# Topos Theory

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# Preface

These lecture notes were written during a *Mastermath* (Dutch national programme for master-level courses in mathematics) course, taught in the fall of 2018.

The main sources I used are:

1) My course notes *Basic Category Theory and Topos Theory* ([8]), material for the lecture course to which the present course is a sequel. Referred to in the text as the *Basic Course*.

2) MacLane's *Categories for the Working Mathematician* ([5]). Referred to as "MacLane".

3) Peter Johnstone's *Topos Theory* ([3]). This is referred to in the text by **PTJ**.

4) MacLane and Moerdijk's *Sheaves in Geometry and Logic* ([6]). Referred to by **MM**.

5) Francis Borceux's *Handbook of Categorical Algebra* ([1]).

6) Peter Johnstone's *Sketches of an Elephant* ([4]). Referred to by **Elephant**.

7) Moerdijk's *Classifying Spaces and Classifying Topoi* ([7]).

8) Olivia Caramello's *Theories, Sites, Toposes* ([2]).

9) Jaap van Oosten's *Realizability: an Introduction to its Categorical Side* ([9]).

There is no original material in the text, except for a few exercises and some proofs.

Conventions: in a categorical product, the projections are usually denoted by $p_0$ and $p_1$, so $p_1$ is the *second* projection.

## 0.1 The plural of the word "topos"

Everyone knows the quip at the end of the Introduction of [3], which asks *those toposophers who persist in talking about topoi whether, when they go out for a ramble on a cold day, they carry supplies of hot tea with them in thermoi.* Since then, everyone has to declare what, in his or her view, is the plural of "topos". The form "topoi", of course, is the plural of the ancient Greek word for “place”. However, Topology is not the science of places, and the name Topology is what inspired Grothendieck to introduce the word Topos.

Someone (I forget who) proposed: the word “topos” is *French*, and its plural is “topos”. True, but English has adopted many French words, which are then treated as English words. The French plural of “bus” is “bus”, but in English it is “buses”.

I stick with “toposes”.

# Contents

# Introduction

We start by recalling some basic definitions from the course *Category Theory and Topos Theory*, which is a prerequisite for this course. For motivation, we start by exhibiting the elementary notions at work in the example of sheaves on a topological space. Later in this introductory chapter, we review a few definitions and results from Category Theory.

**Definition 0.1** A *topos* is a category with finite limits which is cartesian closed and has a *subobject classifier*. A subobject classifier is an arrow $t : 1 \to \Omega$ such that every monomorphism is a pullback of $t$ in a unique way: for every mono $m : X \to Y$ there is a unique arrow $\chi_m : Y \to \Omega$ (the *classifying map*, or *characteristic map* of $m$) such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "m"'] & 1 \arrow[d, "t"] \\ Y \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

is a pullback.

In the introductory course we have seen that the category $\widehat{\mathcal{C}}$ of presheaves on $\mathcal{C}$ (for a small category $\mathcal{C}$) is a topos, and if $J$ is a Grothendieck topology on $\mathcal{C}$ then the full subcategory of $\widehat{\mathcal{C}}$ on the sheaves for $J$, usually denoted $\mathrm{Sh}(\mathcal{C}, J)$, is also a topos. The pair $(\mathcal{C}, J)$ is called a *site*; and a topos of the form $\mathrm{Sh}(\mathcal{C}, J)$ is called a *Grothendieck topos*.

Let us see a concrete example, in order to illustrate some of the themes which are important in Topos Theory.

## 0.2 Sheaves on Spaces

Given a topological space $X$ with set of opens $\mathcal{O}_X$, we view $\mathcal{O}_X$ as a (posetal) category, and form the topos $\widehat{\mathcal{O}_X}$ of *presheaves on* $X$ (as it is usually called). For an open $U \subseteq X$, a sieve on $U$ can be identified with a set $S$ of open subsets of $U$ which is *downwards closed*: if $V \subseteq W \subseteq U$ and $W \in S$, then also $V \in S$. There is a very natural and straightforward Grothendieck topology on $\mathcal{O}_X$: declare a sieve $S$ on $U$ to be covering if $\bigcup S = U$. The category of sheaves on $\mathcal{O}_X$ for this Grothendieck topology is simply called the category of *sheaves on* $X$, and denoted $\mathrm{Sh}(X)$.

Let $F$ be a presheaf on $X$; an element $s \in F(U)$ is called a *local section of $F$ at $U$*. For the action of $F$ on local sections, that is: $F(V \subseteq U)(s) \in F(V)$

(where $V$ is a subset of $U$ and the unique morphism from $V$ to $U$ is denoted by the inclusion), we write $s{\restriction}V$.

Now let $x$ be a point of the space $X$. We consider an equivalence relation on the set $\{(s,U)\,|\,x\in U, s\in F(U)\}$ of local sections defined at $x$, by stipulating: $(s,U)\sim_x (t,V)$ iff there is some neighbourhood $W$ of $x$ such that $W\subseteq U\cap V$ and $s{\restriction}W = t{\restriction}W$. An equivalence class $[(s,U)]$ is called a *germ at* $x$ and is denoted $s_x$; the set of all germs at $x$ is $G_x$, the *stalk of* $x$.

Define a topology on the disjoint union $\coprod_{x\in X} G_x$ of all the stalks: a basic open set is of the form

$$\mathcal{O}^U_s \;=\; \{(y,s_y)\,|\,y\in U\}$$

for $U\in\mathcal{O}_X$ and $s\in F(U)$. This is indeed a basis: suppose $(x,g)\in\mathcal{O}^U_s\cap\mathcal{O}^V_t$. then $g=s_x=t_x$, so there is a neighbourhood $W$ of $x$ such that $W\subseteq V\cap U$ and $s{\restriction}W=t{\restriction}W$. We see that

$$(x,g)\in\mathcal{O}^W_{s\restriction W}\subseteq\mathcal{O}^U_s\cap\mathcal{O}^V_t.$$

We have a map $\pi:\coprod_{x\in X}G_x\to X$, sending $(x,g)$ to $x$. If $U\in\mathcal{O}_X$ and $(x,g)=(x,s_x)\in\pi^{-1}(U)$ then $(x,s_x)\in\mathcal{O}^U_s\subseteq\pi^{-1}(U)$, so $\pi$ is continuous. Moreover, $\pi(\mathcal{O}^U_s)=U$, so $\pi$ is an open map.

The map $\pi$ has another important property. Let $(x,g)=(x,s_x)\in\coprod_{x\in X}G_x$. Fix some $U$ such that $x\in U$ and $s\in F(U)$. The restriction of the map $\pi$ to $\mathcal{O}^U_s$ gives a bijection from $\mathcal{O}^U_s$ to $U$. Since this bijection is also continuous and open, it is a homeomorphism. We conclude that every element of $\coprod_{x\in X}G_x$ has a neighbourhood such that the restriction of the map $\pi$ to that neighbourhood is a homeomorphism. Such maps of topological spaces are called *local homeomorphisms*, or *étale maps*.

Let Top denote the category of topological spaces and continuous functions. For a space $X$ let $\mathrm{Top}/X$ be the slice category of maps into $X$, and let $\mathrm{Et}(X)$ be the full subcategory of $\mathrm{Top}/X$ on the local homeomorphisms into $X$. We have the following theorem in sheaf theory:

**Theorem 0.2** *The categories* $\mathrm{Et}(X)$ *and* $\mathrm{Sh}(X)$ *are equivalent.*

**Proof**. [Outline] For an étale map $p:Y\to X$, define a presheaf $\mathcal{F}$ on $X$ by putting:

$$\mathcal{F}(U)\;=\;\{s:U\to Y\,|\,s \text{ continuous and } ps=\mathrm{id}_U\}.$$

This explains the terminology *local sections*. Then $\mathcal{F}$ is a sheaf on $X$. Conversely, given a sheaf $F$ on $X$, define the corresponding étale map as the map $\pi:\coprod_{x\in X}G_x\to X$ constructed above. These two operations are, up to isomorphism in the respective categories, each other's inverse. ∎

**Exercise 1** Show that for a presheaf $F$ and the associated local homeomorphism $\pi : \coprod_{x\in X} G_x \to X$ that we have constructed, the following holds: every morphism of presheaves $F \to H$, where $H$ is a sheaf, factors uniquely through the sheaf corresponding to $\pi : \coprod_{x\in X} G_x \to X$. Conclude that $\pi : \coprod_{x\in X} G_x \to X$ is the associated sheaf of $F$.

Next, let us consider the effect of continuous maps on categories of sheaves. First of all, given a continuous map $\phi : Y \to X$ we have the inverse image map $\phi^{-1} : \mathcal{O}_X \to \mathcal{O}_Y$ and hence a functor

$$\phi_* = \mathrm{Set}^{(\phi^{-1})^{\mathrm{op}}} : \widehat{\mathcal{O}_Y} \to \widehat{\mathcal{O}_X}$$

and the functor $\phi_*$ restricts to a functor $\mathrm{Sh}(Y) \to \mathrm{Sh}(X)$.

There is also a functor in the other direction: given a sheaf $F$ on $X$, let $\mathcal{F} \to X$ be the corresponding étale map. It is easy to verify that étale maps are stable under pullback, so if

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ Y \arrow[r, "\phi"'] & X \end{tikzcd}$$

is a pullback diagram in Top, let $\phi^*(F)$ be the sheaf on $Y$ which corresponds to the local homeomorphism $\mathcal{G} \to Y$. This defines a functor $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$.

**Proposition 0.3** *We have an adjunction $\phi^* \dashv \phi_*$; moreover, the left adjoint $\phi^*$ preserves finite limits.*

**Definition 0.4** Let $\mathcal{E}$ and $\mathcal{F}$ be toposes. A *geometric morphism*: $\mathcal{F} \to \mathcal{E}$ consists of functors $f_* : \mathcal{F} \to \mathcal{E}$ and $f^* : \mathcal{E} \to \mathcal{F}$ satisfying: $f^* \dashv f_*$ and $f^*$ preserves finite limits. The functor $f_*$ is called the *direct image functor* of the geometric morphism, and $f^*$ the *inverse image functor*.

