CFT and algebra in braided $\otimes$-categories II

Ingo Runkel

1. Bulk algebra
2. Module category
3. Outlook – logarithmic conformal field theories (nonsemisimple).

Recap:

data: [bicat: V a v.o.a, $C = \mathrm{Rep}\,V$, a MTC

C a MTC $\rightsquigarrow$ bicategory – where objects are special symm. Frob. algebras $A, B, \ldots$

– morphisms are A-B-bimodules, defect lines

[$A \in \mathrm{Rep}\,V$.

Frob. algebras occur elsewhere:

$(U, \varphi \in \underbrace{\mathrm{Hom}_A(A \otimes U, A)}\ )$

$\cong \mathrm{Hom}(U, A)$.

i.e. there's a space of boundary fields / open states,

$H_{A,A} = A$. i.e. A represents $U \mapsto \mathrm{Hom}(U, A)$.

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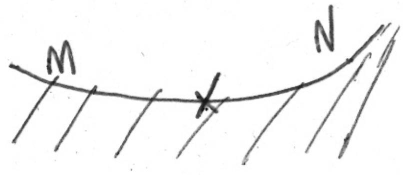
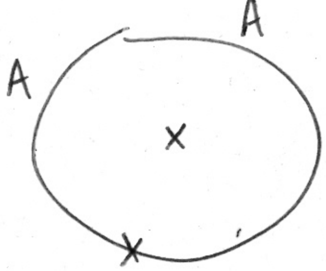


$\psi \in \mathrm{Hom}_A(M \otimes U, N)$

$H_{MN} \cong M^{\vee} \otimes_A N.$

What happens with:

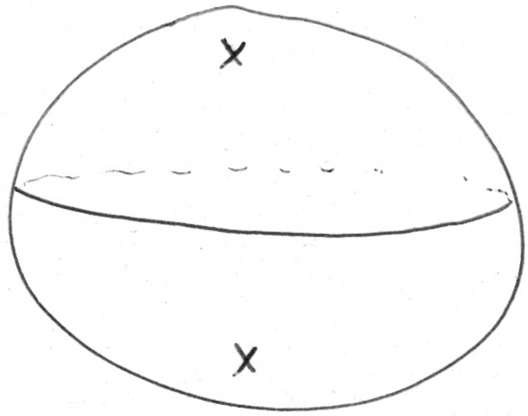


?

becomes

two marked points

$\hat{X}$



$$\begin{array}{ccc} A \otimes_{\mathbb{C}} C & \longrightarrow & \mathbb{C} \\ \uparrow \quad\quad \uparrow & & \\ \in \mathrm{Rep}V \quad \in \mathrm{Rep}V \boxtimes \mathrm{Rep}V & & \end{array}$$

expect :

1) $C \in \mathcal{C}_T \boxtimes \mathcal{C}_-$ — inverse braiding, inverse twist

2) C a <u>commutative</u> sym. Frobenius algebra.

Functor:

$$R : C \longrightarrow C_+ \boxtimes C_-$$

$$V \longmapsto \bigoplus_{i \in I} \underbrace{V \otimes U_i^{\vee}}_{\in C_+} \times \underbrace{U_i}_{\in C_-}$$

the simple objects

think of it as the adjoint of

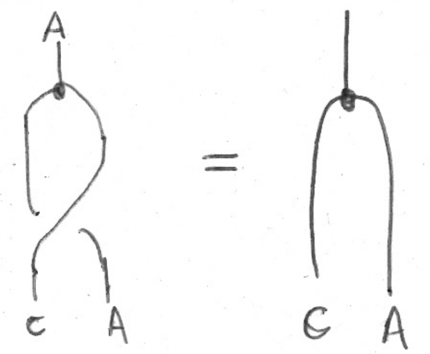
$$C_+ \boxtimes C_- \longrightarrow C$$

$$U \times V \longmapsto U \otimes V.$$

properties: if A is a $\underbrace{\text{ssFa}}_{\text{special sym. Frob. algebra}}$ in C, then $R(A)$ is a ssFA in $C_+ \boxtimes C_-$.

