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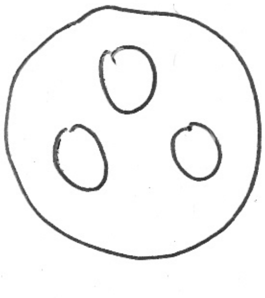
Alternative to yesterday's axioms:

Replace $B(M)$ by embeddings $(\bar{D}^n, M)$



$$\rightsquigarrow \text{Embeddings}(\bar{D}^n, M) \times \text{Emb}(\underbrace{\bar{D}^n \sqcup \cdots \sqcup \bar{D}^n}_{k \text{ times}}, \bar{D}^n)$$

Basic idea

Factorization algebras form a symmetric monoidal category.

If F, F' are factorization algebras, then

$$(F \otimes F')(B) = F(B) \otimes F'(B^*).$$

Def^n: A <u>classical factorization algebra</u> is a commutative algebra in the category of factorization algebras.

[Recall, an E_∞ object in E_n algebras is an E_∞-algebra.]

<u>Idea</u>: We want to associate a classical fact. alg to a classical field theory as follows:

Suppose we have a classical field theory, eg. space of fields is sections of v. bundle $E \longrightarrow M$.

$$S : \Gamma(M, E) \longrightarrow \mathbb{R}$$

is the classical action.

S is <u>local</u>: obtained by $\int$ of a Lagrangian.

If $B \subseteq M$ is a ball, let

$$EL(B) = \left\{ \phi \in \Gamma\left(\text{Interior } B, E\right) \text{ which satisfy the Euler-Lagrange equations} \right\}$$

Dan Freed: You're working in Euclidean signature?
Costello: Yes. We hope we can Wick rotate later.

<u>Rough idea</u>

The classical f. algebra $\mathcal{F}_S$ associated to S assigns to B the algebra of functions on the set ~~space~~ of solutions to EL.

$$\mathcal{O}(EL(B)).$$

We want maps

$$\mathcal{F}_s(B_1) \otimes \cdots \otimes \mathcal{F}_s(B_n) \longrightarrow \mathcal{F}_s(B_{n+1})$$

if $B_1 \amalg \cdots \amalg B_n \subseteq B_{n+1}$.

We have a map

$$EL(B_{n+1}) \longrightarrow EL(B_1) \times \cdots \times EL(B_n)$$

This yields a map

$$\mathcal{O}(EL(B_1)) \otimes \cdots \otimes \mathcal{O}(EL(B_n)) \longrightarrow \mathcal{O}(EL(B_{n+1}))$$

as desired.

<u>Simple example</u>:

Fields are C^∞-functions on M.

$$S(\phi) = \int_M \phi \Delta \phi.$$

Euler-Lagrange eq^n is $\Delta\phi = 0$.

$EL(B)$ = Harmonic functions on Int B

$$\mathcal{O}(EL(B)) := \prod_{n \geq 0} \mathrm{Hom}\left(EL(B)^{\otimes n}, \mathbb{R}\right)^{S_n}.$$

i.e. formal power series.

where Hom means continuous linear maps, $\otimes$ is completed.

Later, we'll see we really need to take the derived space of EL sol^{n}s.

Why does this classical factorization algebra want to become just a fact. algebra?

Fact. algebras form a symmetric monoidal category.

the operad of 0-dim discs.

The E_0 operad is defined by $E_0(n) = \phi$ if $n \geq 1$, and $E_0(0) = pt$.

An E_0-algebra in Vector spaces is just a vector space with an element.

Forgot to mention that fact. algebras need to have a unit, a section of F on $B(M)$, which is a unit for the product.

So: an E_0-algebra in Fact. alg. is just a Fact. algebra!

Classical	Quantum
comm. algebra + $\{\ \}$ deg +1	E_0
Poisson	E_1 algebras = assoc. algebras
comm. algebras + Poisson bracket of deg −1	E_2 " =
comm. algebras + $\{\ \}$ of degree −2	E_3

Beilinson + Drinfeld define an operad over the ring $\mathbb{R}[[\hbar]]$ as follows:

generated by $\bullet$, a comm. product

$\{\ \}$, a Poisson bracket of degree +1

with differential $d(\bullet) = \hbar \{\ \}$.

Call this the BD operad.

$$BD/\hbar BD = \text{operad of comm. algebras } \{\ \} \text{ deg. } +1.$$

$$H_*\left(BD(n) \otimes_{\mathbb{R}[\hbar]} \mathbb{R}((h)) \right) = 0.$$

So, $BD \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}((\hbar)) \simeq E_0$.

[Aside: people call BV operad if framed E_2.

But this has nothing to do with the Batalin-Vilkovisky formalism.

The BV operad is really the BD operad!]

Def^n The P_0 (or Poisson$_0$) operad is the operad of comm. Poisson algebras with $\{\ \}$ of degree $+1$.

so, $P_0 = BD/\hbar$.

General fact

Let M be a manifold, $f : M \longrightarrow \mathbb{R}$.