It is clear that Definition 0.4 gives us a category $\mathcal{T}op$ of toposes and geometric morphisms, and the treatment of categories of sheaves on spaces shows that we have a functor Top $\to \mathcal{T}op$ from topological spaces to toposes. This functor allows us to relate topological properties of a space to category-theoretic properties of its associated topos of sheaves.

Another example of geometric morphism that we have seen, is the one $\mathrm{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$, where the direct image is inclusion as subcategory, and the inverse image is sheafification.

Other examples of geometric morphisms we shall meet during this course, are:

i) Any functor $F : \mathcal{C} \to \mathcal{D}$ between small categories gives rise to a geometric morphism $\widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$.

ii) If $\mathcal{E}$ is a topos and $X$ is an object of $\mathcal{E}$, then the slice category $\mathcal{E}/X$ is a topos; and if $f : X \to Y$ is an arrow in $\mathcal{E}$ then we will have a geometric morphism $\mathcal{E}/X \to \mathcal{E}/Y$.

iii) If $\mathcal{E}$ is a topos and $H : \mathcal{E} \to \mathcal{E}$ is a finite-limit preserving comonad on $\mathcal{E}$, then the category $\mathcal{E}_H$ of coalgebras for $H$ in $\mathcal{E}$ is a topos, and there is a geometric morphism $\mathcal{E} \to \mathcal{E}_H$.

There is another important notion of "morphism between toposes": logical functors.

**Definition 0.5** A *logical functor* between toposes is a functor which preserves the topos structure, that is: finite limits, exponentials and the subobject classifier.

## 0.3 Notions from Category Theory

First, let us deal with a subtlety we overlooked in the *Category Theory and Topos Theory* course. There, we said (following MacLane) that a functor $F : \mathcal{C} \to \mathcal{D}$ *creates limits of type* $J$ if for every diagram $M : J \to \mathcal{C}$ and every limiting cone $(D, \mu)$ for $FM$ in $\mathcal{D}$, there is a *unique* cone $(C, \nu)$ for $M$ in $\mathcal{C}$ which is mapped by $F$ to $(D, \mu)$, and moreover the cone $(C, \nu)$ is a limiting cone for $M$.

For an adjunction $F \dashv G : \mathcal{C} \to \mathcal{D}$ (so $G : \mathcal{C} \to \mathcal{D}, F : \mathcal{D} \to \mathcal{C}$) we have a comparison functor $K : \mathcal{C} \to \mathcal{D}^{GF}$, where $\mathcal{D}^{GF}$ is the category of algebras for the monad $GF$ on $\mathcal{D}$. MacLane, consistently, defines the functor $G$ to be *monadic* if $K$ is an isomorphism of categories. It follows that every monadic functor creates limits.

However, we defined the functor $G$ to be monadic if $K$ is an *equivalence*. And whilst the forgetful functor $U^T : \mathcal{C}^T \to \mathcal{C}$ always creates limits (here $\mathcal{C}^T$ denotes the category of algebras for a monad $T$), with the strict definition we gave this is no longer guaranteed if $U^T$ is composed with an equivalence of categories. Yet, there are good reasons to consider "monadic" functors where the comparison is only an equivalence, and we would like to have a "creation of limits" definition which is stable under equivalence. For example, the "Crude Tripleability Theorem" (0.10) below only ensures an equivalence with the category of algebras.

**Definition 0.6 (Creation of Limits)** A functor $F : \mathcal{C} \to \mathcal{D}$ creates limits of type $J$ if for any diagram $M : J \to \mathcal{C}$ and any limiting cone $(X, \mu)$ for $FM$ in $\mathcal{D}$ the following hold:

i) There exists a cone $(Y, \nu)$ for $M$ in $\mathcal{C}$ such that its $F$-image is isomorphic to $(X, \mu)$ (in the category of cones for $M$).

ii) Any cone $(Y, \nu)$ for $M$ which is mapped by $F$ to a cone isomorphic to $(X, \mu)$, is limiting.

We say that the functor $F$ *creates limits* if $F$ creates limits of every small type $J$.

For the record:

**Theorem 0.7** *Let $\mathcal{C} \overset{G}{\to} \mathcal{D}$ monadic. Then $G$ creates limits.*

The following remark appears on the first pages of Johnstone's *Sketches of an Elephant*, and is very useful.

**Remark 0.8 (Elephant A1.1.1)** Let $A \underset{U}{\overset{F}{\leftrightarrows}} C$ be an adjunction with $F \dashv U$. If there is a natural isomorphism between $FU$ and the identity on $A$, then the counit is a natural isomorphism. Of course, by duality a similar statement holds for units.

**Definition 0.9** A parallel pair of arrows $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is a *reflexive pair* if $f$ and $g$ have a common section: a morphism $s : Y \to X$ for which $fs = gs = \mathrm{id}_Y$. A category is said to have *coequalizers of reflexive pairs* if for every reflexive pair the coequalizer exists.

**Theorem 0.10 (Beck's "Crude Tripleability Theorem")** *Let*

$$A \underset{U}{\overset{F}{\leftrightarrows}} C$$

*be an adjunction with $F \dashv U$; let $T = UF$ be the induced monad on $C$. Suppose that $A$ has coequalizers of reflexive pairs, that $U$ preserves them, and moreover that $U$ reflects isomorphisms. Then the functor $U$ is monadic.*

**Proof**. We start by constructing a left adjoint $L$ to the functor $K$. Recall that $K : A \to C^T$ sends an object $Y$ of $A$ to the $T$-algebra $UFUY \overset{U(\varepsilon_Y)}{\to} UY$.

Let $UFX \xrightarrow{h} X$ be a $T$-algebra. We have that $\eta_X$ is a section of $h$ by the axioms for an algebra, and $F(\eta_X)$ is a section of $\varepsilon_{FX}$ by the triangular identities for an adjunction. So the parallel pair

$$FUFX \underset{\varepsilon_{FX}}{\overset{F(h)}{\rightrightarrows}} FX$$

is reflexive with common section $F(\eta_X)$; let $FX \xrightarrow{e} E$ be its coequalizer. We define $L(h)$ to be the object $E$. Clearly, this is functorial in $h$.

Let us prove that $KL(h)$ is isomorphic to $h$. Note that the underlying object of the $T$-algebra $KL(h)$ is $UE$. By construction of $L(h)$ and the assumptions on $U$, the diagram

$$UFUFX \underset{U(\varepsilon_{FX})}{\overset{UF(h)}{\rightrightarrows}} UFX \xrightarrow{U(e)} UE$$

is a coequalizer. By the associativity of the algebra $h$, the map $h$ coequalizes the pair $(UF(h), U(\varepsilon_{FX}))$; so we have a unique $\xi : UE \to X$ satisfying

$$\xi \circ U(e) = h.$$

We also have the map $U(e) \circ \eta_X : X \to UE$. It is routine to check that these maps are each other's inverse, as well as that $\xi$ is in fact an algebra map. This shows that $KL(h)$ is naturally isomorphic to $h$.

Let us show that $L \dashv K$. Maps in $A$ from $E = L(h)$ to an object $Y$ correspond, by the coequalizer property of $E$, to arrows $f : FX \to Y$ satisfying $f \circ F(h) = f \circ \varepsilon_{FX}$. Transposing along the adjunction $F \dashv U$, these correspond to maps $\bar{f} : X \to UY$ satisfying $\bar{f} \circ h = U(\varepsilon_Y) \circ UF(\bar{f})$; that is, to $T$-algebra maps from $h$ to $K(Y)$. This establishes the adjunction and applying Johnstone's remark 0.8 we conclude that the unit of the adjunction is an isomorphism.

In order to show that also the counit of $L \dashv K$ is an isomorphism, we recall that for an object $Y$ of $A$, $LK(Y)$ is the vertex of the coequalizer diagram

$$FUFUY \underset{\varepsilon_{FUY}}{\overset{FU(\varepsilon_Y)}{\rightrightarrows}} FUY \xrightarrow{w} W$$

Since also $\varepsilon_Y$ coequalizes the parallel pair, we have a unique map $W \xrightarrow{v} Y$ satisfying $vw = \varepsilon_Y$. Itis now not too hard to prove that $U(v)$ is an isomorphism; since $U$ reflects isomorphisms, $v$ is an isomorphism, and we are done. ∎

The following theorem is called "Adjoint lifting theorem".

**Theorem 0.11 (Adjoint Lifting Theorem; PTJ 0.15)** *Let $T$ and $S$ be monads on categories $\mathcal{C}$ and $\mathcal{D}$ respectively. Suppose we have a commutative diagram of functors*

$$\begin{tikzcd}
\mathcal{C}^T \arrow[r, "\bar{F}"] \arrow[d, "U^T"'] & \mathcal{D}^S \arrow[d, "U^S"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}$$

*where $U^T, U^S$ are the forgetful functors. Suppose $F$ has a left adjoint $L$. Moreover, assume that the category $\mathcal{C}^T$ has coequalizers of reflexive pairs. Then the functor $\bar{F}$ also has a left adjoint.*

**Proof**. *[Sketch]* Let $(T, \eta, \mu)$ and $(S, \iota, \nu)$ be the respective monad structures on $T$ and $S$. Our first remark is that every $S$-algebra is a coequalizer of a reflexive pair of arrows between free $S$-algebras. For an $S$-algebra $SX \xrightarrow{h} X$, consider the parallel pair

$$\begin{tikzcd}
S^2X \arrow[r, shift left, "Sh"] \arrow[r, shift right, "\nu_X"'] & SX
\end{tikzcd}$$

This is a diagram of algebra maps $F^S(SX) \to F^S(X)$: $\nu_X S^2 h = Sh\nu_{SX}$ by naturality of $\nu$, and $\nu_X \nu_{SX} = \nu_X S(\nu_X)$ by associativity of $\nu$. The two arrows have a common splitting $S(\iota_X)$ which is also an algebra map since it is $F^S(\iota_X)$. That is: we have a reflexive pair in $S$-Alg. It is easy to see that $h : SX \to X$ coequalizes this pair: this is the associativity of $h$ as an algebra. If $a : F^S(X) \to (\xi : SY \to Y)$ is an algebra map which coequalizes our reflexive pair then $a$ factors through $h : F^S(X) \to (h : SX \to X)$ by $a\iota_X : (SX \xrightarrow{h} X)_{\to}(SY \xrightarrow{\xi} Y))$ and the factorization is unique because the arrow $h$ is split epi in $\mathcal{C}$.

