then our algorithm for constructing Y gives $Y(\langle g^{-1}\rangle) = fX(g^{-1})f'$, so $Y(\langle g^{-1}\rangle)Y(\langle g\rangle) \simeq Y(\langle 1_G\rangle) \simeq 1_Y$. We thus do have $Y(\langle g\rangle)$ is a self equivalence (auto-equivalence) of Y, in our case, a self homotopy equivalence, but we could be in another context, e.g. Cat, and the same basic argument would work.

This is not the end of the example. We have

$$Y : S(G[1]) \to \mathcal{B},$$

but therefore we have a simplicial description of Y as

$$Y : Ner(G[1]) \to Ner_{h.c.}(\mathcal{B}).$$

We know what $Ner(G[1])$ is. It is what we have denoted BG, the classifying space of G. (Unlike the other contexts where we have met it, however, it is the domain not the codomain of the relevant map.)

That gives us an additional intuition on several themes that we have met earlier, but there are others that are closely related where it is not so clear how it might help.

11.5.2 Lax, Op-lax and pseudo- functors

As we have mentioned lax functors several times informally, we need to give a more formal definition[5].

Our earlier discussion, for instance in section 9.1.3, related to a 'functor-like' mapping from a *category*, $\mathbb{A}$, into a 2-category, usually the 2-category Cat. We will give, below, a more general[6]definition for when we have a 2-category, $\mathcal{A}$, as domain and a general 2-category, $\mathcal{B}$, as codomain for our generalised functor. To be able to relate back to the earlier case, it is useful to have some terminology to handle that situation.

Definition: Suppose $\mathcal{A}$ is a 2-category. We say that it is a *locally discrete* 2-category or is *locally 2-discrete* if, for each pair of objects, A_0, A_1 in $\mathcal{A}$, the category $\mathcal{A}(A_0, A_1)$ is a discrete category, (*i.e.*, really just a set, so $\mathcal{A}$ has no non-identity 2-cells).

This will, thus, allow us to think of an ordinary category as being a 2-category, and it gives an embedding of Cat into $2-Cat$. We will shortly be considering a 2-category as an $\mathcal{S}$-category (as on page 498). We also will use such phrases as 'since $\mathbb{A}$ has no non-identity 2-cells' to indicate that we are considering $\mathbb{A}$ as a locally discrete 2-category, without making a fuss about it or denoting that version of $\mathbb{A}$ by some changed symbol. The natural tendency is then to extend this to saying that a 2-category, $\mathcal{A}$, 'has no non-identity 3-cells', although we have not considered 3-categories at all as yet.

If the 2-category is a locally discrete one, then, naturally, the resulting $\mathcal{S}$-category is a locally discrete $\mathcal{S}$-category, as well.

Suppose now that $\mathcal{A}$ and $\mathcal{B}$ are both 2-categories.

Definition: A *lax functor*, $\mathcal{F} = (F, c) : \mathcal{A} \to \mathcal{B}$, assigns

[5]especially as the basic idea is clearly closely related to homotopy coherence in some 'intuitive' way

[6]There are still more general versions in which the 2-categories are replaced by bicategories *aka* weak 2-categories.

- to each object A of $\mathcal{A}$, an object, $F(A)$, of $\mathcal{B}$;
- to each pair of objects, A_0, A_1, of $\mathcal{A}$, a functor,

$$F : \mathcal{A}(A_0, A_1) \to \mathcal{B}(FA_0, FA_1);$$

- to each composable pair of 1-cells / morphisms, (f, g) of $\mathcal{A}$, a 2-cell,

$$c_{f,g} : F(g)F(f) \Rightarrow F(gf),$$

depending naturally on f and g, and to each object A of $\mathcal{A}$, a 2-cell, $c_A : id_{FA} \Rightarrow F(id_A)$,

such that the coherence conditions, below, are satisfied:

- for any composable triple, (f, g, h), of 1-cells / morphisms of $\mathcal{A}$, the diagram

$$\begin{tikzcd}
F(hg)F(f) \arrow[r, Rightarrow, "c_{f,hg}"] & F(hgf) \\
F(h)F(g)F(f) \arrow[u, Rightarrow, "c_{g,h}\circ F(f)"] \arrow[r, Rightarrow, "F(h)\circ c_{f,g}"'] & F(h)F(gf) \arrow[u, Rightarrow, "c_{gf,h}"']
\end{tikzcd}$$

commutes;

- for any 1-cell, $f \in \mathcal{A}(A_0, A_1)$, the diagrams

$$\begin{tikzcd}
F(f)F(id_{A_0}) \arrow[r, Rightarrow, "c_{f,id_{A_0}}"] & F(f \circ id_{A_0}) = F(f) \\
F(f) = F(f) \circ id_{F(A_0)} \arrow[u, Rightarrow, "F(f)c_{A_0}"] \arrow[ur, equal] &
\end{tikzcd}$$

and similarly for id_{A_1} on the other side, commute.

Remarks: (i) Of course, any 2-functor corresponds to a set of data as here, but with each $F(g)F(f) = F(gf)$ and all the $c_{f,g}$s being the relevant identities.

(ii) In some case, for each A, c_A is the identity map, *i.e.*, the lax functor $\mathcal{F}$ preserves identities. In this case the terminology '*normal lax functor* is often used. This is consistent with the use of 'normalised' when referring to constructions such as the bar resolution. Most of the lax functors that we will meet will be 'normal'.

(iii) A quick look forward a few pages to page 593 and the definition of *(lax) monoidal functor* should convince you that the two ideas are closely related. Any 2-category is a 'strict' bicategory and any monoidal category 'is' a bicategory having just a single object, so bicategories (also called *weak 2-categories*) are a common generalisation of both 2-categories and monoidal categories. That being the case, there is a generalisation of lax functor, as defined above, to one, $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, in which $\mathcal{A}$ and $\mathcal{B}$ can be bicategories. (The formulation is **left to you** for later, when you have seen the definition of lax monoidal functor. It needs some more precision on the notion of bicategory so as to introduce notation for the 'associator' 2-cell, and the left and right unit 2-cells, and then a little thought on how to adapt 'lax monoidal functor' to 'lax functor' in that more general sense. Again for now it is left to you to look up the definitions of bicategory / weak 2-category and related notions of functor, natural transformation and then modifications / 3-cells between them. One of the original references is Bénabou, [30], whilst there is a thorough treatment in Borceux, [37], and a useful brief summary by Tom Leinster, [184].) We will, however, look briefly at the pseudo-case as that is useful for some of the theory of stacks, fibred categories, etc.

(iv) The notion of *pseudo-functor* between 2-categories or, more generally, between bicategories is, as was said earlier, the special case of a lax functor in which the two types of 2-cell, both the $c_{f,g}$ and the c_A, are invertible. As said above, we will take a brief look at these ideas in this 'pseudo' case, shortly; see section 11.5.4.

(v) Of importance below will be the notion of an 'op-lax functor', $\mathscr{F} : \mathscr{A} \to \mathscr{B}$, in which the arrow of the 2-cells is reversed, so $c_{f,g} : F(gf) \Rightarrow F(g)F(f)$, etc. This can be accommodated within the system of theory of lax functors by the simple device of forming, from a 2-category, $\mathscr{B}$ (or more generally), a new 2-category, $\mathscr{B}^{(2op)}$, with each $\mathscr{B}^{(2op)}(A,B) = \mathscr{B}(A,B)^{op}$, so reversing the direction of the 2-cells (and hence the notation: '(2op)' = 'opposite on 2-cells'). With this, an op-lax functor, $\mathscr{F} : \mathscr{A} \to \mathscr{B}$, is just a lax functor $\mathscr{F}^{(2op)} : \mathscr{A}^{(2op)} \to \mathscr{B}^{(2op)}$. Of course, if $\mathscr{A}$ is locally discrete, and, thus, has no non-identity 2-cells, then ..., enough said (provided that $\mathscr{F}$ is normal)! Similarly, if $\mathscr{F}$ is a pseudo-functor, then it is both lax and op-lax, or, more precisely, it determines both a lax and an op-lax functor.

Examples: We have already seen some examples of lax, op-lax or pseudo functors, so will not give more here, except, of course the following. We cannot resist it.

Any crossed module gives rise to a 2-category, in fact a 2-group(oid), so it is natural, in the context of our discussion, to look at pseudo-functors between these 2-categories. (Why not 'lax' or 'op-lax', ..., simply that all 2-cells in these 2-categories will be invertible, so the other notions all essentially reduce to 'pseudo', with adjustment being made for the order of composition, etc.) We will examine in some detail what the resulting 'weak morphisms' of crossed modules look like a bit later, but would suggest that **examination of the idea now** and **by you** would at the same time prepare the way for that later discussion *and* give you some experience of handling these ideas if you have not met them in detail before.

11.5.3 ...and nerves for 2-categories

Given all this about lax/op-lax and pseudo-functors, how does this relate to homotopy coherence? To examine this, let us look at homotopy coherent diagrams in a 2-category. We noted earlier (page 498) that any 2-category, $\mathcal{C}$, could be considered as an $\mathcal{S}$-category, $\mathcal{C}_\Delta$. (We should note in passing that, as each $\mathcal{C}(A,B)$ is a category, $\mathcal{C}_\Delta(A,B)$, which is just the nerve of $\mathcal{C}(A,B)$, will *not* usually be a Kan complex, but will always be a weak Kan complex / quasi-category.)

Suppose $\mathbb{A}$ is a category and $\mathscr{B}$ a 2-category (which we will consider as an $\mathcal{S}$-category, $\mathscr{B}_\Delta$, in the above way, but will not write the suffix most of the time). Let $F : S(\mathbb{A}) \to \mathscr{B}_\Delta$ be a $\mathcal{S}$-functor, and thus a homotopy coherent diagram of type $\mathbb{A}$ in $\mathscr{B}$. We have F gives:

- to each object A of $\mathbb{A}$, an object $F(A)$ of $\mathscr{B}$;
- to each pair of objects, A_0, A_1, and each $f : A_) \to A_1$, a morphism / 1-cell, $F(f) : F(A_0) \to F(A_1)$;
- to each composable pair (f,g) in $\mathbb{A}$, ..., what?

A composable pair like this corresponds to a 2-simplex

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "gf"'] & & \cdot \end{tikzcd}$$

in the nerve of $\mathbb{A}$, so to a functor, $\ulcorner(f,g)\urcorner : [2] \to \mathbb{A}$, which will induce $S(\ulcorner(f,g)\urcorner) : S[2] \to S(\mathbb{A})$, and, composing that with F gives

$$\begin{tikzcd} & \cdot \arrow[dr, "F(g)"] & \\ \cdot \arrow[ur, "F(f)"] \arrow[rr, "F(gf)"'] & & \cdot \end{tikzcd}$$

with a 2-cell, $c_{f,g} : F(gf) \Rightarrow F(g)F(f)$. This looks like it is the data for an op-lax functor. We need to check dimension 3, and a composable triple, (f, g, h), gives a diagram $[3] \to \mathbb{A}$, and hence a tetrahedral diagram in $\mathcal{B}$, when mapped by F:

$$S[3] \to S(\mathbb{A}) \to \mathcal{B}.$$

This diagram 'really lives' in the category $\mathcal{B}(F(A_0), F(A_3))$, where $A_0 \xrightarrow{f} A_1 \xrightarrow{g} A_2 \xrightarrow{h} A_3$, and is a square

$$\begin{tikzcd} F(h)F(gf) \arrow[r, Rightarrow] & F(h)F(g)F(f) \\ F(hgf) \arrow[u, Rightarrow] \arrow[ur, Rightarrow] \arrow[r, Rightarrow] & F(hg)F(f) \arrow[u, Rightarrow] \end{tikzcd}$$

with a diagonal, and, as there are no non-trivial 3-cells in $\mathcal{B}$, there are no non-trivial 2-simplices in $\mathcal{B}(F(A_0), F(A_3))$ (either thought of as a category or as the associated simplicial set). As a result, we can conclude that the square commutes.

We thus have that a h. c. functor, $F : \mathbb{A} \to \mathcal{B}_\Delta$, reverting to the full notation, is exactly the same as a *normal* op-lax functor from $\mathbb{A}$, considered as a locally discrete 2-category, to $\mathcal{B}$.

We can note also that this gives a way of defining a *nerve for a 2-category*.

Definition: If $\mathcal{B}$ is a 2-category, we define its *nerve* to be $Ner_{h.c.}(\mathcal{B}_\Delta)$. We will write it $Ner(\mathcal{B})$.

This nerve functor has been studied by Blanco, Bullejos, and Faro, [35] and by Bullejos and Cegarra, [71] and is a specialisation of Duskin's nerve of a bi-category, [110]. Other work on this includes Gurski, [144], who links the construction with Verity's complicial sets, which we mentioned earlier. Here we will explore its properties and applications a bit more. This nerve, and also that extension of it to bicategories, is sometimes called the *Duskin nerve* of the 2-category or sometimes its *geometric nerve*.

Of course, if $\mathcal{B}$ is locally discrete, *i.e.*, is a category masquerading as a 2-category, then $Ner(\mathcal{B})$ is just the nerve of that category.

In general, the vertices of $Ner(\mathcal{B})$ are the objects of $\mathcal{B}$, whilst the 1-simplices are the morphisms. The two simplices are diagrams of the form

$$\begin{tikzcd} & \cdot \arrow[dr] \arrow[d, phantom, "\Uparrow"] & \\ \cdot \arrow[ur] \arrow[rr] & {} & \cdot \end{tikzcd}$$

and the 3-simplices correspond to tetrahedra with one of these 2-simplices in each face, hence together satisfying a cocycle condition. Above that dimension, as we will see, things are determined by their 3-skeletons.

Remarks: We could derive at least two other nerves from this construction, both of which give useful information on $\mathcal{B}$.

(i) We could define a nerve using lax rather than op-lax functors from the various $[n]$ to $\mathcal{B}$. In this case, the basic 2-simplex would look like

$$\begin{tikzcd} & \cdot \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ \cdot \arrow[ur] \arrow[rr] & {} & \cdot \end{tikzcd}$$

This variant does need mentioning, but its detailed treatment will not differ greatly from that of the geometric nerve, since it is $Ner(\mathcal{B}^{(2op)})$. If we need it, we can write it in that form or introduce as a shorthand, $Ner_{lax}(\mathcal{B})$.

(ii) We could also restrict attention to a 'pseudo'-version of this geometric nerve, in which the 2-cell is specified to be invertible:

$$\begin{tikzcd} & \cdot \arrow[dr] \arrow[d, phantom, "\Uparrow\cong"] & \\ \cdot \arrow[ur] \arrow[rr] & {} & \cdot \end{tikzcd}$$

This is related to the 2-nerve of a bicategory as considered by Lack and Paoli, [183]. We will not need to use this explicitly as the nearest we get to it has $\mathcal{B}$ a 2-groupoid - so all its 2-cells are invertible. It is important, however, to note that passing between $Ner_{lax}(\mathcal{B})$ and $Ner_{lax}(\mathcal{B}^{2op})$, one does not get an isomorphic simplicial set. This pheomenon can already be seen for nerves of groupoids. If you take, say, a 2-simplex in the nerve of a groupoid and then form the corresponding 2-simplex with the inverses you get the conjugate 2-simplex and this is not giving an automorphism of the nerve as it is incompatible with the face maps.

What sort of properties does this geometric nerve functor have? What should we intuitively expect, so some idea could guide our investigations?

For a small category $\mathbb{C}$, $Ner(\mathbb{C})$ has some very interesting and useful properties, (see the discussion around about page 508). We pick out that, if we have a k-simplex, σ in $Ner(\mathbb{C})$ with $k > 1$, then σ is completely determined by its 1-skeleton. Its 1-skeleton encodes not only that the various edges fit together, but each triangular face of σ records the fact that the d_1-face is the composite of the other two. We saw this in section 11.4.2. We can formalise this in other terms using the terminology of an earlier section, 5.1.2 (especially page 204). For any $k > 2$, and in any diagram

$$\begin{tikzcd} \partial\Delta[k] \arrow[r] \arrow[d] & Ner(\mathbb{C}) \\ \Delta[k] \arrow[ur, dashed] & \end{tikzcd}$$

there is a unique choice of dotted arrow. Remember that this is referred to as follows:

Lemma 73 *For any small category, $\mathbb{C}$, $Ner(\mathbb{C})$ is a 2-coskeletal simplicial set.*

Proof: Suppose that we have the shell, $x = (x_0, x_1, x_2, x_3)$, of a possible 3-simplex, *i.e.*,

$$x : \partial\Delta[3] \to Ner(\mathbb{C}),$$

then we have the individual 'faces', x_i that fit together correctly. For instance, x_3 is the 'face missing out 3', *i.e.*,

$$\begin{tikzcd} & x(1) \arrow[dr, "x(12)"] & \\ x(0) \arrow[ur, "x(01)"] \arrow[rr, "x(02)"'] & & x(2) \end{tikzcd}$$

and, as this is in $Ner(\mathbb{C})$, this means $x(02) = x(12)x(01)$, and so on. We thus have

$$x(03) = x(23)x(02) = x(23)x(12)x(01).$$

The only 3-simplex that will work is, of course, $\sigma := (x(01), x(12), x(23))$ and so, in the diagram

$$\begin{tikzcd} \partial\Delta[3] \arrow[r, "x"] \arrow[d] & Ner(\mathbb{C}) \\ \Delta[3] \arrow[ur, dashed, "\sigma"'] & \end{tikzcd}$$

this σ works and is the only choice. Of course, the same is true in higher dimension replacing 3 by k. (You are **left to prove** the general form of this, e.g. by induction or directly.) ∎

What about $Ner(\mathcal{C})$, when $\mathcal{C}$ is a 2-category? We might guess the following:

Proposition 118 *For any (small) 2-category, $\mathcal{C}$, $Ner(\mathcal{C})$ is a 3-coskeletal simplicial set.*

Proof: We assume given $x = (x_0, x_1, x_2, x_3, x_4)$, the shell of a potential 4-simplex, and hence

$$\begin{tikzcd} \partial\Delta[4] \arrow[r, "x"] \arrow[d] & Ner(\mathcal{C}) \\ \Delta[4] \arrow[ur, dashed, "?\sigma"'] & \end{tikzcd}$$

and try to see how to build the dotted arrow, σ, so $x_i = d_i\sigma$ for each of the indices, i. The simplest way to do this is to see what makes up such a σ. It is a h.c. diagram of type [4] in $\mathcal{C}$ corresponding, therefore, to an $\mathcal{S}$-functor,

$$\sigma : S[4] \to \mathcal{C},$$

and we discussed $S[4]$ in section 11.4.5. The key diagram is a cube in the *category*, $\mathcal{C}(x(0), x(4))$. That cube needs to commute as there are no non-identity 2-cells in $\mathcal{C}(x(0), x(4))$. We saw (again in section 11.4.5) that, of the 6 faces of this cube, 5 come from the 5 faces of the 4-simplex, hence, if σ is to complete the diagram, these 5 faces must coincide with those specified by the x_i for $i = 0, 1, \ldots, 4$. In other words, we have, within x, the information on all but one face of that cube. Each of those faces is commutative as it comes from a $x_i : S[3] \to \mathcal{C}$. What about the 'extra face'? This is (using the same sort of notation as before):

$$\begin{tikzcd}
x(34)x(23)x(02) \arrow[r, Rightarrow, "x(34)x(23)x(012)"] & x(34)x(23)x(12)x(01) \\
x(24)x(02) \arrow[u, Rightarrow, "x(234)x(02)"] \arrow[r, Rightarrow, "x(24)x(012)"'] & x(24)x(12)x(01) \arrow[u, Rightarrow, "x(234)x(12)x(01)"']
\end{tikzcd}$$

but the commutativity of such a diagram, in general, is equivalent to the interchange law holding in $\mathcal{C}$:

$$\begin{tikzcd}
\cdot \arrow[r, bend left, "x(02)"] \arrow[r, bend right, "x(12)x(01)"'] \arrow[r, phantom, "\Downarrow x(012)"] & \cdot \arrow[r, bend left, "x(24)"] \arrow[r, bend right, "x(34)x(23)"'] \arrow[r, phantom, "\Downarrow x(234)"] & \cdot\ ,
\end{tikzcd}$$

which, of course, it does.

It follows that, given x, we already have all the information needed to specify a unique σ, which completes the proof. ■

The following could have been mentioned much earlier, but was not needed until now:

Proposition 119 *The nerve functor,*

$$Ner : Cat \to \mathcal{S},$$

is full and faithful.

Proof: The 'reason' for this result is that all the information on a (small) category, $\mathbb{C}$, is contained in the first few levels of its nerve, $Ner(\mathbb{C})$. The objects are the vertices and thus form $Ner(\mathbb{C})_0$; the 1-simplices are simply the arrows, so levels 0 and 1 give, together with the face maps and degeneracies, the basic *combinatorial* structure of $\mathbb{C}$. For the composition, one uses $Ner(\mathbb{C})_2$, of course, and the fact the $Ner(\mathbb{C})$ is 2-coskeletal.

That is the 'reason', now for the proof!

We have to examine the function

$$Ner(\mathbb{C})_{\mathbb{C},\mathbb{D}} : Cat(\mathbb{C}, \mathbb{D}) \to \mathcal{S}(Ner(\mathbb{C}), Ner(\mathbb{D})),$$

for $\mathbb{C}$, $\mathbb{D}$ arbitrary small categories. (Check back for 'full' and 'faithful' on page 440 if you have forgotten their meanings.)

This is largely a question of routine checking. If $f : Ner(\mathbb{C}) \to Ner(\mathbb{D})$ is a simplicial map, then f_0 is an assignment

$$f_0 : Ob(\mathbb{C}) \to Ob(\mathbb{D})$$

and f_1, one

$$f_1 : Arr(\mathbb{C}) \to Arr(\mathbb{D})$$

compatibly with source and target maps, so f has the combinatorial structure necessary for a functor. Compatibility with composition is a consequence of f_2 and *its* compatibility with the face maps. Preservation of identities is obvious, and f defines a functor from

$$F : \mathbb{C} \to \mathbb{D}$$

from which, on applying Ner, we get back f itself. We thus have that $Ner(\mathbb{C})_{\mathbb{C},\mathbb{D}}$ is surjective. In fact, better than that, we have constructed an inverse for it, so it is bijective. (Of course, there are some minor **checks to do**, but these are straight forward.) ∎

This says that, in many ways, Cat behaves like a subcategory of $\mathcal{S}$ and this is one of the intuitions that fit well with our categorification process. It motivates quasi-categories and complicial sets, both models for certain classes of weak infinity categories and weak infinity categories are one way of trying to understand cohomology in the general non-Abelian setting.

What about 2-categories? Is the nerve from $2{-}Cat$ to $\mathcal{S}$ full and faithful? In some ways, we should *not* expect it to be. It is defined using lax / homotopy coherent functors, so we should expect it to reflect that somewhere. There is also a less explicit reason for suspecting that it would not be full and faithful. It 'feels' as if $2{-}Cat$ is not a complete 'categorification' of Cat. Categorification' certainly involves replacing sets by categories, functions by functors, etc., as in the passage from Cat to $2{-}Cat$, but also involves weakening 'equality' to 'equivalence'. Composition and identities should become weakened, so bicategories form a fuller categorification of Cat than do 2-categories. Duskin, [110], has given a generalisation of the nerve to bicategories, and this has been pushed further by Lack and Paoli, [183]. We will not go that far. (Further material can be found in the articles [34, 35, 71].)

This suggests, perhaps, that we look at Ner from the point of view of lax / op-lax / pseudo functors.

First recall that a *normal* op-lax functor, $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is an op-lax functor that preserves the identities.

Lemma 74 *A normal op-lax functor, $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, between 2-categories, induces a simplicial mapping, $Ner(\mathcal{F}) : Ner(\mathcal{A}) \to Ner(\mathcal{B})$.*

Proof: We will give this 'as is', *i.e.*, without that much reflection on what makes it work. That we will return to afterwards.

We write $\mathcal{F} = (F, c)$, as above, where F is the assignment on objects, and also denotes, sometimes with suffices, as in F_{A_0,A_1}, the functor between the relevant hom-categories, whilst c assigns 2-cells to composable pairs.

As a lax functor is neatly defined on objects and arrows, there is no problem in defining $Ner(\mathcal{F})_i$ for $i = 0$ and 1. Moreover, as $Ner(\mathcal{A})$ and $Ner(\mathcal{B})$ are 3-coskeletal, if we can define $Ner(\mathcal{F})$ in dimension 2, then it can be automatically generated in higher dimensions, since, for $k \geq 3$, any k-simplex in $Ner(\mathcal{B})$ is determined by its 2-skeleton. We thus have to concentrate on dimension 2.

A 2-simplex, σ, in $Ner(\mathcal{A})$ consists of a 4-tuple, $\sigma = (\sigma(12), \sigma(02), \sigma(01); \sigma(012))$, that is, of three arrows in $\mathcal{C}$ fitting together in a triangle, together with a 2-cell filling that triangle:

$$\begin{tikzcd} & A_1 \arrow[dr, "\sigma(12)"] \arrow[d, phantom, "\Uparrow"] & \\ A_0 \arrow[ur, "\sigma(01)"] \arrow[rr, "\sigma(02)"'] & {} & A_2 \end{tikzcd}$$

with $\sigma(012) : \sigma(02) \Rightarrow \sigma(12)\sigma(01)$ in $\mathcal{A}(A_0, A_2)$. The op-lax functor F assigns to the composable pair, $(\sigma(01), \sigma(12))$, a 2-cell

$$c_{\sigma(01),\sigma(12)} : F(\sigma(01)\sigma(12)) \Rightarrow F(\sigma(01))F(\sigma(12)),$$

and also a functor,

$$F_{02} : \mathcal{A}(A_0, A_2) \to \mathcal{B}(F(A_0), F(A_2)),$$

which, consequently, gives

$$F(\sigma(012)) : F(\sigma(02)) \Rightarrow F(\sigma(12)\sigma(01))$$

These fit together as follows:

$$\begin{tikzcd} & F(A_1) \arrow[dr, "F(\sigma(12))"] \arrow[d, phantom, "\Uparrow"] & \\ F(A_0) \arrow[ur, "F(\sigma(01))"] \arrow[rr, bend left=10] \arrow[rr, bend right=10, "F(\sigma(02))"'] \arrow[rr, phantom, "\Uparrow"] & {} & F(A_2) \end{tikzcd}$$

We look at the composite 2-cell and, of course, it forms, with the other data, a 2-simplex that we take as $Ner(\mathcal{F})(\sigma)$. More formally

$$Ner(\mathcal{F})(\sigma) = (F(\sigma(12)), F(\sigma(02)), F(\sigma(01)); \alpha),$$

where $\alpha = c_{\sigma(01),\sigma(12)} \sharp_1 F(\sigma(012))$.

It is clear that this satisfies the requirements for the face maps of the nerves and the degeneracy maps work as well, since $\mathcal{F}$ is assumed to be a *normal* op-lax functor. ■

Because of this, it is clear that, considered as a functor defined on the category, $2-Cat$, of 2-categories and (strict) 2-functors, Ner cannot be full, but suppose we define a new category $2{-}Cat_{op-lax}$ with the same objects, *but with the normal op-lax functors* as the morphisms between them. The above lemma shows that Ner extends to a functor, $Ner : 2{-}Cat_{op-lax} \to \mathcal{S}$. Is this full and faithful?

Let us examine a simplicial map, $f : Ner(\mathcal{A}) \to Ner(\mathcal{B})$. Can we construct an op-lax functor from it? We certainly have an assignment, F, on objects and on 1-cells, given by f_0 and f_1 respectively. For any pair, $x(01) : A_0 \to A_1$, $x(12) : A_1 \to A_2$, we have a composite $x(02) := x(12)x(01)$ and the identity 2-cell, $id : x(02) \Rightarrow x(12)x(01)$, written in that way for convenience. This gives a 2-simplex, $(x(12), x(02), x(01); id) \in Ner(\mathcal{A})_2$ and hence a 2-simplex,

$f_2(x(12), x(02), x(01); id) \in Ner(\mathcal{B})_2$. We know, since f_2 is compatible with face maps, that this 2-simplex has the form $(f_1x(12), f_1x(02), f_1x(01); y) \in Ner(\mathcal{A})_2$, where y is some 2-cell,

$$y : f_1x(02) \Rightarrow f_1x(12)f_1x(01),$$

and so it is sensible to take $\mathcal{F} = (F, c)$, as suggested above, where, abusing notation slightly, $F(A) = f_0(A)$,

$$F_{A_0,A_1} : \mathcal{A}(A_0, A_1) \to \mathcal{B}(FA_0, FA_1)$$

is defined on objects by f_1, *i.e.*, $F(x) = f_1(A_0 \xrightarrow{x} A_1)$, (but we still need F on 2-cells or, if you prefer, on the arrows in the $\mathcal{A}(A_0, A_1)$), and

$$c_{x(01),x(12)} = y,$$

as in the 2-simplex above.

We are, thus, left to define the F_{A_0,A_1} on the 2-cells and to check that they give a functor, etc.

Suppose

$$\begin{tikzcd} A_0 \arrow[r, bend left, "t(\alpha)"] \arrow[r, bend right, "s(\alpha)"'] \arrow[r, phantom, "\Uparrow\alpha"] & A_1 \end{tikzcd},$$

is a 2-cell of $\mathcal{A}$, then

$$\begin{tikzcd} & A_1 \arrow[dr, equal] & \\ A_0 \arrow[ur, "t(\alpha)"] \arrow[rr, "s(\alpha)"'] & \Uparrow\alpha & A_1 \end{tikzcd}$$

is a 2-simplex, $\sigma = (id, s(\alpha), t(\alpha); \alpha)$, of $Ner(\mathcal{A})$ and we get $f_2(\sigma) = (id, f_1s(\alpha), f_1t(\alpha); F(\alpha))$, defining $F(\alpha)$. (Note we are using that f is compatible with degeneracies here, and can deduce the resulting op-lax functor is going to be a normal one, *i.e.*, identity preserving.)

We have to check that, thus defined, $F_{A_0,A_1} : \mathcal{A}(A_0, A_1) \to \mathcal{B}(FA_0, FA_1)$ *is* a functor. We suppose that we have composable two cells

$$\begin{tikzcd} A_0 \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] \arrow[r, phantom, shift left=3, "\Uparrow\beta"] \arrow[r, phantom, shift right=3, "\Uparrow\alpha"] & A_1 \end{tikzcd},$$

and have to compare $F(\beta\alpha)$ with $F(\beta)F(\alpha)$. To do this, we construct a 3-simplex in $Ner(\mathcal{A})$ that we will call τ, with faces:

$$\begin{aligned} d_0\tau &= (id_{A_1}, id_{A_1}, id_{A_1}; id) \\ d_1\tau &= (id_{A_1}, s(\alpha), t(\alpha); \alpha) \\ d_2\tau &= (id_{A_1}, s(\alpha), t(\beta); \beta\alpha) \\ d_3\tau &= (id_{A_1}, s(\beta), t(\beta); \beta) \end{aligned}$$

which, thus, fit together, diagrammatically, as:

odd numbered faces

$$\begin{tikzcd}
1 \arrow[r, "="] & 2 \arrow[d, "="] \\
0 \arrow[u, "t(\beta)"] \arrow[ur] \arrow[r, "s(\alpha)"'] \arrow[ur, phantom, shift left=4ex, "\Nwarrow\beta"] \arrow[ur, phantom, shift right=4ex, "\Uparrow\alpha"] & 3
\end{tikzcd}$$

even numbered faces:

$$\begin{tikzcd}
1 \arrow[r, "="] \arrow[dr, "="] & 2 \arrow[d, "="] \\
0 \arrow[u, "t(\beta)"] \arrow[r, "s(\alpha)"'] \arrow[ur, phantom, "\beta\alpha\Uparrow"] & 3
\end{tikzcd}$$

As $Ner(\mathscr{A})$ is 3-coskeletal, (or, alternatively, because $\mathscr{A}$ has no non-trivial 3-cells!), this determines a 3-simplex, τ, as promised. Now we map this across to $Ner(\mathscr{B})$ and we get

$$F(\beta\alpha) = F(\beta)F(\alpha),$$

as expected. In other words, F_{A_0,A_1} is a functor.

The obvious question to ask now is whether or not $Ner(\mathscr{F})$ gives us back f. The way $\mathscr{F}$ was constructed on objects and at the object level of each F_{A_0,A_1} gives back f_0 and f_1 fairly obviously, so the crucial examination will be in dimension 2, '3-coskeletal-ness' handling higher dimensions.

Suppose $\sigma = (\sigma(12), \sigma(02), \sigma(01); \alpha)$ is in $Ner(\mathscr{A})$. Consider the 3-simplex, that we will denote by τ, having faces

$$\begin{aligned}
d_0\tau &= (\sigma(12), \sigma(12)\sigma(01), \sigma(01); id) \\
d_1\tau &= (id, \sigma(02), \sigma(12)\sigma(01); \alpha) \\
d_2\tau &= s_1 d_0\sigma = (\sigma(12), \sigma(12), id; id) \\
d_3\tau &= (\sigma(12), \sigma(02), \sigma(01); \alpha) = \sigma,
\end{aligned}$$

(do **check that this is a 3-simplex of** $Ner(\mathscr{A})$). Map it over to $Ner(\mathscr{B})$ using f. The resulting $f(\tau)$ has

$$\begin{aligned}
d_0 f\tau &= (f_1(\sigma(12)), f_1(\sigma(12)\sigma(01)), f_1(\sigma(01)); c) \\
d_1 f\tau &= (id, f_1(\sigma(02)), f_1(\sigma(12)\sigma(01)); F(\alpha)) \\
d_2 f\tau &= s_1 d_0 f(\sigma) = (f_1(\sigma(12)), f_1(\sigma(12)), id; id) \\
d_3 f\tau &= (f_1(\sigma(12)), f_1(\sigma(02)), f_1(\sigma(01)); F(\alpha))\alpha) = f_2(\sigma).
\end{aligned}$$

Here the first use of $F(\alpha)$, as the 2-cell of $d_1 f(\tau)$, is 'by definition', whilst its occurrence as the 2-cell of $d_3 f\tau$ is deduction from the fact that $f(\tau)$ is a 3-simplex of $Ner(\mathscr{B})$. We have proved (bar invoking the 3-skeletal nature of the nerves, so as to complete the final check) that

Proposition 120 *Given any simplicial map $f : Ner(\mathcal{A}) \to Ner(\mathcal{B})$, there is a normal op-lax functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ for which $Ner(\mathcal{F}) = f$.* ■

In fact, as the data for $\mathcal{F}$ is uniquely determined by that for f, and conversely, we have the more detailed statement:

Proposition 121 *The nerve construction gives a full and faithful functor*

$$Ner : 2{-}Cat_{op-lax} \to \mathcal{S}.$$

■

This only addresses the basic level of information. In $\mathcal{S}$, we have a lot of extra 'layers' of structure, homotopies, homotopies between homotopies, etc., as $\mathcal{S}$ is an $\mathcal{S}$-enriched category. The category $2-Cat$ is also $\mathcal{S}$-enriched, as we have been using for some pages now, so what about $2{-}Cat_{op-lax}$? Are there analogues of natural transformations here, as there certainly are in $2{-}Cat$ itself? What are those analogues in this op-lax context? Do they behave nicely with respect to this nerve construction? (Recall that with Cat, natural transformations correspond to homotopies under Ner, so that seems a good question to ask in this wider context.)

11.5.4 Lax, oplax and pseudo-natural transformations

Before continuing with the examples, we will need a little of the basic terminology of 2-category theory and will take the opportunity to sketch out more than is strictly needed at this point, so as not the scatter the material around all over the place.

We first need a definition of a (normal) lax transformation suitable for this setting. (We adapt this from Blanco, Bullejos and Faro, [34], as their treatment is explicitly linked to cohomological applications[7]. The reader, who also wants a more directly categorical treatment, can find an accessible on in Borceux, [37], chapter 7.)

Definition: Given two normal op-lax functors, $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{A} \to \mathcal{B}$, with $\mathcal{F}_i = (F_i, c_i)$ for $i = 1, 2$, a *oplax transformation*, or *oplax natural transformation*, from $\mathcal{F}_1$ to $\mathcal{F}_2$ is a pair, $\alpha = (\alpha, \tau)$, where

(i) α assigns to each object A of $\mathcal{A}$, an arrow,

$$\alpha(A) : F_1A \to F_2A,$$

in $\mathcal{B}$;
and

(ii) τ assigns to each pair of objects, (A_0, A_1) of $\mathcal{A}$, a natural transformation between functors from $\mathcal{A}(A_0, A_1)$ to $\mathcal{B}(F_1A_0, F_2A_1)$, whose value at a 1-cell, $f : A_0 \to A_1$, (which is, thus, an object of the category $\mathcal{A}(A_0, A_1)$), is a 2-cell, τ_f, in $\mathcal{B}$,

$$\tau_f : \alpha(A_1)\sharp_0 F_1(f) \Rightarrow F_2(f)\sharp_0\alpha(A_0),$$

(so the diagram

$$\begin{tikzcd}
F_1(A_0) \arrow[r, "\alpha(A_0)"] \arrow[d, "F_1(f)"'] & F_2(A_0) \arrow[d, "F_2(f)"] \\
F_1(A_1) \arrow[r, "\alpha(A_1)"'] \arrow[ur, phantom, "\tau_f \Nearrow"] & F_2(A_1)
\end{tikzcd}$$

[7] Beware they seem to say 'lax' where the usual terminology would be 'op-lax'.

is filled by τ_f) such that, if $\eta : f \Rightarrow g$ is an arrow in $\mathcal{A}(A_0, A_1)$,

$$(F_2(\eta)\sharp_0\alpha(A_0))\sharp_1\tau_f = \tau_g\sharp_1(\alpha(A_1)\sharp_0 F_1(\eta)),$$

(corresponding to a diagram of the form

$$\begin{tikzcd}
\cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "F_1(\eta)\Uparrow"] \arrow[d, "\alpha(A_0)"'] & \cdot \arrow[d, "\alpha(A_1)"] \\
\cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "F_2(\eta)\Uparrow"] & \cdot
\end{tikzcd}$$

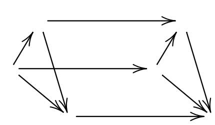
the two sides of the equation being the base and the front sides, and the top and the back, respectively).

These data are to satisfy:

1. $\tau_{1_A} = id_{\alpha(A)}$ (a normalisation condition);

and

2. coherence with the structure maps, c_i, of $\mathcal{F}_i$, for $i = 1, 2$. (This is specified by a prismatic diagram: for given $A_0 \xrightarrow{f} A_1 \xrightarrow{g} A_2$, we get something like



with $c_{1;f,g}$ and $c_{2;f,g}$ on th left and right ends respectively and τ_f, τ_g and τ_{gf} on the three rectangular faces. **You are left to label the diagram yourself and thus to represent this equationally** if you wish, or need, to.)

It is often convenient, since 'op-lax natural transformation' is a bit of a mouthful, to called such a thing simply a *deformation*, (see the use in [71], for instance). We note that the lax form of this reverses the direction of τ_f. We will use *deformation* in all three possible settings. Leaving it up to the reader to sort out which is being used in any particular context. (Sometimes a different convention may be used and unfortunately on meeting the terminology, it is prudent to check back which definition is being called by which term[8]!)

These lax natural transformations compose in a fairly obvious way, using a simple composition on the α_A-parts, and a composition of the τ_f-parts obtained by juxtaposing the resulting squares and 2-cells. This leads to a category, $OpLax(\mathcal{A}, \mathcal{B})$, of normal op-lax functors from $\mathcal{A}$ to $\mathcal{B}$, and normal lax transformations between them. This gives:

Proposition 122 *From the category $2{-}Cat_{op-lax}$, and on further enriching with lax transformations, we get a 2-category.* ∎

[8]In what follows I have tried to be consistent, but do not guarantee that I have succeded!

The details should be more or less clear to you, so are **left to you to complete.**

Remarks about 'pseudo', 'lax' and the direction of τ:

(i) We obtained a 2-category, $OpLax(\mathcal{A}, \mathcal{B})$ above, but could just as well have looked at the 2-categories, $Lax(\mathcal{A}, \mathcal{B})$ or, specialising to the 'pseudo-case', $Pseudo(\mathcal{A}, \mathcal{B})$.

(ii) There is a choice that is made when defining lax natural transformation above. The natural transformation, τ_f, 'measures' the extent to which the naturality square, determined by the αs, $F_1(f)$ and $F_2(f)$, does not commute, but why did it go *from* $\alpha(A_1)F_1(f)$ *to* $F_2(f)\alpha(A_2)$, and not the other way around? The direction is a 'convention'. It is the 'default choice' and why that choice was made is probably 'lost in time'! The opposite choice works just as well, but quite often in the sort of examples we will be considering, the choice is almost completely immaterial as the τ_f are all invertible. This happens when $\mathcal{B}$ is a 2-groupoid, rather than just a 2-category, and we will see examples in which that is the case shortly.

(iii) If one takes the definition and strengthens it by *requiring* that each τ_f be invertible, then we get a version of the definition of a *normalised pseudo-natural transformation*. The case of this where $\mathcal{A}$ is locally discrete (*i.e.*, is just a category) is considered in Borceux and Janelidze, [39]. Of course, if $\mathcal{B}$ is a 2-groupoid, every deformation will be a pseudo-natural transformation, however it is still *important to have a direction* on the 2-cells, even though they are all invertible.

11.5.5 Modifications between pseudo-natural transformations

Although slightly to the side of our main thread, here seems a good place to mention the next level of structure, and to warn that all is not simply a question of following the pattern in then lowest dimensions when going from categories to 2-categories and then .... We will state, and later use, the ideas mainly in the 'pseudo' case, and so review the conventions we are trying to use consistently! It should be noted that the corresponding case in which 'pseudo' is replaced by 'lax' is quite easy to give. It can be found in Borceux, [37], section 7.5.

Remember a pseudo-functor $\mathcal{F} = (F, c) : \mathcal{A} \to \mathcal{B}$, between 2-categories is defined, here on page 531, via the notion of lax functor. The F is an assignment on objects, so given A in $Ob(\mathcal{A})$, we get $F(A)$ in $\mathcal{B}$, together with an assignment of functors to pairs, (A_0, A_1), of objects:

$$F_{A_0,A_1} : \mathcal{A}(A_0, A_1) \to \mathcal{B}(F(A_0), F(A_1)),$$

and then to each composable pair (f, g), the second item, c, assigns a 2-cell

$$c_{f,g} : F(g)\sharp_0 F(f) \Rightarrow F(gf)$$

and a normalisation 2-cell: $c_A : id_{F(A)} \Rightarrow F(id_A)$. That gives the notion of 'lax functor' and 'pseudo functor' assumes just the extra condition that the 2-cells above be invertible, so are isomorphisms in the relevant $\mathcal{B}(F(A_0), F(A_1))$.

We also recall, from our earlier discussion[9] that Grothendieck fibrations / fibred categories over a category, $\mathcal{A}$, correspond to pseudo-functors from that category to Cat. The case, therefore, in which the 2-category, $\mathcal{A}$, here, is 'locally discrete' is important, and for the following we will simplify the exposition by restricting to that case, (but will leave $\mathcal{B}$ as being a general 2-category).

[9] starting in section 9.1.3.

Consider two pseudo-functors, $F_1, F_2 : \mathcal{A} \to \mathcal{B}$, from a *category*, $\mathcal{A}$, (*i.e.*, a 'locally discrete 2-category') to a 2-category, $\mathcal{B}$, together with two pseudo-natural transformations, α and $\alpha' : F_1 \Rightarrow F_2$. (As before α will be shorthand for (α, τ), similarly for α'.)

Definition: A *modification*, $\theta : \alpha \Rrightarrow \alpha'$, from α to α' specifies, for each object, A, of $\mathcal{A}$, a 2-cell,

$$\theta_A : \alpha(A) \Rightarrow \alpha'(A).$$

These two cells must satisfy a coherence condition, namely that, given an arrow / 1-cell, $f : A_0 \to A_1$,

$$\tau'_f \sharp_1 (\theta_{A_1} \sharp_0 F(f)) = (F_2(f) \sharp_0 \theta_{A_0}) \sharp_1 \alpha'(A_0).$$

We will check that this makes sense, breaking up the two composites both algebraically and as two composite 2-cells in a cylindrical diagram. The left hand side is

$$\alpha(A_1)\sharp_0 F_1(f) \xRightarrow{\theta_{A_1}\sharp_0 F_1(f)} \alpha'(A_1)\sharp_0 F_1(f) \xRightarrow{\tau'(f)} F_2(f)\sharp_0 \alpha'(A_0),$$

whilst the right hand side is

$$\alpha(A_1)\sharp_0 F_1(f) \xRightarrow{\tau_f} F_2(f)\sharp_0 \alpha(A_0) \xRightarrow{F_2(f)\sharp_0\theta_{A_0}} F_2(f)\sharp_0 \alpha'(A_0).$$

These correspond to the two parts of a cylinder, much as before:

$$\begin{tikzcd}
A_0 \arrow[d, "f"'] & F_1(A_0) \arrow[r, bend left, "\alpha'(A_0)"] \arrow[d, "F_1(f)"'] \arrow[dr, phantom, "\tau'_f \Nearrow"] & F_2(A_0) \arrow[d, "F_2(f)"] & \equiv & F_1(A_0) \arrow[r, bend left, "\alpha'(A_0)"] \arrow[r, bend right, "\alpha(A_0)"'] \arrow[r, phantom, "\theta_{A_0}\Uparrow"] \arrow[d, "F_1(f)"'] \arrow[dr, phantom, "\tau_f \Nearrow"] & {} \arrow[d, "F_2(f)"] \\
A_1 & F_1(A_1) \arrow[r, bend left, "\alpha'(A_1)"] \arrow[r, bend right, "\alpha(A_1)"'] \arrow[r, phantom, "\theta_{A_1}\Uparrow"] & F_2(A_1) & & F_1(A_1) \arrow[r, bend right, "\alpha(A_1)"'] & F_2(A_1)
\end{tikzcd}$$

The more general case in which $\mathcal{A}$ is a 2-category is given in Borceux's book, [37].

That these definitions give more-or-less what we might expect is amply demonstrated by the following results. Here we have assumed the reader has access to, or can discover, a definition of a (strict) 3-category[10]. The proofs are routine checking and as we will not have much need for the details, they will be omitted[11].

Proposition 123 *There is a 3-category structure given by the following data:*

- *the objects are (strict) 2-categories;*
- *the 1-arrows are pseudo-functors;*

[10] One is given in [37] in case you have difficulty finding one that is explicit enough.
[11] *i.e.*, **left to the reader.**

- *the 2-arrows are pseudo-natural transformations between pseudo-functors;*
- *the 3-arrows are modifications.* ■

Similarly one obtains the analogous result with 'pseudo' replaced by 'lax'.

11.5.6 Somewhat of an aside on lax, and pseudo-limits and colimits

We have sketched out various ways of looking at 'weak 2-limits' of various types, and, in a bit more detail, the corresponding colimits, at several instances earlier in these pages. In most of these instances, the treatment we gave was influenced by the context, e.g. pseudo-functors, pseudo-colimits and the Grothendieck construction, arose from discussions of fibred categories, prestacks and stacks. Naturally, these intertwining threads can make it difficult to extract the key points from some points of view as the discussion tends to get a bit scattered through the text. It, therefore, seems worthwhile to collect up some of the ideas on lax limits, pseudo-limits, etc., in a very short summary section giving a 'context free' categorical summary of some of the main ideas. This is also for convenience as it groups some definitions, and results together in one section, even though closely related definitions and results occur elsewhere in these pages[12].

Suppose, therefore, that we have two 2-categories, strict ones for the moment, $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{A}$ small. For every objects, B of $\mathcal{B}$, there is a 2-functor variously written $cons_B$ or Δ_B from $\mathcal{A}$ to $\mathcal{B}$, which takes the constant value, B. This is a strict 2-functor, so can also be considered as a pseudo-functor or a lax-functor, with the coherence transformations just being the relevant identities.

If $F : \mathcal{A} \to \mathcal{B}$ is a lax-functor there is an 'obvious' way of categorifying the 1-categorical notion of cone[13] namely as follows:

Definition: A *lax cone* on F with (start) vertex, B, is a lax natural transformation, $cons_B \Rightarrow F$.

As we now have a 2-category of lax functors from $\mathcal{A}$ to $\mathcal{B}$, we have that these lax cones form a category, $\mathsf{Lax}{-}\mathsf{Cone}(B, F)$, or in detail

$$\mathsf{Lax}{-}\mathsf{Cone}(B, F) := \mathsf{Lax}(\mathcal{A}, \mathcal{B})(cons_B, F),$$

so the morphisms in $\mathsf{Lax}{-}\mathsf{Cone}(B, F)$ are the (lax) modifications. In the same way:

Definition: A *pseudo-cone* on F with (start) vertex, B, is a pseudo-natural transformation, $cons_B \Rightarrow F$.

We will write $\mathsf{Ps}{-}\mathsf{Cone}(B, F)$ for the category of such pseudo-cones and modifications between them.

Remarks: (i) Clearly we can 'dualise' to look at $\mathsf{Lax}{-}\mathsf{Cocone}(F, B)$, defined as $\mathsf{Lax}(\mathcal{A}, \mathcal{B})(F, cons_B)$, and $\mathsf{Ps}{-}\mathsf{Cocone}(F, B)$, the corresponding pseudo-cocone with base F and final vertex, B. It is fairly

[12] Probably the most accessible source for this material in this form is in Borceux's Handbook, [37], Volume 1.

[13] We will discuss 'cones' here partially for a bit of a change. Earlier, in section 9.4.1, we mainly concentrated on *cocones*.

clear, given the earlier description of these concepts, that this categorical viewpoint gives an equivalent object in both cases. We will shortly look at cones in this way from which discussion, the transition to the cocone form is easy.

(ii) It is clear that $\mathsf{Lax}-\mathsf{Cone}(F,B)$ varies functorially with B. Its variation with respect to F is a bit more difficult to give at the level of the explicit structure, but works well as well.

It is now more or less evident, with one slight caveat, what the definition of a lax limit should be. namely a representing object for the lax cones. More precisely:

Definition: A *(conical) lax limit* of a lax functor, $F : \mathscr{A} \to \mathscr{B}$, between 2-categories, is a pair (L, π), where L is an object of $\mathscr{B}$ and $\pi : cons_L \Rightarrow F$ is a lax cone such that the functor,

$$\pi_* : \mathscr{B}(B, L) \to \mathsf{Lax}-\mathsf{Cone}(B, F),$$

given by post-composition by π, is an isomorphism of categories, naturally in B.

Of course, similarly we have:

Definition: A *(conical) pseudo-limit* of a lax functor, $F : \mathscr{A} \to \mathscr{B}$, between 2-categories, is a pair $(L, \underline{\pi})$, where L is an object of $\mathscr{B}$ and $\underline{\pi} : cons_L \Rightarrow F$ is a pseudo-cone such that the functor,

$$\pi_* : \mathscr{B}(B, L) \to \mathsf{Ps}-\mathsf{Cone}(B, F),$$

given by post-composition by $\underline{\pi}$, is an isomorphism of categories, naturally in B.

Remarks: (i) The level of 'categorification' in these definitions is a bit uneven. The definitions use 'isomorphism' rather that 'equivalence'. The latter would fit better with the general process of 'maximal categorification' by replacing sets by categories , isomorphism and equality by equivalence, and so on. Later we will define weighted 2-limits in which at the corresponding point 'equivalence' *is* used[14].

(ii) We used the term 'conical' here as the generally more versatile weighted or indexed notion will be introduced later on. Our use of lax and pseudo-limits or colimits will often be via end or coend formulae and, with a 'weighted form' used even when a conical form could be used. The conical form, however, is easier on the intuition, so is worth 'taking apart' further.

Starting on that task, and given $F : \mathscr{A} \to \mathscr{B}$, but for simplicity to start with assume that f is strict as a 2-functor. If the lax limit exists then we have an object, L, and a $\underline{\pi} : cons_L \Rightarrow F$. Looking back at lax natural transformations, this $\underline{\pi} = (\underline{\pi}_{Obj}, \underline{\pi}_{Arr})$ will be given by a family,

$$\underline{\pi}_{Obj} = \{\pi_A : L \to FA \mid A \in Ob(\mathscr{A}\},$$

of morphisms in $\mathscr{B}$, together with

$$\underline{\pi}_{Arr} = \{\pi_f \mid f : a \to b \text{ in } \mathscr{A}, \pi_f : F(f)\sharp_0 \pi_A \Rightarrow \pi_{A'}\}$$

[14]A discussion of the various types of (weak) 2-limit is to be found in the nLab, [221], on the pages dealing with 2-limits.

again in $\mathscr{B}$. We can visualise the condition[15] as

$$\begin{tikzcd}
 & FA \arrow[dd, "F(f)"] \\
L \arrow[ur, "\pi_A"] \arrow[dr, "\pi_{A'}"'] \arrow[r, phantom, "\Downarrow \pi_f"] & \\
 & F(A')
\end{tikzcd}$$

These are to be such that $\pi_{id_A} = id_{\pi_A}$ for each A and, if $A_0 \xrightarrow{f} A_1 \xrightarrow{g} A_2$ in $\mathscr{A}$, then

$$\pi_g \sharp_1 (F(g) \sharp_0 \pi_f) = \pi_{gf}$$

the left hand side of which looks like

$$\begin{tikzcd}[row sep=large, column sep=large]
{} \arrow[r, "F(f)"] & {} \arrow[d, "F(g)"] \\
{} \arrow[u, "\pi_{A_0}"] \arrow[ur, "\pi_{A_1}"] \arrow[r, "\pi_{A_2}"'] \arrow[u, phantom, shift right=4ex, "\Downarrow \pi_f"] & {} \arrow[u, phantom, shift right=4ex, "\Downarrow \pi_g"]
\end{tikzcd}$$

with the $\sharp_1$-composite being π_{gf}.

To see this more ‘cohomologically’, or from the viewpoint of homotopy coherence, we now look what happens when F is not strict. In that case, there would have been an additional 2-cell, $c_{f,g}$, linking $F(g)\sharp_0 F(f)$ with $F(gf)$. This fits into a 3-simplex and that hinted-at connection with cocycle conditions and homotopy coherence becomes quite striking[16]:

$$\begin{tikzcd}[row sep=large, column sep=large]
{} \arrow[r, "F(f)"] & {} \arrow[d, "F(g)"] \\
{} \arrow[u, "\pi_{A_0}"] \arrow[ur, "\pi_{A_1}"] \arrow[r, "\pi_{A_2}"'] \arrow[u, phantom, shift right=4ex, "\Downarrow \pi_f"] & {} \arrow[u, phantom, shift right=4ex, "\Downarrow \pi_g"]
\end{tikzcd}
=
\begin{tikzcd}[row sep=large, column sep=large]
{} \arrow[r, "F(f)"] \arrow[dr, "F(gf)"] & {} \arrow[d, "F(g)"] \arrow[d, phantom, shift right=4ex, "\Swarrow c_{f,g}"] \\
{} \arrow[u, "\pi_{A_0}"] \arrow[r, "\pi_{A_2}"'] \arrow[u, phantom, shift left=-4ex, "\Downarrow \pi_{gf}"] & {}
\end{tikzcd}$$

(The odd indexed faces (d_3 and d_1) are on the left, the even ones on the right.) In case the reader needs reminding, these, of course, fit together to form a square, as we have seen before, and will see again shortly. Explicitly we get:

$$\begin{tikzcd}
F(gf)\sharp_0\pi_{A_0} \arrow[r, Rightarrow, "\pi_{gf}"] & \pi_{A_2} \\
F(g)\sharp_0 F(f)\sharp_0\pi_{A_0} \arrow[u, Rightarrow, "c_{f,g}\sharp_0\pi_{A_0}"] \arrow[r, Rightarrow, "F(g)\sharp_0\pi_{gf}"'] \arrow[ur, phantom, "="] & F(g)\sharp_0\pi_{A_1} \arrow[u, Rightarrow, "\pi_g"']
\end{tikzcd}$$

---

[15] N.B. Our previous discussion in section 11.5.4 related to op-lax transformations so there the 2-cell went in the opposite direction.

[16] Compare this with ideas on the nerve of a 2-category (page 530) and earlier, on the $\mathcal{S}$-category, $S[3]$, on page 502 and homotopy coherent diagrams indexed by [3], page 511, as well as numerous cocycle conditions of various types.

in the *category*, $\mathscr{B}(L, F(A_2))$. The two composites must be equal since that category has no 2-cells! Equivalently, $\mathscr{B}$ is a 2-category, so has no non-identity 3-cells. (These last two statements do not quite make sense without some extra comment. Something like: we think of any category as a locally discrete 2-category, and any strict 2-category as a 3-category whose only 3-cells are identity arrows. This may seem a bit strange but is crucial for later developments.) We have used this argument before with cocycle conditions and will have occasion to use it again shortly.

The universal property of (L, π) then says that, given another lax cone, with vertex, B, $\underline{\sigma} = (\underline{\sigma}_{Obj}, \underline{\sigma}_{Arr})$, with $\sigma_A : B \to F(A)$, $\sigma_f : F(f)\sharp_0\sigma_A \Rightarrow \sigma_{A'}$ with analogous properties, there is a *unique* $b : B \to L$ such that $\sigma_A = \pi_A\sharp_0 b$ and $\sigma_f = \pi_f\sharp_0 b$. This gives a condition on objects of $\mathscr{B}(B, L)$, but there will also be a condition on morphisms, so suppose we have $\underline{\sigma}' = (\underline{\sigma}'_{Obj}, \underline{\sigma}'_{Arr})$, another such lax cone, giving $b' : B \to L$. Suppose further that we have a family of 2-cells, $\underline{\theta}_{Obj} = \{\theta_A : \sigma_A \Rightarrow \sigma'_A \mid A \in Ob(\mathscr{A})\}$, such that, if $f, g : A \to A'$ in $\mathscr{A}$ with $\alpha : f \Rightarrow g$, we have a relation expressing that $\underline{\theta} : \underline{\sigma} \Rightarrow \underline{\sigma}'$ is a modification, then there must be a unique 2-cell, $\beta : b \Rightarrow b'$, which is sent by π_* to $\underline{\theta}$.

We will omit the explicit relation here leaving it **up to the reader to work out**[17]. It can be instructive to see how this results from an inverse, θ, to the functor, π_*, so, for instance, given $\underline{\sigma}$ above, b would be $\theta(\underline{\sigma})$. Once the diagrams and the equations, that result from $\theta\pi_*$ and $\pi_*\theta$ being the respective identities, are made ex[licit, it is then feasible to try to modify those conditions to investigate what would be the consequence of relaxing the overall picture to one in which π_* is asked to be an equivalence of categories with θ being a quasi-inverse. The diagrams that result make it clear that there are a lot of simplicial insights to be had here, and also strong connections to the use of cocycle conditions. Some of these connections will become apparent later on when homotopy limits have been more fully explored.

Another remark worth making is that the 'isomorphism' form of the definition allows for a fairly simple analysis of in which 2-categories do lax / op-lax/ pseudo-limits exist. Explicit results on this can be found via the references in the nLab, [221], but also these constructions give *presentations* of the limiting objects in each case. We will shortly see how lax, etc., *ends and coends* and *weighted limits* allow a deeper understanding of these constructions.

11.5.7 Back to the nerve

The above hints at strong connections between 2-category theory and homotopy theory via phenomena linked to nerve-like constructions, so we go 'back to the nerve' to look at such constructions again.

As we have said earlier, functors, which have a natural transformation between them, induce homotopic simplicial maps under the nerve functor. The natural transformation data gives the data for the homotopy. We want to see if anything similar happens with op-lax, lax or pseudo-functors and the corresponding notions of 'deformations'.

By way of a 'warm-up', we will first look at the 1-categorical case. Suppose $\alpha : F_0 \Rightarrow F_1 : \mathbb{A} \to \mathbb{B}$ is a natural transformation between functors from $\mathbb{A}$ to $\mathbb{B}$, then we have simplicial maps, $f_i = Ner(F_i) : Ner(\mathbb{A}) \to Ner(\mathbb{B})$, and want to construct a homotopy,

$$h : Ner(\mathbb{A}) \times \Delta[1] \to Ner(\mathbb{B}) \quad h : f_0 \simeq f_1,$$

[17]... or to look up, for instance, in Borceux, [37], p. 301.

(using α). Of course, α gives us a family $\{\alpha_A\}$ of 1-simplices of $Ner(\mathbb{B})$, so we can use that to define the map, h, that we want on $\langle a_1\rangle \times \Delta[1]$, for a 1-simplex $(a_1 : A_0 \to A_1)$ of $Ner(\mathbb{A})$, by the diagram:

$$\begin{tikzcd}
{} \arrow[r, "f_1(a_1)"] & {} \\
{} \arrow[u, "\alpha_{A_0}"] \arrow[ur] \arrow[r, "f_0(a_1)"'] & {} \arrow[u, "\alpha_{A_1}"']
\end{tikzcd}$$

which commutes (since α is natural), so causes no difficulty on defining the diagonal. For an n-simplex, $\sigma = (A_0 \overset{a_1}{\to} A_1 \to \ldots \overset{a_n}{\to} A_n)$ in $Ner(\mathbb{A})_n$, we just repeat that recipe on each edge, getting a commutative prism, and defining h on $\sigma \times \Delta[1]$. Clearly this works, although we have left out the detailed formulae.

Now replace $\mathbb{A}$ and $\mathbb{B}$ by two 2-categories, F_0 and F_1 by op-lax functors, and α by an op-lax natural transformation. Much of the construction looks as if it works, with some modification. If we write $\alpha = (\alpha, \tau) : \mathcal{F}_0 = (F_0, c_0) \Rightarrow \mathcal{F}_1 = (F_1, c_1) : \mathcal{A} \to \mathcal{B}$, and then put $f_i = Ner(\mathcal{F}_i)$, we can adapt the diagram for h on $\langle a_1\rangle \times \Delta[1]$ (with the same notation as above) to be

$$\begin{tikzcd}
{} \arrow[r, "f_1(a_1)"] \arrow[dr, phantom, near start, "\Nwarrow \tau_{a_1}"] & {} \arrow[dl, phantom, near end, "="] \\
{} \arrow[u, "\alpha_{A_0}"] \arrow[ur] \arrow[r, "f_0(a_1)"'] & {} \arrow[u, "\alpha_{A_1}"']
\end{tikzcd}$$

With that basic change, it is reasonably routine (*i.e.*, a bit of intuition, plus a lot of checking!) to construct h as a homotopy defined on the 1-skeleton of $Ner(\mathcal{A})$. Given the coskeletal propertes of $Ner(\mathcal{B})$, we have to work out how to give h on $Ner(\mathcal{A})_2$, *i.e.*, on the 'cylindrical' prisms of form $(\sigma(12), \sigma(02), \sigma(01); \sigma(12)) \times \Delta[1]$. (This is **left to you**, but first you might glance - in fact, stare, - at the diagram for naturality with respect to 2-cells and the coherence diagram for condition 2 of the definition of op-lax natural transformation.) Once you have done the work, you will have a proof of the following:

Proposition 124 *(see Blanco, Bullejos, and Faro, [35]) Let $\mathcal{F}_0, \mathcal{F}_1 : \mathcal{A} \to \mathcal{B}$ be two normal op-lax functors between 2-categories. Every deformation, $\alpha : \mathcal{F}_0 \Rightarrow \mathcal{F}_1$, induces a homotopy, $h = Ner(\alpha) : Ner(\mathcal{F}_0) \Rightarrow Ner(\mathcal{F}_1)$.* ■

Note: due to a difference in conventions, the above reference states the direction of h to be reversed.

It is clear that, as the construction of h leads to one of the two 2-cells in each of the above diagrams being an equality, and as not every simplicial homotopy between maps from $Ner(\mathcal{A})$ to $Ner(\mathcal{B})$ would have that form, not all such homotopies can be realised by deformations. However, if we are working with 'pseudo' rather than merely 'lax' situations, for instance, if $\mathcal{B}$ is a 2-groupoid, then, in any such square,

$$\begin{tikzcd}
\cdot \arrow[r, "f_1(a_1)"] & \cdot \\
\cdot \arrow[u, "\alpha_{A_0}"] \arrow[r, "f_0(\alpha_1)"'] \arrow[ur] \arrow[u, phantom, shift left=4ex, "\Nwarrow\tau_1"] & \cdot \arrow[u, "\alpha_{A_1}"'] \arrow[u, phantom, shift right=4ex, "\Swarrow\tau_2"]
\end{tikzcd}$$

we have that τ_2 is an invertible 2-cell, so we can build a new square replacing τ_2 by an identity 2-cell and τ_1 by $\tau_1\tau_2^{-1}$, and still giving a homotopy as needed. This suggests the following result (which we **leave to you to prove more formally**).

Proposition 125 *Suppose $\mathcal{F}_i : \mathcal{A} \to \mathcal{B}$, $i = 0, 1$, are two normal op-lax functors with $\mathcal{B}$ a 2-groupoid, then, if there is a homotopy $h : f_0 \simeq f_1$, where $f_i = Ner(\mathcal{F})_i$, then there is a deformation, α, from $\mathcal{F}_0$ to $\mathcal{F}_1$, and the resulting homotopy, $Ner(\alpha)$, is homotopic to the given h.* ∎

'The reader' is left to look at the changes, if any, that would need to be made if 'op-lax' was replaced by 'lax'.

11.5.8 Weak actions of groups

This example is partially a continuation of the general one in the previous section, but, as it is one we have considered before, and is very central to our cohomological theme, it seems a good thing to start a new section for it.

Earlier, in section 6.1, we looked at the way that, in an extension of groups,

$$\mathcal{E}: \quad 1 \to K \to E \xrightarrow{p} G \to 1,$$

a section of p gave a 'lax' action' of G on K. At that point in these notes, we had not a sufficient knowledge of 'lax' or 'pseudo' ideas, nor the concepts and terminology necessary for a fuller treatment. We have now!

We start by recalling (see page 17 for starters) a little of the terminology and notation and the fundamental ideas of actions in the algebraic context. We have a group, G, and so a single object groupoid, $G[1]$. If we have a functor, $\mathcal{K}$, from $G[1]$ to $Grps$, then the functor picks out a group, $K = \mathcal{K}(*)$, where $ObG[1] = \{*\}$, and a mapping

$$\mathcal{K}_{*,*} : G[1](*, *) \to Grps(K, K) = End(K),$$

where $End(K)$ is the monoid of endomorphisms of K. The domain here is, of course, just G and the image will be within the submonoid of invertible endomorphisms, *i.e.*, within $Aut(K)$, the group of automorphisms of K, so we get one of the usual formulations of an action of G on K, namely as a homomorphism from the group G to $Aut(K)$.

Remark: If we start with G being a groupoid, then it already has a set, G_0, of objects, (and we do not need to make G into a groupoid!), then a functor $\mathcal{K} : G \to Grps$ will pick out a *family* $\{K(x) \mid x \in G_0\}$ of groups, and, if $G(x, y)$ is non-empty, morphisms between $K(x)$ and $K(y)$. (Remember G is not necessarily a connected groupoid.) Our discussion for groups extends without problem to groupoids. (A good reference for this is Blanco, Bullejos and Faro, [35], and that has been used as one source for the treatment here.)

We have seen, page 453, that natural transformations between such functors correspond to conjugation by elements of K.

Given our interest in lax and pseudo functors and natural transformations, it is natural to look at such things in this 'action' context and to see if they correspond to anything 'well known'.

We will do this somewhat pedantically, also repeating ideas that were met earlier. We treat G, firstly, as the groupoid, $G[1]$, as before, and then as a (2-)discrete 2-category, which will also be written $G[1]$. We look at *Grps* as a subcategory of *Grpd* and then enrich *Grpd* using the functor category construction, so

$$Grpd(G, H) = H^G = Func(G, H),$$

so making *Grpd* into a 2-category, denoted Grpd. We also will need it as an $\mathcal{S}$-category via the nerves, $Ner(H^G)$.

All 2-cells in Grpd are invertible, so 'lax', 'op-lax' and 'pseudo' more or less coincide. Now for the 'deconstruction' of a lax functor, $\mathcal{K} = (K, \sigma)$,

$$\mathcal{K} : G[1] \to \mathsf{Grpd}.$$

This will correspond, according to the above definition to assignments:

- As $G[1]$ has just one object, we get a group (or more generally a groupoid), $K = \mathcal{K}(*)$, as with an action;

- For any two objects of $G[1]$ (well that is easy, both must be $*$!), a *functor*

$$\mathcal{K}_{*,*} : G[1](*, *) \to \mathsf{Grpd}(K, K),$$

  where $G[1](*, *) = G$, but take care, here. Since the 2-category, $G[1]$, is a locally discrete 2-category, G is also being thought of as a discrete category, that is a *set*; the vertical composition in the 2-category, *i.e.*, of 2-cells, is necessarily trivial, the *horizontal* composition is the multiplication of the group. This just gives a family, $\{K(g) \mid g \in G\}$, of endomorphisms of K. For convenience, if $g \in G$, $K(g)$ is an endomorphism of K and we may write ${}^gk$ for $K(g)(k)$.

- For any three objects of $G[1]$ (no comment this time!), a natural transformation, σ, between 'functors' from $G[1](*, *) \times G[1](*, *)$ to $\mathsf{Grpd}(K, K)$, whose component on a pair, (g_2, g_1), is a 2-cell

$$\sigma_{(g_2,g_1)} : K(g_2g_1) \Rightarrow K(g_2)K(g_1).$$

  Note that (g_2, g_1) is a composable pair of morphisms in $G[1]$! (As usual, we will want $K(1_G)$ to be the identity endomorphism of K, *i.e.*, for $\mathcal{K}$ to be *normal* and also for $\sigma_{(1,g)} = \sigma_{(g,1)} = 1_K$. As we saw when considering 'auto-equivalences', back in section 9.6.11, such a set-up gives that each $K(g)$ is an automorphism of K, not just an endomorphism.)

The pair, $\mathcal{K} = (K, \sigma)$, must satisfy the coherence rule with the associative law, *i.e.*, if $g_3, g_2, g_1 \in G$

(thus are composable maps in $G[1]!$), the diagram

$$\begin{tikzcd}
K(g_3g_2g_1) \arrow[r, Rightarrow, "\sigma_{(g_3g_2),g_1}"] \arrow[d, Rightarrow, "\sigma_{g_3,(g_2g_1)}"'] & K(g_3g_2)K(g_1) \arrow[d, Rightarrow, "\sigma_{g_3,g_2}K(g_1)"] \\
K(g_3)K(g_2g_1) \arrow[r, Rightarrow, "K(g_3)\sigma_{g_2,g_1}"'] & K(g_3)K(g_2)K(g_1)
\end{tikzcd}$$

commutes.

We could take thus apart further, ..., but will leave that for **you to check up** on, as we have done this all before in various forms and guises. Natural transformations correspond to conjugation (page 453) in this context. Autoequivalences are automorphisms (again page 453) and so on. The coherence rule is a cocycle condition, of course.

This gives us the data for an op-lax functor,

$$\mathcal{K} : G[1] \to \mathsf{Grpd},$$

but, of course, only uses a tiny part of Grpd as it only involves one object, namely K. We have a sub 2-category, determined by K, that we will write $\mathsf{End}(K)$ as it is all the endofunctors of K and the natural transformations between them, with composition as the 'horizontal' operation. Within $\mathsf{End}(K)$, we have $\mathsf{Aut}(K)$ (and, yes, this *is* essentially the same notation as that we saw earlier, in our initial discussion of lax actions in section 6.1, and even earlier, way back in section 2.1.1, except that here $\mathsf{Aut}(K)$ is the 2-group, whilst earlier we used the notation for the corresponding crossed module). This is the sub 2-category of $\mathsf{End}(K)$ whose 1-cells are the automorphisms of K. It is, as we just said, a 2-group.

We thus have that our op-lax functor, $\mathcal{K}$, is 'really' an op-lax functor,

$$\mathcal{K} : G[1] \to \mathsf{Aut}(K),$$

and is also a pseudo-functor, as all 2-cells involved are invertible. (We have that last statement was true throughout our recent discussion, of course, as Grpd has all 2-cells invertible.)

Definition: Given groups, G and K, a *lax action* or *weak action* of G on K is an op-lax functor

$$\mathcal{K} : G[1] \to \mathsf{Aut}(K).$$

We can rewrite the above discussion to get more convenient forms of this.

Proposition 126 *(i) A weak action of G on K assigns, to each $g \in G$, an automorphism ${}^g(-) : K \to K$, and to each pair (g_1, g_2) in $G \times G$, an element $k = k(g_1, g_2)$ in K such that, for any $x \in K$,*

$$k.^{(g_2,g_1)}x = {}^{g_2}({}^{g_1}x).k,$$

*(*i.e.*, the inner automorphism by k is the difference between operation with g_2g_1 on the one hand, and with first g_1 and then g_2 on the other);*
and satisfying : for all $x \in K$ and triples (g_3, g_2, g_1) of elements of G

a) ${}^1x = x$;

b) $k(1,1) = 1$;

c) (cocycle condition)

$$k(g_3, g_2)k((g_3g_2), g_1) = {}^{g_3}k(g_2, g_1)k(g_3, g_2g_1).$$

Conversely any such assignment determines a weak action.

(ii) A weak action of G on K determines, and is determined by, a simplicial mapping

$$\mathsf{k} : Ner(G[1]) \to Ner(\mathsf{Aut}(K)).$$

Proof: (i) is just the result of taking apart the definition, and then interpreting the terms in more elementary language, so ... .

(ii) is just a corollary of our earlier result that Ner is full and faithful. ■

This second part deserves some more comment. The domain of k is the classifying simplicial set of G, that which has been written BG in earlier chapters. (As an aside, we should note that often in earlier chapters, G was a sheaf of groups on some space, or, more generally, a group object in some topos. The corresponding theory of lax and pseudo-functors, lax natural transformations, etc., also applies there with minimal disruption / adaptation. Adapting it to the situation in which G and K are bundles of groups, *i.e.*, bringing in a topology on them *is* somewhat harder, but can be done, as can a smooth 'Lie' theory of these.)

Beware: in our earlier discussion, composition order may have been reversed.

The codomain of k is interesting and raises a question. That nerve is of $\mathsf{Aut}(K)$, the 2-group of automorphisms of K, but that is, of course, the 2-group associated to the crossed module, also denoted $\mathsf{Aut}(K) = (K, Aut(K), \iota)$, that we have used so many times. Replacing $\mathsf{Aut}(K)$ by an arbitrary 2-group, $\mathscr{X}(\mathsf{C})$, corresponding to a crossed module, $\mathsf{C} = (C, P, \partial)$, we now have *two* different classifying space objects associated to it, the nerve of the associated 2-group in this 'lax' interpretation and our earlier one going via the nerve of the simplicial group (so the nerve of one of the structures, the internal groupoid one), followed by using $\bar{W}$, (recall this from sections 6.2.3 and 8.3.2). We will return to a more detailed examination of this very shortly.

Another question that was left over from an earlier chapter, (page 246), was of the details of the statement that a section, s, of the epimorphism,

$$p : E \to G,$$

in our extension

$$\mathcal{E} : \quad 1 \to K \to E \xrightarrow{p} G \to 1,$$

gave a lax action of G on K. (Another useful link at this point is to our discussion of fibred categories, for instance, in section 9.1.3. The themes there interact with some of what we will be seeing here.) This is quite well known and is not that hard to provide in detail, so we will leave **you to do this**, but the above discussion should ease the formalisation process. Given a section $s : G \to E$, **you should construct** a lax action **in detail** either as an explicit op-lax functor, or as a simplicial map, perhaps by adapting earlier discussions and using the monadic resolution approach from section 11.2.3, mixed with more recent comments about the relationship between 2-categories and $\mathcal{S}$-categorical methods. The choice is yours and as usual, approaching it in at least two ways can clarify relationships between the approaches. (The reference mentioned above to Blanco, Bullejos, and Faro, [34], may once again help in this.)

This quite naturally, raises other questions - and again investigation is well worth it, and is **left to you**. If we change from the section, s, to another, we clearly should get a lax natural transformation between the weak actions and hence a homotopy between the corresponding simplicial maps. (Again **you are left** to search for, and give, explicit expressions for these and to link them all together into a description in terms of lax / pseudo functors, etc., the cohomology groupoid that they give, and of the equivalence classes of non-Abelian extensions that we looked at in section 6.1.)

The important thing to note is how the different approaches interact and, in fact, intermesh, as this is very useful when generalising and extending things to higher dimensions and to further 'categorification'.

The end result of this investigation would be a version of the results on extensions of G by K, in terms of the set, $[Ner(G[1]), Ner(\mathsf{Aut}(K))]_*$, of (normalised) homotopy classes of pointed simplicial maps. An interesting idea **to follow up** is to link this all up with observations on 'extensions as bitorsors' (page 329, but take care as the extension there uses different notation), the use of classifying spaces in classifying bitorsors and in particular nerves of $\mathsf{Aut}(K)$, then back to the first discussion of 'lax actions' in section 6.1.

11.5.9 Čech and Vietoris complexes

We earlier[18] met the two ways of encoding the information on a relation, $R \subseteq X \times Y$, between two sets. This was an abstraction of the case of a space, X, together with an open cover, $\mathcal{U} = \{U_y : y \in Y\}$, of X. We recall that on choosing a total order on the two sets, we had two simplicial sets:

(i) the *Čech nerve*, $N(\mathcal{U})$ or $N(X, \mathcal{U})$, of $\mathcal{U}$ in which a typical n-simplex was an (ordered) $(n+1)$-tuple, $\langle U_0, \ldots, U_n\rangle$, of open sets from $\mathcal{U}$ such that there is some $x \in \bigcap U_i$, *i.e.*, $x \in U_i$ for all i;

and

(ii) the *Vietoris nerve*, $V(\mathcal{U})$ or $V(X, \mathcal{U})$, of $\mathcal{U}$ in which a typical n-simplex, where an n-simplex is a $(n+1)$-tuple, $\langle x_0, \ldots, x_n\rangle$, such that there is some $U \in \mathcal{U}$ with $x_i \in U$ for all i.

Both of these encode combinatorial information about X at a certain scale or level of detail, which is being specified by $\mathcal{U}$, but is thus limited by what $\mathcal{U}$ is. If we have another open cover, $\mathcal{V}$, say, then how does $N(\mathcal{V})$ relate to $N(\mathcal{U})$, or, more or less equivalently, $V(\mathcal{V})$ to $V(\mathcal{U})$. (To ensure that the open cover is taken into account in notation, we will write $\langle x_0, \ldots, x_n\rangle_{\mathcal{U}}$ for an n-simplex in $V(\mathcal{U})$. To then compare $V(\mathcal{U})$ and $V(\mathcal{V})$, we take the intersection, $\mathcal{U} \wedge \mathcal{V} := \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$, of $\mathcal{U}$ and $\mathcal{V}$. This is clearly 'finer' than both $\mathcal{U}$ and $\mathcal{V}$. In the Vietoris complex of $\mathcal{U} \wedge \mathcal{V}$ the simplices *are 'smaller'* in some sense. What do 'finer' and 'smaller' mean here? We are not assuming a metric structure but the following gives a useful notion of 'finer than', which is both intuitive in its interpretation and highly usable.

Definition: Given two open covers, $\mathcal{U}$ and $\mathcal{V}$, we say that $\mathcal{V}$ is *finer than* $\mathcal{U}$ if, for each $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ with $V \subseteq U$. We write $\mathcal{V} \leq \mathcal{U}$ when this is the case. We also say $\mathcal{V}$ is a *refinement* of $\mathcal{U}$.

[18] starting on page 131

Proposition 127 *If $\mathcal{V} \leq \mathcal{U}$, then there is a natural simplicial map,*

$$V(\mathcal{V}) \to V(\mathcal{U}),$$

given by sending $\langle x_0, \ldots, x_n\rangle_{\mathcal{V}}$ to the corresponding $\langle x_0, \ldots, x_n\rangle_{\mathcal{U}}$.

Proof: This map exists at the level of simplicial complexes, and, for any total order on X, we get a map of simplicial sets. Clearly if $(x_0, \ldots, x_n)$ forms a simplex in $V(\mathcal{V})$, then it also forms one in $V(\mathcal{U})$ as is easy to check. ∎

If we start out with two arbitrary open covers, $\mathcal{U}$ and $\mathcal{V}$, then $\mathcal{U} \wedge \mathcal{V} \leq \mathcal{U}$ and $\mathcal{U} \wedge \mathcal{V} \leq \mathcal{U}$, so there are two simplicial maps from $V(\mathcal{U} \wedge \mathcal{V})$ to $V(\mathcal{U})$ and to $V(\mathcal{V})$ allowing some comparison of these two to be made.

If we define $Cov_{\leq}(X)$ to be the category associated to the poset of all open covers of X ordered by 'finer that', then we get:

Proposition 128 *For any space, X, there is a functor*

$$|V(X)| : Cov_{\leq}(X) \to Top$$

taking values in the subcategory, Poly, of Top consisting of polyhedra and simplicial maps between them. ∎

Here we have $|V(X)|(\mathcal{U}) = |V(X, \mathcal{U})| = |V(\mathcal{U})|$, in our shortened notation. This functor does not depend, up to isomorphism, on the order chosen on the pointset X.

The disadvantage of the Vietoris complex is its size. The set of vertices of each $V(X, \mathcal{U})$ is the set of points of X, and, as usual, to obtain a smaller, more maniable, complex, we replace it by a homotopically equivalent complex, namely the Čech complex, $N(X, \mathcal{U})$, where the set of vertices is in bijection with the set of open sets in the cover $\mathcal{U}$. Of course, the possibility of doing this is given by Dowker's lemma, see page 131 and following pages. We thus get, on using Proposition 117, page 514:

Corollary 25 *For any space X, there is a homotopy coherent diagram,*

$$|N(X)| : Cov_{\leq}(X) \to Top,$$

taking values in Poly. This h.c. diagram is homotopically equivalent to $|V(X)|$. ∎

This result does not give explicit formulae for the homotopy coherence data, but a variant of classical approach to Čech homology was used by Abdul-Kadir, [1], to give such explicit formulae, so we will quickly review that classical style approach.

The classical treatment of Čech homology used the chain complexes, $C(X, \mathcal{U})$, where $C(X, \mathcal{U})$ if the free Abelian chain complex associated to $N(X, \mathcal{U})$, so of the free simplicial Abelian group, $\mathbb{Z}N(X, \mathcal{U})$. If $\mathcal{V} \leq \mathcal{U}$, there is, in general, no *natural* way of defining

$$N(X, \mathcal{V}) \to N(X, \mathcal{U}),$$

as if $V \in \mathcal{V}$, the refinement relation gives the *existence* of a U in $\mathcal{U}$, and thus a $\langle U\rangle$, which could form a suitable image for the vertex $\langle V\rangle$ within $N(X, \mathcal{U})$. It does *not* give uniqueness, nor does it specify the U concerned. It merely says it exists,

Definition: A *refinement map* from $\mathcal{V}$ to $\mathcal{U}$ is a function, $\varphi = \varphi^{\mathcal{V}}_{\mathcal{U}} : \mathcal{V} \to \mathcal{U}$, such that, for all $V \in \mathcal{V}$, $V \subseteq \varphi(V)$.

If φ is a refinement map and $\langle V_0, \ldots, V_n\rangle \in N(X, \mathcal{V})$, then $\langle \varphi(V_0), \ldots, \varphi(V_n)\rangle \in N(X, \mathcal{U})$, as

$$\bigcap_{1=0}^{n} \varphi(V_i) \supseteq \bigcap_{1=0}^{n} V_i,$$

so has nonempty intersection. We thus have that φ induces a 'refinement'

$$\tilde{\varphi} : N(X, \mathcal{V}) \to N(X, \mathcal{U}),$$

which is 'just' φ at the level of the vertices.

As $\varphi^{\mathcal{V}}_{\mathcal{U}}$ is a specification of a choice, the resulting simplicial map, or, if we pass to the corresponding chain complexes, the resulting chain map,

$$\tilde{\varphi}_* : C(X, \mathcal{V}) \to C(X, \mathcal{U}),$$

will also be dependent on the choice, so that if we have a second refinement map $\psi : \mathcal{V} \to \mathcal{U}$, the resulting simplicial and chain maps *will be different*. Will there be a relationship between $\tilde{\varphi}$ and $\tilde{\psi}$? Given the context the obvious relationship to expect would be that they are homotopic. In fact, we have:

Lemma 75 *If φ and ψ are refinement maps from $\mathcal{V}$ to $\mathcal{U}$, then $\tilde{\varphi}$ and $\tilde{\psi}$ are contiguous*[19] *simplicial maps from $N(\mathcal{V})$ to $N(\mathcal{U})$.*

Proof: We first note that, for any vertex $\langle V\rangle$, of $N(\mathcal{V})$, $\langle \varphi(V), \psi(V)\rangle$ is a 1-simplex as

$$V \subseteq \varphi(V) \cap \psi(V),$$

and is thus non-empty. More generally if $\sigma = \langle V_0, \ldots, V_n\rangle$ is an n-simplex of $N(\mathcal{U})$, then $\varphi(\sigma) \cup \psi(\sigma)$ forms a simplex in $N(\mathcal{U})$, so φ and ψ are contiguous maps. ■

It is worth noting that the language of simplicial complexes, as it does not allow for degeneracies, becomes a bit more awkward here. The set $\varphi(\sigma) \cup \psi(\sigma)$ forms a simplex as that family of sets has non-empty intersection, but we do not know its dimension as there may be repeats within the obvious listing $\{\varphi(V_0), \ldots, \varphi(V_n), \psi(V_0), \ldots, \psi(V_n)\}$. If we work with total orders on the open covers, it is also a bit awkward as the order is not necessarily preserved by the refinement maps, and, in any case, the chosen order on $\{\varphi(V_0), \ldots, \varphi(V_n), \psi(V_0), \ldots, \psi(V_n)\}$ will almost certainly not be the one listed.

One can avoid the awkwardness by using the variant of the nerve construction in which no order is given but a n-simplex, $\langle V_0, \ldots, V_n\rangle$, is then a family with non-empty intersection, but this then produces $(n + 1)!$ copies of each such simplex, one for each ordering. The methods give equivalent results after 'invoking homotopy', but each has its positive and negative sides to their use. We will tend to use an informal combination of them.

It is also easy to see directly that

[19] see page 132

Lemma 76 *$\tilde{\varphi}$ is homotopic to $\tilde{\psi}$.*

Proof: To gauge what is needed and to understand the picture a bit more, we look at a 1-simplex $\sigma = \langle V_0, V_1\rangle$ in $N(\mathcal{V})$ and get a 'square' in $N(\mathcal{V}) \times \Delta[1]$. We want a homotopy,

$$h : N(\mathcal{V}) \times \Delta[1] \to N(\mathcal{U}),$$

which will need to be constructed from the two refinement maps and, at the 1-simplex, σ, will link $\tilde{\varphi}(\sigma)$ and $\tilde{\psi}(\sigma)$. The image of the square has thus to look like

$$\begin{tikzcd}
\psi(V_0) \arrow[r] & \psi(V_1) \\
\varphi(V_0) \arrow[r] \arrow[u] \arrow[ur] & \varphi(V_1) \arrow[u]
\end{tikzcd}$$

A bit of easy calculation[20] shows that the two (non-degenerate) 2-simplices in this picture must be $\langle\varphi(V_0), \varphi(V_1), \psi(V_1)\rangle$ and $\langle\varphi(V_0), \psi(V_0), \psi(V_1)\rangle$, having as a common face, $\langle\varphi(V_0), \psi(V_1)\rangle$. This suggests that a general combinatorial rule[21] for the simplicial homotopy, h, on an n-simplex, $\langle V_0, \ldots, V_n\rangle$, will be as a family, $\{h_j : N(\mathcal{V})_n \to N(\mathcal{U})_{n+1}\}$, where

$$h_j(\langle V_0, \ldots, V_n\rangle) = \langle\varphi(V_0), \ldots, \varphi(V_j), \psi(V_j), \ldots, \psi(V_n)\rangle.$$

Of course, it is easy to check that this works and the result follows. ■

11.6 Two nerves for 2-groups

We suggested in the previous section that we have more or less 'by chance' now got two different ways of defining a nerve-like simplicial set for a 2-group, $\mathcal{X}(\mathsf{C})$, associated to a crossed module, C, and hence of assigning a 'nerve' to a crossed module. Discussion of this will take us right back to the basics of crossed modules and so it warrants a section by itself. This will also allow more easy reference to be made to the key ideas here.

