Thomas Schick

Generalised ~~diff~~ smooth cohomology

Smooth (refinement of) cohomology.

Idea: Combine cohomology (an integral theory) and differential forms.

Main diagram:

a functor Man $\to$ P ..., plus natural trans

integral

$$\begin{tikzcd} \hat{H}(M) \arrow[r, two heads, "I"] \arrow[d, "R"'] & H^*(M) \arrow[d, "ch"] \\ \Omega^*_{d=0}(M) \arrow[r] & H^*_{dR}(M) \end{tikzcd}$$

surjective

R for curvature

So ... its a functorial gadget, plus the natural trans. above, plus more.

We want to understand how it differs from ordinary cohomology:

Require a transformation

$$a : \Omega^{*-1}(M)/_{im(d)} \longrightarrow \hat{H}^*(M)$$

action on forms
holonomy

For Urs

1. $\int_X$ tft

2. Witten extended

Defn : If E^* is multiplicative, we say $\hat{E}^*$ is multiplicative if $\hat{E}^*$ really takes values in graded rings, and the transformations are compatible with the multiplication. For the transformation a, this means

$$\underbrace{a(\omega) \vee x}_{\text{multiplication in } \hat{E}^*} = a\big(\omega \wedge R(x)\big) \qquad \forall \omega \in \Omega(M)$$
$$x \in \hat{E}(M)$$

To discuss ~~AA~~ push-forward maps, begin with:

Def: $\hat{E}$ has <u>S^1-integration</u> if there is a natural transformation in M

$$\int : \hat{E}^*(M \times S^1) \longrightarrow \hat{E}^{*-1}(M)$$

compatible with $\int$ of forms and for E, and

- $$\int \circ p^* = 0 \quad \text{for } p : M \times S^1 \longrightarrow M$$

- $$\int \circ \underbrace{\mathrm{id} \times (z \mapsto \bar{z})^*}_{\substack{\text{map from} \\ M \times S^1 \\ \text{to itself}}} = -\int .$$

In Cheeger-Simons, push-down easy. In Deligne model, hard.

Want Exact sequence:

$$\begin{tikzcd} H^{*-1}(M) \arrow[r, "ch"] & \Omega^{*-1}(M)/\mathrm{im}(d) \arrow[r, "a"] \arrow[dr, "d"'] \arrow[dr, phantom, "\text{must commute}"] & \hat{H}(M) \arrow[r, "I"] \arrow[d, "R"] & H^*(M) \arrow[r] & 0 \\ & & \Omega^*_{d=0}(M) & & \end{tikzcd}$$

could write as E since generalized

<u>Defn</u>: Given cohomology theory H^*, a smooth refinement $\hat{H}^*$ is a functor $\hat{H}$: Man $\longrightarrow$ graded abelian groups with transformations I, R, a as above (with the exact sequence and commuting diagram).

Here: Ω^* have to be replaced by $\Omega^*(\cdot, V)$

(forms with values in $V = E^*(pt) \otimes \mathbb{R}$)

as a graded group.

Proof of lemma:

Suffices to show $\mathring{i}_1^*(x) - \mathring{i}_0^*(x) = a\left(\int_{\substack{M\times[0,1] \\ \text{over } M}} R(x) \right)$

$$\forall x \in \hat{E}\left(M \times [0,1]\right)$$

Observe if $x = p^* y$, then LHS $=0$.

Also $\int R(p^* y) = 0$.

For general x, there exists $y \in \hat{E}(M)$ s.t. $x - p^*(y) = a(\omega)$
$\omega \in \Omega(M \times [0,1])$

Now, by Stokes:

$$\mathring{i}_1^*\omega - \mathring{i}_0^*\omega = \int_{[0,1]} d\omega = \int_{[0,1]} R(a(\omega))$$

$$= \int R(x - p^* y)$$

$$= \int R(x)$$

On the other hand,

<u>Lemma</u>: Given $\hat{E}$ smooth coh. theory,

get <u>homotopy formula</u>:

If $h : M \times [0,1] \xrightarrow{\text{smooth}} N$,

$$h_1^*(x) - h_0^*(x) = a\left(\int_{M\times[0,1]/M} h^*(R(x))\right) \quad \forall x \in \hat{E}(N)$$

in classical examples, this is related to Chern-Simons form.

<u>Corollary</u>: $\ker(R)$ is a homotopy invariant functor.

<u>Defn</u> We call $\ker(R)$ the <u>flat part</u> of the smooth cohomology theory.

So exact sequence

$$0 \longrightarrow \hat{E}^*_{\text{flat}}(M) \longrightarrow \hat{E}^*(M) \xrightarrow{R} \Omega^*_{d=0}(M) \longrightarrow 0$$

$$\mathring{i}_1^*(x) - \mathring{i}_0^*(x) = \mathring{i}_1^*\big(a(\omega)\big) - \mathring{i}_0^*\big(a(\omega)\big)$$

$$= a\left(\int R(x) \right).$$

$\square$

Are there things beside Deligne cohomology and Cheeger-Simons which satisfy this ?

<u>Quick calculation</u> : $\hat{H}^1(pt) = \mathbb{R}/\mathbb{Z}$.

ordinary cohomology

$= \hat{K}^1(pt)$

$= \hat{H}^1_{flat}(pt)$

So this shows these smooth versions are the homes of <u>secondary invariants</u> (take values in $\hat{E}^*_{flat}(pt)$).

<u>Theorem</u> (Hopkins-Singer) For each gen. coh. theory E^* ~~[illegible]~~, on $\hat{E}^*$ exists. Moreover, $\hat{E}^*_{flat} = E\,\mathbb{R}/\mathbb{Z}^{*-1}$.

uses abstract homotopy theory

<u>Remark</u> : It's not at all evident how to obtain more structure like multiplication, or integration.

It's best to have explicit models.

<u>Thm</u> ( ) Using geometric models, multiplicative smooth extensions with S^1-integration are constructed for :

- K-theory (Bunke, Schick)

  (based on local index theory. Cycles are family index problems parametrized by M).

- MU-bordism (Bunke, Schröder, Schick, Wethamp)

and from there,

Landweber exact cohomology theories

<u>Uniqueness theorem:</u>

Assume E^* satisfies $E^k(pt) \otimes \mathbb{Q} = 0$ for odd degrees k.

plus technical assumptions.

Then any two smooth extensions with S^1-integration $\hat{E}^*$ and $\tilde{E}^*$ are naturally isomorphic, with a unique iso compatible with S^1-integration.

If $\hat{E}, \tilde{E}$ are multiplicative, the iso is as well.

Example: If we don't require compatibility with S^1- integration, there are "exotic" abelian group structures on $\hat{K}^1$.