

Ulrich Bunke

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Smooth K-theory

1. Cycles, relations

2. Integration

3. ch , $c!$, ψ Adams (? Bruce)

4. Riemann Roch

1. The model : cycles

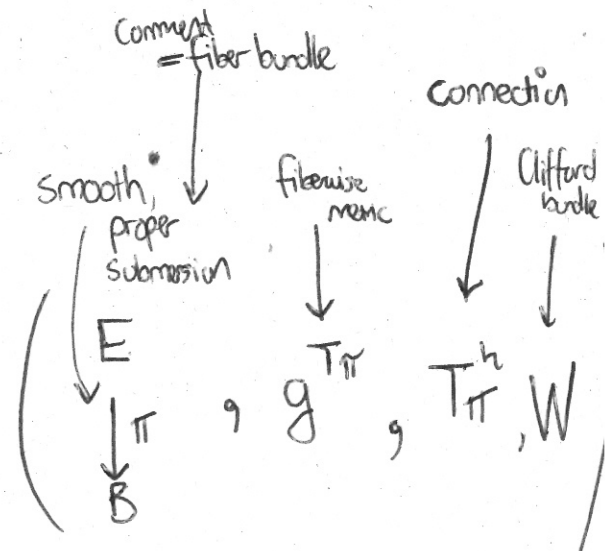
Want to construct

$$(K, R, I, a, S)$$

as in Thomas' talk.

Cycle : (E, p)

E : a geometric family =



example: From $\underline{W} = \left(\begin{array}{c} W \\ \downarrow \\ B \end{array} \right) \begin{array}{l} \text{E } \mathbb{C}\text{-vector} \\ \text{bundle} \end{array}, \quad \left(h^W, \nabla^W \right)$
metric, connectn.

can make a geometric family

$$W = \left(\begin{array}{c} B \\ \downarrow \rho \\ B \end{array}, 0, TB, W \right)$$

0-dim fibers!

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What is ρ ?

$$\rho \in \Omega(B, K) = C^\infty(B, \Lambda^* T^* B \otimes K^*)$$

If you have a geometric family,
have local index form

$$\Omega_r(E) = \int \hat{A} \text{ch}$$

In the W example,

$$\Omega_r(W) = \text{ch}(\nabla^W)$$

usual thing from local index theory.

• disjoint union \sqcup is addition.

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b) Relations :

$H(\mathcal{E})$... family of Hilbert spaces

$$H(\mathcal{E})_b = L^2(E_b, W_{E_b'})$$

$D(\mathcal{E})$ family of Dirac operators

$Q(\mathcal{E})$ family of smoothing operators

s.t. $D(\mathcal{E}) + Q(\mathcal{E})$ invertible

\rightsquigarrow can define η -form

$$\eta(\mathcal{E}_t) \in \Omega(B, K)$$

with property

$$d\eta = \Omega(\mathcal{E}),$$

$$[\Omega(\mathcal{E})] = \text{ch}(\text{index } D(\mathcal{E}))$$

We say

$$(\mathcal{E}, \rho) \sim 0$$

if there exists \mathcal{E}_t such that

$$\rho = \eta(\mathcal{E}_t).$$

A. Henriques: I'm lost.

U. Bunke: Take g. family $(\mathcal{E}, 0) + (\mathcal{E}^{op}, 0) \sim 0$

The claim is that this

$$= (\mathcal{E} \sqcup \mathcal{E}^{op}, 0)$$

Take $\ker(D(\mathcal{E})) \oplus \ker(D(\mathcal{E}^{op}))$



Lemma: $\eta\left(\left(\mathcal{E} \sqcup \mathcal{E}^{op}\right)_t\right) = 0.$

$$R(\mathcal{E}, \rho) = \Omega(\mathcal{E}) - d\rho.$$

$$I(\mathcal{E}, \rho) = \text{index}(D(\mathcal{E})).$$

$$a(\omega) = (\phi, -\omega).$$

What is η -form?