We met, back in section 6.2.3, the classifying 'space' construction, and revisited it in section 8.3.2, which took a crossed module, or its associated 2-group, thought of it as an internal category within the category of groups, constructed the (*internal*) nerve of that (*internal*) category *internally* within *Grps*, so getting a simplicial group, the *simplicial group nerve*, $K(\mathsf{C})$, of C. This was then processed further using $\overline{W}$, to get $\overline{W}(K(\mathsf{C}))$. This was analysed (on page 254) in the slightly more general case when C is a reduced crossed complex. (Take care when reviewing those pages as the $\mathcal{S}$-groupoids are given for the algebraic composition convention.)

We also have the following chain of ideas. A 2-group, $\mathcal{X}(\mathsf{C})$, is a special type of 2-category and any 2-category, as we have just seen, gives an $\mathcal{S}$-category by taking the nerve of each 'hom'. Of course, then the natural thing to do, if we want a nerve, is to take the (homotopy coherent) nerve of that $\mathcal{S}$-category and, again of course, this is the *geometric* nerve of the 2-group. What does *it* look like?

Before we do investigate this more fully, let us see, briefly, why it is important to do so.

[20] using the fact that, in a simplicial complex, simplices are determined by their vertices,

[21] see page 340

The route to a nerve via $\overline{W}$ has important links to simplicial fibre bundle theory; $\overline{W}$ has the Dwyer-Kan 'loop groupoid' functor[22] as a left adjoint and all the mechanisms of twisted Cartesian products, twisting functions, etc., that we looked at in section 6.5 are there for use. The homotopy coherent nerve, on the other hand, opens the way to interpretations of maps as homotopy coherent actions, to links with lax / op-lax / pseudo-category theory, and thus quite directly into the methods of low dimensional non-Abelian cohomology.

We will see that the two nerves are very similar; in fact, they are isomorphic. This suggests many lines of enquiry. Both constructions work for a general $\mathcal{S}$-category, so there are possibilities of links between their extensions to general $\mathcal{S}$-groupoids, or to strict monoidal categories, since they are one object 2-categories. These links have been, in part, investigated in papers by various authors, in particular, Bullejos and Cegarra, [71, 72], Blanco, Bullejos and Faro, [34, 35]. Some of these use, instead of $\overline{W}$, a combination of the nerve on the group structure to get a bisimplicial set, followed by using the diagonal of that 'binerve', a method related to what we saw in section 5.5.1. The $\overline{W}$-construction corresponds to taking the nerve in the 'group direction' followed by using the Artin-Mazur codiagonal, ∇. We will look at this in some detail shortly (starting on page 650). That the resulting constructions are weakly homotopically equivalent follows from the results of Cegarra and Remedios, [78], who prove several results generalising some unpublished work of Zisman.

Back to a detailed look at $Ner(\mathcal{X}(\mathsf{C}))$, we can, of course, just read its details off from our earlier look at $Ner(\mathcal{C})$ for $\mathcal{C}$, a 2-category, together with the description of $\mathcal{X}(\mathsf{C})$ as a 2-category. Because in this sort of calculation, it helps to have eachaspect 'face-up on the table', we will recall $\mathcal{X}(\mathsf{C})$ first, although we have met it many times. (This is mostly important because of the risk of a mix of conventions, for instance, on composition order.)

11.6.1 The 2-category, $\mathcal{X}(\mathsf{C})$

- The 2-category, $\mathcal{X}(\mathsf{C})$, has a single object denoted $*$;
- The set of 1-arrows, $\mathcal{X}(\mathsf{C})(*,*)_0$, is the group, P with $p_1\sharp_0 p_2 = p_1p_2$ as composition and we picture it as
$$* \xrightarrow{p_2} * \xrightarrow{p_1} *,$$
so will use *functional composition order.*
- the set of 2-arrows, $\mathcal{X}(\mathsf{C})(*,*)_1$, is the group $C \rtimes P$. We have that, if $(c,p) \in C \rtimes P$, its source is p and its target is $\partial c.p$. We picture it, in 2-category 'imagery', as
$$\begin{tikzcd} * \arrow[r, bend left, "p"] \arrow[r, bend right, "\partial c.p"'] \arrow[r, phantom, "\Downarrow(c,p)"] & * \end{tikzcd},$$
and have a composition, $\sharp_1$, within the category $\mathcal{X}(\mathsf{C})(*,*)$, given by
$$(c', \partial c.p)\sharp_1(c,p) = (c'c, p).$$

[22] glance back at page 249 if need be

The other composition $\sharp_0$, a 'horizontal' composition, is, as we know, the group multiplication of $C \rtimes P$:

$$(c_2, p_2)\sharp_0(c_1, p_1) = (c_2.{}^{p_2}c_1, p_2p_1),$$

(and the interchange law holds, being equivalent to the Peiffer identity).

11.6.2 The geometric nerve, $Ner(\mathcal{X}(\mathsf{C}))$

- The set of 0-simplices, $Ner(\mathcal{X}(\mathsf{C}))_0$, is the set of objects, so is $\{*\}$. (This nerve will, here, be a reduced simplicial set. Of course, if C was a crossed module of groupoids, then $Ner(\mathcal{X}(\mathsf{C}))_0$ would possibly have more elements.)

- The set of 1-simplices will be the set of arrows of $\mathcal{X}(\mathsf{C})$ and thus is P, *as a set*;

- The 2-simplices of $Ner(\mathcal{X}(\mathsf{C}))$ consist of 4-tuples, $\underline{x} = (x(12), x(02), x(01); x(012))$, as before, where the $x(ij) \in P$ and $x(012) : x(02) \Rightarrow x(12)x(01)$ is a 2-cell.

  The faces of $\underline{x}$ are $d_0\underline{x} = x(12)$, etc, as we saw before, so we will abbreviate $x(12)$ to $x_0 \in P$, etc. Writing $x := x(012)$, we then have x is a 2-cell, $x : x_1 \Rightarrow x_0\sharp_0x_2$, the codomain being just $x_0.x_2$ in different notation, hence x has form (c, x_1) with $\partial c.x_1 = x_0.x_2$,

$$\begin{tikzcd} & \cdot \arrow[dr, "x_0"] \arrow[d, phantom, "\Uparrow x"] & \\ \cdot \arrow[ur, "x_2"] \arrow[rr, "x_1"'] & {} & \cdot \end{tikzcd}$$

  and hence $\partial c = x_0x_2x_1^{-1}$, which is clearly closely related to the form given, page 254, for the $\overline{W}$-based version of the classifying space, but we must check how good that similarity is in detail (and with consistent conventions).

- The 3-simplices of $Ner(\mathcal{X}(\mathsf{C}))$ consist of sets of arrows,

$$\{x(ij) \mid 0 \leq i < j \leq 3\},$$

  and 2-cells,

$$\{x(ijk) \mid 0 \leq i < j < k \leq 3\},$$

  with $x(ijk) : x(ik) \Rightarrow x(jk)x(ij)$, and satisfying a cocycle condition:

$$\begin{tikzcd} x(13)x(01) \arrow[r, Rightarrow, "x(123)\sharp_0 x(01)"] & x(23)x(12)x(01) \\ x(03) \arrow[u, Rightarrow, "x(013)"] \arrow[r, Rightarrow, "x(023)"'] & x(23)x(02) \arrow[u, Rightarrow, "x(23)\sharp_0 x(012)"'] \end{tikzcd}$$

  commutes.

  We again rethink this in terms of C and P, using the fact that

$$d_0\underline{x} = (x(23), x(13), x(12); x(123) : x(13) \Rightarrow x(23)x(12)),$$

  and so on. The i^{th} face is the term that omits i, as usual in these situations.

It is important to note at this point that between them the four faces contain *all* the $x(ij)$ and $x(ijk)$, so completely determine $\underline{x}$ itself. This is, of course, related to the condition that $Ner(\mathcal{X}(\mathsf{C}))$ is 3-coskeletal, but that condition just gives the similar result in higher dimension[23]. This observation says that there is a unique 3-simplex with these faces, not that if you start with four 2-simplices seemingly of the right form then there will automatically exist a 3-simplex with those 2-simplices as its faces, because the 3-cocycle condition intervenes.

Write the four 2-cells as c_0, c_1, c_2, and c_3, corresponding to $d_0\underline{x}$, etc., respectively, so that

- face (123): $\partial c_0 = x(23)x(12)x(13)^{-1}$;
- face (023): $\partial c_1 = x(23)x(02)x(03)^{-1}$;
- face (013): $\partial c_2 = x(13)x(01)x(03)^{-1}$;
- face (012): $\partial c_3 = x(12)x(01)x(02)^{-1}$.

To analyse the commutativity of the square above will require us to look first at the two 'whiskered' terms:

$$x(123)\sharp_0 x(01) = (c_0, x(13))\sharp_0(1, x(01)) = (c_0, x(13)x(01)),$$

whilst

$$x(23)\sharp_0 x(012) = (1, x(23))\sharp_0(c_3, x(02)) = ({}^{x(23)}c_3, x(23)x(02)).$$

The $\sharp_1$-compositions of 2-cells correspond to multiplication in C, so the two routes around the square give

$$\begin{aligned}(x(123)\sharp_0 x(01))\sharp_1 x(013) &= (c_0, x(13)x(01))_1(c_2, x(03))\\ &= (c_0c_2, x(03))\end{aligned}$$

and

$$\begin{aligned}(x(23)\sharp_0 x(012))\sharp_1 x(023) &= ({}^{x(23)}c_3, x(23)x(02))\sharp_1(c_1, x(03))\\ &= ({}^{x(23)}c_3c_1, x(03)).\end{aligned}$$

We thus have a cocycle condition:

$$c_0c_2 = {}^{x(23)}c_3c_1.$$

- Above dimension 3, everything is determined by dimension 3, as we saw that $Ner(\mathcal{X}(\mathsf{C}))$ is 3-coskeletal.

We next turn towards the construction going via the 'internal nerve' or 'simplicial group nerve'. By this route, we first construct a simplicial group, $K(\mathsf{C})$, from C. As above we will repeat that construction in great detail, so as to check consistency of conventions. The simplicial group, $K(\mathsf{C})$, is the internal nerve of the internal groupoid, $\mathcal{X}(\mathsf{C})$, and is constructed within the category of groups. (The relevant earlier discussions are in sections 6.2.2 and 6.2.3.)

The simplicial group, $K(\mathsf{C})$, has:

[23] Check back on the properties of that notion as given by Proposition 51.

- group of 0-simplices, $K(\mathsf{C})_0 = P$;
- group of 1-simplices, $K(\mathsf{C})_1 = C \rtimes P$, with, for (c_1, p), a 1-simplex, $d_0(c_1, p) = \partial c_1.p$, $d_1(c_1, p) = p$ and $s_0(p) = (1, p)$, for $p \in P$;
- group of 2-simplices, $K(\mathsf{C})_2 = C \rtimes (C \rtimes P)$, with, for (c_2, c_1, p), a 2-simplex

$$\begin{aligned} d_0(c_2, c_1, p) &= (c_2, \partial c_1.p), \\ d_1(c_2, c_1, p) &= (c_2.c_1, p), \\ d_2(c_2, c_1, p) &= (c_1, p), \end{aligned}$$

  and degeneracies, $s_0(c_1, p) = (1, c_1, p)$, $s_1(c_1, p) = (c_1, 1, p)$.

  It is useful to repeat the diagram for (c_2, c_1, p):

$$\begin{tikzcd} & \partial c_1.p \arrow[dr, "{(c_2,\partial c_1.p)}"] & \\ p \arrow[ur, "{(c_1,p)}"] \arrow[rr, "{(c_2c_1,p)}"'] & & \partial(c_2c_1).p \end{tikzcd}$$

- for $n \geq 3$, $K(\mathsf{C})_n = C \rtimes K(\mathsf{C})_{n-1}$, with action via the projection to P, and, if $(\underline{c}, p) := (c_n, \ldots, c_1, p)$ is an n-simplex, the face morphisms are given by

$$\begin{aligned} d_0(\underline{c}, p) &= (c_n, \ldots, c_2, \partial c_1.p), \\ d_i(\underline{c}, p) &= (c_n, \ldots, c_{i+1}.c_i, \ldots, p) \quad \text{for } 0 < i < n, \\ d_n(\underline{c}, p) &= (c_{n-1}, \ldots, c_1, p), \end{aligned}$$

  whilst the degeneracy maps insert an identity.

11.6.3 $\overline{W}(H)$ in functional composition notation

We have been operating under the assumption that to hope to obtain fairly simple formulae in cocycles, nerves, etc., it may be a good idea to stick with consistent conventions, so using left actions, function composition order, and so on. This has sometimes worked! It does mean checking through to see that a given formula is consistent with the convention and *that can be tedious!* Does it matter? The answer is 'sometimes'. The mathematical *essence* of the argument *is* fully independent of the notation, but that means that a twisted arcane obscure formula may really represent something simple, and be equivalent to a much simpler transparent one, or it may really reflect some great twisted arcane mathematical form that is impossible to unravel further.

For the $\overline{W}$-construction, we have two or three levels of structure and the order of 'composition' being used is not always in evidence, so giving a consistent convention is quite tricky.

The classifying space of a group is given by the nerve of the corresponding groupoid or, if you prefer, the geometric realisation of that simplicial set. The $\overline{W}$-construction gives a classifying space for a simplicial group (or, more generally, any $\mathcal{S}$-groupoid or small $\mathcal{S}$-category). It is a generalisation of the nerve construction. It can also be derived from the nerve, since, applying the nerve functor to each dimension of a simplicial group gives a bisimplicial set and, as we have mentioned earlier, one can process such an object either using the diagonal functor (as we did in section 5.5.1, page 234) or, using the Artin-Mazur codiagonal that we will meet more formally in the near future (section 13.5.2, page 650).

If G is a groupoid, we can represent an n-simplex of $Ner(G)$ by a diagram

$$x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} \dots \xrightarrow{g_n} x_n,$$

where $t(g_i) = s(g_{i+1})$, and, in 'functional' order, by an n-tuple $\underline{g} = (g_n, \dots, g_1)$ with $d_0\underline{g} = (g_n, \dots, g_2)$, etc. In the $\overline{W}$-construction, we look at an $\mathcal{S}$-groupoid, H, and take 'composable' strings, $\underline{h} = (h_n, \dots, h_1)$, in a similar way, but with $h_i \in H_{i-1}$.

In case you think that we need $h_i \in H_i$, it is worth pausing to discuss the indexing. In a group, G, thought of as the groupoid, $G[1]$, the nerve is a reduced simplicial set, *i.e.*, $Ner(G)_0$ has just one element, and $Ner(G)_1$ is G itself, but the arrows in $G[1]$, as a simplicially enriched groupoid, are thought of as being in dimension 0, so the dimension drops by 1. This sort of conflict of 'rival' indexation ideas is quite usual, quite confusing and quite irritating, but it is also quite easy to accept and to work with. Remember that $\overline{W}$ behaves as if it were a 'suspension' operation, whilst its left adjoint, G, behaves like a 'loops on -' construction, so we should expect shifts in 'geometric' dimension.

The problem is 'what should the face convention be?' If we look at d_0 and define it just to delete the h_1 position, then we get an invalid string, as the dimensions are wrong. The n^{th} face would work alright as that would delete h_n and the resulting string would still be valid. To get around the d_0 problem, we will adopt a definition that (i) is simple, (ii) works and, in fact, (iii) has a neat interpretation, when applied to objects such as $K(\mathsf{C})$. In addition, it seems to be the codiagonal of the bisimplicial nerve construction, but we cannot look at that aspect in detail at the moment, as we do not yet have enough detailed information on the codiagonal.

What is this marvellous convention, ...?

We take H to be an $\mathcal{S}$-groupoid, as usual, with object set, O, say:

- $\overline{W}(H)_0$ is the set, O, of objects of H;
- $\overline{W}(H)_1$ is the set of arrows of the groupoid, H_0;

and, in general,

- $\overline{W}(H)_n$ is the set of all 'composable' strings, $\underline{h} = (h_n, \dots, h_1)$, with $h_i \in H_{i-1}$, and (for 'composable') $t(h_i) = s(h_{i+1})$ for $0 < i < n$.

The face maps are given by:

- $d_0(\underline{h}) = (d_0h_n, \dots, d_0h_2)$;
- $d_i(\underline{h}) = (d_ih_n, \dots, d_ih_{i+1}.h_i, \dots, h_1)$ for $0 < i < n$;
- $d_n(\underline{h}) = (h_{n-1}, \dots, h_1)$.

The degeneracy maps are given by inserting an identity in the appropriate place and using the degeneracies of H to push earlier elements of the string up one dimensions:

- $s_i(\underline{h}) = (s_i(h_n), \dots, s_i(h_{i+1}), id_{x_i}, h_i, \dots, h_1)$.

(Of course, you are **left to check** that this works and gives a simplicial set, etc.)

There are some obvious questions to ask:

• Does this given an isomorphic version of $\overline{W}(H)$? Probably not, as it looks more like a conjugate version of the more standard form. It clearly has the same sort of properties, e.g., being a classifying space for H, classifying principal H-bundles if H is a simplicial group, etc., and has a geometric realisation that is homeomorphic to the standard form.

• Is it easy to visualise the n-simplices? Yes, at least in the case $H = K(\mathsf{C})$, and more generally for any 2-groupoid considered as a $\mathcal{S}$-groupoid. In fact, it works for a 2-category as well:

11.6.4 Visualising $\overline{W}(K(\mathsf{C}))$

First let us see what the 'bottom end' of $\overline{W}(K(\mathsf{C}))$ looks like.

• $\overline{W}(K(\mathsf{C}))_0$ is a point (as we have C is a crossed module of *groups*);

• $\overline{W}(K(\mathsf{C}))_1$ is isomorphic to the *set*, P, as a 1-simplex in $\overline{W}(K(\mathsf{C}))$ is an 'arrow', *i.e.*, an element in $K(\mathsf{C}))_0$, which is the *group* P;

• A 2-simplex of $\overline{W}(K(\mathsf{C}))$ consists of a pair (h_2, h_1) with $h_i \in K(\mathsf{C})_{i-1}$, so $h_2 \in C \rtimes P$, $h_1 \in P$.

In 2-categorical from, this can be thought of as $\underline{h}$ being

$$\begin{tikzcd} {} \arrow[r, "h_1"] & {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow h_2"] & {} \end{tikzcd}$$

and then $d_0(\underline{h})$ deletes h_1, as we want, and takes $d_0(h_2)$ as data from h_2; at the other 'extreme', $d_2(\underline{h})$ just gives us

$$\begin{tikzcd} {} \arrow[r, "h_1"] & {} \end{tikzcd}$$

and, in between, $d_1(\underline{h})$ takes the start of the 2-cell and composes it with h_1 to get $d_1(h_2).h_1$.

It is sometimes useful to draw this as a 'staircase' diagram:

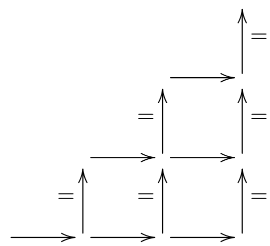
$$\begin{tikzcd} & & & {} \\ & & {} \arrow[r, "d_0h_2"] & {} \arrow[u, "="'] \\ {} \arrow[r, "h_1"'] & {} \arrow[r, phantom] & {} \arrow[u, "="] \arrow[r, "d_1h_2"'] \arrow[ur, phantom, "h_2"] & {} \arrow[u, "="'] \end{tikzcd}$$

and we will see this 'come into its own' importance later when looking at codiagonals.

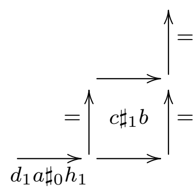
• The 3-simplices, $\underline{h} = (h_3, h_2, h_1)$ with, again, $h_i \in K(\mathsf{C})_{i-1}$, have similar pictures. Remember h_3 is a composable pair of 2-cells, as on the right hand end:

$$\begin{tikzcd} {} \arrow[r, "h_1"] & {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow a"] & {} \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] \arrow[r, phantom, shift left=2ex, "\Uparrow c"] \arrow[r, phantom, shift right=2ex, "\Uparrow b"] & {} \end{tikzcd}$$

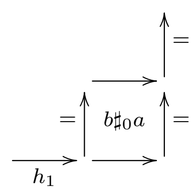
and the staircase, obtained by expanding out the 2-cells:



The staircase shows more clearly the face maps. The d_0 deletes the bottom row completely; d_1 removes the 1st row and 1st column of vertices and composes, where possible, to give



d_2 removes the 2nd row and column and composes:



and d_3 deletes the right hand column (and thus the top row as well).

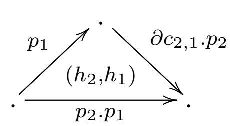
If we go one step further down in the notation, *i.e.*, back to the elements of C and P, then we have that $\underline{h} \in \overline{W}(K(C))_2$ has form

$$\underline{h} = ((c_{2,1}, p_2), p_1)$$

with $h_2 = (c_{2,1}, p_2) \in C \rtimes P$, and so on. The picture of $\underline{h}$ is then

$$\begin{aligned} d_0(\underline{h}) &= \partial c_{2,1}.p_2 \\ d_1(\underline{h}) &= p_2.p_1 \\ d_2(\underline{h}) &= p_1 \end{aligned}$$

giving



If we match that picture with the earlier one (page 549), then

$$\begin{aligned} x_0 &\leftrightarrow \partial c_{2,1}.p_2 \\ x_1 &\leftrightarrow p_2.p_1 \\ x_2 &\leftrightarrow p_1 \end{aligned}$$

and, given $\underline{h}$, we get the geometric nerve 2-simplex,

$$(p_1, p_2.p_1, \partial c_{2,1}.p_2; (c_{2,1}, p_2.p_1)).$$

Working the other way around, given $(x_2, x_1, x_0; (c, x_1))$, gives a $\overline{W}$-based 2-simplex

$$((c, x_1 x_2^{-1}), x_2),$$

and the faces match up. (**Check this all works** - both ways - and do not forget the 'cocycle' conditions relating the x_is.) This looks good. On to dimension 3, ...

If we start with $\underline{h} = (h_3, h_2, h_1)$, where

$$\begin{aligned} h_1 &= p_1 \\ h_2 &= (c_{2,1}, p_2) \\ h_3 &= (c_{3,2}, c_{3,1}, p_3), \end{aligned}$$

we get

$$\begin{aligned} d_0(\underline{h}) &= ((c_{3,2}, \partial c_{3,1}.p_3), (\partial c_{2,1}.p_2)), \\ d_1(\underline{h}) &= ((c_{3,2}.c_{3,1}, p_3), p_2.p_1), \\ d_2(\underline{h}) &= ((c_{3,1}.{}^{p_3}c_{2,1}, p_3 p_2), p_1), \\ d_3(\underline{h}) &= ((c_{2,1}, p_2), p_1), \end{aligned}$$

and now note that given these four faces, we can reconstruct $\underline{h}$ completely, since $d_3(\underline{h})$ gives us h_2 and h_1, and we can use projections onto semi-direct factors of $C \rtimes (C \rtimes P)$ to retrieve h_3 with no bother. This means that there is a unique $\underline{h}$ with this shell - the same phenomenon that we saw with $Ner(\mathcal{X}(\mathsf{C}))$. The isomorphism that we found in levels 0, 1 and 2 can therefore be extended to dimension 3 ..., and above by the fact that we have 3-coskeletal simplicial sets here. (We have not *actually* explicitly checked that $\overline{W}(K(\mathsf{C}))$ *is* 3-coskeletal, but the above calculation linked in with our earlier work (page 204) should **enable you to prove this**.) We have

Proposition 129 *The two classifying spaces, $Ner(\mathcal{X}(\mathsf{C}))$ and $\overline{W}(K(\mathsf{C}))$, are naturally isomorphic.* ∎

This result suggests several questions, *some* of which we will look at shortly, others are **left to you**.

• If C and D are two crossed modules, can we interpret, algebraically, an op-lax morphism between the corresponding 2-groups, since we know that these correspond to simplicial morphisms between the corresponding nerves? This would give a sort of 'weak' morphism between the crossed modules.

• Can we extend the above *isomorphism* to the case where we have 2-categories rather than 2-groupoids? This would look unlikely, since we had to use inverses to check the isomorphism, but perhaps some weaker relationship is possible, cf., for instance, Bullejos and Cegarra, [71]. One important consequence of this is a way of comparing the two obvious ways of assigning a classifying space to a *strict monoidal category*. A monoidal category 'is' a one object bicategory, and a strict one thus corresponds to a one object 2-category. (We will look at monoidal categories is slightly more detail in a coming chapter.) The classical classifying space construction, used by Segal, [243, 244], corresponds to taking the nerve of the category structure and then that of the monoid structure and forming a simplicial set from the resulting bicomplex. The resulting space has a lot of beautiful properties, but we will not go into them here. The relevant papers directly on the comparison between this classical nerve and classifying space and that defined using the homotopy coherent nerve are by Bullejos and Cegarra, [71, 72]. One important point to note is that the Duskin geometric nerve construction which they use is also applicable to bicategories, so some of their results apply also to non-strict monoidal categories.

• Can we find a way of adapting the above proposition to handle some sort of 3-category or 3-groupoid? Perhaps starting with a 2-crossed module, we could form $\overline{W}$ of the corresponding simplicial group, since that is easy, but can we construct the h. c. nerve of such a simplicial group?

More generally:

• If we think of an $\mathcal{S}$-groupoid, G, as an $\mathcal{S}$-category, what is the geometric (h. c.) nerve of that $\mathcal{S}$-category?

11.7 Pseudo-functors between 2-groups

We will look in some detail at the first of these questions.

As crossed modules give rise to 2-groups (or, more generally, 2-groupoids) and these are 2-categories, it is natural to ask what the lax or op-lax functors between two such 2-groups look like. This can be considered both as a good illustrative exercise on (op-)lax functors and thus on homotopy coherence, and also as an important part of the theory of crossed modules that we have yet to explore. We will start with a basic observation and that is that, as 2-groupoids have invertible 1 and 2 arrows, there is no essential difference between lax and op-lax functors and they are both 'the same as' pseudo-functors. Of course, one has to choose a direction for the 2-cells and we will consider 'pseudo = op-lax + invertible', *i.e.*, the structural 2-cells of a pseudo-functor will go from $F(ab)$ to $F(a)F(b)$. These pseudo-functors will be normal ones as usual.

To start with, our study will look at pseudo-functors between two 2-groups, $\mathcal{X}(\mathsf{C})$ and $\mathcal{X}(\mathsf{C}')$, where $\mathsf{C} = (C, P, \partial)$ and $\mathsf{C}' = (C', P', \partial')$, and by analysing them at the level of the groups and actions involved. Later we will examine them at the level of simplicial groups. (As usual the extension to $\mathcal{S}$-groupoids is reasonable easy to do, so will be **left to you**.)

(The material in this section is based, in part, on Noohi's work in [222, 223] (and the correction available as [224]) and with Aldrovandi, [5], for a sheafified version with applications to stacks. There is also a strong link with the Moerdijk-Svensson model category structure on 2-groups, for which see [208] as well as with the papers referred to in the previous section.)

11.7.1 Weak maps between crossed modules

Effectively a *weak map between crossed modules* is what is 'seen', at the level of crossed modules, of a pseudo-functor between the corresponding 2-groups. The abstract definition, as given by Noohi, [222], is:

Definition: Let C and C' be crossed modules, as above. A *weak map*, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$, is a pseudo-functor from $\mathcal{X}(\mathsf{C})$ to $\mathcal{X}(\mathsf{C}')$.

That probably does not say that much to you about what such a thing looks like, so we are going to take the definition apart in various ways so as to get some feel for them.

We first use a direct attack. Consider a normal pseudo-functor:

$$F : \mathcal{X}(\mathsf{C}) \to \mathcal{X}(\mathsf{C}'),$$

then this consists of

- a set map, F_0, on objects (this is 'no big deal' as both $\mathcal{X}(\mathsf{C})$ and $\mathcal{X}(\mathsf{C}')$ have exactly one object);
- a set map, F_1, sending arrows to arrows, so giving a function,

  $$f_0 : P \to P',$$

  which is not necessarily a homomorphism of groups. The obstruction to it being one is given by
- a set map, $\varphi : P \times P \to C' \rtimes P'$, so, if $p_2, p_1 \in P$, $\varphi(p_2, p_1)$ is a 2-cell from $f_0(p_2p_1)$ to $f_0(p_2)f_0(p_1)$;
- a functor

  $$\begin{tikzcd} F_2:\ C \rtimes P \arrow[r] \arrow[d, shift right=2] \arrow[d] & C' \rtimes P' \arrow[d, shift right=2] \arrow[d] \\ P \arrow[r] \arrow[u, shift right=2] & P' \arrow[u, shift right=2] \end{tikzcd}$$

  between the underlying categories, with f_0 at the level of objects. (Importantly, note that this does not mean that this functor preserves horizontal composition, *i.e.*, group multiplication, in either the top or the object levels. This is just $F_2 = F_{*,*} : \mathcal{X}(\mathsf{C})(*, *) \to \mathcal{X}(\mathsf{C}')(*, *)$, as a functor between the corresponding 'hom-categories'.)

Of course, we will have to give some equations and conditions on these, but will explore this little-by-little before giving a résumé of the resulting structure.

First we note that, as we have a normalised pseudo-functor, $f_0(1) = 1$ and $\varphi(1, 1) = 1$. As F_2 is a functor, we have, for $(c, p) \in C \rtimes P$,

$$F_2(c, p) : f_0(p) \to f_0(\partial c.p),$$

but this means that $F_2(c, p)$ has the form,

$$F_2(c, p) = (F_2'(c, p), f_0(p)),$$

for some function $F_2' : C \rtimes P \to C'$. We will set $f_1(c) = F_2'(c, 1)$ and note that $\partial f_1(c) = f_0(\partial c)$.

It will eventually turn out that f_1 is *almost* a group homomorphism and that from f_1 and φ, we will be able to calculate $F_2'(c, p)$ for a general $p \in P$, that is to say, the information needed for F_2 reduces to that for f_1 and φ and, from them, we can reconstruct F_2 itself.

We also have that $\varphi(p_2, p_1)$ is a 2-cell from $f_0(p_2p_1)$ to $f_0(p_2)f_0(p_1)$. It therefore has the form

$$\varphi(p_2, p_1) = (\langle p_2, p_1\rangle, f_0(p_2p_1))$$

for some 'pairing function',

$$\langle\ \ ,\ \ \rangle_\varphi : P \times P \to C'.$$

(We will usually write $\langle\ ,\ \rangle$ instead $\langle\ ,\ \rangle_\varphi$ if no confusion is likely.) We need φ to be 'natural' with respect to pre- and post- whiskering and so will have corresponding conditions on $\langle\ ,\ \rangle$. We first note that, since the target of $\varphi(p_2, p_1)$ is $f_0(p_2)f_0(p_1)$, we have

Lemma 77 ***(Target condition)*** *For any $p_1, p_2 \in P$,*

$$\partial\langle p_2, p_1\rangle = f_0(p_2)f_0(p_1)f_0(p_2p_1)^{-1}.$$

■

The 'associativity' axiom for φ gives a cocycle condition:
for $p_1, p_2, p_3 \in P$, the diagram, in $\mathcal{X}(\mathsf{C}')$

$$\begin{tikzcd}
f_0(p_3p_2p_1) \arrow[r, "\varphi(p_3p_2,p_1)"] \arrow[d, "\varphi(p_3,p_2p_1)"'] & f_0(p_3p_2)f_0(p_1) \arrow[d, "\varphi(p_2,p_1)\sharp_0 f_0(p_1)"] \\
f_0(p_3)f_0(p_2p_1) \arrow[r, "f_0(p_3)\sharp_0\varphi(p_2,p_1)"'] & f_0(p_3)f_0(p_2)f_0(p_1)
\end{tikzcd}$$

is commutative.

Interpreting this at the crossed module level:

Lemma 78 ***(Cocycle condition)*** *For any $p_1, p_2, p_3 \in P$,*

$$\langle p_3, p_2\rangle\langle p_3p_2, p_1\rangle = {}^{f_0(p_3)}\langle p_2, p_1\rangle\langle p_3, p_2p_1\rangle.$$

■

The proof is straightforward. We note that we really do use the formulae for pre- and post-whiskering in terms of the group multiplication. This is just the multiplication on the right or left of (c, p) by some $(1, p')$:

Pre-whisker: $(c, p)\sharp_0(1, p') = (c, pp')$;
Post-whisker: $(1, p')\sharp_0(c, p) = ({}^{p'}c, p'p)$.

As we are considering normalised op-lax and pseudo-functors, we have $\varphi(1, 1) = 1$, so $\langle 1, 1\rangle = 1$ as well, but we can use this together with the cocycle condition to get:

Corollary 26 *For any $p \in P$, $\langle 1, p\rangle$ and $\langle p, 1\rangle$ are both $1_{C'}$.*

Proof: Taking $p_2 = p$, $p_3 = 1$ and $p_1 = p^{-1}$ gives

$$\langle 1, p\rangle\langle p, p^{-1}\rangle = {}^{f_0(1)}\langle p, p^{-1}\rangle\langle 1, 1\rangle,$$

so, as $f_0(1) = 1$ and $\langle 1, 1\rangle = 1$, we have $\langle 1, p\rangle = 1$.
Similarly, try $p_1 = 1$, $p_2 = p$ and $p_3 = p^{-1}$. ■

Remark: It will probably not have escaped your notice that what we have here is very closely related to a weak action of P on C'. This will become more apparent slightly later on.

We next look at the naturality of φ.

If we fix $p \in P$, we get the pre-whiskering

$$-\sharp_0 p : \mathcal{X}(\mathsf{C})(*, *) \to \mathcal{X}(\mathsf{C})(*, *),$$

and the corresponding post-whiskering

$$p\sharp_0 - : \mathcal{X}(\mathsf{C})(*, *) \to \mathcal{X}(\mathsf{C})(*, *).$$

Naturality of φ means that pre- (resp. post-) whiskering in $\mathcal{X}(\mathsf{C})$ is translated into the similar operation in $\mathcal{X}(\mathsf{C}')$.

Pre-whiskering naturality: For any $p_1, p_2 \in P$ and $c \in C$, the diagram

$$\begin{tikzcd}
f_0(p_2p_1) \arrow[r, "\varphi_{p_2,p_1}"] \arrow[d, "F_2(c,p_2p_1)"'] & f_0(p_2)f_0(p_1) \arrow[d, "F_2(c,p_2)\sharp_0 f_0(p_1)"] \\
f_0(p_2'p_1) \arrow[r, "\varphi_{p_2',p_1}"'] & f_0(p_2')f_0(p_1)
\end{tikzcd}$$

in $\mathcal{X}(\mathsf{C}')$ commutes, where $p_2' = \partial c.p_2$.

Using F_2' and $\langle -, - \rangle$, this translates as

Lemma 79 ***(Primitive pre-whiskering condition.)*** *For $p_1, p_2 \in P$ and $c \in C$,*

$$\langle \partial c.p_2, p_1 \rangle.F_2'(c, p_2p_1) = F_2'(c, p_2).\langle p_2, p_1 \rangle.$$

■

We call it 'primitive' as we really want it in terms of f_1 not of F_2'.

Post-whiskering naturality: For any $p_2, p_3 \in P$ and $c \in C$, the diagram

$$\begin{tikzcd}
f_0(p_3p_2) \arrow[r, "\varphi_{p_3,p_2}"] \arrow[d, "F_2({}^{p_3}c,p_3p_2)"'] & f_0(p_3)f_0(p_2) \arrow[d, "f_0(p_3)\sharp_0 F_2(c,p_2)"] \\
f_0(p_3'p_2) \arrow[r, "\varphi_{p_3,p_2'}"'] & f_0(p_3)f_0(p_2')
\end{tikzcd}$$

in $\mathcal{X}(\mathsf{C}')$ commutes, where $p_2' = \partial c.p_2$.

Using F_2' and $\langle -, - \rangle$, this translates as

Lemma 80 ***(Primitive post-whiskering condition.)*** *For $p_2, p_3 \in P$ and $c \in C$,*

$$\langle p_3, \partial c.p_2 \rangle.F_2'({}^{p_3}c, p_3p_2) = {}^{f(p_3)}F_2'(c, p_2).\langle p_3, p_2 \rangle.$$

■

Recall that we wrote $f_1(c)$ for $F_2'(c, 1)$. Using naturality, and from the fact that an arbitrary (c, p) can be written as $(c, 1)\sharp_0(1, p)$, we can derive a rule expressing $F_2'(c, p)$ in terms of $f_1(c)$ and $\langle -, - \rangle$:

Lemma 81 *For any c, p, as above,*

$$F_2'(c,p) = \langle \partial c, p\rangle^{-1} f_1(c).$$

Proof: Pre-whiskering naturality gives

$$\langle \partial c, p\rangle . F_2'(c,p) = F_2'(c,1).\langle 1, p\rangle,$$

but we showed that $\langle 1, p\rangle$ is the identity, so the result follows. ■

Of course, as F_2 is a functor, we also know that $f_1(1) = 1$.

It is thus possible to define $F_2(c,p)$ in terms of the pairing function $\langle -, -\rangle$ together with f_0 and f_1. Of course, we need to be sure that F_2, thus (re-)constructed, has the right properties, mainly as a check that the whole framework holds together, and that we have successfully reduced the data specifying F to a usefully presented description. For instance, $F_2(c,p)$ is to be a 2-cell from $f_0(p)$ to $f_0(\partial c.p)$, *i.e.*, we must have:

Lemma 82 *Thus defined, $F_2'(c,p)$ satisfies $f_0(\partial c.p) = \partial F_2'(c,p).f_0(p)$.*

Proof: (Included really only because it is quite neat. It could have been left to you.)

$$\partial F_2'(c,p) = \partial\langle \partial c, p\rangle^{-1}\partial f_1(c),$$

but we know $\partial f_1(c) = f_0(\partial c)$. We obtain

$$\partial\langle \partial c, p\rangle = f_0(\partial c) f_0(p) f_0(\partial c.p)^{-1},$$

and hence

$$\partial\langle \partial c, p\rangle^{-1} = f_0(\partial c.p) f_0(p)^{-1} f_0(\partial c)^{-1},$$

so

$$\partial F_2'(c,p) = f_0(\partial c.p) f_0(p)^{-1},$$

or

$$f_0(\partial c.p) = \partial F_2'(c,p).f_0(p),$$

as required. ■

Proposition 130 ***Pre-whiskering naturality:*** *For $p_1, p_2 \in P$ and $c \in C$,*

$$^{f_0(\partial c)}\langle p_2, p_1\rangle . f_1(c) = f_1(c).\langle p_2, p_1\rangle.$$

Proof: By calculation after substituting: on substituting $\langle \partial c, p\rangle^{-1} f_1(c)$ for $F_2'(c,p)$, etc., the primitive version gives

$$\langle \partial c.p_2, p_1\rangle . \langle \partial c, p_2.p_1\rangle^{-1} f_1(c) = \langle \partial c, p_2\rangle^{-1} f_1(c)\langle p_2, p_1\rangle.$$

By the associativity cocycle condition,

$$\langle \partial c.p_2, p_1\rangle . \langle \partial c, p_2.p_1\rangle^{-1} = \langle \partial c, p_2\rangle^{-1\, f_0(\partial c)}\langle p_2, p_1\rangle.$$

Cancellation of $\langle \partial c, p_2\rangle^{-1}$ in the combined expression gives the result. ■

Remark: Rearranging the above equation gives

$$^{\partial f_1(c)}\langle p_2, p_1\rangle = f_1(c)\langle p_2, p_1\rangle f_1(c)^{-1},$$

which is related to the Peiffer identity,

$$^{\partial c}c' = c.c'c^{-1},$$

within C' and could have been deduced directly from it.

Back again, this time to Post-Whiskering Naturality, we had

$$\langle p_3, \partial c.p_2\rangle.F_2'(^{p_3}c, p_3p_2) = {}^{f(p_3)}F_2'(c, p_2).\langle p_3, p_2\rangle,$$

and hence

$$\langle p_3, \partial c.p_2\rangle.\langle p_3\partial c.p_3^{-1}, p_3p_2\rangle^{-1} f_1(^{p_3}c) = {}^{f_0(p_3)}\langle \partial c, p_2\rangle^{-1\,f_0(p_3)} f_1(c).\langle p_3, p_2\rangle.$$

Using the 'associativity' cocycle condition gives an expression for the first part of the right hand side as

$$^{f_0(p_3)}\langle \partial c, p_2\rangle = \langle p_3, \partial c\rangle\langle p_3.\partial c, p_2\rangle\langle p_3, \partial c.p_2\rangle^{-1},$$

so we get, after an easy rearrangement:

Proposition 131 ***Post-whiskering naturality:*** *For $p_2, p_3 \in P$ and $c \in C$,*

$$\langle p_3.\partial c.p_3^{-1}, p_3p_2\rangle^{-1} f_1(^{p_3}c) = \langle p_3.\partial c, p_2\rangle^{-1}\langle p_3, \partial c\rangle^{-1\,f_0(p_3)} f_1(c).\langle p_3, p_2\rangle.$$

■

Remarks: (i) This formula, or rather the right action / algebraic composition order form of it, is ascribed to Ettore Aldrovandi in the corrected version of Noohi's notes, [224]. It is worth noting that Noohi uses right actions and a lax functor formulation, so, for instance,

$$\varphi : F(b)F(a) \Rightarrow F(ba).$$

This results in there being no inverse on the pairing brackets, amongst other things.

(ii) If we consider the case $p_3 = p_2^{-1} = p$, say, then we get

$$f_1(^{p}c) = \langle p.\partial c, p^{-1}\rangle^{-1}\langle p, \partial c\rangle^{-1\,f_0(p)} f_1(c)\langle p, p^{-1}\rangle,$$

which is a form of Noohi's 'equivariance condition', cf. [224].

We can use similar arguments to these above to investigate f_1 further.

Proposition 132 *The map $f_1 : C \to C'$ satisfies: for all $c_2, c_1 \in C$,*

$$f_1(c_2c_1) = \langle \partial c_2, \partial c_1\rangle^{-1} f_1(c_2)f_1(c_1).$$

Proof: Using the definition of f_1,

$$\begin{aligned}
(f_1(c_2c_1),1) &= (F_2(c_2c_1,1)\\
&= F_2(c_2,\partial c_1)F_2(c_1,1)\\
&= (\langle \partial c_2,\partial c_1\rangle^{-1}f_1(c_2),\partial c_1)\sharp_1(f_1(c_1),1)\\
&= (\langle \partial c_2,\partial c_1\rangle^{-1}f_1(c_2)f_1(c_1),1)
\end{aligned}$$

as required. ■

We thus have that f_1 is almost a homomorphism. It is 'deformed' by the term $\langle \partial c_2,\partial c_1\rangle$.

We could, as might be expected, derive this also from a combination of pre- and post-whiskering and the interchange law. As the interchange law holds in both $\mathcal{X}(\mathsf{C})$ and $\mathcal{X}(\mathsf{C}')$, and as F_2 is a functor, it must relate these two, preserving 'interchange'.

Suppose we have

$$\alpha : p_1 \Rightarrow p_1',$$
$$\beta : p_2 \Rightarrow p_2',$$

then we have a diagram,

$$\begin{tikzcd}
f_0(p_2p_1)) \arrow[r, Rightarrow, "F(\beta\sharp_0\alpha)"] \arrow[d, Rightarrow, "\varphi_{p_2,p_1}"'] & f_0(p_2'p_1') \arrow[d, Rightarrow, "\varphi_{p_2',p_1'}"] \\
f_0(p_2)f_0(p_1) \arrow[r, Rightarrow, "F(\beta)\sharp_0F(\alpha)"'] & f_0(p_2')f_0(p_1')
\end{tikzcd}$$

which will commute in $\mathcal{X}(\mathsf{C}')$.

We can translate this, as before, in terms of $\langle -,-\rangle$, f_0 and f_1.

Proposition 133 *For $\alpha = (c_1,p_1)$ and $\beta = (c_2,p_2)$,*

$$\langle \partial c_2.p_2,\partial c_1\rangle\langle \partial c_2.p_2\partial c_1p_2^{-1},p_2p_1\rangle^{-1}f_1({c_2}^{p_2}c_1) = \langle \partial c_2,p_2\rangle^{-1}f_1(c_2)^{f_0(p_1)}\langle \partial c_1,p_1\rangle^{-1}.^{f_0(p_1)}f_1(c_1)\langle p_2,p_1\rangle.$$

■

We **leave the proof to you**. The resulting formula reduces to the pre- and post-forms for suitable choices of the variables. In turn, it can be derived by algebraic manipulation from those forms together with the formula for $f_1(c_2c_1)$ in terms of $f_1(c_2)$ and $f_1(c_1)$. The added complexity of the interchange form makes its use less attractive than that of the reduced forms.

Analysing pseudo-functors between 2-groups has thus led us to a list of structure and related properties that we can extract to get the following algebraic form of the definition. As usual, C and C' are two crossed modules.

Definition: Weak map, algebraic form: A *weak map*, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$, is given by the following structure:

- a function, $f_0 : P \to P'$;
- a function, $f_1 : C \to C'$;

- a pairing, $\langle\quad,\quad\rangle : P \times P \to C'$.

These are to satisfy:

W1 (**Normalisation**): $f_0(1) = 1$ and $\langle 1, 1\rangle = 1$;

W2 (**'Almost a homomorphism' for** f_1): for $c_2, c_1 \in C$,

$$f_1(c_2c_1) = \langle \partial c_2, \partial c_1\rangle^{-1} f_1(c_2) f_1(c_1);$$

W3 (**'Almost a homomorphism' for** f_0): for $p_1, p_2 \in P$,

$$f_0(p_2p_1) = \partial\langle p_2, p_1\rangle^{-1} f_0(p_2) f_0(p_1);$$

W4 (**Cocycle**): for $p_1, p_2, p_3 \in P$,

$$\langle p_3, p_2\rangle.\langle p_3p_2, p_1\rangle = {}^{f_0(p_3)}\langle p_2, p_1\rangle.\langle p_3, p_2p_1\rangle;$$

W5 (**Whiskering conditions**):

Pre: for $p_1, p_2 \in P$ and $c \in C$,

$${}^{f_0(\partial c)}\langle p_2, p_1\rangle.f_1(c) = f_1(c).\langle p_2, p_1\rangle;$$

Post: for $p_2, p_3 \in P$ and $c \in C$,

$$\langle p_3.\partial c.p_3^{-1}, p_3p_2\rangle^{-1} f_1({}^{p_3}c) = \langle p_3.\partial c, p_2\rangle^{-1}\langle p_3, \partial c\rangle^{-1\,f_0(p_3)} f_1(c).\langle p_3, p_2\rangle.$$

We then have:

Theorem 35 *(Noohi, [224]) The two definitions of weak map, pseudo-functorial and algebraic, are equivalent.* ■

Remarks: (i) The proof in one direction has been sketched out above, and some indication has been given as to how to go in the other direction. The details of that direction are a **'good exercise for the reader'**.

(ii) In the published form (that is in [223]), the additional assumption that f_1 was a homomorphism was made. This is not a consequence of the pseudo-functorial definition of a weak map. A correction was made available by Noohi, in [224], where the axioms are given in more or less the above form with, however, right actions, etc.

(iii) It should be noted that we have not encoded weak / pseudo-natural transformations in the above. In [224], there is a description of such things within the context of the algebraic definition of weak maps as above. The task of translating that to the notational conventions used here is **left to you**.

(iv) Any morphism of crossed modules gives a weak map between them, with a trivial pairing function, and any weak map with trivial pairing likewise *is* a morphism of crossed modules. With morphisms of crossed modules composition is very easy to do, so what about composition of weak maps? This is again **left as an exercise** for you to investigate. We will shortly see the simplicial description of weak maps and in that description composition is just composition of simplicial maps, so is easy. As a consequence, as yet, no use for a composition formula in the algebraic form of the definition seems to have been found and we will not discuss it further, except to point out that to investigate it yourself can be a useful exercise in linking the 2-group(oid) way of thinking to the crossed module way.

(v) The above algebraic definition is not intended to be in a neatest form. Some of the conditions may be redundant, for instance. The list is inspired both by Noohi's notes, and the form given there, but also by the interpretation of each condition in terms of the pseudo-functorial one.

We observed earlier the similarity between the rules for a weak map, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$, and those for a weak action. To clarify this a bit further, note that if $\mathsf{C} = (1, P, 1)$ is 'really a group', then a weak map, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$, consists just of f_0 and φ, as the only value $f_1(c)$ can take is 1 corresponding to $c = 1$! It is a normalised pseudo-functor from $P[1]$ to $\mathscr{X}(\mathsf{C}')$.

A weak action of P on P' would be a pseudo-functor from $P[1]$ to $\mathsf{Aut}(P')$. The only difference between the two notions is to replace the automorphism 2-group, $\mathsf{Aut}(P')$, by the general 2-group, $\mathscr{X}(\mathsf{C}')$. A weak action of P on P' can thus be thought of as a weak map from P to $\mathsf{Aut}(P')$, (with allowance being made for a deliberate confusion between the 2-group of automorphisms of P' and the corresponding crossed module).

A natural generalisation of weak action of a group is thus a weak action of a crossed module, C, which can be defined to be an op-lax functor from $\mathscr{X}(\mathsf{C})$ to whatever 2-category you like. Equally well, you can make C act weakly on some object in a simplicially enriched setting by using an $\mathcal{S}$-functor from the corresponding simplicial group.

Finally we note the following very interesting and useful result.

Weak maps induce morphisms on homotopy groups.

More precisely,

Proposition 134 *Suppose that* $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ *is a weak map of crossed modules, then* f *induces morphisms,*

$$\pi_i(\mathsf{f}) : \pi_i(\mathsf{C}) \to \pi_i(\mathsf{D}),$$

for $i = 0, 1$.

Proof: There are several different proofs of this. Starting from the algebraic description, we have that f_0 induces a homomorphism from $P/\partial C$ to $P'/\partial C'$. (This looks to be 'immediate' from condition $W3$, but, of course, you do have to check that the apparently induced morphism is 'well-defined'. This is easy since $f_0(\partial c) = \partial f_1(c)$.) That handles the $i = 0$ case.

Suppose next that $c \in Ker\,\partial$, then clearly $f_1(c) \in Ker\,\partial'$. Is the resulting induced mapping a homomorphism? Of course, this follows from $W2$, and we are finished. ∎

There are also easy proofs of this coming from the simplicial description, as we will see.

We have already commented on the link between weak actions and maps between nerves / classifying spaces, and also on the links between extensions, sections and weak actions. We will shortly explore the extension of these links to give us more insight into weak maps.

11.7.2 The simplicial description

Suppose C and D are two crossed modules and $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ a weak map between them in the sense of the definition on page 556. We will rewrite this in a more 'pseudo-functorial' form as a pseudo-functor, $\mathscr{F} = (F, \gamma) : \mathscr{X}(\mathsf{C}) \to \mathscr{X}(\mathsf{D})$, between the corresponding 2-groupoids. By the properties of the nerve construction that we saw earlier in Proposition 124, there is equivalently a simplicial map,

$$f : Ner(\mathscr{X}(\mathsf{C})) \to Ner(\mathscr{X}(\mathsf{D})).$$

In this description, composition of weak maps is no problem, just compose the corresponding simplicial maps. Using the natural isomorphism from Proposition 129, from such an f, we get a corresponding morphism of (reduced) simplicial sets,

$$f : \overline{W}(K(\mathsf{C})) \to \overline{W}(K(\mathsf{D})),$$

and, by the adjunction between $\overline{W}$ and the loop groupoid functor, G, (mentioned back in section 6.2.1, page 249), we get a morphism of simplicial *groups*,

$$\overline{f} : G\overline{W}(K(\mathsf{C})) \to K(\mathsf{D}).$$

The simplicial group, $K(\mathsf{D})$, has a Moore complex of length 1, so $\overline{f}$ factors via a quotient of $G := G\overline{W}(K(\mathsf{C}))$, giving K of the crossed module $M(G, 1)$, *i.e.*, the Moore complex of this quotient will be the crossed module:

$$\partial : \frac{NG_1}{d_0(NG_2)} \to G_0.$$

As G is a free simplicial group, this will have G_0 a free group.

There is a morphism, $G \to K(\mathsf{C})$, corresponding to the identity morphism from $\overline{W}(K(\mathsf{C}))$ to itself, so this is the counit of the adjunction and is a weak equivalence of simplicial groups, *i.e.*, it induces isomorphisms on all homotopy groups. We thus get a *span*[24]

$$K(\mathsf{C}) \overset{\varepsilon_{K(\mathsf{C})}}{\longleftarrow} G \longrightarrow K(\mathsf{D}),$$

or, passing to crossed modules,

$$\mathsf{C} \leftarrow M(G, 1) \to \mathsf{D}.$$

We know that the left hand part of the span is a weak equivalence of crossed modules in the sense of section 3.1 (or of simplicial groups, if we go back a line or two), so what really is this G? It was formed from $\overline{W}(K(\mathsf{C}))$ by applying the loop groupoid functor, G, which is left adjoint to $\overline{W}$ and, as we said above, the natural map, $G\overline{W} \to Id$ is the counit of that adjunction. The results that we mentioned earlier (due to Dwyer and Kan, [113], or originally, as we really are only looking at the reduced case, to Kan, [172]) include that this is a weak equivalence, *i.e.*, it induces isomorphisms on all homotopy groups. (Look up the theory in Goerss and Jardine, [137], for example, if you need more detail.)

This observation gives us a second proof of the result from page 564.

[24]Recall that a *span* in a category, $\mathcal{C}$, is a diagram in $\mathcal{C}$ of form $x \xleftarrow{f} s \xrightarrow{g} y$, and dually a *cospan* is one with the arrows going the other way.

Proposition 135 *(Simplicial version of Proposition 134) Suppose that* $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ *is a weak map of crossed modules, then* f *induces morphisms,*

$$\pi_i(\mathsf{f}) : \pi_i(\mathsf{C}) \to \pi_i(\mathsf{D}),$$

for $i = 0, 1$.

Simplicial Proof: We consider f as the span,

$$\mathsf{C} \leftarrow M(G, 1) \to \mathsf{D}.$$

Now applying π_i, we get

$$\pi_i(\mathsf{C}) \overset{\cong}{\leftarrow} \pi_i(M(G, 1)) \to \pi_i(\mathsf{D}),$$

but the left hand side is a natural isomorphism, and the induced morphism is the composite of that isomorphism's inverse followed by the induced morphism coming from the right hand branch of the span. ∎

We still need to describe G in any detail, and to do this we need to revisit the loop groupoid functor, $G(-)$, and, as we have used the conjugate $\overline{W}$, we must take its conjugate, *i.e.*, the functional composition order version of that construction.

11.7.3 The conjugate loop groupoid

It will be convenient to present the conjugate version of the Dwyer-Kan loop groupoid, that is the one that corresponds to the functional composition order and to the form of $\overline{W}$ that we have just seen, above page 551. The precise description, once we have it, will have an obvious relation with the more standard form that we have seen earlier (page 254), but we will take the opportunity to explore a little why this works and so will pretend to forget that we have seen the other form.

We suppose given a simplicial map, $f : K \to \overline{W}H$ for H an $\mathcal{S}$-groupoid, where we take $\overline{W}$ in the 'functional' form above, (page 551). We want to construct an 'adjoint map', $\overline{f} : G(K) \to H$, but as yet do not have an explicit description of G.

We have $G(K)$ will be some $\mathcal{S}$-groupoid on the object set, K_0, and $\overline{f}$ on objects will just be f_0 (on vertices). We know $G(K)_0$ will be some groupoid and $\overline{f}$, on an arrow $g : x \to y$, must be determined by f_1 on K_1, so the obvious solution is that $G(K)_0$ will be the free groupoid on the non-degenerate 1-simplices. (We must put $s_0(x) = id_x$, for $x \in K_0$. That is needed to get identities to work correctly - for **you to investigate**.) We will use functional composition order in $G(K)_0$, of course.

Defining, for $x \in K_0$, $\overline{x}$ to denote the corresponding object of $G(K)$, then, for $k \in K_1$, we will extend the overline notation and write $\overline{k} : \overline{d_1 k} \to \overline{d_0 k}$ for the corresponding generator of $G(K)_0$ and then $\overline{f}(\overline{k})_0 : f_0 d_1(k) \to f_0 d_0(k)$ in H_0, will be given by $f_1(k)$. (Freeness of $G(K)_0$ guarantees that this $\overline{f}_0$ exists and is unique with the correct universal property.)

The fun starts in dimension 1. Suppose now $k \in K_1$, then

$$f_2(k) = (h_2, h_1) \in \overline{W}(H)_2,$$

and we will write $h_2 = h_2(k)$, $h_1 = h_1(k)$, as these simplices clearly depend on the input k. We have $h_i(k) \in H_{i-1}$ and $s(h_2(k)) = t(h_1(k))$.

We need a groupoid, $G(K)_1$ with K_0 as its set of objects, and a map $\overline{f}_1 : G(K)_1 \to H_1$. (We expect 'freeness' as we have a left adjoint - but free on what? There are several choices to try and several of them work, since we are in a groupoid and, to some extent, we are making a *choice* of generators, so conjugate generators might also give a valid choice and an isomorphic $G(K)_1$.) Writing $\overline{k}$ for the generator corresponding to $k \in K_2$, we do not know what the source and target of $\overline{k}$ should be. Clearly they have to be amongst its vertices! Which ones? There are three of them!

Rather than choose the obvious one with source being the vertex of k corresponding to 0 (*i.e.*, $d_1d_2(k)$) and target being that corresponding to 2 (so $d_0d_0(k)$), we will look at $\overline{f}$ and see if there are advantages with any other choice. Looking at $\overline{f}(\overline{k})_1$, it has to be in H_1 and we already have an element of that groupoid namely $h_2(k)$. This suggests that we try *defining* $\overline{f}(\overline{k})_1$ to be $h_2(k)$ and see what that implies for $\overline{k}$ itself.

We have

$$\begin{aligned} f(d_0(k)) &= d_0(f(k)) = (d_0h_2(k)), \\ f(d_1(k)) &= d_1(f(k)) = (d_1h_2(k).h_1(k)), \\ f(d_2(k)) &= d_2(f(k)) = (h_1(k)), \end{aligned}$$

and, if we take

$$\overline{f}_1(\overline{k}) = h_2(k),$$

then

$$\begin{aligned} d_0\overline{f}_1(\overline{k}) &= d_0h_2(k) = f(d_0(k)), \\ d_1\overline{f}_1(\overline{k}) &= d_1h_2(k) = f(d_1(k)).f(d_2(k))^{-1}, \end{aligned}$$

so as to cancel the $h_1(k)$ term. This suggests that we define $d_0(\overline{k}) = \overline{d_0(k)}$, but $d_1(\overline{k}) = \overline{d_1(k)}(\overline{d_2(k)})^{-1}$. This corresponds to the source of $\overline{k}$ being the target of $\overline{d_2(k)}$, that is the object $\overline{d_0d_2(k)} = \overline{d_1d_0(k)}$, whilst the target of $\overline{k}$ would be the same as that of $\overline{d_0(k)}$, namely the object $\overline{d_0d_0(k)}$.

Those are the natural choices for *that* choice of $\overline{f_1}$. To summarise

- if $k \in K_2$, $s(\overline{k}) = \overline{d_1d_0(k)}$, $t(\overline{k}) = \overline{d_0^{(2)}(k)}$, whilst
  - $d_0(\overline{k}) = \overline{d_0(k)}$,
  - $d_1(\overline{k}) = \overline{d_1(k)}(\overline{d_2(k)})^{-1}$,

  and it works.

  We define $s_i(\overline{k}) = \overline{s_i(k)}$ for $0 \leq i \leq n-1$ and set $\overline{s_n(k)} =$ identity, and do this for all n, although we have not yet looked at $k \in K_n$ for $n > 2$, to which we turn next:

- For $k \in K_n$, in general, we take $\overline{k} \in G(K)_{n-1}$ with
  - $s(\overline{k}) = \overline{d_1d_0^{(n-1)}(k)}$,
  - $t(\overline{k}) = \overline{d_0^{(n)}(k)}$

  with $G(K)_{n-1}$ free on the graph,

$$K_n \underset{t}{\overset{s}{\rightrightarrows}} K_0 \ ,$$

  excepting the edges $s_n(x)$ for $x \in K_{n-1}$.

The face maps are given by

- $d_i(\overline{k}) = \overline{d_i(k)}$ for $0 \leq i < n-1$,
- $d_{n-1}(\overline{k}) = \overline{d_{n-1}(k)}(\overline{d_n(k)})^{-1}$.

It is easy to check that these satisfy the simplicial identities with the degeneracies as given earlier.

We have chosen this source and target, based on a reasonable choice for $\overline{f}$, but there are other choices that could perhaps have been made. For instance, for $(h_2, h_1) \in \overline{W}(H)_2$ with $s(h_2) = t(h_1)$, but that, perhaps, suggests forming $h_2 \cdot s_1(h_1)$, or similar, and this might give another way of defining generators for $G(K)_{n-1}$ and hence a different expression for the elements. We would expect that the result is isomorphic to the G that we have written down, as both *should* be adjoint to $\overline{W}$. The inconvenience of the definition that we have given is that the source and target of $\overline{k}$ seem very strange. It would be nice to have, for instance, for $k \in K_2$, $s(\overline{k}) = \overline{d_1 d_2(k)}$ and $t(\overline{k}) = \overline{d_0 d_0(k)}$ as these, naively, look to be where the simplex starts and ends. Such a choice would make it easier to link it with the left adjoint of the homotopy coherent nerve functor. On the 'plus side', for the G that we have written down (and also for the Dwyer - Kan original version), is that it has an easy unit and counit for the adjunction and a clear link with the twisting function (cf. page 266) for the reduced case. (The other choices suggested may also work *and* the links with twisting function formulations of twisted cartesian products may be as clear in that revised form[25].)

We have stated that this form of G is left adjoint to the 'functional form' of $\overline{W}$ and we launched into this to examine what the idea of 'weak morphism' would give at the 'elementwise' level. Remember, a weak morphism from C to D corresponded to a map of simplicial groups from $G\overline{W}(K(\mathsf{C}))$ to $K(\mathsf{D})$. The counit of the adjunction goes from $G\overline{W}$ to Id and one way to get some data that correspond to a weak morphism is to find some neat way of describing a section of this from $K(\mathsf{C})$ to $G\overline{W}(K(\mathsf{C}))$. That would, we may suppose, correspond to a weak morphism from C to $M(G, 1)$, where $G = G\overline{W}(K(\mathsf{C}))$.)

For this to be feasible, we need to know more about the counit, $\varepsilon : G\overline{W}(H) \to H$, in general, and so may as well look at the unit, $\eta : K \to \overline{W}G(K)$, as well, so as to indicate the structures behind this adjunction.

The unit, $\eta_K : K \to \overline{W}G(K)$: Remember what η_K is. It corresponds, in the adjunction, to the identity on $G(K)$, so one way to derive the following formulae is to work out $\underline{f} : K \to \overline{W}(H)$, when starting with $f : G(K) \to H$.

We have that if $k \in K_n$, then $\underline{f}_n(k)$ will be of the form $(h_n, \ldots, h_1)$ with $h_i \in H_{i-1}$, as before. Looking at $d_n(\underline{f}_n(k) = \underline{f}_{n-1}(d_n(k))$ gives us $(h_{n-1}, \ldots, h_1)$ and allows us to use induction to get all but $h_n \in H_{n-1}$, but we also have that $f_{n-1}(k) \in H_{n-1}$, so we have an obvious candidate for that missing element.

You can easily follow through this process, either for a general $f : G(K) \to H$, or just for $f : G(K) \to G(K)$ being the identity morphism, and this gives η_K.

To write η_K down neatly, it is useful to introduce an abbreviation. If $k \in K_n$, its *last* listed face is $d_n k$ and we will need to iterate this last face construction, $d_{n-1}d_n(k)$ and so on. Rather than have long strings $d_i \ldots d_{n-1}d_n(k)$, we will write 'L' for 'last' and so define

$$d_L^{(m)} = d_{n-m+1} \ldots d_{n-1} d_n$$

[25]I have never seen it explored. Such an exploration would be a **good exercise to do**. If it works well, it could be useful; if it does not work out, why not? **Perhaps some reader will attempt this.** I do not know the answer.

as the m-iterated last face operator. With this notation, for $k \in K_n$,

$$\eta_K k = (\overline{k}, \overline{d_n(k)}, \ldots, \overline{d_L^{(n-1)}(k)}).$$

(You are **left to check the detail**.)

The counit, $\varepsilon_H : G\overline{W}(H) \to H$: We have already seen how to build $\overline{f} : G(K) \to H$ if we start with $f : K \to \overline{W}(H)$, as that was how we sorted out the structure in this version of $G(K)$. Given such an f, where $f_n(k) = (h_n(k), \ldots, h_1(k))$, we had that

$$\overline{f}_{n-1}(\overline{k}) = h_n(k).$$

We thus get, in particular, that if we have $\underline{h} = (h_n, \ldots, h_1)$ in $\overline{W}(H)$, then

$$\varepsilon_H(\overline{\underline{h}}) = h_n,$$

so is almost a 'projection' defined on the generators. (Of course, it resembles even more the counit of the free group(oid) monad which evaluates a word in the elements of a group.)

11.7.4 Identifying $M(G, 1)$

It is not difficult to *start* identifying the Moore complex, $N(G\overline{W}(H))$, in terms of free groups on Moore complex terms from H itself. You can do this with 'bare hands' and it is quite instructive. A complete verification of what you might suspect the terms to be is quite tricky, however, so we will limit ourselves to the case $H = K(\mathsf{C})$ for C, our 'usual' crossed module, $\mathsf{C} = (C, P, \partial)$, as, there, $N(K(\mathsf{C}))_n$ is trivial for $n \geq 2$, and we will even avoid calculating $N(G\overline{W}(K(\mathsf{C})))_1$, as we really need its quotient $M(G\overline{W}(K(\mathsf{C})), 1)$. (We will, as before, write G for $G\overline{W}(K(\mathsf{C}))$, for convenience.)

We will use a neat argument to identify the crossed module, $M(G\overline{W}(K(\mathsf{C})), 1)$, via another route. Before that we *will* look at the bottom terms of the Moore complex of this G.

We write $\underline{h} = (h_n, \ldots, h_1)$, so this defines a generator $\overline{\underline{h}}$ in G_{n-1}. We thus have G_0 is freely generated by the elements of P, *i.e.*, $G_0 \cong FU(P)$, where F is the free group functor and U the underlying set functor.

We can examine a generator, $\overline{\underline{h}}$, for $n = 2$, *i.e.*, in G_1, and

$$d_1(\overline{\underline{h}}) = \overline{d_1(\underline{h})}.\overline{d_2(\underline{h})}^{-1} = \overline{(d_1(h_2).h_1)}.\overline{(h_1)}^{-1}.$$

We immediately can see that such a term will vanish if $d_1(h_2)$ is trivial and with a little more work can show that a word in such terms and their inverses vanishes if d_1 of the h_2-parts of it vanishes. (We will leave this slightly vague as the calculation is **worth doing** and this **is worth pursuing** on your own, so as to get a better 'elementary' understanding of G_1 - and, in fact, of higher G_n in more generality.) This suggests that $N(G)_1$ may be the free group on the underlying set of $NK(\mathsf{C})_1$, but does not by itself prove this (and as we will side-step this calculation shortly, we do not need to do it now).

Of course, $M(G, 1)$ has 'top term' $NG_1/d_0(NG_2)$, so attacking at the elementwise level, the next step would seem to be to work out NG_2 or rather $d_0(NG_2)$ as that is all we need for the moment. We will not, in fact, do this, although, we repeat, it is **worthwhile doing so**, instead we will backtrack a little and review the problem from another direction, one that we visited a few pages back.

We have the counit of the adjunction, giving

$$\varepsilon : G \to K(\mathsf{C}),$$

and, by the construction of the associated crossed complex, $\mathsf{C}(G)$, of the simplicial group G, an adjoint induced map,

$$\mathsf{C}(G) \to \mathsf{C}.$$

This factorises via the map

$$M(G,1) \to \mathsf{C},$$

that we are seeking to understand. For this last step, we are using that $M(G,1)$ is left adjoint to the natural inclusion of the category of crossed modules into that of crossed complexes (both can be 'reduced' or unreduced, it makes no difference).

We also had that $\mathsf{C}(-)$ was left adjoint to the 'inclusion' of crossed complexes (disguised, via K and the Dold-Kan theorem, as group (or groupoid) T-complexes) into all simplicial groups (or $\mathcal{S}$-groupoids). This chain of left adjoints translates into a single universal property, one which is very useful.

If we have any crossed module, E, having $FU(P)$ at its base, and any morphism

$$\mathsf{f} : \mathsf{E} \to \mathsf{C},$$

having that $f_0 : E_0 \to P$ is $\varepsilon_P : FU(P) \to P$, the counit of the free group monad, then we can factor f through the pullback crossed module, $\varepsilon_P^*(\mathsf{C})$:

$$\begin{tikzcd} E_1 \arrow[rr] \arrow[dr] & & E_1 \times_P C \arrow[dl] \\ & E_0 & \end{tikzcd}$$

(see page 43 and note that here $\varepsilon_P^*(\mathsf{C})_1 \cong E_1 \times_P C$). We will generalise this slightly in a moment, but first we introduce some terminology. As before, $\mathsf{C} = (C, P, \partial_{\mathsf{C}})$ and $\mathsf{D} = (D, Q, \partial_{\mathsf{D}})$ are crossed modules:

Definition: (i) A map, $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, of crossed modules is a *fibration* if $f_1 : C \to D$ and $f_0 : P \to Q$ are both epimorphisms of groups.

(ii) A map, f, as above, is a *trivial fibration* if it is a fibration and the induced map,

$$C \to D \times_Q P,$$

is an isomorphism.

Remarks: (i) If $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ is a fibration, it should be obvious that $K(\mathsf{f}) : K(\mathsf{C}) \to K(\mathsf{D})$ is a dimensionwise epimorphism of simplicial groups and hence is a fibration of such (in the sense we discussed in section 1.3.5, page 38). We therefore get a fibration exact sequence of homotopy groups. We set $\mathsf{B} = Ker(\mathsf{f})$, that is, $(Ker(f_1), Ker(f_0), \partial)$ for the restricted $\partial = \partial_{\mathsf{C}}|_{Ker(f_1)}$, and then obtain

$$1 \to \pi_1(\mathsf{B}) \to \pi_1(\mathsf{C}) \to \pi_1(\mathsf{D}) \to \pi_0(\mathsf{B}) \to \pi_0(\mathsf{C}) \to \pi_0(\mathsf{D}) \to 1.$$

This is just the usual $Ker - Coker$ 6-term exact sequence of homological algebra, but in a slightly non-Abelian context.

(**Remember** that in our notation $\pi_1(\mathsf{C}) = Ker\,\partial_{\mathsf{C}}$ and $\pi_0(\mathsf{C}) = Coker\,\partial_{\mathsf{C}} \cong P/\partial_{\mathsf{C}}C$. This is a shift of index from the notation used in some sources, where our $\pi_1(\mathsf{C})$ would be their $\pi_2(\mathsf{C})$, because it is the π_2 of the classifying space of C. Likewise our π_0 is their π_1, so **always check** when comparing results.)

(ii) Suppose now that

$$\begin{tikzcd} C \arrow[r, "f_1"] \arrow[d, "\partial_{\mathsf{C}}"'] & D \arrow[d, "\partial_{\mathsf{D}}"] \\ P \arrow[r, "f_0"'] & Q \end{tikzcd}$$

is a pullback square (which is just saying that $C \to D \times_Q P$ is an isomorphism). It is well known that that implies that the kernels of ∂_{C} and ∂_{D} are isomorphic (via the restricted f_1). That fact is general and has a useful, easy categorical proof, but, none-the-less, we will give an 'element-wise' one, since it shows different aspects that can also be useful. It is equally easy, but slightly less general.

We replace C by $D \times_Q P$, so an element of this is a pair, (d, p), such that $\partial_{\mathsf{D}}d = f_0p$. The description of ∂_{C} is then $\partial_{\mathsf{C}}(d, p) = p$, the second projection morphism. If $(d, p) \in Ker\,\partial_{\mathsf{C}}$, then $p = 1_P$ and $\partial_{\mathsf{D}}d = f_0 1_P = 1_Q$, so the isomorphism claimed associates $(d, 1)$ and d, where $d \in Ker\,\partial_{\mathsf{D}}$.

Going back to the exact sequence, we have that the induced map from $\pi_1(\mathsf{C})$ to $\pi_1(\mathsf{D})$ is an isomorphism in this case (as $\pi_1(\mathsf{B})$ is trivial). We can calculate B explicitly, of course. Identifying C with $D \times_Q P$ once again, $f_1(d, p) = d$, so $(d, p) \in Ker\,f_1$ if $d = 1_D$, and then, of course, $f_0(p) = 1_Q$, so $p \in Ker\,f_0$. The crossed module, B, is thus isomorphic to the crossed module, $(Ker\,f_0, Ker\,f_0, id)$, so, again of course, $\pi_1(\mathsf{B})$ is trivial! It is then clear that $\pi_0(\mathsf{B})$ is also trivial. In other words,

Lemma 83 *A trivial fibration of crossed modules is a weak equivalence.* ■

The particularly useful case of this is the following: Given a crossed module, $\mathsf{C} = (C, P, \partial_{\mathsf{C}})$, pick a free group F together with an epimorphism,

$$\varepsilon : F \to P,$$

(for instance, if given a presentation of P, use the free group on the given set of generators). Form $\varepsilon^*(\mathsf{C}) = (C \times_P F, F, \partial')$, which will be, as we know, a crossed module. There is an induced fibration,

$$\mathsf{f} : \varepsilon^*(\mathsf{C}) \to \mathsf{C},$$

and this will be, by construction, a trivial fibration.

Example: We could take $F = FU(P)$, the free group on the underlying set of P with the counit, ε_P, as the epimorphism. Our earlier discussion suggested that $\varepsilon_P^*(\mathsf{C})$ looks somewhat like our G from there.

This example is what we will need, but it is not the only one around, of course. (That 'looks somewhat like' is vague and we need to do better than that! Here is that in detail.)

Proposition 136 *For* $\mathsf{C} = (C, P, \partial_{\mathsf{C}})$, $G = G\overline{W}(K(\mathsf{C}))$, *and* $\varepsilon_P : FU(P) \to P$, *as before, there is an isomorphism,*

$$M(G,1) \cong \varepsilon_P^*(\mathsf{C}).$$

Proof: We know the base groups are both isomorphic to $FU(P)$, and so have to produce an isomorphism,

$$M(G,1)_1 \cong FU(P) \times_P C,$$

over P, compatibly with the actions.

We certainly have the counit morphism,

$$\begin{tikzcd} M(G,1)_1 \arrow[r] \arrow[d, "\partial"'] & C \arrow[d, "\partial_C"] \\ FU(P) \arrow[r, "\varepsilon_P"'] & P \end{tikzcd}$$

which we will call f, for convenience. We know it is a weak equivalence, since $G\overline{W}(K(\mathsf{C})) \to K(\mathsf{C})$ is a weak equivalence of simplicial groups, so $Ker\,\mathsf{f}$ has trivial homotopy.

We get $\overline{f} : M(G,1)_1 \to FU(P) \times_P C$ by the universal property of pullbacks. Explicitly

$$\overline{f}(h) = (\partial h, f_1(h)).$$

This map, $\overline{f}$, is a morphism of crossed modules by simple general arguments, (*i.e.*, nothing to do with our particular situation here). We thus want to prove $\overline{f}$ is an isomorphism.

We note that $Ker\,\overline{f} \subseteq Ker\,f_1 \cap Ker\,\partial$, but $Ker\,\mathsf{f}$ has trivial π_1, so $Ker\,\overline{f}$ must be trivial and $\overline{f}$ is a monomorphism.

Is $\overline{f}$ an epimorphism? If $(h_0, c_1) \in FU(P) \times_P C$, so $f_0(h_0) = \partial c_1$, then pick $h_1 \in M(G,1)_1$ such that $f_1(h_1) = c_1$, (**check** that f_1 is onto). We have

$$f_0(h_0) = f_0(\partial h_1),$$

so $h_0 = \partial h_1.k_0$ for some $k_0 \in Ker\,f_0$. We also have $\pi_0(Ker\,\mathsf{f})$ is trivial, so there is some $k_1 \in Ker\,f_1$ with $\partial k_1 = k_0$, but then $h' = h_1 k_1$ satisfies

$$\partial h' = h_0, \quad f_1(h') = f_1(h_1) = c_1,$$

so $\overline{f}$ is onto. ■

11.7.5 Cofibrant replacements for crossed modules

In other words, we have identified $M(G,1)$ completely and it has an easy description.

What about the properties of other $\varepsilon^*(\mathsf{C})$ for $\varepsilon : F \to P$, with F free? For the moment, this is prompted by curiosity, but it does provide some useful insights later on.

Our present situation is that a weak map from C to D is given by an actual map of crossed modules,

$$\varepsilon_P^*(\mathsf{C}) \to \mathsf{D},$$

and we also know that the map, $\varepsilon_P^*(\mathsf{C}) \xrightarrow{\simeq} \mathsf{C}$, is 'really' a counit or 'augmentation' of a resolution. We get a span

$$\mathsf{C} \xleftarrow{\simeq} \varepsilon_P^*(\mathsf{C}) \to \mathsf{D}.$$

What about other similar spans,

$$\mathsf{C} \overset{\simeq}{\leftarrow} \varepsilon^*(\mathsf{C}) \to \mathsf{D},$$

with ε, an epimorphism, $\varepsilon : F \to P$, and F a free group? Do they also give weak maps in some way? Of course, this is almost the same question as the previous one.

Before looking at this, we note a nice result:

Proposition 137 *For $\mathsf{C} = (C, P, \partial_{\mathsf{C}})$, with P a free group, the natural morphism $\varepsilon_P^*(\mathsf{C}) \overset{\simeq}{\to} \mathsf{C}$ is a split epimorphism.*

Proof: Of course, $\varepsilon : FU(P) \to P$ is split, since P is free. Let $\sigma_0 : P \to FU(P)$ be a splitting. From σ_0, we can construct

$$\sigma_1 : C \to FU(P) \times_P C,$$

by

$$\sigma_1(c) = (\sigma_0\partial_{\mathsf{C}}(c), c),$$

as being the unique group homomorphism given by the pullback property. It is easy to check that (σ_1, σ_0) defines a crossed module morphism splitting the epimorphism induced by ε_P^*. ■

In fact, this split epimorphism is a trivial fibration, but we will not need this.

We next introduce a bit more of the homotopical terminology as applied to crossed modules, or equivalently to 2-group(oid)s. The ideas are derived from the paper, [208], by Moerdijk and Svensson. We first extend 'fibration' and 'trivial fibration' from crossed modules to 2-group(oid)s via the usual equivalence of categories. We give this in two forms, the first is from Noohi's paper, [224], the second from [208].

Definition: A morphism, $\psi : \mathscr{A} \to \mathscr{B}$, of 2-groupoids is called a *Grothendieck fibration* (or more simply a *fibration*) if it satisfies the following properties:

Fib. 1: for every arrow $b : B_0 \to B_1$ in $\mathscr{B}$ and every object, A_1, in $\mathscr{A}$ over B_1, (so $\psi(A_1) = B_1$), there is a lift $a : A_0 \to A_1$ with codomain, a_1;

Fib. 2: for every 2-arrow, $\beta : b_0 \Rightarrow b_1$ in $\mathscr{B}$, and every arrow a_1 in $\mathscr{A}$ such that $\psi(a_1) = b_1$, there is an arrow a_0 and a 2-arrow $\alpha : a_0 \Rightarrow a_1$ such that $\psi(\alpha) = \beta$.

The fibration is *trivial* if it is also a weak equivalence, *i.e.*, inducing isomorphisms on π_0, π_1 and π_2.

Remark: This is nice as the first condition is a lifting condition for 1-arrows, whilst the second is one for 2-arrows. It is worth noting a slight more or less inconsequential choice is being made here. In covering space theory, it is usual to mention 'unique path lifting'. Recall that this relates to a continuous map of spaces, say $p : Y \to X$, and it requires that if $\lambda : I \to X$ is a path in X and we specify a point y_0 over $x_0 = \lambda(0)$, the starting point of λ, then there is a (unique) lift, $\tilde{\lambda}$, of λ starting at y_0.

In the above definition of fibration for 2-groupoids, no uniqueness is required, but also the specified point is the codomain of the 1-arrow, which intuitively corresponds to the end of the path rather than the start. This does not matter here as in a 2-groupoid both 1- and 2-arrows are invertible, but it is another instance of the lax / op-lax / pseudo 'conflict', so is worth noting that a choice *has* been made here.

Warning about the notation in 'trivial fibration': At the risk of repeating this too often, it should be noted that, if thinking of crossed modules rather than 2-groupoids, the above π_1 is the cokernel of the structure map and π_2 is its kernel. The set of connected components for a *2-group* will be a singleton. The π_1 of the 2-group is the π_0, in our notation, of the corresponding crossed module or simplicial group, and so on.

The alternative definition combines the two conditions in one. It occurs in Moerdijk and Svensson's paper, [208], so will be referred to as the M-S form of the definition.

Definition (alternative M-S form): A morphism, $\psi : \mathscr{A} \to \mathscr{B}$, of 2-groupoids is called a *Grothendieck fibration* (or more simply a *fibration*) if it satisfies the following condition:

for any arrow, $a : A_1 \to A_2$, in $\mathscr{A}$ and any arrows, $b_1 : B_0 \to \psi(A_1)$ and $b_2 : B_0 \to \psi(A_2)$, then any 2-arrow, $\alpha : b_2 \Rightarrow \psi(a) \circ b_1$, can be lifted to a 2-arrow, $\tilde{\alpha} : \tilde{b_2} \Rightarrow a \circ \tilde{b_1}$, (so $\psi(\tilde{\alpha}) = \alpha$, etc.).

Proposition 138 *The two forms of the definition are equivalent.*

Proof: We limit ourselves to a sketch, as the proof is quite easy, once you see that doing a fairly obvious thing is exactly what is needed. (Of course, the details are the **left to you as an exercise**.)

First assume we have a morphism satisfying the alternative (M-S) form of the definition. We must show it to have a lifting property for both 1- and 2-arrows.

Suppose we have $b : B_0 \to B_1$ in $\mathscr{B}$ and an object, A_1, in $\mathscr{A}$ over B_1, (so $\psi(A_1) = B_1$), then, in the alternative form, take $b_1 = b_2 = b$ with $\beta : b_1 \Rightarrow b_1$ the identity 2-arrow. The lift given by the M-S condition gives us a $\tilde{b} : A_0 \to A_1$ (and a $\tilde{\beta}$ that we do not actually need or use).

We thus have: 'M-S' $\Rightarrow$ '1-arrow lifting'.

To derive '2-arrow lifting' from 'M-S', we start with $\beta : b_0 \Rightarrow b_1$ and a_1 such that $\psi(a_1) = b_1$, and need to get some $\tilde{\beta} : \tilde{b_0} \Rightarrow a_1$ over β. This time we choose, in the input to the M-S condition, $a := a_1$, $b_1 := id$, $b_2 := b_0$, so $\beta : b_2 \Rightarrow \psi(a) \circ b_1$, as required, and can read off the lift accordingly. (Beware, you will get an extra lift, say x, of b_1 in your expression that you do not want, and cannot guarantee that it is the identity, however it is invertible, so you can adjust things to fit.)

Given that sketch, the other direction of the equivalence is easy. Assuming 1- and 2-arrow lifting, start with the M-S situation, lift b_1 using 1-arrow lifting, then $b_1 \circ \psi(a) = \psi(\tilde{b_1} \circ a)$, so we can apply 2-arrow lifting to β. ■

The advantage in having these two forms of the definition is that the M-S form is very neat from the categorical context, but the arrow lifting version is more easily seen to be the 2-groupoid version of the definition of fibration of crossed modules that we gave on page 570 and of the 'classical' epimorphism-condition for a 'fibration of simplicial groups'.

Moerdijk and Svensson, [208], also consider *cofibrations*. For the moment, we just need the corresponding condition for an object to be *cofibrant*.

Definitions: (i) A 2-group, $\mathscr{G}$, is *cofibrant in the Moerdijk-Svensson structure*, (we will say *M-S cofibrant*) if every trivial fibration $\mathscr{H} \to \mathscr{G}$, where $\mathscr{H}$ is a 2-groupoid, admits a section.

(ii) A crossed module, C, is *cofibrant* if the corresponding 2-group, $\mathscr{X}(\mathsf{C})$, is M-S cofibrant.

Proposition 139 *(Noohi, [222]) A crossed module* $\mathsf{C} = (C, P, \partial)$*, is cofibrant if and only if* P *is a free group.* ∎

The proof, which is given by Noohi, [222], is similar to that given above for Proposition 137. It can be safely **left to the reader**, except to note that it *does* require the use of the result that subgroups of free groups are free. (Analogues of this result in other categories than that of groups, would need reformulation to avoid the use of the analogous statement which may or may not be true in such settings.)

Example: For any crossed module, C, the pullback crossed module, $\varepsilon_P^*(\mathsf{C})$, or, equivalently $M(G\overline{W}(K(\mathsf{C})), 1)$, is cofibrant. We note also that it depends functorially on C and that there is a natural trivial fibration, $\varepsilon_P^*(\mathsf{C}) \to \mathsf{C}$.

Definition: (i) For C, a crossed module, a *cofibrant replacement for* C is cofibrant crossed module QC, together with a trivial fibration, $\mathsf{q} : \mathsf{QC} \to \mathsf{C}$.

(ii) A *cofibrant replacement functor (for crossed modules)* consists of a functor, $\mathsf{Q} : CMod \to CMod$, together with a natural transformation, $\mathsf{q} : \mathsf{Q} \to Id$, such that for each crossed module, C, $\mathsf{q}_\mathsf{C} : \mathsf{QC} \to \mathsf{C}$ is a cofibrant replacement for C.

The idea of cofibrant replacement given here is just the particular case for the context of crossed modules of a general notion from homotopical algebra. (We suggest that you look at a standard text on model categories and other ideas of homotopical algebra for further details. One such is Hovey's [158].) In a model category, as considered there, there are notions of weak equivalence, fibration and cofibration and thus of fibrant and cofibrant objects. For example, in the category of simplicial sets, considered with its usual model category structure, weak equivalences are what we would expect, that is, simplicial maps inducing isomorphisms of π_0 and all higher homotopy groups for all possible choices of base points. Fibrations are Kan fibrations and cofibrations are simplicial inclusions. All objects are cofibrant, but only the Kan complexes are fibrant. For simplicial groups, fibrations are the morphisms that are epimorphisms in each dimension, and the cofibrant objects are the simplicial groups that are free in each dimension.

For any model category, one can define cofibrant replacements as above, and, dually, fibrant replacements, and can prove that they always exist. They are the model categoric analogues of the projective and injective resolutions of more classical *homological* algebra and are similarly used to define *derived functors*. These, of course, are intimately related to cohomology theory, but we will not follow that link very far here, as our main use for this here is as an illustration and example of homotopy coherence.