We need to take its center to get a commutative ssFa.

Defn The left centre of an alg. A in C is the maximal subobject C_L of C of A such that



we made a choice of braiding!

Also have right centre C_R of A. Need not have $C_L \cong C_R$, or even Morita-equivalent.

<u>Def</u> The full centre $Z(A)$ of a ssFa A in $\mathcal{C}$ is $C_\ell(R(A))$.

$$\in \mathcal{C}_+ \boxtimes \mathcal{C}_-.$$

Need to take left centre for later. This defn more symmetric.

Properties of $Z(A)$:

- $Z(A)$ is commutative ssFA in $\mathcal{C}_+ \boxtimes \mathcal{C}_-$.
- $C_\ell(A) \times \mathbb{1}$ and $\mathbb{1} \times C_R(A)$ are subalgebras of $Z(A)$.
- $$Z(A) = \bigoplus_{i,j \in I} \left(U_i^\vee \times U_j \right)^{\oplus Z_{ij}(A)}$$

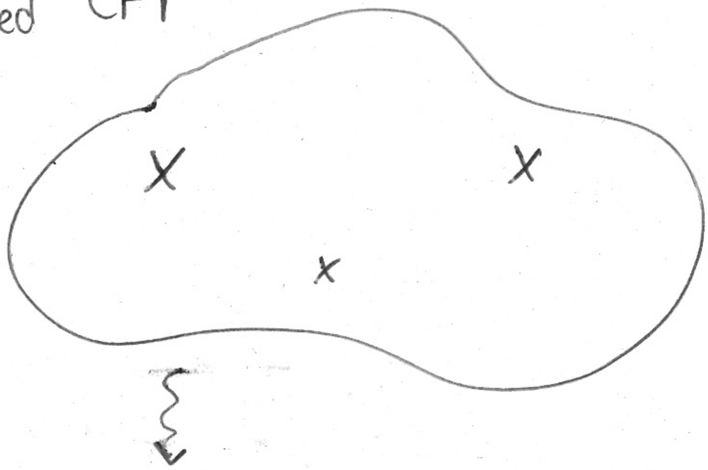
  the modular invariant matrix.

<u>Thm</u> The number of iso-classes of simple A-left modules $= \mathrm{tr}\left[Z_{ij}(A)\right]$

$$\boxed{Z_{ij} = \dim \mathrm{Hom}_{A|A}\left(U_i \otimes^+ A \otimes^- U_j, A\right)}$$

this links to Christoph's talk.

$Z(A)$ defines a closed CFT



correlation function: multilinear map

$$Z(A) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} Z(A) \longrightarrow \mathbb{C}.$$

What have we done

boundary (A), constructed closed CFT $Z(A)$.

from unlabelled M, $M^\vee \otimes_A M$.

Now M and A are Morita-equivalent.

Would like to verify ~~th~~ $Z(A)$ independent of Morita.

<u>Thm</u> (Kong, Runkel 07) $\mathcal{C}$ a mod. tens. cat., A, B simple ssFa, Then

$$A \underset{\text{m.eq.}}{\sim} B \implies Z(A) \underset{\text{iso}}{\simeq} Z(B) \text{ as algebras}$$

(not necessarily as Frob. algebras)

eg eg for ssFa in $\mathrm{Vect}_{\mathbb{C}}$, we have

$$Z(A) \simeq \mathrm{End}(\mathrm{id}_{A\text{-Mod}})$$

$\therefore$ Morita ~~eqv~~ invariant.

The converse holds too: $Z(A) \cong Z(B) \Rightarrow A \sim_{\text{m.e.}} B$.

<u>Thm</u> (Kong, R 08) $\mathcal{C}$ a mod. tens. cat., C a commutative simple ssFa in $\mathcal{C}_+ \boxtimes \mathcal{C}_-$ s.t. $\dim C = \mathrm{Dim}\,\mathcal{C}$ (modular invariance),

then: i) exists ssFA $A \in \mathcal{C}$ s.t. $C \cong Z(A)$.

ii) $T(C) = \bigoplus A_i$,

$T: U \times V \mapsto U \otimes V$

A_i simple ssFa, all Morita equivalent.