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$$\eta(\mathcal{E}_t) = \int \text{Tr} \frac{\partial}{\partial t} A_t(\mathcal{E}_t) e^{-A_t^2(\mathcal{E}_t)}$$

If we have

$$\underline{W} = (W, h^W, \nabla^W)$$

$$\rightsquigarrow [\underline{W}] = (W, 0)$$

ooo >

Interpretation (sorry... Integration)

Take vertical bundle. If it has Spin^c structure, it induces ...

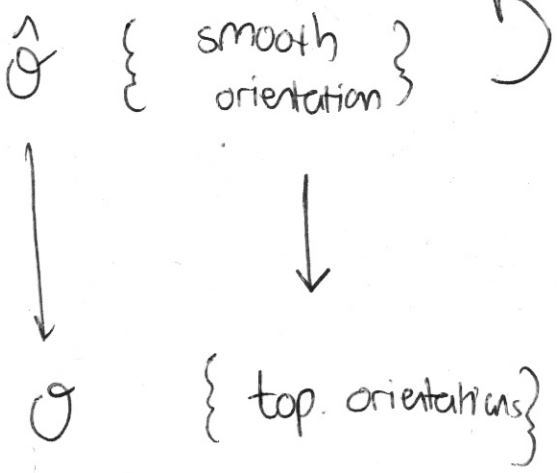
$$p: A \rightarrow B \quad \text{proper submersion}$$

$$T_p^v \text{ spin}^c \rightsquigarrow K\text{-orientation.}$$

Need smooth refinement of orientation!

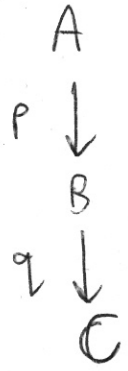
~~For topological orientations~~

Wish list



fibers are torsors for $\Omega^{-1}(A, k) / \text{ind}$

Want also composition



→ compose + pull-back.

André : it is a refinement of the coh. theory $K(\text{smooth orientations})$?

Ulrich : No, I couldn't get that.

$\hat{\circ}$

represented
given by

$$(A, \alpha)$$

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where

$$\begin{array}{c} \uparrow \quad \leftarrow \\ \alpha \in \Omega^{-1}(A, k) \end{array} / \text{3rd.}$$

geometric
family based on p
with Dirac bundle the
 spin^c Dirac bundle

$$W = (T^c p)$$

Relation: $(A, \alpha) \sim (A', \alpha')$

if $\alpha - \alpha' = \hat{A}^c$ transgression of \hat{A}^c
 $= \hat{A} e^{-c_{12}}$

"just the transgression"

This finishes the orientation business.

$$R(\hat{\sigma}) = \hat{A}^c(\hat{\nabla}) - d\alpha \in \Omega^0(A, k)$$

$$\hat{P}_!^{\hat{\theta}} : K^*(A) \longrightarrow K^{*-m}(B)$$

$$m = \dim T^{\vee} p$$

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$$\hat{P}_!^{\hat{\theta}}(\mathcal{E}, p) = \left(p_! \mathcal{E}, \int_{A/B} \hat{A}^c(\tilde{\nabla}) \wedge p + \int \alpha \wedge \mathcal{R}(\mathcal{E}, p) \right)$$

actually only true in the adiabatic limit
where you scale down fibers.

Theorem (Bunke, Schick)

This is \circ

- well-defined
- functorial
- compatible with products

This is the natural home of invariants.

Eg Assume $T^{\vee} p$ is stably framed.

Prop p has a canonical smooth orientation.

"add counterterm"

$$\hat{G} = \left[(A, \tilde{A}^c(\tilde{\nabla}, \tilde{\nabla}^\pi)) \right] \quad (9)$$

Can look at

$$e(\pi) = \hat{p}_1^{\hat{G}}(1) \in \hat{K}_{\text{flat}}^{-m}(B) \\ \cong K \mathbb{R} / \mathbb{Z}^{m-1} \quad (B)$$

If B is a point, m odd,

$$e(\pi) = \text{Adams } e\text{-invariant} \in \mathbb{R} / \mathbb{Z} \\ (\text{uses APS interpretation})$$

Because of nice bordism formula,
easy to calculate in many cases.