For some of the theory of cofibrant replacements and total derived functors, look at the book by Hovey, [158], which is also an excellent introduction to the wider theory of model categories. (It is also useful to glance back at the original sources on homotopical algebra, in particular Quillen's original [231] and the related [232].)

If $\mathcal{C}$ is a model category and Q is a cofibrant replacement functor, the idea is that the value of the derived functor of some functor, $F : \mathcal{C} \to \mathcal{D}$, at an object, C, is obtained by looking at $F(QC)$ 'up to homotopy'. That *is* vague, but, in our context of weak maps, we have, for any given crossed module, D, a functor $CMod(-, \mathsf{D})$ from $CMod^{op}$ to ..., where? Actually 'to the category of groupoids' would be a suitable choice, as we have not only morphisms between crossed modules, but homotopies between them. There is also a groupoid of weak maps from C to D with weak natural transformations as the arrows. (This is **left to you** to look up in Noohi's papers, [222, 224], or to investigate yourselves.) As our functorial Q, given explicitly by $M(G\overline{W}(K(-),1))$, naturally gives weak maps, we come back to our question from earlier, which we can now ask with more exact terminology:

Suppose $\mathsf{q} : \mathsf{QC} \to \mathsf{C}$ *is a cofibrant replacement for* C*, and* $\psi : \mathsf{QC} \to \mathsf{D}$ *is a map of crossed modules, does* ψ *induce a weak map from* C *to* D*?*

We write $\mathsf{QC} = (QC_1, F, \partial_Q)$, and find that, as $\mathsf{q} : \mathsf{QC} \to \mathsf{C}$ is a trivial fibration, $QC_1 \cong F \times_P C = q_0^*(\mathsf{C})_1$. We thus have a lot of information about QC.

Next, apply the functorial construction to $\mathsf{q} : \mathsf{QC} \to \mathsf{C}$ to get

$$\begin{tikzcd}
\varepsilon_F^*(\mathsf{QC}) \arrow[r] \arrow[d] & \varepsilon_P^*(\mathsf{C}) \arrow[d] \\
\mathsf{QC} \arrow[r, "\mathsf{q}"'] & \mathsf{C}
\end{tikzcd}$$

as the two vertical morphisms and the bottom one are weak equivalences, so is the top. It is also a fibration. (In fact, it is the induced map which at level 1 is the obvious map,

$$FU(F) \times_F (F \times_P C) \to FU(P) \times_P C,$$

so is easily checked to be one.) It is thus a trivial fibration with cofibrant codomain. It is therefore split by some section,

$$\sigma : \varepsilon_P^*(\mathsf{C}) \to \varepsilon_F^*(\mathsf{QC}).$$

We can compose this with the natural morphism, $\mathsf{q}_{\mathsf{QC}} : \varepsilon_F^*(\mathsf{QC}) \to \mathsf{QC}$.

Now suppose $\psi : \mathsf{QC} \to \mathsf{D}$ is a morphism of crossed modules, then it gives a composite,

$$\varepsilon_P^*(\mathsf{C}) \xrightarrow{\sigma} \varepsilon_F^*(\mathsf{QC}) \xrightarrow{\mathsf{q}_{\mathsf{QC}}} \mathsf{QC} \xrightarrow{\psi} \mathsf{D}.$$

Clearly, there may be many sections of the map from $\varepsilon_F^*(\mathsf{QC})$ to $\varepsilon_P^*(\mathsf{C})$, so many different 'weak maps' would seem to correspond to a single $\psi : \mathsf{QC} \to \mathsf{D}$, but these weak maps only depend on ψ in the 'last composition'. If we look slightly more deeply, it becomes clear that they correspond to sections of $FU(F) \to FU(P)$, *i.e.*, to choices of transversals for $FU(F) \to P$. This is known, 'standard', even 'classical' territory, and will be **left to you to explore**. The point is that two weak maps coming from different sections, σ and σ', are likely to be 'homotopic' in some sense. (This is explored in the work of Noohi that we referred to earlier.) We summarise the above in the following:

Proposition 140 *If* $\mathsf{q} : \mathsf{QC} \to \mathsf{C}$ *is any cofibrant replacement for a crossed module, any crossed module morphism,* $\psi : \mathsf{QC} \to \mathsf{D}$*, induces a (usually non-unique) weak map of crossed modules from* C *to* D*.* ■

11.7.6 Weak maps: from cofibrant replacements to the algebraic form

It is not hard to start with a weak map, $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, described as a pseudo-functor from $\mathscr{X}(\mathsf{C})$ to $\mathscr{X}(\mathsf{D})$, and to convert that description, via the nerves, to the algebraic description of f. (For instance, as the nerve of $\mathscr{X}(\mathsf{C})$ has P in one dimension and $C \times P \times P$ in the next, the values of f on these should give the f_0, f_1, and the pairing without too much bother.) **Leaving you to investigate** that later by yourself, let us pass further into the simplicial description and use the functorial cosimplicial replacement, $\varepsilon_P^*(\mathsf{C})$, so that we specify f by a crossed module morphism,

$$\mathsf{f} : \varepsilon_P^*(\mathsf{C}) \to \mathsf{D}.$$

(We will write $\mathsf{D} = (D_1, D_0, \partial_{\mathsf{D}})$.) This gives us a square

$$\begin{tikzcd} F \times_P C \arrow[r, "f_1"] \arrow[d, "\partial"'] & D_1 \arrow[d, "\partial_{\mathsf{D}}"] \\ F \arrow[r, "f_0"'] & D_0 \end{tikzcd}$$

where we have written F for $FU(P)$. The elements of $F \times_P C$ are pairs (ω, c), where $\varepsilon_P(\omega) = \partial_{\mathsf{C}} c$, thus ω is a word in generators corresponding to elements of P. We will write (p) for the generator coming from $p \in P$.

Surprisingly enough the f_0 in this corresponds almost exactly to the f_0 in the usual algebraic description. There is a small difference, $f_0(p)$ in the latter description is $f_0((p))$ in the former one, so is the composite of the cofibrant replacement's f_0 with the set theoretic section, η_P, of the epimorphism, $\varepsilon_P : F \to P$, given by 'p goes to (p)', in other words, with the unit of the free-forget adjunction.

Notationally we need to distinguish the two, so will write f_i^{cr} for the different levels of the crossed module morphism, $f : \varepsilon_P^*(\mathsf{C}) \to \mathsf{D}$, the superfix 'cr' standing for 'cofibrant replacement', of course. This notation will be a temporary one. We thus have

$$f_0(p) = f_0^{cr}((p)).$$

We need to obtain $\langle -, - \rangle : P \times P \to D_1$, and $f_1 : C \to D_1$ and these must satisfy certain rules; see the definition on page 562. The basic ones are $\partial_{\mathsf{D}} f_1 = f_0 \partial_{\mathsf{C}}$, and the two 'almost a homomorphism' conditions. The one for f_0 gives

$$f_0(p_2 p_1) = \partial \langle p_2, p_1 \rangle^{-1} f_0(p_2) f_0(p_1).$$

This gives us a lever to get at $\langle p_2, p_1 \rangle$. For any pair of elements, p_2, p_1 in P, we have a cocycle

$$(p_2)(p_1)(p_2 p_1)^{-1} \in FU(P) = F$$

and this is in the kernel of ε_P. As a result, there is an element

$$\{p_2, p_1\} = ((p_2)(p_1)(p_2 p_1)^{-1}, 1) \in F \times_P C.$$

We look at

$$f_1^{cr}\{p_2, p_1\} \in D_1.$$

We have

$$\partial_{\mathsf{D}} f_1^{cr}\{p_2, p_1\} = f_0^{cr} \partial\{p_2, p_1\} = f_0(p_2) f_0(p_1) f_0(p_2 p_1)^{-1},$$

so if we take $\langle p_2, p_1 \rangle := f_1^{cr}\{p_2, p_1\}$, we get the 'almost a homomorphism' condition for f_0.

What about that for f_1? Well, we have yet to write down some f_1 in terms, perhaps, of f_1^{cr}, but if we have $c \in C$, then we clearly have an element $((\partial_{\mathsf{C}} c), c) \in F \times_P C$, so it is a fairly safe bet that $f_1(c)$ will be $f_1^{cr}((\partial_{\mathsf{C}} c), c)$, (or possibly its inverse, since directions can easily get reversed with the different conventions, and it does not pay to be too sure in advance of detailed checking!) The obvious thing to do is to try it in the W2 'almost a homomorphism' condition for f_1, again see the discussion around page 562. In fact, we note

$$\begin{aligned}((\partial_{\mathsf{C}} c_2), c_2).((\partial_{\mathsf{C}} c_1), c_1) &= ((\partial_{\mathsf{C}} c_2)(\partial_{\mathsf{C}} c_1), c_2 c_1)\\ &= ((\partial_{\mathsf{C}} c_2)(\partial_{\mathsf{C}} c_1)(\partial_{\mathsf{C}}(c_2 c_1))^{-1}, 1)((\partial_{\mathsf{C}}(c_2 c_1), c_2 c_1),\end{aligned}$$

so, mapping this via f_1^{cr} gives

$$f_1(c_2) f_1(c_1) = \langle \partial c_2, \partial c_1 \rangle . f_1(c_2 c_1),$$

as required.

Of course, we will need to check the other two conditions, but that is **left to you**. (The cocycle condition is easy to check, the whiskering conditions do require some work. You might start by checking what the action of F on $F \times_P C$ is.) We have proved (modulo your checking):

Proposition 141 *Given a morphism $f^{cr} : \varepsilon_P^*(\mathsf{C}) \to \mathsf{D}$, the structure*

- $f_0 : P \to D_0$ *given by* $f_0(p) = f_0^{cr}((p))$;
- $f_1 : C \to D_1$ *given by* $f_1^{cr}(\partial_{\mathsf{C}} c), c)$;
- $\langle -, - \rangle : P \times P \to D_1$ *given by* $\langle p_2, p_1 \rangle := f_1^{cr}\{p_2, p_1\}$, *where* $\{p_2, p_1\} = ((p_2)(p_1)(p_2 p_1)^{-1}, 1)$,

specifies a weak map, $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, *(in the algebraic description format).* ■

11.7.7 Butterflies

We have, when discussing the algebraic definition of a weak map, pointed out the similarities of certain structure with the cocycle description of group extensions and, thus, of group cohomology. For instance, f_0 and $\langle -, - \rangle$ together yield something very like a weak action of P (on D). The cocycle condition, also, is very reminiscent of the conditions on the factor set, $f : G \times G \to K$, that ensure associativity of the multiplication if reconstructing the middle term of the extension from the two ends, together with the weak action and the factor set. This suggests that there should be an extension associated with a weak map.

Collecting up evidence, we have our 'factor set'-like pairing, $\langle -, - \rangle$, going, in our typical situation, from $P \times P$ to D_1. This would correspond to a group extension

$$D_1 \xrightarrow{\iota} E \xrightarrow{\rho} P,$$

and the cocycle condition suggests that we use $f_0 : P \to D_1$ to get a weak action of P on D_1, that is, looking at the cocycle condition and comparing it with the factor set condition (page 56), we need to get P to 'act' on D_1, and we can use f_0 to get from P to D_0 and then use the action of D_0 on D_1 to get something that might work. In other words, we will interpret ${}^{f_0(p)}x$ for $p \in P$ and $x \in D_1$ as the analogue of the weak action in the extension.

To construct the middle term, E, (as in section 2.3.1), we take the *set* $D_1 \times P$ and give it a multiplication

$$(x_1, p_1)(x_2, p_2) = (x_1.{}^{f_0(p_1)}x_2.\langle p_1, p_2\rangle, p_1p_2).$$

The checking that this is associative, etc., is quite easy, but we will give it in some detail as it is neat and shows how the properties of the pseudo-functor defining the weak map are transformed into quite usual properties of the object, E. This checking is, of course, quite standard in the theory of group extensions.

Lemma 84 *The above multiplication is associative.*

Proof: We calculate

$$\begin{aligned} (x_1,p_1)((x_2,p_2)(x_3,p_3)) &= (x_1,p_1)(x_2.{}^{f_0(p_2)}x_3\langle p_2,p_3\rangle, p_2p_3) \\ &= (x_1.{}^{f_0(p_1)}x_2.{}^{f_0(p_1)f_0(p_2)}x_3.{}^{f_0(p_1)}\langle p_2,p_3\rangle\langle p_1,p_2p_3\rangle, p_1p_2p_3). \end{aligned}$$

(It is worth noting that terms that exist in the cocycle condition for $\langle -, -\rangle$ are occurring naturally here.) The 'other side' gives

$$\begin{aligned} ((x_1,p_1)(x_2,p_2))(x_3,p_3)) &= (x_1.{}^{f_0(p_1)}x_2\langle p_1,p_2\rangle, p_1p_2)(x_3,p_3) \\ &= (x_1.{}^{f_0(p_1)}.\langle p_1,p_2\rangle.{}^{f_0(p_1p_2)}x_3\langle p_1p_2,p_3\rangle, p_1p_2p_3). \end{aligned}$$

Comparing the two expressions, we can match up corresponding parts leaving, in the first expression,

$${}^{f_0(p_1)f_0(p_2)}x_3.{}^{f_0(p_1)}\langle p_2,p_3\rangle\langle p_1,p_2p_3\rangle,$$

which rewrites, using 'cocycle', to

$${}^{f_0(p_1)f_0(p_2)}x_3.\langle p_1,p_2\rangle\langle p_1p_2,p_3\rangle.$$

The last term matches with one in the equivalent position in the second expression. We then attack $f_0(p_1)f_0(p_2)$, using 'almost a homomorphism', giving $\partial\langle p_1,p_2\rangle f_0(p_1,p_2)$. We finally use the Peiffer identity, so

$$\begin{aligned} {}^{f_0(p_1)f_0(p_2)}x_3.\langle p_1,p_2 &= {}^{\partial\langle p_1,p_2\rangle f_0(p_1,p_2)}x_3.\langle p_1,p_2\rangle \\ &= \langle p_1,p_2\rangle.{}^{f_0(p_1p_2)}x_3.\langle p_1,p_2\rangle^{-1}.\langle p_1,p_2\rangle \\ &= \langle p_1,p_2\rangle.{}^{f_0(p_1p_2)}x_3, \end{aligned}$$

as hoped. ■

The identity for the multiplication is clearly $(1,1)$, so we certainly have a monoid. What about inverses? We are given (x,p), and so need to solve

$$(y,q).(x,p) = 1.$$

This gives $q = p^{-1}$ and

$$y = \langle p^{-1}, p\rangle^{f_0(p^{-1})}x^{-1}$$

and so

$$(x,p)^{-1} = (\langle p^{-1}, p\rangle^{f_0(p^{-1})}x^{-1}, p^{-1}).$$

Remark: Of course, we know by standard elementary arguments that this 'left inverse' is also a 'right inverse', but **it is quite interesting to calculate the product**, showing

$$(x,p)(\langle p^{-1},p\rangle^{f_0(p^{-1})}x^{-1},p^{-1}) = (1,1)$$

directly. 'Interesting'? Yes, because it presents some useful calculations that otherwise would not come to the surface this early in an investigation. For instance, we have both $\langle p^{-1},p\rangle$ and $\langle p,p^{-1}\rangle$, occurring in the formulae. What is their relationship?

Lemma 85

$$\langle p,p^{-1}\rangle = {}^{f_0(p)}\langle p^{-1},p\rangle.$$

■

The proof follows from the cocycle condition using $p_1 = p_3 = p$ and $p_2 = p^{-1}$.

Another such result is

Lemma 86

$$f_0(p)f_0(p^{-1}) = \partial\langle p,p^{-1}\rangle.$$

■

This is, of course, an immediate consequence of 'almost a homomorphism' and 'normalization', but, for calculations, is very useful to have explicitly stated.

We have now verified that E is a group - which was obvious from the classical theory of factor sets and has nothing specific to do with weak maps or crossed modules. We record the structural maps for convenience:

in $D_1 \xrightarrow{\iota} E \xrightarrow{\rho} P$, *the maps are given by* $\iota(x) = (x,1)$, $\rho(x,p) = p$.

These are easily seen to be homomorphisms.

All that is standard Schreier theory of factor sets and extensions and gives us a diagram, (a 'partial butterfly'),

$$\begin{tikzcd}
C \arrow[dd, "\partial"'] & & D_1 \arrow[dl, "\iota"'] \arrow[dd, "\partial_{\mathsf{D}}"] \\
& E \arrow[dl, "\rho"'] & \\
P & & D_0
\end{tikzcd}$$

In Noohi's theory of papillons (butterflies), (cf. [222] and [5]), we have the following definition:

Definition: Let $\mathsf{C} = (C,P,\partial)$ and $\mathsf{C}' = (C',P',\partial')$ be two crossed modules. By a *papillon*, or *butterfly*, from C to C', we mean a commutative diagram of groups

$$\begin{tikzcd}
C \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & C' \arrow[dl, "\iota"'] \arrow[dd, "\partial_{\mathsf{C}'}"] \\
& E \arrow[dl, "\rho"'] \arrow[dr, "\lambda"] & \\
P & & P'
\end{tikzcd}$$

in which the diagonals are complexes of groups (so $\lambda\kappa$ and $\rho\iota$ are trivial homomorphisms), the NE-SW sequence,

$$C' \xrightarrow{\iota} E \xrightarrow{\rho} P,$$

is short exact (hence is a group extension), $Ker\,\rho = Im\,\iota$, and, moreover, for all $e \in E, c \in C$ and $c' \in C'$, we have

$$\iota({}^{\lambda(e)}c') = e\iota(c')e^{-1},$$

and

$$\kappa({}^{\rho(e)}c) = e\kappa(c')e^{-1}.$$

As 'papillons' are introduced, in [222] and [5], as a way to handle weak maps, we should be able to complete our partial butterfly to a full one by defining a NW-SE complex. The first map, $\kappa : C \to E$, must be something like $\kappa(c) = (f_1(c), \partial c)$, as the usual rule in these situations is 'build it simply from the parts that you have'. That, however, does not quite work. (This may be due to a question of conventions when representing elements of E in the form (x, p), and some different choice might result in the 'fault' disappearing, however I doubt it, but have no evidence 'one way or t'other', - **it is left as a challenge to the reader to shed some light on this!**) Surprisingly enough, what happens with that attempt gives us the clue to resolving the problem.

(To simplify notation slightly, we will usually write ∂ for the boundary in all the crossed modules involved. Context in each case diminishes the risk of confusion.)

Define $\kappa(c) = (f_1(c)^{-1}, \partial c)$.

Proposition 142 *Defined by this, $\kappa : C \to E$ is a homomorphism satisfying*

$$\kappa({}^{\rho(e)}c) = e\kappa(c)e^{-1}.$$

Proof: (This is another of the calculatory verification proofs that could be very safely left to the reader - but, because of strange inversion in the first factor of κ, it is interesting to see how this works.)

We take $c_1, c_2 \in C$,

$$\begin{aligned}\kappa(c_2c_1) &= (f_1(c_2c_1)^{-1}, \partial c_2c_1)\\ &= ((\langle\partial c_2, \partial c_1\rangle^{-1}f_1(c_2)f_1(c_1))^{-1}, \partial c_2\partial c_1)\\ &= (f_1(c_1)^{-1}f_1(c_2)^{-1}\langle\partial c_2, \partial c_1\rangle, \partial c_2\partial c_1),\end{aligned}$$

whilst

$$\begin{aligned}\kappa(c_2)\kappa(c_1) &= (f_1(c_2)^{-1}, \partial c_2)(f_1(c_1)^{-1}, \partial c_1)\\ &= (f_1(c_2)^{-1}.{}^{f_0(\partial c_2)}f_1(c_1)^{-1}\langle\partial c_2, \partial c_1\rangle, \partial c_2\partial c_1).\end{aligned}$$

Using that $f_0\partial = \partial f_1$, and the Peiffer identity completes the proof that these are equal.

To prove the second condition, it helps to note the following lemma.

Lemma 87 *For any $c \in C$, $c' \in C'$, $[\iota(c'), \kappa(c)] = 1$.*

Proof: We note $\iota(c') = (c', 1)$, whilst $\kappa(c) = (f_1(c)^{-1}, \partial c)$. Now

$$(c', 1)(f_1(c)^{-1}, \partial c) = (c' f_1(c)^{-1}, \partial c),$$

since $\langle 1, \partial c\rangle = 1$ and $f_0(1) = 1$. On the other hand,

$$(f_1(c)^{-1}, \partial c)(c', 1) = (f_1(c)^{-1}.{}^{f_0(\partial c)}c', \partial c),$$

but, as we have used so many times, $f_0\partial = \partial f_1$, so the Peiffer identity gives ${}^{f_0(\partial c)}c' = f_1(c)c' f_1(c)^{-1}$ and the lemma follows. ■

Because of this and the fact that any $(x, p) \in E$ can be decomposed as $(x, 1)(1, p)$, it suffices to prove the result for $e = (1, p)$. This is quite easy and goes as follows:

We first work out $\kappa({}^p c)$. This is $(f_1({}^p c)^{-1}, p.\partial c.p^{-1})$, so we first need $f_1({}^p c)$, but the formula from earlier gave

$$f_1({}^p c) = \langle p.\partial c, p^{-1}\rangle^{-1}\langle p, \partial c\rangle^{-1} f_1(c)\langle p, p^{-1}\rangle,$$

so our 'target formula' should be

$$\kappa({}^p c) = (f_1({}^p c)^{-1}, p\partial c p^{-1}) = (\langle p, p^{-1}\rangle^{-1} f_1(c)^{-1}\langle p, \partial c\rangle\langle p.\partial c, p^{-1}\rangle, p.\partial c.p^{-1}).$$

We thus have to show that this is the result of conjugating $\kappa(c)$ by $(1, p)$. Now

$$\begin{aligned}
(1,p)(f_1(c)^{-1}, \partial c)(1,p)^{-1} &= (1,p)(f_1(c)^{-1}, \partial c)(\langle p^{-1}, p\rangle^{-1}, p^{-1})\\
&= (1,p)(f_1(c)^{-1}.{}^{f_0(\partial c)}\langle p^{-1}, p\rangle^{-1}\langle \partial c, p^{-1}\rangle, \partial c.p^{-1})\\
&= (1,p)(f_1(c)^{-1}.{}^{\partial f_1(c)}\langle p^{-1}, p\rangle^{-1}\langle \partial c, p^{-1}\rangle, \partial c.p^{-1})\\
&= (1,p)(f_1(c)^{-1}.f_1(c)\langle p^{-1}, p\rangle^{-1} f_1(c)^{-1}\langle \partial c, p^{-1}\rangle, \partial c.p^{-1}) \quad \text{by Peiffer}\\
&= (1,p)(\langle p^{-1}, p\rangle^{-1} f_1(c)^{-1}\langle \partial c, p^{-1}\rangle, \partial c.p^{-1})\\
&= ({}^{f_0(p)}\langle p^{-1}, p\rangle^{-1}.{}^{f_0(p)}f_1(c)^{-1}.{}^{f_0(p)}\langle \partial c, p^{-1}\rangle\langle p, \partial c.p^{-1}\rangle, p.\partial c.p^{-1}),
\end{aligned}$$

but ${}^{f_0(p)}\langle p^{-1}, p\rangle^{-1} = \langle p, p^{-1}\rangle^{-1}$, as we saw earlier, and the cocycle rule tells us that

$$\langle p, \partial c\rangle\langle p.\partial c, p^{-1}\rangle = {}^{f_0(p)}\langle \partial c, p^{-1}\rangle\langle p, \partial c.p^{-1}\rangle,$$

so the verification is complete. ■

We next need $\lambda : E \to P'$. If $e = (x, p) \in E$, both x and p map easily into P' and, as there is nothing to choose between them, ..., we use them both and try $\lambda(x, p) = \partial x.f_0(p)$.

Lemma 88 *Thus defined, $\lambda : E \to P'$ is a homomorphism, and $\lambda\kappa$ is the trivial homomorphism, (so NW-SE is a group complex).*

Proof: Left to you. ■

We must also check the validity of ι's credentials!

Proposition 143 *Defining $\iota : C' \to E$ by $\iota(x) = (x, 1)$, ι is a homomorphism, satisfying: for all $e \in E$, and $c' \in C'$*

$$\iota({}^{\lambda(e)}c') = e\iota(c')e^{-1},$$

Proof: The first part is easy, since $\iota(x_2x_1) = (x_2x_1, 1)$, whilst the multiplication formula in E gives the same thing for $\iota(x_2)\iota(x_1)$.

We next note that, if $e = (x, p)$, then $\lambda(e) = \partial x.f_0(p)$, so

$$\iota({}^{\lambda(e)}c') = ({}^{\partial x.f_0(p)}c', 1) = (x.{}^{f_0(p)}c'.x^{-1}, 1),$$

whilst

$$\begin{aligned}(x,p)(c',1)(x,p)^{-1} &= (x.{}^{f_0(p)}c', p)(\langle p^{-1}, p\rangle^{-1}.f_0(p^{-1})x^{-1}, p^{-1}) \\ &= (x.{}^{f_0(p)}c'.{}^{f_0(p)}\langle p^{-1}, p\rangle^{-1}.f_0(p)f_0(p^{-1})x^{-1}\langle p, p^{-1}\rangle, 1).\end{aligned}$$

We have $f_0(p)f_0(p^{-1}) = \partial\langle p, p^{-1}\rangle^{-1}$, so this simplifies to

$$(x.{}^{f_0(p)}c'.{}^{f_0(p)}\langle p^{-1}, p\rangle^{-1}\langle p, p^{-1}\rangle x^{-1}\langle p, p^{-1}\rangle^{-1}\langle p, p^{-1}\rangle, 1),$$

and using that ${}^{f_0(p)}\langle p^{-1}, p\rangle = \langle p, p^{-1}\rangle$ gives the result. ∎

We summarise:

Proposition 144 *From a weak map, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$, the above construction gives a papillon, $\mathbb{f}$,*

$$\begin{tikzcd}
C \arrow[dr, "\kappa"] \arrow[dd, "\partial"'] & & C' \arrow[dl, "\iota"'] \arrow[dd, "\partial_{C'}"] \\
& E \arrow[dl, "\rho"] \arrow[dr, "\lambda"'] & \\
P & & P'
\end{tikzcd}$$

from C to C'. ∎

What about a converse to this? Does a papillon yield a weak map in some nice way? Recalling that the NE-SW sequence is a group extension, if we pick a section for ρ and compose it with λ, we should get a possible $f_0 : P \to P'$, and a 'factor set' pairing, $\langle -, \rangle : P \times P \to C'$. We will also obtain a decomposition of E as a product of P and C' at the underlying set level, and hence can use κ and the set theoretic projection to C' to obtain a suitable f_1. We will leave the investigation of this as **an extended exercise for you.**

Of course, different sections of ρ may yield different f_0s, so we need a notion of morphisms of papillons and there is an obvious candidate.

Definition: If C, and C' are two crossed modules and $\mathbb{f}$ and $\mathbb{f}'$ are two papillons from C to C' (with central group E' in $\mathbb{f}'$, and with 'primes' on the morphisms, κ', etc.), then a morphism from $\mathbb{f}$ to $\mathbb{f}'$ is a homomorphism, $\varphi : E \to E'$, such that $\kappa' = \varphi\kappa$, etc., thus making the evident diagram commute.

Such diagrams compose in the obvious way. This gives a category, in fact, a groupoid because of the following:

Lemma 89 *Any morphism, $\varphi : \mathbb{f} \to \mathbb{f}'$, between two papillons, $\mathbb{f}$ to $\mathbb{f}'$, as above, is an isomorphism.*

Proof: This is clear from the fact that φ yields a map of extensions

$$\begin{tikzcd} 1 \arrow[r] & C' \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\varphi"] & P \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & C' \arrow[r] & E' \arrow[r] & P \arrow[r] & 1 \end{tikzcd}$$

and any such φ must be an isomorphism by the usual 5-lemma argument on short exact sequences. (**Really you should check that the inverse of φ (as a group homomorphism) gives a *morphism of papillons* inverse to φ itself,** but that is more or less obvious.) ■

The category of papillons from C to C' is thus a groupoid, but so is the category of weak maps and 'weak natural transformations' between them. It may be useful to **investigate the relationships between them**. This is one of the themes of Noohi's work, [224]. His joint work with Aldrovandi, [5], further explores this in the context of stacks (of groupoids) and so is also highly relevant to our overall themes.

11.7.8 ...and the strict morphisms in all that?

As we noted much earlier, any morphism of crossed modules gives a 2-functor of the corresponding 2-groups, that is, a strict, rather than an op-lax, '2-functor'. It would be very bizarre if the fact that a given 'weak morphism' was actually a 'strict' one was not evident in the descriptions. That is not to claim that we should be necessarily able to glance at some weak map and decide quickly if it is actually a strict one. No, we should perhaps expect to have to do a little work, to test 'things' somewhat. What 'things' however?

We start with the description via nerves. Any strict $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ induces a simplicial map,

$$Ner(\mathsf{f}) : Ner(\mathsf{C}) \to Ner(\mathsf{D}),$$

both for $Ner(\mathsf{C})$ interpreted as $Ner_{h.c.}(\mathscr{X}(\mathsf{C}))$ and as $\overline{W}(K(\mathsf{C}))$. Does $Ner(\mathsf{f})$ have any identifiable property over arbitrary simplicial maps between two nerves (and thus over weak maps)?

The secret identifier is 'preservation of thinness'. We have had several definitions of the nerve of a crossed module. We had $\overline{W}(K(\mathsf{C}))$, $Ner_{h.c.}(\mathscr{X}(\mathsf{C}))$, but also $Crs(\pi(-), \mathsf{C})$, that is, the simplicial set of crossed complex maps from the various $\pi(n)$ to C, where this $\pi(n)$ is the free crossed complex on the n-simplex, $\Delta[n]$, as was briefly discussed on page 256. That '*singular complex*' version is very useful, and we have not yet exhausted its possibilities, far from it, but neither have we really done it justice, yet!

These various nerves are isomorphic, and so are all T-complexes. The thin elements in the last description are those $\tau : \pi(n) \to \mathsf{C}$, which map the generator corresponding to ι_n, the top level non-degenerate n-simplex of $\Delta[n]$, to an identity element. The elements of each $Ner(\mathsf{C})_n$ for $n > 2$ are all thin since, as a crossed module, C is trivial in dimensions greater than 2. (Beware of indexing conventions! Yes, we do need 2, here not 1.)

If we use the h. c. / geometric nerve form, a general 2-simplex, τ in $Ner(\mathsf{C})$ has form,

$$\tau = (x_0, x_1, x_2; x(012) : x_1 \Rightarrow x_0x_2),$$

where, thus, $x(012) = (c, x_1)$ with $\partial c.x_1 = x_0x_2$. The interpretation of the condition that $\tau(\iota_2)$ be the identity is that c is the identity of C, *i.e.*, the 2-simplex is 'really' in $Ner(P)$, in other words, it commutes, $x_1 = x_0x_2$.

The thin 1-simplices will be the degenerate ones and have just looked at the thin 2-simplices. What about thin 3-simplices? We know $Ner(\mathsf{C})$ is 3-coskeletal, and this came out to be because there were no non-identity 3-cells in the 2-groupoid, $\mathcal{X}(\mathsf{C})$, and, yes, that means that any $\tau : \pi(3) \to \mathsf{C}$ must send the generator corresponding to ι_3 to the identity element, 'there ain't nothing else there to map it to!'. We thus have all 3-simplices are thin, as are all higher dimensional simplices.

Remark: It is a good **exercise** to **define** thinness for these simplices in this way (*i.e.*, without explicit reference to crossed complexes or to $\pi(n)$), and then to check directly that the result is a T-complex (definition and discussion starting on page 38 if you need it). Another **useful exercise** is to write down what $\pi(n)$ is **in 'gory' detail** and to explore the isomorphisms that we mentioned above between the descriptions of $Ner(\mathsf{C})$ given here and the crossed complex based one as a 'singular complex'.

To continue this exploration of 'strictness' of morphisms, we probably need a definition:

Definition: A simplicial map, $f : Ner(\mathsf{C}) \to Ner(\mathsf{D})$, between the geometric nerves of two crossed modules, *preserves thin elements* or, more simply, *preserves thinness* if, for each n, and each thin n-simplex, $t \in Ner(\mathsf{C})_n$, $f_n(t)$ is thin in $Ner(\mathsf{D})$.

Remark: We should comment that preservation of thinness really devolves down to checking that a map preserves thin 2-simplices. The thin 1-simplices are just the degenerate ones, so they will be preserved by any simplicial map, whilst, above dimension 2, all simplices are thin, so preservation is automatic!

We showed (Proposition 531) how a simplicial map, $f : Ner(\mathsf{C}) \to Ner(\mathsf{D})$, induced the data for a pseudo-functor,

$$\mathscr{F} = (F, \varphi) : \mathcal{X}(\mathsf{C}) \to \mathcal{X}(\mathsf{D}).$$

(We will not need to use the detailed notation from there for the limited discussion that we will give here, so will abuse notation enormously!) Translating that data, in the algebraic / combinatorial format, we look at $(p_0, p_0p_2, p_2; id) \in Ner(\mathsf{C})$ and obtain

$$f_2(p_0, p_0p_2, p_2; id) = (f_0(p_0), f_0(p_0p_2), f_0(p_2); (\langle p_2, p_0\rangle, f_0(p_0p_2)))$$

with $\partial\langle p_2, p_0\rangle f_0(p_0p_2) = f_0(p_0)f_0(p_2)$.

If f preserves thinness, then $\langle p_2, p_0\rangle$ is trivial, *i.e.*, the identity in D, so f_0 is a homomorphism, as is f_1, and, by the post-whiskering axiom, $f_1({}^pc) = {}^{f_0(p)}f_1(c)$, so f is a (strict) morphism of crossed modules, as expected.

Clearly, if $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ is a crossed module morphism, then it preserves thinness (in all dimensions). (Just **check it**.)

This raises an interesting **question**. Is there a simple example of a weak (and not strict) morphism of crossed modules, having both f_0 and f_1 group homomorphisms? In such a case, all the $\partial\langle p_1, p_2\rangle$ and $\langle \partial c_1, \partial c_2\rangle$ would be trivial, but would it be possible to have some $\langle p_1, p_2\rangle$ non-trivial? The obvious place to look first would be with modules thought of as crossed modules, so the various ∂ would be trivial.

The above more or less indicates what a strict morphism has that a weak one does not, from the point of view of nerves. What about defining weak maps via cofibrant replacements? If we start with a strict morphism, $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, and a cofibrant replacement, $\mathsf{q} : \mathsf{Q} \to \mathsf{C}$, then there is clearly a morphism,

$$\mathsf{fq} : \mathsf{Q} \to \mathsf{D},$$

which will be a weak map from C to D, or, more exactly, will be one if Q is the natural functorial cofibrant replacement, and, more generally, will *give* a weak map, determined up to equivalence. Conversely, given some $\mathsf{g} : \mathsf{Q} \to \mathsf{D}$, it will correspond to a strict map if g factors through q giving a 'complementary' morphism, $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, ... Uniqueness, etc, of the factorisation is **left to you** to analyse.

Finally, what sort of papillon / butterfly corresponds to a strict morphism, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$? We know that f corresponds to a pairing, $\langle -,- \rangle : P \times P \to C'$, which, here, is trivial. It follows that the NE-SW extension of the papillon will be split, with, as a result, $E \cong C' \rtimes P$, since $\langle -,- \rangle$ was a factor set for it.

This gives a papillon:

$$\begin{tikzcd} C \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & C' \arrow[dl, "\iota"'] \arrow[dd, "\partial_{\mathsf{C}'}"] \\ & C' \rtimes P \arrow[dl, "\rho"] \arrow[dr, "\lambda"'] & \\ P & & P' \end{tikzcd}$$

in which ρ is a split epimorphism.

Now we can go back. First the obvious definition:

Definition: A papillon, as above, in which the NE-SW extension is split (with given splitting) will be called a *split papillon*.

Suppose we have such a split papillon, with $s : P \to C' \rtimes P$, the chosen splitting. (Of course, as soon as we choose a splitting, we are choosing an isomorphism of the central object, E, of the papillon and a semidirect product representation of it. Consequently, if we write $C' \rtimes P$ for the centre term of a papillon, we are not only identifying that group, but are specifying the splitting (namely $s(p) = (1, p)$) and a host of other information. This does lead to a certain redundancy of notation and, perhaps, of terminology, but, hopefully, is clearer in terms of the exposition.) The decomposition as $C' \rtimes P$ also gives us a set theoretic projection from $C' \rtimes P$ to C', which we will denote by d. (This satisfies

$$d((c'_1, p_1)(c'_2, p_2)) = c'_1 . {}^{p_1}c'_2,$$

whilst, of course, $d(c'_1, p_1)d(c'_2, p_2) = c'_1 . c'_2$, so d is not a homomorphism. It is a derivation.) We want to construct a morphism of crossed modules,

$$\mathsf{f} : \mathsf{C} \to \mathsf{C}'.$$

There is an obvious $f_0 : P \to P'$, given by λs, but what about an $f_1 : C \to C'$?

There seem to be only a few possibilities handed to us if we are to use just the 'building blocks' provided. We know that the left 'wing' of the papillon commutes, so $\kappa(c) = (k(c), \partial c)$ and perhaps this mapping, $k : C \to C'$, is what we need.

Before we go further, however, we should look back at how we went *from* weak maps *to* 'papillons'. We took $\kappa(c) = (f_1(c)^{-1}, \partial c)$, so *that* suggests that $k(c)$ is not exactly what we want, rather $k(c)^{-1}$ should be the thing we look at.

(If we look at the fact that κ itself *is* a homomorphism, then k satisfies a derivation type formula,

$$k(c_2c_1) = k(c_2).^{\partial c_2}k(c_1),$$

rather than being a homomorphism. We are in the context of crossed modules, so action by a boundary element, such as ∂c_2, easily converts to conjugation, but the above seems to then end up with the wrong order for things to cancel as we might hope. This again suggests that the idea of the 'inverse of k' is a good one to follow up.)

Given this, we will bravely set $f_1(c) := k(c)^{-1}$ and charge into the attack! First, however, let us make a cunning observation. The above choice looks good, as we said, since then

$$\kappa(c) = (f_1(c)^{-1}, \partial c)$$

as before, so

$$\kappa(c) = (f_1(c)^{-1}, 1)(1, \partial c) = \iota(f_1(c))^{-1}.s(\partial c).$$

Rearranging this gives

$$\iota(f_1(c)) = s(\partial c)\kappa(c)^{-1},$$

we further note that (i) ι is a monomorphism, and (ii), and, in all generality, $[\iota(c'), \kappa(c)] = 1$, since $\rho\iota(c') = 1$ implies that

$$\iota(c')\kappa(c)\iota(c')^{-1} = \kappa({}^{\rho\iota(c')}c) = \kappa(c).$$

(In case you are wondering, it should be noted, that we had previously checked this only for a papillon coming from a weak map, so we *did* need to check it independently!)

Proposition 145 *Given a split papillon, as above, defining $f_0 = \lambda s$ and f_1 given by $\iota f_1(c) = s(\partial c)\kappa(c)^{-1}$, then (f_1, f_0) gives a morphism, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$.*

Proof: We have to check three things:

(a) $\partial f_1 = f_0\partial$;

(b) f_1 is a homomorphism (as we have already checked that f_0 is one);

(c) for all $c \in C$ and $p \in P$,

$$f_1({}^pc) = {}^{f_0(p)}f_1(c).$$

Starting with (a), we have

$$\partial f_1(c) = \lambda\iota f_1(c) = \lambda s(\partial c),$$

since $\lambda\kappa$ is trivial, hence $\partial f_1(c) = f_0\partial(c)$.

Now (b), let c_1, $c_2 \in C$,

$$\begin{aligned}\iota f_1(c_2c_1) &= s\partial(c_2c_1).\kappa(c_2c_1)^{-1}\\ &= s\partial(c_2)s\partial(c_1)\kappa(c_1)^{-1}\kappa(c_2)^{-1}.\end{aligned}$$

(We know what we want this to be, so force it into the right shape with a rewrite.) It equals

$$s\partial(c_2)\kappa(c_2)^{-1}(\kappa(c_2)s\partial(c_1).\kappa(c_1)^{-1}\kappa(c_2)^{-1}) \quad = \quad s\partial(c_2)\kappa(c_2)^{-1}.\kappa(c_2).\iota f_1(c_1).\kappa(c_2)^{-1},$$

but κ and ι "commute", as we saw, so this is $\iota f_1(c_2)\iota f_1(c_1)$, as hoped for.

Finally (c), we take $p \in P$, $c \in C$

$$\begin{aligned} \iota f_1({}^p c) &= s\partial({}^p c).\kappa({}^p c)^{-1} \\ &= s(p\partial c.p^{-1}).\kappa({}^{\rho s(p)} c)^{-1}, \end{aligned}$$

since $p = \rho s(p)$. We use the condition on κ relative to the action of the $\rho(e)$s to get that this is

$$\begin{aligned} s(ps(\partial c)s(p)^{-1}.(s(p)\kappa(c)^{-1}s(p)^{-1}) &= s(p)(s(\partial c).\kappa(c)^{-1})s(p)^{-1} \\ &= s(p)\iota f_1(c)s(p)^{-1}. \end{aligned}$$

We now invoke the condition on ι relative to the action of the $\lambda(e)$s. This becomes $\iota({}^{\lambda s(p)} f_1(c))$, *i.e.*, $\iota({}^{f_0(p)} f_1(c))$. Using that ι is a monomorphism, we get

$$f_1({}^p c) = {}^{f_0(p)} f_1(c),$$

as required. ■

We thus have strict morphisms correspond to split papillons. To be complete in this, we must note that a split papillon may have different splittings, so does a split papillon correspond to several *different* weak morphisms? Clearly, if it does, then these should be equivalent / homotopic. This is **left to you** to check up on and to investigate further. The papers, [222, 224] and [5], will give some ideas about what to expect, but do not expect them to provide all the answers!

It should also be clear that a weak equivalence of crossed modules should correspond to a papillon in which the NW-SE sequence is also exact. Noohi's discussion in [224] goes into this, and this is **suggested as another investigation**. His treatment does not take quite the same route through the ideas as we have, so there are quite a few details to supply ... **over to you**.

Chapter 12

Other enrichments, other versions of (homotopy) coherence?

To understand simplicially based homotopy coherence more fully, it is useful to study other, but related, forms of enrichment and the relations between those and our intuitions about homotopy coherence and more general forms of categorical coherence. This comparison can be very useful when looking at different versions of a given cohomology construction.

We have already looked in some detail above at some aspect of Cat-enrichment and then have sketched some of the links between h. c. and the lax / op-lax theory for 2-categories and bicategories[1] and have also met groupoid enrichments.

12.1 Other enrichments?

Two other obvious examples would be enriching over chain complexes or cochain complexes. The second of these is particularly important for applications in mathematical physics and in representation theory where 'dg-categories' are frequently used. A dg-category is, loosely speaking, a category enriched over differential graded modules. It thus also includes differential graded algebras as the special 'one object' case. There are quite a few other useful enrichments that we can examine.

12.1.1 Enriched categories

We have been using simplicially enriched categories extensively in the above, and now wish to look at other enrichments. That suggests that it would be a good thing to have, a bit more formally, a general definition of enriched category. We do not need an extensive treatment. The obvious detailed reference is Kelly's book, [177], but we will give brief summaries[2] in any case.

First a monoidal category, $\mathcal{V}$, is specified by $\mathcal{V}$ itself, together with the tensor, $\otimes = \otimes_{\mathcal{V}}$ if more precision is needed, and the unit object $I = I_{\mathcal{V}}$, plus, the composition with *associator* transformation, α, and left and right unit isomorphisms[3], $\ell_V : I \otimes V \xrightarrow{\cong} V$, and $r_V : V \otimes I \xrightarrow{\cong} V$, satisfying

[1] We will return to this later in this chapter giving a more indepth treatment.

[2] The reader should check on in Kelly's book for the coherence axioms as these can be important when handling the detailed levels of argument behind the intuitions.

[3] These are sometimes called the *left and right unitors*.

some fairly clear coherence axioms. We will often abbreviate this data to $\mathcal{V} = (\mathcal{V}, \otimes_{\mathcal{V}}, I_{\mathcal{V}})$.

We will often need that $\mathcal{V}$ is symmetric:

Definition: (i) A *symmetry*, c, for a monoidal category, $\mathcal{V}$, is a natural isomorphism, $c_{XY} : X \otimes Y \to Y \otimes X$, satisfying the coherence axioms given on page 14 of [177].

(ii) A monoidal category, $\mathcal{V}$, together with a symmetry is called a *symmetric monoidal category*.

(iii) A monoidal category, $\mathcal{V}$, is *closed*, if the functor, $- \otimes X : \mathcal{V} \to \mathcal{V}$, has a right adjoint[4], which we will denote by $[X, -] : \mathcal{V} \to \mathcal{V}$, or sometimes $\underline{\mathcal{V}}(X, -)$.

The idea of using a $\mathcal{V}$ is, as you have probably assumed, to replace the 'sets' of morphisms in the definition of a category by objects from $\mathcal{V}$. We give a slightly informal definition, in as much as **we leave the diagrams for associativity and unit / identity axioms for you to construct**.

Definition: A $\mathcal{V}$-category, or *category enriched over* $\mathcal{V}$, $\mathcal{A}$, consists of a collection, $Ob(\mathcal{A})$, of 'objects', a 'hom-object', $\mathcal{A}(a, b) \in \mathcal{V}$, for each pair, (a, b), of objects of $\mathcal{A}$, a composition morphism,

$$m_{abc} : \mathcal{A}(b, c) \otimes \mathcal{A}(a, b) \to \mathcal{A}(a, c)$$

for each triple of objects[5] , and an identity element, $j_a : I \to \mathcal{A}(a, a)$, for each object, a, of $\mathcal{A}$; subject to the 'evident' associativity and unit axioms.

We will see many examples of enrichments later on, but would mention, in addition to the simplicially enriched categories that have been predominant in this chapter, *2-categories*, which are $(Cat, \times)$-enriched categories.

It will sometimes be convenient, if $\mathcal{A}$ is an enriched category, to use alternative notations for the collection of objects. We thus may write something like $\mathcal{A}_{ob}$ or $\mathcal{A}_0$ instead of $Ob(\mathcal{A})$.

12.1.2 A little bit of enriched category theory

Later we will need some basic ideas from general enriched category, so as to extend them to handle homotopy coherent phenomena. We refer, once again, to Kelly, [177], as the obvious detailed source. It will however be convenient to have some definitions, discussions and results at hand.

Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-enriched categories with $(\mathcal{V}, \otimes, I)$ a monoidal category as above. The following can be seen to be a natural encoding of 'functor' adapted to this enriched setting.

Definition: A *$\mathcal{V}$-enriched functor*, or simply *$\mathcal{V}$-functor*, $F : \mathcal{C} \to \mathcal{D}$ consists of

- a function, $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$ between the underlying collections of objects[6];

---

[4]This acts like an internal enriched hom-construction.

[5]For convenience we will usually miss the suffices indicating the objects off m when they can be retrieved from the context.

[6]For convenience we write F here although some other notation such as F_{ob} might be less abusive of notation.

- a $(Ob(\mathcal{C}) \times Ob(\mathcal{C}))$-indexed family of morphisms of $\mathcal{V}$:
$$F(x,y) : \mathcal{C}(x,y) \to \mathcal{D}(Fx,Fy),$$
where, as usual, $\mathcal{C}(x,y)$ denotes the hom-object of morphisms from x to y in $\mathcal{C}$, etc.

These are to respect composition and units, *i.e.*,

(i) given objects, x, y, z in $\mathcal{C}$, the diagram

$$\begin{tikzcd}
\mathcal{C}(y,z) \otimes \mathcal{C}(x,y) \arrow[r, "m^{\mathcal{C}}"] \arrow[d, "F(y,z)\otimes F(x,y)"'] & \mathcal{C}(x,z) \arrow[d, "F(x,z)"] \\
\mathcal{D}(Fy,Fz) \otimes \mathcal{D}(Fx,Fy) \arrow[r, "m^{\mathcal{D}}"] & \mathcal{D}(Fx,Fz)
\end{tikzcd}$$

commutes,

and

(ii) for each object, x of $\mathcal{C}$,

$$\begin{tikzcd}
& I \arrow[dl, "j_x^{\mathcal{C}}"'] \arrow[dr, "j_{Fx}^{\mathcal{D}}"] & \\
\mathcal{C}(x,x) \arrow[rr, "F(x,x)"'] & & \mathcal{D}(Fx,Fx)
\end{tikzcd}$$

commutes.

As we now have $\mathcal{V}$-enriched functors, we *naturally* should look for $\mathcal{V}$-enriched natural transformations!

Definition: For $F, G : \mathcal{C} \to \mathcal{D}$, two $\mathcal{V}$-functors, a *$\mathcal{V}$-natural transformation*, $\eta : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$, is a family, $\{\alpha_x : I \to \mathcal{D}(Fx,Gx) \mid x \in Ob(\mathcal{C})\}$, satisfying the naturality condition expressed by the commutativity of

$$\begin{tikzcd}
& I \otimes \mathcal{C}(x,y) \arrow[r, "\eta_y \otimes F"] & \mathcal{D}(Fy,Gy) \otimes \mathcal{D}(Fx,Fy) \arrow[dr, "m"] & \\
\mathcal{C}(x,y) \arrow[ur, "\ell^{-1}"] \arrow[dr, "r^{-1}"'] & & & \mathcal{D}(Fx,Fy) \\
& \mathcal{C}(x,y) \otimes I \arrow[r, "G \otimes \eta_x"] & \mathcal{D}(Gx,Gy) \otimes \mathcal{D}(Fx,Gx) \arrow[ur, "m"'] &
\end{tikzcd}$$

It is easily checked that $\mathcal{V}$-natural transformations compose in a natural way, so making $\mathcal{V}-\mathsf{Cat}$ into a 2-category, which in the case that $\mathcal{V} = Sets$ is just the usual 2-category of categories, functors and natural transformations.

The next idea is give a means to compare this 2-category, $\mathcal{V}-\mathsf{Cat}$, with the 2-category Cat. Any $\mathcal{V}$-category gives rise to an 'underlying category'. The construction is very simple. We will give it in its most elementary form for the moment, but will meet it again in the next section as it is a simple example of the 'change of enrichment' situation studied there. Let us first look at some examples of enrichments $\mathcal{V}$ which have an 'underlying' functor[7] to *Sets*.

[7]We say 'underlying functor' rather than 'forgetful functor' as the latter term tends to be used where there is a natural left adjoint to the construction, and that need not be the case here, although sometimes it is.

- $\mathcal{V} = Ab = \mathbb{Z}-Mod$, $\otimes = \otimes_{\mathbb{Z}}$, $I = \mathbb{Z}$: If A is an Abelian group, its underlying set (in the usual sense) can be identified with $Ab(\mathbb{Z}, A)$ as, for each $a \in A$, there is a homomorphism $\ulcorner a \urcorner : \mathbb{Z} \to A$, 'naming' the element, and given by $\ulcorner a \urcorner(n) = na$. (It is quite usual to use an outrageous abuse of notation and to write $a : \mathbb{Z} \to A$ instead of $\ulcorner a \urcorner$.) Other categories of modules can be treated similarly.

- $\mathcal{V} = Cat$, $\otimes = \times$, $I = [0]$, the category with a single morphism: For a small category C, $Cat(I, C)$ is the set of objects of C. The case of $\mathcal{V} = Grpd$ is similar.

- $\mathcal{V} = Top$, $\otimes = \times$, $I = \{*\}$, a singleton space, and then, for a space, X, $Top(\{*\}, X)$ is in bijection with the underlying set of X, whilst if we consider pointed spaces and the corresponding category, Top_*, then we need I to be the discrete space, $\{0, 1\}$, pointed at 0.

- $\mathcal{V} = \mathcal{S}$, $\otimes = \times$, $I = \Delta[0]$, and, for K a simplicial set, $\mathcal{S}(\Delta[0], K) = K_0$.

The actual intuition and interpretation in each case is perhaps different in its detail, but in each case, $\mathcal{V}(I, x)$ gives some idea of an 'underlying set' associated to the object x, so it is not unreasonable to see if the corresponding representable functor $\mathcal{V}(I, -) : \mathcal{V} \to Sets$ has nice enough properties to allow one to form a (Set-enriched) category, $\mathcal{A}_0$, from a $\mathcal{V}$-enriched one, $\mathcal{A}$. It does, and there is!

Given $\mathcal{V}$-enriched category, $\mathcal{A}$, we take the category $\mathcal{A}_0$ to have the same objects as $\mathcal{A}$, whilst a morphism, $f : A \to B$, in $\mathcal{A}_0$ is just a morphism $f : I \to \mathcal{A}(A, B)$, so is an element in the 'underlying set', $\mathcal{V}(I, \mathcal{A}(A, B))$, of $\mathcal{A}(A, B)$.

We define composition as follows: given $f : A \to B$ and $g : B \to C$ in $\mathcal{A}_0$, (so $f : I \to \mathcal{A}(A, B)$ and $g : I \to \mathcal{A}(B, C)$), then, noting that $I \cong I \otimes_{\mathcal{V}} I$, we can form the composite:

$$I \cong I \otimes_{\mathcal{V}} I \xrightarrow{g \otimes f} \mathcal{A}(B, C) \otimes_{\mathcal{V}} \mathcal{A}(A, B) \xrightarrow{m} \mathcal{A}(A, C).$$

We take this as the definition of the composite, $gf : A \to C$, of f and g in $\mathcal{A}_0$ and we can check that this composition is associative using the natural associativity axioms for $\otimes_{\mathcal{V}}$, and then we need units. For an object, A, of $\mathcal{A}$, the unit on A in $\mathcal{A}$ is already given as a morphism, $j_A : I \to \mathcal{A}(A, A)$, so this is already in the right form for $\mathcal{A}_0$, and the unit axioms in $\mathcal{A}$ readily show that these $j_A : A \to A$ in $\mathcal{A}_0$ are units for the composition, so $\mathcal{A}_0$ is a category.

Definition: Given a $\mathcal{V}$-category, $\mathcal{A}$, its *underlying category* is the category defined above.

Amongst the above examples, the case of $\mathcal{S}$-categories is especially useful for us as it shows that within an $\mathcal{S}$-category, one has a category structure made up of all the 'stuff' specified at its zeroth level.

Just as $\mathcal{V}$-categories give rise to 'underlying categories', so also do $\mathcal{V}$-functors give rise to 'underlying functors' between the corresponding underlying categories of their domain and codomain $\mathcal{V}$-categories, and similarly for the $\mathcal{V}$-natural transformations between $\mathcal{V}$-functors. We refer the reader to Kelly, [177], section 1.3, for a bit more on this. The conclusion should be evident, namely that:

Proposition 146 *The construction $(-)_0$ is a 2-functor from $\mathcal{V}-\mathsf{Cat}$ to Cat* ■

In general this 2-functor throws away some of the information in $\mathcal{V}-\mathsf{Cat}$, but in some cases such as when $\mathcal{V} = Ab$, not much is lost; again see Kelly's discussion, mentioned above.

Notation: Although useful, the $(-)_0$ notation can get cumbersome and we will follow Riehl, [234] in adopting a simplifying convention which seems to be consistent with our previous usage in the case of simplicially enriched categories, cf. section 11.2.2. There we used an underline to signify the enriched version of a hom-set, so $\underline{\mathcal{S}}(K, L)$ was the simplicial set of morphism from K to L and $\mathcal{S}(K, L)$ would be simply the set of simplicial morphism (thus morphisms in the category $\mathcal{S}$) from K to L. We extend that idea as follows: if an unenriched object is denoted with a letter previous assigned an enriched notion, e.g., a functor $F : \mathcal{M} \to \mathcal{N}$ in the presence of a V-functor $\underline{F} : \underline{\mathcal{M}} \to \underline{\mathcal{N}}$, the former will be the underlying object of the latter, so essentially $(\underline{\mathcal{M}})_0 = \mathcal{M}$.

12.1.3 Changing the enrichment

We first need to look at some generalities on functors between monoidal categories and the corresponding situation for categories enriched over them.

Now let $\mathcal{V}$ and $\mathcal{W}$ be two monoidal categories.

Definition: A *(lax) monoidal functor*, $F : \mathcal{V} \to \mathcal{W}$, is a functor, F, between the underlying categories equipped with a natural transformation,

$$\varphi_{A,B} : FA \otimes_{\mathcal{W}} FB \to F(A \otimes_{\mathcal{V}} B),$$

and a morphism, $\eta : I_{\mathcal{W}} \to F(I_{\mathcal{V}})$, such that

- for any triple of objects A, B, C in $\mathcal{V}$, the diagrams

$$\begin{tikzcd}
(FA \otimes_{\mathcal{W}} FB) \otimes_{\mathcal{W}} FC \arrow[r, "\alpha_{\mathcal{W}}"] \arrow[d, "\varphi_{A,B} \otimes_{\mathcal{W}} FC"'] & FA \otimes_{\mathcal{W}} (FB \otimes_{\mathcal{W}} FC) \arrow[d, "FA \otimes_{\mathcal{W}} \varphi_{B,C}"] \\
F(A \otimes_{\mathcal{V}} B) \otimes_{\mathcal{W}} FC \arrow[d, "\varphi_{A \otimes B, C}"'] & FA \otimes_{\mathcal{W}} F(B \otimes_{\mathcal{V}} C) \arrow[d, "\varphi_{A, B \otimes C}"] \\
F((A \otimes_{\mathcal{V}} B) \otimes_{\mathcal{V}} C) \arrow[r, "F\alpha_{\mathcal{V}}"'] & F(A \otimes_{\mathcal{V}} (B \otimes \mathcal{V} C))
\end{tikzcd}$$

-
$$\begin{tikzcd}
FA \otimes_{\mathcal{W}} I_{\mathcal{W}} \arrow[r, "FA \otimes \eta"] \arrow[d, "\rho_{\mathcal{W}}"'] & FA \otimes_{\mathcal{W}} FI_{\mathcal{V}} \arrow[d] \\
FA & F(A \otimes_{\mathcal{V}} I_{\mathcal{V}}) \arrow[l, "F\rho_{\mathcal{V}}"]
\end{tikzcd}$$

  and

- similarly for the λs,

all commute.

A *strong monoidal functor* is one in which the coherence maps are invertible. It is *strict* if they are identities.

Example: Picking up a context from the previous section, suppose $\mathcal{V} = (\mathcal{V}, \otimes_V, I)$ is as above, then we have the representable 'underlying set functor',

$$\mathcal{V}(I, -) : \mathcal{V} \to Sets$$

which we will examine to see if it is a lax monoidal functor or not. Here we are taking $Sets$ with the obvious monoidal category structure with tensor the (Cartesian) product, $\times$ and with a singleton set as I. We suppose V and W are two objects of $\mathcal{V}$ and look for a natural transformation

$$\varphi_{V,W} : \mathcal{V}(I, V) \times \mathcal{V}(I, W) \to \mathcal{V}(I, V \otimes W).$$

In fact, we have almost given this before as we will see. If $f : I \to V$ and $g : I \to W$, we can form their tensor product, $f \otimes_{\mathcal{V}} g : I \otimes_{\mathcal{V}} I \to V \otimes_{\mathcal{V}} W$ and then pre-compose with the unitor isomorphism, $I \xrightarrow{\cong} I \otimes_{\mathcal{V}} I$, to get an element of $\mathcal{V}(I, V \otimes W)$. It is worth noting that the right and left unit isomorphisms both give $I \otimes_{\mathcal{V}} I \cong I$, but in fact the axioms for a monoidal category show that $\ell_I = r_I$, *i.e.*, that we can use either as they agree on I. We then have after a bit of fairly easy checking[8]:

Proposition 147 *The functor $\mathcal{V}(I, -) : (\mathcal{V}, \otimes_{\mathcal{V}}, I) \to (Sets, \times, \{*\})$ is lax monoidal.* ■

Back to the general situation, suppose now that we have a $\mathcal{V}$-category, $\mathcal{C}$, then we can use a lax monoidal functor, $F : \mathcal{V} \to \mathcal{W}$, to construct a $\mathcal{W}$-category from $\mathcal{C}$. For instance, it might be convenient to change a simplicially enriched category into a chain complex enriched one using some monoidal functor from $\mathcal{S}$ to the category of chain complexes. We take the same objects as in our original category, $\mathcal{C}$, and for x, y objects of $\mathcal{C}$ we try $F_*(\mathcal{C})(x, y) := F(\mathcal{C}(x, y))$. Is there a suitable $\mathcal{W}$-enriched composition? We have a lot of ingredients, so try the 'obvious' thing:
for x, y, z, objects of $\mathcal{C}$, take

$$F_*(\mathcal{C})(x, y) \otimes_{\mathcal{W}} F_*(\mathcal{C})(y, z) \xrightarrow{\varphi} F((\mathcal{C}(x, y)) \otimes_{\mathcal{V}} \mathcal{C}(y, z)) \xrightarrow{F(c_{x,y,z})} F(\mathcal{C}(x, z)) \stackrel{def}{=} F_*(\mathcal{C})(x, z).$$

This looks good. Now we need, for any object x, a morphism

$$id_x : I_{\mathcal{W}} \to F_*(\mathcal{C})(x, x),$$

but we have $id_x : I_{\mathcal{V}} \to \mathcal{C}(x, x)$, since $\mathcal{V}$ is a $\mathcal{V}$-category and we can form the composite

$$id_x : I_{\mathcal{W}} \to F(I_{\mathcal{V}}) \to F(\mathcal{C}(x, x)) = F_*(\mathcal{C}(x, x)),$$

so this looks good as well, as a candidate for the identity. (Of course, the second morphism here is the image of id_x under F.) As usual with these things, there is a lot of checking to do, (and **you are left to do this**), but it looks as if $F_*(\mathcal{C})$ with this structure should give a $\mathcal{W}$-category.

Proposition 148 *(i) If $F : \mathcal{V} \to \mathcal{W}$ is a lax monoidal functor, and $\mathcal{C}$ is a $\mathcal{V}$-category, then $F_*(\mathcal{C})$, as specified above, is a $\mathcal{W}$-category.*

(ii) The assignment to $\mathcal{C}$ of $F_*(\mathcal{C})$ gives a functor from the category, $\mathcal{V}-Cat$, of small $\mathcal{V}$-categories to that, $\mathcal{W}-Cat$, of small $\mathcal{W}$-categories. ■

[8]which is **left to the reader**.

The proof is left to you to check or to find in the literature.

Example (continued): If we apply the above to the functor $F = \mathcal{V}(I, -)$, then[9] then for a $\mathcal{V}$-category, $\mathcal{A}$, $F_*(\mathcal{A})$ is exactly $\mathcal{A}_0$, the underlying category of $\mathcal{A}$.

We can apply this to the situation, for $\mathcal{V}$ is $\mathcal{S}$ and $\mathcal{W}$ being something like the monoidal category of chain complexes so as to change a simplicially enriched category into a chain complex enriched one and then to work with the new form of homotopy coherence. The idea is simply that if $\mathbb{A}$ is a small category and $X : S(\mathbb{A}) \to \mathcal{C}$ is a $\mathcal{S}$-functor, (and hence is a h.c. diagram in $\mathcal{C}$), then given a lax monoidal $F : \mathcal{S} \to \mathcal{W}$, we can apply F_* both to $S(\mathbb{A})$ and to $\mathcal{C}$ to get a new form of homotopy coherence, now based on $\mathcal{W}$. We will see this in several cases.

12.1.4 Enrichment over $Simp.K{-}Mod$

We earlier considered the category $Simp.K{-}Mod$ of simplicial K-modules, with K a commutative ring. We mentioned that this had an $\mathcal{S}$-category structure but, of course, it also has an enrichment over itself. For that to make sense, we need a monoidal category structure on it.

Suppose M and N are two simplicial modules over K.

Definition: The tensor product of M and N is defined to be the simplicial module, $M \otimes_K N$ (or simply $M \otimes N$ if K is understood or is $\mathbb{Z}$) such that $(M \otimes N)_n = M_n \otimes N_n$, and the face and degeneracy maps are defined 'component-wise', so, for instance,

$$d_i(m \otimes n) = d_i(m) \otimes d_i(n),$$

of course, extended linearly to the rest of the module.

Recalling the category of modules over K and its tensor product, the *K-linearisation functor from Sets to $K{-}Mod$* sends the product of sets to the tensor product. If we write $K^{(X)}$ for the free K-module on the set X, and hence the 'K-linearisation' of X, then, of course, $K^{(X\times Y)} \cong K^{(X)} \otimes K^{(Y)}$ with the obvious natural isomorphism. The unit of the Cartesian monoidal structure on $Sets$ is 'the' one point set, $\{1\}$ and $K^{\{1\}} \cong K$, as K-modules, is the unit in the monoidal category structure on $K{-}Mod$. The K-linearisation is, thus, a strong lax monoidal functor from $(Sets, \times, \{1\})$ to $(K{-}Mod, \otimes, K)$. It should now be clear that, as we have assumed earlier, $\mathcal{S}$ is a *Cartesian* monoidal category therefore with $\times$ as the multiplication and with $\Delta[1]$ as unit, whilst $\mathcal{S}_K := (Simp.K{-}Mod, \otimes, K)$ is monoidal, where we write K for the constant simplicial K-module with value K. (This is to avoid $K(K, 0)$ as a notation! Unfortunately K is a very useful letter.)

The following should now be routine **for you to prove**:

Lemma 90 *The K-linearisation functor, $K(-) : X \mapsto K^{(X)}$, is a strong monoidal functor from $\mathcal{S}$ to $\mathcal{S}_K$.* ■

This means that there is an induced functor, K_*, from $\mathcal{S}{-}Cat$ to $\mathcal{S}_K{-}Cat$, which may also be called the 'K-linearisation'. Of special interest is : if $X : S(\mathbb{A}) \to \mathcal{B}$ is a h.c. diagram in an $\mathcal{S}$-category, $\mathcal{B}$, then $K_*(X) : K_*S(\mathbb{A}) \to K_*(\mathcal{B})$. Of course, the individual simplicial K-modules that appear in $K_*(X)$ are all free simplicial K-modules, and it would be unwise just to consider

[9] as is fairly obvious if you write things down

such 'diagrams' as being the only homotopy coherent diagrams that can exist in a category enriched over $\mathcal{S}_K$. We consider an arbitrary $\mathcal{S}_K$-category, $\mathcal{C}$.

Definition: If $\mathbb{A}$ is a small category, a *homotopy coherent diagram of shape* $\mathbb{A}$ *in* $\mathcal{C}$ is an $\mathcal{S}_K$-functor $X : K_*S(\mathbb{A}) \to \mathcal{C}$.

We leave you to explore the ways this can be interpreted in simple examples such as $\mathbb{A} = [n], [1]^n$ or $G[1]$, for G some simple example of a group. These are quite interesting and relate well to the simplicial case. Perhaps too well! In fact, we should ask if this really gives us anything new. We saw that any h.c. diagram in $\mathcal{B}$ gave a h.c. diagram in $K_*(\mathcal{B})$ in the $\mathcal{S}_K$ sense, so is there some way of going from an $\mathcal{S}_K$-functor from $K_*S(\mathbb{A})$ to $\mathcal{C}$ and obtaining an $\mathcal{S}$-functor from $S(\mathbb{A})$ to ... what? To go from $\mathcal{S}$-categories to $\mathcal{S}_K$-categories, we used the K-linearisation K_*, built from a monoidal functor from $\mathcal{S}$ to $\mathcal{S}_K$. As a functor, this has a right adjoint, namely the forgetful functor, U from $Simp.K{-}Mod$ to simplicial sets. Can we use that to get from an $\mathcal{S}_K$-category to an $\mathcal{S}$-category? It is not monoidal as $U(M \otimes N)$ and $U(M) \times U(N)$ are not nicely related, in the way needed except in really trivial cases.

We will return to this later as it is a fairly generally occurring situation and it would help to have other examples to play with, and to examine for extra structure if needs be.

12.1.5 Enrichment over chain complexes

A somewhat obvious example of a useful enrichment is that of chain complexes. Of course, we have seen that these come in various flavours, but first we should consider differential $\mathbb{Z}$-graded modules and that has a nice monoidal structure via the tensor product.

We saw this tensor product already in section 8.2.1, and, as usual, we briefly recall the main point of its definition. We have two differential graded modules, C and D, over a commutative ring K, and form $\mathsf{C} \otimes \mathsf{D}$, first as a graded module by

$$(\mathsf{C} \otimes \mathsf{D})_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and then as a differential graded module by defining a boundary / differential, $\partial : \mathsf{C} \otimes \mathsf{D} \to \mathsf{C} \otimes \mathsf{D}$, on generators by

$$\partial(c \otimes d) = (\partial c) \otimes d + (-1)^{|c|} c \otimes (\partial' d),$$

where $|c|$ is the degree of c.

The tensor product gives $dg{-}K{-}Mod$, a monoidal structure, where the unit is the ring, K, considered as a K-module and concentrated in degree zero. This is the chain complex analogue of $K(K, 0)$ for simplicial K-modules. We will sometimes use interchangeably the notations, $dg{-}K{-}Mod$, and Ch_K for this monoidal category, although strictly speaking the latter will refer to the positively graded part of the former.

We have a monoidal category and hence can discuss categories enriched over it, however we usually we need to start with the cases corresponding either to positive (and thus chain) complexes or negative (cochain) complexes. This is partially because those are the ones that come, initially, from the topological / simplicial context, but, as in many other parts of mathematics, it is important to be able to shift a graded module up or down via a suspension operator, too rigid a bound on where the non-zero information sits tends to be silly. We however will mostly discuss chain complex

- cochain complex enrichments as these are nearest to the hypercohomology ideas that we need for our discussion of cohomology, so let us turn to the case given in the title of this section.

(We will 'cheat' and write 'positively' rather than 'non-negatively' graded.)

Definition: A *positively graded dg-category* or *dg^+-category* is a category enriched over the (symmetric) monoidal category Ch^+_K of positively graded chain complexes.

If we are to link this to homotopy coherence, we need a way to go between simplicial sets or simplicial K-modules and chain complexes. We saw earlier that the Moore complex of a simplicial module was a chain complex and that the Dold-Kan theorem in its classical form says that this is an equivalence of categories.

Remark: The Moore complex functor is sometimes called the *normalised complex functor* in this 'Abelian' context. This is to distinguish it from the simple alternating sum complex, which we met briefly when discussing the homology of simplicial complexes.

In that, we have a simplicial K-module, M, say, and form a chain complex simply by taking $C(M)_n = M_n$ and

$$\partial : C(M)_n \to C(M)_{n-1}$$

to be

$$\partial = \sum (-1)^i d_i^n.$$

The Moore complex, as it uses $N(M)_n = \bigcap_{i=1}^n Ker\, d_i$, is a subcomplex of $C(M)$, but it is not hard to see that the inclusion of $N(M)$ into $C(M)$ induces an isomorphism on homology groups.

The Moore complex functor is (lax) monoidal. To see why look at the tensor product of two simplicial modules, A and B. We have $(A\otimes B)_n = A_n \otimes B_n$, $d_i(a\otimes b) = d_i(a)\otimes d_i(b)$. Now suppose $d_i(a) = 0$ for $i = 1, 2, \ldots, p$, whilst $d_i(b) = 0$ for $i = p+1, \ldots, n$, then clearly $d_i(a \otimes b) = 0$ for all $i \geq 1$. Clearly, also, we could have 'shuffled' the indices for which $d_i(a)$ is trivial amongst the remainder and still have got $d_i(a\otimes b) = 0$. Suppose that we have a not in A_n, but in $N(A)_p \subseteq A_p$, we could find a degenerate 'version' of a up in A_n by using q degeneracies, where $p + q = n$. In fact we can do this in many ways. Any 'shuffle' of p elements through q elements yields a potential element $s_\alpha a \otimes s_\beta b \in (A \otimes B)_n$ and if $b \in NB_q$, then we can (and will later) show that the sum of these (with suitable ± 1 weightings) is in $N(A \otimes B)$. (We will look at shuffles very shortly and in some detail.) The end result is a map

$$\nabla : NA \otimes NB \to N(A \otimes B),$$

which serves to give N the important part of the lax monoidal functor structure. The final part of the structure is the fact that the unit of $Simp.K{-}Mod$ is the constant K-value simplicial module, whilst N of this is the complex with K in dimension 0 and zero elsewhere, so N preserves the units.

We have a 'reverse' equivalence from the Dold-Kan Theorem, so might expect that it was monoidal, but it is 'comonodial' in as much as there is a simplicial morphism

$$K(A \otimes B) \to K(A) \otimes K(B)$$

for chain complexes, A and B. Here one can give this explicitly, but we will look at this aspect in a more general setting later. Here we record that the Dold-Kan equivalence *does* give an equivalence between $\mathcal{S}_K-cat$ and dg^+-cat. (A proof is given in Tabuada, [256].) We thus have a definition of homotopy coherence optimised for work with dg^+-categories. For the sake of completeness, we will give the definition:

Let $\mathcal{C}$ be a dg^+-category.

Definition: If $\mathbb{A}$ is a small category, a homotopy coherent diagram of shape $\mathbb{A}$ in $\mathcal{C}$ is a dg^+-functor, $X : NK_*(S(\mathbb{A})) \to \mathcal{C}$.

Because of the equivalence we mentioned above, this idea does not give any more expressive power than by working with $\mathcal{S}_K$-categories, however there can be a more natural structural description of coherence via this, for instance, where it is more natural to consider dg^+-categories than to convert everything to being simplicial.

12.2 From simplicially enriched to chain complex enriched

We can thus go from $\mathcal{S}$-based homotopy coherence to dg^+-based via N applied to the K-linearisation functor. This is quite instructive to look at in much more detail, as it explains in another way the reason why shuffles come into the picture and, although this is a linear case, and so the order of terms does not matter, we can learn a lot from that for later use in other non-linear situations. We would expect to find, for simplicial sets, X and Y, some natural map of chain complexes from $NK^{(X)} \otimes NK^{(Y)}$ to $NK^{(X\times Y)}$. For convenience, we will write $C_N(X)$ instead of $NK^{(X)}$, etc., and so want

$$C_N(X) \otimes C_N(Y) \to C_N(X \times Y).$$

There is such a map and it is known as the *Eilenberg-Mac Lane map*.

12.2.1 Shuffles

To see how to construct this map, we look at the 'archetypal' and universal case, $\Delta[p] \times \Delta[q]$, and thus at trying to get a map,

$$C_N(\Delta[p]) \otimes C_N(\Delta[q]) \to C_N(\Delta[p] \times \Delta[q]).$$

Writing $n = p + q$, an n-simplex of $\Delta[p] \times \Delta[q]$ is given by a functor from $[n]$ to $[p] \times [q]$. The objects of $[p] \times [q]$ are pairs of integers $\begin{pmatrix} i \\ j \end{pmatrix}$ with $0 \leq i \leq p$, $0 \leq j \leq q$.

When we have seen what happens in this case, the general case will be much clearer.

Let us keep things simple and assume $q = 1$ to see what happens (and, hopefully, to awaken memories of where we have seen this before!) Now $n = p + 1$, and, if we restrict to non-degenerate simplices, then the pattern corresponding to an n-simplex will be an increasing sequence that we can think of as an array

$$\begin{pmatrix} i_0 & \dots & i_n \\ j_0 & \dots & j_n \end{pmatrix}$$

of objects, where each is less than the next in the product order on $[p] \times [1]$, (so both the top and bottom rows are increasing). That means that there must be exactly one repeat in the top line, and, at that point, the js jump from 0 to 1. (The possibilities for $p = 2$ were listed on page 513.)

Now if we replace $q = 1$ by the general case, the n-simplex will be an array, as above, except that the top row will repeat at q positions, exactly those positions at which the bottom row jumps up one, i.e., something like

$$\begin{pmatrix} 0 & \dots & 0 & 1 & \dots & 1 & 2 & \dots & p \\ 0 & \dots & i & i & \dots & j & j & \dots & q \end{pmatrix}.$$

The top row is thus some degeneracy $s_\alpha \iota_p$ of the 'universal' p-simplex in $\Delta[p]$, and similarly, the bottom row is $s_\beta \iota_q$. If we represent multiple degeneracies in the usual way, then $\alpha \cap \beta = \emptyset$. For instance, for

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

we have $\alpha = (1)$, $\beta = (2, 0)$. Each such n-simplex thus yields a partition of $\{0, \dots, p + q - 1\}$ into two distinct sets, μ and ν, with $\mu_1 < \dots < \mu_p$ and $\nu_1 < \dots < \nu_q$ and conversely such a partition yields a simplex in the (p, q)-prism. (The relationships between α, β and these partitioning sets will be clearer in a moment.)

Suppose we have an array

$$\begin{pmatrix} i_0 & i_1 & \dots & i_n \\ j_0 & j_1 & \dots & j_n \end{pmatrix}$$

with $0 = i_0 \leq i_1 \leq \dots \leq i_n = p$, then if $i_k = i_{k+1}$, we put k into the second set ν, otherwise put k into μ. For instance, in our old favorite,

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

yields $\nu = \{1\}$, so $\mu = \{0, 2\}$. In fact, this neat and simple recipe yields the sequences for the usual degeneracy representation of the s_α and s_β, but without the (slight) annoyance of trying to remember what the natural way to write multiple degeneracies in this context is!

We now state a proposition before explaining the terminology further. The proof will be given in the next section.

Proposition 149 *For simplicial sets X and Y, there is a natural morphism*

$$\mathbf{b}_{X,Y} : C_N(X) \otimes C_N(Y) \to C_N(X \times Y),$$

which is defined on generators by, if $a \in C_N(X)_p$ and $b \in C_N(Y)_q$,

$$a \otimes b \mapsto \sum (-1)^{sgn(\sigma)} (s_{\sigma_0} a, s_{\sigma_1} b),$$

where $\sigma \in$ Shuff(p, q).

(This $\mathbf{b}$ will later on be the 'φ' of a lax monoidal functor structure, so although we will later use $\mathbf{b}$, we will here refer to the transformation as φ in accordance with our earlier usage.) We call this map the *shuffle map* as well as the *Eilenberg - Mac Lane map*

We must thus explain what *Shuff*(p, q) is, and what the sign, *sgn*, of a shuffle is.

12.2.2 Shuffles in more detail

For the moment, we will write $\mathbf{k} = \{0, 1, \ldots, k-1\}$ and consider the two inclusions

$$i_0 : \mathbf{q} \to \mathbf{p} + \mathbf{q}, \quad i_0(r) = p + r,$$

$$i_1 : \mathbf{p} \to \mathbf{p} + \mathbf{q}, \quad i_1(r) = r,$$

for p, $q \geq 0$.

Definition: A (p,q)-*shuffle* is a permutation σ of the set $\mathbf{p}+\mathbf{q}$ such that both $\sigma_0 = i_0 \circ \sigma$ and $\sigma_1 = i_1 \circ \sigma$ are monotone increasing. We write $\mathit{Shuff}(p,q)$ for the set of such shuffles and sgn for the sign of the corresponding permutation.

In case you have not met the idea before, think of $\mathbf{p}$ and $\mathbf{q}$ as two packs of cards, each in order, perhaps of different colours. To start with we have $\mathbf{p}$ first and $\mathbf{q}$ second in a larger joint pack. We spilt the pack and shuffle the cards, permuting the big pack, but so that the cards from $\mathbf{p}$ stay in the same order, similarly for the cards from $\mathbf{q}$.

How does one get σ to correspond to some α, β or to a partition as before? Look at $\sigma_1 : \mathbf{p} \to \mathbf{p}+\mathbf{q}$. This gives us a set $\{\sigma_1(0), \ldots, \sigma_1(p-1)\}$ of elements of $\mathbf{p}+\mathbf{q}$, and, of course, $\sigma_1(0) < \sigma_1(1) < \ldots < \sigma_1(p-1)$. This gives $s_{\sigma_1(p-1)} \ldots s_{\sigma_1(0)}$ as a multiple degeneracy. Similarly $\sigma_0 : \mathbf{q} \to \mathbf{p}+\mathbf{q}$ gives $\sigma(p) < \sigma(p+1) < \ldots < \sigma(p+q-1)$ and a corresponding degeneracy $s_{\sigma(p+q-1)} \ldots s_{\sigma(p)}$.

From a shuffle, σ, we get a partition of $\mathbf{p}+\mathbf{q}$ into a p element set, $\sigma_1(\mathbf{p})$, and a q element set, $\sigma_0(\mathbf{q})$, and conversely, from a partition of that form, we build a shuffle. The preceding discussion then shows that (p,q)-shuffles correspond bijectively to the top dimensional simplices of $\Delta[p]\times\Delta[q]$. What is even more important for many of the uses of shuffles is that the bijection relates much more. For instance, the $p+q$ simplices of $\Delta[p] \times \Delta[q]$ fit together along $p+q-1$ faces. That can also be tracked in $\mathit{Shuff}(p,q)$. This is important when using shuffles in a less Abelian context. Adjacent $(p+q)$-simplices in $\Delta[p]\times\Delta[q]$ correspond to permutations which are closely related, but will have different 'sign', i.e., will be expressed by an odd number of transpositions if the 'initial' one was even and *vice versa.*

Pause for insight: Hopefully the idea of the sign of a shuffle is clear, but to make it even clearer, look at a very simple example. We have three 3-simplices in $\Delta[2] \times \Delta[1]$, which were, in our array notation,

$$(i) \quad \begin{pmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (ii) \quad \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad (iii) \quad \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

which, in partition notation, gives: (i) $\{0,1\}$, $\{2\}$; (ii) $\{0,2\}$, $\{1\}$; and (i) $\{1,2\}$, $\{0\}$, listing $\sigma_1(\mathbf{p})$ and then $\sigma_0(\mathbf{q})$. We therefore have that, for the corresponding shuffles, (i) is the identity, (ii) is the transposition exchanging 1 and 2, and (iii) is the 3-cycle sending 0 to 1, 1 to 2, and 2 back to 0. The signs of each shuffle are thus easy to workout being +1, -1 and +1 respectively. What do they 'mean'? Perhaps **it is worthwhile** drawing a picture of a prism, marking the three simplices on it, then **staring** at it! The 'orientation' of the second simplex, (ii), is opposite to that of the other two. (This is very easy to see with the two 2-simplices in $\Delta[1] \times \Delta[1]$, but needs a bit more thought, for our case, as three dimensional 'orientation' is a bit less easy to imagine.)

Another way of interpreting this sign is, to start with, very pragmatic. Homology is quite closely related to integration in both its origins and its applications - although this is not always obvious. The simplices in (i) and (ii) have a common d_2-face, those in (ii) and (iii) a common d_1-face. These faces must contribute opposite signed terms, so as to cancel out in the overall sum / integral. (Think of integrating some vector field over a surface element corresponding to some flow through that element.) The transition from (i) to (ii) multiplies the permutation / shuffle for (i) by the transposition $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$, that from (ii) to (iii) changes the shuffle by changing 0 to 1, i.e., post-composing with $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$.

It is very useful to have a multi-facetted intuition about these combinatorial structures, as they seem to occur, for closely related reasons, in many parts of cohomology theory. Their full potential seems far from being used as yet, except perhaps in parts of the interface of algebraic topology with algebraic geometry.

12.2.3 The Eilenberg - Mac Lane map: proof of concept

Proof of the proposition: Back to the proof of our earlier proposition, we will give quite a lot of detail. The idea is geometrically simple, but is extremely useful to have it explicitly available. (A variant will be used shortly when looking at enrichments over crossed complexes. In fact we will adapt our discussion from that in Tonks, [264, 265], which is actually of that case.)

What we have to do first is to check naturality - because *that* is reasonably easy! We suppose $f : X \to X'$ and $g : Y \to Y'$, so get $C_N(f) \otimes C_N(g)$ on the left hand side and $C_N(f \times g)$ on the right. The image of $a \otimes b$ will thus be $f(a) \otimes g(b)$ and we track that across to $C_N(X' \times Y')$ to get the sum of terms $(s_{\sigma_0} f(a), s_{\sigma_1} g(b))$, but, as those are the $C_N(f \times g)(s_{\sigma_0} a, s_{\sigma_1} b)$, we are 'home'.

The next claim is that, if a and b are in the Moore complex, then so is $\varphi_{X,Y}(a \otimes b)$. To check this we have to examine the faces of $\varphi_{X,Y}(a \otimes b)$. Of course,

$$d_i \varphi_{X,Y}(a \otimes b) = \sum (-1)^{sgn(\sigma)} (d_i s_{\sigma_0} a, d_i s_{\sigma_1} b).$$

We thus need to analyse *Shuff*(p,q) to see what these faces correspond to.

For i with $0 \leq i \leq p+q$, any (p,q)-shuffle satisfies exactly one of the following four conditions:

1. $\{i-1, i\} \subseteq \{-1\} \cup Im(\sigma_1) \cup \{p+q\}$;
2. $\{i-1, i\} \subseteq \{-1\} \cup Im(\sigma_0) \cup \{p+q\}$;
3. $i-1 \in Im(\sigma_1)$ and $i \in Im(\sigma_0)$;
4. $i-1 \in Im(\sigma_0)$ and $i \in Im(\sigma_1)$.

The first two give the two possibilities if the pair $\{i-1, i\}$ are together in one of the two sets of the partition; the second two give the corresponding possibilities when they are apart. The point being that $d_i s_i = d_i s_{i-1} = id$, so the d_i will 'annihilate' either one or two degeneracies in all the terms involved in the expression for $\varphi_{X,Y}(a \otimes b)$.

Write $S_r^{(i)}(p,q)$ for $r = 1, \ldots, 4$, for the set of (p,q)-shuffles that satisfy condition r for the given i.

It is clear that there is a bijection

$$\gamma : S_3^{(i)}(p,q) \leftrightarrow S_4^{(i)}(p,q),$$

where $\gamma\sigma$ is the composite of σ with the transposition exchanging $i-1$ and i. We thus have $sgn(\gamma\sigma) = -sgn(\sigma)$ and

$$d_i(s_{(\gamma\sigma)_0}(a), s_{(\gamma\sigma)_1}(b)) = d_i(s_{\sigma_0}(a), s_{\sigma_1}(b)),$$

as the d_i will kill off s_i in one of the two terms and s_{i-1} in the other, on each side of the equation, thus the corresponding terms in $d_i\varphi_{X,Y}(a \otimes b)$ will cancel in pairs.

This leaves those terms that correspond to shuffles of types 1 and 2. We first look at $i > 0$ and we need $d_i\varphi_{X,Y}(a \otimes b)$ to be zero, so as to check $\varphi_{X,Y}(a \otimes b) \in C_N(X \times Y)$. We look at a term $d_i(s_{\sigma_0}(a), s_{\sigma_1}(b))$, when $\sigma \in S_1^{(i)}(p,q)$, then $\{i-1, i\} \subset Im(\sigma_1) \cup \{p+q\}$. As $\sigma_1 = \{\sigma(0), \ldots, \sigma(p-1)\}$, we can find a t, depending on σ and i, such that $\sigma(t) = i$ and $0 < t \leq p-1$. (We have $s_{\sigma_1} = s_{\sigma(p-1)} \ldots s_{\sigma(0)}$, and $d_i s_k = s_{k-1} d_i$ as long as $i < k$, so here we need $k = \sigma(\ell)$, with $t < \ell \leq p-1$, and then d_i will 'pass through' those parts of σ_1, pushing each index one to the left. Intuitively, this should give a new shuffle of smaller size - it may help to look at a prism and its faces at this point.) Define a $(p-1,q)$-shuffle, $\tau := \tau(\sigma, i)$ by

$$\tau(\sigma,i)_0(j) = \begin{cases} \sigma_0(j) & \text{if } \sigma_0(j) < i \\ \sigma_0(j) - 1 & \text{if } \sigma_0(j) > i \end{cases}$$

$$\tau(\sigma,i)_1(j) = \begin{cases} \sigma_1(j) & \text{if } \sigma_1(j) < i \\ \sigma_1(j+1) - 1 & \text{if } \sigma_1(j) > i, \end{cases}$$

then

$$d_i(s_{\sigma_0}a, s_{\sigma_1}b) = (s_{\tau(\sigma,i)_0}(d_t a), s_{\tau(\sigma,i)_1}b),$$

and $sgn(\tau(\sigma,i)) = (-1)^{i+t} sgn(\sigma)$.

