Outline

- language for some elementary topology
- application to generalizations of Hochschild complexes
- examples : a) invariants on mapping spaces
  - b) relation to known constructions related to deformations of Laplacian

Defn / Lemma (What is a DGA?) A commutative DGA is

a monoidal functor

$$\begin{tikzcd} F \arrow[r] & \mathrm{Ch} \end{tikzcd}$$

finite sets
maps of sets
$\amalg$

Ch
Chain complexes,
chain maps
$\otimes$

Why? Write eg. j for set w/ j elements,

objects

$$\emptyset \longmapsto 1$$
$$1 \longmapsto A$$
$$k \longmapsto A^{\otimes k}$$

morphisms

$$\begin{tikzcd} 2 \arrow[d] & A^{\otimes 2} \arrow[d] \\ 1 & A \end{tikzcd}$$

Generalisation :

Defn A partial DGA (a partial algebra) is a lax monoidal functor

$$\mathcal{F} \xrightarrow{A} Ch$$

$$j \longmapsto A(j)$$

i.e. ∃ a natural equivalence

$$A(j \sqcup k) \xrightarrow[\text{quasi-iso}]{T} A(j) \otimes A(k)$$

respecting certain strict coherence properties.

These pop up in homotopy theory all the time.

Generalize : (LHS)
1) co-algebras
2) any operad
3) Note that $\mathcal{F}_*$ (based finite sets) is a module over $\mathcal{F}$, can generalize to modules, co-modules, etc.

Then [W] partial algebras can be functorially replaced by E_∞-algebras.

<u>Example</u> X be a space. Given a function

$$j \xrightarrow{f} k$$

$$X^j = \mathrm{Map}(j, X) \leftarrow \mathrm{Map}(k, X) = X^k$$

Take chains or cochains, get

$$C_*(X^j) \leftarrow C_*(X^k)$$

$$C^*(X^j) \rightarrow C^*(X^k)$$

if you take PL chains, you get this.

Have Kunneth map

$$C_*(X^j) \otimes C_*(X^k) \longrightarrow C_*(X^{j+k})$$

and dual.

This gives a partial coalgebra on ~~the chains~~ $C_*(X)$ and $\underbrace{\text{partial algebra on } C^*(X)}$

Model secretly an E_∞-algebra, and determines the integral homotopy type if X is simply connected.

$$\begin{tikzcd}
S \times \mathbb{R}^{0|1} \arrow[r] \arrow[d, dashed] & M \\
\mathbb{R}^{0|0} \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & C^\infty(\mathbb{R}^{0|1}) \arrow[d, phantom, "\parallel"] \\
C^\infty(M) \arrow[r] & \mathbb{R}[\theta] \otimes C^\infty(S).
\end{tikzcd}$$

$$f \longmapsto \delta(f) + \underbrace{\zeta(f)\,\theta}_{=0}$$

2. Applications

Let Y be any finite simplicial space, A a partial algebra.

Q : What are these PL chains?

A : Take triangulations, then take colimit.

(ie. Y is a simplicial set)

some partial algebra

$$\begin{tikzcd}\Delta \arrow[r, "Y"] & \mathcal{F} \arrow[r, "A"] & Ch\end{tikzcd}$$

This is a simplicial object in Ch, so total complex

$$CH^{Y}(A)$$

In fact, this is a generalization of the Hocschild cohomology, defined for any partial algebra A.

[joint with T. Tradler + M. Zinadior]

Geometrically,

for $A = \Omega(X)$, $CH^{X}(A)$ computes the cohomology of X^{Y} is X is simply connected.

Also, a model due to Jim Maclure doing this for transverse sections.

Example Say A is a strict algebra, $Y = S^1$.

Then

$$CH^{S^1}(A) = \text{usual Hochschild complex}$$
$$= \prod_{n \geq 0} A \otimes A^n$$

differential has 2 terms: internal + multiply

$$a_0 \quad a_1 \quad \cdots \quad a_n$$

As a partial algebra, this is the shuffle product.

This implies there is an exponential map.