Thm Have natural lifts

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$$\hat{c}_i : \hat{K} \longrightarrow \widehat{HZ}^{2i}$$

$$\hat{c}_i^{\text{odd}} : \hat{K}^1 \longrightarrow \widehat{HZ}^{2i-1}$$

$$1 + \hat{c}_i^{\text{odd}} \circ s =: \epsilon$$

preserves group structure.

$$\int \hat{c}_i = \hat{c}_i^{\text{odd}} \int$$

Also, in older Bunka-Schick paper, have

$$\hat{ch} : \hat{K} \longrightarrow \widehat{HQ}$$

More recently, have Adams operations

$$\hat{\psi}^k : \hat{K} \left[\frac{1}{k} \right] \longrightarrow \hat{K} \left[\frac{1}{k} \right]$$

only on compact

$$\hat{\psi}^k \circ \hat{\psi}^l = \hat{\psi}^{kl}$$

Andre: Okay, it's okay ...

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Ulrich: It's Istanbul! (?)

Here, this is \mathbb{Z} -graded.

Riemann-Roch

$$A \xrightarrow[\hat{\mathcal{O}}]{P} B$$

Classical

$$\begin{array}{ccc}
 K(A) & \xrightarrow{\text{ch}} & H^0(A) \\
 \downarrow p_1 & & \downarrow \int A^c (T_V) \cup \dots \\
 K(B) & \xrightarrow{\text{ch}} & H^0(B)
 \end{array}$$

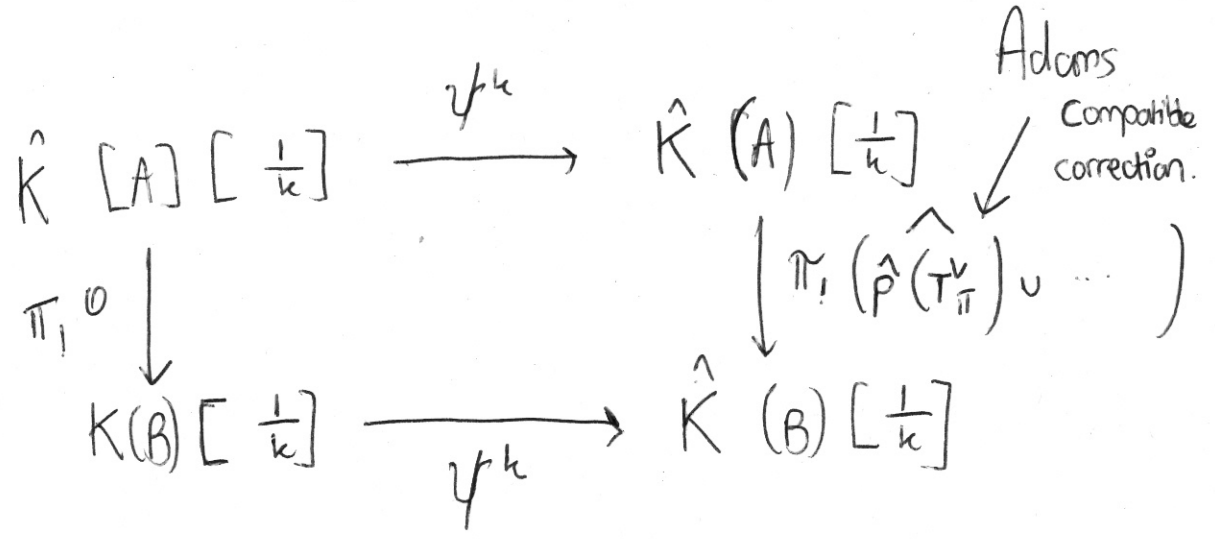
Now we have extensions. Put hats over every thing.

exists $\hat{H}^0(A)$. Need \hat{A} .

Thm (Riemann-Roch) [B-S]. This diagram commutes.

One proves such a thing by showing that the difference couldn't exist!

How about Adams operation?



some equation concerning
Freed-Melros, ?

One consequence:

Prop If $\hat{\theta}$ comes from a stable framing, then
 $\hat{p}(\hat{\theta}) = 1$

Cor $\hat{\psi}^k e(p) = e(p)$
 in special case: $k^L (k^r - 1)$, $(m=2r-1) \dots \rightarrow$