If we are given i and σ, we can work out t and τ, and conversely, so there is a bijection

$$\bigcup_{i=0}^{p+q} (S_1^{(i)}(p,q) \times \{i\}) \leftrightarrow \mathit{Shuff}(p-1,q) \times \{0,1,\ldots,p\}.$$

(Actually we have not looked at the case $i = 0$ - but that is **left to you**.) Similarly, if we have type 2 shuffles, then we get a relationship linking the $S_2^{(i)}(p,q)$ with $\mathit{Shuff}(p,q-1)$.

We also have, for type 2, shuffles, a formula for $d_i(s_{\sigma_0}a, s_{\sigma_1}b)$ in terms of $(s_{\tau'(\sigma,i)_0}(a), s_{\tau'(\sigma,i)_1}(d_{t'}b))$, for suitably defined t' and τ'.

If $i > 0$, as we have assumed, then $t > 0$, (similarly for t'), as, if $\sigma(0) = i$, then $i-1$ must be in 'the other part' of the partition. We thus have, if $a \in C_N(X)_p$, $d_t a = 0$ and all type 1 terms will vanish; similarly, if $b \in C_N(X)_q$, $d_{t'}b = 0$ and the type 2 terms vanish, hence $\varphi(a \otimes b) \in C_N(X \times Y)_{p+q}$.

The only thing remaining to check is what the differentials are, since φ has to be a chain map,

$$\varphi_{X,Y} : C_N(X) \otimes C_N(Y) \to C_N(X \times Y).$$

The differential on the left is that in the tensor product, so

$$\begin{aligned} \partial(a \otimes b) &= \partial a \otimes b + (-1)^p a \otimes \partial b \\ &= d_0 a \otimes b + (-1)^p a \otimes d_0 b, \end{aligned}$$

and, on applying φ,

$$\varphi(\partial(a \otimes b)) = \sum_{\sigma \in Shuff(p-1,q)} (s_{\sigma_0} d_0 a, s_{\sigma_1} b) + (-1)^p \sum_{\sigma \in Shuff(p,q-1)} (s_{\sigma_0} a, s_{\sigma_1} d_0 b).$$

We need to compare this with

$$d_0(\varphi(a\otimes b)) = \ldots .$$

Yes, the d_0 passes through in both type 1 and type 2 terms giving what we hoped for, and, for the other shuffles of types 3 and 4, the terms cancel in pairs.

The morphism, φ, is thus a chain map as claimed. ■

We also need to check 'associativity' of φ. This is quite amusing, that is if you have built up enough intuition about shuffles.

Proposition 150 *For simplicial sets, X, Y and Z, the diagrams*

$$\begin{tikzcd}
C_N(X)\otimes C_N(Y)\otimes C_N(Z) \arrow[r, "\varphi\otimes C_N(Z)"] \arrow[d, "C_N(X)\otimes\varphi"'] & C_N(X\times Y)\otimes C_N(Z) \arrow[d, "\varphi"] \\
C_N(X)\otimes C_N(Y\times Z) \arrow[r, "\varphi"'] & C_N(X\times Y\times Z)
\end{tikzcd}$$

commutes.

Proof: First we should say that each φ is shorthand for a different map depending on the context, for instance, that on the right hand side is clearly $\varphi_{X\times Y,Z}$, and so on.

We have to check commutativity and so can do so on generators, $a\otimes b\otimes c$. (The diagram hides the associators for $\otimes$ and $\times$!) To do this we introduce (p,q,r)-shuffles. We have sets $\mathbf{p}$, $\mathbf{q}$, as before, and now also $\mathbf{r}$, where $a\in C_N(X)_p$, etc. There are injections,

$$j_0 : \mathbf{p}\to\mathbf{p}+\mathbf{q}+\mathbf{r},$$

given by $j_0(k) = k$; similarly $j_1(k) = p+k$, $k\in\mathbf{q}$, and $j_2(k) = p+q+k$, with $k\in\mathbf{r}$.

Definition: A (p,q,r)*-(multi-)shuffle* is a permutation, σ, of $\mathbf{p}+\mathbf{q}+\mathbf{r}$ such that each of the composites, $\sigma\circ j_i$, $i = 0,1,2$, is monotone increasing. Write *Shuff*(p,q,r) for the set of (p,q,r)-shuffles.

Lemma 91 *For any p, q, r, there are bijections*

$$\mathit{Shuff}(p,q,r)\leftrightarrow\mathit{Shuff}(p,q+r)\times\mathit{Shuff}(q,r),$$

and

$$\mathit{Shuff}(p,q,r)\leftrightarrow\mathit{Shuff}(p+q,r)\times\mathit{Shuff}(p,q).$$

Proof: This is sort of clear. Split the $\mathbf{p}+\mathbf{q}+\mathbf{r}$ set in two parts $\mathbf{p}$ and $\mathbf{q}+\mathbf{r}$. Shuffle the second set with some (q,r)-shuffle, then shuffle the whole set with a $(p,q+r)$ shuffle. The result will be a (p,q,r)-shuffle, and any such can be arrived at in this way - and similarly for the second part. (The details are easy to **provide for yourself**.) ■

The proof of the proposition now is routine. You do have to check on the signs of the permutations, but that is fairly obvious. (You can think of this as defining a single diagonal map in the square

$$\varphi_{X,Y,Z} : C_N(X)\otimes C_N(Y)\otimes C_N(Z)\to C_N(X\times Y\times Z)$$

using multishuffles and comparing the two composites with it - again the fine detail is **left to you.**) ■

Although this is not immediately necessary for what we need these maps for, some of the properties that they satisfy are very important. We could ask whether it is an injection, or, for instance, does it have an inverse on the right, or the left, and so on. In fact, historically in cohomology, this map, and some related ones, provided enormous insights into the structure of cohomology (in the Abelian case). We will thus turn aside from the immediate needs of our enquiry homotopy coherence to consider these related maps and the overall properties of this context.

We have concentrated on the passage all the way from simplicial sets to chain complexes, as that was the main immediate use we have for this theory, however it is probably clear to you that the above proof of the existence of this Eilenberg-Mac Lane map using shuffles can apply to a much more general situation.

Suppose we have a simplicial K-module, A, then we can form the corresponding unnormalised associated chain complex, using alternating sums of face maps to get a differential. We will denote this unnormalised chain complex by (A, ∂), usually omiting any mention of A on the notation for the differential if there is little danger of confusion. Given two such simplicial modules, A and B, we can form $(A, \partial) \otimes (B, \partial)$ and also $(A \otimes B, \partial)$. This should be reasonably easy to analyse by now, but let us just note that in dimension n, the first consists of $\bigoplus_{p+q=n}(A_p \otimes B_q)$, whilst the second has $(A_n \otimes B_n)$.

Proposition 151 *For simplicial modules, A and B, there is a natural morphism*

$$\mathbf{b}_{A,B} : (A, \partial) \otimes (B, \partial) \to (A \otimes B, \partial),$$

which is defined on generators by, if $a \in A_p$ and $b \in B_q$,

$$a \otimes b \mapsto \sum (-1)^{sgn(\sigma)}(s_{\sigma_0}a, s_{\sigma_1}b),$$

where $\sigma \in Shuff(p, q)$.

Proof: The proof is very like that of our earlier proposition so we will limit ourselves to pointing out the differences. Most obviously we do not need to check compatibility with the Moore complex, (not yet but there is a corollary below!) That analysis, however, needs to be adapted to show that $\mathbf{b}_{A,B}$ is a chain map. This requires care, but no new ideas, so it **left to you**. ■

This generalisation has a normalised form as well. This is important as it gives a link between the monoidal structures of the categories of simplicial modules and positive chain complexes which are comparable via the Dold-Kan equivalence. We will investigate this further shortly. As usual NA is the Moore complex of the simplicial module A.

Corollary 27 *For simplicial modules, A and B, there is a natural morphism*

$$\mathbf{b}_{A,B} : (NA, \partial) \otimes (NB, \partial) \to (N(A \otimes B), \partial),$$

which is defined on generators by, if $a \in A_p$ and $b \in B_q$,

$$a \otimes b \mapsto \sum (-1)^{sgn(\sigma)}(s_{\sigma_0}a, s_{\sigma_1}b),$$

where $\sigma \in Shuff(p, q)$. ■

The proof should be clear as it reinstates the check of compatibility with Moore complexes used in the earlier proposition.

These results suggest some questions. We have already mentioned one of them. The Dold-Kan equivalence uses a reconstruction of A from its Moore complex and gives us a functor from the category of positively graded chain complexes to that of simplicial K-modules, (cf. page 252, where the functor is denoted K). The above gives that the Moore complex functor is lax monoidal. What about this Dold - Kan inverse / reconstruction functor? How does that react with respect to tensor products?

We have also seen that there is a Dold - Kan equivalence between group T-complexes and crossed complexes. Is there an Eilenberg - Mac Lane map at that level and what about the much more difficult case of general simplicially enriched groupoids and the corresponding Moore complexes / hypercrossed complexes (cf. page 216)? The analysis of shuffles in these non-Abelian situations is much more difficult as order becomes very important, but the fact that the Samelson and Whitehead products in the simplicial context, (see Curtis, [97], p. 197), are definable in terms of shuffles using a very similar formula is highly suggestive that there is something interesting here.

(To be continued)

Chapter 13

More simplicial and categorical constructions!

The previous two chapters started exploring the interaction of homotopy coherence with enriched category theory, what has been called 'enriched homotopy theory' by Shulman, [247]. Before we can go further with our analysis of the transition from simplicial sets to chain complexes as an enrichment, however, we need to have a few more tools from categorical treatments of simplicial theory. We will give the details of more than is strictly needed at this point, since a lot of the next level 'down' of the theory is of direct relevance to our central themes. We will return to the topics that we have just left shortly.

13.1 Total complex constructions: part 1

When working with the transition between simplicial objects and chain complexes of various types, there are various other basic properties of simplices that need to be kept in mind. We have seen the use of shuffles in analysing the Eilenberg-Mac Lane map, and shuffles involve simplices in prisms, $\Delta[p] \times \Delta[q]$. The Eilenberg-Mac Lane shuffle map gave a map *from* a tensor product of chain complexes, and, as its definition uses loads of degeneracies, it looks as if it might be a monomorphism, even a split one (perhaps by a cunning combination of face maps). There is such a map splitting it. This is the Alexander-Whitney map. We will look at it in a lot of detail a little bit further on. For us here, the important thing to note is that it has to be a map whose *codomain* is a tensor product, $(A, \partial) \otimes (B, \partial)$, of chain complexes derived from simplicial modules. This suggests that it is a good point at which to look at the construction of these tensor products in a bit more detail.

We recall, yet again, that if C and D are chain complexes over our commutative ring K, their tensor product is given as a graded modules by

$$(\mathsf{C} \otimes \mathsf{D})_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

with differential

$$\partial(c \otimes d) = (\partial c) \otimes d + (-1)^{|c|} c \otimes (\partial' d),$$

where $|c|$ is the degree of c.

This construction can be usefully viewed as an example of a *total complex of a bicomplex.* First we need the definition of a bicomplex.

You will probably have noted, or previously met, the idea that to define notions such as that of a chain complex, one needs very little structure, at least to start with. In both the development of 'classical' homological algebra and in its applications, the progression in the direction of abstraction went from chain complexes as differential graded vector spaces or Abelian groups, through the obvious generalisation to dg modules, thence to sheaves of these and finally into various forms of additive or Abelian category. (Of course, one can try to develop this further, going to dg groups that need not be Abelian and so on, but in those cases it is often, but perhaps not always, expedient to pass to the corresponding simplicial theory.) In particular the category of chain complexes in such contexts is itself of a form in which we can consider chain complexes of such objects, *i.e.*, we can form chain complexes whose objects in degree n are themselves chain complexes, that is we can form 'bicomplexes'. It is useful to have a formal definition. To show the simplicity of the idea as an extension of what we have seen before, we will, in part, 'cut, paste and edit' bits of our earlier discussion of graded vector spaces, etc.

Definition: (i) A $\mathbb{Z}$-*bigraded vector space* (bi-gvs) is a $\mathbb{Z}\times\mathbb{Z}$-indexed family, $\{V_{i,j}\}_{(i,j)\in\mathbb{Z}\times\mathbb{Z}}$, of vector spaces.

(ii) A morphism, $f : \mathsf{V} \to \mathsf{W}$, of bigraded vector spaces is *homogeneous* if $f(V_{i,j}) \subseteq W_{i+p,j+q}$ for all (i,j) and some common (p,q), called the *bidegree of* f.

(iii) An endomorphism, $d^h : \mathsf{V} \to \mathsf{V}$, of bidegree (-1,0) is called a *horizontal differential* or *boundary* (which is used depending largely on the context) if $d \circ d = 0$.. One, d^v, of bidegree (0,-1) will similarly be called a *vertical differential* or *boundary*..

(iv) A *bicomplex*, $\mathsf{C} = \{C_{i,j}, d^h, d^v\}$, is a bigraded vector space (or more generally module) together with one vertical and one horizontal boundary / differential satisfying

$$d^v d^h = d^h d^v,$$

that is, the two boundaries are independent of each other.

The idea that a bicomplex is a complex of complexes should now be clear, but there is no 'privileged direction' inherent in the definition, so you can think of the above as a complex (going horizontally) whose objects are complexes of modules written vertically, or the other way around. The commutativity or independence condition on the two differentials is to ensure that the differentials in one direction are morphisms of the complexes in the other.

The use of 'horizontal' and 'vertical' is, of course, due to the way in which bicomplexes are often represented in print. The individual modules, $C_{p,q}$, are placed at the vertices of an integer grid,

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
\cdots & C_{p-1,q+1} \arrow[l] \arrow[d] & C_{p,q+1} \arrow[l] \arrow[d] & C_{p+1,q+1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & C_{p-1,q} \arrow[l] \arrow[d] & C_{p,q} \arrow[l] \arrow[d] & C_{p+1,q} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & C_{p-1,q-1} \arrow[l] \arrow[d] & C_{p,q-1} \arrow[l] \arrow[d] & C_{p+1,q-1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & \cdots & \cdots & \cdots &
\end{tikzcd}$$

and then the differentials of bidegree (-1,0) become horizontal arrows and those of bidegree vertical. The conventions of whether these go to the left or right vary from source to source, and, of course, are usually just a matter of taste!

If a bicomplex is a complex of complexes, then there is no reason why we should not have tri-complexes, or general 'multicomplexes, defined as complexes of complexes of ..., and so on, and these can be very useful, however bicomplexes are the most often used because the uses for more general forms of multicomplex can usually be reduced to a repeat use of bicomplex constructions. When we go to the non-Abelian context and replace bicomplexes by bisimplicial sets etc., then there are specific instances where multisimplicial sets are needed, and we have seen one of these earlier when handling algebraic models for n-types. When working with higher 'order' multicomplexes, it is, of course, necessary to use slightly different terminology and notation as 'horizontal' and 'vertical' will not be enough! That is not, however, a good enough reason to discard that terminology in the bicomplex case.

One word of caution, there are situations in which it is useful to require, not that the differentials in the two directions commute, but rather that they 'anti-commute', that is to say, $d^v d^h + d^h d^v = 0$. This is the convention used, for instance, by Mac Lane, [191], (in his section on bicomplexes in Chapter XI). This *simplifies* certain constructions, but the idea that a bicomplex is simply a complex of complexes is harder to justify.

Going back to tensor products, we can take two chain complexes, C and D, and form a bicomplex E, with $E_{i,j} = C_i \otimes D_j$ with the two differentials inherited from C and D respectively. To get from this bicomplex to $\mathsf{C} \otimes \mathsf{D}$, we take the direct sum of all terms $E_{p,q}$ with $p + q = n$ and put that in dimension n. We know what the differential has to look like and so can write that down in terms of the individual 'horizontal' and 'vertical' differentials in E. (That is sort of cheating as it does not really tell us why there is a alternating sign on the second part of the expression. Unfortunately there are several ways of getting this and various 'explanations' of why that is the right way of doing things. There are various subtleties depending on the approach being taken. It *is* a good idea to **look up** some of the treatments of 'total complexes' in standard homological algebra books, but this is **left to you to worry out** as it will not really be needed in what follows and would be a further diversion from our main themes.)

A similar idea occurs in working with $Hom(\mathsf{C}, \mathsf{D})$, where the individual bicomplex terms are the various $Hom(C_p, D_q)$ in bidimension $(-p, q)$. (Why the '$-p$'? Simply that C_p is in the contravariant position, so the degree -1 morphism ∂^{C} induces a map $Hom(C_{p-1}, D_q) \to Hom(C_p, D_q)$, of degree

+1 in p. To get the correct degree, that is, -1, we take $-p$ not p.) The chain complex $Hom(\mathsf{C},\mathsf{D})$ has, as module of n-simplices, the product

$$Hom(\mathsf{C},\mathsf{D})_n = \prod_{p=-\infty}^{\infty} Hom(C_p, D_{p+n}),$$

according to our discussion on page 361. We reindex this by

$$\mathsf{H}_{r,s} = Hom(C_{-r}, D_s),$$

giving a bicomplex, and a total complex, $Hom(\mathsf{C},\mathsf{D})$, collecting up terms along the lines $r+s=n$.

Definition: For a bicomplex, $\mathsf{C} = \{C_{i,j}, d^h, d^v\}$, the *total complex of* C is the complex, $\mathsf{X} = Tot(\mathsf{C})$, defined by

$$X_n = \bigoplus_{p+q=n} C_{p,q},$$

$$d_n = \{d^h_p + (-1)^p d^v_q)\}.$$

A word needs to be said about the notation here. If we have an element $c_{p,q}$ in $C_{p,q}$, it will have two boundary components, one in $C_{p-1,q}$, the other in $C_{p,q-1}$. Both will contribute to $Tot(C)_{n-1}$, of course, where $n = p+q$, but will themselves be in different summands of that object. Combining the two boundaries, as in the above formula, gives a total boundary for a term $\underline{x} = (x_p)_{p\in\mathbb{Z}}$, where $x_p \in X_{p,n-p}$, of

$$d\underline{x} = (d^h x_{p+1} + (-1)^p d^v x_p)_{p\in\mathbb{Z}}.$$

The result of applying d twice to $c_{p,q}$ will result in terms in three bidimensions, $(p-2,q)$, $(p,q-2)$, and $(p-1,q-1)$. The first two of these terms are zero because they involve $d^v d^v$ or $d^h d^h$, the third involves $(-1)^p d^v d^h$ and $(-1)^{p-1} d^h d^v$ and this is also zero, because of the sign. There is clearly a second possible definition of a differential using the vertical direction to influence the sign in the formula. The theory gives equivalent results. Note that we have, in this discussion, essentially checked that (X, d) is a complex.

Of course, all this depends heavily on the fact that we are working in an Abelian setting. If we go to a non-Abelian setting, simplicial objects will be better at handling things.

13.2 Ordinal sum

13.2.1 Initiation

Chain complexes only involve degrees that are 'numerals'. Simplicial objects have 'degrees' or 'dimensions' that are better viewed as being 'ordinals'. This is not only for the formal reason that a simplicial object is defined as a functor from the opposite of the category, $\mathbf{\Delta}$, of finite ordinals to whatever category you are working in, but also for the more intuitive and pragmatic reason that, when working with simplicial objects, you have to use a collection of face and degeneracy operators and these involve orders on the sets of indices used. (It may be helpful to recall that to go from a simplicial complex to a simplicial set, we needed to pick an *order* on the set of vertices. The residue of these orders can be seen in the alternating sum construction using an odd / even split on the faces. In non-Abelian settings, it is sometimes useful to replace chain complexes with 'parity complexes' in which there are two maps from degree n to degree $n-1$ satisfying various rules. Another way to get around the question of 'order' is to use cubical complexes. We will not go into either of these here.) The ordinal sum is a very natural construction that gives a replacement for the simple '+' on degrees when collecting terms on passing from *bisimplicial* objects to simplicial ones.

Definition: The *ordinal sum*

$$\oplus : \mathbf{\Delta} \times \mathbf{\Delta} \to \mathbf{\Delta}$$

is given by : $[p] \oplus [q] = [p+q+1]$ and if $f : [p] \to [p']$, $g : [q] \to [q']$, then $f \oplus g : [p] \oplus [q] \to [p'] \oplus [q']$ is given by :

$$f \oplus g(k) = \begin{cases} f(k) & \text{if } 0 \le k \le p \\ g(k-p-1) + q' + 1 & \text{if } p+1 \le k \le p+q+1. \end{cases}$$

Thus $\oplus$ concatenates two ordinals : $[p] \oplus [q]$ is *first* $[p]$ *then* $[q]$.

Remarks: (i) If the notation $\oplus$ is read as in Latex, this is 'oplus' and 'o' reminds one of 'ordinal' and 'plus' of 'sum'.

(ii) Ordinal sum is a special case of the join operation, $X \wedge Y$, for partially ordered sets, where everything in X is less than everything in Y, but otherwise elements are compared as in the individual posets, X and Y. Sort of conversely, there is a join operation for (augmented) simplicial sets which is defined using the ordinal sum; see [119] for a brief introduction.

(iii) The ordinal sum functor almost makes $\mathbf{\Delta}$ into a monoidal category. It is clear, for instance, that $([p] \oplus [q]) \oplus [r] = [p+q+r+2]$, as is $[p] \oplus ([q] \oplus [r])$, but there is no unit as that would need a -1 'dimensional' object. By adding the empty ordinal into $\mathbf{\Delta}$, we *do* get a monoidal category, $\mathbf{\Delta}_\emptyset$, with remarkable properties. This, in turn, gives the category of presheaves $\mathcal{C}^{\mathbf{\Delta}_\emptyset^{op}}$, a closed monoidal structure. The monoidal tensor is the join operator above and the 'closed' (*i.e.*, enriched 'object of morphisms') structure uses the décalage. (This is a general construction and is often called the *Gray monoidal structure* on a presheaf category, $\mathcal{C}^{\mathbb{M}^{op}}$, when the domain, $\mathbb{M}$ has a monoidal structure.) We will see shortly that $\mathcal{C}^{\mathbf{\Delta}_\emptyset^{op}}$ can be interpreted as a category of '*augmented simplicial objects in* $\mathcal{C}$'.

13.2.2 Dec and *diag*

Recall that a simplicial simplicial set is called a bisimplicial set. If Y is a bisimplicial set, then it is a functor from $\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$ to *Sets*. It is sometimes useful to draw a diagram in the form of a 2-dimensional array with the horizontal and vertical face and degeneracy maps shown, as on page 32.

To link the ordinal sum with other constructions, we need various functors between $Sets^{\mathbf{\Delta}^{op}}$ and $Sets^{\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}}$:

Definition: The *total décalage functor*, **Dec**, from $Sets^{\mathbf{\Delta}^{op}}$ to $Sets^{\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}}$ is given by composition along $\oplus^{op} : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathbf{\Delta}^{op}$. If X is in $Sets^{\mathbf{\Delta}^{op}}$, then $(\mathbf{Dec}X)_{p,q} = X_{p+q+1}$.

Definition: The diagonal functor $\mathbf{\Delta}^{op} \to \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$ induces a functor, $diag$, from $Sets^{\mathbf{\Delta}^{op}\times\mathbf{\Delta}^{op}}$ to $Sets^{\mathbf{\Delta}^{op}}$, which restricts each functor to the diagonal copy of $\mathbf{\Delta}^{op}$ in $\mathbf{\Delta}^{op}\times\mathbf{\Delta}^{op}$, thus $(diag\,Y_{\cdot,\cdot})_n = Y_{n,n}$.

Remarks: (i) This diagonal functor has a useful description as a coend:

$$(diag\,Y)_q \cong \int^p Y_{p,q} \cdot (\Delta[p]),$$

where $A \cdot B$ denotes the A-fold copower of B, *i.e.*, the coproduct of A-many copies of the object B, where A is a set. (For a brief discussion of ends and coends, see below, section 13.4.)

(ii) There is nothing special about using bisimplicial and simplicial *sets* here. If $\mathcal{C}$ is any category, then the analogues of $diag$ and **Dec** clearly exist for simplicial and bisimplicial objects in $\mathcal{C}$. We have already met **Dec** for simplicial *groups*, when discussing models for n-types, (see the comment below), although this was not made explicit.

(iii) Both $diag$ and **Dec** have left and right adjoints and these will be reviewed if and when needed. If we are working with (bi)simplicial objects in a category other than $Sets$, then completeness or cocompleteness will be need to ensure that the relevant adjoint exists.

The definitions of $diag$ and **Dec** are neat, but quite 'dry'. They do not tell you what these things 'look like' to any great extent. It is therefore useful to explore them in low dimensions to see what they do. In particular, in both definitions we have given the form of the object in given dimensions, but have not looked at the face and degeneracy maps. Clearly this is a useful thing to do now.

For the diagonal of a bisimplicial object, Y, an n-simplex is an element $y_{n,n}$ in $Y_{n,n}$. We have $d_i(y_{n,n}) = d_i^v d_i^h(y_{n,n})$, and similarly for the degeneracies. The simplicial identities are easy to check.

For the 'total Dec', it is a good idea to sketch out the diagram for the bottom part of this bisimplicial object.

- $Dec(X)_{0,0} = X_1$;
- $Dec(X)_{1,0} = Dec(X)_{0,1} = X_2$, with $d_i^h = d_i$ and $d_j^v = d_{j+1} : X_2 \to X_1$;

and, in general,

- $Dec(X)_{p,q} = X_{p+q+1}$ with $d_i^h = d_i$, and $d_j^v = d_{p+1+j}$.

It is **left to you** to write down the corresponding degeneracies, and to justify the description in terms of $\mathbf{Dec}(X) = X\oplus$. It may also help to draw an array diagram for the low dimensions along the lines of that on page 32.

Before we analyse these functors and their adjoints in any detail, it is convenient to digress and to look at various related aspects of the theory. We have introduced a décalage, Dec, earlier (cf. page 384) and should clearly examine the relationship between Dec and **Dec**. The other point we will examine is how to think of ends and coends as these will be increasingly needed in what follows. As in previous sections, this will be principally concerned with understanding these generalisations of limits and colimits. **You will be left to search out** a more 'in-depth' treatment for when you need it.

13.3 First Interlude: Augmentations, Décalage and Resolutions

(Good sources for some of this are Illusie's thesis, [163, 164], and Jack Duskin's Memoir, [107]. Some of the material in this part of the notes was greatly aided by unpublished notes of Duskin and Van Osdol. A treatment of it was given in the M.Sc. and Ph.D. theses of Phil Ehlers, [117] and [118], with related material in [119, 120].)

The simple décalage of a simplicial set, simplicial group, or more generally, Y, was given by $(Dec\,Y)_p = Y_{p+1}$, with $d_i^{Dec\,Y} = d_i$, etc.; (refer back to page 384 for this). The total décalage is given by $(\mathbf{Dec}\,Y)_{p,q} = Y_{p+q+1}$, with corresponding face and degeneracy maps in the two directions. We thus have $(Dec\,Y)_p = (\mathbf{Dec}\,Y)_{p,0}$, which is a nice simple relationship. (**You should just check** that face and degeneracy maps coincide.)

There is more to this than meets the eye however. We have met *augmented simplicial objects* several times; for instance, their first appearance was on page 35. We must now look at them in some detail.

13.3.1 Augmented Simplicial Objects

There are at least three or four ways in which augmented simplicial objects can be viewed. All are useful and, as is usual in such situations, each gives a clearer intuition of some aspects of the notion and of its interactions with other theory. We will repeat the basic definition that we gave on page 35, adjusting, and extending, notation and terminology for our current needs, and will add a little more detail in the process. As before, we let $\mathcal{C}$ be some category such as that of sets, or groups or sheaves of such on some space or site.

Definition (more-or-less a repeat): An *augmented simplicial object*, $\mathbf{X}$, in $\mathcal{C}$ consists a simplicial object, $X_\cdot$, together with a morphism, $d_0 : X_0 \to X_{-1}$, to an object in $\mathcal{C}$, that satisfies $d_0 d_0 = d_0 d_1$:

$$\begin{tikzcd} \cdots \arrow[r, shift left=2, "d_0"] \arrow[r] \arrow[r, shift right=2, "d_2"'] & X_1 \arrow[l, bend right=60, "s_0"'] \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \arrow[l, bend right=60, "s_0"'] \arrow[r, "d_0"] & X_{-1}. \end{tikzcd}$$

It is usual to consider the augmentation object as occurring in dimension -1, (the reason for this will become apparent shortly).

A *morphism,* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$*, of augmented simplicial objects* is, of course, an augmented simplicial morphism, that is, an ordinary simplicial morphism, $f_\cdot : X_\cdot \to Y_\cdot$, together with an additional morphism, $f_{-1} : X_{-1} \to Y_{-1}$, in dimension -1, such that $d_0 f_0 = f_{-1} d_0$.

Notationally it may sometimes be useful to write q_X instead of $d_0 : X_0 \to X_{-1}$, or similar, for instance, if we are thinking of X_{-1} as being separate or additional data. For instance, the same simplicial object, $X_\cdot$, may occur with different augmentations and using the same notation d_0 for all of them is clearly not optimal use of notation! It is also useful to consider the augmented object with X^+ being the non-negatively graded part, together with another object, X, and appropriate augmentation map having X as its codomain. This is most useful when X is given initially, perhaps being fixed during some discussion, and X^+ is, in this case, the additional data. The earlier use of simplicial resolutions of a group is an instance of this. A useful notation for an augmented simplicial object, in this case, would be $\mathbf{X} = (X^+, X, d_0)$ or (X^+, X) for short. The notation $\mathbf{f} = (f^+, f)$ for a morphism would then be appropriate.

We will write $AugSimp(\mathcal{C})$ for the category of augmented simplicial objects in $\mathcal{C}$ with $Aug\mathcal{S}$, or $\mathcal{AS}$ for the important case of $\mathcal{C} = Sets$.

If we have a fixed augmentation object, X, then we get a subcategory, $AugSimp(\mathcal{C})_X$, consisting of those (X^+, X) with augmentation object the fixed X and in which the morphisms are the (f^+, id_X), having the identity morphism in dimension -1.

Lemma 92 *For any object X in $\mathcal{C}$, we have an equivalence of categories*

$$AugSimp(\mathcal{C})_X \simeq Simp(\mathcal{C}/X).$$

Proof: This is more or less obvious if you draw the augmentation as a vertical arrow:

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[drr] & X_1 \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[dr] & X_0 \arrow[d, "d_0"] \\ & & & X_{-1}, \end{tikzcd}$$

making everything 'over X'. The slanting arrows are the composites of the face maps of X^+ with the augmentation map. Any such composite from X_i to X_{-1} is equal to $d_0^{(i+1)}$, that is $X_i \xrightarrow{d_0} X_{i-1} \xrightarrow{d_0} \ldots \xrightarrow{d_0} X_0 \xrightarrow{d_0} X_{-1}$ by the simplicial identities together with the augmentation condition, $d_0 d_0 = d_0 d_1$. ■

Remark: This is very useful, since, for instance, if $\mathcal{C}$ has (finite) limits, then the slice category, $\mathcal{C}/X$, has finite products, and hence so will $AugSimp(\mathcal{C})_X$. There is a functor from $AugSimp(\mathcal{C})$ to $\mathcal{C}$, which picks out the augmentation object, *i.e.*, it sends (X^+, X) to X. If $\mathcal{C}$ has pullbacks, this will be a fibred category.

Another alternative view of $AugSimp(\mathcal{C})_X$ is obtained by replacing X by the constant simplicial object, $K(X, 0)$, which has X in all dimensions and in which all face and degeneracy maps are the identity on X. We redraw the above diagram by replacing the $X = X_{-1}$ by $K(X, 0)$, and, of course, spreading it out,

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "d_0^{(3)}"] & X_1 \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d, "d_0^{(2)}"] & X_0 \arrow[d, "d_0"] \\ \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X \arrow[r, shift left=1] \arrow[r, shift right=1] & X \end{tikzcd}$$

giving a map, $\mathbf{d} : X^+ \to K(X, 0)$, where $\mathbf{d}$ in dimension i is, once again, $d_0^{(i+1)} : X_i \to X$. This process is reversible and, from such a map, we get an augmented object. We thus get:

Lemma 93 *For any object X in $\mathcal{C}$, we have an equivalence of categories*

$$AugSimp(\mathcal{C})_X \simeq Simp(\mathcal{C})/K(X, 0),$$

the category of simplicial objects over $K(X, 0)$. ■

Examples: Any simplicial set gives rise to two augmentations, which only coincide if the simplicial set is connected. Similar results hold in other categories provided that coequalisers and a terminal object exist. We illustrate this for *Sets*.

Given a simplicial set, we can form the set, $\pi_0(X)$, of connected components of X by forming the coequaliser,

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \arrow[r, dashed, "q_X"] & \pi_0(X). \end{tikzcd}$$

Definition: The *canonical augmentation of a simplicial set*, X, is given by $X_{-1} = \pi_0(X)$ with q_X being the natural quotient of the coequaliser.

For this construction, the quotient map, q_X, corresponds to the minimal equivalence relation on X_0, so that d_0z and d_1z are equivalent for all $z \in X_1$. Of course, the construction will work for simplicial objects in any category having coequalisers.

For the other augmentation, we make all elements of X_0 equivalent regardless of what happens in X_1, and so the quotient is a singleton set, which, of course, is the terminal object in the category of sets.

Definition: The *trivial augmentation of a simplicial set*, X, is given by $X_1 = \{*\}$ a singleton set with q_X being the natural unique map.

If $\mathcal{C}$ is any category having a terminal object, 1, the above can be adapted to give a trivial augmentation for any simplicial object in $\mathcal{C}$.

As was said above, the two types of augmentation coincide if X is connected.

One final way of analysing augmentation, as a process, is in the 'models'. We have our category, $\mathbf{\Delta}$, of finite non-empty ordinals, but there is also $\mathbf{\Delta}_\emptyset$ with one more object, the empty ordinal, together with the order preserving maps as before. There is, of course, a unique order preserving map from $\emptyset$ to $[n]$ for any n, as $\emptyset$ is initial in *Sets* and the empty function that results is, of course, order preserving. We think of $\emptyset$ as being $[-1]$ and have used that notation above.

Lemma 94 *There is a natural equivalence*

$$AugSimp(\mathcal{C}) \simeq \mathcal{C}^{\mathbf{\Delta}_\emptyset^{op}}.$$

Proof: Of course, the objects are just as we have defined them earlier, and the morphisms have that one extra component, f_{-1}, compatible with the augmentations, just as before. ∎

The inclusion of $\mathbf{\Delta}$ into $\mathbf{\Delta}_\emptyset$ induces the functor, $^+$, from $AugSimp(\mathcal{C})$ to $Simp(\mathcal{C})$ that forgets the augmentation. It is not surprising and **easy to verify** that, when it exists, the canonical augmentation is left adjoint to this forgetful functor, whilst the trivial augmentation functor is right adjoint, when it exists.

Remarks and asides on Kan extensions: This is a good example of behaviour typical for 'presheaf categories'. We have already seen it when looking at base change for sheaves in sections 7.3.8 and geometric morphisms in section 7.3.11. It will be useful to look briefly at the general categorical theory. We will need to use these ideas several times and in various forms later and have already met them several times. As usual in such situations, we will attempt to suggest the main ideas, to ease the process of attacking them in detail, but will not attempt to give a thorough treatment as there are excellent detailed treatments in category theory texts.

If $\mathcal{C}$ and $\mathcal{D}$ are small categories, and $K : \mathcal{C} \to \mathcal{D}$ is a functor, then, using the alternative notation $Sets^{\mathcal{C}}$ instead of $[\mathcal{C}, Sets]$ that we used in section 7.3.11, pre-composition gives an induced functor, $K^* : Sets^{\mathcal{D}} \to Sets^{\mathcal{C}}$, which has both right and left adjoints. These can be constructed by the method of *Kan extensions*.

Suppose $F : \mathcal{C} \to Sets$ is a functor, then we have a diagram, (the solid arrows),

$$\begin{tikzcd} \mathcal{C} \arrow[r, "K"] \arrow[d, "F"'] & \mathcal{D} \arrow[dl, dashed] \\ Sets & \end{tikzcd}$$

and search for natural or 'universal' ways of filling it to get a functor (dashed) from $\mathcal{D}$ to $Sets$. There is usually no functor that make the diagram commute, but we can hope for a functor G together with, perhaps, a natural transformation $F \Rightarrow GK$, or $GK \Rightarrow F$, and among these we could look for universal ones.

Definition: The *right Kan extension*, $Ran_K F$, of F along K is a functor, $Ran_K F : \mathcal{D} \to Sets$, together with a natural transformation,

$$\varepsilon : Ran_K FK \Rightarrow F,$$

inducing, for each $R : \mathcal{D} \to Sets$, a natural isomorphism

$$Nat(RK, F) \cong Nat(R, Ran_K F).$$

There is, of course, a dual notion of left Kan extension, which is **left for you** to consider, check up on, etc. If it is needed in any detail, we will assume that the reader has an acquaintance with this dual notion! (Some references will be given slightly later.)

To construct $Ran_K F$, we need to define it on objects of $\mathcal{D}$, and, if d is such, we can form the *comma category*[1], $d \downarrow K$, which consists of 'approximations on the right'[2] to d by objects in the image of K. More precisely, the objects are pairs, $(f : d \to Kc, c)$, so f is a morphism in $\mathcal{D}$, and the morphisms are between the $\mathcal{C}$-components compatibly with the 'approximating' morphisms:

$$\gamma : c \to c' \in \mathcal{C}$$

such that

$$\begin{tikzcd} & d \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ Kc \arrow[rr, "K\gamma"'] & & Kc' \end{tikzcd}$$

[1] This is not the most general form of comma category which relates to a cospan of functors; see page 698.

[2] Although we have already given a brief 'recall' of the related notion of 'over category' (see page 413 for a slightly simpler case), to make this section more 'stand alone' we will give an independent 'recall' for $d \downarrow K$, here.

commutes. (Note that the comma category records the object of $\mathcal{C}$, as well as the morphism f, so that, if K identifies two objects of $\mathcal{C}$, the same f may occur with different cs. This is very important.) There is a natural functor from $d \downarrow K$ to $\mathcal{C}$ given by the 'codomain', *i.e.*, it sends (f, c) to c. Finally we take the composite of this with F and then form the limit:

$$Ran_K F(d) = Lim((d \downarrow K) \to \mathcal{C} \xrightarrow{F} Sets).$$

If $d = Kc$ for some (unique-up-to-isomorphism) c in $\mathcal{C}$, then $d \downarrow K$ has an initial object namely (id_{Kc}, c). (If the c is not unique, **what happens**?) In this case, $Ran_K F(d) \cong Fc$, and this tells us what the natural transformation, ε, will be.

This construction is at the heart of various quite classical methods of (co-)homology theory, e.g. Čech *homology* is a Kan extension. The theory of total derived functors from homotopical algebra is another instance of this type of construction. We leave it aside for the moment, but note that $F \mapsto Ran_K F$ gives a functor and this is right adjoint to $K^* : Sets^{\mathcal{D}} \to Sets^{\mathcal{C}}$.

The method of Kan extensions in various forms is extremely useful and is one of those parts of category theory that is encountered time and time again in lots of different contexts. It does **need examining** in much more detail, but is well represented in category theory texts, so will not be investigated further here, as such. As mentioned above, we will however use it and so will assume some basic knowledge of it from now on, so if you have not met it in detail, ...

Right Kan extensions have the above limit description (as usual, up to isomorphism) and also a very useful description as an end

$$Ran_K F(d) = \int_c Sets(\mathcal{D}(d, Kc), Fc).$$

(Those of you who have met ends before will recognise this as being $Nat(\mathcal{D}(d, K-), F)$.) The general theory and these formulae also make sense, and are valid, if we replace $Sets$ as the codomain of the functors by some other category, $\mathcal{E}$, provided only that the relevant limits exist.

Dually one obtains left Kan extensions with a colimit formula and also a coend one. Of course, chasing that up is **left to you**. Useful material can be found in Mac Lane, [192], and other standard category theory texts, and there is a discussion of various slightly less standard aspects of the theory in Cordier - Porter, [89]. It is also useful to look back at the section that was mentioned above on change of base (section 7.3.8) and to see how that fits in this Kan extension setting. That does require a little bit of thought as there are several different things happening at once in that theory. Finally, but crucially, you are **left to check** the statement, near the start of this 'aside', that the Kan extensions give right and left adjoints to the 'precomposition functor'.

Returning to the inclusion of $\mathbf{\Delta}$ into $\mathbf{\Delta}_\emptyset$, we get formulae for the right and left adjoints of the induced (forgetful) functor from $\mathcal{AS}$ to $\mathcal{S}$, and more generally from $AugSimp(\mathcal{C})$ to $Simp(\mathcal{C})$. We do already know what these are, but, in fact, the secondary use for this interlude, and the above aside, is to get used to handling the Kan extension formulae, initially when the answer is known and then when it needs more work. If we look just at the right Kan extension, then the 'translation' from the general case above is $\mathcal{C} = \mathbf{\Delta}^{op}$ and $\mathcal{D} = \mathbf{\Delta}_\emptyset^{op}$. (The 'op' does make it necessary to go slightly more carefully and slowly!) The functor, K, is here inclusion and most of the objects of $\mathbf{\Delta}_\emptyset^{op}$ are in the subcategory, $\mathbf{\Delta}^{op}$, so the corresponding comma categories have initial objects and the limit used for working out $Ran_K F[n]$ will just evaluate F at the object, $[n]$. This is as expected as the adjoints leave the main body of the simplicial set untouched. That leaves us dealing with

$n = -1$ and the empty ordinal. An approximation to $\emptyset$ will have a morphism $f : \emptyset \to [n]$ in $\mathscr{D}$, now comes the bit that is easy to get wrong, yet which is really very simple (and that is why it is easy to get wrong!). The category $\mathscr{D} = \mathbf{\Delta}_\emptyset^{op}$, so our f is in an opposite category and corresponds to $f : [n] \to \emptyset$ in $\mathbf{\Delta}_\emptyset$. (We will not bother to use a different symbol for the morphism considered in the opposite category, although it can sometimes be a good idea to do so.) The only such morphism is the identity on $\emptyset$, *i.e.*, the case of $n = -1$, as the only function with codomain the empty set is the empty function, but that is not possible here as its domain is not in the image of K. Thus our $d \downarrow K$ is the empty category, but the limit of a functor to $Sets$, indexed by an empty category, is a singleton set. We thus have that the right adjoint of the forgetful functor from $\mathscr{AS}$ to $\mathscr{S}$ is, as we knew, the trivial augmentation functor. (No prizes for guessing what the other Kan extension is ..., but it is a good exercise **for the reader** to check through, as it uses the left Kan extension formula and that has already been left for the reader.)

13.3.2 Splittings

The simple décalage functor naturally gives an augmented simplicial set, since, if Y is a simplicial set, $Dec\,Y$ comes equipped with a map $q : (Dec\,Y)_0 \to Y_0$, namely the d_0^1 of Y. The two maps, d_0^1, d_1^1 of $Dec\,Y$ are the old d_0^2, d_1^2 of Y, so $qd_0^1 = qd_1^1$ by the simplicial identities. As we mentioned in our earlier discussion of décalage, (section 8.3.5), the 'left over' s_0 splits q and gives a homotopy equivalence between $Dec\,Y$ and $K(Y_0, 0)$. (In fact, the image of Y_0 in each Y_n, given by composite degeneracies, gives a constant simplicial subset of $Dec\,Y$ isomorphic to $K(Y_0, 0)$ and which is a strong deformation retract of $Dec\,Y$, the homotopy being built from all the s_{last} maps. We will look at this shortly in some detail.) In other words, $Dec\,Y$ is quite special as an augmented simplicial set. It is 'split'. We can find this structure also at the level of the ordinals. First we should establish, and explain, some useful terminology.

The usual meaning in Topology of 'contractible' for a space, X, is that X is homotopy equivalent to a point. This is fine as far as it goes, but it fails to handle the case of a space which is the disjoint union of contractible components. There does not really seem to be an accepted neat term used for this, yet, for instance, the underlying simplicial set of a simplicial resolution of a group, G, is an example of exactly the simplicial analogue of that situation. The point with contractibility, in the standard sense, may be that, in the category of topological spaces, the terminal object is the singleton space and a space X is contractible if the unique map to that terminal object is a homotopy equivalence. In the case of the simplicial resolution, we are working 'over G' and the resolution is a simplicial object *in the category of groups over* G. This looks (and is) a situation that is more easily handled in the augmented situation. The category, $AugSimp(\mathscr{C})_X$, of simplicial objects augmented over some fixed object, X, has a terminal object, namely the augmented object, $(K(X, 0), X)$, so it would seem reasonable to say that a general, (X^+, X), should be considered to be contractible if the corresponding unique map to the terminal object is a homotopy equivalence. If we are in the category of sets and the augmentation is the canonical one, then this says X^+ is the disjoint union of contractible connected components more or less in the classical sense.

Let us formalise this (following the treatment in Duskin's Memoir, [107]).

Definition: If $\mathbf{X} = (X^+, X)$ is an augmented simplicial object, then $\mathbf{X}$ is said to be *contractible* *above, or over,* X, if the associated map

$$\mathbf{d} : X_\cdot^+ \to K(X,0)$$

is a homotopy equivalence, *i.e.*, there is a simplicial map, $\mathbf{s} : K(X,0) \to X_\cdot^+$, such that $\mathbf{ds}$ and $\mathbf{sd}$ are homotopic to the respective identities.

To see what this means, we will first see what $\mathbf{s} : K(X,0) \to X^+$ must look like. (First note that we can keep life simple, since $K(X,0)$ has identity maps as both its face and its degeneracy maps. We shall resist the temptation to introduce a notation other than id_X for all these maps! We will discuss this in some detail a bit later on.) In dimension 0, we have $\mathbf{s}_0 : X \to X_0$ (or from X_{-1} to X_0, if you prefer.) This determines $\mathbf{s}_n : X \to X_n$ in all dimensions, since, for example, $\mathbf{s}_1 : X \to X_1$ has to satisfy $\mathbf{s}_1 = s_0\mathbf{s}_0$ (leaving out the s_0 of $K(X,0)$ on the left as it is the identity), and also $d_i\mathbf{s}_1 = \mathbf{s}_0 d_i$ for $i = 0,1$. The first of these thus specifies $\mathbf{s}_1$ and does this so that the second family of equations is satisfied automatically. In general, $\mathbf{s}_0 : X \to X_n$ will be $s_0^{(n)}\mathbf{s}_0$, the composite of an n-fold composite of s_0 of X with the map, $\mathbf{s}_0$, or, put more sloppily, $s_0^{(n+1)}$. (Note and **check** that it does not matter what composite of degeneracies you take from X_0 to X_n as they are all equal.)

The notation $\mathbf{s}_n$ with a bold s is perhaps easy to confuse with s_n, an n^{th} degeneracy map, so often we will expand it out as $s_0^{(n)}\mathbf{s}_0$ or $s_0^{(n+1)}$. Similarly we noted that $\mathbf{d}_n$ was $d_0^{(n+1)}$ and we should add that this is also $\mathbf{d}_0 d_0^{(n)}$. The various notations, of course, correspond to the different equivalent views that we saw just now. None is without its drawbacks, so we will swap between them as seems best in any discussion and hope the result is reasonably clear.

There is a problem that arises here. To make precise what a contraction is we have to specify the homotopies involved between $\mathbf{sd}$ and the identity on X^+, and similarly between $\mathbf{ds}$ and the identity on $K(X,0)$. However, in saying 'between', we are making no assumption as to whether there is a *direct* homotopy linking them nor in what direction such a direct homotopy might go, if it exists. If the simplicial objects concerned satisfy some sort of Kan filler condition, then homotopy would be transitive and symmetric as a relation, and then the direction and the directness, or not, would not really matter. We will not assume such conditions however and so will refer to 'right' (or 'left') homotopies and contractions. This occasionally makes life a bit more complicated, but the addition complication is quite 'handleable'. Usually we will work with the following. There are variants, for instance, in which $h : \mathbf{sd} \simeq id$, that are **left to you** to think about.

To specify a (right) contraction, we also have the homotopies, $h : id \simeq \mathbf{sd}$ and $k : id \simeq \mathbf{ds}$. We will use both of these in the combinatorial form (see page 340). For convenience, we repeat (and adapt) that description here, for instance:

the homotopy, $h : id \simeq \mathbf{sd}$, is given by a family, $\{h_j^n : X_n \to X_{n+1} \mid 0 \le j \le n\}_{n\ge 0}$, satisfying

$$d_0 h_0^n = s_0^{(n+1)} d_0^{(n+1)}, \qquad\qquad d_{n+1} h_n^n = id_{X_n},$$

$$\begin{array}{rcll} d_i h_j &=& h_{j-1} d_i & \text{for } i < j, \\ d_{j+1} h_{j+1} &=& d_{j+1} h_j, & \\ d_i h_j &=& h_j d_{i-1} & \text{for } i > j+1, \end{array}$$

and the corresponding degeneracy rules are

$$s_i h_j = h_{j+1} s_i, \qquad i \le j,$$

$$s_i h_j = h_j s_{i-1}, \qquad i > j;$$

similarly for k, with $\{k_j^n : X \to X \mid 0 \leq j \leq n\}_{n \geq 0}$, with **sd** replaced by **ds**, so $d_0 k_0^n = d_0^{(n+1)} s_0^{(n+1)}$, and id_{X_n} by id_X, so $d_{n+1} k_n^n = id_X$. In fact, we will first look at k and will use what seems an almost silly argument, ..., but one that works!

As all the face maps in $K(X, 0)$ are the identity on X, $k_n^n = id_X$ from the last equation, whilst $k_0^n = d_0^{(n+1)} s_0^{(n+1)}$. The face relations for k^n, $d_{j+1} k_{j+1} = d_{j+1} k_j$, show that all the k_j^n are the same, so $d_0^{(n+1)} s_0^{(n+1)} = id_X$.

Remarks on Constant Simplicial Objects: In fact, the above can be read off for $n = 0$ from the conditions for h_0^0 and then it follows for general n from the simplicial identities within X^+. This is worth **checking** as it may be clearer than the quicker direct argument in all dimensions at once, since, although true, $k_0^n = d_0^{(n+1)} s_0^{(n+1)}$ does look slightly wrong. It is true, because by saying that $K(X, 0)$ has X in each dimension and the identities on X for its faces and degeneracies, we gain in simplicity most of the time, but we also have to pay, in as much as any given point, $x \in K(X, 0)_n$ has dimension n, but really we just have $x \in X$ and there is no record of its 'dimension' there. There are various simple notational ways around this, at least when working with $\mathcal{C} = Sets$. For instance, if we define $K(X, 0)$ to have $X \times \{n\}$ in dimension n with $d_i(x, n) = (x, n-1)$ and so on. Such a use would perhaps clarify certain arguments which may seem 'sleight-of-hand', but the notational price elsewhere would be quite heavy and it would not be obvious how to adapt that 'trick' in more general settings, that is with other $\mathcal{C}$, as it would seem to need the natural numbers as labels.

We will leave this aside, but note that the conversion of a simplicial complex to a simplicial set uses a total order on the vertices, so that then the degeneracies are given by repeating an element. If we apply this to a set, X, thought of as a zero dimensional simplicial complex so, then the resulting simplicial set is isomorphic to $K(X, 0)$, *but* has an *isomorphic copy* of X in each dimension. The geometric, philosophical and foundational questions that this raises are left to the reader to worry about if they so wish. Essentially it does not really matter most of the time, and is merely an expositional nicety, but like 'isomorphism' versus 'equivalence' of categories, it does need thinking about.

From that look at k, we deduce that the map, $\mathbf{d} : X^+ \to K(X, 0)$, is a split epimorphism and, most usefully, that $d_0 s_0 = id_X$.

We now can state a result that is central to the monadic approach to cohomology; cf. Duskin [107]. First a definition:

Definition: If $\mathbf{X} = (X^+, X)$ is an augmented simplicial object in $\mathcal{C}$, then a (right) *splitting* of $\mathbf{X}$ is a family $\mathbf{s} = \{s_{n+1} : X_n \to X_{n+1}\}_{n \geq -1}$ of morphisms satisfying the 'extra degeneracy relations':

$$\begin{aligned} d_i s_n &= s_{n-1} d_i && \text{for } 0 \leq i < n \\ d_n s_n &= id && \\ s_i s_n &= s_{n+1} s_i && \text{for } 0 \leq i < n \\ s_{n+1} s_n &= s_n s_n. && \end{aligned}$$

Example: The augmented décalage, $(Dec\,Y, Y_0)$, of a simplicial object, Y, is an augmented simplicial object that has a splitting, namely the left-over degeneracy

$$\{s_{last} : (Dec\,Y)_n \to (Dec\,Y)_{n+1}\}_{n\geq -1}.$$

Proposition 152 *An augmented simplicial object, (X^+, X), is contractible over X if, and only if, it has a splitting.*

Proof: Suppose that $\mathbf{d} : X^+_{\cdot} \to K(X,0)$, $\mathbf{s} : K(X,0) \to X^+_{\cdot}$, $h : id \simeq \mathbf{sd}$, $k : id \simeq \mathbf{ds}$ specifies a contraction over X, *i.e.*, a set of homotopy equivalence data showing that $\mathbf{d}$ is a homotopy equivalence.

We again use the combinatorial description of h as a collection of families of maps, $h^n_j : X_n \to X_{n+1}$, satisfying the conditions we recalled above. We set $s_0 = \mathbf{s}_0 : X \to X_0$ and take $s_{n+1} := h^n_n : X_n \to X_{n+1}$ for $n \geq 0$. We claim these s_{n+1} give a splitting.

This is mostly a trivial routine check and so is, of course, **left to the reader** for the detail, however to aid in that checking, let us look at the low dimensional situation. We always have

$$\begin{aligned} d_0 h^n_0 &= s_0^{(n)} \mathbf{s}_0 \mathbf{d}_0 d_0^{(n)}, \\ d_{n+1} h^n_n &= id, \end{aligned}$$

but, when $n = -1$, $\mathbf{d}_0 \mathbf{s}_0 = id$ by our earlier calculations and, for $n = 0$,

$$\begin{aligned} d_0 s_1 &= d_0 h^0_0 = \mathbf{s}_0 \mathbf{d}_0, \\ d_1 s_1 &= d_1 h^0_0 = id, \end{aligned}$$

since $h : id \simeq \mathbf{sd}$, whilst the degeneracy rules follow from those for h.

For $n = 1$, $s_2 := h^1_1 : X_1 \to X_2$ and

$$\begin{aligned} d_0 s_2 &= d_0 h^1_1 = h^0_0 d_0 = s_1 d_0, \\ d_1 s_2 &= d_1 h^1_1 = d_1 h^1_0 = h^0_0 d_1 = s_1 d_1, \\ d_2 s_2 &= id, \end{aligned}$$

and so on. In each case, the splitting rules follow from the rules for h being a homotopy. Thus any (right) contractible augmented simplicial object has a (right) splitting. (We have shown this for right contractions, the other forms go across to give analogous results.)

Conversely, suppose $\{s_{n+1} : X_n \to X_{n+1}\}_{n\geq -1}$ is a splitting and we want to construct suitable $\mathbf{s}$, h and k. The first part is more or less clear. We take $\mathbf{s}_0 := s_0 : X_{-1} \to X_0$, then there is no choice: $\mathbf{s}_n = s_0^{(n)} \mathbf{s}_0$. We have $\mathbf{d}_0 \mathbf{s}_0 = id$, so $\mathbf{d}_n \mathbf{s}_n = id$ and we can choose k to be the identity/trivial homotopy. That leaves us with h. We clearly should set $h^n_n := s_{n+1} : X_n \to X_{n+1}$, so need to define the other h^n_j for $0 \leq j < n$. Looking at low dimensions shows how to proceed (and, of course, you will be **left to check the details for the general case**).

For $n = 0$, we have $h^0_0 = s_1$ and that is all we need.

For $n = 1$, we have $h^1_1 = s_2$ and need h^1_0. There is no unique single choice in general, so we just try to find one solution. The specifications are $d_0 h^1_0 = \mathbf{s}_1 \mathbf{d}_1 = s_0 \mathbf{s}_0 \mathbf{d}_0 d_0$, $d_1 h^1_0 = d_1 h^1_1$ and

$d_2h_0^1 = h^0d_1 = s_1d_1$. (Note the problem is *not* one of filling a horn as we have *all* the faces. We have to spot the possible solutions.) We know h_1^1, so can find $d_1h_0^1 = d_1h_1^1 = d_1s_2 = s_1d_1$, so h_0^1 has two faces equal, the first and the second. This suggests a probable solution, namely we can try $h_0^1 = s_1d_1s_2 = s_1^{(2)}d_1$ as this will match those two faces. (Here remember the intuitions concerning degeneracies.) We check $d_0s_1d_1s_2 = ... = s_0^{(2)}d_0^{(2)}$, where the ... stands for a series of basic simplicial manipulations. We hence have h_1^1. We will do one more level.

For $n = 2$, $h_2^2 := s_3 : X_2 \to X_3$ and we need h_1^2 and h_0^2. We use $d_2h_2^2 = d_2h_1^2$ to get information on h_1^2 and try $h_1^2 := s_2d_2h_2^2$. This checks with other information from the splitting. We retrieve $d_1h_0^2 = d_1h_1^2$ and try $h_0^2 = s_1d_1h_1^2$. This works.

The pattern is set and can be safely **left to you to complete and check.** ■

What we have called a splitting could equally well be called a *contraction*. The choice of 'splitting' is due to the fact that it allows some nice terminology:

Definition: A *split augmented simplicial object* is a pair consisting of an augmented simplicial object and a splitting of it. Morphisms of such objects are assumed to respect the given splittings.

The category of split augmented simplicial objects in $\mathcal{C}$ will be denoted $SplitAugSimp(\mathcal{C})$.

We note that with a contractible augmented simplicial object, the existence of a splitting is required, with a split one it is part of the data. A split object is analogous, in this respect, to a homotopy coherent diagram, rather than merely a homotopy commutative diagram, which would more nearly correspond to a contractible object.

These split simplicial objects have a simple description in terms of the models. Let $\mathbf{\Delta}_\ell$, (ℓ for 'last'), be the subcategory of $\mathbf{\Delta}$ consisting of *all* the objects, but in which $\mu : [m] \to [n]$ is in $\mathbf{\Delta}_\ell$ if and only if $\mu(m) = n$, that is, the morphisms preserve the maximum of each ordinal. We write $inc : \mathbf{\Delta}_\ell \to \mathbf{\Delta}$ for the inclusion. It induces, by precomposition, a functor from $\mathcal{S}$ to $Sets^{\mathbf{\Delta}_\ell^{op}}$.

What do the objects of $Sets^{\mathbf{\Delta}_\ell^{op}}$ look like? First it pays to see what $\mathbf{\Delta}_\ell$ itself looks like. The basic morphisms of $\mathbf{\Delta}$ are, as we know, generated by the coface maps, $\delta_i^n : [n-1] \to [n]$, and codegeneracies $\sigma_i^n : [n] \to [n-1]$. The first of these are all injections missing the element i and the second are surjections covering i twice. The only ones of these generators not in $\mathbf{\Delta}_\ell$ are thus the δ_n^n as they send the maximal element $n-1$ to itself in $[n]$, $\delta_n^n(n-1) = n-1$, but, of course, that is no longer maximal. We thus have

Lemma 95 *The wide subcategory $\mathbf{\Delta}_\ell$ of $\mathbf{\Delta}$ is that generated by all δ_i^n with $0 \leq i < n$ and all σ_i^n, $0 \leq i \leq n$, and for all n.* ■

We can view $\mathbf{\Delta}_\ell$ diagrammatically as

$$\begin{tikzcd}
{[0]} \arrow[r, "\delta_0"] & {[1]} \arrow[l, bend left=50, "\sigma_0"] \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] & {[2]} \arrow[l, bend left=40, "\sigma_0"] \arrow[l, bend left=70, "\sigma_1"] \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & \cdots
\end{tikzcd}$$

and that looks remarkably like the dual of the diagram for a split augmented simplicial object! . . . , but with a shift in dimension. Suppose, therefore, that we take $A \in \mathcal{C}^{\mathbf{\Delta}_\ell^{op}}$ and define an augmented simplicial object, $c(A)$, in $\mathcal{C}$, by

$$\begin{array}{rcll} c(A)_n & = & A_{n+1} & \text{for } n \geq -1, \\ d_i^n = c(A)(\delta_i^n) & = & A(\delta_i^{n+1}) & \text{for } 0 \leq i \leq n, \\ s_i^n = c(A)(\sigma_i^n) & = & A(\sigma_i^{n+1}) & \text{for } 0 \leq i \leq n, \end{array}$$

then defining $s_n = A(\sigma_{n+1}^{n+1})$ gives a splitting.

Proposition 153 *The construction, c, above, gives an isomorphism of categories,*

$$\mathcal{C}^{\mathbf{\Delta}_\ell^{op}} \cong SplitAugSimp(\mathcal{C}).$$

■

The proof is **left to you** as it is just 'dotting is and crossing ts' with respect to the construction, c, e.g. checking functoriality, etc.

13.3.3 Back towards the décalage

This functor is clearly closely linked to *Dec*. Our functor, $c : \mathcal{C}^{\mathbf{\Delta}_\ell^{op}} \to SplitAugSimp(\mathcal{C})$, however whilst, from our definition of *Dec*, that goes from $Simp(\mathcal{C})$ to itself. You probably have realised that this is not quite fair! The décalage of a simplicial object *is* just a simplicial object, as defined, but it comes together with a natural augmentation, so is '*really*' an augmented simplicial object, but, hold on, that augmented simplicial object has a natural splitting, so *Dec really* is in $SplitAugSimp(\mathcal{C})$ after all. Which of these views is 'really' right? The answer has to be at the same time 'all of them' and 'none of them'. In other words, there are, at least, three versions of décalage and it depends what you want *Dec* to do which is best for your context. We will be sloppy and where possible just write *Dec* for any of them, but can add a subscript as in Dec_{AS} or Dec_{SAS} for the augmented and split augmented variants if needed.

It is quite fun to see how all of this is reflected in the 'models'. It also shows some extra bits of insight that may help later.

We have inclusions $\mathbf{\Delta}_\ell \hookrightarrow \mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_\emptyset$, and hence precomposition induced functors

$$\begin{tikzcd} \mathcal{C}^{\mathbf{\Delta}_\emptyset^{op}} \arrow[r] \arrow[d, dash, "="'] & \mathcal{C}^{\mathbf{\Delta}^{op}} \arrow[r] \arrow[d, dash, "="'] & \mathcal{C}^{\mathbf{\Delta}_\ell^{op}} \arrow[d, "c"', "\cong"] \\ AugSimp(\mathcal{C}) & Simp(\mathcal{C}) & SplitAugSimp(\mathcal{C}) \end{tikzcd}$$

The first of these forgets the augmentation, the second forgets the last face. The functor, c, then shifts dimension, so the induced functor from $Simp(\mathcal{C})$ to $SplitAugSimp(\mathcal{C})$ is effectively the richest form, Dec_{SAS}, of décalage.

Lemma 96 *The functor $inc : \mathbf{\Delta}_\ell \hookrightarrow \mathbf{\Delta}_\emptyset$ has a left adjoint.*

Proof: Define $b : \mathbf{\Delta}_\emptyset \to \mathbf{\Delta}_\ell$ by

$$b[n] = [n+1],$$

and, for $\mu : [m] \to [n]$,

$$b[\mu](i) = \begin{cases} i & \text{if } i < m+1 \\ n+1 & \text{if } i = m+1. \end{cases}$$

We claim $b \dashv inc$, *i.e.*, that b is the desired left adjoint. It is now fairly easy to show that there is a bijection

$$\theta : \mathbf{\Delta}_\ell(b[n],[m]) \to \mathbf{\Delta}_\emptyset([n], inc[m]),$$

which send $\mu : b[n] = [n+1] \to [m]$ to its restriction to $[n]$ as a subset of $b[n]$, so $\theta(\mu)(i) = \mu(i)$, for $0 \leq i \leq n$. We thus have $b \dashv inc$, as claimed. ■

This completes the proof, but, as is usually the case with adjunctions, it is useful to make the unit and counit explicit whilst we are about it. We have

$$\theta : \mathbf{\Delta}_\ell(b[n], b[n]) \to \mathbf{\Delta}_\emptyset([n], inc.b[n]),$$

and the unit $\eta : id \to inc.b$ is the image of the identity, more precisely, $\eta[n] = \theta(id)$. We thus have that $\eta_{[n]}$ is precisely the last coface operator, δ^{n+1}_{n+1}. For the counit, we use the isomorphism

$$\theta : \mathbf{\Delta}_\ell(b.inc[m],[m]) \to \mathbf{\Delta}_\emptyset(inc[m], inc[m]),$$

and, of course, the counit, ε is the reverse image of $id_{inc[m]}$, *i.e.*, $\theta(\varepsilon_{[m]}) = id_{[m]}$, but then we obtain that

$$\varepsilon_{[m]}(i) = \begin{cases} i & \text{if } i \leq m \\ m & \text{if } i = m+1 \end{cases}$$

and that is precisely the codegeneracy, σ^{m+1}_{m+1}, (but, remember, m can be -1 in both cases).

We should remark that we can restrict b to $\mathbf{\Delta}$ to get a right adjoint to $inc : \mathbf{\Delta}_\ell \hookrightarrow \mathbf{\Delta}$ as well.

We now need to see the impact of the existence of b on the various categories of simplicial objects. We have $inc^* : SplitAugSimp(\mathcal{C}) \to AugSimp(\mathcal{C})$, modulo identifying $SplitAugSimp(\mathcal{C})$ with $\mathcal{C}^{\mathbf{\Delta}_\ell^{op}}$, and b^* going in the opposite direction. As $b \dashv inc$, we might expect that b^* would be adjoint on one side or the other to inc^*. We will look at b^* and see if it has a right adjoint, ..., since we have a Kan extension formula that we can try out. (Of course, there is a more profound reason, which will become apparent when we have looked at the formula in detail.) We have, say, $A \in AugSimp(\mathcal{C})$, or more usefully, $A : \mathbf{\Delta}_\emptyset^{op} \to \mathcal{C}$, so we have a diagram:

$$\begin{tikzcd} \mathbf{\Delta}_\emptyset^{op} \arrow[r, "b^{op}"] \arrow[d, "A"'] & \mathbf{\Delta}_\ell^{op} \arrow[dl, dashed] \\ \mathcal{C} & \end{tikzcd}$$

We have that, if $Ran_{b^{op}}A$ exists, then it is given by the limit formula

$$(Ran_{b^{op}}A)[n] = Lim(([n] \downarrow b^{op}) \to \mathbf{\Delta}_\emptyset^{op} \xrightarrow{A} \mathcal{C}.$$

Once again, we have to be careful, since the functor b^{op} is between opposite categories. We examine the comma category $([n] \downarrow b^{op})$. Its objects are pairs, $(\mu : [n] \to b[m], [m])$, but in which μ is a morphism in $\mathbf{\Delta}_\ell^{op}$, so is, in 'reality', $\mu : b[m] \to [n]$ in $\mathbf{\Delta}_\ell$. As $b \dashv inc$, $\mu : b[m] \to [n]$ corresponds to a unique $\tilde{\mu} : [n] \to inc[n]$. Using this we can set up an isomorphism between $([n] \downarrow b^{op})$ and $(inc[n] \downarrow \mathbf{\Delta}_\ell^{op})$. This latter category has an initial object, namely $(inc[n] \to [inc[n], inc[n])$ and hence $([n] \downarrow b^{op})$ also has an initial object. This will be $(\varepsilon_{[n]} : b.inc[n] \to [n], inc[n])$. (Careful beware of getting confused by the arrow's direction; here we have given the corresponding arrow in $\mathbf{\Delta}_\ell$, not in $\mathbf{\Delta}_\ell^{op}$.) We thus have:

Lemma 97 *For any $A \in AugSimp(\mathcal{C})$, $Ran_{b^{op}}A$ exists and is given by*

$$(Ran_{b^{op}}A)_n \cong A.inc[n],$$

i.e.

$$Ran_{b^{op}}A \cong inc^*A,$$

so $b^* \dashv inc^*$. ■

Remark: Here there are no conditions whatsoever on $\mathcal{C}$. If $\mathcal{C}$ is complete and cocomplete, (so all limits and colimits exist), then inc^* will also have a right adjoint and b^* will have a left adjoint, given in each case by Kan extensions. (If $\mathcal{C}$ is just complete or cocomplete then one of these will exist, but perhaps not the other.)

As we have that $\mathcal{C}^{\mathbf{\Delta}_\ell^{fop}}$ and $SplitAugSimpl(\mathcal{C})$ are isomorphic, we can look at the corresponding adjoint functors:

$$c.inc^* : AugSimp(\mathcal{C}) \to SplitAugSimp(\mathcal{C})$$

and

$$b^*.c^{-1} : SplitAugSimp(\mathcal{C}) \to AugSimp(\mathcal{C}).$$

First suppose $A \in AugSimp(\mathcal{C})$, then $inc^*(A)_n = A_n$ and the face and degeneracy maps are the same as those in A, except that $inc^*(A)$ does not have the last face maps, d_n^n, of A in its structure. Taking c of this, we get $c.inc^*(A) \in SplitAugSimp(\mathcal{C})$ with $c.inc^*(A)_n = A_{n+1}$, with all the same face and degeneracies, but the 'last-degeneracy' family being considered as a splitting, ..., yes, we have:

Proposition 154 *There is an isomorphism*

$$Dec_{SAS}(A^+) \cong c.inc^*(A).$$

Proof: The only point left to examine is the $^+$. In going from A to $inc^*(A)$, we throw away A_{-1} and the augmentation, and then the last faces, think of restricting from $\mathcal{C}^{\mathbf{\Delta}_\emptyset^{op}}$ first to $\mathcal{C}^{\mathbf{\Delta}^{op}}$ and then to $\mathcal{C}^{\mathbf{\Delta}_\ell^{op}}$. ■

What about $b^*.c^{-1}$? Suppose that $X \in SplitAugSimp(\mathcal{C})$, so, in addition to a simplicial object, X^+, we have an augmentation $d_0 : X_0 \to X_{-1}$ and a splitting $\{s_{n+1} : X_n \to X_{n+1}\}_{n\geq -1}$. We first write down $c^{-1}X \in \mathcal{C}^{\mathbf{\Delta}_\ell^{op}}$. This has

$$c^{-1}(X)_n = X_{n-1}$$

with the indices of both face and degeneracies adjusted. Now, for any $T \in \mathcal{C}^{\mathbf{\Delta}_\ell^{op}}$, $(b^*Y)_n = Yb[n] = Y_{n+1}$, with adjustments on the face and degeneracies - **left to you to make precise**. This gives $b^*c^{-1}(X)_n = X_n$ with the overall effect of reinstating the indices as they originally were, *but* forgetting the splitting. We have:

Proposition 155 *The functor, $Dec_{SAS}(-^+)$, is right adjoint to the functor from $SplitAugSimp(\mathcal{C})$ to $AugSimp(\mathcal{C})$ that* forgets *the splitting.* ■

We thus have for X in $SplitAugSimp(\mathcal{C})$, and A in $AugSimp(\mathcal{C})$,

$$AugSimp(\mathcal{C})(forget(X), A) \cong SplitAugSimp(\mathcal{C})(X, Dec(A^+)).$$

In particular, if $\mathcal{C}$ has a terminal object, then the functor, $(-)^+$, that forgets augmentation, has itself a right adjoint, the trivial augmentation functor, and composing we will get that $Dec_{SAS} : Simp(\mathcal{C}) \to SplitAugSimp(\mathcal{C})$ has a left adjoint $(-)^+$ that forgets both the splitting and the augmentation. In fact, we will examine this result more closely, firstly because it is true without the requirement that $\mathcal{C}$ have a terminal object and secondly it is quite easy to give the unit and counit explicitly, which is useful for some considerations later on. We first set the situation up in a slightly simplified notation. (We adapt the treatment given in Duskin's memoir, [107], for this.)

We have $Dec_{SAS} : Simp(\mathcal{C}) \to SplitAugSimp(\mathcal{C})$ and $(-)^+ : SplitAugSimp(\mathcal{C}) \to Simp(\mathcal{C})$, which takes $(X^+, X, \mathbf{s})$ to X^+, *i.e.*, forgets both the augmentation and the splitting.

Proposition 156 *The functor Dec_{SAS} is right adjoint to $(-)^+$.*

Proof: (We will leave details to you, but will sketch the structure being used.)

In forming the simplest form of Dec, we forgot the last face and recall, from our use of this in section 8.3.5, that the last face map gives a simplicial map

$$d_{last} : Dec(X) \to X.$$

This is what we take as the counit of the adjunction, since $Dec(X)$ is also $(Dec_{SAS}(X))^+$. We thus have

$$\varepsilon_X : (Dec_{SAS}(X))^+ \to X$$

is given by

$$(\varepsilon_X)_n = d^{n+1}_{n+1} : X_{n+1} \to X_n,$$

for all $n \geq 0$. (Remember that if X was a simplicial group, this map is the key fibration used in obtaining cat^n-group models for $(n+1)$-types; cf. the discussion starting on page 384.) Clearly ε_X is a natural transformation.

The unit $\eta_A : A \to Dec_{SAS}(A^+)$ will, in dimension n, be a map from A_n to A_{n+1}. Here $A = (A^+, A_{-1}, \mathbf{S})$ and a good guess for η_A would be the splitting $\mathbf{s} = \{s_{n+1} : A_n \to A_{n+1}\}_n$. This works. (**You should check the adjunction rules** and see to what they correspond.)

So as to remind the reader of how the unit-counit information yields the usual natural isomorphism of 'hom-sets', let us go through this in this example. The desired isomorphism will be between $Simp(\mathcal{C})(A^+, X)$ and $SplitAugSimp(\mathcal{C})(A, Dec_{SAS}(X))$, so consider $f : A^+ \to X$, apply Dec_{SAS} to get $Dec_{SAS}(f) : Dec_{SAS}(A^+) \to Dec_{SAS}(X)$, then precompose with η_A. (It helps to draw some diagrams to see what this process does.) Going the other way, if $g : A \to Dec_{SAS}(X)$, strip off the augmentation using $(-)^+$ to get

$$g^+ : A^+ \to Dec_{SAS}(X)^+,$$

(and recall $Dec_{SAS}(X)^+$ is the 'original' Dec), now post-compose with ε_X. (Again drawing some diagrams to see what is happening **is a good idea**.) The resulting morphism is thus

$$A_n \stackrel{g_n}{\to} X_{n+1} \stackrel{d^{n+1}_{n+1}}{\to} X_n.$$

Checking that these two processes are inverses to each other is now easy. (It is quite interesting to see how the simplicial identities come in.) ∎

13.3.4 Monads and Comonads, Algebras and Coalgebras

There is more that can be said about the adjunction that we have just seen, but for that we need to explore monads, comonads and related ideas from basic category theory. (Again we will skim over the surface of this as the theory is well known and is well represented in most category theory texts. It provides insights on algebra and on cohomology theory in general.)

Definition: Let $\mathcal{A}$ be a category. A *monad* $T = (T, \eta, \mu)$ in (or 'on') $\mathcal{A}$, consists of a functor $T : \mathcal{A} \to \mathcal{A}$ and two natural transformations,

$$\eta : ID_{\mathcal{A}} \to T \quad \text{and} \quad \mu : T^2 \to T,$$

called the *unit* and *multiplication* respectively, such that the diagrams

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, dash, "="'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, dash, "="] \\ & T & \end{tikzcd}$$

and

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

commute.

The first of these diagrams is to be thought of as the analogue of the equations for the unit in a *monoid* that say $1.m = m = m.1$. The second is saying 'μ is associative'. Because of this, they are sometimes known as the (monadic) unit and associativity axioms.

Before we give examples, we formally note the dual notion, since that is what we used earlier (page 90).

Definition: A *comonad* (L, ε, δ) in $\mathcal{A}$ consists of a functor $L : \mathcal{A} \to \mathcal{A}$, and two natural transformations

$$\varepsilon : L \to Id_{\mathcal{A}} \quad \text{and} \quad \delta : L \to L^2,$$

the *counit* and *comultiplication*, such that

$$\begin{tikzcd} & L \arrow[dl, dash, "="'] \arrow[d, "\delta"] \arrow[dr, dash, "="] & \\ L & L^2 \arrow[l, "\varepsilon L"] \arrow[r, "L\varepsilon"'] & L \end{tikzcd}$$

and

$$\begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "\delta"'] & L^2 \arrow[d, "L\delta"] \\ L^2 \arrow[r, "\delta L"'] & L^3 \end{tikzcd}$$

commute. These are the counit and coassociativity axioms.

(A notational point is that in the first diagram here $\varepsilon : L \to Id$, so $L\varepsilon : L^2 \to L.Id = L$ and similarly, $\varepsilon L : L^2 \to Id.L = L$. The dual remark could have been made for the first diagram in the definition of monad.)

The first example of a monad, that we will give, takes $\mathcal{A} = Sets$, takes a monoid, M in the ordinary sense of the word, and defines $T(X) = M \times X$, the product of the underlying set of M with the input set, X, in this case. As there is a unit element $1 \in M$, we can define

$$\eta_X : X \to TX$$

by $\eta_X(x) = (1, x)$. As there is a multiplication, we can take

$$\mu_X : T^2X \to TX,$$

that is, from $M \times M \times X$ to $M \times X$, to be $\mu_X(m_2, m_1, x) = (m_2.m_1, x)$, *i.e.*, the map induced by the multiplication in M. The unit rule for the monad is implied by the unit axiom for the monoid, and similarly, the associativity of the monad follows from the associativity of the multiplication in M.

The second example is a generic one that we will be needing for our discussion on augmentations, etc. Suppose $F : \mathcal{A} \to \mathcal{B}$, and $G : \mathcal{B} \to \mathcal{A}$ are adjoint functors, $F \dashv G$, with $\eta : Id \to GF$, $\varepsilon : FG \to Id$, the unit and counit respectively. We set $T = GF : \mathcal{A} \to \mathcal{A}$, keep η as is, and use ε to build $G\varepsilon F :: G(FG)F \to GF$, which is the multiplication $\mu : T^2 \to T$. (**You are left to check that this works** or to look it up. It is in Mac Lane, [192], for instance.) To illustrate what is happening here, take the adjunction with $F : Sets \to Groups$ being the free group functor, and $G = U$, the underlying set function. For a set, X, $\eta_X : X \to UF(X)$ is the insertion of the generators. To understand μ, first note that, for a group, G, $\varepsilon_G : FU(G) \to G$ is the natural epimorphism from the free group on the *set*, $U(G)$, back down onto G by remembering the composition in G. (If you like thinking in terms of presentations, $FU(G)$ has presentation $(U(G) : \emptyset)$, whilst G has one as $(U(G) : G \times G)$, as we saw earlier (page 47), so is a quotient on introducing the relation $(g_1)(g_2)(g_1g_2)^{-1}$. Now $\mu_X : UFUF(X) \to UF(X)$ encodes the multiplication in the free group, $F(X)$.

Definition: If $(F, G, \eta, \varepsilon)$ is an adjoint pair with $F : \mathcal{A} \to \mathcal{B}$, etc. as above, the monad $T = (GF, \eta, \mu)$ with $\mu = G\varepsilon F$ is called the *monad defined by the adjunction*.

The particular case with the free-forget adjunction on groups is the *free group monad*.

An adjoint pair also defines a comonad. In the above notion, (L, ε, δ), is given by

$$\begin{aligned} L &= FG, \\ \varepsilon &= \varepsilon, \\ \delta &= F\eta G. \end{aligned}$$

This will be called the *comonad defined by the adjunction*

Historically the question was asked early on in the development of the theory of monads: *is every monad defined by an adjunction?* The answer was yes, up to equivalence, of course. The methods used in the solution are relevant to our discussion on the adjunction $(+, Dec)$, as well as shedding light on the algebraic nature of monads, even when they seem to be merely combinatorially based. There were two separate and distinct constructions that appeared. One due to Heinrich Kleisli gave a category of 'free algebras' that was the minimal solution. It has just enough objects to allow an obvious 'forgetful' functor to have a left adjoint with the right properties, so the objects are 'free'. The construction is very neat. One starts with the category, $\mathcal{A}$, and a monad, (T, η, μ), on it, and constructs the *Kleisli category*, $\mathcal{A}_T$, of T by taking as the objects those of the category, $\mathcal{A}$, but with $\mathcal{A}_T(x, y) = \mathcal{A}(x, Ty)$. The unit $\eta_x : x \to Tx$, then fulfils the role of the identity on x for the composition defined in the following elegant way:

if $f : x \to Ty$ and $g : y \to Tz$, then applying T to g gives $Tg : Ty \to T^2z$ and composition in $\mathcal{A}$ then yields $tg \circ f : x \to T^2z$. To get back into $\mathcal{A}(x, Tz)$, we compose that with the natural 'multiplication' $\mu_z : T^2z \to Tz$.

(If you have not seen this before, it is quite fun to see how on earth this construction works. Why is η_x the identity on x? Why is composition associative? Why is it referred to as a category of 'free algebras'? That last question is nice to look at when $T = UF$ is the free group monad in *Sets*. We have here $\mathcal{A} = Sets$, so $\mathcal{A}_T(X, Y) = Sets(X, UF(Y)) \cong Groups(FX, FY)$, since we have $F \dashv U$. Of course, you can replace free groups by free whatever-you-likes, but do check that the Kleisli category does give the sub-category of free whatever-you-likes as far as composition etc. is concerned.)

The forgetful functor, $\mathcal{A}_T \to \mathcal{A}$, has a left adjoint, and, of course, the monad this adjoint pair defines is isomorphic to the monad with which we started. (It is **left to you** to look up or find, how to define the forgetful functor and the corresponding left adjoint; again this is in standard category theory texts such as Mac Lane, [192].)

The second construction, due to Eilenberg and Moore, gives a notion of T-algebra for a monad, T, in $\mathcal{A}$, so that again the category of T-algebras has a pair of adjoint functors to $\mathcal{A}$, which again give back the given monad up to isomorphism. This category, denoted $\mathcal{A}^T$, is 'maximal' with respect to that property, whilst the Kleisli category is 'minimal'; see the discussion in, for instance, Mac Lane, [192], for more on these results. (It is clear that Kleisli's construction must be the smallest in some sense as it just has the free algebras and they have to be there to get the left adjoint to have images of every object in $\mathcal{A}$.) What is a T-algebra?

Definition: Given a monad, $T = (T, \eta, \mu)$, on a category $\mathcal{A}$, a *T-algebra*, (A, a), consists of an object A of $\mathcal{A}$ and a morphism, $a : TA \to A$, called the *structure map of the algebra*, which is such that the diagrams

$$\begin{tikzcd} T^2A \arrow[r, "Ta"] \arrow[d, "\mu A"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, no head, "="'] & TA \arrow[d, "a"] \\ & A \end{tikzcd}$$

commute.

A *morphism of T-algebras*, $f : (A, a) \to (B, b)$, is a morphism $f : A \to B$ in $\mathcal{A}$ commuting with the structure maps:

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

We denote by $\mathcal{A}^T$, the category of T-algebras for a given T in $\mathcal{A}$. It is also known as the *Eilenberg-Moore category* of the monad, T.

There is an obvious forgetful functor from $\mathcal{A}^T$ to $\mathcal{A}$ that sends (A, a) to A, This has a left adjoint sending A to (TA, μ), since, of course, if you substitute TA for A and μ for a in the above diagrams, you get the associativity and unit rules for the monad (T, η, μ). If we write $G^T : \mathcal{A}^T \to \mathcal{A}$ for $G^T(A, a) = A$, the forgetful functor and $F^T : \mathcal{A} \to \mathcal{A}^T$, $F(A) = (TA, \mu)$, then it is reasonably clear that $F^T \dashv G^T$ and that $G^T F^T \cong T$ with the corresponding units and multiplication matching up, so we retrieve the monad T up to equivalence.

Given any adjunction $(F : \mathcal{A} \to \mathcal{B}, G : \mathcal{B} \to \mathcal{A}, \eta, \varepsilon)$, we can form the corresponding monad $T = (GF, \eta, \mu)$ and then form $\mathcal{A}^T$. There is a comparison functor,

$$K : \mathcal{B} \to \mathcal{A}^T,$$

given by $K(B) = (GB, G\varepsilon)$. (It is easy to **check this works**.) This functor is compatible with both the free and forgetful functors. (Again this is standard material, so is **left for you** to follow up elsewhere.)

Definition: In an adjunction, $(F, G, \eta, \varepsilon)$, the functor, $G : \mathcal{B} \to \mathcal{A}$, is said to be *monadic* (and $\mathcal{B}$ to be *monadic over* $\mathcal{A}$) if the comparison functor $K : \mathcal{B} \to \mathcal{A}^T$ is an equivalence of categories.

Remarks: (i) Of course, if $L = (L, \varepsilon, \delta)$ is a comonad, then there are categories of *coalgebras* for L and a *co-Kleisli category* of co-free coalgebras, together with comparison theorems and comonadicity results.

A *coalgebra for L* is a pair, (A, c), where A is an object of $\mathcal{A}$ and $c : A \to LA$ is a morphism satisfying counit and coassociativity axioms that are given by the dual diagrams to those for T-algebras. The pair, $(LA, \delta : LA \to L^2A)$, is a L-coalgebra, which is *cofree* in a suitable sense.

In both the monadic and comonadic situations, there are well known results characterising the situation in terms of conditions on the functor concerned. These can be found in many category theory texts and as our use of monadicity is relatively simple, we will not need them, so they are left to you for further study.

(ii) It may help, before we leave the topic, to look at $\mathcal{A}^T$, when $\mathcal{A} = Sets$ and $TA = M \times A$, for some monoid A. The T-algebras then consist of a set, A, together with a function $a : M \times A \to A$. The unit and associativity diagrams then tell us that a is simply an action of M on A. Morphisms of T-algebras are exactly the M-equivariant functions. The free T-algebra on a set, A, is $TA = M \times A$ with the action given by multiplication on the first factor. Another example that is worth worrying out is that for the free group monad, which, of course, has T-algebras that are groups.

We will revisit comonads again shortly, but next examine the monad induced on $SplitAugSimp(\mathcal{C})$ by the adjoint pair, $+ \dashv Dec_{SAS}$. This monad will have $T = Dec_{SAS}(-^+)$, η given by the splitting and, as ε is the 'last-face' map, $\mu : T^2A \to TA$ will, in fact, be $d_{last-but-one}$ of the original A. What are the T-algebras?

We will have a split augmented simplicial object, A, together with a map, $a : TA \to A$, satisfying the unit and associativity rules. In fact, it is easier to see what this looks like if we cross over to $\mathcal{C}^{\mathbf{\Delta}_\ell^{op}}$, and look at the corresponding monad, defined by $b^* \dashv inc^*$, there. We will denote this monad by $(T', \varepsilon', \delta')$.

