M. Hopkins

The Kervaire Invariant

Thm (Hill, H, Ravenel)

If M is a stably framed manifold of Kervaire invariant 1, then $\dim M$ is

$$2, 6, 14, 30, 62, \boxed{126}.$$

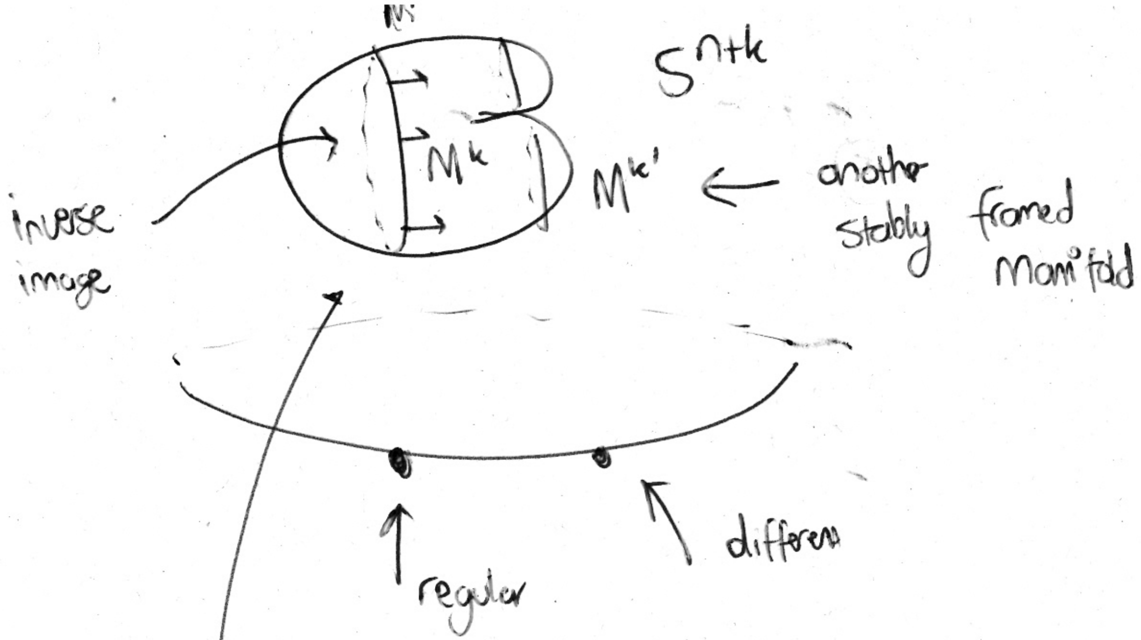
Don't know what's going on! Should be 6! (God made one every day)

The story of the problem

We start off by thinking about h. classes of maps

$$S^{n+k} \longrightarrow S^n$$

Pontryagin asked: what if we have more variables?



Get triv. of normal bundle inside sphere
"stably framed manifold".

Pontryagin set up an amazing relationship between geometry and homotopy theory:

1930's $\quad \pi_{n+k} S^n =$ cobordism group of stably framed k-manifolds

He <u>used</u> the classification of manifolds in dim 0, 1, 2 to understand homotopy groups of spheres!