By the way,
$CH^Y(A)$
is a partial algebra itself!
Related to string topology

Calculate

factorials vanish because we sum our shuffles.

$$e^{1 \otimes x} \in 1 \otimes A.$$

$$= 1 + 1 \otimes x + 1 \otimes x \otimes x + 1 \otimes x \otimes x \otimes x$$
$$+ \cdots$$

Compute

$$\overset{\text{the differential}}{D} e^{1\otimes x} = \Big(1 \otimes \underbrace{dx + x\cdot x}_{\text{Maurer-Cartan eqn}}\Big) \overset{\text{shuffle product.}}{\cdot} e^{1\otimes x}$$

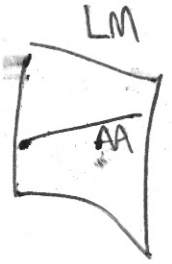
<u>Consequences</u>:
- If $dx + x\cdot x = 0$ then $De^{1\otimes x} = 0$.
- Reminds us of curvature of connections.

In fact, can be taken further:

~~$\begin{tikzcd} & CH^{S^1} \arrow[d] \\ K(M) \arrow[r, "ch"] & \Omega(M) \end{tikzcd}$~~

(more room)

$$\begin{tikzcd} & CH^{S^1}(A) \arrow[d] & \left(\substack{\text{forms on}\\ \text{the loop space}} \simeq \Omega(M^{S^1})\right) \\ K(M) \arrow[ur, bend left, "\text{algebraic Hochschild proof due to Getzler, □, □}"] \arrow[r, "ch"'] \arrow[urr, bend left] & \Omega^1(M) & \end{tikzcd}$$



$\uparrow$ v.bundles w connection

closed class lifts to closed class on loop space.

Bismut showed analytically that there's a class here, which when restricted to const. loops, makes diag commute.

Can do for other mapping spaces, eg. there is a map

$$\begin{tikzcd} CH^{S^1\times S^1}(A) \arrow[d] \\ CH^{S^1}(A) \end{tikzcd}$$

making everything commute.

Ex 2 $Y = I$.

Observation:

$CH^{I}(A, M, N)$ turns out to be the two-sided Bar construction

left module   right module.

$$= \prod_{n\geq 0} M \otimes A^{\otimes n} \otimes N$$

$* : \Omega^{j} \to \Omega^{n-j}$

case when: $A = \Omega$(Riemannian manifold), d, $\wedge$

$M = \Omega$, d, usual module structure

$N = \Omega$, d^*, dual module   cap product

$(x,y) \in A \otimes M \longmapsto *^{-1}(x \wedge *y)$

It is miraculous that such a structure exists at the level of chains!

Let D be the differential on CH^{I} for usual algebra both sided module structures

D^{*} " " for A, M, N described here.

$$\Delta = [D, D^{*}]$$

Calculation: $\Delta\left(x \cdot e^{s(1 \otimes x \otimes 1)} \cdot x\right)$

$$= x e^{1 \otimes s x \otimes 1}\left(\Delta x + \overbrace{d{*}(x \wedge {*}x)}^{\|x\|^{2}} + s{*}(x \wedge {*}dx)\right.$$

$$\left. + s^{2} \underbrace{{*}(x \wedge {*}x)}_{|x|^{2}} \cdot x\right)$$

usual Laplacian $[d, d^{*}]$

This is the term that appears in Witten's deformation of the Laplacian.

The nonlinear term corresponds to an interesting PDE

$$\Delta x + dp + \frac{1}{2} d{*}(x \wedge {*}x) + (x \wedge {*}dx)$$

In $\mathbb{R}^3$, this is the

Navier Stokes equation !!

$$\dot{x} = \Delta x + dp + s\tfrac{1}{2}d*(x\wedge *x)$$

$$+ \; s\underbrace{*(x\wedge \underbrace{*dx}_{\text{curl } x})}_{X \times \text{curl } X}$$ In $\mathbb{R}^3$,

s is the viscosity

$x \in \Omega^1(M) \times \mathbb{R}$

$p \in \Omega^0(M) \times \mathbb{R}$ $\quad p = \frac{1}{2}*(x\wedge * x)$

~~$= \frac{1}{2} d * x$~~ $\quad = \frac{1}{2}\|x\|^2$