Now $A' : \mathbf{\Delta}_\ell^{op} \to \mathscr{A}$ will denote $c^{-1}(A)$, so $A'_n = A_{n-1}$, etc. We have $(T'A')_n = A'_{n+1}$, so $a' : T'A' \to A'$ is given by some $\{a'_n : A'_{n+1} \to A'_n\}_n$, or $\{a_n : A_n \to A_{n-1}\}_n$. It is now fairly routine to check that the a'_n act as a 'last-face' operator, since the unit axiom gives the behaviour with respect to the splitting and the associativity that with respect to the 'last-but-one' face. The other simplicial identities are a consequence of naturality, *i.e.*, that a' is a simplicial map. In other words, a T'-algebra structure will allow us to 'replace' the missing last face, extending A' from the sub-category $\mathbf{\Delta}_\ell^{op}$ to the whole of $\mathbf{\Delta}^{op}$, giving us a simplicial object in $\mathcal{C}$. Conversely, any simplicial object in $\mathcal{C}$ will 'restrict' to give a T'-algebra. The details are quite fun, but are fairly routine checking - **left to you**. This verification proves

Proposition 157 *The category of simplicial objects in $\mathcal{C}$ is monadic over that of the corresponding split augmented simplicial objects.* ■

This result is not just a curiosity. For a start, it seems to say something very subtle about the relationship of general 'homotopy types' in $\mathcal{C}$ to the contractible ones. The contractible ones in some sense generate all the other, *provided you keep account of the contractions / splittings involved.* It plays an important role in Bourn's proof of a Dold-Kan theorem for use in 'semi-Abelian' categories, for which see Bourn, [40].

13.3.5 More on comonadic resolutions

We next look at comonads and recall how a comonad gives us a way of building simplicial resolutions. We saw this in section 3.5.3, but will repeat some of that material for convenience, as now we can give more details of the results and in more generality. In particular, we now have a much greater range of terminology, etc., for handling the augmented objects that result. (We use the same conventions for faces and degeneracies as in the original introduction to this back on page 90, rather than the alternative 'dual' ordering that we used for $S(\mathscr{A})$ earlier in this chapter.)

We suppose that we have a comonad, $L = (L, \varepsilon, \delta)$, with $L : \mathscr{A} \to \mathscr{A}$, so that $\varepsilon : L \to I$ is the counit of the comonad whilst $\delta : L \to L^2$ is the comultiplication. Suppose A is an object of $\mathscr{A}$ and set $\mathsf{L}(A)_i = L^{i+1}(A)$. The counit gives, in each dimension, face morphisms,

$$d_i = L^{n-i}\varepsilon L^i(A) : L^{n+1}(A) \to L^n(A),$$

for $i = 0, \ldots, n$, whilst the comultiplication gives degeneracies,

$$s_i : L^n(A) \to L^{n+1}(A)$$

$$s_i = L^{(n-i)-1}\delta L^i(A),$$

for $i = 0, \ldots, n-1$, satisfying the simplicial identities. In fact, we also have that the two face maps from $\mathsf{L}A_1$ to $\mathsf{L}A_0$ are the morphisms ε_{LA} and $L\varepsilon_A$ and the co-associativity axiom for (L, ε, δ) gives that $\varepsilon_A.\varepsilon_{LA} = \varepsilon.L\varepsilon_A$, so the counit gives an augmentation of $\mathsf{L}A$, that is, we have:

Lemma 98 *The simplicial object,* $\mathsf{L}A$*, is naturally augmented over* A*, and* L *gives a functor,*

$$\mathsf{L} : \mathcal{A} \to AugSimpl(\mathcal{A}).$$

■

This suggests that a definition would be useful!

Definition: Given a comonad $L = (L, \varepsilon, \delta)$ with $L : \mathcal{A} \to \mathcal{A}$, the augmented simplicial object, $\mathsf{L}A$, defined above, is called the *standard or comonadic resolution* of A (relative to L).

Not only did we see this when discussing simplicial resolutions of groups back in section 3.5.3, but we used it in 11.2.3, when constructing the simplicially enriched category, $S(\mathbb{A})$, from a small category, $\mathbb{A}$. Of course, the definition used the term resolution and we should check that in general the usual attributes of a simplicial resolution are satisfied here.

What should those 'usual attributes' be? Perhaps 'contractible' over A?

Given our extensive examination of split augmented simplicial objects, it is natural to enquire if such a resolution is, not only 'contractible', but 'split'. The case of the comonad defined by the free-forget adjunction between *Groups* and *Sets* gives a clear idea of the situation. The basic functor here is FU, (see page 90 and following pages), the 'free group on the underlying set' of a group. The counit, $\varepsilon_A : FU(A) \to A$, for a group, A, will only be split if A is itself a free group, yet if the augmented object, $(\mathsf{L}A, A)$ is split, then, in particular, the augmentation will be split, so clearly the answer to our enquiry must be negative. However, ε_A is an epimorphism of groups, therefore is surjective on the underlying sets and in *Sets*, surjections / epimorphisms are split (by the Axiom of Choice). That is misleading, however, and if pursued would give a classic case of using 'a sledge-hammer to crack a nut'. It suggests an idea, that somehow the underlying augmented simplicial set is split, but there are numerous useful instances where epimorphisms of some structure do not split. An important case of this would be if given a sheaf version of the above. We can have an epimorphism of sheaves of groups, yet the underlying epimorphism of sheaves of sets need not split. The invocation of the Axiom of Choice would be (i) unnecessary and (ii) unhelpful as we really could do with a *splitting* and use of the Axiom of Choice will not give us that!

None-the-less, $U\varepsilon$ is naturally split and the reason is simple, ..., look at the adjunction rules. We have η, the unit of the adjunction and with ε, it satisfies the triangle axioms, one of which is that

$$\begin{tikzcd} UFU \arrow[d, "U\varepsilon"'] & U \arrow[l, "\eta U"'] \arrow[dl, no head, "="] \\ U & \end{tikzcd}$$

commutes, so $U\varepsilon_A$ is split by η_{UA}. We get an explicit splitting of the underlying map of the counit and this looks general in its applicability as it just uses the adjunction.

We take this idea and the comparison theorem for comonads to write $L = FG$ for some adjoint pair, $G : \mathcal{A} \to \mathcal{B}$, etc. (Note: as we have $L : \mathcal{A} \to \mathcal{A}$, we will reverse the roles of the symbols $\mathcal{A}$ and $\mathcal{B}$ from our earlier discussion. Here, think $\mathcal{A}$ for 'algebras', $\mathcal{B}$ for 'base'!)

These ideas are encoded as follows:

Definition: (i) An augmented simplicial object, $\mathbf{X} = (X^+, X)$ in $\mathcal{A}$ is said to be *G-splittable* if $G(\mathbf{X}) = (GX^+, GX)$, its image under the right adjoint G, is contractible over GX in $AugSimp(\mathcal{B})$.

(ii) A *G-split augmented simplicial object* in $\mathcal{A}$ is a pair consisting of a G-splittable, $\mathbf{X}$, as above, together with a given G-splitting, *i.e.*, a family $\mathbf{s} = \{s_{n+1} : GX_n \to GX_{n+1}\}_{n\geq -1}$.

(iii) A *morphism* of G-split augmented simplicial objects, $\mathbf{f} : (\mathbf{X}, \mathbf{s}) \to (\mathbf{Y}, \mathbf{t})$, is a morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in $AugSimp(\mathcal{A})$, which is compatible with the G-splittings, *i.e.*, so that $G(\mathbf{f}) : (G(\mathbf{X}), \mathbf{s}) \to (G(\mathbf{Y}), \mathbf{t})$ is a morphism in $SplitAugSimp(\mathcal{B})$.

We may also use the term '*coherent morphism*' for such maps (following Duskin's terminology in [107]). These make up a category, G–$SplitAugSimp(\mathcal{A})$ or G–$SAS(\mathcal{A})$, for short, and subcategories over a fixed augmentation object, X, denoted $G-SplitAugSimp(\mathcal{A})_X$ or $G-SAS(\mathcal{A})_X$.

We note that we can think of a G-split, $\mathbf{X}$, in several alternative ways, that is, $s : K(GX, 0) \to GX^+$ is a homotopy inverse for $G\mathbf{d} : GX^+ \to K(GX, 0)$, etc., similarly to those for a split object. The right adjoint $G : \mathcal{A} \to \mathcal{B}$ induces functors from $Simp(\mathcal{A})$ to $Simp(\mathcal{B})$ and from $AugSimp(\mathcal{A})$ to $AugSimp(\mathcal{B})$, which we have used in the above, and clearly any split augmented simplicial object in $\mathcal{A}$ gives a split augmented one in $\mathcal{B}$ and hence is itself G-split. These, however, are not the interesting ones. What we get, say, from the free-group comonad is a G-split object, which is not split unless the X being 'resolved' is itself 'free'. This is a general result.

Proposition 158 *Let A be an object of $\mathcal{A}$ and $\mathbf{L}A = (\mathsf{L}A, A)$, the augmented simplicial object in $\mathcal{A}$ over A given by the comonadic resolution construction. Define*

$$\mathbf{s} = \{s_{n+1} : GL(A)_n \to GL(A)_{n+1}\}_{n\geq -1}$$

by

$$s_{n+1} = \eta GL^{n+1}(A) : GL(A)_n \to GL(A)_{n+1},$$

then $\mathbf{s}$ *is a G-splitting.*

Proof: We will check that this makes sense, but will **leave much of the later details to you**.

The unit goes from the identity to GF, so

$$\eta(X) : X \to GF(X).$$

If we apply this to $GL^{n+1}(A)$, we have $s_{n+1} : GL^{n+1}(A) \to GL^{n+2}(A)$, so that 'type checks' correctly, that is, the parts each make sense and are of the right 'type', fitting together properly.

We note $d_n : \mathsf{L}_n(A) \to \mathsf{L}_{n-1}$ is $\varepsilon L^n(A)$ and, adjusting indices,

$$s_n = \eta GL^n(A).$$

(Indices here *do* get confusing, so, when checking, take a lot of care!) We need to look at $Gd_n.s_n$. This is $(G\varepsilon L^n(A)).(\eta GL^n(A)) = (G\varepsilon.\eta G)(L^n(A))$, *i.e.*, the composite natural transformation $G\varepsilon.\eta G$ evaluated at $L^n(A)$. The 'triangle rules' for the adjunction, $F \vdash G$, include that $G\varepsilon.\eta G$ is the identity (do check up in almost any category theory text, if you are not sure of this), so $Gd_n.s_n = id$.

Most of the other equations for a splitting follow either from naturality of the transformations involved combined with (co)associativity of the comultiplication, and, as was said above, are **left to you**. ■

We note that the bottom level of $\mathbf{s}$ is $\eta G(A)$.

Because the G-splitting is defined using the unit, η, of the adjunction, we will follow Duskin, [107], in denoting it if needed by a bold $\boldsymbol{\eta}$. We call the resulting $(LA^+, A, \boldsymbol{\eta})$ the *G-split comonadic resolution of A.*

The comonadic resolution is very special. For a start it is functorial, but it is also, in some sense, 'universal'.

Suppose $(A^*, A, \mathbf{s})$ is a G-split simplicial object over A, then, in the bottom dimension, we have a G-splitting

$$s_0 : GA \to GA_0,$$

which corresponds, by adjunction, to a morphism

$$\widetilde{s_0} : FG(A) \to A_0.$$

To get $\widetilde{s_0}$, we first apply F to s_0 giving $F(s_0) : FG(A) \to FG(A_0)$, then compose with $\varepsilon(A_0) : FG(A_0) \to A_0$. We thus have

$$\widetilde{s_0} = \varepsilon(A_0).F(s_0).$$

We know $G(d_0)s_0 = id$, so $d_0\widetilde{s_0} = d_0\varepsilon(A_0)F(s_0) = \varepsilon(A).FG(d_0).F(s_0) = \varepsilon(A)$, so $\widetilde{s_0}$ is a morphism over A:

$$\begin{tikzcd} FG(A) \arrow[r, "\widetilde{s_0}"] \arrow[d, "\varepsilon(A)"'] & A_0 \arrow[d, "d_0"] \\ A \arrow[r, "="'] & A \end{tikzcd}$$

(This is thinking of $FG(A)$ as being $L\mathsf{L}_0(A)$ with augmentation $\varepsilon(A)$.) Does it respect the G-splitting? Does

$$\begin{tikzcd} GFG(A) \arrow[r, "G\widetilde{s_0}"] & G(A_0) \\ G(A) \arrow[u, "\eta G(A)"] \arrow[r, "="'] & G(A) \arrow[u, "s_0"'] \end{tikzcd}$$

commute? We check

$$G(\widetilde{s_0}).\eta G(A) = G\varepsilon(A_0)GF(s_0)\eta G(A) = s_0.G\varepsilon(A).\eta G(A) = s_0,$$

so : Yes.

That looks so natural that we should try to extend $\widetilde{s_0}$ to a coherent simplicial map,

$$\tilde{s} : \mathsf{L}(A) \to \mathbf{A},$$

of G-split things. We start by looking for $\widetilde{s_1}$, which we want to be from $\mathsf{L}(A)_1 = L^2(A)$, in other words, $FGFG(A)$, to A_1. The only 'interesting' map from somewhere to $G(A_1)$ is $s_1 : G(A)_0 \to G(A)_1$, *i.e.*, *that* part of the G-splitting (and we clearly have to use that somewhere). That will be adjoint to something from $FG(A_0)$ to A_1. We need to get from a 'thing' related to A to $FG(A_0)$ and an obvious map to try is $FG(\widetilde{s_0}) : FG(FG(A)) \to FG(A_0)$. The 'picture' now looks like

$$L^2(A) \xrightarrow{L(\widetilde{s_0})} L(A_0) \xrightarrow{F(s_1)} L(A_1) \xrightarrow{\varepsilon(A_1)} A_1,$$

where the last two maps compose to give the adjoint to s_1, again using the formula for the adjunction isomorphism in terms of the counit. This gives

$$\widetilde{s_1} = \varepsilon(A_1)F(s_1)FG(\widetilde{s_0}).$$

(**You can check** that this 'works', *i.e.*, is compatible with the face and degeneracy maps that are relevant: $d_1\widetilde{s_1} = \widetilde{s_0}d_i$, $i = 0, 1$ and $\widetilde{s_1}.s_0 = s_0.\widetilde{s_1}$. We will see this in general shortly, but it is a **good exercise** to do it in this simple case first.)

This suggests giving an inductive formula for $\widetilde{s_n}$. We already have a map $\widetilde{s_0} : \mathsf{L}(A)_0 \to A_0$, so assume that we have defined

$$\widetilde{s_{n-1}} : \mathsf{L}(A)_{n-1} \to A_{n-1},$$

making up a truncated simplicial map, compatible with the augmentations and G-splittings. Define

$$\widetilde{s_n} : \mathsf{L}(A)_n \to A_n,$$

by

$$\widetilde{s_n} = \varepsilon(A_n)F(s_n)FG(\widetilde{s_{n-1}}).$$

We first check that this composite makes sense:

$$L^{n+2}(A) = (FG)(L^{n+1}(A)) \xrightarrow{FG(\widetilde{s_{n-1}})} FG(A_{n-1}) \xrightarrow{F(s_n)} FG(A_n) \xrightarrow{\varepsilon} A_n.$$

(We note that the composite is the adjoint of $s_n.G(\widetilde{s_{n-1}})$.)

Lemma 99 *Thus defined,* $\tilde{\mathbf{s}} : \mathsf{L}(A) \to A^+$ *is a coherent morphism of G-split augmented simplicial objects over A.*

Proof: We will check some of what is required, but some will be **left to you**.

The compatibility with regard to the simplicial operators requires us to check

$$d_i\widetilde{s_n} = \widetilde{s_{n-1}}d_i, \qquad\qquad 0 \le i \le n.$$

The case $i = n$ is different, so we will do that first:

$$\begin{aligned} d_n\widetilde{s_n} &= d_n.\varepsilon(A_n).F(s_n)FG(\widetilde{s_{n-1}}) \\ &= \varepsilon(A_{n-1}).FG(d_n).F(s_n).FG(\widetilde{s_{n-1}}) \\ &= \varepsilon(A_{n-1}).F(G(d_n).s_n).FG(\widetilde{s_{n-1}}) \\ &= \varepsilon(A_{n-1}).FG(\widetilde{s_{n-1}}) \\ &= \widetilde{s_{n-1}}\varepsilon((FG)^n(A)) \\ &= \widetilde{s_{n-1}}d_n, \end{aligned}$$

using naturality of ε several times, together with $G(d_n)s_n = id$, and finally that $d_n = \varepsilon(L^n(A))$ for the n^{th} face map of $\mathsf{L}(A)$.

For $i < n$, the argument is slightly simpler as it just uses naturality of ε and the induction hypothesis. The relevant diagram is

$$\begin{tikzcd} (FG)^{n+1}(A) \arrow[r, "FG(\widetilde{s_{n-1}})"] \arrow[d, "FG(d_i)=d_i"'] & FG(A_{n-1}) \arrow[r, "F(s_n)"] \arrow[d, "FG(d_i)"] & FG(A_n) \arrow[r, "\varepsilon(A_n)"] \arrow[d, "FG(d_i)"] & A_n \arrow[d, "d_i"] \\ (FG)^n(A) \arrow[r, "FG(\widetilde{s_{n-2}})"] & FG(A_{n-2}) \arrow[r, "F(s_{n-1})"] & FG(A_{n-1}) \arrow[r, "\varepsilon(A_{n-1})"] & A_{n-1} \end{tikzcd}$$

The left hand square is the image of a square that commutes by induction hypothesis, so it also commutes. The middle square commutes by the rules on G-splittings and the last one by the naturality of ε. The left hand vertical map is $FG(d_i) = L(L^{n-i}\varepsilon L^i(A)) = (L^{n+1-i}\varepsilon L^i)(A) = d_i^{n+1}$, so that works.

The proof for the degeneracies is very similar.

Compatibility with the G-splittings follows from the G-splitting rules and the fact that $\widetilde{s_n}$ is the adjoint of $s_n.G(\widetilde{s_{n-1}})$, so is uniquely determined. ■

Slightly more is true, since, if

$$\mathbf{t} : \mathsf{L}(A) \to \mathbf{A}$$

is a coherent maps over A, then $\mathbf{t} = \tilde{\mathbf{s}}$. The point being that compatibility with the G-splitting forces $G(t_0).\eta(G(A)) = s_0$, but then it is easy to check $\widetilde{s_0} = t_0$. This primes an inductive proof that $\widetilde{s_k} = t_k$ for all k. We summarise the result:

Proposition 159 *Given any G-split augmented simplicial object, $\mathbf{A} = (A^+, A, \mathbf{s})$ over A, there is a unique coherent map*

$$\tilde{\mathbf{s}} : \mathsf{L}(A) \to \mathbf{A}$$

in $G{-}SAS(\mathcal{A})_A$. ■

Corollary 28 *Given $\mathbf{A} = (A^+, A, \mathbf{s})$, as above, and a morphism, $f : B \to A$, in $\mathcal{A}$, there is a unique coherent map in $G{-}SAS(\mathcal{A})$*

$$\tilde{t} : \mathsf{L}(B) \to \mathbf{A}$$

*over f, (*i.e., $t_{-1} = f$*).*

Proof: The map f defines $\mathsf{L}(f) : \mathsf{L}(B) \to \mathsf{L}(A)$, so we set $\tilde{t} = \tilde{\mathbf{s}}.\mathsf{L}(f)$. The uniqueness proof follows the same pattern as in the proposition. ■

Duskin, [107], uses the neat terminology: $\tilde{\mathbf{s}}$ is the *adjoint of the splitting map*, $\mathbf{s}$. This relationship is very like an adjunction, but we will not pursue that line here. It is clear that the proposition implies that $G{-}SAS(\mathcal{A})_A$ has an initial object, namely $\mathsf{L}(A)$.

Before we return to the décalage, we should mention what this theory looks like if one handles the alternative comonadic resolution. This is very important as both forms occur in the literature.

We first note that there is another subcategory of $\mathbf{\Delta}$ that is 'dual' to $\mathbf{\Delta}_\ell$; this is $\mathbf{\Delta}_0$, the wide subcategory of $\mathbf{\Delta}$ consisting of those morphisms that fix 0. The theory can be developed with $\mathbf{\Delta}_0$ instead of $\mathbf{\Delta}_\ell$. The inclusion of $\mathbf{\Delta}_0$ into $\mathbf{\Delta}_\emptyset$ has an adjoint. The objects in $\mathcal{C}^{\mathbf{\Delta}_0^{op}}$ can be identified with augmented simplicial objects with a *left splitting*. (It is a **good exercise** to see what such must be.) There are obvious adjoint pairs and simplicial objects are monadic over the resulting category. One of the functors involved here is the alternative *Dec*, which strips off the d_0 and s_0 and shifts the indices 'left'. The s_0s left over yield a left splitting of the result, (and that should tell you what a left splitting must satisfy: it is an extra degeneracy s_{-1}.)

Turning to the alternative form of the comonadic resolution, this is naturally left G-split and is universal for left G-split augmented simplicial objects in $\mathcal{A}$.

This is a good opportunity to examine the simplicially enriched category, $S(\mathbb{A})$, (cf. page 499) from this point of view. We defined this using the adjoint pair given by the forgetful functor,

U, from Cat, the category of small categories, to $DGrph_0$, the category of directed graphs[3] with distinguished vertex loops, and its left adjoint, which is a free category construction on the non-distinguished edges of the graph, and the distinguished loops are then identified with the identity loops of that free category. The forgetful functor, U, thus just forgets the composition, but not that the identity loops are 'special'. The resulting alternative simplicial category resolution of $\mathbb{A}$ is naturally a $\mathcal{S}$-category, as we saw (Lemma 68, page 500). The category, $\mathbb{A}$, itself can be considered as an $\mathcal{S}$-category, $K(\mathbb{A}, 0)$, with constant simplicial sets of morphisms between objects. There is an augmentation $\mathcal{S}$-functor from $S(\mathbb{A})$ to $\mathbb{A}$, or, if you prefer, an $\mathcal{S}$-functor $d_0 : S(\mathbb{A}) \to K(\mathbb{A}, 0)$. This is U-split, *i.e.*, each $S(\mathbb{A})(a, a') \to K(\mathbb{A}(a, a'), 0)$ is a homotopy equivalence. The U-splittings send a morphism, $f : a \to a'$, to the corresponding generating 0-simplex. This does *not* give a functor. (**Have a look at the case** $\mathbb{A} = [2]$, if you need convincing.) Understanding the augmentation and the U-splitting comes in *very* useful when working with homotopy limits and colimits, or, more generally, homotopy ends and coends; see the paper by Cordier and Porter, [88]. It is often the case that, provided the diagram indexed by $\mathbb{A}$ is made up of Kan complexes, using the augmentation allows one to say extra about the homotopy limit or colimit of the diagram. (If you need diagrams of things other than simplicial sets, a condition on the objects can usually be found that works in a similar way.)

Aside: All this is also closely related to our discussion (see 565 and the following pages) of weak maps of crossed modules. Recall that this applied the homotopy coherent nerve of a 2-groupoid, and ended up with a map from $G\overline{W}(K(\mathsf{C}))$ to $K(\mathsf{D})$. This $G\overline{W}(K(\mathsf{C}))$ was a cofibrant replacement for $K(\mathsf{C})$ and we investigated what other cofibrant replacements gave.

Finally, after that longish detour, we get back to the total décalage. We saw that $Simp(\mathcal{C})$ was monadic over $SAS(\mathcal{C})$. The 'positive part' functor, $(-)^+$, which forgets the augmentation and the splitting, defines a left adjoint to $Dec\,(= Dec_{SAS})$. What will be the G-split comonadic resolution of a simplicial object, A? (We first have to check the notational correspondence: $(-)^+$ is left adjoint, so is 'free', *i.e.*, is F, whilst Dec is right adjoint, so behaves like a 'forget', *i.e.*, G.) The comonad, $L = (L, \varepsilon, \delta)$, generated by $(+, Dec)$, is then based on $Dec_{SAS}(-)+$, which is simply, 'take décalage, and forget the splitting and the augmentation', ..., but that is just 'original décalage', *i.e.*, Dec as we introduced back in section 8.3.5. What is the counit? We observed earlier (page 626) that this was the 'last-face' map, $d_{last} : Dec(A) \to A$, (..., so this is our $\varepsilon : FG \to Id$). The unit of the adjunction was similarly identified earlier. It took an object $\mathbf{X} = (X^+, X, \mathbf{s})$ in $SAS(\mathcal{C})$ and turned $\mathbf{s}$ into a morphism $\mathbf{X} \to Dec(X^+)$, *i.e.*, the unit, $\eta(\mathbf{X}) = \mathbf{s} = \{s_{n+1} : X_n \to X_{n+1}\}_n$. The comultiplication of the comonad, $\delta : L \to L^2$, is $F\eta G$, *i.e.*,

$$Dec(A) \to Dec^2(A)$$

being the splitting of $Dec(A)$, which is the last degeneracy of the original A, that is, s_{last}.

Next look at $Dec^p(A)_q$, the q^{th} level of the p^{th} iterate. If $p = 0$, we just have A_{q+1}; if $p = 1$, A_{q+2}, and so on. In other words

$$Dec^p(A)_q = A_{p+q+1}.$$

We leave **the checking of the indexation to you**, but we get:

Proposition 160 *For any A in $Simp(\mathcal{C})$, there is an isomorphism between the total décalage of A and the comonadic resolution of A for the $(+, Dec_{SAS})$-comonad.* ■

[3] This is the same as $Quiv_0$, but, following the literature, we here will use the 'directed graph' terminology.

Remark: If we have a monad, $T = (T, \eta, \mu)$ on a category $\mathcal{A}$, then, not surprisingly, we get a cosimplicial co-resolution of any object. This can sometimes be very useful, but we do not explore it here as it really is subsumed in our discussion by replacing $\mathcal{A}$ by $\mathcal{A}^{op}$, as a monad on $\mathcal{A}$ gives a comonad on the opposite category.

The aim of this interlude has been to give a thorough discussion of the total décalage, and, in the process, to visit some of the categorical constructions that come into its theory. These have been discussed for their own sake, and that discussion has not been limited just to this particular case. This was because several of them will give additional insight into future (and past) constructions. To do this, however, it is very convenient to use the simplification that comes from the use of ends and coends.

13.4 Second Interlude: Ends and coends

This second interlude will be a bit shorter than the previous one, but the material is also very important and useful.

As usual in our treatment of (purely) categorical material, we will try to present enough to give some additional intuition and understanding for the non-category theorist, for instance, by discussing further examples of relevance to our main themes. The reader will most likely need to refer to category theory texts for further details and examples and, in particular, for proofs of the deeper results.

13.4.1 From sets of natural transformations towards 'ends'

The first example we will examine will be the set of natural transformations between two functors, $F, G : \mathcal{C} \to \mathcal{D}$, with $\mathcal{C}$ a small category (to avoid foundational difficulties). We know well that a natural transformation, $\eta : F \to G$, is a family $\{\eta(c) : F(c) \to G(c) \mid c \in Ob)\mathcal{C})\}$ of morphisms in $\mathcal{D}$ such that, if $\gamma : c \to c'$ is a morphism in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(c) \arrow[r, "\eta(c)"] \arrow[d, "F(\gamma)"'] & G(c) \arrow[d, "G(\gamma)"] \\ F(c') \arrow[r, "\eta(c')"'] & G(c') \end{tikzcd}$$

is commutative.

We thus start with elements in the various $\mathcal{D}(F(c), G(c))$ and see what happens if we apply $\gamma : c \to c'$. We get induced maps that fit in a diagram

$$\begin{tikzcd} & \mathcal{D}(F(c), G(c)) \arrow[dr, "G(\gamma)_*"] & \\ 1 \arrow[ur, dashed, "\eta(c)"] \arrow[dr, dashed, "\eta(c')"'] & & \mathcal{D}(F(c), G(c')) \\ & \mathcal{D}(F(c'), G(c')) \arrow[ur, "F(\gamma)^*"'] & \end{tikzcd}$$

The diagonal dashed arrows are the 'names' of the various components of η, that is, the arrows from the singleton terminal object, 1, in *Sets* to the various sets involved, that pick out the named elements.

The condition for naturality is satisfied for η if and only if all such diagrams, for all morphisms, γ, are commutative. We can rewrite this slightly by forming the product of all the $\mathcal{D}(F(c), G(c))$, indexed by the objects of $\mathcal{C}$ and another product of the $\mathcal{D}(F(c), G(c'))$, this time indexed by all the morphisms in $\mathcal{C}$. (You can see why we wanted $\mathcal{C}$ to be small, otherwise these products might be 'unmanageable' and, depending on your set theoretic outlook, perhaps even 'non-existent'! Beware of thinking, however, that $\mathcal{C}$ has to be small for a construction or idea to work. Sometimes the explicit construction looks 'large' and probably 'invalid', yet because of special circumstances in a particular situation, it does work.)

There are, as usual, projection maps from these products:
for each object $c \in \mathcal{C}$

$$\pi_c : \prod_{x \in \mathcal{C}} \mathcal{D}(F(x), G(x)) \to \mathcal{D}(F(c), G(c)),$$

and, for each morphism, $\gamma : c \to c'$,

$$\pi_\gamma : \prod_{\alpha : x \to y} \mathcal{D}(F(x), G(y)) \to \mathcal{D}(F(c), G(c')).$$

We now specify two maps

$$a, b : \prod_{x \in \mathcal{C}} \mathcal{D}(F(x), G(x)) \to \prod_{\alpha : x \to y} \mathcal{D}(F(x), G(y))$$

by specifying what they do on projection:

$$\begin{aligned} \pi_\gamma a &= G(\gamma)_* \pi_c \\ \pi_\gamma b &= F(\gamma)^* \pi_{c'}. \end{aligned}$$

(**Check these make sense**. They relate to the totality of all the diagrams above, indexed over all the morphisms.) We note that there is a map from $Nat(F, G)$ to $\prod_{x \in \mathcal{C}} \mathcal{D}(F(x), G(x))$, given by sending η to the family $(\eta(x))_{x \in Ob(\mathcal{C})}$. This map equalises a and b, that is, $Nat(F, G) \cong Eq(a, b)$.

The two functors, F, G, and the 'hom' in $\mathcal{D}$ gave us a functor $\mathcal{D}(F(-), G(-)) : \mathcal{C}^{op} \times \mathcal{C} \to Sets$ and we could repeat the above, with minor changes, replacing $\mathcal{D}(F(-), G(-))$ by a general two variable functor, T, from $\mathcal{C}^{op} \times \mathcal{C}$ to *Sets*.

In fact, why should we restrict the codomain of T to being *Sets*! The construction would work in any category, $\mathcal{B}$, with enough limits of the right sort. In other words, if $\mathcal{B}$ has all equalisers of diagrams of the form

$$\prod_x T(x, x) \overset{a}{\underset{b}{\rightrightarrows}} \prod_{\gamma : x \to y} T(x, y),$$

where a and b are the obvious generalisations of those defined earlier, then the equaliser, $Eq(a, b)$, is a generalisation of the construction of $Nat(F, G)$. It is what we will call the *end* of T, written $\int_x T(x, x)$ or $\int_{\mathcal{C}} T(x, x)$. (We will give a fuller definition shortly, so consider this as the idea to go on with for the moment.)

Purely as a result of the definition / construction, we have

Proposition 161 *For any functors $F, G : \mathcal{C} \to \mathcal{D}$, there is an natural isomorphism*

$$Nat(F,G) = \int_c \mathcal{D}(F(c), G(c)).$$

■

This gives a construction of $\int_x T(x,x)$ as an object of $\mathcal{B}$. It does not emphasise its universal properties. Oh yes, it is only determined up to natural isomorphism as is $Eq(a,b)$, of course, but the resulting universal property is a bit 'squashed up' and difficult to interpret. It is very like defining a limit of a functor by its construction as a subobject of a product obeying some rules. That works, but obscures the universality property and it is really *that* which determines the limit.

13.4.2 Dinatural transformations

The key to understanding limits is a form of natural transformation, namely the cones, those with a constant functor as domain. Similarly the customary approach to ends uses a nice generalisation of natural transformation called *dinatural transformation* and the corresponding notion of *wedge*, being a dinatural transformation with a constant functor as its domain. (We will use Mac Lane, [192], as a source, but many other texts give this, and you can try Wikipedia as well.) Again we assume that $\mathcal{C}$ is small.

Definition: Let $S, T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{B}$ be two functors. A *dinatural transformation*, $\alpha : S \to T$, is a family, $\{\alpha_c : S(c,c) \to T(c,c) \mid c \in Ob(\mathcal{C}\}$, of morphisms in $\mathcal{B}$, such that, for any $f : c \to c'$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} & S(c,c) \arrow[r, "\alpha_c"] & T(c,c) \arrow[dr, "T(c,f)"] & \\ S(c',c) \arrow[ur, "S(f,c)"] \arrow[dr, "S(c',f)"'] & & & T(c,c') \\ & S(c',c') \arrow[r, "\alpha_{c'}"'] & T(c',c') \arrow[ur, "T(f,c')"'] & \end{tikzcd}$$

commutes.

If $\tau : S \to T$ is a natural transformation, then we would have morphisms, $\tau_{(c,c')} : S(c,c') \to T(c,c')$, and it is easy to **check** that restricting to the $\tau_{(c,c)}$ gives a dinatural transformation, but not all dinatural transformations need arrive in this way.

We leave you to read more on dinatural transformations and their properties. At the moment, we need them, in detail, only for the definitions of ends (and coends) by means of universal properties.

Definition: Let $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{B}$ be a functor.

(i) A *wedge on T with vertex, b*, is a dinatural transformation from the constant functor, having value b, from $\mathcal{C}^{op} \times \mathcal{C}$ to $\mathcal{B}$ with codomain the given functor, T.

(ii) An *end* is a pair, (b, ω), where b is an object of $\mathcal{B}$ and $\omega : b \to T$ is a wedge on T such that if (b', β) is another such pair, there is a unique arrow, $h : b' \to b$ in $\mathcal{B}$ with $\beta_c = \omega_c.h$ for all $c \in \mathcal{C}$.

Let us take this definition apart. We take $S = b$ in our previous diagram, (and use a shorthand, as before, that the constant functor with a given value is denoted by that value), so the left hand part of the diagram collapses as it is made up of equality maps. We can therefore simplify the diagram for a wedge to be: for $f : c \to c'$ in $\mathcal{C}$, we have

$$\begin{tikzcd} & T(c,c) \arrow[dr, "T(c,f)"] & \\ b \arrow[ur, "\omega_c"] \arrow[dr, "\omega_{c'}"'] & & T(c',c') \\ & T(c',c) \arrow[ur, "T(f,c')"'] & \end{tikzcd}$$

commutes.

The end of T, if it exists is written $\int_c T(c,c)$ or just $\int T$ as above and the morphism $\omega_c : \int T \to T(c,c)$ is the 'projection' to $T(c,c)$.

The general wedge, (b', β), looks similar and we get as diagram corresponding to the universal property:

$$\begin{tikzcd} & & T(c,c) \arrow[dr, "T(c,f)"] & \\ b' \arrow[urr, "\beta_c"] \arrow[r, dashed, "\exists! h"'] \arrow[drr, "\beta_{c'}"'] & b \arrow[ur, "\omega_c"] \arrow[dr, "\omega_{c'}"] & & T(c',c') \\ & & T(c',c) \arrow[ur, "T(f,c')"'] & \end{tikzcd}$$

($\exists! h =$ "there exists a unique h", in case you have not met use of this notation before.)

Example: right Kan extensions as ends. We earlier looked at Kan extensions. We had a situation

$$\begin{tikzcd} \mathcal{C} \arrow[r, "K"] \arrow[d, "F"'] & \mathcal{D} \arrow[dl, dashed, "Ran_K F"] \\ \mathcal{E} & \end{tikzcd}$$

(in particular, we had $\mathcal{E} = Sets$), and sought a way to fill the triangle with a functor and a natural transformation.

There is an end formula for $Ran_K F$ constructed as follow:

(i) For any set, X and object E of $\mathcal{E}$, we denote by E^X, the X-fold power of E, *i.e.*, we take copies of E indexed by the elements of X and then form their product. (We, of course, need these products and, more generally, any limits to exist in $\mathcal{E}$ in the usual way.)

(ii) For each d in $\mathcal{D}$ and c in $\mathcal{C}$, we form the set $\mathcal{D}(d, Kc)$ and then set[4]

$$T(c, c') = (Fc')^{\mathcal{D}(d,Kc)},$$

[4]**You should check** that this is a functor, $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$. The variation in c may need thinking about to see how it is defined and that it is contravariant.

and finally,

(iii) the result:

Proposition 162 *For any F and K, as above,*

$$(Ran_K F)(d) \cong \int_c T(c,c).$$

Proof: We will, in fact, 'prove' this as it uses lots of neat 'tricks' and results that make life a lot easier. Some of these 'tricks' require proofs that will not be given - you will need to **track them down** or prove them yourself.

The defining property of $Ran_K F$ is that, for any $S : \mathcal{D} \to \mathcal{E}$,

$$Nat(S, Ran_K F) \cong Nat(SK, F).$$

(You should check that you know how this determines $Ran_K F$ and the natural transformation as that is crucial.) We write $R = \int_c T(c,c)$ and use the end formula for natural transformations:

$$\begin{aligned}
Nat(S,R) &= \int_d \mathcal{E}(Sd, Rd) \\
&= \int_d \mathcal{E}(Sd, \int_c Fc^{\mathcal{D}(d,Kc)}) \\
&= \int_d \int_c \mathcal{E}(Sd, Fc^{\mathcal{D}(d,Kc)}) && (check) \\
&= \int_c \int_d \mathcal{E}(Sd, Fc^{\mathcal{D}(d,Kc)}) && (Fubini) \\
&= \int_c \int_d Sets(\mathcal{D}(d,Kc), \mathcal{E}(Sd,Fc)) && \text{(property of powers)} \\
&= \int_c \mathcal{E}(SKc, Fc) && (Yoneda) \\
&= Nat(SK, F).
\end{aligned}$$

Just a word about the cryptic remarks on the right of some of the lines. 'Check' means that you should do just that! 'Fubini's theorem' refers to the result on interchange of the order of evaluating ends when there are 'two' parameters, that you can find in standard category theory texts. The 'property of powers' referred to is that $\mathcal{E}(E,E')^X \cong Sets(X, \mathcal{E}(E,E'))$, which you should again 'check out'.

Finally if $G : \mathcal{D}^{op} \to Sets$ is any functor, then $\int_d Sets(\mathcal{D}(d,Kc), Gd) \cong GK(c)$. This is just a form of the Yoneda lemma[5]. Here we apply it to $Gd = \mathcal{E}(Sd, Fc))$, as a functor of d.

Putting this all together gives the desired isomorphism between R and $Ran_K F$. ∎

Coends:

Dually one has coends.

Definition: If $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{B}$ is a functor, a *coend* is a pair (b, ω), where b is an object of $\mathcal{B}$ and $\omega : S \to b$ is a dinatural transformation ('co-wedge') from S to the constant functor with

[5] You might try to prove it independently of the usual proof, directly from the construction of ends. This is quite instructive.

value b, universal among such dinatural transformations from S to a constant functor. The object, b, when it exists, is written $\int^c S(c,c)$ or simply $\int^{\mathcal{C}} S$.

We can construct coends by a coequaliser construction using coproducts[6] (and the details are relatively easy to give so are **left to you**).

13.4.3 More examples of ends and coends and their uses:

(i) **Contracted products:** A very instructive example of a coend is given by the contracted product of G-sets (or of torsors, of course, but that is a bit harder to give in detail).

Let G be a group, X a right G-set, Y a left G-set. We form the set, $X \times Y$, and make it into a functor, S, from $G^{op} \times G$ to $Sets$. (Of course, and as usual, we are thinking of G as a single object groupoid.) Explicitly, $S(*,*) = X \times Y$ and $S(g,h)(x,y) = (x.g, h.y)$. We look at the coend, $\int^G S(*,*)$, (Here the second notation for coends comes in useful. The first would give $\int^* S(*,*)$, which, somehow, is nowhere near as clear in this case, as it is the morphisms of $G[1]$, not the objects, that matter!) This coend is a quotient of $\coprod_* S(*,*)$, (*i.e.*, $X \times Y$), by the equivalence relation generated from the diagram:

$$\coprod_{g\in G} S(*,*) \underset{b}{\overset{a}{\rightrightarrows}} \coprod_{*\in Ob(G)} S(*,*) \ ,$$

where $a((x,y),g) = (x.g,y)$ and, of course, $b((x,y),g) = (x,g.y)$, *i.e.*, $\int^G S(*,*)$ is just the contracted product, $X \wedge^G Y$, of X and Y.

(ii) **Geometric realisation as a coend:** Another example of a coend is given by the geometric realisation functor from $\mathcal{S}$ to Top. We have a simplicial set, $K : \mathbf{\Delta}^{op} \to Sets$; consider each K_n as a discrete space and form a functor $S : \mathbf{\Delta}^{op} \times \mathbf{\Delta} \to Top$ by $S([m],[n]) = K_m \times \Delta^n$, where Δ^n is the standard topological affine n-simplex. (If you do not like making K_n into a space, perhaps think of $S([m],[n])$ as the K_n-fold copower of Δ^n. This has some advantages, but is not the usual treatment. It is, of course, completely equivalent.) The geometric realisation of K, denoted $|K|$, is $\int^{[n]} S([n],[n])$. It takes lots of copies of the various dimensioned Δ^n, indexed by the simplices of K, and glues them together according to the combinatorial instructions encoded in K.

(iii) **Enriched functor categories and the enriched Yoneda lemma:** (A full treatment is to be found in Kelly, [177], §2. 2. We just sketch the main points here.)

If $F, G : \mathcal{C} \to \mathcal{D}$ are two functors, then we have seen that there is an end formula for $Nat(F,G)$ and we also know, (if $\mathcal{C}$ is small) that one has a functor category, $[\mathcal{C},\mathcal{D}]$, sometimes written $\mathcal{D}^{\mathcal{C}}$, with the functors from $\mathcal{C}$ to $\mathcal{D}$ as objects and with such $Nat(F,G)$ as the corresponding 'hom-sets'. If, now, $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$-categories, then one can adapt that idea in an obvious way to give a $\mathcal{V}$-enriched category of $\mathcal{V}$-functors. The only condition is that certain ends exist in $\mathcal{V}$ and so, generally, that $\mathcal{V}$ be complete, so as to have all limits and hence all ends.

[6]Notationally, it is sometimes convenient to use a tensor product sign for 'elements' of a coend; here we might write $x \otimes y$ for the equivalence class determined by (x,y). The tensor product of modules is a coend and there the notation is the standard one

Definition: Let $\mathcal{V}$ be a complete monoidal category and $\mathcal{A}$ and $\mathcal{B}$ $\mathcal{V}$-categories with $\mathcal{A}$ small. For $\mathcal{V}$-functors, $S, T : \mathcal{A} \to \mathcal{B}$, we write

$$[\mathcal{A}, \mathcal{B}](S, T) = \int_{A \in \mathcal{A}} \mathcal{B}(SA, TA),$$

(provided the end on the right hand side exists[7]). We write

$$E_{A,S,T} : [\mathcal{A}, \mathcal{B}](S, T) \to \mathcal{B}(SA, TA)$$

for the projection of the end given by evaluation at A.

Note that the set of elements, $[\mathcal{A}, \mathcal{B}](S, T)_0$, of $[\mathcal{A}, \mathcal{B}](S, T)$ is the set of $\mathcal{V}$-natural transformations, $\alpha : S \Rightarrow T : \mathcal{A} \to \mathcal{B}$, from S to T.

When $[\mathcal{A}, \mathcal{B}](S, T)$ exists for all $S, T : \mathcal{A} \to \mathcal{B}$, it is the hom-object of a $\mathcal{V}$-category $[\mathcal{A}, \mathcal{B}]$ having the $\mathcal{V}$-functors, $S : \mathcal{A} \to \mathcal{B}$, as its objects. The composition law is given by[8] the commutativity of

$$\begin{tikzcd}
{[\mathcal{A}, \mathcal{B}](S, T) \otimes [\mathcal{A}, \mathcal{B}](R, S)} \arrow[r, "m"] \arrow[d, "E_A \otimes E_A"'] & {[\mathcal{A}, \mathcal{B}](R, T)} \arrow[d, "E_A"] \\
\mathcal{B}(SA, TA) \otimes \mathcal{B}(RA, SA) \arrow[r, "m"'] & \mathcal{B}(RA, TA)
\end{tikzcd}$$

given the universal property of the defining ends and the projections / evaluations. Similarly the ‘identity elements’, $j_S : I \to [\mathcal{A}, \mathcal{B}](S, S)$, are determined by the equations:

$$E_A j_S = j_{SA}.$$

Lemma 100 *With these definitions, $[\mathcal{A}, \mathcal{B}]$ is a $\mathcal{V}$-category.* ∎

The proof is given in Kelly, [177], p. 30.

Definition: The $\mathcal{V}$-category, $[\mathcal{A}, \mathcal{B}]$, is called the *$\mathcal{V}$-functor category*[9] or the *enriched functor category* for this context.

The underlying category of $[\mathcal{A}, \mathcal{B}]$ is the ordinary category of all $\mathcal{V}$-functors from $\mathcal{A}$ to $\mathcal{B}$ and all $\mathcal{V}$-natural transformations between them.

Provided that $\mathcal{V}$ is a closed symmetric monoidal category, $\mathcal{V}$ will itself be a $\mathcal{V}$-category[10] and then amongst the $\mathcal{V}$-functors from a $\mathcal{V}$-category $\mathcal{A}$ are the *representable* enriched functors, $\mathcal{A}(A, -) : \mathcal{A} \to \mathcal{V}$. More generally any $\mathcal{V}$-functor from $\mathcal{A}$ to $\mathcal{V}$ which is isomorphic to such a $\mathcal{A}(A, -)$ will be said to be representable.

Given any $\mathcal{V}$-functor, $\underline{F} : \mathcal{A} \to \mathcal{V}$, and an object, A of $\mathcal{A}$, then to each $\mathcal{V}$-natural transformation, $\alpha : \mathcal{A}(A, -) \to F$, we can form the composite,

$$I \xrightarrow{j_A} \mathcal{A}(A, A) \xrightarrow{\alpha_A} FA$$

giving us an element of FA.

[7]which it will since $\mathcal{A}$ is supposed to be small and $\mathcal{V}$ is assumed to be complete, but this might exist in special cases even without these assumptions.

[8]and is uniquely determined by,

[9]of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations, from $\mathcal{A}$ to $\mathcal{B}$, if more precision is needed.

[10]see Kelly, [177], §1.6.

Proposition 163 *((Weak) enriched Yoneda lemma) The assignment of $\alpha_A j_A$ to α gives a natural bijection between the set $\mathcal{V}-\mathsf{Cat}(\mathcal{A}(A,-),\underline{F})$ and the set, $FA = \mathcal{V}(A,\underline{F}A)$, of elements of $\underline{F}A$.* ■

The proof is given in section 1.9 of Kelly, [177], and can be seen as a good illustration of the use of 'elements' in the enriched setting to replace ordinary elements in the £Set£-enriched classical case. As this gives a natural *bijection*

$$([\mathcal{A},\mathcal{V}](\mathcal{A}(A,-),\underline{F}))_0 \xrightarrow{\cong} (\underline{F}(A)_0,$$

it is natural to ask if there is a 'lifting' of this bijection to an isomorphism between the corresponding objects before applying $(-)_0$. This does work if $\mathcal{V}$ is symmetric monoidal closed (and, of course, complete to ensure that the functor category is defined.) This *strong* form of the enriched Yoneda lemma is then:

Proposition 164 *Let $\mathcal{V}$ be a complete symmetric monoidal closed category, $\underline{F} : \mathcal{A} \to \mathcal{V}$, a $\mathcal{V}$-functor and A, an object of $\mathcal{A}$, then there is a natural isomorphism*

$$\underline{F}A \xrightarrow{\cong} [\mathcal{A},\mathcal{V}](\mathcal{A}(A,-),\underline{F}).$$

■

We will, again, not give a detailed proof as one is readily available in [177], §2.4, but we will note down how this isomorphism is constructed as this is not easily seen as being adapted from the classical setting. It uses the extra properties that have been demanded of $\mathcal{V}$.

First note that, as $\mathcal{V}$ is closed, the 'hom', $\mathcal{V}(B_1,B_2)$ is the set of elements of the 'internal hom' $[B_1,B_2]$ coming from the closed structure, so, for instance, any morphism $B_1 \otimes B_2 \to B_3$ in $\mathcal{V}$ corresponds to a unique morphism $B_1 \to [B_2,B_3]$, and, as $\mathcal{V}$ is also assumed symmetric, we can 'flip' the two inputs of the tensor, $B_1 \otimes B_2$, and get a second morphism $B_2 \to [B_1,B_3]$, Now the $\mathcal{V}$-functor $\underline{F}$ is given by assignment morphisms

$$F_{A_1,A_2} : \mathcal{A}(A_1,A_2) \to [FA_1,FA_2]$$

and hence to $\mathcal{A}(A_1,A_2)\otimes FA_1 \to FA_2$ (giving an 'action-like' version of the definition). Using the 'flip' of the tensor and passing to the adjoint form this gives

$$FA_1 \to [\mathcal{A}(A_1,A_2),FA_2].$$

We use this with $A_1 = A$, the given object of $\mathcal{A}$, varying A_2 and using naturality to obtain a wedge

$$\{FA \to [\mathcal{A}(A,A_2),FA_2] \mid A_2 \in Ob(\mathcal{A})\},$$

and hence the morphism

$$FA \to \int_{A_2} [\mathcal{A}(A,A_2),FA_2] = [\mathcal{A},\mathcal{V}](\mathcal{A}(A,-),\underline{F})$$

required for the statement of the result. The rest of the proof consists of checking that the wedge is universal so this natural morphism is an isomorphism.

(iv) **The 'co-Yoneda lemma' and 'Yoneda reduction':** Before launching into this, it is important to note the use of *powers* and *copowers* that was made previously. We are initially working, for this example, with ordinary standard 'set-enriched' categories and a good intuition about these constructions is useful, even needed, when adapting them for use with enriched categories.

In our earlier discussions, we met the idea of tensors and cotensors in an $\mathcal{S}$-category, (see page 505 in section 11.3.2). Recall that if K was a simplicial set and A an object in an $\mathcal{S}$-category, then the tensor, $K\overline{\otimes}A$, was defined by the following isomorphism of simplicial sets[11]:

$$\underline{\mathcal{A}}(K\overline{\otimes}A, B) \cong \underline{\mathcal{S}}(K, \underline{\mathcal{A}}(A, B)),$$

whilst the cotensor was given by an object, $\overline{\mathcal{A}}(K, A)$ satisfying

$$\underline{\mathcal{A}}(B, \overline{\mathcal{A}}(K, A)) \cong \underline{\mathcal{S}}(K, \underline{A}(B, A)).$$

In our discussion, above, of right Kan extensions as ends, we used a property of powers,

$$\mathcal{E}(E, (E')^X) \cong Sets(X, \mathcal{E}(E, E')).$$

We need to understand this thoroughly. If we look at the family $\{E'_x \mid x \in X\}$, with $E'_x = E'$ for all x in X, and take its product, we get what we have denoted by $(E')^X$, but any morphism $f : E \to (E')^X$ will be determined by the family of its composites with the projections $p_x : (E')^X \to E'$, since $(E')^X$ is, after all, a product. These composites, $f_x = p_x f : E \to E'$, determine a set map / function from X to $\mathcal{E}(E, E')$, indexing the family $\{f_x \mid x \in X\}$. In other words, the power is the '$Sets$-enriched' version of the cotensor and its identification as such is exactly the characterisation of the power as a (particular type of) product.

Dually, we have for a copower,

$$\mathcal{E}(X \cdot E, E') \cong Sets(X, \mathcal{E}(E, E')),$$

because a map from X to $\mathcal{E}(E, E')$ *is* a family $f_x : E \to E' \mid x \in X\}$, and thus uniquely corresponds to a map from the copower, $X \cdot E$, to E'. We have : 'copower' as 'tensor' in the $Sets$-enriched setting.

Although the definition of powers and copowers in a general $\mathcal{V}$-category is probably clear, for the purposes of reference later on, it will be helpful to give the definition explicitly.

Let $\mathcal{V}$ be a closed monoidal category (so that there are internal 'hom-objects') and let $\mathcal{C}$ be a $\mathcal{V}$-category.

Definition: The *power* or *cotensor* of an object, y, in $\mathcal{C}$ by an object, v, in $\mathcal{V}$ is an object, $\overline{\mathcal{C}}(v, y)$, with a natural isomorphism

$$\underline{\mathcal{C}}(x, \overline{\mathcal{C}}(v, y) \cong \underline{\mathcal{V}}(v, \underline{\mathcal{C}}(x, y)),$$

where $\underline{\mathcal{C}}(-,-)$ is the $\mathcal{V}$-valued hom of $\mathcal{C}$ and $\underline{\mathcal{V}}(-,-)$ is that of $\mathcal{V}$.

Alternative notations are sometimes used. As it plays the role of a 'hom', it is sometimes written y^v and, in the defining isomorphism, the right hand side can be written $[v, \underline{\mathcal{C}}(x, y)]$.

[11] so the $\mathcal{A}$-hom is to be the enriched one, as is the $\mathcal{S}$ one, hence the underlining in the formula

We note that, when it exists, the cotensor $\overline{\mathcal{C}}(v, y)$ is determined up to isomorphism. If all such power objects exist in $\mathcal{C}$, then it is said to be *powered* or *cotensored*.

Dually one has the notion of a copower or tensor[12] of an object in $\mathcal{C}$ by one in $\mathcal{V}$.

Definition: The *copower* or *tensor* of an object, $x \in \mathcal{C}$, by an object, k, of $\mathcal{V}$ is an object $k\overline{\otimes}x$ with a natural isomorphism

$$\underline{\mathcal{C}}(k\overline{\otimes}x, y) \cong \underline{\mathcal{V}}(k, \underline{\mathcal{C}}(x, y)).$$

If all copower objects exist in $\mathcal{C}$, then it is said to be *copowered*.

Remark: The terminology of powers and copowers often seems better than that of cotensors and tensors as these latter terms have multiple other uses, which can end up in conflict with their use in this context.

To return to the title given to this example, we will show ends and powers and coends and copowers at worktogether by looking at a version of the well known result that every presheaf on a small category can be canonically presented as a colimit of representable functors. We will look at this in the unenriched, more classical, setting to start with.

Proposition 165 *Let $K : \mathcal{C}^{op} \to Sets$ and $H : \mathcal{C} \to Sets$ be functors with $\mathcal{C}$ a small category, then there are canonical isomorphisms,*

(i) $K \cong \int^c Kc \times \mathcal{C}(-, c)$;

(ii) $K \cong \int_c Kc^{\mathcal{C}(c,-)}$;

(iii) $H \cong \int^c Hc \times \mathcal{C}(c, -)$;

(iv) $H \cong \int_c Hc^{\mathcal{C}(c,-)}$.

Proof: We look at (i) only as (ii) is similar and clearly (iii) and (iv) are versions of (i) and (ii) obtained by replacing $\mathcal{C}$ by its dual.

We use the end/coend calculus as this illustrates this useful style of argument which generalises to both the enriched setting and later to homotopy coherent / ∞-categorical settings. We examine $Sets(\int^c Kc \times \mathcal{C}(-, c), -)$ and compare it with $Sets(K-, -)$, evaluating both on an object y:

$$\begin{aligned} Sets(\int^c Kc \times \mathcal{C}(x, c), y) &\cong \int_c Sets(Kc \times \mathcal{C}(x, c), y) \\ &\cong \int_c Sets(\mathcal{C}(x, c), Sets(Kc, y)), \end{aligned}$$

but this is just $Nat(\mathcal{C}(x, -), Sets(K-, y)$ and the Yoneda lemma says this is $Sets(Kx, y)$. Hence the two representable functors are isomorphic and, as usual, this means the two representing objects are themselves naturally isomorphic. ■

(v) **Left Kan extensions are given by a coend formula:**

[12] This, then, generalises the idea of product.

Let $K : \mathcal{A} \to \mathcal{B}$, $F : \mathcal{A} \to \mathcal{E}$, and write $L = Lan_K F$ for the left Kan extensions of F along K (if it exists).

If E is an object of $\mathcal{E}$ and X is a set, we will use $X \cdot E$ to denote the X-fold *copower* of E, that is the coproduct of the X-indexed family, $\{E_x \mid x \in X\}$, where each $E_x = E$, so it is a constant family.

Proposition 166 *In the above notation, if b is an object of $\mathcal{B}$,*

$$Lb \cong \int^a \mathcal{B}(Ka, b) \cdot Fa.$$

■

The proof is dual to the earlier one for $Ran_K F$, so is **left to you**.

This suggests that there should be enriched Kan extensions using tensors and cotensors to replace copowers and powers respectively. We will not pursue that yet, although we may need to later on.

13.5 Ordinal sum, revisited

With a bisimplicial object, Y, in a category, $\mathcal{C}$, the domain of the functor, which *is* the object Y, is $\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$ and, for simplicial objects, it is $\mathbf{\Delta}^{op}$. We have our ordinal sum functor, $\oplus : \mathbf{\Delta} \times \mathbf{\Delta} \to \mathbf{\Delta}$, (and thus $\oplus^{op} : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathbf{\Delta}^{op}$), which will induce the functor $\mathbf{Dec} : Simp(\mathcal{C}) \to BiSimp(\mathcal{C})$ as $\oplus^*$. Our earlier discussion of such induced functors and their adjoints as given by Kan extensions tells us that provided $\mathcal{C}$ has enough limits and / or colimits, **Dec** have right and / or left adjoints. (We often have $\mathcal{C}$ is both complete and cocomplete, so then it will have both.) For the moment, we want the right adjoint of **Dec**. This will need $\mathcal{C}$ to be complete, so that the end formula works.

13.5.1 The right adjoint of Dec

We set $RY = Ran_{\oplus^{op}} Y$ for Y, a bisimplicial object in $\mathcal{C}$. The relevant diagram is

$$\begin{tikzcd}
\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \arrow[r, "\oplus^{op}"] \arrow[d, "Y"'] & \mathbf{\Delta}^{op} \arrow[dl, dashed, "RY"] \\
\mathcal{C} &
\end{tikzcd}$$

(Note the $\mathbf{\Delta}^{op}$ - not $\mathbf{\Delta}$ - and, as before, take care!)

Our formula for RY gives, for $[n]$,

$$(RY)_n = RY[n] = \int_{[p],[q]} (Y_{p,q})^{\mathbf{\Delta}^{op}([n],[p]\oplus[q])}$$

so we have to start looking at $\mathbf{\Delta}^{op}([n], [p] \oplus [q])$. (Aha! again remember the 'op'!) This is $\mathbf{\Delta}([p] \oplus [q], [n])$, varying with p and q. We need to look at these sets for all p and q.

To make things easier, we start by looking at the case $n = 2$, as that shows how things work.

Suppose $f : [p] \oplus [q] \to [2]$, then the image of p is less than or equal to the images of any element in the $[q]$ part. We must have $f(p) = 0, 1$ or 2, of course. (It will be convenient to write $[p] \oplus [q]$ as the ordinal

$$0 < 1 < \ldots < p < 0' < 1' < \ldots < q'.$$

The 'dashes' or 'primes' are there simply to indicate 'belonging to the second part', so $1'$ means the 1 in $[q]$, whilst 1 means the 1 in $[p]$.

Any such f will factor through at least one of the following three maps:

(i)	(ii)	(iii)
$[0] \oplus [2] \to [2]$	$[1] \oplus [1] \to [2]$	$[2] \oplus [0] \to [2]$
$0 \to 0$	$0 \to 0$	$0 \to 0$
$0' \to 0$	$1 \to 1$	$1 \to 1$
$1' \to 1$	$0' \to 1$	$2 \to 2$
$2' \to 2$	$1' \to 1$	$0' \to 2$

To see why, suppose that $f(p) = 0$, then (a) if $f(0') = f(p)$, f factors as

$$[p] \oplus [q] \overset{f_1 \oplus f_2}{\to} [0] \oplus [2] \overset{(i)}{\to} [2],$$

where $f_1[p] \to [0]$, of course, maps everything to 0, and $f_2 : [q] \to [2]$ with $f_2(0') = 0'$, and (b) if $f(0') > f(p)$, f factors similarly, but with $f_2(0') = 1'$ or $2'$. (**Examine the other possible situations:** $f(0) = 1$ or 2, and in the general case n.) This factorising is not always unique, for instance, $[1] \oplus [0]$ is 'the same as' $[2]$, and we have the identity map

$$\begin{array}{c} 0 \to 0 \\ 1 \to 1 \\ 0' \to 2 \end{array}$$

This factors via (ii)

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \\ 1 \arrow[r] & 1 \arrow[r] & 1 \\ 0' \arrow[dr] & 0' \arrow[r] & 2 \\ & 1' \arrow[r] & 2 \end{tikzcd}$$

but also via (iii),

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \\ 1 \arrow[r] & 1 \arrow[r] & 1 \\ 0' \arrow[dr] & 2 \arrow[r] & 2 \\ & 0' \arrow[r] & 2 \end{tikzcd}$$

Suppose we now look at the equaliser diagram determining the end for RY_2. The elements are determined by three components: $Y_{0,2}$, $Y_{1,1}$, and $Y_{2,0}$. To see why, suppose $\underline{y}$ is in RY_2 as defined, then $\underline{y}$ is an element of the product, $\prod Y_{p,q}$, but the various components are related by the requirement that if $f : [p] \to [p']$ and $g : [q] \to [q']$, then $Y(f \oplus g) : Y_{p',q'} \to Y_{p,q}$ sends $y_{p',q'}$ to $y_{p,q}$. We saw that any $f : [p] \oplus [q] \to [2]$ factors through one of those three cases, hence the $y_{0,2}$, $y_{1,1}$ and $y_{2,0}$ components of y determine *all* the rest. The non-uniqueness of the factorisation implies that these are not independent. The two elements $y_{1,1}$ and $y_{2,0}$ are, for instance, linked by $d_0^v y_{1,1} = d_2^h y_{2,0}$. (Note this important non-uniqueness is always at the interface of the two parts of the sum.)

Of course, something similar happens in the general case as we will see shortly.

13.5.2 The Artin-Mazur codiagonal

This structure, which we have come to via ordinal sum and décalage, seems first to have been noticed by Artin and Mazur, in their paper, [10], on van Kampen's theorem. As it collects up terms along the 'codiagonal' of the 'array', $\{Y_{p,q}\}$, it is sometimes called the *Artin-Mazur codiagonal functor*. This also indicates its relation with the diagonal functor, although the full relationship is somewhat complicated; see the excellent papers by Cegarra and Remedios, [78, 79], but be aware that their $\overline{W}$ is our ∇, defined below. We give a stand-alone definition of this codiagonal functor as, for many of its properties, the combinatorial form of its definition is more immediately useful than the end formula. This provides a good geometric flavoured viewpoint for understanding this functor. As usual, the two viewpoints, combinatorial and categorical, interact and combine to give a clear view of the construction and what makes it 'tick'.

As before, we let $Y = \{Y_{p,q}\}$ be a bisimplicial object in some category, $\mathcal{C}$. We will, more or less, concentrate on the special case of $\mathcal{C} = Sets$, as this is the first one that we will need, but most of the insights are not restricted to that case.

We define a simplicial object, $\nabla(Y)$, as follows:

$$\nabla(Y)_n = \{\underline{y} = (y_0, \ldots, y_n) \mid y_i \in Y_{i,n-i}, d_0^v y_i = d_{i+1}^h y_{i+1} \text{ for } 0 \leq i \leq n-1\}.$$

We thus have $\nabla(Y)_n \subseteq \prod_{i=0}^n Y_{i,n-i}$. This formula works for $\mathcal{C} = Sets$; for other cases, we assume, of course, that $\mathcal{C}$ has finite limits, and then we can form $\nabla(Y)_n$ as a multiple pullback. For example, $\nabla(Y)_2$ is given by pulling back the diagram:

$$\begin{tikzcd}
 & & Y_{0,2} \arrow[d, "d_0^v"] \\
 & Y_{1,1} \arrow[r, "d_1^h"] \arrow[d, "d_0^v"] & Y_{0,1} \\
Y_{2,0} \arrow[r, "d_2^h"] & Y_{1,0} &
\end{tikzcd}$$

For the face and degeneracy maps,

$$d_i(\underline{y}) = (d_i^v y_0, d_{i-1}^v y_1, \ldots, d_1^v y_{i-1}, d_i^h y_{i+1}, \ldots, d_i^h y_n),$$

and

$$s_i(\underline{y}) = (s_i^v y_0, s_{i-1}^v y_1, \ldots, s_0^v y_i, s_i^h y_i, s_i^h y_{i+1}, \ldots, s_i^h y_n).$$

The cases $i = 0$ and n for the face formula, perhaps, need some comment or 'interpretation'. When $i = 0$, the vertical terms are not there, as the last term of the vertical part of the formula is a d_1^v term. This gives

$$d_0(\underline{y}) = (d_0^h y_1, \ldots, d_0^h y_n).$$

Likewise when $i = n$, the 'horizontal terms' do not make sense as they concern parts of $\underline{y}$ that are *not there*! This gives

$$d_n(\underline{y}) = (d_n^v y_0, d_{i-1}^v y_1, \ldots, d_1^v y_{n-1}).$$

Similar comments apply to the degeneracies.

Proposition 167 *For each bisimplicial object, Y, in a complete category, $\mathcal{C}$, there is a natural isomorphism between $\nabla(Y)$ and $Ran_{\oplus^{op}}Y$.* ■

We could almost say: *"The proof is more or less routine, and will be **left to the reader**."* There are, however, several steps to go through which need noting if the properties of ∇ are to be fully appreciated. In fact, there are several distinct ways of proving this and it is suggested that they each be 'tried out'. Some of these need some work and thought, and some of that *will* be **left to the reader**. Later on in this section we will examine some of these in a bit more detail, but it will help if you have tried each of the methods prior to that.

Preliminary point: Check that $\nabla(Y)$ *is* a simplicial set, then:

Method 1: Prove that ∇, as defined above, is right adjoint to **Dec**. To do this look at a bisimplicial morphism

$$f_{..} : \mathbf{Dec}(X)_{..} \to Y_{..},$$

and construct an adjoint morphism

$$\underline{f}_{.} : X_{.} \to \nabla(Y)_{.},$$

and conversely, starting with a

$$g_{.} : X_{.} \to \nabla(Y)_{.},$$

construct an adjoint

$$\overline{g}_{..} : \mathbf{Dec}(X)_{..} \to Y_{..}.$$

Of course, the constructions should be 'natural' in both the standard and the categorical senses of the term, and be mutually inverse.

Method 2: Alternatively, try to construct, directly, the unit and counit of the adjunction:

$$\eta_X : X \to \nabla\mathbf{Dec}(X)$$

and

$$\varepsilon_Y : \mathbf{Dec}\nabla(Y) \to Y.$$

We will return to this in some more detail shortly.

Method 3: We have the end formula for $Ran_{\oplus^{op}}Y$ and we have sketched an argument that shows

$$(RY)_n = (Ran_{\oplus^{op}}Y)_n \subseteq \prod_{i=0}^{n} Y_{i,n-i}.$$

A bit more detailed look at this will show that the two subsets $(RY)_n$ and $\nabla(Y)_n$ coincide. The next step is to examine what the face and degeneracy maps look like. For this, given an indexing map, $f : [p] \oplus [q] \to [n]$, which can be assumed to be one of those 'special' ones with $p + q = n$, compose f with one of the standard, $\delta_i : [n] \to [n+1]$ or $\sigma_i : [n] \to [n-1]$, and check the factorisations. The rest is routine checking.

Remark: This codiagonal is one of several 'total complex' constructions on bisimplicial sets. Another is the diagonal; cf. page 234. We will also meet and need yet other total complexes, which start with a cosimplicial simplicial object, or similar. We will introduce these slightly later.

The formula for the codiagonal given by the above proposition makes it very feasible to calculate such codiagonals in some important and interesting cases. The first that we will look at can be thought of as an amusing example, but, in fact, has some very useful consequences that will be needed later on.

Definition: If we have two simplicial sets, K and L, we can form a bisimplicial set, $K \times^{(2)} L$, simply by taking

$$(K \times^{(2)} L)_{p,q} = K_p \times L_q$$

with

$$d_i^h(x, y) = (d_i x, y)$$
$$d_j^v(x, y) = (x, d_j y)$$

and similarly for the degeneracies. This will be called the *double product of K and L.*

This is not that good a term for this construction, but it is difficult to find a better one! There are variants of this construction in other contexts. As a first generalisation we could replace *Sets* by any category, $\mathcal{C}$, with finite products (or, better, finite limits as we will hope to have a ∇ as well). We will extend the term 'double product' to handle this case as well.

Another further generalisation replaces $\mathcal{C}$ by a monoidal category, $(\mathcal{C}, \otimes, I)$, (again with finite limits, so as to be able to construct ∇).

Definition: Given a monoidal category, $(\mathcal{C}, \otimes, I)$, and two simplicial objects in $\mathcal{C}$, K and L, their *double tensor product*, $K \otimes^{(2)} L$, is the bisimplicial object having

$$(K \otimes^{(2)} L)_{p,q} = K_p \otimes L_q$$

with

$$d_i^h = d_i \otimes id_L,$$
$$d_j^v = id_K \otimes d_j,$$
$$s_i^h = s_i \otimes id_L,$$
$$s_j^v = id_K \otimes s_j.$$

Examples: (i) Of course, taking $\mathcal{C}$ to have its product as its monoidal structure, we just retrieve the double product.

(ii) Of particular use is the case of $\Bbbk$-modules, (for a commutative ring, $\Bbbk$), with the usual tensor product. This is useful because of the following, which is more or less immediate:

Lemma 101 *The bisimplicial $\Bbbk$-linearisation functor,* i.e.*,*

$$\Bbbk(-) : Bi\mathcal{S} \to BiSimp.\Bbbk\text{−}Mod$$

sends $X \times^{(2)} Y$ to $\Bbbk^{(X)} \otimes^{(2)} \Bbbk^{(Y)}$, up to isomorphism. ∎

Going back to the *Sets*-based case, we have our bisimplicial set, $K \times^{(2)} L$, for simplicial sets K and L. It is interesting, therefore, to ask what is $\nabla(K \times^{(2)} L)$.

Proposition 168 *For simplicial sets, K and L, there is a natural isomorphism,*

$$K \times L \stackrel{\cong}{\to} \nabla(K \times^{(2)} L).$$

Proof: We start by examining an element $\underline{y} = (y_0, \ldots, y_n) \in \nabla(K \times^{(2)} L)$, where the $y_i \in (K \times^{(2)} L)_{i,n-i} = K_i \times L_{n-i}$ and which satisfy those conditions that we listed earlier. We thus have $y_i = (k_i, \ell_{n-i})$. The conditions on the y_i include, for instance, that $d_1^h y_1 = d_0^v y_0$, but this means

$$(d_1 k_1, \ell_{n-1}) = (k_0, d_0 \ell_n),$$

so $k_0 = d_1 k_1$ and $\ell_{n-1} = d_0 \ell_n$. Continuing like this, one sees that

$$k_i = d_{i+1} k_{i+1} = d_{i+1} d_{i+2} k_{i+2} = \ldots = d_{i+1} \ldots d_n k_n,$$

whilst $\ell_{n-i} = (d_0)^i \ell_n$, and so $\underline{y}$ is completely determined by $(k_n, \ell_n) \in (K \times L)_n$.

We now define the map from $K \times L$ to $\nabla(K \times^{(2)} L)$ by sending $(k_n, \ell_n) \in (K \times L)_n$ to $\underline{y} = (y_0, \ldots, y_n)$, where $y_i = (k_i, \ell_{n-i})$ and $k_i = d_{i+1} \ldots d_n k_n$, $\ell_{n-i} = (d_0)^i \ell_n$. It is routine to check that this is a simplicial map and it is clear that it is an isomorphism by our earlier calculation. ∎

We can extend the above to give

Proposition 169 *For simplicial $\Bbbk$-modules, A and B, there is a natural isomorphism,*

$$A \otimes B \stackrel{\cong}{\to} \nabla(A \otimes^{(2)} B).$$

∎

The idea of the proof is essentially 'the same' as the previous one. In fact, there is a morphism, more generally, for simplicial objects, A and B, in a general monoidal category $\mathcal{C}$, from $A \otimes B$ to $\nabla(A \otimes^{(2)} B)$. In many useful cases, this is found to be an isomorphism, but it somehow seems unlikely that it is one in general without some special conditions on $\mathcal{C}$ and its tensor.

The second example of the use of the codiagonal formula is in a situation that we have met before. The formulae for $\nabla(Y) \subseteq \prod Y_{p,q}$ look very like those for the classifying space, $\overline{W}(G)$, for a simplicial group or $\mathcal{S}$-groupoid, G. (We saw these earlier, initially in section 6.2.3, page 254, then in section 8.3.2, starting page 377, for use with the long Puppe sequence method. We have given the usual formula twice, so will resist the temptation to give it a third time! We have also looked at a conjugate form, in section 11.6.3, starting on page 551.) That construction would seem to combine ideas from the nerve construction with those of the codiagonal. In fact, if G is a constant simplicial group(oid) with 'value' also denoted by G, then $\overline{W}G$ is simply $Ner(G)$ as all the face and degeneracy maps are the identity morphism of G.

Putting that aside for a moment, we could approach this from another direction. If we have a group or groupoid, G, we *can* construct $Ner(G)$, which is a simplicial set. Of course, Ner is a functor from $Grpd$ to $\mathcal{S}$. Any simplicial group or simplicially enriched groupoid (or, more generally, any simplicial groupoid) gives, or, perhaps better, *is* a functor $G : \mathbf{\Delta} \to Grpd$. If we compose this with $Ner : Grpd \to \mathcal{S}$, we get a bisimplicial set. All we have done here is to apply the nerve functor to each dimension of G, so we have

$$Ner(G)_{p,q} = Ner_p(G_q)$$

or, equally validly, $Ner_q(G_p)$, picking whichever order seems more appropriate or accords better with other formulae that you are using at the time.

For us here, it is very natural to try to calculate $\nabla Ner(G)$.

We will assume, for simplicity, that we have a simplicial group, G, as the groupoid case is very slightly more complex. We will examine an element $\underline{y} = (y_0, \ldots, y_n) \in \nabla Ner(G)_n$. We have $y_i \in Ner(G_i)_{n-i}$, so we can write it as a sequence,

$$y_i = (g_{i,1}, \ldots, g_{i,n-i})$$

of elements of G_i. (Note that the case $i = n$ is special, as $Ner(G_n)_0$ is a single point, namely the single object of 'G as a category'.) We have that $d_0^v y_i = d_{i+1}^h y_{i+1}$. We thus have

$$(g_{i,2}, \ldots, g_{i,n-i}) = (d_{i+1} g_{i+1,1}, \ldots, d_{i+1} g_{i+1,n-i-1}),$$

so the only component of y_i not predetermined by y_{i+1} is $g_{i,1}$. Leaving the $i = n$ case **for you** to consider, we have that $\underline{y}$ corresponds to a sequence $(g_{0,1}, g_{1,1}, \ldots, g_{n-1,1})$ with $g_{i,1}$ an element of G_i, so it corresponds to an element of $\overline{W}(G)_n$. We will also **leave to you** the task of checking that the face and degeneracy operators *do* match up, thus proving

Proposition 170 *There is a natural isomorphism*

$$\nabla Ner(G) \xrightarrow{\cong} \overline{W}(G).$$

■

Remark: How does one get the conjugate form of this? **This is left to you to investigate.**

We now return to the unit, $\eta_X : X \to \nabla\mathbf{Dec}(X)$, of the adjunction of Proposition 167 as this is useful and tells us quite a lot about the reasons that ∇ works.

We look at η_X in low dimensions to see how it might be constructed. This examination shows, in fact, that, as one might expect, examination of what it must do and its 'naturality', both in the intuitive and categorical sense, end up by handing us its definition 'on a plate' and telling us about the key role of various features of the construction of ∇.

In dimension 0, $(\eta_X)_0 : X_0 \to X_1$ and so 'must' be s_0, as that is the only thing in sight that has the right domain and codomain. (That is not a proof, but *is* a good indication.)

In dimension 1, $\nabla(\mathbf{Dec}(X))$ is given by a pullback. We first display this for a general bisimplicial object, Y, in place of $Dec(X)$. The diagram is

$$\begin{tikzcd} & Y_{0,1} \arrow[d, "d_0^v"] \\ Y_{1,0} \arrow[r, "d_1^h"'] & Y_{0,0} \end{tikzcd}$$

The pullback of this is $\nabla(Y)_1$. Now, substituting $Dec(X)$ for Y and using our earlier identification of the faces and degeneracies of that bisimplicial object, we obtain the corresponding diagram, which is

$$\begin{tikzcd} & X_2 \arrow[d, "d_1"] \\ X_2 \arrow[r, "d_1"'] & X_1, \end{tikzcd}$$

with $\nabla(Dec(X))_1$ being its pullback. We are looking for a map from X_1 to this pullback and so need a pair of maps from X_1 to X_2 making the resulting square commute. Given the suggested map at the zeroth level, we here try the pair (s_0, s_1). It works and the verification comes down to the simplicial identity: $d_1s_0 = d_1s_1 = id$. In other words,

$$(\eta_X)_1 : X_1 \to (\nabla\mathbf{Dec}(X))_1$$

works because in $\mathbf{\Delta}$, the identity on [1] factorises in these two different ways. That reminds one of the fact that $id_{[1]} : [0] \oplus [0] \to [1]$ factorises in two ways via the maps of form $[p] \oplus [q] \to [1]$ with $p + q = 1$ and from our 'list', which in this dimension has just two maps in it. This is not a coincidence, as the simplicial identity $d_is_i = d_is_{i-1} = id_i$ is essentially the same fact as the factorisation that we noted earlier. This observation gives us the key to the construction in general. It suggests that we try

$$(\eta_X)_n : X_n \to (\nabla\mathbf{Dec}(X))_n$$

as being given by, for $x \in X_n$,

$$\eta_X(x) = (s_0x, \ldots, s_nx).$$

It is easy to check that this does map into $(\nabla\mathbf{Dec}(X))_n$, *i.e.*, that it does satisfy $d_0^v y_i = d_{i+1}^n y_{i+1}$, and it is useful to see this both as part of the limit / multiple pullback defining $(\nabla\mathbf{Dec}(X))_n$ and as a consequence of the factorisation of the various identities in $\mathbf{\Delta}$. To make this last point even clearer, we note that in dimension 2, the relevant diagram is a case of the one we saw earlier:

$$\begin{tikzcd}
X_2 \arrow[rr, dashed, "s_2"] \arrow[dr, dashed, "s_1"] \arrow[dd, dashed, "s_0"'] & & X_3 \arrow[d, "d_2"] \\
& X_3 \arrow[r, "d_2"] \arrow[d, "d_1"] & X_2 \\
X_3 \arrow[r, "d_1"] & X_2 &
\end{tikzcd}$$

expressing both the simplicial identities that we noted. This induces a unique map to the limit of the 'stairway' on the bottom right, *i.e.*, to $(\nabla\mathbf{Dec}(X))_2$.

Of course, all that we have done is to show that there is this naturally constructed map, $\eta_X : X \to \nabla(\mathbf{Dec}(X))$. In fact, we have not even done this, as we have to check it is a simplicial morphism, but that is 'routine', *i.e.*, **left to the reader**. It is still necessary, after that, to check that it is the unit of the adjunction.

There are, again, several ways to do this. We will look at one. Using our $\eta_X : X \to \nabla(\mathbf{Dec}(X))$, we can induce a map

$$Bi\mathcal{S}(\mathbf{Dec}(X), Y) \to \mathcal{S}(X, \nabla Y)$$

by sendiing

$$f = (f_{p,q} : \mathbf{Dec}(X)_{p,q} \to Y_{p,q})$$

to

$$\tilde{f} : X \to \nabla Y,$$

where $\tilde{f}_n : X_n \to (\nabla Y)_n$ is given by

$$\tilde{f}_n(x) = (f_{0,n}s_0(x), \ldots, f_{p,n-p}s_p(x), \ldots, f_{n,0}s_n(x)).$$

(Checking that this is a simplicial map is routine. This is **left to you**.)

We also need to check that it ends up where we want it to. This again is easy, but as we will need it explicitly in a moment, we will give this. We have to check the defining condition $d_0^v y_i = d_{i+1}^h y_{i+1}$. Here this comes out to be an examination of the $d_0^v f_{p,n-p} s_p(x)$ and we can calculate, using our knowledge of the vertical face maps of **Dec**: as usual $n = p + q$,

$$\begin{aligned} d_0^v f_{p,n-p} s_p(x) &= f_{p,n-p-1} d_0^v s_p(x) \\ &= f_{p,n-p-1} d_{p+1} s_p(x) \\ &= f_{p,n-p-1}. \end{aligned}$$

We can also calculate $d_{p+1}^h f_{p+1,n-p-1} s_{p+1}(x)$ and it is fairly obviously the same. Is this correspondence sending f to $\tilde{f}$ a bijection? Note that $\tilde{f}$ can also be written as

$$\tilde{f} = \nabla(f)\eta_X,$$

so if η_X is the unit of the adjunction, this correspondence will be, in fact, *must be*, a bijection. We therefore, in part, have to see if, given $\tilde{f}$, we can retrieve f, but our calculation just now suggests how to do this. Given $\tilde{f}$, we can calculate $d_0^v \tilde{f}_{p,q+1}$. This has component $\tilde{f}_{p,n+1-p} : Dec(X)_{q,n+1} \to Y_{p,q+1}$ and so $d_0^v \tilde{f}_{p,q+1} = f_{p,q} : Dec(X)_{p,q} \to Y_{p,q}$.

The correspondence is thus one-to-one. We can use the same means to show it is onto, but this is **left to you**. Note that what we have really done is we have calculated the counit of the adjunction

$$\varepsilon : \mathbf{Dec}\nabla Y \to Y$$

and have used that. If you look at $\mathbf{Dec}\nabla Y_{p,q}$, it is ∇Y_{p+q+1}, and so an element of this will be a string, $(y_0, \ldots, y_{p+q+1})$, with $y_k \in Y_{k,p+q+1}$ and $d_0^v y_i = d_{i+1}^h y_{i+1}$. We therefore have $y_p \in Y_{p,q+1}$, and we need to get to $Y_{p,q}$. The obvious thing to do is to use a vertical face to get there. The only sensible choice is the d_0^v. Why? Simply that we could also have argued that $y_{p+1} \in Y_{p+1,q}$ and have used a horizontal face. This would have given lots of feasible answers to the problem, but the condition $d_0^v y_i = d_{i+1}^h y_{i+1}$ tells us that two of these coincide, namely the $d_0^v y_p$ and $d_{p+1}^h y_{p+1}$. This therefore is the sensible thing to check if it works. (It would be very strange if it did not!) As usual, this just suggests what ε should be. The details that this suggestion is the right one still need checking through, but, of course, **that can be left to you**. We record the unit and counit for future use:

Lemma 102 *In the adjunction between* **Dec** *and* ∇*, the unit*

$$(\eta_X) : X_n \to (\nabla\mathbf{Dec}(X))$$

is being given by, for $x \in X_n$*,*

$$\eta_X(x) = (s_0 x, \ldots, s_n x),$$

and the counit,

$$\varepsilon : \mathbf{Dec}(\nabla Y) \to Y$$

by

$$\varepsilon(y_0, \ldots, y_{p+q+1}) = d_0^v(y_p).$$

■

13.6 Total complex constructions: part 2

In many situations in homotopical and homological algebra, and also in homotopy theory as such, constructions lead to multicomplexes or multisimplicial sets. Most often 'multi-' is 'bi-', that is to say, either bicomplexes, as we have seen earlier, or one of the three variants: bisimplicial sets, cosimplicial simplicial sets or bi-cosimplicial sets, although this last is fairly rare in what we will be looking at. Other instances of the same sort of objects would be simplicial or cosimplicial objects in simplicially enriched categories, since, if $\mathcal{B}$ is such a category with B, an object, and X, a simplicial object in $\mathcal{B}$, then $\mathcal{B}(B,X)$ will be a bisimplicial set. If X had been a cosimplicial object, then $\mathcal{B}(B,X)$ would be a cosimplicial simplicial set, and on. Of particular interest, for our main themes, are the cases of simplicial chain complexes and simplicial crossed complexes, as they are slightly simpler than more general situations, yet have a good range of applications.

In all these cases it is useful to find a simplicial object that encodes the 'important' information on the 'multi-objetc', that is, a total complex of some sort. We have already seen several of these.

In this section we will look at some of the better known examples of total complexes and also at some important applications of their construction.

13.6.1 Total complexes of cosimplicial simplicial sets

As we mentioned in our earlier discussion of total complexes, which was centred on the bicomplex situation, a 'total complex' construction reduces the 'bi-thing' to a 'thing'. The case we saw then was from bicomplexes to complexes, but clearly $\nabla : Bi\mathcal{S} \to \mathcal{S}$ and $diag : Bi\mathcal{S} \to \mathcal{S}$ are other examples of the same type.

We will start this second section on total complexes with a way to get from cosimplicial simplicial sets to $\mathcal{S}$ itself.

Definition: (i) Suppose $X : \mathbf{\Delta} \to \mathcal{S}$ is a cosimplicial object in $\mathcal{S}$, then we define

$$Tot(X) = Nat_{\mathcal{S}}(\Delta, X),$$

the simplicial set of natural transformations from the functor $\Delta : \mathbf{\Delta} \to \mathcal{S}$ to X. (The functor, Δ, sends $[n]$ to $\Delta[n]$.) The simplicial set, $Tot(X)$, is called the *Bousfield-Kan total complex* of X.

We have

$$Tot(X) = \int_{[n]} \underline{\mathcal{S}}(\Delta[n], X^n_*).$$

Following our comments above, we can extend this definition as follows:

Definition: (ii) Suppose that $\mathcal{B}$ is a complete simplicially enriched category (hence cotensored) and X is a cosimplicial object in $\mathcal{B}$, then we set

$$Tot(X) = \int_{[n]} \overline{\mathcal{B}}(\Delta[n], X^n_*).$$

This is called the *enriched total object* of X.

We will need this total complex construction later when we come to discuss homotopy limits, especially with regard to '(co)-simplicial replacement'.

13.6.2 ∇ and Tot

We mentioned that ∇ should also be considered to be a total complex construction. It applies to bisimplicial objects and, in the case of bisimplicial Abelian groups, we clearly should look for connections between it and the 'Tot' for bicomplexes, since $Simp.Ab$ and $Ch_{\mathbb{Z}}$ are so closely related by the classical Dold-Kan theorem, (page 252). Also recall that one of our aims in examining these total complex constructions was to understand the passage from simplicial enrichments, *i.e.*, enrichment over the monoidal category, $(\mathcal{S}, \times, \Delta[0])$, to that over $Ch_{\mathbb{Z}} = dg{-}Ab = (dg^+, \otimes, \mathbb{Z})$.

We first look at passing from $Bi\mathcal{S}$ to $BiCh_{\mathbb{Z}}$. Let X be a bisimplicial set. We form a bicomplex, $C^{(2)}(X)$, as follows:

$$C^{(2)}(X)_{p,q} = \mathbb{Z}(X_{p,q}), \text{ or, if you prefer, } \mathbb{Z}^{(X_{p,q})},$$

that is the free Abelian group on the set, $X_{p,q}$, of (p,q)-bisimplices;

$$\partial^h(x) = \sum_{i=0}^{p}(-1)^i d_i^h(x),$$

and

$$\partial^v(x) = \sum_{j=0}^{q}(-1)^j d_j^v(x),$$

for $x \in X_{p,q}$.

Of course, here we are really abusing notation by identifying an element of $X_{p,q}$ with the corresponding basis element of $C^{(2)}(X)_{p,q}$, but this is customary and should cause no problem.

It is clear that $C^{(2)}(X)$ is a bicomplex. We can thus form $Tot(C^{(2)}(X))$. This is particularly interesting if $X = K \times^{(2)} L$, the 'double product' of two simplicial sets, K and L, since then

$$\begin{aligned} C^{(2)}(K \times^{(2)} L)_{p,q} &\cong \mathbb{Z}(K_p \times L_q) \\ &\cong C(K)_p \otimes C(L)_q, \end{aligned}$$

and checking up on boundaries,

$$C^{(2)}(K \times^{(2)} L) \cong C(K) \otimes^{(2)} C(L),$$

the double tensor product of $C(K)$ and $C(L)$.