Any one can be used, i.e.

$$C \cong Z(\text{summand in } T(C)).$$

Every modular invariant CFT with left/right chiral sym given by V is part of an open/closed CFT.

2. Module categories

R a ring, a right module is $M \times R \longrightarrow M$.

categorify.

<u>Defn</u> $\mathcal{C}, \mathcal{M}$ be abelian, $\mathbb{C}$-linear categories. $\mathcal{C}$ a tensor cat.

$\mathcal{M}$ is a right <u>module category</u> over $\mathcal{C}$ if:

- bifunctor $\odot \quad \mathcal{M} \times \mathcal{C} \longrightarrow \mathcal{M}$

  s.t. associative, $1 \in \mathcal{C}$ acts as unit.

  up to coherent iso. (mixed pentagon, triangle)

Examples: $*$ $\mathcal{M} = \mathcal{C}$, $\odot = \otimes$.

$*$ A an algebra in $\mathcal{C}$, A-mod is a right-module cat over $\mathcal{C}$.

$$ {}_A M \odot U = {}_A M \underset{\substack{\uparrow \\ \text{in } \mathcal{C}}}{\otimes} U $$

Thm (Ostrik 01) $\mathcal{C}$ a mtc (don't need braiding), $\mathcal{M}$ finitely semisimple, indecomposable ($\neq \mathcal{M}_1 \oplus \mathcal{M}_2$). Then

$\mathcal{M} \simeq A$-mod for A an algebra in $\mathcal{C}$.

Get A via

Internal homs :

$\mathcal{M}$ a module cat over $\mathcal{C}$.

$\cancel{M}$, $M, N \in \mathcal{M}$,

$\underline{\mathrm{Hom}}(M,N)$ is object in $\mathcal{C}$ representing the functor

$$U \longmapsto \mathrm{Hom}_{\mathcal{M}}(M \odot U, N).$$

- associative composition

$$\underline{\mathrm{Hom}}(M,N) \otimes \underline{\mathrm{Hom}}(K,M) \longrightarrow \underline{\mathrm{Hom}}(K,N)$$

in part $\underline{\mathrm{End}}(M)$ is an algebra in $\mathcal{C}$.

This is how you get A.

In fact,

$$\mathrm{Fun}(\mathcal{M}, \mathcal{N})$$

$\uparrow$

demand $F(M) \odot U \cong F(M \odot U)$

$\mathrm{Fun}(A\text{-mod}, B\text{-mod}) \cong$ B-A mod as $\otimes$-cat.

objects are

- "good" module cat over $\mathcal{C}$,

  i.e. equiv to A-mod for some ssFa A in C.

- morphisms are $\mathrm{Fun}(\mathcal{M}, \mathcal{N})$.

space of boundary fields

$M, N \in \mathcal{M}$ = collection of <u>all</u> boundary conditions of the given CFT compatible with V.

Urs: want to think of, alá Ben-Zvi;

$$C\text{-Mod} \hookrightarrow \mathrm{Mod}_C$$

objects are C-enriched categories, profunctors between them, ...

$\mathcal{C}$ a ~~mtc, $\mathcal{M}$ a 'good'~~ m

$\mathcal{C}$ a mtc, M a 'good' module cat over $\mathcal{C}$.

braided induction

$$\alpha^{\pm} : \mathcal{C} \longrightarrow \mathrm{End}(M)$$

$$\alpha^{\pm}(U) = (M \longmapsto M \otimes U)$$

The $\pm$ refers to equipping it as an endofunctor of M, (need braiding).

$$\mathrm{End}(M) \times \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathrm{End}(M)$$

$$F \times (U \times V) \longmapsto \alpha^{+}(U) \cdot F \cdot \alpha^{-}(V).$$

statement: $Z_M = \underline{\mathrm{End}}(\mathrm{id}_M)$

internal End of identity functor on M.

$$\boxed{\textbf{Thm} \quad Z_{A\text{-mod}} \simeq Z(A).}$$

so you can recover