This construction is functorial. Given an $S$-algebra map $f : (SX \xrightarrow{h} X) \to (SY \xrightarrow{k} Y)$ the diagram

$$\begin{tikzcd}
S^2X \arrow[r, shift left, "\nu_X"] \arrow[r, shift right, "Sh"'] \arrow[d, "S^2f"'] & SX \arrow[d, "Sf"] \\
S^2Y \arrow[r, shift left, "\nu_Y"] \arrow[r, shift right, "Sk"'] & SY
\end{tikzcd}$$

commutes serially (i.e., $Sf\nu_X = \nu_Y S^2 f$ and $SfSh = SkS^2f$). So, we have a functor $R$ from $S$-Alg to the category of diagrams of shape $\circ \rightrightarrows \circ$ in $S$-Alg, with the properties:

i) The vertices of $R(h)$ are free algebras.

ii) $R(h)$ is always a reflexive pair.

iii) The colimit of $R(h)$ is $h$.

Our second remark is that since $\bar{F}$ is a lifting of $F$ ($U^S\bar{F} = FU^T$) there is a natural transformation $\lambda : SF \to FT$ constructed as follows. Consider $F(\eta) : F \to FT = FU^TF^T = U^S\bar{F}F^T$ and let $\tilde{\lambda} : F^SF \to \bar{F}F^T$ be its transpose along $F^S \dashv U^S$. Define $\lambda$ as the composite

$$SF = U^SF^SF \xrightarrow{U^S\tilde{\lambda}} U^S\bar{F}F^T = FU^TF^T = FT.$$

*Claim*: The natural transformation $\lambda$ makes the following diagram commute:

$$\begin{tikzcd}
F \arrow[r, "\iota_F"] \arrow[dr, "F(\eta"'] & SF \arrow[d, "\lambda"'] & S^2F \arrow[l, "\nu_F"'] \arrow[d, "S\lambda"] \\
& FT & SFT \arrow[d, "\lambda T"] \\
& & FT^2 \arrow[ul, "F\mu"]
\end{tikzcd}$$

Now we are ready for the definition of $\bar{L}$ on objects: if $\bar{L}$ is going to be left adjoint to $\bar{F}$ then, by uniqueness of adjoints and the fact that adjoints compose, $\bar{L}F^S = F^TL$, so we know what $\bar{L}$ should do on free $S$-algebras $F^SY$. Now every $S$-algebra $\xi : SY \to Y$ is coequalizer of a reflexive pair of arrows between free $S$-algebras, and as a left adjoint, $\bar{L}$ should preserve coequalizers. Therefore we expect $\bar{L}(\xi)$ to be coequalizer of a reflexive pair

$$F^TLSY = \bar{L}F^S(SY) \overset{f_\xi}{\underset{g_\xi}{\rightrightarrows}} \bar{L}F^S(Y) = F^TLY$$

between free $T$-algebras. It is now our task to determine $f_\xi$ and $g_\xi$.

By our first remark we have a coequalizer

$$F^S(SY) \overset{S\xi}{\underset{\nu_Y}{\rightrightarrows}} F^SY \xrightarrow{\xi} (\xi)$$

and the topmost arrow of the reflexive pair is in the image of the functor $F^S$, so we can take $F^TL(\xi)$ for $f_\xi$. The other map – $\nu$ – is not in the image of $F^S$ and needs a bit of doctoring using the adjunction $L \dashv F$ and the natural transformation $\lambda$ we constructed. Let $\alpha$ be the unit of the adjunction $L \dashv F$. Consider the arrow

$$SY \overset{S(\alpha_Y)}{\longrightarrow} SFL(Y) \overset{\lambda_{L(Y)}}{\longrightarrow} FTL(Y)$$

This transposes under $L \dashv F$ to a map $LS(Y) \to TL(Y) = U^T F^T L(Y)$, and this in turn transposes under $F^T \dashv U^T$ to a map

$$F^T LS(Y) \to F^T L(Y)$$

which we take as our $g_\xi$.

Note that the construction is natural in $\xi$, so if $k : \xi \to \zeta$ is a map of $S$-algebras, we obtain a natural transformation from the diagram of parallel arrows $f_\xi, g_\xi$ to the diagram with parallel arrows $f_\zeta, g_\zeta$. Hence we also get a map from the coequalizer of the first diagram, which is $\bar{L}(\xi)$, to the coequalizer of the second one, which is $\bar{L}(\zeta)$. And this map between coequalizers will be $\bar{L}(k)$.

There is still a lot to check. This is meticulously done in Volume 2 of Borceux's *Handbook of Categorical Algebra*, section 4.5. There the proof takes 10 pages. ∎

**Remark 0.12** There is a better theorem than the one we just partially proved: the *Adjoint Triangle Theorem*. It says that whenever we have functors $\mathcal{B} \overset{R}{\to} \mathcal{C} \overset{U}{\to} \mathcal{D}$ such that $\mathcal{B}$ has reflexive coequalizers and $U$ is *of descent type* (that is: $U$ has a left adjoint $J$ and the comparison functor $K : \mathcal{C} \to UJ{-}\mathrm{Alg}$ is full and faithful), then $UR$ has a left adjoint if and only if $R$ has one.

Note, that given the diagram of Theorem 0.11, the diagram

$$\mathcal{C}^T \overset{\bar{F}}{\to} \mathcal{D}^S \overset{U^S}{\to} \mathcal{D}$$

satisfies the conditions of the Adjoint Triangle Theorem. Since the composition $U^S\bar{F}$, which is $F^T L$, has a left adjoint, we conclude that $\bar{F}$ has a left adjoint. Note in particular that we do not use that $\mathcal{C}^T$ is monadic.

**Definition 0.13** A diagram $a \overset{f}{\underset{g}{\rightrightarrows}} b \overset{h}{\longrightarrow} c$ in a category is called a *split fork* if $hf = hg$ and there exist maps

$$a \overset{t}{\longleftarrow} b \overset{s}{\longleftarrow} c$$

such that $hs = \mathrm{id}_c$, $ft = \mathrm{id}_b$ and $gt = sh$.

**Exercise 2** Show that every split fork is a coequalizer diagram, and moreover a coequalizer which is preserved by any functor (this is called an *absolute* coequalizer).

**Exercise 3** Suppose $D_1$ is the diagram $a \underset{g}{\overset{f}{\rightrightarrows}} b \xrightarrow{h} c$ in a category $\mathcal{C}$, and $D_2$ is the diagram $a' \underset{g'}{\overset{f'}{\rightrightarrows}} b' \xrightarrow{h'} c'$ in $\mathcal{C}$. Assume that $D_2$ is a retract of $D_1$ in the category of diagrams in $\mathcal{C}$ of type $\bullet \rightrightarrows \bullet \longrightarrow \bullet$. Prove that if $D_1$ is a split fork, then so is $D_2$.

**Definition 0.14** In a category, a family of arrows $\{f_i : A_i \to B \,|\, i \in I\}$ is called epimorphic if for every parallel pair of arrows $u, v : B \to C$ the following holds: if $uf_i = vf_i$ for all $i \in I$, then $u = v$.

**Exercise 4** If the ambient category has $I$-indexed coproducts, a family $\{f_i : A_i \to B \,|\, i \in I\}$ is epimorphic if and only if the induced arrow from the coproduct $\sum_{i \in I} A_i$ to $B$ is an epimorphism.

We shall also have to deal with *comonads*; a comonad on a category $\mathcal{C}$ is a monad on $\mathcal{C}^{\mathrm{op}}$. Explicitly, we have a functor $G : \mathcal{C} \to \mathcal{C}$ with natural transformations $\varepsilon : G \Rightarrow \mathrm{id}_{\mathcal{C}}$ (the "counit")) and $\delta : G \Rightarrow G^2$ (the "comultiplication") which make the following (coassociativity and counitarity) diagrams commute:

$$
\begin{tikzcd}
G \arrow[r, "\delta"] \arrow[d, "\delta"'] & G^2 \arrow[d, "\delta G"] \\
G^2 \arrow[r, "G\delta"'] & G^3
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G \arrow[d, "\delta"] & \\
G \arrow[ur, "\mathrm{id}"] & G^2 \arrow[l, "\varepsilon G"] \arrow[r, "G\varepsilon"'] & G \arrow[ul, "\mathrm{id}"']
\end{tikzcd}
$$

Dual to the treatment for monads, we have the category $G$-Coalg of $G$-coalgebras, the notion of a functor being "comonadic", etcetera. We have the forgetful functor $V : G{-}\mathrm{Coalg} \to \mathcal{C}$ which has a *right* adjoint $C : \mathcal{C} \to G{-}\mathrm{Coalg}$, the "cofree coalgebra functor". Without proof we record the following theorem:

**Theorem 0.15 (Eilenberg-Moore; MM V.8.1-2; PTJ 0.14)** *Suppose $T$ is a monad on a category $\mathcal{C}$, such that the functor $T$ has a right adjoint $G$. Then there is a unique comonad structure $(\varepsilon, \delta)$ on $G$ such that the categories*

*$T$-Alg and $G$-Coalg are isomorphic by an isomorphism which commutes with the forgetful functors:*

$$\begin{tikzcd}
T-\mathrm{Alg} \arrow[rr, "L"] \arrow[dr, "U"'] & & G-\mathrm{Coalg} \arrow[dl, "V"] \\
& \mathcal{C} &
\end{tikzcd}$$

**Corollary 0.16** *If $(T, \eta, \mu)$ is a monad on $\mathcal{C}$ and the functor $T$ has a right adjoint $G$, then the forgetful functor $T - \mathrm{Alg} \to \mathcal{C}$ has both a left and a right adjoint.*

# 1 Elementary Toposes

In this chapter we are going through Chapter 1 of P.T. Johnstone's *Topos Theory*, expanding the proofs a bit when necessary.