Going back to the general case, this gives us *two* ways of going from a bisimplicial set to a chain complex, since we could equally well go from X to $\nabla(X)$ and then form $C(\nabla(X))$, which raises the interesting question of the relationship between them.

Proposition 171 *For X, a bisimplicial set, there is a natural transformation*

$$\theta_X : C(\nabla(X)) \to Tot(C^{(2)}(X)),$$

given, in dimension n, by

$$\theta_X(x_0, \ldots, x_n) = (x_0, \ldots, x_n).$$

Proof: Naturality will be clear if this formula makes sense!

We start by seeing what it means:

On the left, $\underline{x} = (x_0, \ldots, x_n) \in \nabla X_n$, so $x_i \in X_{i,n-i}$, and is a generator of $C(\nabla(X))_n$.

On the right,

$$Tot(C^{(2)}(X)) = \bigoplus_{j=0}^{n} \mathbb{Z}(X_{j,n-j}),$$

so an element of this is an $(n+1)$-tuple, $(c_0, c_1, \ldots, c_n)$ with $c_j \in \mathbb{Z}(X_{j,n-j})$. We thus have that the right hand side of the formula for θ_X interprets the simplex x_j as a generating element of the Abelian group, $\mathbb{Z}(X_{j,n-j})$, so, as a graded morphism, the formula for θ_X does make sense. We just have to check boundaries.

The boundary of the generator, $\underline{x}$, will be

$$\partial \underline{x} = \sum (-1)^i d_i \underline{x},$$

so we need to recall $d_i \underline{x}$ for each i. The general form was

$$d_i \underline{x} = (d_i^v x_0, d_{i-1}^v, \ldots, d_1 x_{i-1}^v, d_i^h x_{i+1}, \ldots, d_i^h x_n),$$

where the exceptional cases of $i = 0$ and $i = n$ need a bit of care, so $d_0 \underline{x} = (d_0^h x_1, \ldots, d_i^h x_n)$, and $d_n \underline{x} = (d_n^v x_0, d_{i-1}^v, \ldots, d_1 x_{n-1}^v)$.

We thus have

$$\partial \underline{x} = (y_0, \ldots, y_{n-1}),$$

where

$$y_p = \sum_{j=0}^{p} (-1)^j d_j^h x_{p+1} + (-1)^p \Big(\sum_{k=1}^{n-p} (-1)^k d_k^v x_p \Big).$$

For the other expression, *i.e.*, for $\partial \theta(\underline{x})$, we have that it has form $(z_0, \ldots, z_{n-1})$, where

$$\begin{aligned} z_p &= \partial^h x_{p+1} + (-1)^p \partial^v x_p \\ &= \sum_{j=0}^{p+1} (-1)^j d_j x_{p+1} + (-1)^p \sum_{k=0}^{n-p} (-1)^k d_k^v x_p. \end{aligned}$$

Although there initially seems to be two extra terms in this, if we remember that $d_0^v x_p = d_{p+1}^h x_{p+1}$, we see that they cancel, so θ_X *is* a chain map. ∎

13.6.3 The Alexander-Whitney map and its applications

Corollary 29 *There is a natural chain map,*

$$a_{K,L} : C(K \times L) \to C(K) \otimes C(L).$$

Proof: This just puts the pieces together for the case $X = K \times^{(2)} L$. We saw that $\nabla(K \times^{(2)} L) \cong K \times L$ and that $Tot(C^{(2)}(K \times^{(2)} L)) \cong C(K) \otimes C(L)$. ∎

This natural map is usually called the *Alexander-Whitney map*.

We can use our earlier calculations to give this explicitly. A generator of $C(K\times L)_n$ is of form (k_n,ℓ_n) and the corresponding element of $\nabla(K\times^{(2)}L)$ is $(x_0,\ldots,x_n)$, where $x_i=(k_i,\ell_{n-i})$ with $k_i=d_{i+1}\ldots d_nk_n$ and $\ell_{n-i}=(d_0)^i\ell_n$. This gives

$$a_{K,L}(k_n,\ell_n)=\sum_{i=0}^{n}d_{i+1}\ldots d_nk_n\otimes(d_0)^i\ell_n,$$

which is the usual formula for the Alexander-Whitney map that one finds, for instance, in Mac Lane's text on Homology, [191], or May's book on Simplicial Homotopy Theory, [198]. It provides, in the case $K=L$, an 'approximation to the diagonal', since $K\to K\times K$, given by the obvious map sending x to (x,x), yields $C(K)\to C(K\times K)$ and composing with $a_{K,K}$ gives a 'pseudo-diagonal', $C(K)\to C(K)\otimes C(K)$. This gives rise to the 'cup product' in cohomology, (see the discussion in Mac Lane, [191], for instance) and will also be useful slightly later in this section.

Starting with, instead of simplicial sets, K and L, two simplicial R-modules, A and B, and, as before, write (A,∂) for the corresponding chain complex, *i.e.*, the same graded module with differential $\partial=\sum(-1)^id_i$. The above formulae give us a natural map

$$a_{A,B}:(A\times B,\partial)\to(A,\partial)\otimes(B,\partial).$$

There are several questions that need raising here - not all of which we will answer, since answers are relatively easy to find in the literature. The first is whether this is an 'associative' type of operator. By this vague wording, we mean the following:
If we have three simplicial sets, K, L and M, then we have various possible ways to go from $C(K\times L\times M)$ to $C(K)\otimes C(L)\otimes C(M)$, even ignoring the 'associators' on the Cartesian and tensor products. First we could bracket the product as $(K\times L)\times M$ and go

$$C(K\times L\times M)\to C(K\times L)\otimes C(M)\to C(K)\otimes C(L)\otimes C(M),$$

or we could use $K\times(L\times M)$. A third, perhaps more interesting or amusing approach would be to form a trisimplicial set with $(K\times L\times M)_{p,q,r}=K_p\times L_q\times M_r$, develop a functor generalising ∇ and then, using a tricomplex and a total complex construction for such, go *directly* from $C(K\times L\times M)$ to $C(K)\otimes C(L)\otimes C(M)$. This would undoubtably work - although one would be deprived of 'h' and 'v' as indicators of direction in the formulae! We will not explore this. We content ourselves with a summary proposition whose proof is **left to you** to provide. (It can be found in several of the sources already mentioned, but is not too difficult in any case.)

Proposition 172 *For simplicial sets, K, L and M, the following diagram commutes:*

$$\begin{tikzcd}
C(K\times L\times M) \arrow[r, "a_{K\times L,M}"] \arrow[d, "a_{K,L\times M}"'] & C(K\times L)\otimes C(L) \arrow[d, "a_{K,L}\otimes id"] \\
C(K)\otimes C(L\times M) \arrow[r, "id\otimes a_{L,M}"'] & C(K)\otimes C(L)\otimes C(M)
\end{tikzcd}$$

■

The proof is a routine exercise in manipulating terms.

When we looked at the shuffle map, (page 599),

$$b_{K,L} : C(K) \otimes C(L) \to C(K \times L),$$

we were able to check that, if we restricted to the normalised complexes, $C_N(K)$ etc., then $b_{K,L}$ gave a morphism on these, that is,

$$b_{K,L} : C_N(K) \otimes C_N(L) \to C_N(K \times L).$$

This suggests a second question: does $a_{K,L}$ do something similar? Does it pass to the normalised complexes? The answer is both no and yes! If we start with an element of $C_N(K \times L)$, then the result of applying $a_{K,L}$ may result in terms other than those in the subcomplex $C_N(K) \otimes C_N(L)$ of $C(K) \otimes C(L)$, so we have to work harder to analyse that situation.

First note that $C(K)_n$ is isomorphic to $C_N(K)_n \oplus D(K)_n$, where $D(K)_n$ is the subgroup generated by the degenerate elements. (Remember, $C_N(K)$ is the same as the Moore complex of the simplicial Abelian group, $\mathbb{Z}(K)$, and the Dold-Kan decomposition gives the direct sum decomposition that we are using here, cf. page 252. We saw this in the non-Abelian case with the Conduché decomposition, again on page 251, of G_n as a semidirect product of degenerate terms with the Moore complex term, NG_n. The situation in our Abelian case is much less complex ..., but that raises interesting questions about adapting this to at least a slightly non-Abelian context.)

We can identify $C_N(K)$ with $C(K)/D(K)$ and can realise $C_N(K) \otimes C_N(L)$ as

$$(C(K) \otimes C(L))/(D(K) \otimes C(L) \oplus C(K) \otimes D(L)).$$

If we now look at $a_{K,L}$, it does not matter if $a_{K,L}(x_n, y_n)$ has component terms that involve degeneracies provided that, if $(x_n, y_n) = s_i(x_{n-1}, y_{n-1})$ is itself a degenerate element, then $a_{K,L}(x_n, y_n)$ is also a sum of 'degenerate' terms. In other words, we do not need $a_{K,L}$ to *restrict* to a map from $C_N(K \times L)$ to $C_N(K) \otimes C_N(L)$ for it *induce* such a map. The condition on images of degenerate elements is then needed to ensure well definition of the induced map. We have:

Proposition 173 *The Alexander-Whitney map, $a_{K,L}$, induces a natural transformation (which will also be denoted $a_{K,L}$) from $C_N(K \times L)$ to $C_N(K) \otimes C_N(L)$.*

Proof: Following on from the argument preceding the statement of the result, we need to check that if (x, y) is degenerate, then its image under the original $a_{K,L}$ will be a sum of terms which are in $D(K) \otimes C(L) \oplus C(K) \otimes D(L)$, so that $a_{K,L}$ will be a well defined map on the quotient $C(K \times L)/D(K \times L)$.

Suppose that $(x_n, y_n) \in (K \times L)_n$ is degenerate, say, $(x_n, y_n) = (s_k x_{n-1}, s_k y_{n-1})$ and we consider it as a generating element of $C(K \times L)_n$. We examine the i^{th} term of $a_{K,L}(x_n, y_n)$:

- If $i \geq k$, we have
$$d_{i=1} \ldots d_n s_k x_{n-1} = s_k d_i \ldots d_{n-1} x_{n-1},$$
and so is in $D(K)_n$. Hence these terms of $a_{K,L}(x_n, y_n)$ are in $D(K) \otimes C(L)$.

- If, on the other hand, $i < k$,
$$(d_0)^i s_k y_{n-1} = s_{k-i}(d_0)^i y_{n-1}$$
is in $D(L)$, so the corresponding term of $a_{K,L}(x_n, y_n)$ is in $C(K) \otimes D(L)$,

thus $a_{K,L}(x_n, y_n)$ is within the 'degenerate' part of $C(K) \otimes C(L)$, and $a_{K,L}$ induces a map on the normalised complexes. ■

Remark: This proof is well known and can easily be found in many sources, including those we have given above. We could therefore have left it to the reader to look up, however the approach we have given has been chosen so as to make the transition from this Abelian case to that of crossed complexes as smooth as possible. This then prepares us better for the challenging and as yet unresolved problems of less 'Abelian' situations. We will give a résumé of the crossed complex theory shortly.

13.6.4 The classical Eilenberg-Zilber theorem

The classical Eilenberg-Zilber theorem relates $C_N(K \times L)$ and $C_N(K) \otimes C_N(L)$, or, more or less equivalently, the corresponding unnormalised complexes. We will not give a complete proof of that theorem as (i) it is easy to find in many textbooks, (ii) the usual proofs use the method of acyclic models, which although useful, can also be found in many sources, and (iii) we will not need the details of the proof, much beyond what we have already provided, in what follows. Some parts of the theorem are worth noting in some detail, however:

Proposition 174 *If K and L are simplicial sets, then the two maps, $a_{K,L}$ and $b_{K,L}$, on normalised complexes satisfy $a_{K,L} \circ b_{K,L} = id$.*

Proof: Suppose that $x_p \otimes y_q$ is a generator of $C(K) \otimes C(L)$ in dimension $n = p+q$, then $a(b(x_p \otimes y_q))$ is given by a sum of terms of form

$$(\pm 1)\, d_{i+1} \dots d_n s_{\sigma_0} x_p \otimes (d_0)^i s_{\sigma_1} y_q,$$

where $\sigma \in \mathit{Shuff}(p, q)$, and σ_0 and σ_1 are the sequences corresponding to $\sigma(\mathbf{q})$ and $\sigma(\mathbf{p})$. (Look back at section 12.2.2 if you need to.)

For the $(d_0)^i s_{\sigma_1} y_q$ term to be non-degenerate requires that $\sigma(k) \leq i - 1$ for $k \leq p - 1$ and for $d_{i+1} \dots d_n s_{\sigma_0} x_p$ to be non-degenerate $\sigma(k) \geq i$ for $k > p$. The whole term is thus non-degenerate only if σ is the identity shuffle and $i = p$. In this case $sgn(\sigma) = 1$, of course, and the term is $x_p \otimes y_q$. As, in the normalised complex, degenerate terms will be trivial, this completes the proof. ■

We thus have our two maps, $a_{K,L}$ and $b_{K,L}$, satisfying the above, so $b_{K,L}$ is a monomorphism, split by $a_{k,L}$, and the only other feature of the (normalised version of the) Eilenberg-Zilber theorem is the fact the other composite $b_{K,L} \circ a_{K,L}$ is *homotopic* to the identity on $C_N(K \times L)$. (In the un-normalised form of the theorem both this composite and $a_{K,L} \circ b_{K,L}$ are homotopic to the respective identities. This form can be retrieved from the normalised one, since $C_N(K) \to C(K)$ is a homotopy equivalence.) We will not give the homotopy, but refer to the literature for it. It is usually shown just to exist by the acyclic models method, but Eilenberg and Mac Lane gave an in-depth analysis of that and produced a formula for it. Some indications on how to do this are given in Mac Lane's book, [191], whilst a crossed complex version of the argument is given by Tonks in [264].

Theorem 36 *(Eilenberg-Zilber theorem: normalised form) For simplicial sets, K and L, the maps, $a_{K,L}$ and $b_{K,L}$, make $C_N(K) \otimes C_N(L)$ a strong deformation retract of $C(K \times L)$.* ■

(In case you have forgotten, a *strong deformation retract* is a subspace / subcomplex / subobject,... with a retraction from the 'super-object' such that the other composite is homotopic to the identity, by a homotopy which restricts to the identity on the subobject, cf. page 368.)

It should be pointed out (i) that there is an un-normalised version of this theorem, and (ii) there are generalisations to simplicial modules, linking the associated chain complex of a product with the tensor of the associated chain complexes. These can be found in 'the literature' without much bother, so we will not look at them further.

In the above, the original purpose, for us, was to look at the Eilenberg Mac Lane shuffle map as it enabled the transition from $\mathcal{S}$-categories to dg^+-categories. The Eilenberg-Zilber theorem then say something about the change in the information encoded by those categories and furthermore tells us something about the classical Dold-Kan correspondence as far as the monoidal structures on the categories is concerned. Briefly, one can think of 'Eilenberg-Zilber' as saying that, in the passage from $\mathcal{S}$ to dg^+, the linearisation process is where all the change is taking place, not in the passage from simplicial Abelian groups to chain complexes.

There is another facet to the data making up the Eilenberg-Zilber theorem and that is that the Alexander-Whitney map is needed for the specification of the $\mathcal{S}$-category structure on Ch^+_R, the category of positive chain complexes of modules over a commutative ring R. When we introduced this early in this chapter, we linked that structure to that of simplicial modules and that is true, but it does hide the role that the Alexander-Whitney map plays in the definition of composition.

Effectively we are linking Ch^+_R with $\mathcal{S}_R$, the category of simplicial R-modules, and so are using the Dold-Kan equivalence, and hence the Moore complex / normalised chain complex functor.

We need to recall from section 11.2.2, the details of the $\mathcal{S}$-category structure that we have been using on $\mathcal{S}$ itself. As we have said several times, the fact that $\mathcal{S}$ is a category of presheaves on $\mathbf{\Delta}$ automatically give an 'internal hom', $\underline{\mathcal{S}}(K,L)$, where

$$\underline{\mathcal{S}}(K,L)_n := \mathcal{S}(\Delta[n]\times K, L).$$

Composition is then given by: for $f\in\underline{\mathcal{S}}(K,L)_n$, $g\in\underline{\mathcal{S}}(L,M)_n$, that is, $f:\Delta[n]\times K\to L$, $g:\Delta[n]\times L\to M$, we have

$$g\circ f := (\Delta[n]\times K \xrightarrow{diag\times K} \Delta[n]\times\Delta[n]\times K \xrightarrow{\Delta[n]\times f} \Delta[n]\times L \xrightarrow{g} M).$$

Here $diag:\Delta[n]\to\Delta[n]\times\Delta[n]$ is the diagonal map, which exists because $\times$ is a categorical product.

If, as we want to do, we replace simplicial sets as our objects by (positive) chain complexes, then the homotopy theory of those gadgets that we saw and used in section 8.2.1, corresponds to taking $\mathsf{C}[n]=C_N(\Delta[n])$ and to having an $\mathcal{S}$-structure on Ch^+_R, where, for chain complexes, A and B, the simplicial 'hom' between them satisfies

$$\underline{Ch^+_R}(\mathsf{A},\mathsf{B})_n = Ch^+_R(\mathsf{C}[n]\otimes\mathsf{A},\mathsf{B}),$$

but we now need to look at composition. Naively we might write

$$\mathsf{C}[n]\otimes\mathsf{A}\xrightarrow{?\otimes\mathsf{A}}\mathsf{C}[n]\otimes\mathsf{C}[n]\otimes\mathsf{A}\xrightarrow{\mathsf{C}[n]\otimes f}\mathsf{C}[n]\otimes\mathsf{B}\xrightarrow{g}\mathsf{C}),$$

but what is the map labelled '?'? It must be a map $\mathsf{C}[n] \to \mathsf{C}[n] \otimes \mathsf{C}[n]$, but tensor products of complexes are *not* categorical products, so will not automatically have something resembling a diagonal map. Of course, we have

$$C_N(\Delta[n]) \to C_N(\Delta[n] \times \Delta[n])$$

induced by the simplicial diagonal map, and we also have the Alexander-Whitney map,

$$a_{\Delta[n],\Delta[n]} : C_N(\Delta[n] \times \Delta[n]) \to C_N(\Delta[n]) \otimes C_N(\Delta[n]),$$

so composing we get an 'approximation to the diagonal',

$$\mathsf{C}[n] \to \mathsf{C}[n] \otimes \mathsf{C}[n].$$

Our earlier results on associativity imply that this is 'coassociative', (in fact, $\mathsf{C}[n]$ is a coalgebra with this structure, as it also has a counit). This means that the diagram

$$\begin{tikzcd}
\mathsf{C}[n] \arrow[r] \arrow[d] & \mathsf{C}[n] \otimes \mathsf{C}[n] \arrow[d] \\
\mathsf{C}[n] \otimes \mathsf{C}[n] \arrow[r] & \mathsf{C}[n] \otimes \mathsf{C}[n] \otimes \mathsf{C}[n]
\end{tikzcd}$$

commutes and this in turn means that defining composition of f and g, as suggested above with ? being this 'pseudo-diagonal', gives an associative composition, *i.e.*, a simplicial map,

$$\underline{Ch_R^+}(\mathsf{A}, \mathsf{B}) \times \underline{Ch_R^+}(\mathsf{B}, \mathsf{C}) \to \underline{Ch_R^+}(\mathsf{A}, \mathsf{C}),$$

which shows Ch_R^+ is an $\mathcal{S}$-category, ..., well almost, but not quite, as we have left out some details above and we have to handle the identities – but these points will be **left to you to check**.

13.7 Crs as an $\mathcal{S}$-category

We gave quite a lot of time and space to the detailed discussion of the Eilenberg - Zilber theorem set up in the case of chain complexes. This was because we will need the corresponding results and theory for crossed complexes, so as (i) to be able to examine homotopy coherent diagrams in that context, and, in particular, 'homotopy coherent actions' of one group on another, which will require an $\mathcal{S}$-category structure on Crs, and (ii) to indicate a Crs-category based theory of homotopy coherence as that will reduce considerably the size of the data needed to encode the diagrams found in (i).

As in some of the earlier sections, here we will omit quite a lot of detail, since otherwise there would need to be a long discussion of the detailed structure of Crs as a symmetric monoidal closed category. We will limit ourselves to giving a résumé of much of this as there are good sources for this material and duplication of it would seem unnecessary. In particular, the tensor product material can be found in Brown - Higgins - Sivera, [64] and the extensive work by Brown and Higgins on which it is based, in particular, [61]. The sketch of the Eilenberg - Zilber theorem is based on the work of Tonks, [264, 265], that has already been used in the previous sections. It is important to note that the conventions used in those sources include the use of right actions, so formulae would need adapting. (This is another reason for not including a large section repeating that material with left actions!)

13.7.1 Crossed complexes recalled and revisited

Most of our previous discussion of crossed complexes has concentrated on *reduced* crossed complexes, that is, corresponding to simplicial *groups* rather that simplicially enriched groupoids, with, of course, those thin filler conditions that we discussed in sections 1.3.6 and 3.5.1. We will recall the basic material and examples for convenience, but suggest that you do look back at the various earlier sections for slightly more detail.

A crossed complex, C, is a sequence

$$\begin{array}{ccccccccc} \longrightarrow C_n & \xrightarrow{\delta} & C_{n-1} & \longrightarrow & \ldots & \longrightarrow & C_2 & \xrightarrow{\delta} & C_1 \\ &&&&&&&& \delta^0 \Big\downarrow \Big\downarrow \delta^1 \\ &&&&&&&& C_0 \end{array}$$

where $(C_1, C_0, \delta^0, \delta^1)$ is a groupoid, (C_2, C_1, δ) is a crossed module of groupoids (*i.e.*, C_2 is a family of groups indexed by C_0 with an action of C_1 on it satisfying the equivariance and Peiffer relations for δ), then the rest of the structure is a chain complex of C_0-indexed families of Abelian groups with an action of C_1, so, for $n \geq 3$, each C_n is a groupoid with object set, C_0, but with $C_n(x, y)$ empty if $x \neq y$ and an Abelian group if $x = y$. There are then two compatibility axioms, namely that each $\delta : C_n \to C_{n-1}$ is equivariant and that δC_2 acts trivially on C_n for $n \geq 3$.

As we indicated earlier, an important example of such a non-reduced crossed complex, comes from homotopy theory. This is the fundamental crossed complex of a filtered space, cf. [64] and here page 69 in section 3.1.2.

Definition: A *filtered space*, $\underline{X}$, consists of a space, X, together with an increasing sequence of subspaces, $\{X_n\}_{n\in\mathbb{N}}$:

$$\underline{X} := X^0 \subseteq X^1 \subseteq \ldots \subseteq X^n \subseteq \ldots \subseteq X.$$

The sequence is called a *filtration* of the space X.

A continuous map, $f : X \to Y$, between the underlying spaces of two filtered space is called *filtration preserving* if $f(X_n) \subseteq Y_n$ for all $n \geq 0$.

(We will not normally need conditions such as that $\bigcup X_n = X$.)

Let $\underline{X} = \{X_n\}_{n\in\mathbb{N}}$ be a filtered space. In the case of most immediate importance, the space has a CW-complex structure and the filtration is by skeletons, so $X_n = X^n$, the n-skeleton of X. We have (cf. page 69), the *fundamental crossed complex*, $\pi(\underline{X})$, of $\underline{X}$, which has $C_0 = X_0$, $(C_1, C_0, \delta^0, \delta^1)$ is the fundamental groupoid of X^1 based at the points of X_0, that is, $\Pi_1 X_1 X_0$, and if $n \geq 2$, $C_n = (\pi_n(X_n, X_{n-1}, x))_{x\in X_0}$, the family of relative n^{th} homotopy groups of X_n relative to X_{n-1}, and based at the points of X_0. The action of C_1 on C_n is by change of base point and the boundary homomorphisms, $\delta : C_n \to C_{n-1}$, are the composites,

$$\pi_n(X_n, X_{n-1}, x) \to \pi_{n-1}(X_{n-1}, x) \to \pi_{n-1}(X_{n-1}, X_{n-2}, x),$$

from the long exact sequences of the pairs (X_n, X_{n-1}) and (X_{n-1}, X_{n-2}).

If K is a simplicial set, its geometric realisation, $|K|$, is a CW-complex and, giving it the skeletal filtration, we get a crossed complex, $\pi(K) := \pi(\underline{|K|})$, called the *fundamental crossed complex of K*

In particular, we define $\pi(n) := \pi(\Delta[n])$, the fundamental crossed complex of the n-simplex. (This is a free crossed complex generated by single generator in dimension n, cf. [64].)

The topological route from simplicial sets to crossed complexes is equivalent to a more purely algebraic one. Given any such K, we first form $G(K)$, the Dwyer-Kan $\mathcal{S}$-groupoid of K (as we examined in section 6.2.1, and then take the associated crossed complex, $C(G(K))$, by taking the quotient of the Moore complex, $N(G(K))$, by the intersection of that with the subobject generated by the degenerate elements, to get

$$C(G(K))_n = \frac{NG(K)_{n-1}}{(NG(K)_{n-1} \cap DG(K)_{n-1})d_0(NG(K)_n \cap DG(K)_n)} \quad \text{for } n \geq 2,$$

but remember this indexation as we discussed in that earlier section.

We saw, in section 6.2.3, a fair amount of detail on the nerve of a crossed complex. One of the approaches mentioned briefly there was as a 'singular complex', so that

$$Ner(\mathsf{C})_n = Crs(\pi(n), \mathsf{C})$$

with face and degeneracy maps induced in the standard way. Standard results on this sort of construction suggest the following result:

Proposition 175 *The functor, $\pi : \mathcal{S} \to Crs$, sending K to $\pi(K)$, is left adjoint to the nerve functor, $Ner : Crs \to \mathcal{S}$.* ∎

A proof can be found in [63].

This gives a third way of defining $\pi(K)$. Since

$$K = \int^{[n]} K_n \cdot \Delta[n]$$

and left adjoints preserve coends,

$$\pi(K) \cong \int^{[n]} K_n \cdot \pi(n).$$

(Of course, this needs Crs to be cocomplete, which it is as it is essentially a category of algebras for a many sorted algebraic theory; alternatively look up the construction of colimits within Crs either in the papers of Brown and Higgins, or in the monograph, [64]. It is often very useful to know how they are constructed and not just their existence, so in many ways this second course of action is the better one.)

We also recall (from section 6.2.3) that $Ner(\mathsf{C})$ is a T-complex, in the sense of section 1.3.6, where a simplex $f : \pi(n) \to \mathsf{C}$ will be *thin* if it is trivial on the top dimensional generator, ι_n, of $\pi(n)$.

The category, Crs, is, as has been mentioned several times, symmetric monoidal closed, as shown by Brown and Higgins, [61], and, as usual, [64]. This means that we need tensor products, $\mathsf{A} \otimes \mathsf{B}$, and an 'internal hom', $\textsc{Crs}(\mathsf{B}, \mathsf{C})$, such that there is a natural isomorphism,

$$Crs(\mathsf{A} \otimes \mathsf{B}, \mathsf{C}) \cong Crs(\mathsf{A}, \textsc{Crs}(\mathsf{B}, \mathsf{C})).$$

If we set $\mathsf{A} = \pi(n)$, we then get

$$Ner(\textsc{Crs}(\mathsf{B}, \mathsf{C}))_n \cong Crs(\pi(n) \otimes \mathsf{B}, \mathsf{C})$$

and the expression on the right is what we used to give a simplicial enrichment for Crs, *i.e.*, we defined a simplicial set, $\underline{Crs}(\mathsf{B}, \mathsf{C})$, by

$$\underline{Crs}(\mathsf{B}, \mathsf{C}) = Crs(\pi(n) \otimes \mathsf{B}, \mathsf{C}).$$

This thus shows how we have an interacting set of ideas, none of which we have actually given in detail!

From this we can predict what the various components of this picture should look like. For a start, as $\pi(0)$ is just a single object, single morphism groupoid, and the vertices of the nerve of a crossed complex are its set of objects, we immediately have that $\textsc{Crs}(\mathsf{B}, \mathsf{C})$ has to have $Crs(\mathsf{B}, \mathsf{C})$ as its set of objects.

The free crossed complex on one generator, c_n, in dimension n is not $\pi(n)$, which is $\pi(\Delta[n])$, but the crossed complex of the n-ball with an obvious filtration. We will denote this by $\mathsf{F}(n)$. The filtered standard n-ball consists of the n-ball,

$$E^n = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\},$$

with the $n-1$-sphere

$$S^{n-1} = \{x \in \mathbb{R}^n \mid ||x|| = 1\},$$

as $(E^n)_{n-1}$ if $n > 1$, and the point $x = (1, 0, ..., 0)$ as base point. This filtered space, thus, has just the base point up to filtration degree $n-1$, in that degree has an $n-1$-sphere and in degree n the n-ball. Note that the case $n = 1$ is a bit different since for $n = 1$, E^1 has two elements in the bottom degree.

Let us next look at this concept of a '*free crossed complex on a single generator*' and why this can be identified with a n-ball based model. (You are recommended, here, to look up a thorough treatment of this, either in the original sources or in [64]. We will be giving merely a sketch of the theory, and whilst crossed complexes, and the equivalent models such as ω-groupoids, represent only a limited number of homotopy types, and hence a limited set of types of non-Abelian gadgets for cohomology, they provide an important and subtle first step, and one in which the corresponding detailed analysis is readily available.)

We will look, as usual, in low dimensions to start with.

- $\mathsf{F}(0)$ has a single 'generating' object! It thus is the same as $\pi(0)$.

- $\mathsf{F}(1)$ has a single generator in dimension 1. This will be an arrow, c_1, between two objects which we may label 0 and 1, so $c_1 : 0 \to 1$ and, as $\mathsf{F}(1)$ is essentially a groupoid (intuitively because there is nothing forcing it to have anything non-trivial in dimensions greater than 1), there is a $c_1^{-1} : 1 \to 0$ as well, plus the identities on 0 and 1 and that is all.

- $\mathsf{F}(n)$ has but a single object, then, in dimension n, it has the generator c_n, giving an infinite cyclic group in that dimension; that generating element, c_n, has to have a boundary, so there is another element, c_{n-1}, in dimension $n-1$ with $\delta(c_n) = c_{n-1}$, and that generates another infinite cyclic group in that dimension. All other dimensions are trivial.

It is hopefully clear why this is $\pi(\underline{E}^n)$.

It is important to see why this defines $\mathsf{F}(n)$ as the free crossed complex on a single generator in dimension n. For a start what does that mean exactly?

If we have some crossed complex C and an element x in dimension n, then we can find a unique morphism of crossed complexes $\hat{x} : \mathsf{F}(n) \to \mathsf{C}$ such that $\hat{x}(c_n) = x$. This is standard for $n = 0$ and 1, and for $n \geq 2$, just needs thinking about and checking. (You are told where c_n is to go, that tells you where c_{n-1} has to go if the assignment is to be a crossed complex morphism. You only need one object in these higher degree $\mathsf{F}(n)$s, since in dimensions 2 and above C_n has no morphisms between distinct objects, so x is in a vertex group of C_n.) It is useful, since then $C_n \cong Crs(\mathsf{F}(n), \mathsf{C})$.

If we apply this to our collection of hoped for 'equations', we obtain a description of $\text{CRS}(\mathsf{B}, \mathsf{C})_n$ as $Crs(\mathsf{F}(n) \otimes \mathsf{B}, \mathsf{C})$, so the elements of degree n in $\text{CRS}(\mathsf{B}, \mathsf{C})$ can be thought of as some sort of n-homotopies. How should we think of these? For $n = 0$, we know that we just have the morphisms. For $n = 1$, we have a cylinder type object and so algebraically expect the 1-homotopies to look like homotopies of groupoid morphisms at the bottom and chain homotopies further up. In other words, they should correspond to some sort of degree 1 morphisms. (Think back to chain homotopies, and how, for chain complexes, a cylinder based homotopy corresponded uniquely to a chain homotopy / degree 1 map.)

Definition: (cf. [64]) Let C, D be crossed complexes and

$$f, g : \mathsf{C} \to \mathsf{D}$$

be two morphisms of crossed complexes. A *homotopy* or *1-homotopy* from f to G,

$$H : f \simeq g$$

is a 'map of degree 1 from C to D over f_0 starting at f and ending at g'.

It is evident that this needs expanding. First 'degree 1' might seem obvious: if we look in dimension n, then such a map will go from C_n to D_{n+1}, but, take care, both C_n and D_{n+1} are families of groups unless $n = 1$, so if $x \in C_n(x_0)$ for some $x_0 \in C_0$, we have to specify in which of the groups in the family will be $H(x)$. This is where the 'over f_0' bit comes in. It means, as you no doubt have guessed, that if $x \in C_n(x_0)$, then $H(x) \in D_{n+1}(f_0(x_0))$. That also makes sense for $n = 1$, but for $n = 0$, we want: if $x \in C_0$, then $H(x) : f_0(x) \to g_0(x)$. Next we need H to preserve the compositions and actions in some sense: in dimension 1, we need H to be a derivation over f, so

$$H(xx') = H(x).^{f(x)}H(x'),$$

and in higher dimensions for it to be linear. Finally it needs to be a chain homotopy in higher dimensions and a groupoid morphism homotopy at the base.

(The above omits details quite extensively, so **do look at** [64]. We also refer to that source for composition of 1-homotopies. Note that throughout the actions are on the right not on the left as we have been using. If you need to use the ideas in detail, it is not too difficult to switch from right to left, provided you first interpret the formulae correctly.)

We can now sketch what an m-homotopy is for $m \geq 2$. These are to form $\text{CRS}(\mathsf{B}, \mathsf{C})_m$ and our aim is to have $\text{CRS}(\mathsf{B}, \mathsf{C})$ is a crossed complex, but that means that, in these dimensions,

$\mathrm{C{\scriptsize RS}}(\mathsf{B},\mathsf{C})_m$ is a family of groups indexed by $\mathrm{C{\scriptsize RS}}(\mathsf{B},\mathsf{C})_0$, which is, of course, just $Crs(\mathsf{B},\mathsf{C})$, so we need to specify a morphism f and an element of the vertex group of the groupoid $\mathrm{C{\scriptsize RS}}(\mathsf{B},\mathsf{C})_m$ at f:

Definition: Let C, D be crossed complexes. An *m-homotopy over a morphism,* $f : \mathsf{C} \to \mathsf{D}$, is a degree m-map from C to D that consists of morphisms of modules over groupoids over the groupoid morphism, f_1. It therefore preserve the actions, is linear, $H_n : C_n(x_0) \to D_{n+m}(f_0(x))$ for $n \geq 2$ and a derivation over f for $n = 1$.

Again you are referred to [64] for the full details and, in particular, a proof of the fact that defining $\mathrm{C{\scriptsize RS}}(\mathsf{B},\mathsf{C})_m$ to be the set of such m-homotopies, we get a crossed complex, $\mathrm{C{\scriptsize RS}}(\mathsf{B},\mathsf{C})$, for suitable notions of boundary and actions.

Definition: Let C, D be crossed complexes. The crossed complex, $\mathrm{C{\scriptsize RS}}(\mathsf{B},\mathsf{C})$, is called the *mapping crossed complex of morphism from* C *to* D or, simply, the *internal hom* of such morphisms.

From other similar situations, and, in particular, the related case of chain complexes, we would expect a composition,

$$\mathrm{C{\scriptsize RS}}(\mathsf{A},\mathsf{B}) \otimes \mathrm{C{\scriptsize RS}}(\mathsf{B},\mathsf{C}) \to \mathrm{C{\scriptsize RS}}(\mathsf{A},\mathsf{C}),$$

but, of course, we *still* have not got the tensor product that we need for this to make sense. We therefore need to head back towards tensors of crossed complexes, as such, but, before we do that, let us look at the relationship between tensors of crossed complexes and tensoring in the $\mathcal{S}$-category structure on Crs (that as yet we have not shown exists!) Formally we have, for $K \in \mathcal{S}$ and $\mathsf{C} \in Crs$ the tensor $K\overline{\otimes}\mathsf{C}$, which satisfies

$$\underline{Crs}(K\overline{\otimes}\mathsf{C},\mathsf{D}) \cong \underline{\mathcal{S}}(K, \underline{Crs}(\mathsf{C},\mathsf{D})),$$

naturally in D. If we restrict attention to $K = |\Delta[n]|$, then our assumption that Crs has a simplicial enrichment as we outlined above, gives

$$\begin{aligned} Crs(\Delta[n]\overline{\otimes}\mathsf{B},\mathsf{C}) &\cong \mathcal{S}(\Delta[n], Ner(\mathrm{C{\scriptsize RS}}(\mathsf{B},\mathsf{C}))) \\ &\cong Ner(\mathrm{C{\scriptsize RS}}(\mathsf{B},\mathsf{C}))_n \\ &\cong Crs(\pi(n)\otimes\mathsf{B},\mathsf{C}), \end{aligned}$$

i.e., even without the formula for tensor products in general, we could retrieve $\pi(n)\otimes B$ as $\Delta[n]\otimes B$. This in turn would give $\pi(K)\otimes\mathsf{B}$ for all K as being $K\otimes\mathsf{B}$. (**Left to you to check.**) There are coend formulae for tensors in some generality, so we might obtain a construction of $\pi(K)\otimes\mathsf{B}$ via this method. This is not the route we will take however and we mention it just to show how interrelated all the structure is at such points as this. It does indicate that once we have a working tensor product for crossed complexes, then the tensoring by simplicial sets is predetermined.

Anyhow that is somewhat premature as we have yet to construct the tensor product and have not shown that there is a composition in the tentative $\mathcal{S}$-category structure that such a composition would give us on Crs. Of course, the example that we have already examined of chain complexes gives us a skeleton of a construction plan.

Suppose $f : \pi(n)\otimes\mathsf{A} \to \mathsf{B}$ and $g : \pi(n)\otimes\mathsf{B} \to \mathsf{C}$, then we want to define $g\circ f$ by the same sort of process as for chain complexes, but the obstruction to doing so for the moment, is the crossed complex analogue of the Alexander-Whitney map. We want

$$a_{\Delta[n],\Delta[n]} : \pi(\Delta[n] \times \Delta[n]) \to \pi(n) \otimes \pi(n),$$

since then we will have an approximation to the diagonal

$$\pi(n) \to \pi(n) \otimes \pi(n).$$

The way to obtain $a_{K,L} : \pi(K \times L) \to \pi(K) \otimes \pi(L)$ is given in detail by Tonks in [264]. We will shortly sketch his method, but will leave the detail for you to check in the source. Before we do that, however, we need to define the tensor product. Once that is done, the above discussion will release a lot of results for our use.

13.7.2 Tensor products of crossed complexes

(As indicated in the earlier discussion, our aim here will be to motivate the definition of the tensor product rather than to give a full exhaustive treatment.)

As well as the sought after isomorphism,

$$Crs(\mathsf{A} \otimes \mathsf{B}, \mathsf{C}) \cong Crs(\mathsf{A}, \mathrm{C{\scriptsize RS}}(\mathsf{B}, \mathsf{C})),$$

there is another factor at play, namely a topological tensor-like structure on the category of filtered spaces.

We can think of filtered spaces as an abstraction of simplicial and CW-complexes with the filtration degree of an element somehow recording its 'entry dimension', as this is related to the interpretation of that degree in the original cases: we filter a simplicial complex or rather its geometric realisation / polyhedron, X, by its skeletons, X_n, where X_n is the union of all the simplices of dimension $\leq n$, so if $x \in X_n$ and $x \notin X_{n-1}$, then it is in the interior of an n-simplex of X. (Of course, that point x *is* a point, so has dimension zero, but it is 'living' inside something of dimension n.) We will usually be a bit sloppy about notation[13], but, when we want to emphasise the filtered nature of the filtered space, we may write it with an underline, $\underline{X} = \{X_n\}_{n\in\mathbb{N}}$.

If we take two cell complexes, X and Y, (as an example we take $X = E^2$, the 2-dimensional cell with skeletons: $X_0 = \{1\}$, $X_1 = S^1$ and $X_2 = E^2$ itself; and $Y = E^1$ with $\{0, 1\}$ and E^1 as the subspaces of degree 0 and 1 respectively, (we will not bother to write higher degree parts as they are all the same), then there is a natural cell structure on $X \times Y$. The product of two open cells $e^k \times e^\ell$ is $e^{k+\ell}$, so the natural n-skeleton of $X \times Y$ is made up from parts $X_k \times Y_\ell$, where $k + \ell = n$. This gives the product cell complex structure

$$(X \otimes Y)_n = \bigcup_{k+\ell=n} X_k \times Y_\ell$$

on $X \times Y$.

This can be applied equally well with any filtered spaces. In [64], $X \otimes Y$ is used for the resulting filtered space and we will adopt this usage. We write $FTop$ for the category of filtered spaces and filtered continuous maps.

[13]especially when it *should be* clear what the situation is and, thus, that no confusion *should* arise!

Definition: If $\underline{X} = \{X_n\}_{n\in\mathbb{N}}$, $\underline{Y} = \{Y_n\}_{n\in\mathbb{N}}$ are two filtered spaces, we denote by $\underline{X} \otimes \underline{Y}$, the filtered space with underlying space $X \times Y$ and filtration given by

$$(X \otimes Y)_n = \bigcup_{k+\ell=n} X_k \times Y_\ell,$$

the union of those subspaces of total filtration degree n.

We record the following result for interest. Here 1 is the one element filtered space.

Proposition 176 *$(FTop, \otimes, 1)$ is a symmetric monoidal category.* ∎

More discussion of this can be found in [64].

Of course, the union over k, and ℓ such that $k+\ell = n$ suggests the chain complex total complex, but we have now passed 'beyond' chain complexes to crossed complexes and note a result of Brown and Higgins.

Proposition 177 *(i) There is a natural morphism*

$$\pi(\underline{X}) \otimes \pi(\underline{Y}) \to \pi(\underline{X} \otimes \underline{Y}).$$

(ii) This is an isomorphism for skeletally filtered CW-complexes, X and Y. ∎

The proof can be found in [61, 63] and in [64].

This still begs the issue, as we have used the tensor product of two crossed complexes in the statement and have not yet defined it, *but* it tells us *how* to define that tensor product in certain cases.

Case study and Examples: We have already met (page 667) the filtered space, E^n, having $(E^n)_1 = \ldots = (E^n)_{n-2} = \{1\}$, $(E^n)_{n-1} = S^{n-1}$, the $(n-1)$-sphere, and $(E^n)_k = E^n$, the n-cell, for $k \geq n$, with obvious changes for the low dimensional cases. We know that, provided we are above the slightly awkward case of $n = 1$, $\pi(E^n)$ is what we have written $\mathsf{F}(n)$, *i.e.*, $\mathsf{F}(n)_n \cong \langle c_n\rangle$, $\mathsf{F}(n)_{n-1} \cong \langle c_{n-1}\rangle$ with $\delta(c_n) = c_{n-1}$, with other dimensions trivial. (This is $\pi_n(E^n, S^{n-1}) \cong \langle c_n\rangle$, $\pi_{n-1}(S^{n-1}) \cong \langle c_{n-1}\rangle$, so is intimately linked to that cell structure.)

We now look at the cell structure and filtration of $E^m \otimes E^n$. To start with we will assume $m, n \geq 2$ and that $m \leq n$. We have $(E^m \otimes E^n)_0 = \ldots = (E^m \otimes E^n)_{m-2}$ and has exactly one point namely $(1, 1)$. We have,

$$(E^m \otimes E^n)_k = \bigcup_{p+q=k} (E^m)_p \times (E^n)_q,$$

so to continue it is convenient to draw up a simple table. The filtered space E^m has cells only in dimensions 0, $m-1$ and m; similarly for E^n, so

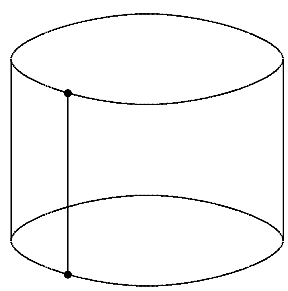
p	0	$m-1$	m	0	$m-1$	m	0	$m-1$	m
q	0	0	0	$n-1$	$n-1$	$n-1$	n	n	n
$p+q$	0	$m-1$	m	$n-1$	$m+n-2$	$m+n-1$	n	$m+n-1$	$m+n$

Of course, this means that the actual number of k-cells in $E^m \otimes E^n$ is going to depend strongly on the relative sizes of m and n. If there are no coincidences (so we do not have, for instance, $m = n$ or $m = n - 1$), then we have single cells in dimensions $m - 1$, m, n, $m + n - 2$ and $m + n$ and two in dimension $m + n - 1$.

Of course, the number of cells is only one part of the cell structure as it does not tell us how each cell is attached to those of lower dimension. The cell structure gives the structure of the crossed complex, $\pi(E^m \otimes E^n)$, and we can see in what ways it is different from the chain complex tensor product of $\pi(E^m)$ and $\pi(E^n)$. (We note that as both E^m and E^n are reduced and simply connected, their crossed complexes look like their chain complexes, except that we have a singleton set in dimension 0, and that is a set not a trivial Abelian group! This may seem a banal point to make, but it is important.)

The crossed complex $\pi(E^m \otimes E^n)$ is thus, sort of, of form $\mathsf{C} \otimes \mathsf{D}$ as if we had chain complexes, with the additional features of single generators in the parts $C_{m-1} \otimes D_0$, $C_m \otimes D_0$, $C_0 \otimes D_{n-1}$ and $C_0 \otimes D_n$, not forgetting the single object that makes up $C_0 \otimes D_0$. (You are left to see what the differential will be from $C_m \otimes D_n$ to $C_{m-1} \otimes D_n \oplus C_m \otimes D_{n-1}$, etc., ..., not that that is routine!)

This was relatively simple because we chose $m, n \geq 2$. If either is 1, then the crossed complex corresponding to that has two objects, so the usual simple ideas of tensor products need a bit of revision. To illustrate, look at $E^1 \otimes E^2$, which looks like a solid cylinder:



The cell structure is easy to see. There are two cells in dimension 0, 3 in dimension 1, 3 in dimension 2 and one in dimension 3. If we write e^2_j for the j-cell in E^2, then we can list them, e.g. in dimension 2,

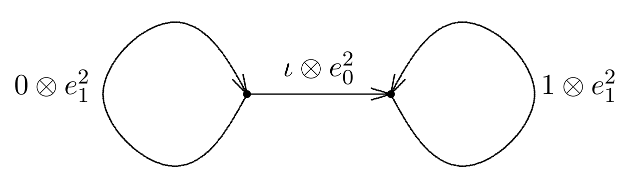
$$0 \otimes e^2_2, \quad 1 \otimes e^2_2, \quad \iota \otimes e^2_1,$$

where $\iota : 0 \to 1$ is the 1-cell in E^1, as before. All together these will generate a length 3 crossed complex with a single generator in dimension 3, 3 in dimension 2, and so on. (But what does this *actually* give us?)

We need, therefore, to see exactly what each object is here.

- $C_0 = \{0 \otimes e^2_0, 1 \otimes e^2_0\}$ is just a set;

- C_1 will be a (free) groupoid having C_0 as objects. It is free on the graph that makes up the 1-skeleton of the space, so flattening that out:



This is best viewed by orienting the graph and labelling it. (In the picture, we have used the pre-existing orientation on ι, together with what amounts to a 'clockwise' orientation on the top and bottom circles viewed from above. The free groupoid on a graph, Γ, makes sense whether or not the graph is directed as each edge γ in Γ yields two arrows, γ and γ^{-1}, in the groupoid, ..., but, for deciding which is which, you need to direct the edge! This is like saying that a singly generated free group is isomorphic to C_∞, but which isomorphism do you take, as there are two of them, equivalently, there are two perfectly good generators one generating c or its inverse. They are equally good. Of course, a similar point can be made in higher dimensions. You need to pick an orientation to be able to handle things more easily, but the choice of orientation does not change the 'thing' being handled.)

It will pay to be a bit careful (dare we say 'pedantic') about the structure of C_1 as simple lessons learnt here can be very useful in higher dimensions. The vertex groups, $C_1(0 \otimes e_0^2)$ and $C_1(1 \otimes e_0^2)$ are free on two generators. (Recall that we are using functional order for composition on paths, etc., and that this is *not* the usual one, but does give a left action convention and that *is* more convenient elsewhere.)

The two generators of $C_1(0\otimes e_0^2)$ are the class of the loop $0\otimes e_1^2$ and the class of the composite path $(\iota \otimes e_0^2)^{-1}(1 \otimes e_1^2)(\iota \otimes e_0^2)$, (and *note the order* - we are reading this as the string / path

$$0 \otimes e_0^2 \xrightarrow[\iota\otimes e_0^2]{} 1 \otimes e_0^2 \xrightarrow[1\otimes e_1^2]{} 1 \otimes e_0^2 \xleftarrow[\iota\otimes e_0^2]{} 0 \otimes e_0^2$$

going from $0 \otimes e_0^2$ to $1 \otimes e_0^2$, then around the loop $1 \otimes e_1^2$ before coming back to $0 \otimes e_0^2$). It is sometimes useful to write this second generator as a conjugate, ${}^{(\iota\otimes e_0^2)^{-1}}(1 \otimes e_1^2)$, of the basic loop $1 \otimes e_1^2$.

Similarly $C_1(1\otimes e_0^2)$ has generators the classes of $1\otimes e_1^2$ and $(\iota\otimes e_0^2)(0\otimes e_1^2)(\iota\otimes e_0^2)^{-1}$ or, written as a conjugate, ${}^{(\iota\otimes e_0^2)}(0\otimes e_1^2)$. There are, of course, also $C_1(0\otimes e_0^2, 1\otimes e_0^2)$ and $C_1(1\otimes e_0^2, 0\otimes e_0^2)$. The elements of these are left to you to think about. Remember that these sets have both left and right actions of the vertex groups on them.

- C_2 will be the 'free C_1-crossed module on 3 generators', but we need to take this apart to see what it means. (It is, in any case, imprecise as there can be many different ways of mapping 3 generators into C_1, and hence many different free C_1-crossed modules on 3 generators, and all but one of them is irrelevant here!) There are two groups, $\pi_2(X_2, X_1, x)$, one for each value of 'base point', x, namely $x = 0\otimes e_0^2$ and $x = 1\otimes e_0^2$, and each has 3 generators, corresponding to the 3 two dimensional cells of $X = E^1 \otimes E^2$. We have to see how to represent these generators in a sensible way and then determine what the boundary map and the actions look like.

  First the action: we have only handled the reduced case earlier, so need to sketch in a bit more detailed structure. We have an action of the groupoid, C_1, on the groupoid C_2. This latter is just a pair of groups and to see what such an action should be, we first recall that a left G-module, for G a group, is an Abelian group with a G-action,

$$G \times M \to M,$$

  so that, for each $g \in G$, action by g is a homomorphism (actually an isomorphism), so

$${}^g(m_1 + m_2) = {}^g m_1 + {}^g m_2,$$

and that ${}^{(g_2g_1)}m = {}^{g_2}({}^{g_1}m)$, with 1_G acting trivially. We have on many occasions above used the alternative formulation obtained by replacing the group G by the corresponding one-object groupoid, $G[1]$, then noting that the G-module, M, can be thought of as a functor,

$$M : G[1] \to Ab,$$

which sends the unique object to our old M and, thinking of $g \in G$ as a loop at the unique object, gives an automorphism, $M(g) : M \to M$, of Abelian groups, (so that $M(g)(m) = {}^gm$, in our old notation).

There are two generalisations that we need to this. First that we can replace $G[1]$ by an arbitrary groupoid, and, in particular, by our C_1 from our example, and secondly that there is nothing here stopping us replace Ab by other categories - and we will need the case of *Grps* here.

As C_1 has two objects $0 \otimes e_0^2$ and $1 \otimes e_0^2$, we will need two groups and these we know already.

(To be continued)

13.8 Complicial and weak complicial sets

We next change direction turning to an alternative model for homotopy coherence and non-Abelian cohomology, that is parallel to that using quasi-categories that we saw earlier. It is also very related to the idea of a T-complex; cf. section 1.3.6. This theory, that of *complicial sets*, was initially sketched by John Roberts, [236, 237], was developed further by Ross Street, [253], and then, together with their weak / lax analogues, extensively studied by Dominic Verity, [266–269][14].

13.8.1 Stratified simplicial sets in the sense of Verity

Just as with the idea of a T-complex that we met back on page 38, the theory of complicial sets is based on the observation that certain simplices, in constructions such as that of the singular complex, $Sing(X)$, of a space, X, or the nerve of a category, look *thin*. Degenerate simplices clearly are thin as, if $\sigma = s_i\tau$, and $\tau \in K_n$, then although σ is an $(n+1)$-dimensional simplex, it is determined by the n-dimensional one, τ, together with an index, i. Similarly fillers in $Sing(X)$, for instance, for an (n,i)-horn, are shown to exist by using a retraction in the topological simplices, plus information from dimension $(n-1)$. An instance of this is the observation that the homotopies used to prove associativity of composition in the fundamental groupoid, ΠX, occur within the 'trace' of the paths being composed. The homotopies do not 'sweep out' any area and so are 'thin'[15].

To codify this sort of idea, Roberts and Street introduced stratified simplicial sets[16].

[14] A useful set of lecture notes on some aspects of this theory is available, see Emily Riehl's notes, [235].

[15] In fact the essence of the homotopies is 'within the models', *i.e.*, it is independent of the space, X.

[16] This was done at approximately the same time, but independently of Dakin, and Brown and Higgins introduction of T-complexes in both the simplicial and cubical settings.

Definition: A *Verity-stratified simplicial set* (or just *stratified set*[17] for short) is a simplicial set, X, together with, for each n, a subset $tX_n \subseteq X$ of the set of n-simplices, whose elements are then called *thin n-simplices*. This structure is to satisfy the conditions

- no 0-simplex of X is thin, so $tX_0 = \emptyset$;
- all of the degenerate simplices of X are in tX.

We may sometimes say that the stratified set, (X, tX) is *supported by* X.

Note: Given a simplicial set, X, it is often convenient, in the development of this theory, to consider X as consisting of the set $\coprod_{n\in\mathbb{N}} X_n$, together with a set of actions of the simplicial operations, d_i^n and s_j^n on X. This is the convention used, for instance, by Verity, [266]. It enables one to write $tX \subset X$ without the inconvenience of using suffices for dimension[18].

Definition (continued): A morphism, $f : (X, tX) \to (Y, tY)$, between two stratified simplicial sets, hence called a *stratified simplicial map* or *stratified morphism*, is a simplicial map, $f : X \to Y$, such that $tX \subseteq f^{-1}tY$, so is a simplicial map that preserves *thinness*.

Clearly the composite of two stratified maps is again stratified and the identity simplicial map, $Id_X : X \to X$, is stratified on any (X, tX), supported by X, so we get a category, $Strat$, of stratified sets and stratified maps between them.

Quite often, explicit mention of the set of thin simplices in a stratified set, (X, tX), will be omitted if confusion is not likely to occur.

Lemma 103 *The forgetful functor, $U : Strat \to \mathcal{S}$, defined by $U(X, tX) = X$, etc., has both left and right adjoints. Both left and right adjoints are fully faithful.*

Proof: The right adjoint, R, takes X to $(X, X \setminus X_0)$, so all simplices of dimension > 0 are to be thin, whilst the left adjoint, L, has $L(X) = (X, deg(X))$, where $deg(X)$ is the set of degenerate elements of X, thus making as few simplices thing as possible. ■

It is usual to identify $\mathcal{S}$ with its image under L, so with the category of stratified sets for which every thin simplex is degenerate. This applies, in particular, to the standard simplices and horns, $\Delta[n]$ and $\Lambda^i[n]$, for all n and $0 \leq k \leq n$.

Comment: If we have need of the dual, opposite, or conjugate form of a simplicial construction (reversing the roles of faces, replacing d_0 by d_{last}, etc.[19]), then we obtain a dual simplicial set, X^o or X^*, and this can be extended to stratified sets by defining $(X, tX)^o = (X^o, tX^o)$ where $tX^o = tX$ as sets.

A pair, (Y, tY), is a *stratified simplicial subset* (or, more simply, a *stratified subset* of (X, tX) if

[17] There is also another use of the term 'stratified simplicial set' due to Douteau, [105]. Within their individual highly recognisable contexts, there is little likelihood of confusion between them, so we can safely use 'stratified simplicial set' in both with little risk of confusion and will so do.

[18] As we will be using Verity's papers as the main source for the account here, and as a reference for futher reading, we will use this convention from time to time.

[19] see page 33

- Y is a simplicial subset of X,
- tY contains all of the degenerate simplices of Y,

and

- tY is a subset of tX.

In this case, we write $(Y, tY) \subseteq_s (X, tX)$ and note that the obvious inclusion, $(Y, tY) \to (X, tX)$, is a stratified map.

Intersections and unions of stratified subsets, $(Y_i, tY_i) \subseteq_s (X, tX)$ are given by the obvious formulae:

$$\bigcup_i (Y_i, tY_i) := (\bigcup_i Y_i, \bigcup_i tY_i),$$

and

$$\bigcap_i (Y_i, tY_i) := (\bigcap_i Y_i, \bigcap_i tY_i).$$

Terminology: Various special forms of stratified maps are given special names, (cf. Verity, [266], §6.1).

A stratified map, $f : X \to Y$, is

- **regular** if it *reflects thin simplices*, *i.e.*, if $f(x)$ is thin in Y, then x was thin in X, so $f^{-1}(tY) = tX$;
- **entire** if f is surjective on simplices, so f is a surjective simplicial map,
- an **inclusion** if its underlying simplicial map is injective.

If we denote the classes of these types of morphisms by $\mathcal{R}$, $\mathcal{E}$ and $\mathcal{I}$, then f is a stratified isomorphism if, and only if, $f \in \mathcal{R} \cap \mathcal{E} \cap \mathcal{I}$.

Suppose, now, that $f : X \to Y$ is a stratified map.

Definition:
- The *image under* f of a stratified subset $X' \subseteq_s X$ is $f(X') := (f(X'), f(tX')$.
- The *inverse image under* f of a stratified subset, $Y' \subseteq_s Y$ is $f^{-1}(Y') := (f^{-1}(Y'), f^{-1}(tY'))$.

Observations: a) For any f, the inverse image of a regular subset, $Y' \subseteq_r Y$, of Y will be a regular subset of X, where we write $Y' \subseteq_r$ if the associated inclusion map $Y' \to Y$ is a regular morphism, or, equivalently, when $tY' = Y' \cap tY$.

b) For any regular stratified map, $f : X \to Y$, the image $f(X')$ of a regular subset of X will be a regular subset of Y.

Extending the above terminology, $X' \subseteq_e X$, *i.e.*, X' is an entire subset of X if $X' \xrightarrow{\subseteq} X$ is an entire morphism (so the underlying simplicial sets are the same).

Chapter 14

Indexed / weighted limits and colimits

We used the idea of a total complex of various types above as part of the machinery for passing between simplicially enriched and chain complex enriched categories. We will need them again later as a means for handling more general descent situations, but the main theme underlying our recent chapters was an exploration of homotopy coherence, and its various forms, in terms of enriched categories. One aspect of that is the subject of homotopy limits and colimits. We have touched on homotopy colimits several times in earlier sections and need to firm that knowledge up. We have also met homotopy limit type constructions. As these use other total complex constructions *and* illustrate how to work with homotopy coherence *and* are very useful, it would seem appropriate to explore aspects of their construction at the same time.

We will consider the homotopy coherence properties of various constructions that are analogues of limits and colimits in this context. We adopt the expedient terminology of calling these homotopy limits and colimits, although it might be better to refer to them as homotopy coherent limits and colimits. What is the difference? There is another very good approach to homotopy limits that considers that such objects are only defined up to homotopy equivalence or weak equivalence (within a homotopy context such as that of Quillen's model category theory). They are then objects that have slightly different universal properties than the ones we will be considering. In most interesting cases, the homotopy coherent limit is just a definite *representative* of the homotopy class being considered as the homotopy limit in the other approach. The link between the two approaches has been studied, initially by Vogt, [271], (as Vogt's theorem on the link between homotopy coherent diagrams and localisations can be viewed as being essentially this) and, thoroughly from a modern Quillen model category viewpoint, by Shulman, [247].

We will follow the idea that homotopy limits and colimits are best viewed as indexed limits and colimits. These are the appropriate form of limits for use with enriched category theory and so fit neatly into our description, however it is not just expediency that dictates this choice of approach. Indexed limits and colimits are also *very* useful for descriptions of various total object constructions and in simplicial descriptions of descent. It does not seem sensible to leave such a useful conceptual and calculatory tool aside, given that we will need the concepts they provide in several situations later.

14.1 Enriched Limits and Colimits?

Just as we recalled earlier, for colimits as universal cocones on a functor, we know that, in the usual set-based category theory, a limit interprets as a 'universal cone' on the functor. For instance, if $F : \mathbb{I} \to \mathcal{C}$ is a functor, a cone of F consists of an object, c, of $\mathcal{C}$ and a natural transformation from the constant functor, $cons(c)$, from $\mathbb{I}$ to $\mathcal{C}$ having value c, to the functor F. Because all the maps making up the diagram, $const(c)$, are the identity on c, we can collapse the diagram that represents a natural transformation, $\eta : cons(c) \to F$, into a cone shaped diagram,

$$\begin{tikzcd}
 & \, \\
 & F(i) \arrow[u] \\
c \arrow[uur, dashed] \arrow[ur, "\eta_i"] \arrow[dr, "\eta_j"'] \arrow[ddr, dashed] & \\
 & F(j) \arrow[uu] \\
 & \, \arrow[u]
\end{tikzcd}$$

and then a limit for F consists of a universal such cone, so that all others factorise through it in a unique way. If we write $Cone(c, F) = Nat(cons(c), F)$, then if $\ell = lim\, F$ exists, it satisfies $Cone(c, F) \cong \mathcal{C}(c, \ell)$, *i.e.*, it *represents* the functor, $Cone(-, F) : \mathcal{C}^{op} \to Sets$.

Limits within enriched categories may exist, but ideally their properties should represent something like an *object of cones on* F, not just a set as before. In general, it is thus better to use a variant of the idea of limit namely that of an indexed or weighted limit. As these, and the corresponding indexed colimits, include many constructions that we have been meeting and using, it seems a good idea to go through some of the basic notions of their theory. (In this section, we will often be following the viewpoint of Bourn and Cordier, [41].)

14.1.1 Bousfield Kan Homotopy Limits

We start with a particular case, both of an indexed limit and, in fact, of a homotopy limit, namely the *Bousfield-Kan homotopy limit*. These were introduced in [42]. About the same time, various other types of homotopy limit were introduced, notably by Vogt, [271], whilst Illusie, [163, 164], considered the *total derived functors of the limit*, which amounted to the same thing, but in a slightly different context. We will look at some of these variants later, but the Bousfield-Kan homotopy limit has the advantage of being fairly easy to work with, using a minimum of tools.

First note that, for any two functors $X, Y : \mathbb{I} \to \mathcal{S}$, we can form their 'simplicial set of natural transformations'. This can also be thought of as the 'space of functions' between the two diagrams. It is denoted by $\underline{\mathcal{S}}^{\mathbb{I}}(X, Y)$ and is defined by means of an end formula:

$$\underline{\mathcal{S}}^{\mathbb{I}}(X, Y) = \int_i \underline{\mathcal{S}}(X(i), Y(i)).$$

(This type of construction will work in any enriched category, provided the enrichment is over a complete base category, $\mathcal{V}$. We will use it more generally shortly.)

There is a natural functor, $\mathbb{I} \downarrow - : \mathbb{I} \to Cat$, namely the functor which to i assigns $\mathbb{I} \downarrow i$, the category of objects over i, which we have met in particular cases several times. Applying Ner, we can convert this into a functor taking values in $\mathcal{S}$. (Some sources write this also as $\mathbb{I} \downarrow -$ rather than the fuller $Ner(\mathbb{I} \downarrow -)$, and we may sometimes do the same, however, if the greater precision seems advisable, we will use the full form. In fact, there is a slight complication, namely that some sources use the conjugate form of nerve, so, when working with these ideas, **ideally, always check** and, as the ideal is not always what we do, **less than ideally**, if you think the formula that you need is possibly in the conjugate form ..., *i.e.*, you keep on getting a different answer from 'them' ..., then **check**.)

Remark on notation: As Bousfield and Kan use the 'slice category / category over i' notation $\mathbb{I}/-$, where we have used $\mathbb{I}\downarrow i$. Of course, both stand for a comma category (cf. page 698).

What do the simplices in $Ner(\mathbb{I}\downarrow i)$ look like? If $\sigma \in Ner(\mathbb{I}\downarrow i)_n$, then it corresponds to a composible sequence of n maps over i and so to a diagram,

$$i_0 \stackrel{\alpha_1}{\to} i_1 \stackrel{\alpha_2}{\to} \ldots \stackrel{\alpha_n}{\to} i_n \stackrel{\alpha}{\to} i,$$

since the composites look after the 'over i' aspect automatically. We will denote this by $\sigma = \langle\alpha_0, \alpha_1, \ldots, \alpha_n; \alpha\rangle$. Most of the face and degeneracy maps are as you would expect in a nerve, but the d_n is slightly more worthy of comment:

$$d_n\langle\alpha_0, \alpha_1, \ldots, \alpha_n; \alpha\rangle = \langle\alpha_0, \alpha_1, \ldots, \alpha_{n-1}; \alpha\alpha_n\rangle.$$

Of course, any n-simplex,

$$\langle\alpha_0, \alpha_1, \ldots, \alpha_n; \alpha\rangle,$$

is thus the n-face of an $n+1$-simplex,

$$\langle\alpha_0, \alpha_1, \ldots, \alpha_n, \alpha; id_i\rangle.$$

This is a very useful idea to remember if working with these 'beasties', as it is closely related to the Yoneda lemma.

Now we can define the Bousfield-Kan version of the homotopy limit.

Definition: For a functor $F : \mathbb{I} \to \mathcal{S}$, its *Bousfield-Kan homotopy limit* is

$$holim\, F = \underline{\mathcal{S}}^{\mathbb{I}}(\mathbb{I}\downarrow -, F) = \int_i \underline{\mathcal{S}}(\mathbb{I}\downarrow i, F(i)).$$

14.1.2 Homotopy pullback, a simple case study

Although this is easy to define, it is less easy to see how and why it works, so we will look at a simple example. We will look at the homotopy pullback construction as a homotopy limit. For this, we take $\mathbb{I}$ to be the category

$$\begin{tikzcd} & 1 \arrow[d, "\alpha"] \\ 2 \arrow[r, "\beta"'] & 0 \end{tikzcd}$$

and hence F to be given by a diagram

$$\begin{tikzcd} & F(1) \arrow[d, "F(\alpha)"] \\ F(2) \arrow[r, "F(\beta)"'] & F(0) \end{tikzcd}$$

The diagram, $\mathbb{I}\downarrow -$, has $\mathbb{I}\downarrow 0\cong \mathbb{I}$, (since 0 is terminal in $\mathbb{I}$) and $\mathbb{I}\downarrow 1$ and $\mathbb{I}\downarrow 2$ are trivial categories. We can represent the nerve of $\mathbb{I}\downarrow 0$ by $\Delta[1]_{d_0}\sqcup_{d_0}\Delta[1]$, the result of gluing two copies of $\Delta[1]$ along the d_0-vertex:

$$1 \to 0 \leftarrow 2$$

The two morphisms, $\mathbb{I}\downarrow\alpha : \Delta[0]\to\Delta[1]_{d_0}\sqcup_{d_0}\Delta[1]$, and similarly $\mathbb{I}\downarrow\beta$, are the obvious ones, sending the point of $\Delta[0]$ to 1 or 2, respectively. We next look at the parts making up $holim\, F$. Those parts indexed by 1 and 2 pick out simplices in $F(1)$ and $F(2)$, resp., as they are just $\underline{\mathcal{S}}(\Delta[0], F(i))$, for $i = 1, 2$, so are isomorphic to $F(i)$, in this case. The component $\underline{\mathcal{S}}(\mathbb{I}\downarrow 0, F(0))$ gives two homotopies in $F(2)$, *i.e.*, $\underline{\mathcal{S}}(\Delta[1]_{d_0}\sqcup_{d_0}\Delta[1], F(0))$, which is isomorphic to $\underline{\mathcal{S}}(\Delta[1], F(0))_{d_0}\times_{d_0}\underline{\mathcal{S}}(\Delta[1], F(0))$, *i.e.*, to $F(0)^I{}_{d_0}\times_{d_0} F(0)^I$, two copies of the cocylinder on $F(0)$, glued along the 'target maps': $F(0)^I\to F(0)$. The next step is to form the end / limit, but this is simply the pullback,

$$\begin{tikzcd}
holim\, F \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & F(1) \arrow[d, "F(\alpha)"] \\
& F(0)^I{}_{d_0}\times_{d_0} F(0)^I \arrow[r, "d_0"'] \arrow[d, "d_0"'] & F(0) \\
F(2) \arrow[r, "F(\beta)"'] & F(0) &
\end{tikzcd}$$

We thus have a diagram:

$$\begin{tikzcd}
holim\, F \arrow[r] \arrow[d] \arrow[dr] \arrow[r, phantom, shift right=3, "\Uparrow"] & F(1) \arrow[d] \\
F(2) \arrow[r] \arrow[r, phantom, shift left=3, "\Downarrow"] & F(0)
\end{tikzcd}$$

This gives a homotopy coherent cone on F in the obvious sense. (You are **left to explore the universality** of this square, analogous to the universal property of pullback squares. Note there are *two* 2-cells / homotopies in this diagram.)

We can describe $(holim\, F)_n$ quite explicitly, and in elementary terms, because the pullback, like every limit in $\mathcal{S}$, is calculated 'dimension-wise'.

- $n = 0$, a vertex of $holim\, F$ consists of a triple

$$(x_1, (y_1, y_2), x_2),$$

where $x_1 \in F(1)_0$, $x_2 \in F(2)_0$, $y_1, y_2 \in (F(0)^I)_0 = F(0)_1$ and satisfy $d_0(y_1) = d_0(y_2)$ and where $d_1(y_1) = F(\alpha)(x_1)$, $d_1(y_2) = F(\beta)(x_2)$.

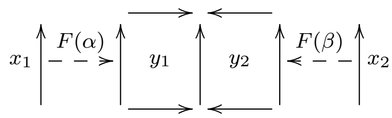
To ease exposition and interpretation later, it may help to draw a schematic diagram of form

$$x_1. \overset{F(\alpha)}{\dashrightarrow} . \xrightarrow{y_1} . \xleftarrow{y_2} . \overset{F(\beta)}{\dashleftarrow} .x_2\ .$$

- $n = 1$, a 1-simplex of $holim\, F$ consists of a triple

$$(x_1, (y_1, y_2), x_2),$$

where $x_1 \in F(1)_1$, $x_2 \in F(2)_1$, $y_1, y_2 \in (F(0)^I)_1$. The condition on y_1, y_2 is best illustrated diagrammatically, and we will incorporate the relationships with x_1 and x_2 in the same diagram:



so the 'd_0-faces' of y_1 and y_2 are equal, whilst $d_1(y_1) = F(\alpha)(x_1)$, $d_1(y_2) = F(\beta)(x_2)$. Of course, $y_i \in (F(0)^I)_1$, so can be represented by a square: $y_i \in \mathcal{S}(\Delta[1], F(0)^I) \cong \mathcal{S}(\Delta[1] \times I, F(0))$, and the '$d_0$-face' refers to the face of the square in the I-direction.

- $n = 2$, we have $x_i \in F(i)_2$ and $y_i : \Delta[2] \times I \to F(0)$ giving prisms, which agree on the e_0-face in the I-direction, and which are $F(\alpha)(x_1)$ and $F(\beta)(x_2)$ on the d_1-face in that direction.
- $n = 3$, etc., ..., and so on. The pattern continues.

It is worth noting that, if we replace $\mathbb{I} \downarrow -$ by $cons(\Delta[0])$, we would have the pullback square and the limit of F, rather than the homotopy limit, so replacing $cons(\Delta[0])(i)$ by the *contractible* $Ner(\mathbb{I} \downarrow i)$ gives the change from $lim\, F$ to $holim\, F$. There is an important, but simple, point here. For each i, there is a unique simplicial morphism,

$$\mathbb{I} \downarrow i \to \Delta[0],$$

of course, since $\Delta[0]$ is terminal. This is split by sending the unique non-degenerate 0-simplex, ι_0, of $\Delta[0]$ to the identity map on i in $\mathbb{I} \downarrow i$. This splitting is part of a contracting homotopy for $\mathbb{I} \downarrow i$, but is not compatible with composition, *i.e.*, if $\alpha : i \to j$, we have $\mathbb{I} \downarrow \alpha : \mathbb{I} \downarrow i \to \mathbb{I} \downarrow j$, and a commutative diagram

$$\begin{tikzcd}
\mathbb{I} \downarrow i \arrow[r, shift right] \arrow[d, "\mathbb{I}\downarrow\alpha"'] & \Delta[0] \arrow[l, dashed, shift right, "s(i)"'] \arrow[d, "="] \\
\mathbb{I} \downarrow j \arrow[r, shift right] & \Delta[0] \arrow[l, dashed, shift right, "s(j)"']
\end{tikzcd}$$

but $\mathbb{I} \downarrow \alpha \circ s(i)(\iota_0) = \langle \alpha \rangle$, the vertex of $\mathbb{I} \downarrow j$ corresponding to the object $\alpha : i \to j$ in $\mathbb{I} \downarrow j$. On the other hand, $s(j)(\iota_0) = \langle id_j \rangle$, so the splitting is not compatible with the other maps. This is reminiscent of the U-split resolutions, where, here, $U : \mathcal{S}^{\mathbb{I}} \to \mathcal{S}^{\mathbb{I}_0}$ forgets the morphisms between the objects in $\mathbb{I}$. (More precisely, $\mathbb{I}_0$ here is the discrete category on the objects of $\mathbb{I}$, so there is a functor, $inc : \mathbb{I}_0 \to \mathbb{I}$, and $U(F) = F \circ inc$.) The morphism, $(\mathbb{I} \downarrow -) \to \Delta[0]$, is U-split, but not, usually, split.

14.2 Profunctors / Distributors

Turning back towards the general discussion of indexed limits and colimits, it is very convenient to introduce a categorical idea which simplifies greatly both the intuition and the calculations. We will call these things *distributors*, or, more often, *profunctors*, (as they 'stand in' for functors when such may not be there). They are also called *(bi)modules*, but that *does* seem an overuse of the term, at least in our context. They are a categorification of relations. We will need them in an enriched setting, but will first look at them in the ordinary context of standard set-based category theory. (References for this section are to Bénabou's notes on *distributeurs*, [31], Borceux's first volume, [37] and the notes [32] by Thomas Streicher of a course given in Darmstadt by Bénabou. A brief introduction to the subject is also given in Cordier-Porter, [89].)

14.2.1 Definition

Suppose we have a functor, $F : \mathcal{A} \to \mathcal{B}$, between two categories, then we can form two other functors,

$$\varphi_F : \mathcal{B}^{op} \times \mathcal{A} \to Sets,$$

defined by $\varphi_F(B, A) = \mathcal{B}(B, FA)$, and a second one,

$$\varphi^F : \mathcal{A}^{op} \times \mathcal{B} \to Sets,$$

given by $\varphi^F(A, B) = \mathcal{B}(FA, B)$. These are the two *profunctors induced from F*. We write $\varphi_F : \mathcal{A} \nrightarrow \mathcal{B}$ and $\varphi^F : \mathcal{B} \nrightarrow \mathcal{A}$. More generally, these constructions give two profunctors in the enriched context as well.

We assume given a symmetric monoidal category, $\mathcal{V}$, with tensor, $\otimes = \otimes_{\mathcal{V}}$, and unit, $I = I_{\mathcal{V}}$, and two $\mathcal{V}$-categories, $\mathcal{A}$ and $\mathcal{B}$.

Definition: A *$\mathcal{V}$-profunctor*, or *$\mathcal{V}$-distributor*, $\varphi : \mathcal{A} \nrightarrow \mathcal{B}$, is a $\mathcal{V}$-functor, $\varphi : \mathcal{B}^{op} \times \mathcal{A} \to \mathcal{V}$.

The set of profunctors from $\mathcal{A}$ to $\mathcal{B}$ will be denoted $\mathcal{V}-Prof(\mathcal{A}, \mathcal{B})$. This is, in fact, the set of objects of a category in a natural way, once we have defined:

A *natural transformation* or *2-cell* between two *$\mathcal{V}$-profunctors* is just a $\mathcal{V}$-natural transformation between the corresponding functors.

Example: If $t : F \Rightarrow G$ is a natural transformation between two $\mathcal{V}$-functors, $F, G : \mathcal{A} \to \mathcal{B}$, then there is an induced 2-cell, $\varphi_t : \varphi_F \Rightarrow \varphi_G$ between the corresponding profunctors.

Composition of natural transformations makes $\mathcal{V}-Prof(\mathcal{A}, \mathcal{B})$ into a category internal to $\mathcal{V}$. This is a 'vertical composition'. There is also another 'horizontal' composition given by a sort of tensor.

14.2.2 Composition?

Imagine that $\mathcal{V} = Ab$, the category of Abelian groups with the usual tensor, and $\mathcal{A}$ and $\mathcal{B}$ were thus additive categories. (In fact if they have just one object, then they are just rings in the ordinary mathematical sense. This is the additive analogue of the categorification of groups to give single object groupoids.) An Ab-profunctor, $\varphi : \mathcal{A} \nrightarrow \mathcal{B}$, is then given by a bimodule with left $\mathcal{A}$ action and right $\mathcal{B}$ action. Suppose, again in this additive case, we had also $\psi : \mathcal{B} \nrightarrow \mathcal{C}$, *i.e.*, $\psi : \mathcal{C}^{op} \times \mathcal{B} \to Ab$, a left $\mathcal{B}$, right $\mathcal{C}$ bimodule. We want to compose the profunctors to get a profunctor, $\psi \otimes \varphi : \mathcal{A} \nrightarrow \mathcal{C}$, and thus a left $\mathcal{A}$, right $\mathcal{C}$ bimodule. In this situation, the obvious and natural construction would be a 'tensor product' of the bimodules, *i.e.*, to form

$$\psi \otimes \varphi(C, A) = \coprod_{B \in Ob(\mathcal{B})} \psi(C, B) \otimes \varphi(B, A)/\sim,$$

where $\sim$ identifies elements linked via the two actions of $\mathcal{B}$,

$$(g.b, f) \sim (g, b.f).$$

This is, effectively, a coend formula and will work for any $\mathcal{V}$ provided it is cocomplete. (It is also the contracted product idea, yet again.)

Definition: Suppose $\mathcal{V}$ is cocomplete. Let $\varphi : \mathcal{A} \nrightarrow \mathcal{B}$ and $\psi : \mathcal{B} \nrightarrow \mathcal{C}$ be $\mathcal{V}$-profunctors, then the *composite profunctor*, $\psi \otimes \varphi : \mathcal{A} \nrightarrow \mathcal{C}$, is given by

$$\psi \otimes \varphi(C, A) = \int^B \psi(C, B) \otimes_{\mathcal{V}} \varphi(B, A).$$

In perhaps more elementary terms, we get, as above, but with more precision,

$$\psi \otimes \varphi(C, A) = \coprod_{B \in Ob(\mathcal{B})} \psi(C, B) \otimes \varphi(B, A)/R,$$

where R is the equivalence relation generated by the relation

$$(\psi(C, b)(\beta), \alpha)R(\beta, \varphi(b, A)(\alpha)),$$

for $\alpha \in \varphi(B, A)$, $\beta \in \psi(B', C)$ and $b \in \mathcal{B}(B, B')$. If we need to, we will write $y \otimes x$ for the equivalence class determined by (y, x) with $x \in \varphi(B, A)$ and $y \in \psi(B, C)$. This element-wise formulation works best for the set-based case, but has an obvious sense in other cases as well. Of course, the coend version works regardless of whether or not $\mathcal{V}$ has 'elements', which can be important in sheaf based situations. In any case, one can often use 'generalised elements', that is morphisms to the objects concerned, exactly like local sections in a sheaf based situation.

We mention the following more for the techniques used in its proof, than for its use later in our discussions:

Lemma 104 *If $F : \mathcal{A} \to \mathcal{B}$, and $G : \mathcal{B} \to \mathcal{C}$, then*

$$\varphi_{G \circ F} \cong \varphi_G \otimes \varphi_F.$$

Proof: This is really just calculation. First,

$$\varphi_{G \circ F}(C, A) = \mathcal{C}(C, G(F(A))).$$

Now look at

$$(\varphi_G \otimes \varphi_F)(C, A) = \int^B \mathcal{C}(C, GB) \otimes_V \mathcal{B}(B, FA).$$

We will write the detailed calculation just for the set-based case as that contains the essence of the argument. As usual, the main point to note is that we have very little information on the objects $\mathcal{B}(B, FA)$ except in the one case when $B = FA$, when we know there is something there, namely the identity map on FA!

Suppose $y \in \mathcal{C}(C, GB)$ and $x \in \mathcal{B}(B, FA)$, then $x = \mathcal{B}(x, FA)(id_{FA})$, so

$$\begin{aligned} y \otimes x &= y \otimes \mathcal{B}(x, FA)(id_{FA}) \\ &= \varphi_G(C, x)(y) \otimes id_{FA} \\ &= (Gx \circ y) \otimes id_{FA}, \end{aligned}$$

and, of course, $Gx \circ y \in \mathcal{C}(C, G(F(A)))$. This sets up the isomorphism, but do note that it is an isomorphism, *not* an equality.

To generalise this to the general case, you may need to think of the coend in terms of the diagrams for a dinatural transformation. (This is **left to you**.) ■

14.2.3 The bicategory of $\mathcal{V}$-profunctors

We thus have what looks to be a 2-category with $\mathcal{V}$-profunctors between $\mathcal{V}$-categories, generalising, perhaps, the 2-category of small categories. This is not, however, quite correct. The composition is not associative, in general, so, just as the category of bimodules, or of bitorsors, gives us a monoidal category and *not* a strict monoidal category, $\mathcal{V}-Prof$ is a bicategory and not a 2-category. In fact, the composition is given by a coend, and such a thing is only defined *up to isomorphism*, so we have the composition is only defined *up to isomorphism* as well. This is not a defect and causes very little extra work, but does need to be kept in mind. (As with monoidal categories, it is a good thing to have your own favorite reference for ideas on bicategories, but, as before, this will be **left to you**. As we said earlier, the original reference is to Bénabou, [30], whilst there is a thorough treatment in Borceux, [37].)

There are some neat facts about profunctors / distributors, which makes them very useful. In Cat, we know how to say that two functors are adjoint. This situation can be encoded within the 2-categorical language of Cat, and so can be reformulated in a general 2-category, so we can say that two 1-arrows / morphisms between objects in a 2-category, $\mathcal{C}$, are adjoint if there are some 2-cells satisfying some equations. (Look up the details, say, in Borceux, [37].) We can generalise this, further, to bicategories and thus to $\mathcal{V}-Prof$, where we find the following amusing, and sometimes useful, fact:

Proposition 178 *For any $\mathcal{V}$-functor, $F : \mathcal{A} \to \mathcal{B}$, the corresponding $\mathcal{V}$-profunctor, $\varphi_F : \mathcal{A} \nrightarrow \mathcal{B}$ has a right adjoint, namely φ^F.* ■

We will not give a proof here, as there is a more useful result.

14.2.4 Right Kan extensions and profunctors

We know right adjoints correspond to right Kan extensions, so what about analogues of Kan extensions. For this, we will assume that $\mathcal{V}$ is *closed*, *i.e.*, that the functor $- \otimes X : \mathcal{V} \to \mathcal{V}$ has a right adjoint (an internal enriched hom-construction), which we will denote by $[X, -] : \mathcal{V} \to \mathcal{V}$. We will use this several times in the coming pages. Of course, for $\mathcal{V} = \mathcal{S}$, $[X, Y]$ is just the 'function space' simplicial set, $\underline{\mathcal{S}}(X, Y)$, that we have seen several times. (In many cases of $\mathcal{V}$, there are well established notations for this 'function space' object, as here for $\mathcal{V} = \mathcal{S}$, and we will, in such contexts, tend to use the specific notation rather than this generic one.)

Proposition 179 *If $\varphi : \mathcal{A} \nrightarrow \mathcal{B}$ and $\chi : \mathcal{A} \nrightarrow \mathcal{D}$ are two $\mathcal{V}$-profunctors, then the right Kan extension, $[\![\varphi, \chi]\!]$ of χ along φ is given by the end*

$$[\![\varphi, \chi]\!](D, B) = \int_A [\varphi(B, A), \chi(D, A)],$$

Proof: First note that $[\![\varphi, \chi]\!](D, B)$ is the same thing as $\mathcal{V} - Nat(\varphi(B, -), \chi(D, -))$, - just look at the 'end' descriptions. The enriched analogue of the right Kan extension property would be an isomorphism

$$[\![\psi \otimes \varphi, \chi]\!] \cong [\![\psi, [\![\varphi, \chi]\!]]\!],$$

natural in ψ, but this is clear from the 'end calculus' and the adjointness of $- \otimes Y$ and $[-, Y]$. ■

This looks wonderful. Kan extensions always exist and are given by an easy formula. Where is the snag? In some sense, there is none. It all works beautifully. From another perspective, however, there is one, namely, you usually start with a functor and want a functor out at the end. The profunctor method gives you some beautiful formulae that are very easy to manipulate, but one needs one final definition to bring it back to where we need it.

Definition: A $\mathcal{V}$-profunctor $\varphi : \mathcal{A} \nrightarrow \mathcal{B}$ is said to be *representable* if there is a functor, $F : \mathcal{A} \to \mathcal{B}$, such that φ is naturally isomorphic to φ_F.

Again we refer to the literature for some of the consequences of this notion. We will see it in use shortly. The trick is to prove things within the distributor / profunctor context and then to look for when the result is representable, so as to 'collapse down' from the 'virtual' world of profunctors to the 'real' world of categories and functors.

14.3 Indexed / weighted limits and colimits

Returning to 'indexed limits' as such, we consider a symmetric monoidal category, $\mathcal{V}$, as before, and will continue to assume that it is closed. In addition, we will need it to be complete and cocomplete. In general, it is often convenient to use the term *Bénabou cosmos* as follows:

Definition: A *(Bénabou) cosmos*, $\mathcal{V}$, is a complete and cocomplete closed symmetric monoidal category.

We then can simply ask that $\mathcal{V}$ be a cosmos[1].

We can build a $\mathcal{V}$-category, $\mathbb{1}$, having a single object, $*$, and with $\mathbb{1}(*, *) = I$, the unit object of $\mathcal{V}$. For the standard case of $\mathcal{V} = Sets$, there is a unique functor from any category, $\mathcal{A}$, to $\mathbb{1}$ and this allows the unambiguous construction of constant functors on $\mathcal{A}$. For a general $\mathcal{V}$, there is no longer such a unique $\mathcal{V}$-functor, and so no single notion of limit. For instance, take $\mathcal{V}$ to be $(Vect_K, \otimes, K)$. The unit object is the field K as a 1-dimensional vector space. It thus has many automorphisms. If $\mathcal{A}$ is a $(Vect_K, \otimes)$-category, there are many functors from $\mathcal{A}$ to $\mathbb{1}$, in fact even

[1] As there are also notions of 'cosmos' due to Street and an ∞-categorical version due to Riehl and Verity. We will not need this level of generality 'for the moment', but, if the needs arise, we will refer to the version due to Bénabou as a Bénabou cosmos to distinguish it from the other notions.

many functors from $\mathbb{1}$ to itself. The job done by a functor $\varphi : \mathcal{A} \to \mathbb{1}$ in the set-based case will be done by a functor, or, more generally, a profunctor, $\varphi : \mathcal{A} \nrightarrow \mathbb{1}$, in the enriched case. Such a $\mathcal{V}$-profunctor is just an ordinary $\mathcal{V}$-functor, $\varphi : \mathcal{A} \to \mathcal{V}$. We will use such a φ to 'index' or 'weight' our limits and colimits.