$\underline{k=0}$

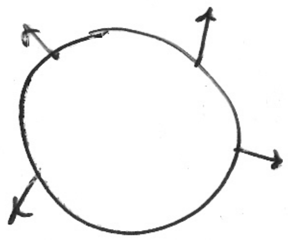
+ point

− point

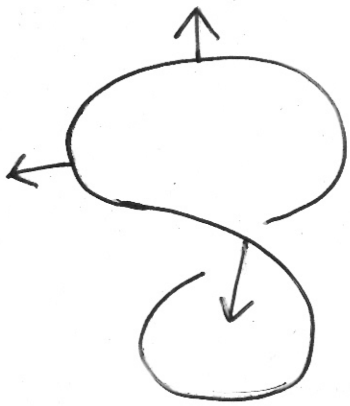
reproduces <u>degree</u> of
a map,

so
$$\pi_n S^n = \mathbb{Z}.$$

$\underline{k=1}$



also



so $\boxed{\pi_{n+1} S^n = \mathbb{Z}/2.}$ ✱

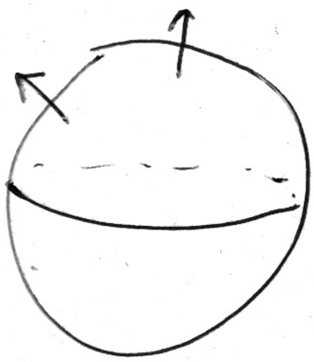
Nice story: generalized notion of <u>degree</u> of a map
$$M \longrightarrow \text{Sphere}.$$

Thinking about maps
$$M^{n+1} \longrightarrow S^n$$
was due to Steenrod. This one extra case ✱

led to the development of all htpy theory!

<u>k=2</u> Pontryagin made a mistake!

g=0.



bounds a disc.

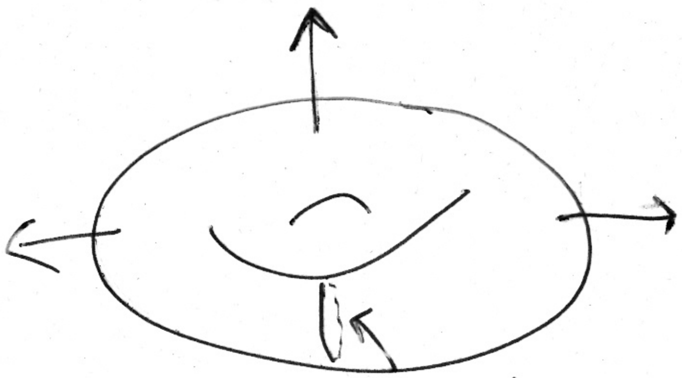
change of framing = map into general linear 2-group

$\uparrow$

$\pi_2(\dots) = 0$

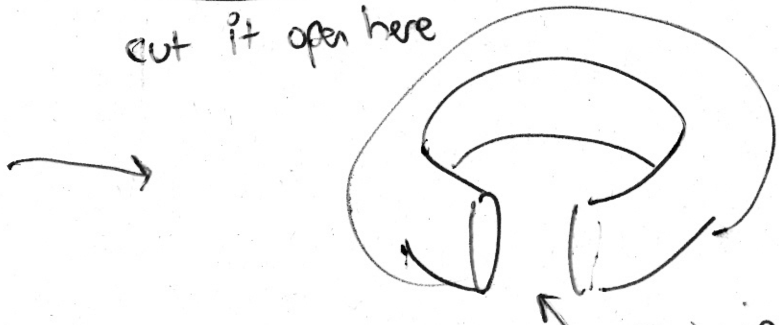
so if genus = 0, must be trivial

g=1



<u>Brilliant idea</u>

cut it open here



sew in 2 discs to form sphere.

But obstruction!

$$\phi : H_1(\Sigma, \mathbb{Z}) \longrightarrow \mathbb{Z}/2.$$

Pontryagin said dim is even, so....
≥ 2

so always something in kernel, so you can lower genus. Therefore concluded

$$\boxed{\pi_{n+2} S^n = 0.}$$

mistake!

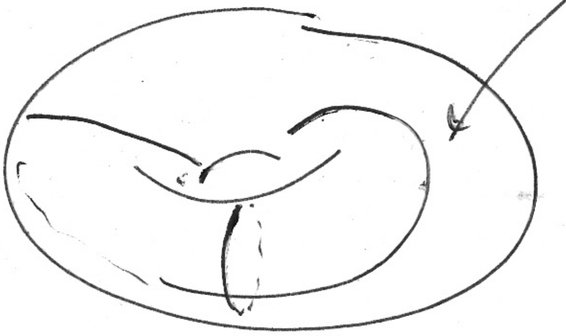
mid 1930's.

1940's Whitehead calculated cyclic of order two.

~~A~~ Andrew asked Ranicki : Took 5 years

The mistake is the map ϕ is nonlinear in general!