**Example 1.1** Let $\mathcal{G}$ be a group. In the topos $\widehat{\mathcal{G}}$ of right $\mathcal{G}$-sets (sets $X$ with $\mathcal{G}$-action $X \times \mathcal{G} \to X$, written $(x, g) \mapsto x{\cdot}g$) we have:

i) the subobject classifier $1 \xrightarrow{t} \Omega$ is the map from $\{*\}$ to $\{0, 1\}$ which sends $*$ to 1; here $\{0, 1\}$ has the trivial $\mathcal{G}$-action.

ii) The exponent $Y^X$ of two $\mathcal{G}$-sets $X$ and $Y$ is the set of *all* functions $X \xrightarrow{\phi} Y$, with $\mathcal{G}$-action:

$$(\phi{\cdot}g)(x) \;=\; (\phi(x{\cdot}g^{-1})){\cdot}g$$

We see at once that the forgetful functor $\widehat{\mathcal{G}} \to \mathrm{Set}$ is logical, as is the functor $\mathrm{Set} \to \widehat{\mathcal{G}}$ which sends a set $X$ to the set $X$ with trivial $\mathcal{G}$-action.

We can also consider the category $\mathrm{Set}_{\mathrm{f}}^{\mathcal{G}^{\mathrm{op}}}$ of *finite* $\mathcal{G}$-sets; and we see that this is also a topos (even if $\mathcal{G}$ itself is not finite); the inclusion functor $\mathrm{Set}_{\mathrm{f}}^{\mathcal{G}^{\mathrm{op}}} \to \widehat{\mathcal{G}}$ is logical.

**Lemma 1.2 (PTJ 1.21)** *In a topos, every mono is regular.*

**Proof**. Every mono is a pullback of $1 \xrightarrow{t} \Omega$, and $t$ is split mono, so regular. ∎

**Corollary 1.3 (PTJ 1.22)** *Every map in a topos which is both epi and mono is an isomorphism (one says that a topos is* balanced*).*

**Definition 1.4** In a category with finite limits, an *equivalence relation on an object* $X$ is a subobject $R$ of $X \times X$ for which the following statements hold:

i) The diagonal embedding $X \to X \times X$ factors through $R$.

ii) The composition $R \to X \times X \xrightarrow{\mathrm{tw}} X \times X$ factors through $R$, where tw denotes the twist map

$$\langle p_1, p_0 \rangle : X \times X \to X \times X.$$

(Here $p_0, p_1 : X \times X \to X$ are the projections)

iii) The map $\langle p_0 s, p_1 t\rangle : R' \to X \times X$ factors through $R$, where we assume that the subobject $R$ is represented by the arrow $\langle r_0, r_1\rangle : R \to X \times X$, and the maps $s$ and $t$ are defined by the pullback diagram

$$\begin{tikzcd} R' \arrow[r, "t"] \arrow[d, "s"'] & R \arrow[d, "r_0"] \\ R \arrow[r, "r_1"'] & X \times X \end{tikzcd} .$$

The subobject $R'$ is the "object of $R$-related triples".

Equivalently, a subobject $R$ of $X \times X$ is an equivalence relation on $X$ if and only if for every object $Y$, the relation

$$\{(f, g) \mid \langle f, g\rangle : Y \to X \times X \text{ factors through } R\}$$

is an equivalence relation on the set of arrows $Y \to X$.

Clearly, for every arrow $f : X \to Y$, the kernel pair of $f$, seen as a subobject of $X \times X$, is an equivalence relation on $X$. Equivalence relations which are kernel pairs are called *effective* (don't ask me why).

**Proposition 1.5 (PTJ 1.23)** *In a topos, every equivalence relation is effective, i.e. a kernel pair.*

**Proof**. Let $\phi : X \times X \to \Omega$ classify the subobject $\langle r_0, r_1\rangle : R \to X \times X$, and let $\bar{\phi} : X \to \Omega^X$ be its exponential transpose (in Set, $\bar{\phi}(x)$ will be the $R$-equivalence class of $x$). We claim that the square

$$\begin{tikzcd} R \arrow[r, "r_1"] \arrow[d, "r_0"'] & X \arrow[d, "\bar{\phi}"] \\ X \arrow[r, "\bar{\phi}"'] & \Omega^X \end{tikzcd}$$

is a pullback, so that $R$ is the kernel pair of $\bar{\phi}$. To see that it commutes, we look at the transposes of the compositions $\bar{\phi} r_i$, which are maps

$$R \times X \xrightarrow{r_i \times \mathrm{id}} X \times X \xrightarrow{\phi} \Omega$$

Both these maps classify the object $R'$ of $R$-related triples, seen as subobject of $R \times X$, so they are equal. To see that the given diagram is a pullback, suppose we have maps $f, g : U \to X$ satisfying $\bar{\phi} f = \bar{\phi} g$. Then $\phi(f \times \mathrm{id}_X) =$

$\phi(g \times \mathrm{id}_X) : U \times X \to \Omega$. Composing with the map $\langle \mathrm{id}_U, g\rangle : U \to U \times X$ we get that the square

$$\begin{tikzcd} U \arrow[r, "\langle f,g\rangle"] \arrow[d, "\langle g,g\rangle"'] & X \times X \arrow[d, "\phi"] \\ X \times X \arrow[r, "\phi"'] & \Omega \end{tikzcd}$$

commutes. Now $\phi$ classifies $R$ and by reflexivity of $R$ the map $\langle g, g\rangle$ factors through $R$, so $\phi\langle g, g\rangle$ is the composite map $U \xrightarrow{!} 1 \xrightarrow{t} \Omega$; so this also holds for the other composite and therefore also $\langle f, g\rangle$ must factor through $R$, which says that the given diagram is indeed a pullback. ∎

The least equivalence relation on an object $X$ is the *diagonal* $\delta = \langle \mathrm{id}_X, \mathrm{id}_X\rangle : X \to X \times X$. It is classified by some $\Delta : X \times X \to \Omega$; let $\{\cdot\} : X \to \Omega^X$ be the exponential transpose of $\Delta$. The map $\{\cdot\}$ is of course thought of as the *singleton map* from $X$ to its power object.

**Exercise 5** Prove that the map $\{\cdot\}$ is monic. [Hint: for an arrow $f : Y \to X$, the composite $\{\cdot\}\circ f$ transposes to a map which classifies the graph of $f$ (as subobject of $Y \times X$).]

**Definition 1.6** A *partial map* from $X$ to $Y$ is an arrow from a subobject of $X$ to $Y$. More precisely, it is a diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "m"'] & Y \\ X & \end{tikzcd}$$

with $m$ mono.

We write $f : X \rightharpoonup Y$ to emphasize that the map is partial.

We say that *partial maps are representable* if for each object $Y$ there is a monomorphism $\eta_Y : Y \to \tilde{Y}$ with the property that for every partial map $X \xleftarrow{m} U \xrightarrow{f} Y$ from $X$ to $Y$ there is a *unique* arrow $\tilde{f} : X \to \tilde{Y}$, making the square

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "m"'] & Y \arrow[d, "\eta_Y"] \\ X \arrow[r, "\tilde{f}"'] & \tilde{Y} \end{tikzcd}$$

a pullback.

Let us spell out what this means for $Y = 1$: we have an arrow $\eta : 1 \to \tilde{1}$ such that for every mono $m : U \to X$ there is a unique map $X \to \tilde{1}$ making the square

$$\begin{tikzcd} U \arrow[r] \arrow[d, "m"'] & 1 \arrow[d, "\eta"] \\ X \arrow[r] & \tilde{1} \end{tikzcd}$$

a pullback. But this is just the definition of a subobject classifier; we conclude that $1 \xrightarrow{\eta} \tilde{1}$ is $1 \xrightarrow{t} \Omega$.

**Theorem 1.7 (PTJ 1.26)** *In a topos, partial maps are representable.*

**Proof**. Let $\phi : \Omega^Y \times Y \to \Omega$ classify the graph of the singleton map:

$$\begin{tikzcd} Y \arrow[r, "\langle\{\cdot\},\mathrm{id}\rangle"] \arrow[d] & \Omega^Y \times Y \arrow[d, "\phi"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$

and let $\bar{\phi} : \Omega^Y \to \Omega^Y$ be its exponential transpose.

Let

$$E \xrightarrow{e} \Omega^Y \underset{\mathrm{id}}{\overset{\bar{\phi}}{\rightrightarrows}} \Omega^Y$$

be an equalizer. We shall show that we can take $E$ for $\tilde{Y}$. Think of $E$ as the "set"

$$\{\alpha \subseteq Y \mid \forall y(y \in \alpha \leftrightarrow \alpha = \{y\})\},$$

that is: the set of subsets of $Y$ having at most one element. We consider the pullback diagram

$$\begin{tikzcd} Y \arrow[r, "\mathrm{id}"] \arrow[d, "\delta"'] & Y \arrow[d, "\langle\{\cdot\},\mathrm{id}\rangle"] \\ Y \times Y \arrow[r, "\{\cdot\}\times\mathrm{id}_Y"'] & \Omega^Y \times Y \end{tikzcd}$$

Composing this with the diagram defining $\phi$, we obtain pullbacks

$$\begin{tikzcd}
Y \arrow[r, "\delta"] \arrow[d, "\mathrm{id}"'] & Y \times Y \arrow[d, "\{\cdot\}\times\mathrm{id}_Y"] \\
Y \arrow[r, "\langle\{\cdot\},\mathrm{id}_Y\rangle"] \arrow[d] & \Omega^Y \times Y \arrow[d, "\phi"] \\
1 \arrow[r, "t"'] & \Omega
\end{tikzcd}$$

from which we conclude that $\phi(\{\cdot\} \times \mathrm{id}_Y)$ classifies the diagonal map on $Y$; hence its exponential transpose, which is $\bar{\phi}\circ\{\cdot\} : Y \to \Omega^Y$, is equal to $\{\cdot\}$. Therefore the map $\{\cdot\} : Y \to \Omega^Y$ factors through the equalizer $E$ above; so we have the required map $Y \to E = \tilde{Y}$ (which is monic since $\{\cdot\}$ is).