14.3.1 Indexed cones

Let now $F : \mathcal{A} \to \mathcal{B}$ be a $\mathcal{V}$-functor:

Definition: The *φ-indexed cone* functor on F, denoted $[\![\varphi, F]\!] : \mathcal{B}^{op} \to \mathcal{V}$, or, equivalently, $[\![\varphi, F]\!] : \mathbb{1} \nrightarrow \mathcal{B}$, is the right Kan extension of F along φ, given by the end

$$[\![\varphi, F]\!](B) = \int_A [\![\varphi(A), \mathcal{B}(B, F(A))]\!].$$

This formula is worth 'taking apart': First, if we had had functors rather than profunctors, $[\![\varphi, F]\!] : \mathbb{1} \to \mathcal{B}$ would, naively, have given us an object of $\mathcal{B}$, which does not seem quite right, so let us look at the set-based case, where $\mathbb{1}(*, *)$ is just the singleton set. The expression inside the end is then $[\![1, \mathcal{B}(B, F(A))]\!]$, which is the *set* of morphisms, $\mathcal{B}(B, F(A))$, and, yes, this end makes this into the *set* of cones on F and gives a functor, $Cone(-, F) : \mathcal{B}^{op} \to Sets$, *i.e.*, a *profunctor* $\mathbb{1} \nrightarrow \mathcal{B}$.

As an example of an indexed limit, take $\mathcal{V} = \mathcal{S}$ and $\varphi : \mathbb{I} \to \mathcal{S}$ to be $\varphi(i) = Ner(\mathbb{I} \downarrow i)$, where we have an ordinary category, $\mathbb{I}$, thought of as a trivially enriched $\mathcal{S}$-category. We take $\mathcal{B} = \mathcal{S}$ and $F : \mathbb{I} \to \mathcal{B}$, a diagram of simplicial sets. (Taking $\mathbb{I}$ to be our 'pullback template' is a good exercise if you feel the need to go into more 'concrete' examples.) What $[\![\varphi, F]\!](B)$ is is the simplicial set,

$$\begin{aligned} [\![\varphi, F]\!](B) &= \int_i \underline{\mathcal{S}}(Ner(\mathbb{I} \downarrow i), \underline{\mathcal{S}}(B, F(i))) \\ &= \int_i \underline{\mathcal{S}}(B \times Ner(\mathbb{I} \downarrow i), F(i)), \end{aligned}$$

which gives homotopy coherent cones on F with vertex B in dimension 0, homotopies of such cones in dimension 1, and so on. (To see this in a simple example, take $\mathbb{I}$ to be the category with two objects and a single arrow between them, so, essentially, the ordinal $[1] = \{0 < 1\}$ considered as a category, then $Ner(\mathbb{I} \downarrow 0) \cong \Delta[0]$, $Ner(\mathbb{I} \downarrow 1) \cong \Delta[1]$, F is $F(0) \overset{F(01)}{\to} F(1)$, and a vertex of $[\![\varphi, F]\!](B)$ consists of a morphism, $B \overset{f(0)}{\to} F(0)$, together with a homotopy, $f(1) : B \times \Delta[1] \to F(1)$, agreeing with $F(01) \circ f(0)$ on the initial end, (*i.e.*, $f(1) \circ e_0$), thus giving a diagram

$$\begin{tikzcd} & F(1) \\ B \arrow[ur, "g(0)"] \arrow[dr, "f(0)"'] \arrow[r, phantom, "f(1)\ \Nearrow"] & \\ & F(0) \arrow[uu, "F(01)"'] \end{tikzcd}$$

where $g(0) = f(1) \circ e_1$.

A 1-simplex of $[\![\varphi, F]\!](B)$ will be a homotopy between two such h. c. cones, fixed on the diagram corresponding to F.

Of course, we are really interested in the case where $[\![\varphi, F]\!](-)$ is representable.

14.3.2 Indexed / weighted limits: the definition

Definition: We say that the $\mathcal{V}$-functor, F, admits a *φ-indexed limit* or *φ-weighted limit* if $[\![\varphi, F]\!]$ is representable, so there is an object, $\varphi{-}lim\, F$, of $\mathcal{B}$ such that

$$[\![\varphi, F]\!](B) \cong \mathcal{B}(B, \varphi{-}lim\, F)$$

naturally in B.

We also need 'φ-indexed colimits':

Definition: The $\mathcal{V}$-functor, $L : \mathcal{A}^{op} \to \mathcal{B}$, has a *$\varphi$-indexed colimit*, denoted $\varphi - colim\, L$ if $L^{op} : \mathcal{A} \to \mathcal{B}^{op}$ has a φ-indexed limit.

This definition may seem a bit convoluted, but this is only due to φ being a profunctor from $\mathcal{A}$ to $\mathbb{1}$, not from $\mathcal{A}^{op}$ to $\mathbb{1}$. The fact that we are working with L^{op} and $\mathcal{B}^{op}$ merely means that a bit more care has to be taken in the calculations.

14.3.3 Some first examples:

(1) Clearly ordinary limits are indexed limits. For them, $\mathcal{V} = Sets$ and φ is the profunctor corresponding to the *unique* functor from $\mathcal{A}$ to $\mathbb{1}$. We then have, for $F : \mathcal{A} \to \mathcal{B}$,

$$\begin{aligned}
[\![\varphi, F]\!](B) &= \int_A Sets(\varphi(A), \mathcal{B}(B, F(i))) \\
&= \int_A \mathcal{B}(B, F(A)) \\
&= Nat(const(B), F) \\
&= cone(B, F) \\
&\cong \mathcal{B}(B, lim\, F).
\end{aligned}$$

(2) For $\mathcal{V} = \mathcal{S}$, and $\varphi = \mathbb{I} \downarrow - : \mathbb{I} \to \mathcal{S}$, then, for $F : \mathbb{I} \to \mathcal{S}$,

$$\begin{aligned}
[\![\varphi, F]\!](B) &= \int_i \underline{\mathcal{S}}(\mathbb{I} \downarrow i, \underline{\mathcal{S}}(B, F(i))) \\
&= \int_i \underline{\mathcal{S}}(B \times \mathbb{I} \downarrow i, F(i)), \qquad \text{as before,} \\
&= \int_i \underline{\mathcal{S}}(B, \underline{\mathcal{S}}(\mathbb{I} \downarrow i, F(i))) \\
&= \underline{\mathcal{S}}(B, \int_i \underline{\mathcal{S}}(\mathbb{I} \downarrow i, F(i))),
\end{aligned}$$

so $(\mathbb{I} \downarrow -){-}lim\, F = holim\, F$, the Bousfield-Kan homotopy limit.

Whilst we are in this context, we can look at φ-indexed colimits as well. For the moment, we will stick with the same φ, although this is a bit awkward and usually not 'optimal'. We take $F : \mathbb{I} \to \mathcal{S}$, as before, but now $\mathcal{A}$ is the dual category $\mathbb{I}^{op}$, so $\varphi(i) = \mathbb{I}^{op}/i$ or more precisely, $Ner(\mathbb{I}^{op}/i)$. An n-simplex of $\varphi(i)$, thus looks like $i \to i_n \to \ldots \to i_0$ back in $\mathbb{I}$. The form of the calculation of $\varphi{-}colim\, F$ is actually independent of the particular φ that we use. It goes like this, using a bit of 'end calculus' and the adjointness relations for $\times$ and the 'internal' function space functor:

$$\begin{aligned} [\![\varphi, F]\!](B) &= \int_i \underline{\mathcal{S}}(\varphi(i), \underline{\mathcal{S}}^{op}(B, F(i))) \\ &= \int_i \underline{\mathcal{S}}(\varphi(i), \underline{\mathcal{S}}(F(i), B))) \\ &= \int_i \underline{\mathcal{S}}(\varphi(i) \times F(i), B) \\ &\cong \underline{\mathcal{S}}(\int^i \varphi(i) \times F(i), B), \end{aligned}$$

so $\varphi{-}colim\, F \cong \int^i \varphi(i) \times F(i)$.

The indexation we used is the *conjugate* of the usual one, which is $\varphi(i) = Ner(i/\mathbb{I})$. For that indexation, the φ-colimit is the *Bousfield-Kan homotopy colimit* introduced by them in [42].

(3) The Yoneda embedding from a category, $\mathcal{C}$, into $Sets^{\mathcal{C}^{op}}$, specialises to one from $\mathbf{\Delta}$ to $\mathcal{S}$, which we will denote Yo, thus $Yo([n]) = \Delta[n]$, the usual n-simplex model in $\mathcal{S}$.

If $X : \mathbf{\Delta} \to \mathcal{S}$ is a cosimplicial simplicial set, then the enriched hom, $\mathcal{S}^{\mathbf{\Delta}}(Yo, X)$, is the same as $Yo{-}lim\, X$.

The following is 'for the record'. It is just a question of comparison.

Proposition 180 *If $X : \mathbf{\Delta} \to \mathcal{S}$ is a cosimplicial simplicial set, its* Bousfield-Kan total complex *of X is the Yo-indexed limit of X.* ∎

We will consider a generalisation of this in a moment, but first note that, dually, if X was a bisimplicial set, then we could work out its Yo-colimit. The calculation we made in the previous example adapts to this situation to show

$$Yo{-}colim\, X \cong \int^n Yo([n]) \times X([n], .),$$

and hence is exactly the diagonal, $diag X$, of X.

Back to Yo-limits, if we have that $\mathcal{B}$ is a complete $\mathcal{S}$-category, then, as we mentioned back on page 341, we have cotensors, denoted $\overline{\mathcal{B}}(K, B)$, in $\mathcal{B}$. We can then form, as in section 13.6.1, the *enriched total object* or *enriched total complex*:

$$Tot(X) = \int_n \overline{\mathcal{B}}(Yo([n]), X^n).$$

A variant of this is taken up in the next section.

(4) A final example is to take $\mathcal{A}$ to be a small category, $F : \mathcal{A} \to Cat$, a functor, and $\varphi(A) = A/\mathcal{A} = A \downarrow \mathcal{A}$, the category of objects under A. The φ-colimit of F is more or less the Grothendieck construction on F, giving the fibred category over $\mathcal{A}$ corresponding to F. The details are omitted and the statement *is* slightly simplified above. The basic formula therefore looks something like

$$\int^A A/\mathcal{A} \times F(A),$$

but we will return to this later, so will not expand on it here.

14.3.4 Total category of a cosimplicial category

Let $\mathcal{C} : \mathbf{\Delta} \to Cat$ be a cosimplicial category and, as before, $Yo : \mathbf{\Delta} \to \mathcal{S}$, the Yoneda embedding with $Yo([n]) = \Delta[n]$. As Cat is a complete $\mathcal{S}$-category, we can form a *total category* of this cosimplicial category, $\mathcal{C}$. It will be useful, later on, to have quite a thorough and explicit description of the resulting category, $Tot(\mathcal{C})$, so we will expand the structure bit-by-bit.

Firstly, recall that the $\mathcal{S}$-category structure on Cat is given by taking the 2-category structure (which is just the Cat-enriched structure that considers each $Cat(C, D)$ as D^C, the category of functors from C to D) and then applying the nerve functor to each 'hom-category'. This uses that Cat is Cartesian closed, but, if we use that once more, we can mirror earlier results, obtaining

$$\begin{aligned} Ner(\underline{Cat}(C, D))_n &= Cat([n], \underline{Cat}(C, D)) \\ &= Cat([n] \times C, D), \end{aligned}$$

as before, page 497. We may sometimes write $Cat_{\mathcal{S}}$ for Cat with this $\mathcal{S}$-category structure.

Next, we need to get the cotensor structure. Suppose we look at what $\overline{Cat}(\Delta[n], D)$ has to be. It has to satisfy

$$Cat(C, \overline{Cat}(\Delta[n], D)) \cong \mathcal{S}(\Delta[n], \underline{Cat}(C, D)),$$

but this is $Ner(\underline{Cat}(C, D))_n$, which is $Cat([n] \times C, D)$ and thus is $Cat(C, \underline{Cat}([n], D))$. In other words,

$$\overline{Cat}(\Delta[n], D) \cong D^{[n]} = [[n] \to D],$$

adopting another frequently used notation, which is easier to use below. (We could also work out $\overline{Cat}(K, D)$, for a general simplicial set, K, but **leave that to you** as we will not, in fact, need it.)

We thus need to examine

$$Tot(\mathcal{C}) = \int_n [[n] \to C^n].$$

(As a small category can be reconstructed from its nerve, the calculation here will be very similar to our earlier one with a homotopy pullback. We can calculate things dimension-wise and so will adopt a simplicial-like notation.)

- $Tot(\mathcal{C})_0$, that is the set of objects of $Tot(\mathcal{C})$, consists of sequences (x_n) with $x_n : [0] \times [n] \to C^n$ and satisfying some cosimplicial relations. (Of course, $[0] \times [n] \cong [n]$.) We thus have
  - $x_0 : [0] \times [0] \to C^0$ is an object of C^0;

- $x_1 : [0] \times [1] \to C^1$ is an arrow of C^1 such that $x_1 : d^1 x_0 \to d^0 x_0$. (We are, as usual, using d^i for the *co*-face operators of the *co*-simplicial category, $\mathcal{C}$.)
- $x_2 : [0] \times [2] \to C^2$ *is* a triangular diagram in C^2 of form:

$$\begin{tikzcd} & \bullet \arrow[dr, "d^0x_1"] & \\ \bullet \arrow[ur, "d^2x_1"] \arrow[rr, "d^1x_1"'] \arrow[rr, phantom, shift left=4ex, "x_2"] & & \bullet \end{tikzcd}$$

so this is a *cocycle condition* which just states that

$$d^1 x_1 = d^0 x_1 \cdot d^2 x_1$$

within C^2;
- the x_n, for $n \geq 3$, are completely determined by (x_0, x_1, x_2), since, for instance, x_3 is a tetrahedral diagram in C^3, which commutes precisely because each of its four faces commutes, but these are images, by the cofaces, of x_2.

* $Tot(\mathcal{C})_1$, the set of morphisms of $Tot(\mathcal{C})$ has, as elements, sequences, $\underline{y} = (y_n)$, where $y_n : [1] \times [n] \to C^n$, and again satisfying some compatibility identities. Here we have

- $y_0 : [1] \times [0] \to C^0$, so is an arrow in C^0, written $y_0 : s(y_0) \to t(y_0)$;
- $y_1 : [1] \times [1] \to C^1$ and is a commutative square diagram in C^1:

$$\begin{tikzcd} \bullet \arrow[r, "d^0y_0"] & \bullet \\ \bullet \arrow[u, "s(y_1)"] \arrow[r, "d^1y_0"'] \arrow[ur, phantom, "y_1"] & \bullet \arrow[u, "t(y_1)"'] \end{tikzcd}$$

such that, in the second direction, $y_1 : d^1 y_0 \to d^0 y_0$;
- $y_2 : [1] \times [2] \to C^2$ is a prismatic diagram in C^2, again commutative. As the triangular ends are $s(y_2)$ and $t(y_2)$, this guarantees that the source and target of y_2 satisfy the cocycle condition that we saw above, and imposes no extra constraints on the square faces as they are the $d^i y_1$ for $i = 0, 1, 2$, and so are known already to be commutative;
- for $n \geq 3$, $y_n : [1] \times [n] \to C^n$, and is determined by the lower dimensional parts of $\underline{y}$.

* The source and target maps of the various C^n give

$$s, t : Tot(\mathcal{C})_1 \to Tot(\mathcal{C})_0$$

(You are left to check that s and t applied to a $\underline{y}$ give a sequence in $Tot(\mathcal{C})_0$. Nearly all the details are given above, but you may feel that they need 'linking' together more than has been done there.)

- To complete the description of $Tot(\mathcal{C})$ as a small category, we need to note (i) the identities (hopefully it is clear that i on the various C^n induces a map from $Tot(\mathcal{C})_0$ to $Tot(\mathcal{C})_1$, which satisfies $si = ti =$ identity on $Tot(\mathcal{C})_0$), and (ii) to produce a composition

$$Tot(\mathcal{C})_{1s} \times_t Tot(\mathcal{C})_1 \to Tot(\mathcal{C})_1.$$

There is an evident candidate, corresponding to 'horizontal' composition of the sequences, $\underline{y}$, (where 'horizontal' refers to the diagrams that we have used above). We would expect the nerve of $Tot(\mathcal{C})$ to be related to the total complex of the cosimplicial simplicial set $Ner(\mathcal{C})$, so the composition should also be evident in a calculation of $Tot(\mathcal{C})_2$. The elements here are sequences, $\underline{z} = (z_n)$, with

$$z_n : [2] \times [n] \to C^n$$

and $z_0 : [2] \times [0] \to C^0$ gives the above 'composition' as expected. (**You are left** to investigate any significance behind z_1, etc.)

Before passing to more examples and generalisations, it is worth looking back at the definition of the descent category, $\mathsf{Des}(\mathcal{U}, F)$, on page 437, for comparison. We will return to consider this in more detail in a later chapter, (see page **??** and the development there).

14.3.5 Lax and pseudo-(co)ends

We need to use some of the ideas above to provide more tools for handling pseudo-functors / fibred categories (and thus prestacks and stacks). We will be working with strict 2-categories as well as with weak ones (bicategories) plus strict, lax, op-lax and pseudo-functors. The main tools we need are the lax, etc., versions of ends and coends (and then weighted / indexed (co)limits) as this will give an entry point to handling the corresponding versions of Kan extensions[2]. We start by looking at lax versions of basic ideas the we have met earlier in section 13.4, on ends and coends, on page 640. We will be using, several times, the idea that a 2-category can be considered to be a 3-category in which the only 3-cells are identities[3].

To approach the necessary lax formulas starting from a classical viewpoint, we assume given two strict 2-categories, $\mathcal{A}$ and $\mathcal{B}$, and a strict 2-functor, $T : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{B}$ and let b be an object of $\mathcal{B}$. We know what a *wedge with base T and vertex, b*, looks like in the classical case[4], so we imagine how to 'laxify' it. We have a diagram in $\mathcal{B}$, back there, which commutes, so we replace it by one with a 2-cell 'measuring' how the two composites are related. We thus have 1-cells, $\omega_a : b \to T(a, a)$, one for each object, a in $\mathcal{A}$, and for each 1-cell, $f : a \to a'$, in $\mathcal{A}$, a 2-cell

$$\omega_f : T(a, f)\sharp_0\omega_a \Rightarrow T(f, a')\sharp_0\omega_{a'},$$

[2] A useful informal survey article on, or introduction to, these generalised ends / coends is to be found in the notes of Loregian, [188], and we will be mixing his treatment with the ideas originally developed by Bozapalides, in his thesis, [44] and papers [45, 46].

[3] This is the same idea as when showing that, for instance, an n-cocycle condition must be satisfied as there are no non-idenitity / non-trivial $(n + 1)$-cocycles around.

[4] look back at page 640.

This then looks like:

$$\begin{tikzcd}
& T(a,a) \arrow[dr, "T(a,f)"] & \\
b \arrow[ur, "\omega_a"] \arrow[dr, "\omega_{a'}"'] \arrow[rr, phantom, "\Downarrow \omega_f"] & & T(a,a') \\
& T(a',a') \arrow[ur, "T(f,a')"'] &
\end{tikzcd}$$

If, now, $\lambda : f \Rightarrow f'$ is a 2-cell in $\mathscr{A}$, then we get such a 2-cell, as above, for both f and f', as well as the corresponding images of

$$\begin{tikzcd}
a \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\lambda"] & a'
\end{tikzcd} .$$

This gives us a 3-dimensional diagram which can be thought of as being made up of two pieces: a top,

$$\begin{tikzcd}
& T(a,a) \arrow[dr, bend left, "T(a,f)"] & \\
b \arrow[ur, "\omega_a"] \arrow[dr, "\omega_{a'}"'] \arrow[r, phantom, "\Downarrow \omega_f"] & {} & T(a,a') \\
& T(a',a') \arrow[ur, bend left, "T(f,a')"] \arrow[ur, bend right, "T(f',a')"'] \arrow[ur, phantom, "\Downarrow T(\lambda,a')"] &
\end{tikzcd}$$

and a bottom,

$$\begin{tikzcd}
& T(a,a) \arrow[dr, bend left, "T(a,f)"] \arrow[dr, bend right, "T(a,f')"'] \arrow[dr, phantom, "\Downarrow T(a,\lambda)"] & \\
b \arrow[ur, "\omega_a"] \arrow[dr, "\omega_{a'}"'] \arrow[r, phantom, "\Downarrow \omega_{f'}"] & {} & T(a,a') \\
& T(a',a') \arrow[ur, bend right, "T(f',a')"'] &
\end{tikzcd}$$

and the composite 2-cells must be the same[5]. We can write this down algebraically as:

$$(T(\lambda, a')\sharp_0\omega_{a'})\sharp_1\omega_f = \omega_{f'}\sharp_1(T(a,\lambda)\sharp_0\omega_a).$$

That handles compatibility of the ω_fs with the 2-cells in $\mathscr{A}$, but what about their compatibility with basic composition of 1-cells. Suppose, in addition to $f : a \to a'$, we have a second 1-cell, $g : a' \to a''$ and thus the composite $gf : a \to a''$. Both g and gf give us two cells:

$$\omega_g : T(a', g)\sharp_0\omega_{a'} \Rightarrow T(g, a'')\sharp_0\omega_{a''},$$

[5] As always if we are thinking in higher dimensions, as $\mathscr{B}$ is a 2-category,thought of as anhigher dimensional category, there are no *non-identity* 3-cells to compare these composites.

and

$$\omega_{gf} : T(a, gf)\sharp_0\omega_a \Rightarrow T(gf, a'')\sharp_0\omega_{a''},$$

respectively. This gives

$$\begin{tikzcd}
 & T(a,a) \arrow[r, "T(a,f)"] \arrow[d, phantom, "\Downarrow\omega_f"] & T(a,a') \\
b \arrow[ur, "\omega_a"] \arrow[r, "\omega_{a'}"] \arrow[dr, "\omega''_a"'] & T(a',a') \arrow[ur, "T(f,a')"'] \arrow[dr, "T(a',g)"] \arrow[d, phantom, "\Downarrow\omega_g"] & \\
 & T(a'',a'') \arrow[r, "T(g,a'')"'] & T(a',a'')
\end{tikzcd}$$

which can be completed to give

$$\begin{tikzcd}
 & T(a,a) \arrow[r, "T(a,f)"] \arrow[d, phantom, "\Downarrow\omega_f"] & T(a,a')) \arrow[dr, "T(a,g)"] & \\
b \arrow[ur, "\omega_a"] \arrow[r, "\omega_{a'}"] \arrow[dr, "\omega''_a"'] & T(a',a') \arrow[ur, "T(f,a')"'] \arrow[dr, "T(a',g)"] \arrow[d, phantom, "\Downarrow\omega_g"] & & T(a,a'') \\
 & T(a'',a'') \arrow[r, "T(g,a'')"'] & T(a',a'') \arrow[ur, "T(f,a'')"'] &
\end{tikzcd}$$

and the right hand square commutes as it is the image (under T) of a commuting square in $\mathscr{A}$.

(This looks like the usual picture of one half of a cube, and so we might expect that the other half is around somewhere!)

For gf, we have

$$\begin{tikzcd}
 & T(a,a) \arrow[dr, "T(a,gf)"] & \\
b \arrow[ur, "\omega_a"] \arrow[dr, "\omega_{a''}"'] \arrow[r, phantom, "\Downarrow\omega_{gf}"] & & T(a,a'') \\
 & T(a'',a'') \arrow[ur, "T(gf,a'')"'] &
\end{tikzcd}$$

but T is a functor, so $T(gf, a'') = T(g, a'')\sharp_0 T(f, a'')$, etc., and we can complete this diagram to get

$$\begin{tikzcd}
 & T(a,a) \arrow[r, "T(a,f)"] \arrow[dr, "T(a,gf)"] & T(a,a') \arrow[dr, "T(a,g)"] & \\
b \arrow[ur, "\omega_a"] \arrow[dr, "\omega_{a''}"'] \arrow[r, phantom, "\Downarrow\omega_{gf}"] & & T(a,a'') \arrow[r, "id_{T(a,a'')}"] & T(a,a'') \\
 & T(a'',a'') \arrow[ur, "T(gf,a'')"'] \arrow[r, "T(g,a'')"'] & T(a',a'') \arrow[ur, "T(f,a'')"'] &
\end{tikzcd}$$

The two parts glue together to give a cube, and[6] so the composite 2-cells in the two parts must be equal. If we had just had that T was just a pseudo-functor then the commutativity of the two

[6]again because there are no 3-cells, or, if you prefer, because the only 3-cells are identities

'cells' in the second part of the cube would be replaced by suitable 2-cells giving the isomorphisms. The picture would be similar if we considered a lax or op-lax functor for T, but the composites are more messy to write down[7].

Here we can easily read off:

$$(T(f,a'')\sharp_0\omega_g)\sharp_1(T(a,g)\sharp_0\omega_f) = \omega_{fg}.$$

Adding in a normalisation condition that $\omega_{id_a} = id_{\omega_a}$, we get

Definition: Let $T : \mathcal{A}^{op}\times\mathcal{A} \to \mathcal{B}$ be a strict 2-functor (as above), and let b be an object of $\mathcal{B}$. A *lax wedge*, $\underline{\omega}$, with base T and vertex b is a pair, $\underline{\omega} := \{\underline{\omega}_{Obj}, \underline{\omega}_{Arr}\}$, where $\underline{\omega}_{Ob} = \{\omega_a \mid a \in Ob(\mathcal{A})\}$ and $\underline{\omega}_{Arr} = \{\omega_f \mid f \in Arr(\mathcal{A})\}$ are families[8] of respectively 1-cells and 2-cells of $\mathcal{B}$ satisfying the conditions mentioned above. It will be convenient to write $\underline{\omega} : b \Rightarrow T$ to indicate a lax wedge with base T and vertex b.

The *set* of lax wedges with base, T, and vertex, b, forms the set of objects of a category in which the morphisms are *modifications* between such wedges.

Definition: A *modification*, $\underline{m} : \underline{\omega} \Rrightarrow \underline{\sigma}$, between two lax wedges with the same base and vertex, consists of a collection, $\underline{m} := \{m_a : \omega_a \Rightarrow \sigma_a \mid a \in Ob(\mathcal{A})\}$, of 2-cells such that the two composite 2-cells in the following diagrams:

$$\begin{tikzcd}
& T(a,a) \arrow[dr, "T(a,f)"] & \\
b \arrow[ur, bend left, "\omega_a"] \arrow[dr, bend right, "\sigma_{a'}"'] \arrow[dr, bend left, "\omega_{a'}"] \arrow[dr, phantom, "\Downarrow m_{a'}"] & \Downarrow \omega_f & T(a,a') \\
& T(a',a') \arrow[ur, "T(f,a')"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& T(a,a) \arrow[dr, "T(a,f)"] & \\
b \arrow[ur, bend left, "\omega_a"] \arrow[ur, bend right, "\sigma_a"'] \arrow[ur, phantom, "\Downarrow m_a"] \arrow[dr, bend right, "\sigma_{a'}"'] & \Downarrow \sigma_f & T(a,a') \\
& T(a',a') \arrow[ur, "T(f,a')"'] &
\end{tikzcd}$$

are equal, *i.e.*, 'algebraically':

$$(T(f,a')\sharp_0 m_{a'})\sharp_1\omega_f = \sigma_f\sharp_1(T(a,f)\sharp_0 m_a).$$

The definition of modification in this context is modelled on that of modifications between lax and pseudo-natural transformations that we looked at in section 11.5.5. Although there, we

[7] One of these several cases is considered in Bozapalides, [45].

[8] There is a tacit assumption here that $\mathcal{A}$ be small.

limited ourselves to the case of 'pseudo' rather than '(op)lax' natural transformations[9] we could have considered the lax or the op-lax case with a bit of effort. In any case, our contention is that thinking of things not at the 2-categorical level, but ∞-categorically, is often advantageous. (That being said, in applications of this theory, the 2-categorical or bicategorical setting is the form in which the problems and structures arise, so it is important to be able to handle them in that form as well.)

As we have seen earlier, there is a 3-category $2-\mathsf{Cat}$, of 2-categories, having 2-functors as its 1-cells, 2-natural transformations as 2-cells with the modifications then forming the 3-cells. There are pseudo and lax / oplax forms of this as well, however these hit problems, so there is a 3-category consisting of 2-categories, (strong or strict) 2-functors, pseudo-natural transformations, and modifications, but no laxness is possible at this level (without "laxifying" the notion of 3-category), see the n-Lab, [221] for discussion[10].

Given a lax wedge, $\underline{\omega}$, with base, T, and vertex, b', say, and a morphism $\varphi : b \to b'$, we can form a lax wedge on T with vertex b by precomposing the various families of cells defining $\underline{\omega}$ with φ, giving us $\underline{\omega}\sharp_0\varphi$. The collection of lax wedges with given (b, T) with the modifications between them, form a category, $\mathsf{LaxWedge}(b, T)$, and the above construction of $\underline{\omega}\sharp_0\varphi$ can easily be shown to make this functorial in b. The definition of lax end for T then essentially asks that this functor be representable.

Definition: Let $T : \mathscr{A}^{op} \times \mathscr{A} \to \mathscr{B}$ be, as before, a 2-functor. A lax wedge, $\underline{\omega}$, is called a *lax end* of T and its vertex denoted $\oint_a T(a,a)$, or, more briefly, $\oint T$, (so $\underline{\omega} : \oint T \Rightarrow T$), if, for any other lax wedge, $\underline{\sigma} : b' \Rightarrow T$, there exists a unique 1-cell, $x : b' \to \oint T$, such that $\underline{\sigma} = \underline{\omega}\sharp_0 x$, and, if every modification, $\theta : \underline{\sigma} \Rrightarrow \underline{\sigma}'$ induces a unique 2-cell, $\lambda : x \Rightarrow x'$ such that $\lambda\sharp_0\omega_a = \theta_a$ for each a in $\mathscr{A}$.

We note that this is merely a statement of the representability of $\mathsf{LaxWedge}(-, T)$ as, to a lax wedge, $\underline{\sigma} : b' \Rightarrow T$, one can assign the corresponding $x : b' \to \oint T$, whilst the condition on modifications extends this to a functor. This then gives:

Proposition 181 *If $\oint_a T(a,a)$ exists, then there us a natural isomorphism of categories,*

$$\mathsf{LaxWedge}(b', T) \cong \mathscr{B}(b', \oint T)$$

■

Remarks: (i) We note that this being an *isomorphism* rather than an equivalence is rather against the spirit of the 'categorification' process and this is not the most general form we could use[11]. It is however the form that is often found in the sources. A similar point occurs with homotopy limits and colimits. A construction will give something that have an evident universal property built into it. There will be, however, a definition using a homotopy universal property which is often 'instantiated' or 'presented' by the construction. In any case, it is worth stressing that specifying an equivalence is more than just saying that the two categories are equivalent, and in specifying the equivalence one has to take care of the coherence between the data involved.

---

[9] and even concentrated on the case in which the domain 2-category was 'locally discrete',

[10] The problems that occur are closely related to the inability of strict 3-groupoids to model all 3-types, and to structures involving the Gray-tensor products and analogues in homotopy theory. This is a complex area, but it quite fascinating! We will see similar problems arising when considering homotopy coherent transformations later on.

[11] An even weaker form would be to ask that there be an adjunction between these two categories.

(ii) There is a 'pseudo' and an 'op-lax' form of this as we mentioned, and also a lax, etc., coend version obtained by dualising.

(iii) Trying to laxify everything in sight hits the snag that 'whiskering' does not in general work for lax natural transformations and lax functors. This difficulty will not be important to us as the instances for which we need the ideas do not involve that level of laxness.

We will next write down a presentation[12] of $\oint_a T(a,a)$, due to Bozapalides, [45], where $T : \mathcal{A}^{op} \times \mathcal{A} \to \mathsf{Cat}$, as in this frequently occurring an explicit description of a model for $\oint T$ is fairly easy to give and this should help the understanding of what $\oint T$ looks like in general. It gives a recipe which can be mimicked or adapted to handles the general case provided $\mathcal{B}$ is sufficiently complete. (It also exhibits a clear joint family resemblance with the sort of construction which we have been considering in a simplicial context.)

We start with $T : \mathcal{A}^{op} \times \mathcal{A} \to \mathsf{Cat}$ and give a description of a category, $LE(T)$,[13] which will *be* a lax end of T. This has, as objects, pairs, $(\underline{x}_{Obj}, \underline{x}_{Arr})$, where

$$\underline{x}_{Obj} = \{x_a \mid a \in Ob(\mathcal{A})\}$$

for $x_a \in Ob(T(a,a))$, and

$$\underline{x}_{Arr} = \{x_f \mid f : a \to b \text{ in } \mathcal{A}\}$$

and, when $f : a \to b$, x_f is a morphism in $T(a,b)$ from $T(a,f)(x_a)$ to $T(f,b)(x_b)$. These pairs are to be such that

(i) for any 2-cell, $\lambda : f \Rightarrow g : a \to b$ in $\mathcal{A}$, the diagram

$$\begin{tikzcd} T(a,f)(x_a) \arrow[r, "T(a,\lambda_a)(x_a)"] \arrow[d, "x_f"'] & T(a,g)(x_a) \arrow[d, "x_g"] \\ T(f,b)(x_b) \arrow[r, "T(\lambda_b,b)(x_b)"'] & T(g,b)(x_b) \end{tikzcd}$$

commutes,

and

(ii) for any composable pair of 1-cells, $a \xrightarrow{f} b \xrightarrow{g} c$ in $\mathcal{A}$

$$x_{gf} = T(f,c)(x_g)\sharp_0 T(a,g)(x_f).$$

(N.B. This uses that T is a strict 2-functor, so $T(a,g)T(a,f) = T(a,gf)$ and $T(f,c)T(g,c) = T(gf,c)$.)

(iii) We note that a normalisation relation holds automatically:
for every object, a in $\mathcal{A}$, $T(id_a,a)(x_a) = T(a,id_a)(x_a) = x_a$. This holds because T is a (strict) 2-functor.

[12] even an explicit construction / description
[13] standing for **L**ax **E**nd of T.

The morphisms of $LE(T)$ from $\underline{x} = (\underline{x}_{Obj}, \underline{x}_{Arr})$ to $\underline{y} = (\underline{y}_{Obj}, \underline{y}_{Arr})$ are to be families, $\underline{m} = \{m_a : x_a \to y_a \mid a \in Ob(\mathcal{A})\}$, of 1-cells of $\mathcal{A}$ such that, for $f : a \to b$ in $\mathcal{A}$, the square

$$\begin{tikzcd}
T(a,f)(x_a) \arrow[r, "T(a,f)(m_a)"] \arrow[d, "x_f"'] & T(a,f)(y_a) \arrow[d, "y_f"'] \\
T(f,b)(x_b) \arrow[r, "T(f,b)(m_b)"] & T(f,b)(y_b)
\end{tikzcd}$$

commutes.

It is easily checked that $LE(T)$ is a category and that, for $a \in \mathcal{A}$, the projections

$$pr_a : LE(T) \to T(a,a)$$

defined by

$$pr_a(\underline{x}) = x_a$$

$$pr_a(\underline{m}) = m_a$$

are functors from $LE(T)$ to $T(A,a)$ and, given a 1-cell $f : a \to b$ in $\mathcal{A}$

$$pr_f : T(a,f)\sharp_0 pr_a \Rightarrow T(f,b)\sharp_0 pr_b,$$

defined by

$$pr_f(\underline{x}) = (x_f : T(a,f)(x_a) \to T(f,b)(x_b),$$

is a natural transformation. Finally one obtains:

Proposition 182 *There is a natural isomorphism,*

$$LE(T) \cong \oint_a T(a,a).$$

■

The proof is just checking the universal property, but that is built into the construction, so this is **left as an exercise for the reader**.

Examples of lax / pseudo ends or coends

1. **Comma objects:** In 'ordinary' category theory, it is quite an easy exercise, on introducing ends, to examine the end of a functor:

$$T : [1]^{op} \times [1] \to \mathcal{C}$$

as T determines a *cospan*[14]:

$$T(0,0) \xrightarrow{T(0,\iota)} T(0,1) \xleftarrow{T(\iota,1)} T(1,1).$$

[14] *i.e.*, some diagram of form $A \xrightarrow{f} C \xleftarrow{g} B$.

It is clear that $\oint T$, if it exists, is specified by a universal wedge which reduces to a pullback diagram:

$$\begin{tikzcd}
\oint T \arrow[r, "pr_1"] \arrow[d, "pr_0"'] & T(1,1) \arrow[d, "T(\iota,1)"] \\
T(0,0) \arrow[r, "T(0,\iota)"'] & T(0,1)
\end{tikzcd}$$

where $\iota : 0 \to 1$ is the unique such morphism, also written $0 < 1$, in $[1]$.

If $\mathcal{C}$ has an initial object, $\perp$, then any cospan yields a functor by setting $T(1,0) = \perp$, and then matching th rest of the square diagram in an obvious way.

If we have $\mathcal{C}$ being a 2-category such as Cat itself, then, given a cospan, $A \xrightarrow{f} C \xleftarrow{g} B$, in $\mathcal{C}$, the *comma object*, $(f \downarrow g)$, of the cospan is an objects, also denoted $(f \downarrow g)$, together with projections, $p : (f \downarrow g) \to A$, $q : (f \downarrow g) \to B$, and a 2-cell

$$\begin{tikzcd}
(f \downarrow g) \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\swArrow_\alpha"] & A \arrow[d, "f"] \\
B \arrow[r, "q"'] & C
\end{tikzcd}$$

from fp to gq, which is universal as a 2-limit[15].

A special case of this is in the 2-category, Cat. Here we have A, B, and C are categories and f, and g functors and the comma object, $(f \downarrow g)$, is the usual comma category of f and g. Recall that this has as objects, the triples (a, γ, b), where a is an object of A, b one of B and $\gamma : f(a) \to g(b)$ is a morphism of C going between the two images[16]. A morphism of $(f \downarrow g)$ from (a, γ, b) to (a', γ', b') consists of a pair of morphisms, $(u : a \to a', v : b \to b')$ such that

$$\begin{tikzcd}
f(a) \arrow[r, "f(u)"] \arrow[d, "\gamma"'] & (a') \arrow[d, "\gamma'"] \\
g(b) \arrow[r, "g(v)"'] & g(b')
\end{tikzcd}$$

commutes. The composition and identities are hopefully clear. The functors, $p : (f \downarrow g) \to A$, and $q : (f \downarrow g) \to B$ are the obvious 'projections' $p(a, \gamma, b) = a$and $q(a, \gamma, b) = b$, whilst for any object (a, γ, b), $\alpha(a, \gamma, b) = \gamma$, (so it is clear why α needs not to be a natural isomorphism in general). It should be clear that the description of $\oint T$ as $LE(T)$ is, in this case in which $\mathcal{A} = [1]$, precisely that of the comma category $(T(0, \iota) \downarrow T(\iota, 1))$, so, in this case:

$$\oint T \cong (f \downarrow g).$$

This makes it feasible that $\mathcal{C}$ will have lax ends as soon as it has comma objects and Cat-enriched limits. (The proof is **left for you to find for yourself.**)

[15]N. B. the 2-cell, α, is not required to be invertible.

[16]but not that there is a definite direction on γ and it is not specified to be an isomorphism.

2. **Categories of Lax transformations:** We started our discussion on ends (back in section 13.4), by examining the set, $Nat(F, G)$, of natural transformations between two functors, $F, G : \mathcal{C} \to \mathcal{D}$ and so justifying the result that, if we define $T : \mathcal{C}^{op} \times \mathcal{C} \to Sets$ by $T(c, c') = \mathcal{D}(Fc, Gc')$, then $\int_c T(c, c) \cong Nat(F, G)$, and this is, more-or-less, by construction. Given this we will see if the lax version of ends satisfies a similar result. Now $\mathcal{C}$ and $\mathcal{D}$ will be 2-categories and $F, G :: \mathcal{C} \to \mathcal{D}$, will be 2-functors[17]. We set $T : \mathcal{C}^{op} \times \mathcal{C} \to \mathsf{Cat}$ by $T(c, c') = \mathcal{D}(Fc, Gc')$, then writing $LaxTrans(F, G)$ for the category of lax transformation from F to G and modifications between them, we have

Proposition 183

$$\oint_c T(c, c) \cong LaxTrans(F, G).$$

Proof (Sketch): This just requires the reader (or prover) to feed $\mathcal{D}(Fc, Gc')$ into the $LE(T)$-machine given above and then to arduously check through the definition of lax transformation and modifications to see they match up[18]. ■

14.4 Simplicial Replacement Schemes

In their original work on homotopy limits, Bousfield and Kan developed a way of replacing the varying categories used as 'templates' for the diagrams, (and thus domains for the corresponding functors), uniformly by a simplicial or cosimplicial construction. There are both simplicial and cosimplicial replacement schemes. Which to use depends on whether we are interested in homotopy limits or homotopy colimits. As that for homotopy limits is a bit more complicated, we treat that in detail first.

14.4.1 The simple case

We again consider $F : \mathbb{I} \to \mathcal{S}$ and want to calculate $[\![\mathbb{I} \downarrow -, F]\!](B)$ for $B \in \mathcal{S}$. As $\mathcal{S}$ is a category of presheaves on $\mathbf{\Delta}$, every object is a colimit of representable objects as we have seen earlier[19]. Here, of course, the representable objects are just the $\Delta[n]$, as these were defined by $\Delta[n] = \mathbf{\Delta}(-, [n])$. The statement that every object is a colimit of such is then the fairly obvious: if $S \in \mathcal{S}$,

$$S = \int^{[n]} S_n \times \Delta[n],$$

which states that $\mathcal{S}$ is built up by taking copies of the various $\Delta[n]$ and gluing them together. We can feed this into the end formula for $[\![\mathbb{I} \downarrow -, F]\!](B)$ and it gives:

$$\begin{aligned}
[\![\mathbb{I} \downarrow -, F]\!](B) &= \int_i \underline{\mathcal{S}}(\mathbb{I} \downarrow i, \underline{\mathcal{S}}(B, F(i))) \\
&= \int_i \int_n \underline{\mathcal{S}}(\Delta[n] \times (\mathbb{I} \downarrow i)_n, \underline{\mathcal{S}}(B, F(i))) \\
&= \int_n \underline{\mathcal{S}}(\Delta[n], \underline{\mathcal{S}}(B, \int_i \prod_{(\mathbb{I} \downarrow i)_n} F(i))).
\end{aligned}$$

[17] We will restrict to them being strict 2-functors. The case in which they are lax functors is a bit more complex, and would need slightly different conditions to involve the relevant type of modifications. For the sake of the exposition, we therefore limit to the simpler case.

[18] The author thinks all the conventions fixed in earlier chapters match up, but ...!

[19] or look in standard texts / n-Lab

We can rewrite the innermost end as $\prod_{i_0,\ldots,i_n} F(i_n)$, since $(\mathbb{I} \downarrow i)_n = \coprod \mathbb{I}(i_0, i_1) \times \ldots \times \mathbb{I}(i_n, i)$, where the coproduct is taken over all sequences, $(i_0, \ldots, i_n)$, of length $n+1$, of objects in the small category, $\mathbb{I}$. 'Integrating' with respect to i in $\int_i F(i)^{\mathbb{I}(i_n,i)}$ has the effect of evaluating F at i_n. (This is basically the Yoneda lemma.)

This means that the indexed cone is given as a total object of a cosimplicial simplicial set, namely $\underline{\mathcal{S}}(B, \prod^* F)$, where $\prod^n F := \prod_{i_0,\ldots,i_n} F(i_n)$.

We need to look a bit more at the way this thing, $\prod^* F$, varies cosimplicially. We have a product indexed by $Ner(\mathbb{I})_n$, so suppose $\sigma = (i_0 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_n} i_n) \in Ner(\mathbb{I})_n$ and $\underline{x} = (x_\sigma)$ with $x_\sigma \in F(i_n)_q$ for some q. It is important first to emphasise where the cofaces, etc., go from and go to. If we are, at the moment, in dimension n, and so are looking at elements which are indexed sets of simplices from the various $F(i_n)$, the indexation is by n-simplices, $\sigma = (i_0 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_n} i_n)$ as above. We are then looking at $\prod^n F$ and the various cofaces are maps to $\prod^{n+1} F$. The cofaces of $\underline{x}$ are easy to give, at least for $0 \leq i \leq n$, as they just use the corresponding faces d_i of $Ner(\mathbb{I})$ to change the index without changing the component, *i.e.*, if $\tau = (i_0 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_{n+1}} i_{n+1})$, $d^i(\underline{x})_\tau = (x_{d_i(\tau)}) \in F(i_{n+1})$. (Although this looks easy to understand, for use for working with the final coface, it **pays to check this through in some detail**. We know $\underline{x}$ on n-simplices; we need $d^i(\underline{x})$ on $n+1$-simplices, and we get it on an $(n+1)$ simplex, τ, by mapping τ to its i-face and then applying $\underline{x}$ to the result, which is fine because we know the values of $\underline{x}$ on such n-simplices as $d_i(\tau)$, ... Everything thus works well.)

For $i = n+1$, however, we have $d_{n+1}(\tau)$ ends with i_n rather than i_{n+1}. We do know $x_{d_{n+1}(\tau)}$, of course, but it is in $F(i_n)$ rather than $F(i_{n+1})$. We therefore use the $\alpha_{n+1} : i_n \to i_{n+1}$, and the corresponding $F(\alpha_{n+1}) : F(i_n) \to F(i_{n+1})$, to map it 'onwards' to $F(i_{n+1})$, that is, we define $d^{n+1}(\underline{x})_\tau = (F(\alpha_{n+1})x_{d_{n+1}(\tau)})$.

As degeneracies in $Ner(\mathbb{I})$ just insert identities in the chain, the corresponding codegeneracies cause no such similar problem.

We note that we could have applied this construction to any functor, $F : \mathbb{I} \to \mathcal{B}$, to a simpicially enriched category, $\mathcal{B}$, provided $\mathcal{B}$ is complete (so, in particular, had cotensors). It gives a *cosimplicial replacement* for F, namely a cosimplicial object, $\prod^* F$, in $\mathcal{B}$, such that the $(\mathbb{I} \downarrow -)$-indexed cone is

$$[\![\mathbb{I} \downarrow -, F]\!](B) = Tot(\mathcal{B}(B, \prod{}^* F).$$

The total complex construction uses an end, so works for cosimplicial objects in $\mathcal{B}$. In fact, we can extract the homotopy limit of F from this, giving

$$holim\, F \cong Tot(\prod{}^* F).$$

The original description of $[\![\mathbb{I} \downarrow -, F]\!](B)$ as a total complex yields a clear description of the homotopy coherent nature of the cones, τ, that it has as its vertices. This description is, of course, equivalent to the one we sketched out earlier, but is easier to read off from the formula, and so can be more easily pushed further:

- for each object i in $\mathbb{I}$, we have a morphism $\tau_i : B \to F(i)$;
- for each morphism $\alpha : i_0 \to i_1$, a homotopy, $\tau(\alpha) : B \times \Delta[1] \to F(i_1)$, with $d_1(\tau(\alpha) = F(\alpha) \circ \tau(i_0)$, and $d_0(\tau(\alpha)) = \tau(i_1)$;

- for each $\sigma = (\alpha_2, \alpha_1) \in Ner(\mathbb{I})_2$, a '2-homotopy', $\tau(\sigma) : B \times \Delta[2] \to F(i_2)$, as shown,

$$\begin{tikzcd} & F(\alpha_2)\tau(_1) \arrow[dr, Rightarrow, "\tau(\alpha_2)"] \arrow[d, phantom, "\tau(\sigma)"] & \\ F(\alpha_2)F(\alpha_1)\tau(i_0) \arrow[ur, Rightarrow, "F(\alpha_2)\tau(\alpha_1)"] \arrow[rr, Rightarrow, "F(\alpha_2\alpha_1)"'] & & F(i_2); \end{tikzcd}$$

- and so on, for higher dimensional $\sigma \in Ner(\mathbb{I})_n$, $n \geq 3$, giving $\tau(\sigma) : B \times \Delta[n] \to F(i_n)$.

We, of course, also have higher dimensional simplices in $[\![\mathbb{I} \downarrow -, F]\!](B)$, which give homotopies and then higher homotopies between these h. c. cones.

If we are interested in homotopy colimits, rather than homotopy limits, then, again, there is a replacement scheme, a simplicial one. The basic formula for the homotopy cocone on F, can be turned around using the same sort of idea, so as to produce a bisimplicial set, $\coprod_* F$, which has diagonal the homotopy colimit. How is this defined? In dimension, (p, q), it is $\coprod_{\mathbb{I}_p} F(i_p)_q$. If we think of the elements as being 'tensors', $\sigma_p \otimes x_q$, that is, 'index' $\otimes$ 'element', then $d^h_k(\sigma_p \otimes x_q) = d_k(\sigma_p) \otimes x_q$, for $k < p$, whilst $d^h_p(\sigma_p \otimes x_q) = d_p(\sigma_p) \otimes F(\alpha_n)(x_q)$, where $\sigma = i_n \overset{\alpha_p}{\to} \ldots \overset{\alpha_1}{\to} i_0$ (because of the slightly strange conventions on defining the indexed colimits!). Degeneracies cause no problems. The vertical faces and degeneracies are just inherited from $F(i_q)$. (**You are left to check** that $diag \coprod_* F$ is the homotopy colimit.) Of course, we call $\coprod_* F$, the *simplicial replacement of F*

14.4.2 The enriched case

We have seen that a homotopy coherent diagram in $\mathcal{B}$ with domain a small category, $\mathbb{I}$, can be considered, or, even better, defined, to be an $\mathcal{S}$-functor from $S(\mathbb{I})$ to $\mathcal{B}$. We want to extend homotopy limits and colimits to handle homotopy coherent diagrams, so need to look at a more $\mathcal{S}$-enriched version of the above. The $\mathcal{S}$-category, $S(\mathbb{I})$, is free in all dimensions, but we may want to allow for the initial data to be more 'constrained' by equations, as, for instance, if we have the action of a simplicial group, G, on a simplicial set. We might also need homotopy coherent such actions. This sort of context is clearly necessary for handling situations in cohomology, so we place ourselves in a wide enough context to handle both it and the original setting of a simple 'ordinary' small category. We will work, therefore, with general simplicially enriched categories.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are simplicially enriched categories and F is a simplicially enriched functor, $F : \mathcal{A} \to \mathcal{B}$. We want to define an analogue of $\underline{\mathcal{S}}(B, \prod^* F)$, extending the construction in the previous section. We start with a way of generalising homotopy cones.

We have, as we have used previously, the set of n-simplices of the nerve of a category, $\mathbb{I}$,

$$Ner(\mathbb{I})_n = \coprod \mathbb{I}(i_0, i_1) \times \ldots \times \mathbb{I}(i_{n-1}, i_n),$$

with the coproduct being over all sequences, $\underline{i} := (i_0, \ldots, i_n)$, of objects in $\mathbb{I}$. If we replace $\mathbb{I}$ by $\mathcal{A}$, the analogous construction would be

$$\coprod \mathcal{A}(A_0, A_1) \times \ldots \times \mathcal{A}(A_{n-1}, A_n),$$

with the coproduct being over all sequences $\underline{A} = (A_0, \ldots, A_n)$. This is a *simplicial set*, not just a set, as each $\mathcal{A}(A, A')$ is one.

For the indexation of the Bousfield-Kan homotopy limit, we fitted the $(\mathbb{I} \downarrow i)_n$ together via face and degeneracy maps to get $Ner(\mathbb{I} \downarrow i)$ as the indexation, and then via the end $\int_i \underline{\mathcal{S}}(B, \prod_{(\mathbb{I}\downarrow i)_n} F(i)) \cong \underline{\mathcal{S}}(B, \prod_{i_0 \to \cdots \to i_n} F(i_n))$ to get homotopy cones. We will come back to the indexation later, but consider the variation of our coproduct with n first.

We have various maps:

- $k_0 : \mathcal{A}(A_0, A_1) \times \mathcal{A}(A_1, A_2) \to \mathcal{A}(A_1, A_2)$ will be the natural projection;
- for $0 < i < n$, $k_i : \mathcal{A}(A_{i-1}, A_i) \times \mathcal{A}(A_i, A_{i+1}) \to \mathcal{A}(A_{i-1}, A_{i+1})$ will be the (simplicial) composition within $\mathcal{A}$;

and

- for k_n, we need the analogue of the operation that goes from index $\sigma \in Ner(\mathbb{I})_n$ and $x \in F(i_n)$ to $d^n(\underline{x}) = (F(\alpha_n)x_{d_n(\sigma)})$. The product will be replaced by a 'function space' construction and, in some sense, we want to use the 'action'
$$\mathcal{A}(A_{n-1}, A_n) \times \mathcal{B}(B, FA_{n-1}) \to \mathcal{B}(B, FA_n)$$
given by mapping $\mathcal{A}(A_{n-1}, A_n)$ to $\mathcal{B}(FA_{n-1}, FA_n)$ and then using composition. It is actually slightly more convenient to replace this by its adjoint:
$$k_n : \mathcal{B}(B, FA_{n-1}) \to \underline{\mathcal{S}}(\mathcal{A}(A_{n-1}, A_n), \mathcal{B}(B, FA_n)).$$

Mapping our coproduct into the $\mathcal{B}(B, FA)$s, we get the enriched analogue of $\underline{\mathcal{S}}(B, \prod^* F)$, namely, writing $\underline{A} = (A_0, \ldots, A_n)$, with consequent $d_i\underline{A} = (A_0, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$, etc. when it helps:

Definition: Given $F : \mathcal{A} \to \mathcal{B}$, and an object B of $\mathcal{B}$, denote by $\prod^*(B, F)$, the cosimplicial simplicial set with
$$\prod\nolimits^n(B, F) = \prod_{A_0, \ldots, A_n} \underline{\mathcal{S}}(\mathcal{A}(A_0, A_1) \times \ldots \times \mathcal{A}(A_{n-1}, A_n), \mathcal{B}(B, FA_n))$$
with cofaces and codegeneracies defined by:

- for $0 \leq i < n$,
$$p_{\underline{A}} d^i = \underline{\mathcal{S}}(id \times \ldots \times k_i \times \ldots id, \mathcal{B}(B, FA_n))p_{d_i\underline{A}},$$
where $p_{\underline{A}}$ is the projection of the product onto the factor indexed by $\underline{A}$, etc.;
- for $i = n$,
$$p_{\underline{A}} d^n = \underline{\mathcal{S}}(\mathcal{A}(A_0, A_1) \times \ldots \times \mathcal{A}(A_{n-2}, A_{n-1}), k_n)p_{d_n\underline{A}};$$

and

- for the codegeneracies,
$$p_{A_0, \ldots, A_{n-1}} s^i = \underline{\mathcal{S}}(id \times \ldots \times h_i \times \ldots id, \mathcal{B}(B, FA_{n-1}))p_{A_0, \ldots, A_{i-1}, A_i, A_i, A_{i+1} \ldots, A_{n-1}},$$
where $h_i : \Delta[0] \to \mathcal{A}(A_i, A_i)$ is the morphism that picks out the identity morphism on A_i as a 0-simplex of $\mathcal{A}(A_i, A_i)$.

The *homotopy cone* with vertex B and base F is $Tot(\prod^*(B,F))$. It will sometimes be denoted by $hocone(B,F)$.

Varying B gives a profunctor, $\mathbb{1} \nrightarrow \mathcal{B}$, as before, and it is natural to define:

Definition: The *homotopy limit*, *holim* F, of F is a representative of the (pro)functor

$$hocone(-,F) = Tot(\prod{}^*(B,F)) : \mathcal{B}^{op} \to \mathcal{S}.$$

Proposition 184 *If* $\mathcal{B} = \mathcal{S}$, *then* $holim\, F = Tot(\prod^* F)$, *where* $\prod^* F$ *is the cosimplicial simplicial set,* $\prod^*(\Delta[0], F)$.

Proof: For any simplicial set, B,

$$\underline{\mathcal{S}}(B, Tot(\prod{}^* F)) \cong Tot(\prod{}^*(B,F)) = hocone(B,F).$$

■

We can do better than this.

In general, for $\mathcal{B}$, a complete $\mathcal{S}$-category, we have a cosimplicial object in $\mathcal{B}$, given by replacing $\underline{\mathcal{S}}$ in the formula for $\prod^* F$ by $\overline{\mathcal{B}}$, the cotensor construction of $\mathcal{B}$, thus we have, in this enriched setting,

$$\begin{aligned}\prod{}^*(B,F) &= \prod_{A_0,\ldots,A_n} \underline{\mathcal{S}}(\mathcal{A}(A_0,A_1)\times\ldots\times\mathcal{A}(A_{n-1},A_n), \mathcal{B}(B,FA_n))\\ &= \mathcal{B}(B, \prod_{A_0,\ldots,A_n} \overline{B}(\mathcal{A}(A_0,A_1)\times\ldots\times\mathcal{A}(A_{n-1},A_n), F(A_n)))\end{aligned}$$

and we take $\prod^* F$, here, to be the cosimplicial object occupying the right hand part of this last expression, *i.e.*, $\prod^n F = \prod_{A_0,\ldots,A_n} \overline{B}(\mathcal{A}(A_0,A_1)\times\ldots\times\mathcal{A}(A_{n-1},A_n), F(A_n)))$. The enriched total complex construction that we saw before, page 657, then gives a description of a homotopy limit in this case, that is:

Proposition 185 *If* $\mathcal{B}$ *is a complete* $\mathcal{S}$*-category, and* $F : \mathcal{A} \to \mathcal{B}$, *then*

$$holim\, F \cong Tot(\prod{}^* F).$$

■

The proof is essentially the same as that of the previous proposition. This leads to the following:

Definition: Given $F : \mathcal{A} \to \mathcal{B}$, as above, the *cosimplicial replacement of* F is the cosimplicial object in $\mathcal{B}$, denoted above by $\prod^* F$.

14.5 Homotopy limits as indexed limits

We thus have a definition of a homotopy limit in the enriched context, given in terms of a total complex of a cosimplicial object. We should be able to run 'our machine' backwards to find a suitable indexation, so that this 'holimit' is given as an indexed limit. In fact, the previous section was based on the simple idea of viewing $Ner(\mathbb{I})_n$, as a coproduct of a product of sets and we can exploit that directly to get a feasible indexation that extends our earlier one.

We start with an $\mathcal{S}$-category, $\mathcal{A}$, and an object, A in $\mathcal{A}$, and construct a bisimplicial set, $\mathcal{A}/A$, with

$$(\mathcal{A}/A)_{n,*} = \coprod_{A_0,\ldots,A_n} \mathcal{A}(A_0, A_1)_* \times \ldots \times \mathcal{A}(A_{n-1}, A_n)_* \times \mathcal{A}(A_n, A)_*$$

with horizontal face and degeneracy maps given by

- $d_0(z_1, \ldots, z_n, \alpha) = (z_2, \ldots, z_n, \alpha)$;

- $d_i(z_1, \ldots, z_n, \alpha) = (z_1, \ldots, z_{i+1} \cdot z_i, \ldots, z_n, \alpha)$ for $0 < i < n$,

- $d_n(z_1, \ldots, z_n, \alpha) = (z_1, \ldots, z_{n-1}, \alpha \cdot z_n)$,

and

- $s_i(z_1, \ldots, z_{n-1}, \alpha) = (z_1, \ldots, z_i, id, z_{i+1}, \ldots, z_{n-1}, \alpha)$.

This gives an $\mathcal{S}$-functor,

$$\mathcal{A}/- : \mathbf{\Delta}^{op} \times \mathcal{A} \to \mathcal{S},$$

where $\mathbf{\Delta}$ is trivially enriched in $\mathcal{S}$, and hence gives an $\mathcal{S}$-profunctor, $\mathcal{A}/- : \mathcal{A} \nrightarrow \mathbf{\Delta}$.

This seems clearly to be on the same track as our earlier constructions and this impression is borne out by the following:

Proposition 186 *The $\mathcal{S}$-profunctor $\prod^*(-, F) : \mathbf{\Delta} \nrightarrow \mathcal{B}$ is the right Kan extension of F along $\mathcal{A}/-$.*

Proof: We first look again at $\prod^*(-, F)$. Given B and n, we get a simplicial set, $\prod^n(B, F)$, and the variation in these 'variables' corresponds to a functor from $\mathcal{B}^{op} \times \mathbf{\Delta}$ to $\mathcal{S}$, *i.e.*, a profunctor $\mathbf{\Delta} \nrightarrow \mathcal{B}$, and that is the right 'type' to be the right Kan extension. (The relevant diagram is

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathcal{A}/-", "|" marking] \arrow[d, "F"'] & \mathbf{\Delta} \arrow[dl, "F_1", "|" marking] \arrow[dl, phantom, bend right=40, "\Nwarrow"] \\ \mathcal{B} & \end{tikzcd}$$

where F_1 is the right Kan extension.) We use the formulae that we obtained earlier:

$$\begin{aligned}
F_1(B,n) &= \int_A \underline{\mathcal{S}}((\mathcal{A}/A)_n, \mathcal{B}(B, FA)) \\
&= \underline{Nat}((\mathcal{A}/-)_n, \mathcal{B}(B, F-)) \\
&= \underline{Nat}(\coprod_{\underline{A}} \mathcal{A}(A_0, A_1) \times \ldots \times \mathcal{A}(A_n, A_-), \mathcal{B}(B, F-)) \\
&= \prod_{\underline{A}} \underline{Nat}(\mathcal{A}(A_n, -), \underline{\mathcal{S}}(\mathcal{A}(A_0, A_1) \times \ldots \times \mathcal{A}(A_{n-1}, A_n), \mathcal{B}(B, F-)) \\
&= \prod_{\underline{A}} \underline{\mathcal{S}}(\mathcal{A}(A_0, A_1) \times \ldots \times \mathcal{A}(A_{n-1}, A_n), \mathcal{B}(B, FA_n)) \\
&\cong \prod{}^n(B, F),
\end{aligned}$$

as promised. ■

This gives a very neat description of $\prod^n(B,F)$. To get a useful one of $hocone(B,F)$, we use the total complex functor, which we saw was indexed by $Yo : \mathbf{\Delta} \nrightarrow \mathbb{1}$. We thus have

$$\mathcal{A} \overset{\mathcal{A}/-}{\nrightarrow} \mathbf{\Delta} \overset{Yo}{\nrightarrow} \mathbb{1},$$

and can form the composite profunctor, $Yo \otimes \mathcal{A}/-$. We denote this by $H_{\mathcal{A}}$ and can easily calculate it (as a functor $\mathcal{A} \to \mathcal{S}$).

Lemma 105 *For any A in $\mathcal{A}$, there is a natural isomorphism*

$$H_{\mathcal{A}}(A) \cong diag(\mathcal{A}/-).$$

Proof: For A, an object of $\mathcal{A}$,

$$\begin{aligned}
H_{\mathcal{A}}(A) &= = (Yo \otimes \mathcal{A}/-)(A) \\
&= \int_n \mathbf{\Delta}[n] \times (\mathcal{A}/A)_{n,*} \\
&= diag(\mathcal{A}/A).
\end{aligned}$$

■

The importance of this is that it shows which construction from bisimplicial objects to simplicial objects to use in this case.

Proposition 187 *If $\mathcal{A}$ is an $\mathcal{S}$-category, $H_{\mathcal{A}}$ is the indexation for the homotopy limit for $\mathcal{S}$-functors with domain $\mathcal{A}$.*

Proof: Let F be an $\mathcal{S}$-functor from $\mathcal{A}$ to $\mathcal{B}$. We have a diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathcal{A}/-", "|" marking] \arrow[d, "F"'] & \mathbf{\Delta} \arrow[r, "Yo", "|" marking] & \mathbb{1} \\
\mathcal{B}
\end{tikzcd}$$

and the $H_{\mathcal{A}}$-indexed limit will be a representing object for the right Kan extensions of F along $H_{\mathcal{A}}$, but calculating this, first as that along $\mathcal{A}/-$ and then along Yo, is exactly the process of forming $F_1 = \prod^*(-, F)$, and then taking its total complex, *i.e.*, forming $hocone(-, F)$. ■

It is now just a small step from $hocone(-, F)$ to $holim\, F$. We have the following, extending the earlier result with $\mathcal{B} = \mathcal{S}$.

Proposition 188 *If $F : \mathcal{A} \to \mathcal{B}$ is an $\mathcal{S}$-functor with $\mathcal{B}$ a complete $\mathcal{S}$-category, then we have*

$$\begin{aligned} holim\, F &\cong Tot(\textstyle\prod^* F) \\ &\cong \int_A \overline{\mathcal{B}}(H_{\mathcal{A}}(A), F(A)). \end{aligned}$$

Proof: This is merely a collecting up of parts from earlier results and discussions, together with minor changes, so ... . ■

14.6 Homotopy colimits and simplicial replacement

The homotopy colimit can be treated dually.

14.6.1 Homotopy colimit, the enriched case

Suppose $F : \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$ cocomplete, so has tensor products with objects of $\mathcal{S}$. We can use $H_{\mathcal{A}^{op}}$ as an indexation and take as definition:

Definition: The *homotopy colimit* of F is the $H_{\mathcal{A}^{op}}$-indexed colimit of F, namely

$$hocolim\, F = \int^A F(A) \otimes H_{\mathcal{A}^{op}}(A).$$

14.6.2 Simplicial replacement

We have a simplicial object in $\mathcal{B}$, given by

$$(\coprod F)_n = \coprod_{\underline{A}} F(A_n) \otimes (\mathcal{A}(A_n, A_{n-1}) \times \ldots \times \mathcal{A}(A_1, A_0))$$

with fairly obvious face and degeneracy maps. This is called the *simplicial replacement of F*.

Proposition 189 *For $\mathcal{B}$ and F as above, there is a natural isomorphism*

$$hocolim\, F \cong diag(\coprod F)_*,$$

where, for a simplicial object, K, in $\mathcal{B}$, $diag\, K := \int^{[n]} K_n \otimes \Delta[n]$.

Proof: This can safely be **left as an exercise**. It is the dual of arguments that we have already looked at in some detail. ■

This 'diagonal' may be called a *realisation* of the simplicial object, as, in the case of $\mathcal{B}$ being the category of spaces, this is just the geometric realisation. The construction is also well known for the case of small categories.

14.6.3 The Grothendieck construction and homotopy colimits

We looked at the Grothendieck construction in section 9.2, so will briefly recall it.

(MORE TO GO HERE.)

14.6.4 The homotopy orbit space / Borel construction

(MORE TO GO HERE!)

14.6.5 G-spaces and fibrations over BG

(... AND HERE.)

14.7 Homotopy Kan extensions

We have seen homotopy limits and colimits represented by indexed limits and colimits. The important idea here is that these constructions *represent* the homotopy limit or colimit, that is, an object determined up to homotopy by a certain universal property. The main 'trick' was to use an indexation that reflected the 'geometry' of the domain, and such that the 'resolution' of the constant functor, $const : \mathcal{B} \to \mathcal{B}^{\mathbb{I}}$, used to model the other side of the 'constant functor / limit' adjunction, was going to given a 'local equivalence'. The typical case was to replace each object i, in a domain category $\mathbb{I}$, by the simplicial set $Ner(\mathbb{I} \downarrow i)$, which is contractible. Each $Ner(\mathbb{I} \downarrow i) \to i$ is a homotopy equivalence, but its homotopy inverse is not functorial nor 'natural' with respect to change in the index, ..., in other words, it is 'locally' a homotopy equivalence.

One aim is to obtain a limit-like adjunction at the homotopy level. Usually this will look something like:

$$Ho(\mathcal{B}^{\mathbb{I}})(c(K), X) \cong Ho(\mathcal{B})(K, holim\, X),$$

but we could demand more since, in very many ways, $Ho(\mathcal{B})$ and $Ho(\mathcal{B}^{\mathbb{I}})$ are just the surface structure of a much richer homotopy set up.

Perhaps we should look for more. Suppose $\varphi : \mathbb{I} \to \mathbb{J}$ is a functor on the index categories, (which might be enriched as well), then there will be an induced functor

$$\mathcal{B}^{\varphi} : \mathcal{B}^{\mathbb{J}} \to \mathcal{B}^{\mathbb{I}},$$

and, as we have seen (in sections 13.3.1 and 14.2.4), provided $\mathcal{B}$ is sufficiently complete (resp. cocomplete) this functor will have a right (resp. left) adjoint. If we have a need for a homotopy structure of some sort on $\mathcal{B}$, then we could hope for a 'homotopy adjoint' generalising homotopy limits, *i.e.*, we would hope for 'homotopy Kan extensions'.

The need for such a theory, and the importance of the methods involved, were realised by various researchers from the 1980s onwards. For instance, Heller, in [148, 149] uses all the diagram categories in his development of what he envisages a homotopy theory to be. At about the same time Grothendieck wrote another long document, '*Les Dérivateurs*', [142], which looked at an essentially similar notion, but based more on the context of Ch-enriched categories and derived functor theory. (This has been further developed by various authors and there are numerous articles on that area, for which you should search the web, but first see below.)

To define, construct and, most crucially, study Kan extensions, we saw, earlier, how useful ends and coend were. Similarly, as we are concentrating on a simplicial approach to homotopy coherence, we need a simplicial approach to 'coherent' ends and coends, that is, *simplicially coherent ends and coends*. With that tool in hand, we can then look at homotopy coherent Kan extensions and can use them to get useful practical tools to simplify later discussions.

(The main reference we will use is Cordier and Porter, [87, 88], supplemented by some other texts such as Shulman's [247]. Some mention will be made of *Derivators* and, for that, the papers on George Maltsiniotis' webpage are amongst the best starting points,

(http://people.math.jussieu.fr/∼maltsin/groth/Derivateurs.html),

in particular, [196] and [142].)

14.7.1 Simplicially coherent ends and coends.

As pointed out briefly above, ends and coends are basic constructions in category theory. Mac Lane, [192], shows, for example, how the important notion of Kan extension can be encoded in the language of ends and coends and then that formulation can be used, together with the 'end-calculus' and results such as 'Fubini's theorem', to give quick, elegant proofs of some of the important results in the development of that subject.

Variants of these coherent ends have been proposed by several authors (Segal, [244], Meyer, [201], Heller, [148], Dwyer, and Kan, [112], as well as Cordier, [83], and Cordier and Porter, [84]). They generalise homotopy limits, and can be used with good effect to construct a homotopy coherent version of category theory, see [88]. We will give some of that theory, that part that will be useful for later applications here.

Let $\mathcal{A}$ be a small $\mathcal{S}$-category. For A, B in $\mathcal{A}$, form the bisimplicial set, $X(A,B)$, defined by

$$X(A,B)_{n,\star} = \coprod_{A_0\ldots A_n} \mathcal{A}(A,A_0)_\star \times \mathcal{A}(A_0,A_1)_\star \times \ldots \times \mathcal{A}(A_n,B)_\star,$$

where

$$d_j : X(A,B)_{n,\star} \to X(A,B)_{n-1,\star}$$

is defined by composition in $\mathcal{A}$,

$$\mathcal{A}(A_{i-1}A_i) \times \mathcal{A}(A_i,A_{i+1}) \to \mathcal{A}(A_{i-1},A_{i+1}),$$

(we write $A_{-1} = A$, $A_{n+1} = B$ for the purposes of this definition), and $s_i : X(A,B)_n \to X(A,B)_{n+1}$ is induced by the morphism,

$$\Delta[0] \to \mathcal{A}(A_i,A_i),$$

the simplicial morphism 'constant on Id_{A_i}'.

Now set $\hat{\mathcal{A}}(A,B) = diag X(A,B)$, the diagonal simplicial set. We recall (see page 688) that, in general, for a bisimplicial set, $X_{\bullet,\star}$,

$$diag\, X_\star \cong \int^{[n]} \Delta[n] \times X_{n,\star}.$$

Example: Let $\mathcal{A}$ be a small category, A, and B, objects of $\mathcal{A}$. Consider $\mathcal{A}$ as a locally discrete $\mathcal{S}$-category. Let $A \downarrow \mathcal{A} \downarrow B$ be the category of objects under A and over B, *i.e.*, a double comma category, then $X(A,B) \cong Ner(A \downarrow \mathcal{A} \downarrow B)$, the nerve of that category.

This example suggests the use we will make of the $X(A,B)$, especially when we think back to the use of the comma categories, $\mathcal{A} \downarrow A$, by Bousfield and Kan, [42], and the extension to $\mathcal{S}$-enriched indexing categories, as in [41], that we looked at above.

Let $\mathcal{B}$ be a complete $\mathcal{S}$-category, $\mathcal{A}$, as above, a (small) $\mathcal{S}$-category and $T : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{B}$, an $\mathcal{S}$-functor. As $\mathcal{B}$ is complete, it is cotensored.

Definition: The *simplicially coherent end* of T will be the object $\oint_A T(A,A)$ of $\mathcal{B}$ defined by

$$\oint_A T(A,A) = \int_{(A',A'')} \overline{\mathcal{B}}(\hat{\mathcal{A}}(A',A''), T(A',A'')),$$

where $(A',A'') \in \mathcal{A}^{op} \times \mathcal{A}$.

Remark: We will be consistently using $\hat{\mathcal{A}}$ to index these coherent ends, however it is important to remember that if we were to replace $X(A,B)$ by $X(A,B)^*$, where $d_i^{n*} = d_{n-i}^n$, $s_i^{n*} = s_{n-i}^n$, we would obtain a parallel theory.

Given any cosimplicial object Y in $\mathcal{B}$ (so $Y : \Delta \to \mathcal{B}$), we know that we can form the *(enriched) total object* of Y in $\mathcal{B}$. This was defined above (page 657) by

$$\int_{[n]} \overline{\mathcal{B}}(\mathbf{\Delta}[n], Y^n).$$

In the case when $\mathcal{B} = \mathcal{S}$, Y is a cosimplicial simplicial set and the total complex, denoted $Tot(Y)$ or $\hom(\Delta, Y)$, is $\int_{[n]} \underline{\mathcal{S}}(\Delta[n], Y^n)$. In the previous sections, this construction was used together with a cosimplicial replacement formula to provide a useful reformulation of the homotopy limit functor. Here we will give a similar description of $\oint_A T(A,A)$.

Again let $T : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{B}$ and set

$$Y(T)^n = \prod_{A_0,\ldots,A_n} \overline{\mathcal{B}}(\mathcal{A}(A_0,A_1) \times \ldots \times \mathcal{A}(A_{n-1},A_n), T(A_0,A_n)).$$

These objects form the basis for the cosimplicial object, $Y(T)$, where the coface and codegeneracy maps of $Y(T)$ are defined as follows: First let

$$p_{A_0,\ldots A_n} : Y(T)^n \to \overline{\mathcal{B}}(\mathcal{A}(A_0,A_1) \times \ldots \times \mathcal{A}(A_{n-1},A_n), T(A_0,A_n))$$

denote the projection;

- for $0 < i < n$, c_i will denote the map from

  $$\mathcal{A}(A_0,A_1) \times \ldots \times \mathcal{A}(A_{i-1},A_i) \times \mathcal{A}(A_i,A_{i+1}) \times \ldots \times \mathcal{A}(A_{n-1},A_n)$$

  to

  $$\mathcal{A}(A_0,A_1) \times \ldots \times \mathcal{A}(A_{i-1},A_{i+1}) \times \ldots \times \mathcal{A}(A_{n-1},A_n)$$

  induced by the composition from $\mathcal{A}(A_{i-1},A_i) \times \mathcal{A}(A_i,A_{i+1})$ to $\mathcal{A}(A_{i-1},A_{i+1})$;

- for $0 \leq i \leq n-1$, k_i will denote the map
$$\mathcal{A}(A_0, A_1) \times \ldots \times \mathcal{A}(A_{n-2}, A_{n-1}) \to \mathcal{A}(A_0, A_1) \times .. \times \mathcal{A}(A_i, A_i) \times \ldots \times \mathcal{A}(A_{n-2}, A_{n-1})$$
induced by the map $\Delta[0] \to \mathcal{A}(A_i, A_i)$ which picks out the identity map, (*i.e.*, the "name of the identity map");
- for any objects $A, A^{'}, A^{''}$;
$$T(-,A)_{A^{'},A^{''}} : \mathcal{A}(A^{'}, A^{''}) \to \mathcal{B}(T(A^{''}, A), T(A^{'}, A))$$
and
$$T(A,-)_{A^{'},A^{''}} : \mathcal{A}(A^{'}, A^{''}) \to \mathcal{B}(T(A, A^{'}), T(A, A^{''}))$$
will denote the maps corresponding to the fact that $T(-, A)$ and $T(A, -)$ are $\mathcal{S}$-functors.

Now the codegeneracies, s^i, of $Y(T)$ are given by
$$p_{A_0,\ldots,A_{n-1}} s^i = \overline{\mathcal{B}}(k_i, T(A_0, A_{n-1})) p_{A_0,\ldots,A_i,A_i,\ldots,A_{n-2}}, \quad \text{for } 0 \leq i \leq n-1,$$
whilst, for $0 < i < n$, the coface map, d^i, is given by
$$p_{A_0,\ldots,A_n} d^i = \overline{\mathcal{B}}(c_i, T(A_0, A_{n-1})) p_{A_0,\ldots,A_{i-1},A_{i+1},\ldots,A_n}$$
(*i.e.*, s^i and d^i are induced by the inclusion of identities and the composition in the usual way. The slightly more complicated formulation is forced on us by the setting.) This leaves d^0 and d^n. These are slightly more difficult to specify. The map $T(-,A)_{A^{'},A^{''}} : \mathcal{A}(A^{'}, A^{''}) \to \mathcal{B}(T(A^{''}, A), T(A^{'}, A))$ gives an element in
$$\underline{\mathcal{S}}(\mathcal{A}(A^{'}, A^{''}), \mathcal{B}(T(A^{''}, A), T(A^{'}, A)))_0$$
and hence in
$$\mathcal{B}(T(A^{''}, A), \overline{\mathcal{B}}(\mathcal{A}(A^{'}, A^{''}), T(A^{'}, A)))_0,$$
(as, by assumption, $\mathcal{B}$ is cotensored). This is thus a map
$$T(-,A)_{A^{'},A^{''}} : T(A^{''}, A) \to \overline{\mathcal{B}}(\mathcal{A}(A^{'}, A^{''}), T(A^{'}, A))_0.$$
We also need the isomorphism
$$\begin{aligned}\tau : \ & \overline{\mathcal{B}}(\mathcal{A}(A_0, A_1) \times \ldots \times \mathcal{A}(A_{n-1}, A_n), T(A_0, A_n)) \\ & \longrightarrow \quad \overline{\mathcal{B}}(\mathcal{A}(A_1, A_2) \times \ldots \times \mathcal{A}(A_{n-1}, A_n), \overline{\mathcal{B}}(\mathcal{A}(A_0, A_1), T(A_0, A_n))).\end{aligned}$$
Define d^0 by $p_{A_0,\ldots,A_n} d^0 = \tau \circ \overline{\mathcal{B}}(\mathcal{A}(A_1, A_2) \times \ldots \times \mathcal{A}(A_{n-1}, A_n), T(-, A_n)_{A_0,A_1})$. Similarly $T(A,-)_{A',A''} : \mathcal{A}(A', A'') \to \mathcal{B}(T(A, A'), T(A, A''))$ gives us
$$T(A,-)_{A',A''} : T(A, A') \to \overline{\mathcal{B}}(\mathcal{A}(A', A''), T(A, A'')),$$
and
$$p_{A_0,\ldots,A_n} d^n = \tau^{'} \circ T(A_0, -)_{A_{n-1},A_n},$$
where $\tau^{'}$ is the analogous isomorphism. In practice, we pretend that τ and $\tau^{'}$ are identities, so that
$$p_{A_0,\ldots,A_n} d^n = \overline{\mathcal{B}}(c_0, T(-, A_n)_{A_0,A_1}) p_{A_1,\ldots A_n},$$
and
$$p_{A_0,\ldots,A_n} d^n = \overline{\mathcal{B}}(c_n, T(A_0, -)_{A_{n-1},A_n}) p_{A_0,\ldots,A_{n-1}},$$
where c_0 and c_n are the obvious projections. If you **take this to pieces**, you will see it gives a very similar formulation to that in the case of Bousfield and Kan's cosimplicial replacement functor.

Proposition 190 *(Cosimplicial replacement) Let T be an $\mathcal{S}$-functor from $\mathcal{A}^{op} \times \mathcal{A}$ to a complete $\mathcal{S}$-category $\mathcal{B}$, then the simplicially coherent end of T is isomorphic to the total object of the cosimplicial object, $Y(T)$, and hence has the following universal property:*
for $n \in \mathbb{N}$, let $p_n : N = \oint_A T(A, A) \to \overline{\mathcal{B}}(\Delta[n], Y(T)^n)$ be the canonical projection, then for $\mu : [n] \to [m] \in \mathbf{\Delta}$,

$$\overline{\mathcal{B}}(\Delta[n], Y(T)^{\mu})p_n = \overline{\mathcal{B}}(\Delta[\mu], Y(T)^m)p_m$$

and if N' is in $\mathcal{B}$, and $\{q_n : N' \to \overline{\mathcal{B}}(\Delta[n], Y(T)^n) \mid n \in \mathbb{N}\}$ is a family of morphisms so that, for each $\mu : [n] \to [m]$,

$$\overline{\mathcal{B}}(\Delta[n], Y(T)^{\mu})q_n = \overline{\mathcal{B}}(\Delta[\mu], Y(T)^m)q_m,$$

then there is a unique morphism, $b : N' \to N$ in $\mathcal{B}$, such that for all n, $p_n b = q_n$.

Proof: We start by noting that, by the usual construction of the enriched end applied to the functor from $\mathbf{\Delta}^{op} \times \mathbf{\Delta}$ to $\mathcal{B}$, which assigns to $([n], [m])$, the object $\overline{\mathcal{B}}(\Delta[n], Y(T)^m)$, the total object of $Y(T)$ is the kernel of the pair of morphisms,

$$\prod_n \overline{\mathcal{B}}(\Delta[n], Y(T)^n) \rightrightarrows \prod_{\mu:[n]\to[m]} \overline{\mathcal{B}}(\Delta[n], Y(T)^m),$$

given by $\overline{\mathcal{B}}(\Delta[n], Y(T)^{\mu})$ and $\overline{\mathcal{B}}(\Delta[\mu], Y(T)^m)$. This implies that the universal property will follow, once the first part is proved.

The first part is however merely an exercise in the end calculus (using "Fubini" (cf. MacLane, [192]) several times). We will include it because it is quite 'fun':

$$\begin{aligned}
&\oint_A T(A, A) \\
&= \int_{A'} \int_{A''} \overline{\mathcal{B}}(\int^n \Delta[n] \times \coprod \mathcal{A}(A', A_0) \times \ldots \times \mathcal{A}(A_n, A''), T(A', A'')) \\
&= \int_n \int_{A'} \int_{A''} \prod \overline{\mathcal{B}}(\Delta[n] \times \mathcal{A}(A', A_0) \times \ldots \times \mathcal{A}(A_{n-1}, A_n), \overline{\mathcal{B}}(\mathcal{A}(A_n, A''), T(A', A''))) \\
&= \int_n \int_{A'} \prod \overline{\mathcal{B}}(\Delta[n] \times \mathcal{A}(A', A_0) \times \ldots \times \mathcal{A}(A_{n-1}, A_n), \int_{A''} \overline{\mathcal{B}}(\mathcal{A}(A_n, A''), T(A', A'')))
\end{aligned}$$

where the indexation of the (co)products are over all $A_0, \ldots, A_n$ in $\mathcal{A}$.

The "enriched Yoneda lemma" (see Kelly, [177]), allows one to evaluate the last part of this end. The classical Yoneda lemma tells one that, given a functor $F : A \to Sets$, the natural transformations from a 'hom-set' functor $A(a, -)$ to F are in bijective correspondence with the elements of $F(a)$. This is easily encoded into the language of ends, and then the natural way of enriching this gives the following isomorphism in our context:

$$\int_{A''} \overline{\mathcal{B}}(\mathcal{A}(A_n, A''), T(A', A'')) \cong T(A', A_n).$$

Continuing we get

$$\begin{aligned}
&\oint_A T(A,A)\\
&\quad\cong \int_n \int_{A'} \prod \overline{\mathcal{B}}(\Delta[n] \times \mathcal{A}(A', A_0) \times \ldots \times \mathcal{A}(A_{n-1}, A_n), T(A', A_n))\\
&\quad\cong \int_n \prod \overline{\mathcal{B}}(\Delta[n] \times \mathcal{A}(A_0, A_1) \times \ldots \times \mathcal{A}(A_{n-1}, A_n), \int_{A'} \overline{\mathcal{B}}(\mathcal{A}(A', A_0), (T(A', A_n)))\\
&\quad\cong \int_n \overline{\mathcal{B}}(\Delta[n], \prod \overline{\mathcal{B}}(\mathcal{A}(A_0, A_1) \times \ldots \times \mathcal{A}(A_{n-1}, A_n), T(A_0, A_n)))\\
&\quad\cong \int_n \overline{\mathcal{B}}(\Delta[n], Y(T)^n),
\end{aligned}$$

as required. ■

Examples: 1. Let $\mathcal{A}$ be an $\mathcal{S}$-category and $T : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{S}$, an $\mathcal{S}$-functor, then

$$\oint_A T(A,A) = Tot(Y(T)),$$

in the original sense of the Bousfield and Kan total complex, *i.e.*, Y is "cosimplicial replacement".

One can handle the elements of $\oint_A T(A,A)$ as follows, at least in low dimensions:

if $\sigma \in Y(T)^0_n$, then $p(A)\sigma \in \underline{\mathcal{S}}(\Delta[0], T(A,A))_n = \mathcal{S}(\Delta[n], T(A,A))$ and so can be represented by a map from $\Delta[n]$ to $T(A,A)$.

The coface $d^0\sigma \in Y(T)^1_n$ projects via $p(A_0, A_1)$ to an element in

$$\mathcal{S}(\mathcal{A}(A_0, A_1), T(A_0, A_1))_n,$$

thus is represented by some map

$$p(A_0, A_1)d^0\sigma : \mathcal{A}(A_0, A_1) \times \Delta[n] \to T(A_0, A_1).$$

This, if one interprets the general coface formula above, is the composite

$$\mathcal{A}(A_0, A_1) \times \Delta[n] \to \mathcal{A}(A_0, A_1) \times T(A_1, A_1) \to T(A_0, A_1),$$

where the first map is $\mathcal{A}(A_0, A_1) \times p(A_1)\sigma$ and the second is the left action of $\mathcal{A}(A_0, A_1)$ on $T(A_1, A_1)$, *i.e.*, "precomposition". Similarly for $p(A_0, A_1)d^1\sigma$, which thus is the composite

$$\mathcal{A}(A_0, A_1) \times \Delta[n] \to \mathcal{A}(A_0, A_1) \times T(A_0, A_0) \to T(A_0, A_1)$$

involving $\mathcal{A}(A_0, A_1) \times p(A_0)\sigma$ and the "post composition" map or right action.

With longer strings, $A_0, A_1, \ldots, A_n$, the intermediate cofaces cause no problems and, following the detailed description given earlier, it is easy to write down $d^i\sigma$ for all $\sigma \in Y(T)^m_n$.