Could do surgery



weird loop.

Somehow makes ϕ not linear.

In fact, ϕ is quadratic!

$$\phi(x+y) - \phi(x) - \phi(y) = \underbrace{I(x,y)}_{\text{intersection pairing}}$$

In our example

$$\mathbb{Z}/2 \quad \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$$

$$\mathbb{Z}/2$$

2 quadratic refinements.

Beautiful invariant of a quadratic function:

$$\mathrm{Arf}(\phi) \in \mathbb{Z}/2$$

: there are <u>two</u> framed Riemann surfaces so $\pi_{n+2} S^n = \mathbb{Z}/2$.

In modern terms, Pontryagin was trying to use surgery to convert mfld into sphere.

Q In which dimension does every framed coboordism class contain a homotopy sphere? (in dim > 5, this = homeomorphic to sphere)

Answer: In every dim except those six!

Pontryagin only had a $6/\infty$ chance of making a mistake!

Invariants of quadratic forms / functions are deep and subtle!

1956 : Milnor showed there were exotic 7-spheres.

A completely amazing thing!

1960 : Kervaire studied Pontryagin's mistake.

Defined a quadratic refinement

$$\phi : H_n(M^{2n}, \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$$

when n is an odd number, and M stably framed.

$$\boxed{\text{Kervaire invariant} : \quad \Phi(M) = \mathrm{Arf}(\phi).}$$

Kervaire did two things.

- Showed
$$\Phi(M^{10}) = 0.$$
- showed it could be defined when M not smooth constructed framed PL manifold N^{10}, and found
$$\Phi(N^{10}) \neq 0.$$

So an example of a PL manifold that couldn't be smoothed! Beautiful homotopy theory ideas.

~~Q~~ At this point:

<u>Question</u>: In which dimensions can $\Phi(M)$ be nonzero?

Kervaire – Milnor (announced at international congress 1958, published 1963)

(Nathaniel argues).

This is a beautiful paper. Introduced group

$$\Theta_n = \text{group of smooth structures on } S^n$$

under connected sum.

Related to htpy groups of spheres, Bernoulli numbers, ... great differential topology.

When n even: signature $\longrightarrow$ Bernoulli

odd: $\longrightarrow$ Kervaire invariant

Determined ~~in~~ terms of $\pi_n S^0$ up to
Θ_n

a factor of 2. ~~that~~, which depended

on the Kervaire invariant!

So now we know: most of the time there are
twice as many exotic spheres as we used to
know!

There were 2 papers that approached this via htpy theory.

1966 Brown - Petersen.

$$\Phi(M^{8k+2}) = 0 \quad k \geq 1.$$

Used spin structures and techniques of K-theory.

Design a cohomology theory which produces an arithmetic
sequence... then eliminate it! We used the same technique.

1969 Browder ... very deep.

$$\Phi(M^n) = 0 \text{ unless } n = 2^{j+1} - 2.$$

7

$\exists\ M^{2^{j+1}-2}$ with $\Phi(M) \neq 0$

$\iff \exists\ \theta_j \in \pi_{2^{j+1}-2} S^0$

representing an h_j^2 in the Adams spectral sequence.

$h_j \longleftrightarrow$ Hopf m valued 1 class

$h_1 \longleftrightarrow S^3$ $\quad \theta_1 = S^1 \times S^1$

$h_2 \longleftrightarrow S^3$ $\quad \theta_2 = S^2 \times S^2$

$h_3 \longleftrightarrow S^7$ $\quad \theta_3 = S^3 \times S^3$

By relating it to Adams spectral sequence, this was a game-changer.

André : Is it conceivable you could use some techniques to eliminate 126?

M. Hopkins : We might be able to eliminate it, but we couldn't verify its existence.

Barratt – Mahowald – Jones

1968, 1984

θ_4 exists ← geometric construction (Jones)

θ_5 exists.

Very far from geometry, algebraic, you calculate the Adams spectral sequence and eliminate things.

The mindset was that these all exist.

It's a real game changer to say:

"we're looking for some construction which works in dim 2^j-2 and its beautiful"

No response.