In order to show that the constructed mono $Y \to \tilde{Y}$ indeed represents partial maps into $Y$, let

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "m"'] & Y \\
X &
\end{tikzcd}$$

be a partial map $X \rightharpoonup Y$, so $m$ is monic. Consider the graph of $f$: $U \xrightarrow{\langle m,f\rangle} X \times Y$. It is classified by a map $\psi : X \times Y \to \Omega$; let $\bar{\psi} : X \to \Omega^Y$ be the exponential transpose of $\psi$. We have a commutative diagram

$$(*) \qquad \begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "\langle m,f\rangle"'] & Y \arrow[d, "\langle\{\cdot\},\mathrm{id}\rangle"] \\
X \times Y \arrow[r, "\bar{\psi}\times\mathrm{id}"'] \arrow[d] & \Omega^Y \times Y \arrow[d] \\
X \arrow[r, "\bar{\psi}"'] & \Omega^Y
\end{tikzcd}$$

The lower square is a pullback, so the outer square is a pullback if and only if the upper square is. We prove that the outer square is a pullback. Suppose $V \xrightarrow{a} X$, $V \xrightarrow{b} Y$ are maps such that $\{\cdot\}b = \bar{\psi}a$. Then by transposing, the square

$$\begin{tikzcd}
V \times Y \arrow[r, "b\times\mathrm{id}"] \arrow[d, "a\times\mathrm{id}"'] & Y \times Y \arrow[d, "\Delta"] \\
X \times Y \arrow[r, "\psi"'] & \Omega
\end{tikzcd}$$

commutes (recall that $\Delta$ classifies the diagonal $Y \to Y \times Y$). Composing with the map $V \xrightarrow{\langle \mathrm{id}, b\rangle} V \times Y$ gives

$$\begin{array}{rcccl} \psi\circ\langle a,b\rangle & = & \Delta\circ\langle b,b\rangle & = & \text{(by definition of } \Delta) \\ & & & = & V \to 1 \xrightarrow{t} \Omega \end{array}$$

So $\psi\circ\langle a,b\rangle$ factors through $t$, and since $\psi$ classifies the graph of $f$, the map $V \xrightarrow{\langle a,b\rangle} X \times Y$ factors through $U$; we conclude that the outer square of $(*)$ is indeed a pullback. Hence the upper square of $(*)$ is a pullback.

Now since

$$\begin{tikzcd} \Omega^Y \times Y \arrow[r, "\phi"] & \Omega \\ Y \arrow[u, "\langle\{\cdot\},\mathrm{id}\rangle"] \arrow[r] & 1 \arrow[u] \end{tikzcd}$$

is a pullback by definition of $\phi$, composing with the upper square of $(*)$ yields pullbacks

$$\begin{tikzcd} X \times Y \arrow[r, "\bar\psi\times\mathrm{id}"] & \Omega^Y \times Y \arrow[r, "\phi"] & \Omega \\ U \arrow[u, "\langle m,f\rangle"] \arrow[r] & Y \arrow[u, "\langle\{\cdot\},\mathrm{id}\rangle"'] \arrow[r] & 1 \arrow[u, "t"'] \end{tikzcd}$$

So the graph of $f$ is classified by $\phi\circ(\bar\psi \times \mathrm{id})$. It follows that $\phi\circ(\bar\psi \times \mathrm{id}) = \psi$, and by transposing we get $\bar\phi\bar\psi = \bar\psi : X \to \Omega^Y$. So $\bar\psi : X \to \Omega^Y$ factors through $\tilde Y \to \Omega^Y$ by a map $\tilde f : X \to \tilde Y$. The factorization is unique since $\tilde Y \to \Omega^Y$ is monic. Summarizing, we have

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dd, "m"'] & Y \arrow[d, "\eta_Y"] \\ & \tilde Y \arrow[d] \\ X \arrow[ur, "\tilde f"] \arrow[r, "\bar\psi"'] & \Omega^Y \end{tikzcd}$$

where the outer square is a pullback (it is the outer square of $(*)$), and since $\tilde Y \to \Omega^Y$ is monic the upper square is a pullback too.

From the uniqueness of $\tilde f$ we can prove that the assignment $Y \Rightarrow \tilde Y$, together with the maps $\eta_Y : Y \to \tilde Y$, gives a functor $\mathcal{E} \to \mathcal{E}$ (where $\mathcal{E}$

denotes the ambient topos): given a map $f : X \to Y$, let $\tilde{f} : \tilde{X} \to \tilde{Y}$ represent the partial map

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \tilde{X} \\ Y & \end{tikzcd}$$

By uniqueness we see that $\tilde{g}\tilde{f} = \widetilde{gf}$. We also see that $\eta$ is a natural transformation $\mathrm{id}_{\mathcal{E}} \Rightarrow \widetilde{(\cdot)}$. It has the special property that all naturality squares are pullbacks. ∎

An object of the form $\tilde{X}$ is called a *partial map classifier*.

**Proposition 1.8 (PTJ 1.27)** *The partial map classifiers $\tilde{Z}$ are injective.*

**Proof**. Given a diagram

$$\begin{tikzcd} X' & \\ X \arrow[u, "m"] \arrow[r, "f"'] & \tilde{Z} \end{tikzcd}$$

with $m$ mono, we need to find a map $X' \to \tilde{Z}$ making the triangle commute. To this end, form the pullback

$$\begin{tikzcd} X \arrow[r, "f"] & \tilde{Z} \\ Y \arrow[u, "n"] \arrow[r, "g"'] & Z \arrow[u, "\eta_Z"'] \end{tikzcd}$$

Let the partial map $X' \rightharpoonup Z$ given by $X' \xleftarrow{mn} Y \xrightarrow{g} Z$ be represented by $\tilde{g} : X' \to \tilde{Z}$. It is left to you to verify that the square

$$\begin{tikzcd} X \arrow[r, "\tilde{g}m"] & \tilde{Z} \\ Y \arrow[u, "n"] \arrow[r, "g"'] & Z \arrow[u, "\eta_Z"'] \end{tikzcd}$$

is a pullback. We see that the arrows $f$ and $\tilde{g}m$ represent the same partial map, hence the triangle commutes. ∎

**Corollary 1.9 (PTJ 1.28)** *Suppose we are given a pushout square*

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ Z \arrow[r] & T \end{tikzcd}$$

*with $f$ mono. Then $g$ is also mono, and the square is also a pullback.*

**Proof**. Consider the partial map $Z \rightharpoonup Y$ given by the diagram $Z \xleftarrow{f} X \xrightarrow{m} Y$; let it be represented by a map $h : Z \to \tilde{Y}$. Since the original square is a pushout, we have a unique map $T \to \tilde{Y}$ making the diagram

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "f"'] & Y \arrow[d, "g"'] \arrow[ddr, "\eta_Y"] & \\ Z \arrow[r] \arrow[drr, "h"'] & T \arrow[dr] & \\ & & \tilde{Y} \end{tikzcd}$$

commute. Then $g$ is mono because $\eta_Y$ is mono, and the outer square is a pullback, so the inner square is a pullback too. ∎

**Remark 1.10** Proposition 1.8 shows, in particular, that a topos *has enough injectives*: that is, for every object $X$ there is a mono from $X$ into an injective object. The following exercise elaborates on this.

**Exercise 6** a) Show that, in a topos, an object is injective if and only if it is a retract of $\Omega^Y$ for some $Y$.

b) Suppose $\mathcal{A} \xrightarrow{G} \mathcal{B}$ be a functor with left adjoint $\mathcal{B} \xrightarrow{F} \mathcal{A}$. Show that if $F$ preserves monos, $G$ preserves injectives; and that the converse holds if $\mathcal{A}$ has enough injectives.

In the following, let $\mathcal{E}$ be a topos. We start by considering the category $\mathcal{E}^{\mathrm{op}}$. We have a functor $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$: on objects, $PX = \Omega^X$ and for maps $X \xrightarrow{f} Y$ we have $Pf : \Omega^Y \to \Omega^X$, the map which is the exponential transpose of the composition $\Omega^Y \times X \xrightarrow{\mathrm{id}\times f} \Omega^Y \times Y \xrightarrow{\mathrm{ev}} \Omega$.

Note that the same data define a functor $P^* : \mathcal{E} \to \mathcal{E}^{\mathrm{op}}$, and we have:

**Lemma 1.11** $P^* \dashv P$.

**Proof**. We have natural bijections

$$\begin{array}{rcccl} \mathcal{E}^{\mathrm{op}}(P^*X, Y) & = & \mathcal{E}^{\mathrm{op}}(\omega^X, Y) & = & \mathcal{E}(Y, \Omega^X) \simeq \mathcal{E}(Y \times X, \Omega) \\ \simeq & \mathcal{E}(X \times Y, \Omega) & \simeq \mathcal{E}(X, \Omega^Y) & = & \mathcal{E}(X, PY) \end{array}$$

■

Hence, we have a monad $T = PP^*$ on $\mathcal{E}$, and thus a comparison functor $K : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}^T$.

For a mono $g : W \to Z$ we also have a map $\exists g : \Omega^W \to \Omega^Z$: it is the transpose of the map $\widetilde{\exists g} : \Omega^W \times Z \to \Omega$ which classifies the mono

$$\in_W \longrightarrow \Omega^W \times W \xrightarrow{\mathrm{id} \times g} \Omega^W \times Z$$

(here $\in_W$ is the subobject of $\Omega^W \times W$ classified by the evaluation map $\mathrm{ev}_W : \Omega^W \times W \to \Omega$).

We have that the square

$$\begin{tikzcd} \in_W \arrow[r] \arrow[d] & \Omega^W \times W \arrow[r, "\mathrm{id} \times g"] & \Omega^W \times Z \arrow[d, "\widetilde{\exists g}"] \\ 1 \arrow[rr, "t"'] & & \Omega \end{tikzcd}$$

is a pullback; hence, since $g$ is mono, also the square

$$\begin{tikzcd} \in_W \arrow[r] \arrow[dd] & \Omega^W \times W \arrow[d, "\mathrm{id} \times g"] \\ & \Omega^W \times Z \arrow[d, "\widetilde{\exists g}"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$

is a pullback. We see that $\widetilde{\exists g} \circ (\mathrm{id} \times g)$ classifies the mono $\in_W \to \Omega^W \times W$, and we conclude that $\widetilde{\exists g} \circ (\mathrm{id} \times g) = \mathrm{ev}_W$.