If $\sigma \in (\oint_A T(A,A))_n$, then $\sigma : \Delta \times \Delta[n] \to Y(T)$. We will consider only $n = 0$ and 1, as these cases are, here, quite typical of the general one. The extension to the general case poses no problems.

We have $\sigma = (\sigma^m)$, where $\sigma^m : \Delta[m] \to Y(T)^m$, and, for each m and $0 \leq i \leq m$, $\sigma^m d^i = d^i \sigma^{m+1}$, similarly for degeneracies.

We examine $\sigma^1 d^0 = d^0\sigma^0$ in detail. As both maps end up in $Y(T)^1$, we project via $p(A_0, A_1)$ and get that $p(A_0, A_1)d^0\sigma^0$ is as we have written earlier, whilst $p(A_0, A_1)d^0\sigma^1$ is

$$\mathcal{A}(A_0, A_1) \longrightarrow \mathcal{A}(A_0, A_1) \times \Delta[1] \stackrel{p(A_0,A_1)\sigma^1}{\longrightarrow} T(A_0, A_1),$$

where the first map is the inclusion into the top of the cylinder, *i.e.*, interpreting $p(A_0, A_1)d^0\sigma^1$ as $d_0(p(A_0, A_1)\sigma^1)$. We have

$$d_0(p(A_0, A_1)\sigma^1) = p(A_0, A_1)d^0\sigma^0,$$

and similarly

$$d_1(p(A_0, A_1)\sigma^1) = p(A_0, A_1)d^1\sigma^0,$$

i.e., $p(A_0, A_1)\sigma^1$ is a homotopy between $p(A_0, A_1)d^1\sigma^0$ and $p(A_0, A_1)d^0\sigma^0$.

2. Consider a 2-category, $\mathcal{A}$, as an $\mathcal{S}$-category, as before, by taking the nerve of each hom-category and let $T : \mathcal{A}^{op} \times \mathcal{A} \to Cat$ be a 2-functor, again considered as being $\mathcal{S}$-enriched rather than Cat-enriched, then the above description of $\oint_A T(A, A)$ shows that this simplicially coherent end construction is isomorphic to the lax end construction of Bozapalides, [44], cf. [46].

3. Let A be an ordinary small category considered as a locally discrete $\mathcal{S}$-category, and T : $A^{op} \times A \to Top$ be a functor, then

$$\oint_A T(A, A) = \int_n \prod \overline{Top}(\Delta^n, T(A_0, A_n)),$$

where the product is over all n-simplices, $A_0 \stackrel{f_1}{\to} A_1 \to ... \stackrel{f_n}{\to} A_n$ in $NerA$.

This sort of construction is well known in special cases, cf. Cordier, [83] or Vogt, [271], and interprets geometrically as saying that the elements of $(\oint_A T(A, A))_0$ are the families of functions

$$h(f_1, \ldots, f_n) : \Delta^n \to T(A_0, A_n)$$

such that, on writing a point in Δ^n as an n-tuple $(v_1, \ldots, v_n)$ with

$$0 \leq v_1 \leq \ldots \leq v_n \leq 1,$$

we have:

$$h(f_1, \ldots f_n)(v_1 \leq \ldots \leq v_n) =$$
$$\begin{cases} T(f_1, A_n)h(f_2, \ldots, f_n)(v_2 \leq \ldots \leq v_n) & \text{if } v_1 = 0 \\ h(f_1, \ldots, f_{i+1}f_i, \ldots, f_n)(v_1 \leq \ldots v_i \leq v_{i+2} \leq \ldots \leq v_n) & \text{if } v_i = v_{i+1} \\ T(A_0, f_n)h(f_1, \ldots, f_{n-1})(v_1 \leq \ldots \leq v_{n-1}) & \text{if } v_n = 1 \\ h(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n)(v_1 \leq \ldots \leq v_{i-1} \leq v_{i+1} \leq \ldots v_n) & \text{if } f_i = id. \end{cases}$$

There is a fairly obvious dual definition and construction.

Let $\mathcal{A}$ be an $\mathcal{S}$-category and T an $\mathcal{S}$-functor from $\mathcal{A}^{op} \times \mathcal{A}$ to a *cocomplete* $\mathcal{S}$-category $\mathcal{B}$, tensored by $-\overline{\otimes}- : \mathcal{S} \times \mathcal{B} \to \mathcal{B}$.

Definition: The *simplicially coherent coend* of T will be the object

$$\oint^A T(A, A) = \int^{A',A''} (\widehat{\mathcal{A}^{op}})(A', A'')\overline{\otimes}T(A', A'')$$

where $(\widehat{\mathcal{A}^{op}})(-, -)$ is the $\mathcal{S}$-functor from $\mathcal{A} \times \mathcal{A}^{op}$ to $\mathcal{S}$ given by $(\widehat{\mathcal{A}^{op}})(A', A'') = diag\, X(A'', A')$.

Proposition 191 *(dual form)*

$$\oint^A T(A,A) = \int^n \Delta[n]\overline{\otimes}((\coprod \mathcal{A}(A_0,A_1)\times\ldots\times\mathcal{A}(A_{n-1},A_n))\overline{\otimes}T(A_n,A_0)),$$

where, as usual, the indexation of the coproduct is over all $A_0,\ldots,A_n$ in $\mathcal{A}$. ■

Remarks: 1. As we already have noted, if Y is a bisimplicial set, $diag\,Y$ is given by

$$(diag\,Y)_\bullet = \int^n \Delta[n]\times Y_{n,\bullet},$$

thus the formula, given for $\oint^A T(A,A)$ by the above, is a generalisation of the diagonal, applied to a simplicial object, $Y(T)$, in $\mathcal{B}$, given by the term in brackets. If $\mathcal{B}=\mathcal{S}$, we retrieve a diagonal; if $\mathcal{B}=Cat$, then we have a description similar to that of a lax coend and for $\mathcal{B}=Top$, we get a description in terms of topological higher homotopy coherence data, since the tensor, $K\overline{\otimes}-$, is the product with the geometric realisation, $|K|\times -$. This can be compared with the cobar construction of May, [199], Elmendorf, [124], Meyer, [201], and others.

2. We note that our results on coherent ends and coends are formal in the sense that they do not depend on the existence of the objects supposedly represented by the formula, but only on the formal manipulation of these formulae. If $Y(T)$ is to exist, then it is mostly likely to need the existence of the product, within $\mathcal{B}$, over *all* ordered sets of objects, $\{A_0,\ldots,A_n\}$, in $\mathcal{A}$. Of course, $\oint_A T(A,A)$ might exist, even if, in general, such products do not exist, but, in any case, these considerations are extraneous to our work here. Our results describe how $\oint_A T(A,A)$ reacts when it exists, regardless of questions related to the size of $\mathcal{A}$.

14.7.2 Coherent ends and Kan complexes.

(In this section, full proofs would need some results from the Bousfield-Kan lecture notes, [42], especially on how to handle the interaction of cosimplicial constructions in a category with a homotopy theory on that category. Because we do not otherwise need such results, proofs have been, at most, sketched and sometimes omitted. If omitted then a proof can be found in the literature, usually in [88], with some ideas being needed from [42], as mentioned above.)

In the category of simplicial sets, $\mathcal{S}$, the enriched hom-object, $\underline{S}(X,Y)$ is a Kan complex if Y is Kan, or, adopting the terminology of Quillen model category theories, if Y is *fibrant*. Thus to some extent locally Kan $\mathcal{S}$-categories are like categories in which every object is fibrant. This, of course, greatly enriches the potential of the theory as will be seen later. Partially because of this, it is useful to know conditions that imply that $\oint_A T(A,A)$ will be a Kan complex for $T:\mathcal{A}^{op}\times\mathcal{A}\to\mathcal{S}$. The methods used are based heavily on those developed for use with homotopy limits by Bousfield and Kan, [42].

Proposition 192 *If $\mathcal{A}$ is an $\mathcal{S}$-category and $T:\mathcal{A}^{op}\times\mathcal{A}\to\mathcal{S}$ is an $\mathcal{S}$-functor such that each $T(A',A'')$ is a Kan complex, then $\oint_A T(A,A)$ is a Kan complex.* ■

A proof is to be found in [88].

Corollary 30 *If $T_0,T_1 : \mathcal{A}^{op}\times\mathcal{A}\to\mathcal{S}$ are $\mathcal{S}$-functors such that each $T_0(A',A'')$ and each $T_1(A',A'')$ is a Kan complex and*

$$\{\eta(A',A''): T_0(A',A'')\to T_1(A',A'')\}$$

is an $\mathcal{S}$-natural transformation of $\mathcal{S}$-functors such that each $\eta(A', A'')$ is a homotopy equivalence, then η induces a homotopy equivalence,

$$\oint_A \eta(A, A) : \oint_A T_0(A, A) \to \oint_A T_1(A, A).$$

Proof: It is immediate that η induces a map of cosimplicial simplicial sets, $Y(\eta) : Y(T_0) \to Y(T_1)$, and that $Y(\eta)$ is a weak equivalence in the sense (Bousfield and Kan ([42], Ch.X §4) that for each $n \geq 0$, $Y(\eta)^n : Y(T_0)^n \to Y(T_1)^n$, is a homotopy equivalence.

(In fact, they only require that each $Y(\eta)^n$ be a *weak* equivalence, but as $Y(T_0)^n$ and $Y(T_1)^n$ are Kan complexes, this implies that each $Y(\eta)^n$ is a homotopy equivalence.) As a consequence we can conclude, [42], Ch.X §5, that $Tot\, Y(\eta)$ is a homotopy equivalence in $\mathcal{S}$, but $Tot\, Y(\eta)$ is, of course, $\oint_A \eta(A, A)$. ■

In the next corollary, we extend the proposition above to an arbitrary complete $\mathcal{S}$-category $\mathcal{B}$.

Definition: An object B in $\mathcal{B}$ is *fibrant* if $\mathcal{B}(X, B)$ is a Kan complex for all objects X in $\mathcal{B}$.

A consequence of this is that, if $\mathcal{B}$ is locally Kan, all objects are fibrant. The method we use below will be used many times later in this section.

Corollary 31 *Let $T : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{B}$ be an $\mathcal{S}$-functor where $\mathcal{B}$ is a complete $\mathcal{S}$-category and, for each A', A'' in $\mathcal{A}, T(A', A'')$ is fibrant, then $\oint_A T(A, A)$ is fibrant.*

Proof: Let X be an arbitrary object of $\mathcal{B}$. One easily checks

$$\mathcal{B}(X, \oint_A T(A, A)) \cong \oint_A \mathcal{B}(X, T(A, A))$$

since $\mathcal{B}(X, \_)$ preserves limits, but, as $\mathcal{B}(X, T(\_, \_)) : \mathcal{A}^{op} \times \mathcal{A} \to \underline{\mathcal{S}}$ takes "Kan" values, $\oint_A \mathcal{B}(X, T(A, A))$ is Kan. As X was arbitrary, this interprets as saying $\oint_A T(A, A)$ is fibrant. ■

The internal homotopy theory in a locally Kan $\mathcal{S}$-category has been mentioned earlier (section 11.3.1). We can take those ideas a little further with a fairly obvious definition.

Definition: A morphism, $f : B \to B'$, in an $\mathcal{S}$-category $\mathcal{B}$, is said to be a *homotopy equivalence* if for all X in $\mathcal{B}$, $\mathcal{B}(X, f)$ is a homotopy equivalence.

This *external* idea of homotopy equivalence can be *internalised* provided $\mathcal{B}$ is locally Kan and is a full sub $\mathcal{S}$-category of a complete (or cocomplete) $\mathcal{S}$-category.

Corollary 32 *If $T_0, T_1 : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{B}$, $\eta : T_0 \to T_1$ is a natural homotopy equivalence, and T_0, T_1 take fibrant values, then $\oint_A \eta$ is a homotopy equivalence in $\mathcal{B}$.* ■

The proof should now be clear as it combines the methods of earler results.

14.7.3 Coherent transformations and the coherent Yoneda lemma.

To express the universal properties of Kan extensions, one needs $Nat(F, G)$, the *set* of natural transformations between two functors. Following the *yoga* of categorification, but with a homotopy coherent flavour, we will probably need some notion of a *simplicial set* of *homotopy coherent transformations* between two $\mathcal{S}$-functors, and it is to that we turn next.

It is routine that if $\mathbb{A}$ and $\mathbb{B}$ are categories and $F, G : \mathbb{A} \to \mathbb{B}$ are functors, the set of natural transformations from F to G is given by $\int_A \mathbb{B}(FA, GA)$. (It is well known, but slightly less routine, that in the analogous 2-categorical situation, one can use lax ends to get the lax analogue.) This suggests the following:

Definition: Let $F, G : \mathcal{A} \to \mathcal{B}$, be two S-functors, then the *simplicial set of coherent transformations* from F to G, denoted $\mathrm{Coh}(\mathcal{A}, \mathcal{B})(F, G)$, is defined to be

$$Coh(\mathcal{A}, \mathcal{B})(F, G) = \oint_A \mathcal{B}(FA, GA).$$

This is thus given as a total object by the formula,

$$\int_n \mathcal{S}(\Delta[n], \prod_{A_0,\ldots,A_n} \mathcal{S}(\mathcal{A}(A_0, A_1) \times \ldots \times \mathcal{A}(A_{n-1}, A_n), \mathcal{B}(FA_0, GA_n))).$$

Example: Let $\mathcal{A} = \mathbb{A}$ be an ordinary category, that is, trivially $\mathcal{S}$-enriched, $F, G : \mathbb{A} \to Top$ be two ordinary functors, then

$$Coh(A, Top)(F, G) = \int_n Top(\Delta^n \times \coprod_{A_0 \xrightarrow{f_1} \ldots \xrightarrow{f_n} A_n} FA_0, GA_n).$$

Specifying a coherent morphism

$$m : F \to G,$$

thus corresponds to a 0-simplex of $Coh(A, Top)(F, G)$, that is, to the specification of a higher 'homotopy',

$$h(f_1, \ldots, f_n) : FA_0 \times \Delta^n \to GA_n,$$

for each n-tuple of composable morphisms, $A_0 \xrightarrow{f_1} A_1 \to \ldots \xrightarrow{f_n} A_n$. These homotopies are compatible with each other in the following sense (essentially read off from the face and degeneracy information encoded in the coherent end):
$h(f_1, \ldots, f_n)(x, v_1 \leq \ldots \leq v_n)$ is
$h(f_2, \ldots, f_n)(F(f_1)x, v_2 \leq \ldots \leq v_n)$ if $v_1 = 0$;
$h(f_1, \ldots, f_{i+1}f_i, \ldots, f_n)(x, v_1 \leq \ldots \leq v_i \leq v_{i+2} \leq \ldots \leq v_n)$ if $v_i = v_{i+1}$ for $0 < i < n$;
$G(f_n)h(f_1, \ldots, f_{n-1})(x, v_1 \leq \ldots \leq v_{n-1})$ if $v_n = 1$;
and
$h(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n)(x, v_1 \leq \ldots \leq v_{i-1} \leq v_{i+1} \leq \ldots \leq v_n)$ if $f_i = id$.

Remarks:

1) These are only the 0-simplices. There are higher order simplices giving homotopies between these coherent morphisms and so on.

2) It is simple to extend this detailed description to the case where F, G are homotopy coherent diagrams of type A. One replaces A by $\mathcal{S}(A)$, the $\mathcal{S}$-category resolving A, (see section 11.2.3). The data for a coherent transformation as defined combinatorially / 'geometrically' there is slightly more general than that used here. The two simplicial sets have the same homotopy type, however, since the "receiving category", **Top**, is locally Kan.

Proposition 193 *Given $F, G : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is locally Kan, then $Coh(\mathcal{A}, \mathcal{B})(F, G)$ is a Kan complex.* ∎

The proof is immediate from the definition given the results of section 14.7.2. Later on, we will examine the question of composition of coherent transformations, but before that we will look at a representation of coherent transformations as being $\mathcal{S}$-natural transformations between related $\mathcal{S}$-functors.

The augmentation of the indexation. A technical tool we shall use several times in the following is the augmentation of the indexation functor.

The bisimplicial set, $X(A, B)$, comes with a natural augmentation

$$d_0 : X(A, B)_0 \to \mathcal{A}(A, B)$$

given by composition:

$$X(A, B)_0 = \coprod_{A_0} \mathcal{A}(A, A_0) \times \mathcal{A}(A_0, B) \to \mathcal{A}(A, B).$$

This has a homotopy inverse given by s_{-1}, *i.e.*, it sends $g \in \mathcal{A}(A, B)_n$ to (id_A, g). These two maps pass to the diagonal to give

$$d_0 : \hat{\mathcal{A}}(A, B) \to \mathcal{A}(A, B),$$

and

$$s_{-1} : \mathcal{A}(A, B) \to \hat{\mathcal{A}}(A, B),$$

which are both homotopy equivalences. The important difference between them is that whilst d_0 is natural in A and B, s_{-1} is natural in B, but not in A. Furthermore whilst $d_0 s_{-1}$ *is* the identity, $s_{-1} d_0$ is *homotopic* to the identity by a homotopy which, again, is natural in B, but not in A. In fact both s_{-1} and this homotopy are coherent in A as is easily checked.

Now suppose $\mathcal{B}$ is a complete locally Kan $\mathcal{S}$-category and that $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{A} \to \mathcal{S}$ are S-functors. We define the *coherent mean cotensor* of F and G by

$$\overline{B^A}(G, F) = \oint_A \overline{\mathcal{B}}(GA, FA).$$

In particular, we get a new functor $\overline{F} : \mathcal{A} \to \mathcal{B}$, defined by $\overline{F}(A) = \overline{B^A}(\mathcal{A}(A, \_), F(\_))$.

Remark: This coherent mean cotensor could equally well be termed the *coherent mean power*, since, as we have remarked earlier, the cotensor is a generalisation of the 'power' functor, sending a set, X, and and object A in a complete category, to A^X the X-fold product of A with itself.

For $\mathcal{S}$-functors, $F, G : \mathcal{A} \to \mathcal{B}$, $\mathcal{B}^{\mathcal{A}}(G, F) = \int_A \mathcal{B}(GA, FA)$ is the simplicial set of natural transformations, so this coherent mean cotensor generalises this, both by replacing G by $G : \mathcal{A} \to \mathcal{S}$, and by replacing the end by a coherent end.

Dually if $G : \mathcal{A}^{op} \to \mathcal{S}$, then the *coherent mean tensor* or *coherent mean copower*, $G\overline{\otimes}F$, will exist if $\mathcal{B}$ is cocomplete:

$$G\overline{\otimes}F = \oint^A GA\overline{\otimes}FA.$$

In particular we will write $\underline{F} : \mathcal{A} \to \mathcal{B}$ for the functor given by $\underline{F}(A) = \mathcal{A}(\_, A)\overline{\otimes}F$.

These functors $\overline{F}$ and $\underline{F}$ "absorb coherence" as follows:

Proposition 194 *(i) If $\mathcal{B}$ is complete, there is a natural isomorphism*

$$Coh(\mathcal{A}, \mathcal{B})(F, G) \cong \mathcal{B}^{\mathcal{A}}(F, \overline{G})$$

for any $F, G : \mathcal{A} \to \mathcal{B}$, where, as above, $\mathcal{B}^{\mathcal{A}}(F, \overline{G})$ is the simplicial set of natural transformations from F to $\overline{G}$.

(ii) If $\mathcal{B}$ is cocomplete, there is a natural isomorphism

$$Coh(\mathcal{A}, \mathcal{B})(F, G) \cong \mathcal{B}^{\mathcal{A}}(\underline{F}, G).$$

Proof

We will only give the proof of (ii). That of (i) is dual.

$$\begin{aligned}
\mathcal{B}^A(\underline{F}, G) &= \int_A \mathcal{B}(\underline{F}A, GA) \\
&= \int_A \mathcal{B}(\int^{A', A''} (\widehat{\mathcal{A}^{op}})(A', A'')\overline{\otimes}(\mathcal{A}(A', A)\overline{\otimes}FA''), GA) \\
&\cong \int_{A, A', A''} \mathcal{S}(diag\, X(A'', A') \times \mathcal{A}(A', A), \mathcal{B}(FA'', GA)) \\
&\cong \int_{A, A''} \mathcal{S}(\hat{\mathcal{A}}(A'', A), \mathcal{B}(FA'', GA)) \\
&= Coh(\mathcal{A}, \mathcal{B})(F, G)
\end{aligned}$$

■

Examples. 1. For fixed A in $\mathcal{A}$, let $F = \mathcal{A}(A, \_) : \mathcal{A} \to \mathcal{S}$, then

$$\begin{aligned}
\underline{F}(B) &= \int^{A', A''} (\widehat{\mathcal{A}^{op}})(A', A'') \times \mathcal{A}(A', B) \times \mathcal{A}(A, A'') \\
&\cong \int^{A', A''} \mathcal{A}(A, A'') \times \hat{\mathcal{A}}(A'', A') \times \mathcal{A}(A', B) \\
&\cong \hat{\mathcal{A}}(A, B)
\end{aligned}$$

The isomorphism of (ii), above, specialises to give

$$\mathcal{S}^{\mathcal{A}}(\hat{\mathcal{A}}(A, \_), \mathcal{A}(A, \_)) \cong Coh(\mathcal{A}, \mathcal{S})(\mathcal{A}(A, \_), \mathcal{A}(A, \_))$$

and the $\mathcal{S}$-natural transformation corresponding to the identity coherent transformation is d_0, as it is induced by composition.

2. Reversing the rôles of $\hat{\mathcal{A}}(A,\_)$ and $\mathcal{A}(A,\_)$, we get that, to the identity in

$$\mathcal{S}^A(\hat{\mathcal{A}}(A,\_), \hat{\mathcal{A}}(A,\_)),$$

there corresponds a *coherent* transformation, σ, in

$$Coh(\mathcal{A},\mathcal{S})(\mathcal{A}(A,\_), \hat{\mathcal{A}}(A,\_))_0.$$

Direct calculation of σ shows it to be given by a natural map

$$\sigma : \hat{\mathcal{A}}(A'',\_) \to \mathcal{S}(\mathcal{A}(A,A''), \hat{\mathcal{A}}(A,\_))$$

which is adjoint to the $\mathcal{A}$-action,

$$\overline{\sigma} : \mathcal{A}(A,A'') \times \hat{\mathcal{A}}(A'',\_) \to \hat{\mathcal{A}}(A,\_),$$

given by composition. To the identity string in $\hat{\mathcal{A}}(A'',A'')$, there, thus, corresponds a natural map, $\sigma(id) : \mathcal{A}(A,A'') \to \hat{\mathcal{A}}(A,A'')$, and if $g \in \mathcal{A}(A,A'')_n$, then $\sigma(id)(g) = (id_A, g)$. In other words, $\sigma(id)$ is the homotopy inverse to the augmentation and is, in fact, a natural transformation in the variable A''.

3. In general, if $\mathcal{B}$ is complete and $F : \mathcal{A} \to \mathcal{B}$, then there is a natural transformation,

$$\eta_F : F \to \overline{F},$$

corresponding, by (i) of the above proposition, to the identity coherent transformation on F. Dually, if $\mathcal{B}$ is cocomplete, there is a natural transformation,

$$\eta^F : \underline{F} \to F.$$

Proposition 195 *(i) If $\mathcal{B}$ is complete, then $\eta_F : F \to \overline{F}$ is a level-wise homotopy equivalence, i.e. for each A in $\mathcal{A}$, $\eta_F(A)$ is a homotopy equivalence.*

(ii) If $\mathcal{B}$ is cocomplete, then $\eta^F : \underline{F} \to F$ is a levelwise homotopy equivalence.

Before we prove this, we extend and adapt some ideas and results from Cordier-Porter, [84].

Suppose $F, F' : \mathcal{A} \to \mathcal{S}$, $G : \mathcal{A} \to \mathcal{B}$ are $\mathcal{S}$-functors, $f_0, f_1 : F \to F'$ are two natural transformations and $h : F \times \Delta[1] \to F'$ is a natural homotopy between them. Suppose that $\mathcal{B}$ is cotensored, then we can form $\int_A \overline{\mathcal{B}}(FA, GA)$ and $\int_A \overline{\mathcal{B}}(F'A, GA)$ and f_0, f_1 induce natural transformations

$$f_0^{\#}, f_1^{\#} : \int_A \overline{\mathcal{B}}(F'A, GA) \to \int_A \overline{\mathcal{B}}(FA, GA).$$

Proposition 196 *The natural homotopy, $h : f_0 \simeq f_1$, induces a natural homotopy, $h^{\#} : f_0^{\#} \simeq f_1^{\#}$.*

Proof: The natural transformation h induces a map

$$\int_A \overline{\mathcal{B}}(hA, GA) : \int_A \overline{\mathcal{B}}(F'A, GA) \to \int_A \overline{\mathcal{B}}(FA \times \Delta[1], GA),$$

(which we will normally shorten to $\int_A h$), but

$$\begin{aligned} \int_A \overline{\mathcal{B}}(FA \times \Delta[1], GA) &\cong \int_A \overline{\mathcal{B}}(\Delta[1], \overline{\mathcal{B}}(FA, GA)) \\ &\cong \overline{\mathcal{B}}(\Delta[1], \int_A \overline{\mathcal{B}}(FA, GA)) \end{aligned}$$

and it is easily checked that $h^\# = \int_A h$ gives the required homotopy. ■

Corollary 33 *Suppose $F, F' : \mathcal{A} \to \mathcal{S}$. If $f : F \to F'$ and $g : F' \to F$ are natural homotopy inverses,* i.e., *there are natural homotopies, $h : fg \simeq Id$, $k : gf \simeq Id$, then, for any $G : \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$ cocomplete, $\int_A f$ and $\int_A g$ form a homotopy equivalence between $\int_A \overline{\mathcal{B}}(FA, GA)$ and $\int_A \overline{\mathcal{B}}(F'A, GA)$.*

Proof: The proof is a simple application of Proposition 195. ■

There are, of course, dual versions of these results, (Proposition 196 and Corollary 33), whose formulation and proof we **leave 'to the reader', that is, you**.

Proof of Proposition 195 (i): Since by the Yoneda lemma,

$$FA = \int_{A'} \overline{\mathcal{B}}(\mathcal{A}(A, A'), FA'),$$

whilst

$$\overline{F}A \cong \int_{A'} \overline{\mathcal{B}}(\hat{\mathcal{A}}(A, A'), FA'),$$

the natural homotopy inverses, $d_0 : \hat{\mathcal{A}}(A, \_) \to \mathcal{A}(A, \_)$ and $s_{-1} : \mathcal{A}(A, \_) \to \hat{\mathcal{A}}(A, \_)$, provide the solution to constructing a homotopy equivalence from FA to $\overline{F}A$. It remains to check that this is η_F, but this is routine, so is **left to you** to examine using the usual techniques of the proof of the Yoneda lemma.

The proof of Proposition 195 (ii) is dual, using the dual forms of Proposition 196 and Corollary 33. ■

It is important to note that, although η_F is a natural (levelwise) homotopy equivalence, its homotopy inverse, $\int_A s_{-1}$, is not natural.

Corollary 34 *Suppose that $\mathcal{B}$ is a complete cotensored locally Kan $\mathcal{S}$-category and $F, G : \mathcal{A} \to \mathcal{B}$ two $\mathcal{S}$-functors, then the augmentation induces a homotopy equivalence,*

$$Coh(\mathcal{A}, \mathcal{B})(G, F) \to Coh(\mathcal{A}, \mathcal{B})(G, \overline{F}).$$

■

Remark: Noting that $Coh(\mathcal{A},\mathcal{B})(G,F) \cong \mathcal{B}^A(G,\overline{F})$ by Proposition 195(i), this homotopy equivalence is the natural inclusion (up to identification via this isomorphism) of $\mathcal{B}^A(G,\overline{F})$ into $\mathrm{Coh}(\mathcal{A},\mathcal{B})(G,\overline{F})$.

Proof of Corollary 34 First we write: $\overline{F}A \cong \int_{A'} \overline{\mathcal{B}}(\hat{\mathcal{A}}(A,A'),FA')$, by the isomorphism used previously, then

$$\begin{aligned} Coh(\mathcal{A},\mathcal{B})(G,F) &\cong \int_{A,A'} \mathcal{S}(\hat{\mathcal{A}}(A,A),\mathcal{B}(GA,\int_{A''}\overline{\mathcal{B}}(\hat{\mathcal{A}}(A',A''),FA'')) \\ &\cong \int_{A,A'} \mathcal{S}(\hat{\mathcal{A}}(A,A'),\int_{A''}\mathcal{S}(\hat{\mathcal{A}}(A',A''),\mathcal{B}(GA,FA''))). \end{aligned}$$

Similarly

$$Coh(\mathcal{A},\mathcal{B})(G,F) \cong \int_{A,A'} \mathcal{S}(\hat{\mathcal{A}}(A,A'),\int_{A''}\mathcal{S}(\mathcal{A}(A',A''),\mathcal{B}(GA,FA''))).$$

Writing

$$H(A,A') = \int_{A''}\mathcal{S}(\hat{\mathcal{A}}(A',A''),\mathcal{B}(GA,FA''))$$

and

$$K(A,A) = \int_{A''}\mathcal{S}(\mathcal{A}(A',A''),\mathcal{B}(GA,FA'')),$$

then

a) $H,K : \mathcal{A}^{op}\times\mathcal{A} \to \mathcal{S}$ take Kan values, since $\mathcal{B}$ is locally Kan,

b) the augmentation, $d_0 : \hat{\mathcal{A}}(A',A'') \to \mathcal{A}(A',A'')$, induces, by Proposition 33, a homotopy equivalence

$$K(A,A') \to H(A,A'),$$

natural in A and A',

c) $Coh(\mathcal{A},\mathcal{B})(G,F) \cong \oint_A K(A,A)$ and $Coh(\mathcal{A},\mathcal{B})(G,\overline{F}) \cong \oint_A H(A,A)$, so the result follows.■

Remark: Proposition 195(i) is a strong form of what might be called the *homotopy coherent Yoneda lemma.* If $\mathcal{B} = \mathcal{S}$, this then reads as:
for each $F : \mathcal{A} \to \mathcal{S}$, there is a homotopy equivalence

$$\eta_F(A) : F(A) \to Coh(\mathcal{A},\mathcal{S})(\mathcal{A}(A,\_),F).$$

In the next section, we will need to use the description of $\overline{F}A$ as a total object of a cosimplicial object and to this we turn next. If $F : \mathcal{A} \to \mathcal{B}$ and A is in $\mathcal{A}$, then

$$\begin{aligned} \overline{F}A &= \int_{A',A''} \overline{\mathcal{B}}(\hat{\mathcal{A}}(A',A''),\overline{\mathcal{B}}(\mathcal{A}(A,A'),FA'')) \\ &\cong \int_{A',A''} \overline{\mathcal{B}}(\mathcal{A}(A,A')\times\hat{\mathcal{A}}(A',A''),FA'') \\ &\cong \int_{A''} \overline{\mathcal{B}}(\hat{\mathcal{A}}(A,A''),FA'') \\ &= \int_{[n]}\int_{A''} \overline{\mathcal{B}}(X(A,A'')_n\times\Delta[n],FA'') \\ &\cong \int_{[n]} \overline{\mathcal{B}}(\Delta[n],\int_{A''}\overline{\mathcal{B}}(X(A,A'')_n,FA''), \end{aligned}$$

so we take $Y(F,A)_n = \int_{A''} \overline{\mathcal{B}}(X(A,A''), FA'')$ to get a cosimplicial object in $\mathcal{B}$, whose total object, $\hom(\Delta, Y(F,A))$, is $\overline{F}A$. (We leave to you the task of writing this in its product form.)

Given this description, we note that there is a natural map from Δ to $*$, the constant cosimplicial simplicial set with value $\Delta[0]$. This induces a map with codomain $\overline{F}A$ from the object defined by the end

$$\int_n \overline{\mathcal{B}}(*, \int_{A''} \overline{\mathcal{B}}(X(A,A'')_n, FA'')),$$

i.e., $\hom(*, Y(F,A))$. The following proposition is sometimes of use.

Proposition 197 *There is a natural isomorphism,*

$$F(A) \cong \hom(*, Y(F,A)),$$

such that, modulo identifying these two objects, the natural map

$$\hom(*, Y(F,A)) \to \overline{F}(A),$$

induced by $\Delta \to *$, is the map $\eta_F(A)$.

Proof: The proof is simply to note that, as $*$ is constant in n ,

$$\begin{aligned} \int_n \overline{\mathcal{B}}(*, \int_{A''} \overline{\mathcal{B}}(X(A,A'')_n, FA'')) &\cong \overline{\mathcal{B}}(*, \int_{A''} \overline{\mathcal{B}}(colim_n X(A,A'')_n, FA'')) \\ &\cong \overline{\mathcal{B}}(*, \int_{A''} \overline{\mathcal{B}}(\mathcal{A}(A,A''), F(A'')) \\ &\cong FA \end{aligned}$$

by the Yoneda lemma. The identification of the induced map as $\eta_F(A)$ is now easy. ■

Corollary 35 *Suppose $F, G : \mathcal{A} \to \mathcal{B}$ are $\mathcal{S}$-functors. The natural map from Δ to $*$ induces a map,*

$$\hom(*, Y(F,G) \to \hom(\Delta, Y(F,G))$$

which is isomorphic to the natural inclusion,

$$\mathcal{B}^A(F,G) \to Coh(\mathcal{A},\mathcal{B})(F,G).$$

■

The proof of this using $\eta_G : G \to \overline{G}$ and the previous result is clear. There is also a proof establishing the natural isomorphism,

$$\hom(*, Y(F,G)) \cong \mathcal{B}^{\mathcal{A}}(F,G),$$

directly.

We next turn to the problem of 'interpreting' the simplices in the simplicial set,

$$Coh(\mathcal{A},\mathcal{B})(F,G),$$

(which we will usually abbreviate to $Coh(F,G)$ if the context is clear). One expects the 0-simplices of $Coh(F,G)$ to correspond to families of maps,

$$\{f_A : F(A) \to G(A)\},$$

indexed by the objects of $\mathcal{A}$, together with higher homotopy information on the "homotopy commutativity" of various "diagrams". We have already briefly seen this earlier, back in Chapter 11, when we looked at the category $\mathcal{S}$ itself and A, an ordinary category with F, G ordinary functors. Our aim is thus to make precise what it means to say that $\{f_A\}$, as above, is "coherent in A" as one might say "$\{f_A\}$ is natural in A".

Suppose $f \in Coh(F,G)_0$. Using our interpretation of $Coh(F,G)$ as the total object of $Y(F,G)$, we have that $f : \Delta \to Y(F,G)$, and hence $f = (f^n)$, where $f^n : \Delta[n] \to Y(F,G)^n$ and $d^i f^{n+1} = f^n d^i$.

We start by examining the case, $n = 0$. Referring back to our interpretation of the elements of $\oint_A T(A,A)_0$ in section 14.7.1, $f(A_0) = p(A)f^0 \in \mathcal{S}(FA, GA)_0$, so is a map from FA to GA in $\mathcal{B}$, whilst

$$p(A_0, A_1)f^1 : \mathcal{A}(A_0, A_1) \times \Delta[1] \to \mathcal{B}(FA_0, GA_1).$$

For simplicity of interpretation, we assume that $\mathcal{B}$ is tensored, and hence we can rewrite $p(A_0, A_1)f^1$ as a homotopy from $\mathcal{A}(A_0, A_1)\overline{\otimes}FA_0$ to GA_1 . The two ends of this homotopy then correspond to the two composites around the square,

$$\begin{tikzcd}
\mathcal{A}(A_0, A_1)\overline{\otimes}FA_0 \arrow[r] \arrow[d, "\mathcal{A}(A_0,A_1)\overline{\otimes}f(A_0)"'] & FA_1 \arrow[d, "f(A_1)"] \\
\mathcal{A}(A_0, A_1)\overline{\otimes}GA_0 \arrow[r] & GA_1
\end{tikzcd}$$

where the horizontal arrows come from the actions of $\mathcal{A}$ on F, and G, respectively, adjoint to the structure maps, $\mathcal{A}(A_0, A_1) \to \mathcal{B}(FA_0, FA_1)$, etc. Similarly in the next dimension, there are actions

$$\begin{array}{rcl}
(\mathcal{A}(A_0, A_1) \times \mathcal{A}(A_1, A_2))\overline{\otimes}FA_0 & \to & FA_2, \\
(\mathcal{A}(A_0, A_1) \times \mathcal{A}(A_1, A_2))\overline{\otimes}FA_0 & \to & \mathcal{A}(A_1, A_2)\overline{\otimes}FA_1,
\end{array}$$

and

$$\mathcal{A}(A_1, A_2)\overline{\otimes}FA_1 \to FA_2,$$

and similarly for G, which are linked via the various $f(A)s$, and the higher homotopy,

$$p(A_0, A_1, A_2)f^2 : (\mathcal{A}(A_0, A_1) \times \mathcal{A}(A_1, A_2) \times \Delta[2])\overline{\otimes}FA_0 \to GA_2,$$

provides a homotopy linking them. More precisely, the faces of this are:

$$\begin{array}{lll}
d_0 & \text{is} & (\mathcal{A}(A_1, A_2) \times \Delta[1])\overline{\otimes}FA_1 \overset{p(A_1,A_2)f^1}{\to} GA_2, \\
d_1 & \text{is} & (\mathcal{A}(A_0, A_2) \times \Delta[1])\overline{\otimes}FA_0 \overset{p(A_0,A_2)f^1}{\to} GA_2, \\
d_2 & \text{is} & (\mathcal{A}(A_0, A_1) \times \Delta[1])\overline{\otimes}FA_0 \overset{p(A_0,A_1)f^1}{\to} GA_1.
\end{array}$$

Each of these correspond to a square, as above, and these squares fit to form a homotopy coherent prism. It is, however, clearer to think of the higher homotopy, f^2, in the original form as

$$(\mathcal{A}(A_0, A_1) \times \mathcal{A}(A_1, A_2) \times \Delta[2])\overline{\otimes}FA_0 \to GA_2$$

(or

$$\mathscr{A}(A_0, A_1) \times \mathscr{A}(A_1, A_2) \times \Delta[2] \to \mathscr{B}(FA_0, GA_2)),$$

together with the face information. **We leave to you** the joy of writing down the information corresponding to $p(A_0, A_1, A_2, A_3)f^3$, etc.

It should now be clear how to specify what it means for a family, $\{f(A) : FA \to GA\}$, to be coherent in A, namely the existence of homotopies, higher homotopies, etc., linking the $f(A)$ with the iterated actions of $\mathscr{A}$.

Bibliography

[1] M. A. Abdul Kadir, 2000, *Coverings, Hypercoverings and Homotopy Coherence*, Ph.D. thesis, University of Wales, Bangor. 545

[2] H. Abels and S. Holz, *Higher generation by subgroups*, J. Alg, 160, (1993), 311– 341. 123, 125, 134, 135, 143, 150, 151, 152, 430

[3] J. F. Adams, 1972, *Algebraic Topology - A Student's Guide*, number 4 in London Mathematical Society Lecture Note Series, Cambridge University Press. 363, 737

[4] E. Aldrovandi, 2005, *2-Gerbes bound by complexes of gr-stacks, and cohomology*, URL `http://www.arXiv.org:math/0512453`. 248

[5] E. Aldrovandi and B. Noohi, *Butterflies I: Morphisms of 2-group stacks*, Advances in Mathematics, 221, (2009), 687–773. 556, 580, 581, 584, 588

[6] M. André, 1967, *Méthode simpliciale en algèbre homologique et algèbre commutative*, number 32 in Lecture Notes in Maths, Springer-Verlag, Berlin. 89, 91, 93

[7] M. André, 1970, *Homology of simplicial objects*, in *Proc. Symp. on Categorical Algebra*, 15 – 36, American Math. Soc. 89, 91

[8] M. Artin, *On the joins of Hensel rings*, Advances in Math., 7, (1971), 282 – 296. 464

[9] M. Artin, A. Grothendieck and J. L. Verdier, 1973, *Théorie des Topos et Cohomologie Etale des Schémas*, volume 269, 270, 305, of *Springer Lecture Notes in Mathematics*, Springer, available from: http://www.math.mcgill.ca/∼archibal/SGA/SGA.html. 89, 198

[10] M. Artin and B. Mazur, *On the van Kampen Theorem*, Topology, 5, (1966), 179 – 189. 234, 650

[11] M. Artin and B. Mazur, 1969, *Étale homotopy*, number 100 in Lecture Notes in Maths., Springer-Verlag, Berlin. 198, 215

[12] N. Ashley, 1978, *Simplicial T-Complexes: a non abelian version of a theorem of Dold and Kan*, Ph.D. thesis, University of Wales, Bangor, URL `http://ehres.pagesperso-orange.fr/Cahiers/AshleyEM32.pdf`. 39

[13] N. Ashley, *Simplicial T-Complexes: a non abelian version of a theorem of Dold and Kan*, Dissertations Math., 165, (1989), 11 – 58. 39, 40, 251, 254, 256

[14] J. C. Baez and A. S. Crans, *Higher-Dimensional Algebra VI: 2-Lie Algebras*, Theory and Applications of Categories, 12, (2004), 492 – 528. 356, 357

[15] J. C. Baez and J. Dolan, 1998, *Categorification*, URL http://www.arXiv.org:math/9802029. 15, 60

[16] J. C. Baez and A. D. Lauda, *Higher-Dimensional Algebra V: 2-Groups*, Theory and Applications of Cateories, 12, (2004), 423 – 491. 2

[17] A. Bak, *Global Actions: The algebraic counterpart of a topological space*, Uspeki Mat. Nauk., English Translation: Russian Math. Surveys, 525, (1997), 955 –996. 127, 147

[18] A. Bak, 1998, *Topological Methods in Algebra*, in S. Caenepeel and A. Verschoren, eds., *Rings, Hopf Algebras and Brauer Groups*, number 197 in Lect. Notes in Pure and Applied Math, M. Dekker, New York. 127, 147

[19] A. Bak, R. Brown, G. Minian and T. Porter, *Global Actions, Groupoid Atlases and Application*, Journal of Homotopy and Related Structures, 1, (2006), 101 – 167. 125, 127

[20] M. Barr and J. Beck, 1969, *Homology and Standard Constructions*, in *Seminar on triples and categorical homology*, number 80 in Lecture Notes in Maths., Springer-Verlag, Berlin. 89

[21] M. G. Barratt, V. Gugenheim and J. Moore, *On semisimplicial fibre-bundles*, American Journal of Mathematics, 81, (1959), 639–657. 282

[22] H. Bass, *K-Theory and stable algebra*, Publications Mathématiques de l'IHÉS, 22, (1964), 5 – 60. 116

[23] H. Bass, *Covering theory for graphs of groups*, Journal of Pure and Applied Algebra, 89, (1993), 3–47. 161, 166

[24] M. A. Batanin, *Monoidal globular categories as a natural environment for the theory of weak n-categories*, Advances in Math., 136, (1998), 39 – 103. 514

[25] H. J. Baues, 1989, *Algebraic Homotopy*, volume 15 of *Cambridge Studies in Advanced Mathematics*, Cambridge Univ. Press. 67

[26] H. J. Baues, 1991, *Combinatorial homotopy and 4-dimensional complexes*, Walter de Gruyter. 238, 504

[27] H. J. Baues, 1995, *Homotopy Types*, in I. M. James, ed., *Handbook of Algebraic Topology*, 1 – 72, Elsevier. 67, 192, 193

[28] H. Behr, *Explizite Präsentation von Chevalley-gruppen über Z*, Math. Z., 141, (1975), 235 – 241. 123

[29] T. Beke, *Higher Čech Theory*, K Theory, 32, (2004), 293 – 322. 198, 203, 204, 205, 381, 382

[30] J. Bénabou, 1967, *Introduction to bicategories*, in *Reports of the Midwest Category Seminar*, 1 – 77, Springer, Berlin. 522, 684

[31] J. Bénabou, 1973, *Les distibuteurs*, Technical Report 33, Inst. Math. Pure Appl., Univ. Louvain-la-Neuve. 682

[32] J. Bénabou, 2000, *Distributors at Work*, URL http://www.mathematik.tu-darmstadt.de/~streicher/FIBR/DiWo.pdf, (Notes by T. Streicher). 682

[33] G. Biedermann, *On the homotopy theory of n-types*, Homology, Homotopy Appl., 10, (2008), 305 – 325. 209

[34] V. Blanco, M. Bullejos and E. Faro, *Categorical non-abelian cohomology and the Schreier theory of groupoids*, Math. Z., 251, (2005), 41 – 59. 246, 527, 531, 544, 548

[35] V. Blanco, M. Bullejos and E. Faro, *A Full and Faithful Nerve for 2-Categories*, Applied Categorical Structures, 13, (2005), 223 – 233. 523, 527, 539, 540, 548

[36] J. M. Boardman and R. M. Vogt, 1973, *Homotopy Invariant algebraic structures on topological spaces*, number 347 in Lecture Notes in Maths, Springer-Verlag. 503, 509, 511, 514

[37] F. Borceux, 1994, *Handbook of categorical algebra: 1*, volume 50 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge. 306, 310, 311, 522, 531, 533, 534, 535, 538, 682, 684

[38] F. Borceux, 1994, *Handbook of categorical algebra: 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge. 495

[39] F. Borceux and G. Janelidze, 2001, *Galois theories*, volume 72 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press. 280, 295, 297, 299, 300, 301, 305, 393, 411, 533

[40] D. Bourn, 2007, *Moore normalisation and Dold-Kan theorem for semi-Abelian categories*, in *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, 105 – 124, Amer. Math. Soc., Providence, RI. 631

[41] D. Bourn and J.-M. Cordier, *A general formulation of homotopy limits*, J. Pure Appl. Alg., 29, (1983), 129 – 141, ISSN 0022-4049. 678, 709

[42] A. Bousfield and D. Kan, 1972, *Homotopy limits, Completions and localizations*, volume 304 of *Lecture Notes in Maths*, Springer-Verlag. 678, 688, 709, 714, 715

[43] P. Bouwknegt, A. L. Carey, V. Mathai, M. K. Murray and D. Stevenson, *Twisted K-Theory and K-Theory of Bundle Gerbes*, Communications in Mathematical Physics, 228, (2002), 17 – 49. 470

[44] S. Bozapalides, 1976, *Théorie formelle des bicatégories*, Ph.D. thesis, Paris. 691, 713

[45] S. Bozapalides, *Les fins cartésiennes généralisées*, Archivum Mathematicum, 13, (1977), 75–87. 691, 694, 696

[46] S. Bozapalides, *Some remarks on Lax presheafs*, Illinois J. Math., 24, (1980), 676 – 680. 691, 713

[47] G. Bredon, 1972, *Introduction to compact transformation groups*, Pure and applied mathematics, Academic Press, URL http://books.google.co.uk/books?id=qXEl_AIeSBUC. 161

[48] L. Breen, 1990, *Bitorseurs et cohomologie non abélienne*, in *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, 401 – 476, Birkhäuser Boston, Boston, MA. 2, 248, 313, 320, 328, 335, 347, 364, 376, 455

[49] L. Breen, *Théorie de Schreier supérieure*, Ann. Sci. École Norm. Sup. (4), 25, (1992), 465 – 514. 313, 346

[50] L. Breen, *On the classification of* 2*-gerbes and* 2*-stacks*, Astérisque, 225, (1994), 160, ISSN 0303-1179. 248, 313, 321, 453, 490

[51] L. Breen, 2009, *Notes on 1-and 2-gerbes*, in J. Baez and J. May, eds., *Towards Higher Categories*, volume 152 of *The IMA Volumes in Mathematics and its Applications*, 193–235, Springer. 2, 313, 328, 364, 458, 461, 463, 482, 486

[52] L. Breen and W. Messing, *Differential Geometry of Gerbes*, Advances in Mathematics, 198, (2005), 732 – 846, URL http://arxiv.org/abs/math.AG/0106083. 2

[53] S. G. Brick and J. M. Corson, *Dehn functions and complexes of groups*, Glasgow Mathematical Journal, 40, (1998), 33–46. 140

[54] M. Bridson and A. Haefliger, 1999, *Metric Spaces of Non-Positive Curvature*, number 31 in Grundlehren der Math. Wiss, Springer. 135, 140, 151, 161, 162, 163, 175, 177, 178, 179, 181, 186

[55] K. Brown, *Abstract Homotopy Theory and Generalized Sheaf Cohomology*, Trans. Amer. Math. Soc, 186, (1973), 419 – 458. 57

[56] K. Brown, 1982, *Cohomology of groups*, number 87 in Graduate Texts in Mathematics, Springer-Verlag. 27, 73, 171

[57] R. Brown, *Fibrations of groupoids*, Journal of Algebra, 15, (1970), 103–132. 184, 406

[58] R. Brown, 1988, *Topology, a geometric account of general topology, homotopy types and the fundamental groupoid*, Ellis Horwood. 178, 728

[59] R. Brown, 2006, *Topology and Groupoids*, BookSurge, (This is an extended version of [58]). 18, 148, 178, 182, 184, 406, 429, 458

[60] R. Brown and P. J. Higgins, 1982, *Crossed complexes and non-abelian extensions*, in *Int. Conf. Cat. Theory, Gummersbach*, volume 962 of *Lecture Notes in Maths*, Springer-Verlag. 69

[61] R. Brown and P. J. Higgins, *Tensor products and homotopies for ω-groupoids and crossed complexes,*, J. Pure Appl. Alg., 47, (1987), 1 – 33. 67, 365, 664, 666, 671

[62] R. Brown and P. J. Higgins, *Crossed complexes and chain complexes with operators*, Math. Proc. Camb. Phil. Soc., 107, (1990), 33 – 57. 67, 70, 83

[63] R. Brown and P. J. Higgins, *The classifying space of a crossed complex*, Math. Proc. Cambridge Philos. Soc., 110, (1991), 95 – 120. 251, 256, 666, 671

[64] R. Brown, P. J. Higgins and R. Sivera, 2010, *Nonabelian Algebraic Topology: Filtered spaces, crossed complexes, cubical homotopy groupoids*, volume 15 of *EMS Tracts in Mathematics*, European Mathematical Society. 2, 44, 67, 68, 70, 149, 223, 497, 509, 664, 665, 666, 667, 668, 669, 670, 671

[65] R. Brown and J. Huebschmann, 1982, *Identities among relations*, in R. Brown and T. L. Thickstun, eds., *Low Dimensional Topology*, London Math. Soc Lecture Notes, Cambridge University Press. 47, 48, 55, 82, 86

[66] R. Brown and J.-L. Loday, *Homotopical excision, and Hurewicz theorems, for n-cubes of spaces*, Proc. London Math. Soc., (3)54, (1987), 176 – 192. 230, 237

[67] R. Brown and J.-L. Loday, *Van Kampen Theorems for diagrams of spaces*, Topology, 26, (1987), 311 – 337. 212, 230, 237

[68] R. Brown, E. J. Moore, T. Porter and C. D. Wensley, *Crossed complexes, and free crossed resolutions for amalgamated sums and HNN-extensions of groups*, Georgian Math. J., 9, (2002), 623 – 644, ISSN 1072-947X, dedicated to Professor Hvedri Inassaridze on the occasion of his 70th birthday. 150, 430

[69] R. Brown and T. Porter, *On the Schreier theory of non-abelian extensions: generalisations and computations*, Proc. Roy. Irish Acad. Sect. A, 96, (1996), 213 – 227. 247

[70] R. Brown and C. Spencer, *G-groupoids, crossed modules and the fundamental groupoid of a topological group*, Proc. Kon. Ned. Akad. v. Wet, 79, (1976), 296 – 302. 58

[71] M. Bullejos and A. Cegarra, *On the Geometry of 2-Categories and their Classifying Spaces*, K-Theory, 29, (2003), 211 – 229. 523, 527, 532, 548, 555, 556

[72] M. Bullejos and A. M. Cegarra, *Classifying Spaces for Monoidal Categories Through Geometric Nerves*, Canadian Mathematical Bulletin, 47, (2004), 321–331. 548, 556

[73] M. Bullejos, A. M. Cegarra. and J. Duskin, *On cat^n-groups and homotopy types*, J. Pure Appl. Alg., 86, (1993), 135 – 154. 234

[74] S. Buoncristiano, 2003, *Fragments of geometric topology from the sixties*, http://www.emis.de/journals/GT//gtmcontents6.html. 269

[75] S. Buoncristiano and C. Rourke, 2003, *Fragments of geometric topology from the sixties*, volume 6 of *Geometry & Topology Monographs*, Mathematical Sciences Publishers, University of California, Berkeley. 269

[76] P. Carrasco, 1987, *Complejos Hipercruzados, Cohomologia y Extensiones*, Ph.D. thesis, Universidad de Granada. 31, 39, 40, 216

[77] P. Carrasco and A. M. Cegarra, *Group-theoretic Algebraic Models for Homotopy Types*, J. Pure Appl. Alg., 75, (1991), 195 – 235. 31, 39, 216, 251

[78] A. Cegarra and J. Remedios, *The relationship between the diagonal and the bar constructions on a bisimplicial set*, Topology and its Applications, 153, (2005), 21 – 51. 234, 548, 650

[79] A. Cegarra and J. Remedios, *The behaviour of the $\overline{W}$-construction on the homotopy theory of bisimplicial sets*, Manuscripta Math., 124, (2007), 427 – 457, ISSN 0025-2611. 650

[80] M. M. Cohen, 1973, *A Course in Simple-Homotopy Theory*, number 10 in Graduate Texts in Math., Springer-Verlag, URL http://books.google.co.uk/books?id=MFrvAAAAMAAJ. 114, 115

[81] D. Conduché, *Modules croisés généralisés de longueur 2*, J. Pure Appl. Alg., 34, (1984), 155 – 178. 215, 216, 250, 251, 518

[82] D. Conduché, *Simplicial Crossed Modules and Mapping Cones*, Georgian Math. J., 10, (2003), 623 – 636. 221, 234

[83] J.-M. Cordier, *Sur la notion de diagramme homotopiquement cohérent*, Cahiers de Top. Géom. Diff., 23, (1982), 93 – 112. 501, 503, 511, 514, 515, 708, 713

[84] J.-M. Cordier and T. Porter, *Vogt's theorem on categories of homotopy coherent diagrams*, Math. Proc. Cambridge Philos. Soc., 100, (1986), 65 – 90, ISSN 0305-0041. 503, 514, 516, 708, 719

[85] J.-M. Cordier and T. Porter, *Maps between homotopy coherent diagrams*, Top. and its Appls., 28, (1988), 255 – 275. 514, 515

[86] J.-M. Cordier and T. Porter, *Fibrant diagrams, rectifications and a construction of Loday*, J. Pure. Applied Alg., 67, (1990), 111 – 124.

[87] J.-M. Cordier and T. Porter, *Categorical Aspects of Equivariant Homotopy*, Applied Categorical Structures, 4, (1996), 195 – 212. 708

[88] J.-M. Cordier and T. Porter, *Homotopy Coherent Category Theory*, Trans. Amer. Math. Soc., 349, (1997), 1 – 54. 514, 515, 637, 708, 714

[89] J.-M. Cordier and T. Porter, 2008, *Shape Theory, categorical methods of approximation*, Dover (reprint of earlier 1989 edition). 310, 617, 682

[90] J. M. Corson, *Complexes of Groups*, Proc. London Math. Soc., 65, (1992), 199–224. 140, 150, 161, 166, 168

[91] J. M. Corson, 1995, *Groups acting on Complexes and Complexes of Groups*, in R. Charney, M. Davis and M. Shapiro, eds., *Geometric Group Theory*, 79 – 97, de Gruyter, Berlin.

[92] J. M. Corson, *Howie Diagrams and Complexes of Groups*, Comm. Algebra, 23, (1995), 5221 – 5242. 150, 161

[93] J. M. Corson and B. S. Spieler, *The qsf property and group actions via complexes of spaces*, Mathematische Zeitschrift, 228, (1998), 589 – 603. 140, 151

[94] H. S. Coxeter and W. O. Moser, 2013, *Generators and relations for discrete groups*, volume 14, Springer Science & Business Media. 52

[95] R. H. Crowell, *The derived module of a homomorphism*, Advances in Math., 5, (1971), 210 – 238. 72, 76, 78

[96] R. H. Crowell and R. H. Fox, 1977, *Introduction to Knot Theory*, number 57 in Graduate Texts, Springer. 79, 148, 429

[97] E. Curtis, *Simplicial Homotopy Theory*, Advances in Math., 6, (1971), 107 – 209. 30, 252, 268, 282, 290, 363, 605

[98] N. K. Dakin, 1977, *Kan complexes and multiple groupoid structures*, Ph.D. thesis, University of Wales, Bangor. 38

[99] R. Debremaeker, 1976, *Cohomologie met Waarden in een Gekruiste Groepenschoof op ein Situs*, Ph.D. thesis, Katholieke Universiteit te Leuven. 248

[100] R. Debremaeker, *Cohomologie à valeurs dans un faisceau de groupes croisés sur un site. I*, Acad. Roy. Belg. Bull. Cl. Sci. (5), 63, (1977), 758 – 764.

[101] R. Debremaeker, *Cohomologie à valeurs dans un faisceau de groupes croisés sur un site. II*, Acad. Roy. Belg. Bull. Cl. Sci. (5), 63, (1977), 765 – 772.

[102] R. Debremaeker, *Non abelian cohomology*, Bull. Soc. Math. Belg., 29, (1977), 57 – 72. 248

[103] R. Debremaeker, 2017, *Cohomology with values in a sheaf of crossed groups over a site*, URL https://arxiv.org/abs/1702.02128. 248

[104] P. Dedecker, *Les foncteurs Ext_Π, ${H_\Pi}^2$ et ${H_\Pi}^2$ non abéliens*, C. R. Acad. Sci. Paris, 258, (1964), 4891 – 4894. 2, 247

[105] S. Douteau, 2018, *A simplicial approach to stratified homotopy theory*, arXiv preprint arXiv:1801.04797, URL https://arxiv.org/abs/1801.04797. 675

[106] C. H. Dowker, *Homology Groups of Relations*, Annals of Maths, 56, (1952), 84 – 95. 131, 133

[107] J. Duskin, 1975, *Simplicial methods and the interpretation of "triple" cohomology*, number 163 in Mem. Amer. Math. Soc., 3, Amer. Math. Soc. 39, 90, 91, 198, 200, 215, 342, 382, 385, 509, 613, 618, 620, 626, 633, 634, 636

[108] J. Duskin, 1979, *Higher-dimensional torsors and the cohomology of topoi: the abelian theory*, in *Applications of Sheaves*, number 753 in Lecture Notes in Maths., Springer-Verlag. 203, 382

[109] J. Duskin, *An outline of a theory of higher dimensional descent*, Bull. de la Soc. Math. de Belgique, 41, (1989), 249 – 277. 342

[110] J. Duskin, *A simplicial matrix approach to higher dimensional category theory I: nerves of bicategories*, Theory and Applications of Categories, 9, (2002), 198 – 308, URL http://www.tac.mta.ca/tac/volumes/9/n10/9-10abs.html. 198, 209, 523, 527

[111] W. G. Dwyer and D. Kan, *Simplicial Localization of Categories*, J. Pure Appl. Alg., 17, (1980), 267 – 284. 503

[112] W. G. Dwyer and D. M. Kan, *Function complexes for diagrams of simplicial sets*, Proc. Kon. Ned. Akad. Wet., 86, (1983), 86. 708

[113] W. G. Dwyer and D. M. Kan, *Homotopy theory and simplicial groupoids*, Nederl. Akad. Wetensch. Indag. Math., 46, (1984), 379 – 385. 31, 249, 565

[114] W. G. Dwyer, D. M. Kan and J. H. Smith, *Homotopy Commutative Diagrams and their Realizations*, J. Pure Appl. Alg., 57, (1989), 5 – 24. 503

[115] P. Ehlers and T. Porter, *Varieties of simplicial groupoids I: Crossed complexes*, J. Pure Appl. Alg., 120, (1997), 221 – 233. 251

[116] P. Ehlers and T. Porter, *Erratum to "Varieties of simplicial groupoids I: Crossed complexes"*, J. Pure Appl. Alg., 134, (1999), 207 – 209. 251

[117] P. J. Ehlers, 1991, *Simplicial Groupoids as Models for Homotopy Type*, M.Sc. thesis, University College of North Wales, Bangor, URL http://www.maths.bangor.ac.uk/research/ftp/theses/ehlers-msc.pdf. 377, 613

[118] P. J. Ehlers, 1993, *Algebraic Homotopy in Simplicially Enriched Groupoids*, PhD thesis, University of Wales Bangor. 613

[119] P. J. Ehlers and T. Porter, *Joins for (augmented) simplicial sets*, J. Pure Appl. Alg., 145, (2000), 37 – 44. 611, 613

[120] P. J. Ehlers and T. Porter, *Ordinal subdivision and special pasting in quasicategories*, Advances in Math., 217, (2008), 489 – 518. 613

[121] S. Eilenberg and S. M. Lane, *Cohomology theory in abstract groups, II*, Annals of Math., 48, (1957), 326 – 341. 95

[122] G. J. Ellis, *Crossed Squares and Combinatorial Homotopy*, Math. Z., 214, (1993), 93 – 110. 236

[123] G. J. Ellis and R. Steiner, *Higher dimensional crossed modules and the homotopy groups of* $(n+1)$*-ads*, J. Pure Appl. Alg., 46, (1987), 117–136. 230, 231, 234

[124] A. Elmendorf, *Systems of Fixed Point Sets*, Trans. Amer. Math. Soc., 277, (1983), 275 – 284. 714

[125] T. Everaert, M. Gran and T. Van der Linden, *Higher Hopf formulae for homology via Galois Theory*, Advances in Math., 217, (2008), 2231 – 2267. 229

[126] B. Fantechi, L. Göttsche, L. Illusie, S. L. Kleiman, N. Nitsure and A. Vistoli, 2005, *Fundamental algebraic geometry: Grothendieck's FGA explained*, number 123 in Mathematical Surveys and Monographs, American Mathematical Soc. 741

[127] P. H. H. Fantham and E. J. Moore, *Groupoid enriched categories and homotopy theory*, Canad. J. Math., 35, (1983), 385 – 416. 504

[128] Y. Felix, S. Halperin and J.-C. Thomas, 2001, *Rational Homotopy Theory*, volume 205 of *Graduate Texts in Mathematics*, Springer. 368

[129] T. M. Fiore, 2006, *Pseudo Limits, Biadjoints, and Pseudo Algebras: Categorical Foundations of Conformal Field Theory*, volume 182, Amer. Math. Soc., URL http://www.citebase.org/abstract?id=oai:arXiv.org:math/0408298. 432

[130] M. E. Forrester-Barker, 2003, *Representations of crossed modules and cat^1-groups*, Ph.D. thesis, University of Wales Bangor, URL www.math.bangor.ac.uk/research/preprints/04/algtop04.html. 356, 357

[131] R. H. Fox, *Free differential Calculus, I: Derivation in the free group ring*, Ann. of Maths, 57, (1953), 547 – 560. 79

[132] P. Gabriel and M. Zisman, 1967, *Calculus of Fractions and Homotopy Theory*, number 35 in Ergebnisse der Math. und ihrer Grenzgebeite, Springer. 182, 184, 250, 375, 397, 504

[133] N. D. Gilbert and T. Porter, 1996, *Knots and Surfaces*, Oxford University Press. 52, 54, 113, 429

[134] J. Giraud, 1964, *Méthode de la descente*, volume 2 of *Mémoires de la Société Mathématique de France*, Société Mathématique de France. 400

[135] J. Giraud, 1971, *Cohomologie non-abélienne*, number 179 in Grundlehren der mathematischen Wissenschaften, Springer-Verlag. 389, 400

[136] P. Glenn, *Realization of cohomology classes in arbitrary exact categories*, J. Pure. Appl. Alg., 25, (1982), 33 – 105. 203, 382

[137] P. G. Goerss and J. F. Jardine, 1999, *Simplicial Homotopy Theory*, number 174 in Progress in Mathematics, Birkhauser. 209, 282, 377, 380, 382, 565

[138] M. Gran and V. Rossi, *Galois Theory and Double Central Extensions*, Homology, Homotopy and Applications, 6, (2004), 283 – 298. 229

[139] A. Grothendieck, 1958-1960, *Technique de descente et théorèmes d'existence en géométrie algébrique. I. Généralités. Descente par morphismes fidèlement plats*, in *Séminaire Bourbaki, Vol. 5*, Exp. No. 195, 299 – 327, Soc. Math. France (Numdam), Paris. 398, 400

[140] A. Grothendieck, 1983, *Pursuing Stacks*, manuscript, 600 + pages. 275, 364, 503

[141] A. Grothendieck, 2003, *Revêtements étales et groupe fondamental (SGA1)*, Documents Mathématiques (Paris), 3, Société Mathématique de France, Paris, séminaire de géométrie algébrique du Bois Marie 1960–61. Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; (50 #7129)]. 184, 280, 389, 400

[142] A. Grothendieck, ∼ 1990, *Les Dérivateurs*, editors: M. Künzer, J. Malgoire and G. Maltsiniotis. 707, 708

[143] D. Guin-Walery and J.-L. Loday, 1981, *Obstructions à l'excision en K-théorie algèbrique*, in *Evanston Conference on Algebraic K-theory, 1980*, volume 854 of *Lecture Notes in Maths.*, 179 – 216, Springer. 2, 212, 225

[144] N. Gurski, *Nerves of bicategories as stratified simplicial sets*, Journal of Pure and Applied Algebra, 213, (2009), 927–946. 523

[145] A. Haefliger, 1991, *Complexes of Groups and Orbihedra*, in *Group Theory from a Geometric viewpoint*, 504 – 540, ICTP, Trieste, 26 March- 6 April 1990, World Scientific. 140, 150, 151, 152, 161, 162, 163

[146] A. Haefliger, *Extensions of Complexes of Groups*, Annales Inst.Fourier, 42, (1992), 275 – 311. 140, 150, 161, 163

[147] A. Hatcher, 2002, *Algebraic topology*, Cambridge University Press. 194, 205, 206, 209

[148] A. Heller, *Homotopy in Functor Categories*, Trans. Amer. Math. Soc., 272,, (1982), 185 – 202. 707, 708

[149] A. Heller, 1988, *Homotopy Theories*, number 383 in Memoirs Amer. Math. Soc., Amer. Math. Soc. 707

[150] P. J. Higgins, *The fundamental groupoid of a graph of groups*, Journal of the London Mathematical Society, 2, (1976), 145–149. 167

[151] P. J. Higgins, 2005, *Categories and groupoids*, number 7 in Reprints in Theory and Applications of Categories, TAC, reprint of the 1971 original [*Notes on categories and groupoids*, Van Nostrand Reinhold, London] with a new preface by the author. 18, 50, 54, 178, 179, 458

[152] P. Hilton and U. Stammbach, 1971, *A course in Homological Algebra*, Graduate Texts in Mathematics, Springer Verlag. 104

[153] P. J. Hilton, 1953, *An Introduction to Homotopy Theory*, Cambridge University Press. 44

[154] V. Hinich, 2007, *Homotopy Coherent Nerve in Deformation theory*, URL `http://arxiv.org/abs/0704.2503`. 511

[155] N. J. Hitchin, 2001, *Lectures on special Lagrangian submanifolds*, in C. Vafa and S.-T. Yau, eds., *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds*, volume 23 of *Studies in Advanced Mathematics*, 151 – 182, AMS/International Press, Providence. 464

[156] N. J. Hitchin, *What is ... a gerbe?*, Notices Amer. Math. Soc., 50, (2003), 218–219. 464

[157] S. Holz, 1985, *Endliche Identifizier zwishen Relationen*, Ph.D. thesis, Univeristät Bielefeld. 150, 430

[158] M. Hovey, 1998, *Model Categories*, volume 63 of *Mathematical Surveys and Monographs*, Amer. Math. Soc. 375, 503, 575

[159] J. Huebschmann, *Crossed n-fold extensions of groups and cohomology*, Comm. Math. Helv., 55, (1980), 302 – 313. 69, 94

[160] D. Husemoller, 1975, *Fibre bundles*, volume 20 of *Graduate Texts in Mathematics*, Springer-Verlag, New York-Heidelberg, second edition. 279, 280, 281, 466

[161] K. Igusa, *The generalised Grassmann invariant*, preprint, (unpublished). 110

[162] K. Igusa, 1979, *The $Wh_3(\pi)$ obstruction for pseudo-isotopy*, Ph.D. thesis, Princeton University. 110

[163] L. Illusie, 1971, *Complexe cotangent et déformations I*, volume 239 of *Lecture Notes in Maths*, Springer-Verlag. 213, 214, 383, 385, 613, 678

[164] L. Illusie, 1972, *Complexe cotangent et déformations II*, volume 283 of *Lecture Notes in Maths*, Springer-Verlag. 213, 383, 385, 613, 678

[165] G. Janelidze, *What is a double central extension ? (the question was asked by Ronald Brown)*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, 32, (1991), 191 – 201. 229

[166] J. F. Jardine, 1989, *Universal Hasse-Witt classes.*, in *Algebraic K-theory and algebraic number theory*, volume 83 of *Contemporary Mathematics*, 83 – 100, American Mathematical Society, AMS. 383

[167] D. L. Johnson, 1980, *Topics in the theory of group presentations*, number 42 in London Math. Soc Lecture NotesMS Lecture Notes, Cambridge University Press. 431

[168] D. L. Johnson, 1997, *Presentations of groups*, volume 15 of *London Mathematical Society Student Texts*, Cambridge University Press, Cambridge, second edition. 431

[169] A. Joyal, *Quasi-categories and Kan complexes*, J. Pure Applied Alg., 175, (2002), 207 – 222. 507, 509, 510, 514

[170] B. Jurčo, 2005, *Crossed Module Bundle Gerbes; Classification, String Group and Differential Geometry*. 211, 346

[171] K. H. Kamps and T. Porter, 1997, *Abstract Homotopy and Simple Homotopy Theory*, World Scientific Publishing Co. Inc., River Edge, NJ. 30, 38, 210, 249, 338, 341, 365, 369, 407, 497, 503, 506, 511

[172] D. Kan, *Abstract Homotopy IV*, Proc. Nat. Acad. Sci. USA, 42, (1956), 542 – 544. 565

[173] D. Kan, *On homotopy theory and c.s.s groups*, Ann. of Math., 68, (1958), 38 – 53. 30

[174] M. Kapranov and M. Saito, 1999, *Hidden Stasheff polytopes in algebraic K-theory and in the space of Morse functions*, in *Higher homotopy structure in topology and mathematical physics (Poughkeepsie, N.Y. 1996)*, volume 227 of *Contemporary Mathematics*, 191–225, AMS. 109, 111, 112, 144

[175] M. M. Kapranov and V. A. Voevodsky, 1991, *2-categories and Zamolodchikov tetrahedra equations*, in *Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.* 356

[176] M. Karoubi, 1978, *K-theory*, volume 226 of *Grundlehren der math. Wiss. N°*, Springer-Verlag Berlin. 281

[177] G. M. Kelly, 2005, *Basic Concepts of Enriched Category Theory*, 10, Reprints in Theory and Applications of Categories. 589, 590, 592, 643, 644, 645, 711

[178] M. Kervaire, 1972, *Multiplicateurs de Schur et K-théorie*, in A. Haefliger and R. Narasimhan, eds., *Essays in Topology and Related Topics memoires dedies à Georges de Rham*, 212–225, Springer-Verlag. 145

[179] F. Keune, 1972, *Homotopical Algebra and Algebraic K-theory*, Ph.D. thesis, Universiteit van Amsterdam. 91, 93

[180] N. H. Kuiper and R. K. Lashof, *Microbundles and Bundles I. Elementary Theory*, Invent. Math., 1, (1966), 1 – 17. 269

[181] N. H. Kuiper and R. K. Lashof, *Microbundles and Bundles II. Semisimplicial Theory*, Invent. Math., 1, (1966), 243 – 259. 269

[182] S. Lack, 2010, *A 2-categories companion*, in *Towards higher categories*, 105–191, Springer. 412, 446

[183] S. Lack and S. Paoli, *2–nerves for bicategories*, K-Theory, 38, (2008), 153 – 175. 524, 527

[184] T. Leinster, 1998, *Basic Bicategories*, URL https://arxiv.org/pdf/math/9810017.pdf. 522

[185] R. D. Leitch, *The Homotopy Commutative Cube*, J. London Math. Soc., (2) 9, (1974), 23 – 29. 503

[186] J.-L. Loday, *Spaces with finitely many homotopy groups*, J. Pure Appl. Alg., 24, (1982), 179 – 202. 2, 65, 67, 225, 226

[187] J.-L. Loday, 2000, *Homotopical Syzygies*, in *Une dégustation topologique: Homotopy theory in the Swiss Alps*, volume 265 of *Contemporary Mathematics*, 99 – 127, AMS. 48, 55, 83, 109, 110, 518

[188] F. Loregian, 2015, *This is the (co) end, my only (co) friend*, URL https://arxiv.org/abs/1501.02503. 691

[189] E. Lupercio and B. Uribe, *An introduction to gerbes on orbifolds*, Annales mathématiques Blaise Pascal, 11, (2004), 155 – 180. 300, 481

[190] J. Lurie, 2009, *Higher Topos Theory*, number 170 in Annals of mathematics studies, Princeton University Press, URL http://arxiv.org/abs/math.CT/0608040. 511, 515

[191] S. Mac Lane, 1967, *Homology*, number 114 in Grundlehren, Springer. 57, 89, 90, 95, 104, 361, 609, 660, 662

[192] S. Mac Lane, 1978, *Categories for the Working Mathematician*, number 5 in Graduate Texts, Springer. 90, 199, 306, 310, 311, 617, 628, 629, 640, 708, 711

[193] S. Mac Lane, *Historical Note*, J. Alg., 60, (1979), 319 – 320, appendix to D. F. Holt, An interpretation of the cohomology groups, $H^n(G, M)$, J. Alg. 60 (1979) 307-318. 57

[194] S. Mac Lane and I. Moerdijk, 1992, *Sheaves in Geometry and Logic*, Springer-Verlag, New-York. 443

[195] S. Mac Lane and J. H. C. Whitehead, *On the 3-type of a complex*, Proc. Nat. Acad. Sci. U.S.A., 36, (1950), 41 – 48. 46, 99, 197

[196] G. Maltsiniotis, 2001, *Introduction à la théorie des dérivateurs, d'après Grothendieck*, (available from http://people.math.jussieu.fr/∼maltsin/textes.html. 708

[197] W. S. Massey, 1967, *Algebraic Topology, an introduction*, Harcourt, Brace & World. 148

[198] J. P. May, 1967, *Simplicial objects in algebraic topology*, number 11 in Math. Studies, van Nostrand, Princeton. 30, 35, 37, 38, 260, 268, 282, 287, 290, 660

[199] J. P. May, 1975, *Classifying spaces and fibrations*, volume 155 of *Memoirs*, Amer. Math. Soc. 714

[200] J. P. May, 1999, *A Concise Course in Algebraic Topology*, Chicago Lectures in Mathematics, University of Chicago Press. 366, 375

[201] J.-P. Meyer, *Bar and cobar constructions*, J. Pure Applied Alg. J. Pure Applied Alg., 33, (1984), 163 – 207. 708, 714

[202] J. S. Milne, 2003, *Gerbes and abelian motives*, URL `http://www.arXiv.org:math/0301304`. 248

[203] J. W. Milnor, 1956, *On the construction FK*, mimeographed note from Princeton University, in Adams, [3]. 363

[204] J. W. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc., 72, (1966), 358 – 426. 114, 115

[205] J. W. Milnor, 1971, *Introduction to Algebraic K-theory*, Annals of Math. Studies, Princeton University Press. 43

[206] I. Moerdijk, 2002, *Introduction to the Language of Stacks and Gerbes*, URL `http://arxiv.org/abs/math/0212266`. 292, 479, 480

[207] I. Moerdijk and D. A. Pronk, *Orbifolds, Sheaves and Groupoids*, K-theory, 12, (1997), 3–21. 189

[208] I. Moerdijk and J. Svensson, *Algebraic classification of equivariant homotopy 2-types, part I.*, J. Pure Applied Alg., 89, (1993), 187 – 216. 556, 573, 574

[209] E. J. Moore, 2001, *Graphs of groups: word computations and free crossed resolutions.*, Ph.D. thesis, University of Wales Bangor. 167, 170, 430

[210] J. C. Moore, 1956, *Seminar in Algebraic Homotopy*, Princeton. 91

[211] M. K. Murray, *Bundle gerbes*, J. London Math. Soc., 54(2), (1996), 403 – 416. 469, 473

[212] M. K. Murray and D. Stevenson, *Bundle Gerbes: Stable Isomorphism and Local Theory*, J. London Math., 62, (2000), 925 – 937. 477

[213] A. Mutlu and T. Porter, *Applications of Peiffer pairings in the Moore complex of a simplicial group*, Theory and Applications of Categories, 4, (1998), 148 – 173. 216

[214] A. Mutlu and T. Porter, *Freeness Conditions for 2-Crossed Modules and Complexes*, Theory and Applications of Categories, 4, (1998), 174 – 194.

[215] A. Mutlu and T. Porter, *Free crossed resolutions from simplicial resolutions with given CW-basis*, Cahiers Top. Géom. Diff. catégoriques, 50, (1999), 261 – 283. 91

[216] A. Mutlu and T. Porter, *Freeness Conditions for Crossed Squares and Squared Complexes*, K-Theory, 20, (2000), 345 – 368.

[217] A. Mutlu and T. Porter, *Iterated Peiffer pairings in the Moore complex of a simplicial group*, Applied Categorical Structures, 9, (2001), 111 – 130. 216

[218] G. Nan Tie, *A Dold-Kan theorem for crossed complexes*, J. Pure Appl. Alg., 56, (1989.), 177 – 194. 31, 39, 256, 382

[219] G. Nan Tie, *Iterated* $\overline{W}$ *and* T*-groupoids*, J. Pure Appl. Alg., 56, (1989), 195 – 209. 31, 39, 256, 382

[220] A. Neeman, 2001, *Triangulated Categories*, Princeton University Press Press. 375

[221] nLab team, *nLab*, URL http://ncatlab.org/nlab/show/HomePage. 3, 198, 199, 209, 210, 307, 311, 505, 508, 536, 538, 695

[222] B. Noohi, 2005, *On weak maps between 2-groups*, URL https://arxiv.org/abs/math/0506313. 355, 556, 575, 576, 580, 581, 588

[223] B. Noohi, *Notes on 2-groupoids, 2-groups and crossed-modules*, Homotopy, Homology, and Applications, 9, (2007), 75 – 106, URL www.intlpress.com/HHA/v9/n1/contv9n1.pdf. 355, 556, 563

[224] B. Noohi, *Notes on 2-groupoids, 2-groups and crossed-modules (corrected)*, Homotopy, Homology, and Applications, 1, (2007), 75, URL http://arxiv.org/abs/math.CT/0512106v3. 556, 561, 563, 573, 576, 584, 588

[225] S. Paoli, *Weakly globular cat*^{n}*-groups and Tamsamani's model*, Advances in Mathematics, 222, (2009), 621–727. 236

[226] T. Porter, $>$ 2019, *Profinite Algebraic Homotopy*, (uncertain as yet as to publisher). 74, 84, 184, 223, 230, 234, 236, 237, 241, 254

[227] T. Porter, n*-types of simplicial groups and crossed* n*-cubes*, Topology, 32, (1993), 5 – 24. 234, 235

[228] T. Porter, *Abstract Homotopy Theory, the interaction of category theory and homotopy theory*, Cubo Matematica Educacional, 5, (2003), 115 – 165. 511

[229] T. Porter, 2004, $\mathcal{S}$*-categories,* $\mathcal{S}$*-groupoids, Segal categories and quasicategories*, URL http://arxiv.org/abs/math/0401274. 249

[230] V. Prasolov, 2007, *Elements of homology theory*, volume 81 of *Graduate Studies in Mathematics*, Amer Mathematical Society. 161

[231] D. G. Quillen, 1967, *Homotopical algebra*, number 53 in Lecture Notes in Maths., Springer-Verlag. 89, 575

[232] D. G. Quillen, 1970, *On the (co-)homology of commutative rings*, in *Proc. Symp. on Categorical Algebra*, 65 – 87, American Math. Soc. 89, 575

[233] D. G. Quillen, 1973, *Higher algebraic K-theory, I*, in *Higher K-theories.*, volume 341 of *Lecture Notes in Math.*, 85 – 147. 135

[234] E. Riehl, 2014, *Categorical homotopy theory*, number 24 in New Mathematical Monographs, Cambridge University Press. 593

[235] E. Riehl, 2016, *Complicial sets, an overture*, arXiv preprint arXiv:1610.06801, URL https://arxiv.org/abs/1610.06801. 674

[236] J. E. Roberts, 1977, *Mathematical aspects of local cohomology*, in *Algebres d'opérateurs et leurs applications en physique mathématique.(Proc. Colloq. Marseille, 1977)*, 321–332. 674

[237] J. E. Roberts, 1978, *Complicial sets*, Handwritten manuscript. 674

[238] D. Rodelo and T. van der Linden, 2008, *The Third Cohomology group classifies double central extensions*, Pré-Publicacões do Departamento de Matematica 08-55, Universidade de Coimbra. 229

[239] I. Satake, *On a generalisation of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A., 42, (1956), 359–363. 189

[240] I. Satake, *The Gauss–Bonnet theorem for V-manifolds,*, J. Math. Soc. Japan, 9, (1957), 464–492. 189

[241] C. Schommer-Pries, *Central extensions of smooth 2-groups and a finite dimensional string 2-group*, Available as arXiv 0911.2483v3. 211

[242] P. Scott and T. Wall, 1979, *Topological methods in group theory*, in *Homological group theory: proceedings of a symposium, held at Durham in September 1977, on" Homological and combinatorial techniques in group theory"*, volume 36 of *London Mathematical Society Lecture Note Series*, 137, Cambridge Univ Press. 168, 170

[243] G. Segal, *Classifying spaces and spectral sequences*, Publ. Math. Inst. des Haut. Études Scient., 34, (1968), 105 – 112. 508, 556

[244] G. Segal, *Categories and Cohomology Theories*, Topology, 13, (1974), 293 – 312. 503, 556, 708

[245] J.-P. Serre, 1977, *Arbres, amalgames, SL_2*, volume 46 of *Astérisque*, Société mathématique de France. 140, 154, 161, 166

[246] J.-P. Serre, 2003, *Trees*, Springer Monographs in Mathematics, Springer-Verlag Berlin. 52, 140, 154, 161, 166

[247] M. A. Shulman, 2006, *Homotopy limits and colimits and enriched homotopy theory*, arXiv:math.CT/0610194. 607, 677, 708

[248] Y. Song, *The equivalence of two algebraic K-theories*, Kangweon-Kyungki Math. Jour., 5, (1997), 107 – 112. 143

[249] C. Soulé, *Groupes opérants sur un complexe simplicial avec domain fondamental*, C.R. Acad. Sci. Paris, Sér A 276, (1973), 607–609. 123

[250] E. H. Spanier, 1966, *Algebraic Topology*, McGraw Hill. 141, 301, 464, 465

[251] Stacks Project Authors, 2017, *Stacks Project*, http://stacks.math.columbia.edu. 304, 307, 310

[252] N. Steenrod, 1974, *The topology of fibre bundles*, Princeton Univ. Press, Princeton, ninth edition. 279

[253] R. Street, *The algebra of oriented simplexes*, J. Pure Appl. Algebra, 49, (1987), 283 – 335. 39, 674

[254] T. Streicher, 2014, *Fibred Categories à la Jean Bénabou*, http://www.mathematik.tu-darmstadt.de/ streicher/. 389, 401, 405, 406, 413, 417, 422

[255] A. A. Suslin and M. Wodzicki, *Excision in algebraic K-theory*, The Annals of Mathematics, 136, (1992), 51 – 122. 128, 130, 143, 155

[256] G. Tabuada, 2007, *Differential graded versus Simplicial categories*, preprint. 598

[257] Z. Tamsamani, *Sur des notions de n-catégorie et n-groupoïde non-strictes via des ensembles multi-simpliciaux*, K. Theory, 16, (1999), 51 – 99. 236

[258] R. W. Thomason, *Homotopy colimits in the category of small categories*, Math. Proc. Cambridge Philos. Soc., 85, (1979), 91 – 109. 396, 423

[259] W. P. Thurston, 1979, *The geometry and topology of three-manifolds*, number 35 in Princeton Mathematical Series, Princeton University Press. 189

[260] H. Tietze, *Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten*, Monatshefte für Mathematik, 19, (1908), 1–118. 113

[261] J. Tits, 1974, *Buildings of spherical type and finite BN-pairs*, number 386 in Lecture Notes in Mathematics, Springer. 123

[262] J. Tits, 1981, *A local approach to buildings*, in *The geometric vein. The Coxeter Festshrift*, Springer-Verlag Berlin.

[263] J. Tits, *Buildings and group amalgamations*, Proceedings of groups: St. Andrew's 1985, 121, (1986), 110. 123

[264] A. P. Tonks, 1993, *Theory and applications of crossed complexes*, Ph.D. thesis, University of Wales, Bangor. 107, 497, 601, 662, 664, 670

[265] A. P. Tonks, *On the Eilenberg–Zilber theorem for crossed complexes*, J. Pure Appl. Alg., 179, (2003), 199 – 220. 601, 664

[266] D. R. Verity, 2005, *Complicial Sets*, URL http://arxiv.org/abs/math.CT/0410412. 39, 514, 674, 675, 676

[267] D. R. Verity, 2006, *Weak complicial sets I: basic homotopy theory*, available from : arXiv.org:math/0604414. 514

[268] D. R. Verity, 2006, *Weak complicial sets. III. Enriched and internal quasi-category theory*, (in preparation).

[269] D. R. Verity, 2007, *Weak complicial sets. II. Nerves of complicial Gray-categories*, in *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, 441 – 467, Amer. Math. Soc., Providence, RI. 39, 514, 674

[270] A. Vistoli, 2004, *Grothendieck topologies, fibered categories and descent theory, (version 4)*, in [126], URL http://arxiv.org/abs/math.AG/0412512. 389, 413, 417

[271] R. Vogt, *Homotopy limits and colimits*, Math. Z., 134, (1973), 11 – 52. 503, 514, 677, 678, 713

[272] I. Volodin, *Algebraic K-theory as extraordinary homology theory on the category of associative rings with unity*, Izv. Akad. Nauk. SSSR, 35, (Translation: Math. USSR Izvestija Vol. 5 (1971) No. 4, 859-887). 123, 127, 145, 147

[273] J. Wagoner, 1973, *Buildings, stratifications, and higher K-theory*, in H. Bass, ed., *Higher K-Theories*, volume 341 of *Lecture Notes in Mathematics*, 148–165, Springer Berlin / Heidelberg, URL http://dx.doi.org/10.1007/BFb0067054, 10.1007/BFb0067054. 127

[274] C. A. Weibel, 1995, *An Introduction to Homological Algebra*, number 38 in Cambridge Studies in Advanced Mathematics, Cambridge University Press. 104

[275] C. A. Weibel, 1999, *The development of algebraic K-theory before 1980*, in *Algebra, K-theory, groups, and education (New York, 1997)*, volume 243 of *Contemp. Math.*, 211 – 238, Amer. Math. Soc., Providence, RI. 123

[276] J. H. C. Whitehead, *Combinatorial homotopy I*, Bull. Amer. Math. Soc, 55, (1949), 213 – 245. 113

[277] J. H. C. Whitehead, *Combinatorial Homotopy II*, Bull. Amer. Math. Soc., 55, (1949), 453 – 496. 2, 67, 113

[278] J. H. C. Whitehead, 1950, *Algebraic Homotopy Theory*, in *Proc. Int. Cong. of Mathematics, Harvard*, volume 2, 354 – 357. 113

[279] J. H. C. Whitehead, *Simple Homotopy Type*, Amer. J. Math., 72, (1950), 1–57. 114

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