But if you say,

"We're looking for 6 things,"

it's psychologically different! Lie groups, etc.

E_6, E_7, E_8.

Exciting question:

$2, 6, 14, 30, 62, 126$

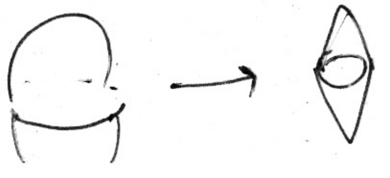
~~F_4~~

$D_4 \quad D_5 \quad E_6 \quad E_7 \quad E_8$

Hans Duistermaat: said he'd seen those numbers pop up in Painleve theory and they correspond to D_4 D_5 E_6 E_7 and E_8!

James (localize at 2)

$$\pi_k(S^n) \xrightarrow{E} \pi_{k+1} S^{n+1}.$$



$$\pi_k(S^n) \xrightarrow{E} \pi_{k+1} S^{n+1} \xrightarrow{H} \pi_{k+1} S^{2n+1}$$

$$\to \pi_{k-1} S^n \longrightarrow \pi_k S^{n+1} \longrightarrow \pi_k S^{2n+1}$$

$$\to \cdots$$

Toda used this sequence to calculate first 14 homotopy groups of spheres. Showed Hopf invariant$_4$ didn't exist.

Must come to grips with first place the S^{2n+1} term is nonzero.

$$\pi_{2n+1} S^{2n+1} \longrightarrow \pi_{2n-1} S^n$$

$$1 \longmapsto [\cancel{\#}\, i, i] \quad \ldots \text{ Whitehead product.}$$

Write $\quad S^n \times S^n = S^n \vee S^n \cup e^{2n}$.

contains the <u>tangent bundle</u> of the sphere.

Think as an exact couple. Two questions:

a) Is Whitehead product divisible by 2?

b) For which j is $[i_n, i_n]$ in the image of E^k?

This breaks into three separate problems.

Question b) is equivalent to the vector fields on spheres problem.
(solved in 60's by Adams using K-theory).

a) when n is even $\Longleftrightarrow$ Hopf invariant one problem
n odd $\Longleftrightarrow$ Kervaire problem

So ^amazing that three fundamental problems in homotopy theory have

---

Consider following space:

$V_2(S^n)$ = space of points (a,b), $a, b \in S^n$
s.t. $a \neq b$, $a \neq -b$
$\overset{\text{homotopy}}{\simeq}$ orthonormal 2 frames in (S^{2n-1}) ?

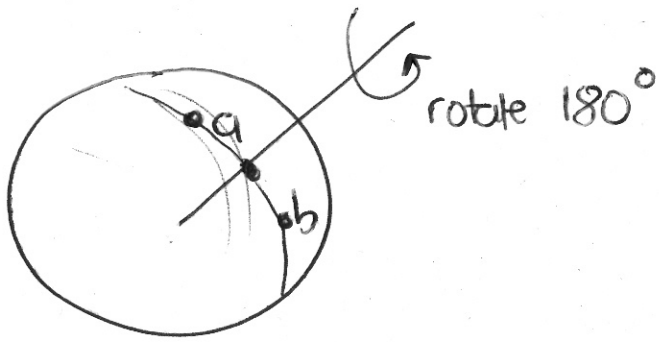
Map from $\overset{\circlearrowright}{V_2(S^n)}$ $\quad (a,b) \longmapsto (b,a)$.

<u>Question</u> Is this map homotopic to the identity?

→ Like asking to divide Whitehead product by two.

eg dim 2:

2-sphere



average them → midpoint.

Can do this every time sphere S^n has almost complex structure.

So at least S^2, $S^6 = G_2/SU(3)$

↑ has almost complex structure because

So an exceptional Lie group comes up. I don't know, maybe something to this!

$$\boxed{\text{Ranicki : Pontryagin and Whitehead both got it independently in 1950. Defined quadratic formula geometrically. See webpage!}}$$

Mark's guess now : that 126 exists,
possibly.
Something about θ_5^2.

In Turkey, has banknote with Arf invariant!