**Lemma 1.12 (PTJ 1.32; "Beck Condition")** *Suppose the square*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ Z \arrow[r, "k"'] & T \end{tikzcd}$$

*is a pullback with the arrows $g$ and $h$ monic. Then the following square commutes:*

$$\begin{tikzcd}
\Omega^Y \arrow[r, "Pf"] \arrow[d, "\exists h"'] & \Omega^X \arrow[d, "\exists g"] \\
\Omega^T \arrow[r, "Ph"'] & \Omega^Z
\end{tikzcd}$$

**Proof**. We look at the exponential transposes of the two compositions. For the clockwise composition $\exists g \circ Pf : \Omega^Y \to \Omega^Z$, its transpose is the top row of

$$\begin{tikzcd}
\Omega^Y \times Z \arrow[r, "Pf\times\mathrm{id}"] & \Omega^X \times Z \arrow[r, "\widetilde{\exists g}"] & \Omega \\
\Omega^Y \times X \arrow[u, "\mathrm{id}\times g"] \arrow[r, "Pf\times\mathrm{id}"'] & \Omega^X \times X \arrow[u, "\mathrm{id}\times g"'] & \\
E \arrow[u] \arrow[r] & \in_X \arrow[u] \arrow[r] & 1 \arrow[uu, "t"']
\end{tikzcd} .$$

We see that this top row classifies the subobject $E \to \Omega^Y \times X \xrightarrow{\mathrm{id}\times g} \Omega^Y \times Z$.

Since $\widetilde{\exists g} \circ (\mathrm{id} \times g) = \mathrm{ev}_X$ as we noted just before the statement of the lemma, the subobject $E \to \Omega^Y \times X$ is classified by the composition $\Omega^Y \times X \xrightarrow{Pf\times\mathrm{id}} \Omega^X \times X \xrightarrow{\mathrm{ev}_X} \Omega$, which equals the composition $\Omega^Y \times X \xrightarrow{\mathrm{id}\times f} \Omega^Y \times Y \xrightarrow{\mathrm{ev}_Y} \Omega$ sich both compositions are transposes of $Pf$. Therefore we have a pullback diagram

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & \Omega^Y \times X \arrow[d, "\mathrm{id}\times f"] \\
\in_Y \arrow[r] & \Omega^Y \times Y
\end{tikzcd}$$

For the counterclockwise composition $Pk \circ \exists h$, its transpose is $\Omega^Y \times Z \xrightarrow{\exists h\times\mathrm{id}} \Omega^T \times Z \xrightarrow{\mathrm{id}\times k} \Omega^T \times T \xrightarrow{\mathrm{ev}_T} \Omega$ which equals $\Omega^Y \times Z \xrightarrow{\mathrm{id}\times k} \Omega^Y \times T \xrightarrow{\exists h\times\mathrm{id}} \Omega^T \times T \xrightarrow{\mathrm{ev}_T} \Omega$.

Now $\mathrm{ev}_T \circ (\exists h \times \mathrm{id})$ and $\widetilde{\exists h} : \Omega^Y \times T \to \Omega$ both transpose to $\exists h$, so these maps are equal. We conclude that $Pk \circ \exists h$ transposes to the composition

$\Omega^Y \times Z \xrightarrow{\mathrm{id}\times k} \Omega^Y \times T \xrightarrow{\widetilde{\exists h}} \Omega$, and we consider pullbacks

$$
\begin{tikzcd}
E \arrow[r] \arrow[d] & \Omega^Y \times X \arrow[r, "\mathrm{id}\times g"] \arrow[d, "\mathrm{id}\times f"] & \Omega^Y \times Z \arrow[d, "\mathrm{id}\times k"] \\
\in_Y \arrow[r] \arrow[d] & \Omega^Y \times Y \arrow[r, "\mathrm{id}\times h"] & \Omega Y \times T \arrow[d, "\widetilde{\exists h}"] \\
1 \arrow[rr] & & \Omega
\end{tikzcd}
$$

Again, we have $\widetilde{\exists h} \circ (\mathrm{id} \times h) = \mathrm{ev}_Y : \Omega^Y \times Y \to \Omega$ and we see that the counterclockwise composition transposes to a map which classifies the same subobject $E \to \Omega^Y \times X \xrightarrow{\mathrm{id}\times g} \Omega^Y \times Z$ as we saw for the clockwise composition.

Therefore the two compositions are equal, and the given diagram commutes. ∎

**Corollary 1.13 (PTJ 1.33)** *If $f : X \to Y$ is mono then $Pf \circ \exists f = \mathrm{id}_{\Omega^X}$.*

**Proof**. Apply 1.12 to the pullback diagram

$$
\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

∎

**Theorem 1.14 (PTJ 1.34)** *The functor $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ is monadic.*

**Proof**. We use the Crude Tripleability Theorem (0.10). We need to verify its conditions:

1) $\mathcal{E}^{\mathrm{op}}$ has coequalizers of reflexive pairs.

2) $P$ preserves coequalizers of reflexive pairs.

3) $P$ reflects isomorphisms.

Verification of 1) is trivial, since coequalizers in $\mathcal{E}^{\mathrm{op}}$ are equalizers in $\mathcal{E}$, and $\mathcal{E}$ has finite limits.

For 2), let $X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z$ be a diagram in $\mathcal{E}$ which is a coequalizer of a reflexive pair in $\mathcal{E}^{\mathrm{op}}$. Since the pair $(g, h)$ is reflexive in $\mathcal{E}^{\mathrm{op}}$ we have an arrow $Z \overset{d}{\to} Y$ satisfying $dg = dh = \mathrm{id}_Y$. This means that $g$ and $h$ are monos, and the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ Y \arrow[r, "h"'] & Z \end{tikzcd}$$

is a pullback. We see that also $f$ is mono, and applying 1.12 we find that $\exists f \circ Pf = Ph \circ \exists g$. Moreover by 1.13 we have the equalities $Pf \circ \exists f = \mathrm{id}_{\Omega^X}, Pg \circ \exists g = \mathrm{id}_{\Omega^Y}$. Using these equalities we see that the $P$-image of the original coequalizer diagram:

$$\Omega^Z \underset{Ph}{\overset{Pg}{\rightrightarrows}} \Omega^Y \xrightarrow{Pf} \Omega^X$$

is a split fork in $\mathcal{E}$, with splittings $\exists g : \Omega^Y \to \Omega^Z$, $\exists f : \Omega^X \to \Omega^Y$. In particular it is a coequalizer in $\mathcal{E}$.

For 3), we observe that for any morphism $f : X \to Y$ in $\mathcal{E}$, the map $Y \overset{\{\cdot\}}{\to} \Omega^Y \overset{Pf}{\to} \Omega^X$ transposes to the map $Y \times X \to \Omega$ which classifies the graph of $f$, i.e. the subobject represented by $\langle f, \mathrm{id}\rangle : X \to Y \times X$. Note that if the graphs of $f$ and $g : X \to Y$ coincide then $f = g$. Therefore, $Pf = Pg$ implies $f = g$ and $P$ is faithful, hence reflects both monos and epis. By Corollary 1.3, $P$ reflects isomorphisms. ∎

**Corollary 1.15 (PTJ 1.36)** *A topos has finite colimits.*

**Proof**. For a finite diagram $M : I \to \mathcal{E}$ consider $M^{\mathrm{op}} : I^{\mathrm{op}} \to \mathcal{E}^{\mathrm{op}}$ and compose with $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$. The diagram $P \circ M^{\mathrm{op}}$ has a limit in $\mathcal{E}$ since $\mathcal{E}$ has finite limits. But $P$, being monadic, creates limits so $M^{\mathrm{op}}$ has a limit in $\mathcal{E}^{\mathrm{op}}$; that is, $M$ has a limit in $\mathcal{E}$. ∎

**Corollary 1.16 (PTJ 1.37)** *Let $T : \mathcal{E} \to \mathcal{F}$ be a logical functor between toposes. Then the following hold:*

*i)* *$T$ preserves finite limits.*

*ii)* *If $T$ has a left adjoint, it also has a right adjoint.*

**Proof**. i) Since $T$ is logical, the diagram

$$\begin{tikzcd} \mathcal{E}^{\mathrm{op}} \arrow[r, "T^{\mathrm{op}}"] \arrow[d, "P"'] & \mathcal{F}^{\mathrm{op}} \arrow[d, "P"] \\ \mathcal{E} \arrow[r, "T"'] & \mathcal{F} \end{tikzcd}$$

commutes up to isomorphism. Proving that $T$ preserves finite colimits amounts to proving that $T^{\mathrm{op}}$ preserves finite limits. So let $M : I \to \mathcal{E}^{\mathrm{op}}$ be a finite diagram, with limiting cone $(D, \mu)$ in $\mathcal{E}^{\mathrm{op}}$. Now $T$ and $P$ preserve finite limits, so $TP(D, \mu)$ is a limiting cone for $TPM$; hence $PT^{\mathrm{op}}(D, \mu)$ is a limiting cone for $PT^{\mathrm{op}}M$ by commutativity of the diagram. Since $P$ creates limits, $T^{\mathrm{op}}(D, \mu)$ is a limitng cone for $T^{\mathrm{op}}M$. We conclude that $T^{\mathrm{op}}$ preserves finite limits.

For ii), we employ the Adjoint Lifting Theorem (0.11) to the same diagram. The assumptions are readily verified, and we conclude that $T^{\mathrm{op}}$ has a left adjoint. But this means that $T$ has a right adjoint. ∎

We now discuss slice categories of toposes. In any category $\mathcal{E}$ with finite limits, for each object $X$ we have the category $\mathcal{E}/X$ whose objects are arrows into $X$ and whose arrows: $(Y \xrightarrow{f} X) \to (Z \xrightarrow{g} X)$ are arrows $Y \xrightarrow{h} Z$ such that $f = gh$. Given an arrow $f : Y \to X$ we have a pullback functor $f^* : \mathcal{E}/X \to \mathcal{E}/Y$, which has a left adjoint $\sum_f$; $\sum_f(Z \xrightarrow{g} Y = (Z \xrightarrow{fg} X)$. In the case of the unique arrow $X \to 1$ we write $X^* : \mathcal{E} \cong \mathcal{E}/1 \to \mathcal{E}/X$ for the pullback functor. Note that $X^*(Y)$ is the projection $Y \times X \to X$. Note also that $X \xrightarrow{\mathrm{id}} X$ is a terminal object of $\mathcal{E}/X$.

The following theorem was dubbed the "Fundamental Theorem of Topos Theory" by Peter Freyd.