Then $\mathcal{O}$(Derived critical locus of f) is a P_0-algebra.

functions

The critical locus $= \overset{\text{zero set}}{Z}(df)$.

So $\mathcal{O}(\text{critical locus}) = \mathcal{O}(M) \Big/ \text{Image}\left(\Gamma(M, TM)^{\vee} \xrightarrow{df} C^\infty(M)\right)$

The <u>derived critical locus</u> has functions the dga

$$\cdots \to \Gamma(M, \Lambda^2 TM) \to \Gamma(M, TM) \xrightarrow{\text{contract with } df} \mathcal{O}(M)$$

If f is Morse, this is equiv. to usual setup.

In general captures more info.

This is the same as polyvector fields $\Gamma(M, \Lambda^{\cdot} TM)$

$\Lambda^k TM$ is in deg $-k$

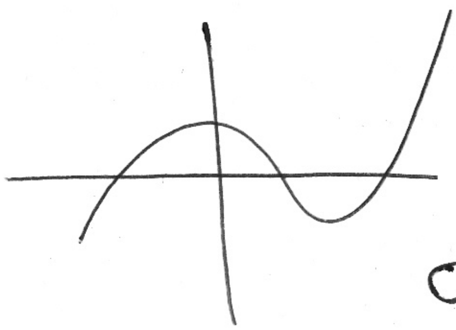
with diff $v\, df$.

Now,

$$\Gamma(M, \Lambda^{\cdot} TM)$$

has Schouten bracket, which is of deg $+1$.

This "wants" to become E_0.



T^*M

graph of df

$$\mathrm{Crit}(f) = \Gamma(df) \cap M$$

Derived critical locus = derived intersection.

Observation :

If M has a <u>measure</u>, then

$$\mathcal{O}(\mathrm{Crit}^{h}(f))$$

has a canonical quantization to an $\underbrace{E_0\text{-algebra}}_{\text{algebra over } BD}$.

The quantization is

$$\left(\Gamma(M, \Lambda^{\bullet} TM) \, , \, \iota_{df} + \hbar \Delta \right)$$

the BV operator arises whenever M has a measure.

$$\Delta X = \mathrm{Div}\, X \quad \text{if } X \text{ is a vector field}$$

This is also done by Kevin Walker (blob homology) or Jacob Lurie (topological chiral homology).

<u>Lemma</u> For a massive scalar field,

$$CH_\bullet(M, \mathcal{F}) \cong \mathbb{R}[[\hbar]]$$

not very exciting!

In general,

$CH_\bullet(M, \mathcal{F})$ looks like measures on the space of critical points of the classical action.

If we perturb around isolated critical points,

$$CH_\bullet(M, \mathcal{F}) \cong \mathbb{R}[[\hbar]]$$

In this situation, correlation functions exist and are unique.

<u>General program</u>: correlation functions define a measure on space of classical solutions which we perturb around.

It's strange: we ~~do~~ don't really perform the path integral, we "quantize", and this does it "automatically".

Q: Where's the propagator?

A: some "QME's written down,
Something about renormalization.

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So far: The derived critical locus of a function is a P_0-algebra, so it wants to quantize to E_0.

If we have a classical field theory, the derived space of solutions to EL yields a P_0 algebra in factorization algebras. So it wants to become a factorization algebra.

Example: $\phi \in C^\infty(M)$, $S(\phi) = \int \phi \Delta \phi$,

Derived space of solutions to EL is the complex

$$\underset{0}{C^\infty(M)} \xrightarrow{\Delta} \underset{1}{C^\infty(M)}$$

If $B \subseteq M$ is a ball, then

$$\mathcal{O}\left(EL^{\text{derived}}(B)\right) = \text{Symmetric algebra on dual}$$

$$= \prod_{n \geq 0} \mathrm{Hom}\left(\left(C^\infty(\mathrm{Int}\,B) \xrightarrow{\Delta} C^\infty(\mathrm{Int}\,B)\right), \mathbb{R} \right)^{S}$$

This is a commutative dga, and defines a commutative factorization algebra.

If $S(\phi) = \int \phi \Delta \phi + \phi^3$

we get the same algebra of functions, but the differential changes.

Yang-Mills : first consider the appropriate derived quotient of $\Omega^1(M) \otimes \mathfrak{g}$ by $\Omega^0(M) \otimes \mathfrak{g}$, and then take derived critical locus of YM action.

In physics literature, this is called the <u>BV formalism</u>.

What we get, when linearized, looks like

$$\mathcal{E} = \underset{-1}{\Omega^0(M)_{\otimes\mathfrak{g}}} \xrightarrow{d} \underset{0}{\Omega^1(M)_{\otimes\mathfrak{g}}} \xrightarrow[\text{d star d.}]{d{*}d} \underset{1}{\Omega^3(M)_{\otimes\mathfrak{g}}} \xrightarrow{d} \underset{2}{\Omega^4(M)_{\otimes\mathfrak{g}}}$$

The algebra of functions is $\prod \operatorname{Hom}(\mathcal{E}^{\otimes n}, \mathbb{R})^{S_n}$ with differential including YM action.