Ulrich Bunke

Smooth K-theory

1. Cycles, relations
2. Integration
3. ch, c!, ψ Adams (? Bruce)
4. Riemann Roch

1. The model: cycles

Want to construct

$$(\hat{K}, R, I; a, \int)$$

as in Thomas' talk.

Comment = fiber bundle

smooth, proper submersion — fiberwise metric — connection — Clifford bundle

Cycle: $(\mathcal{E}, \rho)$

$$\mathcal{E}: \text{a } \underline{\text{geometric family}} = \left(\begin{array}{c} E \\ \downarrow \pi \\ B \end{array}, g^{T\pi}, T^h\pi, W \right)$$

example: From $\underline{W} = \left(\begin{array}{c} W \\ \downarrow \\ B \end{array} \mathbb{C}\text{-vector bundle}, h^W, \nabla^W \right)$

metric, connection.

can make a geometric family

$$\mathbb{W} = \left(\begin{array}{c} B \\ \downarrow \pi \\ B \end{array}, 0, TB, w \right)$$

0-dim fibers!

What is ρ?

$$\rho \in \Omega(B, K) = C^\infty(B, \Lambda^* T^* B \otimes K^*)$$

If you have a geometric family, have <u>local index form</u>

$$\Omega(\mathcal{E}) = \int \hat{A}\, ch$$

In the $\mathbb{W}$ example,

$$\Omega(\mathbb{W}) = ch(\nabla^W)$$

usual thing from local index theory.

- disjoint union $\sqcup$ is addition.

b) Relations :

$H(\mathcal{E})$ .... family of Hilbert spaces

$$H(\mathcal{E})_b = L^2(E_b, W_{E_b})$$

$D(\mathcal{E})$ family of Dirac operators

$Q(\mathcal{E})$ family of smoothing operators

s.t. $D(\mathcal{E}) + Q(\mathcal{E})$ invertible

$\rightsquigarrow$ can define η-form

$$\eta(\mathcal{E}_t) \in \Omega(B, K)$$

with property

$$d\eta = \Omega(\mathcal{E}),$$

$$[\Omega(\mathcal{E})] = ch(\text{index } D(\mathcal{E}))$$

We say

$$(\mathcal{E}, \rho) \sim 0$$

if there exists $\mathcal{E}_t$ such that

$$\rho = \eta(\mathcal{E}_t).$$

A. Henriques: I'm lost.

U. Bunke: Take g. family $(\mathcal{E}, 0) + (\mathcal{E}^{op}, 0) \sim 0$

The claim is that this

$$= (\mathcal{E} \sqcup \mathcal{E}^{op}, 0)$$

Take $\ker(D(\mathcal{E})) \oplus \ker(D(\mathcal{E}^{op}))$

Lemma: $\eta((\mathcal{E} \sqcup \mathcal{E}^{op})_t) = 0.$

$$R(\mathcal{E}, \rho) = \Omega(\mathcal{E}) - d\rho.$$

$$I(\mathcal{E}, \rho) = \operatorname{index}(D(\mathcal{E})).$$

$$a(\omega) = (\phi, -\omega).$$

What is η-form?

$$\eta(\mathcal{E}_t) = \int \mathrm{Tr}\, \frac{\partial}{\partial t} A_t(\mathcal{E}_t)\, e^{-A_t^2(\mathcal{E}_t)}$$

If we have

$$\underline{W} = (W, h^W, \nabla^W)$$

$$\rightsquigarrow [\underline{W}] = (W, 0).$$

ooo〉

<u>Interpretation (sorry... Integration)</u>

Take vertical bundle. If it has $Spin^{\mathbb{C}}$ structure, it induces ...

$$p : A \longrightarrow B \quad \text{proper submersion}$$

$$T^v p \ \ spin^c \rightsquigarrow K\text{-orientation.}$$

Need smooth refinement of orientation!

~~For topological orientations~~

Wish list

fibers are torsors for $\Omega^{-1}(A, k)/_{ind}$

$$
\begin{array}{ccc}
\hat{\vartheta} & \left\{ \begin{array}{c}\text{smooth}\\ \text{orientation}\end{array} \right\} & \circlearrowleft \\
\downarrow & \downarrow & \\
\vartheta & \{ \text{top. orientations} \} &
\end{array}
$$

Want also composition

$$
\begin{array}{c}
A \\
p \downarrow \\
B \\
q \downarrow \\
C
\end{array}
\quad \rightarrow \quad \text{compose + pull-back.}
$$

André: it is a refinement of the coh. theory K(smooth orientations)?

Ulrich: No, I couldn't get that.