**Theorem 1.17 (PTJ 1.42)** *Let $\mathcal{E}$ be a topos and $X$ an object of $\mathcal{E}$. Then $\mathcal{E}/X$ is a topos, and the functor $X^* : \mathcal{E} \to \mathcal{E}/X$ is logical.*

**Proof**. In the case $\mathcal{E} = \mathrm{Set}$, it is useful to view objects of $\mathcal{E}/X$ as "$X$-indexed families of sets" rather than as functions into $X$. This intuition will also guide us in the general case.

Binary products in $\mathcal{E}/X$ are pullbacks over $X$: if we adopt the notation $Y \times_X Z$ for the vertex of the pullback diagram

$$\begin{tikzcd} Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

then in $\mathcal{E}/X$, the product $f \times g$ is the arrow $Y \times_X Z \to X$. Equalizers in $\mathcal{E}/X$ are just equalizers in $\mathcal{E}$. So $\mathcal{E}/X$ has finite limits, and the functor $X^*$ preserves finite limits since it has a left adjoint $\sum_f$ as we remarked.

Monos in $\mathcal{E}/X$ are monos in $\mathcal{E}$, and the diagram

$$\begin{tikzcd} X \arrow[r, "\langle t, \mathrm{id}\rangle"] \arrow[dr, "\mathrm{id}"'] & \Omega \times X \arrow[d, "\pi"] \\ & X, \end{tikzcd}$$

seen as an arrow in $\mathcal{E}/X$, is a subobject classifier in $\mathcal{E}/X$. Note, that this map is $X^*(1 \xrightarrow{t} \Omega)$, so $X^*$ preserves subobject classifiers.

In order to prove cartesian closure, first observe that for $\mathcal{E} = \mathrm{Set}$, the exponent $(Z \xrightarrow{g} X)^{(Y \xrightarrow{f} X)}$ is the $X$-indexed family $(g^{-1}(x)^{f^{-1}(x)})_{x \in X}$, or the projection function from the set $\{(h, x) \mid h : f^{-1}(x) \to g^{-1}(x)\}$ to $X$.

We first construct the exponential $(Z \xrightarrow{g} X)^{(Y \xrightarrow{f} X)}$, then explain its meaning in intuitive terms (as if $\mathcal{E}$ were the topos Set); then we prove that it has the required universal property.

Let $\theta : X \times Y \to \tilde{X}$ represent the partial map $X \xleftarrow{f} Y \xrightarrow{\langle f, \mathrm{id}\rangle} X \times Y$, i.e. let

$$\begin{tikzcd} Y \arrow[r, "\langle f, \mathrm{id}\rangle"] \arrow[d, "f"'] & X \times Y \arrow[d, "\theta"] \\ X \arrow[r, "\eta_X"'] & \tilde{X} \end{tikzcd}$$

be a pullback. Let $\bar{\theta} : X \to \tilde{X}^Y$ be the exponential transpose of $\theta$. Finally, let

$$\begin{tikzcd} E \arrow[r, "q"] \arrow[d, "p"'] & \tilde{Z}^Y \arrow[d, "\tilde{g}^Y"] \\ X \arrow[r, "\bar{\theta}"'] & \tilde{X}^Y \end{tikzcd}$$

be a pullback; the claim is that $E \xrightarrow{p} X$ is the required exponential.

Intuitive explanation: think of $\tilde{X}$ as the set of subsets of $X$ having at most one element. So $\theta(x, y) = \{x \mid f(y) = x\}$. The function $\tilde{g} : \tilde{Z} \to \tilde{X}$ sends subset $\alpha$ of $Z$ to $\{g(z) \mid z \in \alpha\}$. Then, the function $\tilde{g}^Y : \tilde{Z}^Y \to \tilde{X}^Y$ sends a function $h : Y \to \tilde{Z}$ to the function $y \mapsto \{g(z) \mid z \in h(y)\}$. We have $\bar{\theta}(x)(y) = \{x \mid f(y) = x\}$. So the object $E$ can be identified with the set of pairs $(x, h)$ satisfying:

$$\begin{array}{l} x \in X, h : Y \rightharpoonup Z \\ \mathrm{dom}(h) = f^{-1}(x) \\ \text{for all } y \in f^{-1}(x),\ h(y) \in g^{-1}(x). \end{array}$$

That is, $E$ is isomorphic to $\{(h,x) \mid h : f^{-1}(x) \to g^{-1}(x)\}$.

Now we prove that the constructed $E \xrightarrow{p} X$ has the property of the exponential $(Z \xrightarrow{g} X)^{(Y \xrightarrow{f} X)}$.

For an arbitrary object $(T \xrightarrow{k} X)$ of $\mathcal{E}/X$, maps in $\mathcal{E}/X$ from $k$ to $p$ correspond bijectively to maps $T \xrightarrow{l} \tilde{Z}^Y$ satisfying $\tilde{g}^Y l = \bar{\theta} k$. These correspond to maps $T \times Y \xrightarrow{\bar{l}} \tilde{Z}$ satisfying $\theta(k \times \mathrm{id}_Y) = \tilde{g}\bar{l}$. These in turn, correspond to maps $T \times_X Y \xrightarrow{\bar{\bar{l}}} Z$ which satisfy that $g\bar{\bar{l}}$ is the composite $T \times_X Y \to T \times Y \to Y \xrightarrow{f} X$; that is, to maps from $T \times_X Y$ to $Z$ making the triangle

$$\begin{tikzcd}
T \times_X Y \arrow[rr] \arrow[dr] & & Z \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

commute; that is, maps from $k \times f$ to $g$ in $\mathcal{E}/X$.

For the third correspondence in the chain above, suppose we have $T \times Y \xrightarrow{\bar{l}} \tilde{Z}$ satisfying $\theta(k \times \mathrm{id}_Y) = \tilde{g}\bar{l}$. Let

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & T \times Y \arrow[d, "\bar{l}"] \\
Z \arrow[r, "\eta_Z"'] & \tilde{Z}
\end{tikzcd}$$

be a pullback. We then have a commutative diagram

$$\begin{tikzcd}
T \arrow[dd, "k"'] & T \times Y \arrow[l, "\pi"'] \arrow[dd, "k \times \mathrm{id}"'] \arrow[dr, "\bar{l}"] & & W \arrow[ll] \arrow[dd] \arrow[dr] & \\
& & \tilde{Z} \arrow[dd, "\tilde{g}"] & & Z \arrow[ll, "\eta_Z"'] \arrow[dd, "g"] \\
X & X \times Y \arrow[l, "\pi"'] \arrow[rr] \arrow[dr, "\theta"'] & & Y \arrow[dr, "f"] & \\
& & \tilde{X} & & X \arrow[ll, "\eta_X"']
\end{tikzcd}$$

The front face and the top face of the cube are pullbacks, as is the bottom. Hence the back face is a pullback too. Composing the back face with the left hand square reveals that $W$ is $T \times_X Y$.

The fact that $X^*$ preserves exponentials is left as an exercise. ∎

**Exercise 7** Show that $X^*$ preserves exponentials.

**Corollary 1.18 (PTJ 1.43)** *For any arrow $f : X \to Y$ in $\mathcal{E}$ the pullback functor $f^* : \mathcal{E}/Y \to \mathcal{E}/X$ is logical, and has a right adjoint $\prod_f$.*

**Proof.** We now know that $\mathcal{E}/Y$ is a topos, so we can apply Theorem 1.17 with $\mathcal{E}/Y$ in the role of $\mathcal{E}$ and $f$ in the role of $X$. We see that $f^*$ is logical. By Corollary 1.16, $f^*$ has a right adjoint, since it has a left adjoint $\sum_f$.

However, we can also exhibit the right adjoint $\prod_f$ directly: we do this for the case $Y = 1$. Given an object $(Y \xrightarrow{f} X)$ of $\mathcal{E}/X$ let $\ulcorner \mathrm{id} \urcorner : 1 \to X^X$ denote the exponential transpose of the identity arrow on $X$, and let

$$\begin{tikzcd} Z \arrow[r] & Y^X \arrow[r, shift left, "f^X"] \arrow[r, shift right, "\ulcorner \mathrm{id} \urcorner \circ !"'] & X^X \end{tikzcd}$$

be an equalizer diagram. Think of $Z$ as the object of sections of $f$. Now for any object $W$ of $\mathcal{E}$, arrows $g : X^*(W) \to f$:

$$\begin{tikzcd} W \times X \arrow[rr, "g"] \arrow[dr] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

correspond, via the exponential adjunction, to arrows $\tilde{g} : W \to Y^X$ such that $f^X \circ \tilde{g}$ factors through $\ulcorner \mathrm{id} \urcorner$; that is to arrows $W \to Z$. Therefore $Z$ is $\prod_X(f)$. ∎

**Example 1.19** Consider the subobject classifier $1 \xrightarrow{t} \Omega$; let us calculate $\prod_t : \mathcal{E} \to \mathcal{E}/\Omega$. For an object $X$ of $\mathcal{E}$ and an arrow $Y \xrightarrow{m} \Omega$ we have that maps from $m$ to $\prod_t(X)$ in $\mathcal{E}/\Omega$ correspond to maps from $Y'$ to $X$, where $Y'$ is the subobject of $Y$ classified by $m$. That is, to maps $g : Y \to \tilde{X}$ for which the domain (i.e. the map $g^*(\eta_X) : Y' \to Y$) is the subobject of $Y$ classified by $m$. But these correspond to maps in $\mathcal{E}/\Omega$ from $m$ to the arrow $s : \tilde{X} \to \Omega$ which classifies the mono $X \xrightarrow{\eta_X} \tilde{X}$.

**Corollary 1.20 (PTJ 1.46)** *Every arrow $f : X \to Y$ in $\mathcal{E}$ induces a geometric morphism*

$$\begin{tikzcd} f : \mathcal{E}/X \arrow[r, shift right, "\prod_f"'] & \mathcal{E}/Y \arrow[l, shift right, "f^*"'] \end{tikzcd} .$$

*This geometric morphism has the special features that the inverse image functor $f^*$ is logical and has a left adjoint.*

**Definition 1.21** A geometric morphism $f$ for which the inverse image functor $f^*$ has a left adjoint is called *essential*.

Without proof, we mention the following partial converse to corollary 1.20.