$\hat{\mathcal{O}}$ ~~given~~ represented by (A, α) $\leftarrow$ $\alpha \in \Omega^{-1}(A, k) \big/ \mathrm{3nd}.$

where $\cancel{A}$ $\uparrow$ geometric family based on p with Dirac bundle the spin^c Dirac bundle $W = (T^c p)$

Relation: $(A, \alpha) \sim (A', \alpha')$

if $\alpha - \alpha' = \cancel{\hat{A}}$ transgression of $\hat{A}^c$
$= \hat{A} e^{-c_{12}}$.

"just the transgression".

This finishes the orientation business.

$$R(\hat{\sigma}) = \hat{A}^c(\hat{\nabla}) - d\alpha \in \Omega^0(A, k).$$

$$\hat{p}_!^{\hat{o}} : K^*(A) \longrightarrow K^{*-m}(B)$$

$$m = \dim T^V p$$

$$\hat{p}_!^{\hat{o}}(\mathcal{E}, \rho) = \left(p_! \mathcal{E}, \int_{A/B} \hat{A}^c(\tilde{\nabla}) \wedge \rho + \int \alpha \wedge R(\mathcal{E}, \rho) \right)$$

actually only true in the adiabatic limit where you scale down fibers.

> **Theorem (Burke, Schick)**
>
> This is:
>
> - well-defined
> - functorial
> - compatible with products

This is the natural home of invariants.

Eg Assume $T^V p$ is stably framed.

Prop p has a canonical smooth orientation.

"add counterterm"

$$\hat{\mathcal{O}} = \left[\left(A, \tilde{\hat{A}}^c(\tilde{\nabla}, \tilde{\nabla}^{\mathcal{H}}\right) \right]$$

Can look at

$$e(\pi) = \hat{p}_! \hat{\mathcal{O}}(1) \in \hat{K}^{-m}_{flat}(B)$$

$$\cong K\,\mathbb{R}/\mathbb{Z}^{m-1}(B)$$

If B is a point, m odd,

$$e(\pi) = \text{Adams } e\text{-invariant} \in \mathbb{R}/\mathbb{Z}$$

(uses APS interpretation)

Because of nice bordism formula, easy to calculate in many cases.

<u>Thm</u> Have natural lifts

$$\hat{c}_i : \hat{K} \longrightarrow \widehat{H\mathbb{Z}}^{2i}$$

$$\hat{c}_i^{\,odd} : \hat{K}^1 \longrightarrow \widehat{H\mathbb{Z}}^{2i-1}$$

$$1 + \hat{c}_1^{\,s} =: c$$

preserves group structure.

$$\int \hat{c}_i = \hat{c}_i^{\,odd} \int$$

Also, in older Bunke – Schick paper, have

$$\widehat{ch} : \hat{K} \longrightarrow \widehat{H\mathbb{Q}}.$$

More recently, have <u>Adams operations</u>

$$\hat{\psi}^k : \hat{K}\left[\tfrac{1}{k}\right] \longrightarrow \hat{K}\left[\tfrac{1}{k}\right]$$

only on compact

$$\hat{\psi}^k \circ \hat{\psi}^l = \hat{\psi}^{kl}.$$

Andre: Okay, it ordinary K...

Ulrich: It's Istanbul! (?)

Here, this is $\mathbb{Z}$-graded.

Riemann-Roch

$$A \xrightarrow[\hat{\mathcal{O}}]{p} B$$

Classical

$$\begin{tikzcd} K(A) \arrow[r, "ch"] \arrow[d, "p_!"'] & H\mathbb{Q}(A) \arrow[d, "\int \hat{A}^{c}(T_v^{\vee}) \cup \cdots"] \\ K(B) \arrow[r, "ch"'] & H\mathbb{Q}(B) \end{tikzcd}$$

Now we have extensions. Put hats over everything.

exists $\hat{H\mathbb{Q}}(A)$. Need $\hat{\hat{A}}$.

Thm (Riemann-Roch) [B-S]. This diagram commutes.

One proves such a thing by showing that the difference couldn't exist!

How about Adams operation?

$$\begin{tikzcd} \hat{K}[A][\frac{1}{k}] \arrow[r, "\psi^k"] \arrow[d, "\pi_! \,0"'] & \hat{K}(A)[\frac{1}{k}] \arrow[d, "\pi_!(\hat{\rho}(\widehat{T^v_\pi}) \cup \cdots)"] \\ K(B)[\frac{1}{k}] \arrow[r, "\psi^k"'] & \hat{K}(B)[\frac{1}{k}] \end{tikzcd}$$

Adams compatible correction.

$$\boxed{\text{Some equation concerning Freed-Melros , ?}}$$

One consequence:

<u>Prop</u> If $\hat{O}$ comes from a stable framing, then

$$\hat{\rho}(\hat{O}) = 1$$

<u>Cor</u> $\hat{\psi}^k e(\rho) = e(\rho)$

in special case: $k^L(k^r - 1)$, $(m = 2r-1)$ $\cdots \to$