**Theorem 1.22 (PTJ 1.47)** *Let $f : \mathcal{F} \to \mathcal{E}$ be an essential geometric morphism such that $f^*$ is logical and its left adjoint $f_!$ preserves equalizers. Then there is an object $X$ of $\mathcal{E}$, unique up to isomorphism, such that $\mathcal{F}$ is equivalent to $\mathcal{E}/X$ and, modulo this equivalence, the geometric morphism $f$ is isomorphic to the geometric morphism $(X^* \dashv \prod_X)$ of Corollary 1.20.*

We recall from the *Category Theory and Topos Theory* course that a *regular category* is a category with finite limits, which has coequalizers of kernel pairs, and in which regular epimorphisms are stable under pullback. Recall that in such a category, every arrow factors, essentially uniquely, as a regular epimorphism followed by a monomorphism. The construction is as follows: given $f : X \to Y$, let $X \stackrel{e}{\to} E$ be the coequalizer of the kernel pair of $f$, and let $m : E \to Y$ be the unique factorization of $f$ through this coequalizer.

Since pullback functors have right adjoints, they preserve regular epimorphisms, so every topos is a regular category.

**Lemma 1.23 (PTJ 1.53)** *In a topos, every epi is regular.*

**Proof**. Given an epi $f : X \to Y$, let $X \stackrel{e}{\to} E \stackrel{m}{\to} Y$ be its regular epi-mono factorization. Since $f$ is epi, $m$ must be epi; by 1.3, $m$ is an isomorphism. So $f$ is regular epi. ∎

**Definition 1.24** An *exact* category is a regular category in which every equivalence relation is effective.

By 1.5 we have:

**Proposition 1.25** *Every topos is an exact category.*

**Proposition 1.26 (PTJ 1.56)** *In a topos the initial object $0$ is strict; that is, every arrow into $0$ is an isomorphism.*

**Proof**. Given $X \stackrel{i}{\to} 0$, we have a pullback

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "\mathrm{id}_X"'] & 0 \arrow[d, "\mathrm{id}_0"] \\ X \arrow[r, "i"'] & 0 \end{tikzcd}$$

so $\mathrm{id}_X = i^*(\mathrm{id}_0)$. Now $\mathrm{id}_0$ is initial in $\mathcal{E}/0$, so $\mathrm{id}_X$ is initial in $\mathcal{E}/X$ (since $i^*$, having a right adjoint, preserves initial objects). But that means that $X$ is initial in $\mathcal{E}$, since for any object $Y$ of $\mathcal{E}$ there is a bijection between arrows $X \to Y$ in $\mathcal{E}$, and arrows $\mathrm{id}_X \to X^*(Y)$ in $\mathcal{E}/X$. ∎

**Corollary 1.27 (PTJ 1.57)** *In a topos, every coprojection $X \to X+Y$ is monic. Moreover, "coproducts are disjoint": that is, the square*

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & X+Y
\end{tikzcd}$$

*is a pullback.*

**Proof**. From Proposition 1.26 it follows easily that every map $0 \to X$ is monic. Since the given square is always a pushout, the statement follows at once from Corollary 1.9. ∎

**Exercise 8** Prove that for a topos $\mathcal{E}$ and objects $X, Y$ of $\mathcal{E}$ the categories $\mathcal{E}/(X+Y)$ and $\mathcal{E}/X \times \mathcal{E}/Y$ are equivalent.

As a consequence of regularity (and existence of coproducts) we can form *unions of subobjects*: given subobjects $M, N$ of $X$, represented by monos $M \stackrel{m}{\to} X, N \stackrel{n}{\to} X$, its union $M \cup N$ (least upper bound in the poset $\mathrm{Sub}(X)$) is defined by the regular epi-mono factorization

$$M + N \to M \cup N \to X$$

of the map $\begin{bmatrix} m \\ n \end{bmatrix} : M + N \to X$. We have:

**Proposition 1.28** *In a topos, for any object $X$ the poset $\mathrm{Sub}(X)$ of subobjects of $X$ is a distributive lattice. Moreover, for any arrow $X \stackrel{f}{\to} Y$ the pullback functor $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ between subobject lattices has both adjoints $\exists_f$ and $\forall_f$.*

**Proof**. Finite meets in $\mathrm{Sub}(X)$ (from now on called "intersections" of subobjects) are given by pullbacks, and unions by the construction above. Distributivity follows from the fact that pullback functors preserve coproducts and regular epimorphisms. The left adjoint $\exists f$ is constructed using regular epi-mono factorization as in the course *Category Theory and Topos Theory*. The right adjoint $\forall f$ is just the restriction of $\prod_f$ to subobjects: $\prod_f$ preserves monos. ∎

The following fact will be important later on.

**Proposition 1.29 (Elephant, A1.4.3)** *Let $M \overset{m}{\to} X, N \overset{n}{\to} X$ be monos into $X$ (we also write $M, N$ for the subobjects represented by $m$ and $n$). Let the intersection and union of $M$ and $N$ be represented by arrows $M \cap N \to X$, $M \cup N \to X$, respectively. Then the diagram*

$$\begin{array}{ccc} M \cap N & \longrightarrow & M \\ \downarrow & & \downarrow \\ N & \longrightarrow & M \cup N \end{array}$$

*is both a pullback and a pushout in $\mathcal{E}$.*

**Proof**. This proof is not the proof given in **Elephant**.

The partial order $\mathrm{Sub}(X)$ is, as a category, equivalent to the full subcategory $\mathrm{Mon}/X$ of the slice $\mathcal{E}/X$ on the monomorphisms into $X$. Since the given square is a pullback in $\mathrm{Sub}(X)$ hence in $\mathrm{Mon}/X$, and the domain functor $\mathrm{Mon}/X \to \mathcal{E}$ preserves pullbacks, the square is a pullback in $\mathcal{E}$.

Let us define $\mathrm{Sub}_{\leq 1}(X)$ as the set of those subobjects $M \overset{m}{\to} X$ for which the unique map $M \to 1$ is a monomorphism. Note that there is a natural bijection between $\mathrm{Sub}_{\leq 1}(X)$ and $\mathcal{E}(1, \tilde{X})$, where $\tilde{X}$ is the partial map classifier of $X$. Writing $M$ both for a subobject of $X$ and for the corresponding map $1 \to \tilde{X}$, we define the subobject $\mathrm{dom}(M)$ of 1 by the pullback

$$\begin{array}{ccc} \mathrm{dom}(M) & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle M} \\ X & \underset{\eta_X}{\longrightarrow} & \tilde{X} \end{array}$$

Note, that $\mathrm{dom}(M)$ is also the image of the map $M \to 1$. For a subobject $c$ of 1, we define $M{\restriction}c$ by the pullback

$$\begin{array}{ccc} M{\restriction}c & \longrightarrow & M \\ \downarrow & & \downarrow \\ c & \longrightarrow & 1 \end{array}$$

We have the following lemma.

**Lemma 1.30** *Let $M, N \in \mathrm{Sub}_{\leq 1}(X)$, with $\mathrm{dom}(M) = c, \mathrm{dom}(N) = d$. If $M{\restriction}(c \cap d) = N{\restriction}(c \cap d)$ as subobjects of $X$, then $M \cup N \in \mathrm{Sub}_{\leq 1}(X)$.*

**Proof**. We must prove that the map $\phi : M \cup N \to 1$ is monic. Clearly, this map factors through $c \cup d$, so it is enough to prove that $(c \cup d)^*(\phi)$ is monic in $\mathcal{E}/(c \cup d)$.

We have $c^*(M \cup N) = c^*(M) \cup c^*(N)$. Since $c^*(N)$ has domain $c^*(d) = c \cap d$ and $M$ and $N$ agree on $c \cap d$, we have $c^*(N) \leq c^*(M)$, so $c^*(M \cup N) = c^*(M)$ and $c^*(\phi)$ is monic. In a symmetric way, $d^*(M \cup N) = d^*(N)$ and $d^*(\phi)$ is monic.

The topos $\mathcal{E}/(c+d)$ is isomorphic to $\mathcal{E}/c \times \mathcal{E}/d$ by Exercise 8, so we see that $(c+d)^*(\phi)$ is monic. Now $c + d \to c \cup d$ is epi, so the pullback functor $\mathcal{E}/(c \cup d) \to \mathcal{E}/(c+d)$ reflects monomorphisms. We conclude that $(c \cup d)^*(\phi)$ monic, as required. This proves the lemma. ∎

Continuing the proof of Proposition 1.29: as usual, we may do as if $X = 1$. So we have subobjects $c, d$ of 1 and we wish to prove that the square

$$\begin{tikzcd}
c \cap d \arrow[r] \arrow[d] & c \arrow[d] \\
d \arrow[r] & c \cup d
\end{tikzcd}$$

is a pushout. Let $M : c \to X$, $N : d \to X$ be maps which agree on $c \cap d$. Then $M$ and $N$ define elements of $\mathrm{Sub}_{\leq 1}(X)$ for which the hypothesis of Lemma 1.30 holds. Therefore, the map $c \cup d \to X$ which names the subobject $M \cup N$ is a mediating map, which is unique because the maps $\{c \to c \cup d, d \to c \cup d\}$ form an epimorphic family. ∎

## 2 Geometric Morphisms

This section contains material from the book *Sheaves in Geometry and Logic* by MacLane and Moerdijk; hereafter referred to by "MM".

We recall that a *geometric morphism* $\mathcal{F} \to \mathcal{E}$ between toposes is an adjoint pair $f^* \dashv f_*$ with $f^* : \mathcal{E} \to \mathcal{F}$ (the *inverse image functor*), $f_* : \mathcal{F} \to \mathcal{E}$ (the *direct image functor*), with the additional property that $f^*$ preserves finite limits.

A geometric morphism $\mathrm{Set} \to \mathcal{E}$ is called a *point* of $\mathcal{E}$.

**Examples 2.1** 1) In the Introduction we have seen that every continuous function of topological spaces $f : X \to Y$ determines a geometric morphism $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. If the space $Y$ is sufficiently separated (here we shall assume that $Y$ is Hausdorff, although the weaker condition of *sober* suffices) then there is a converse to this: every geometric morphism $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is induced by a unique continuous function. Indeed, let $f$ be such a geometric morphism. In $\mathrm{Sh}(Y)$, the lattice of subobjects of 1 (the terminal object) is in 1-1, order-preserving, bijection with $\mathcal{O}(Y)$, the set of open subsets of $Y$. The same for $X$, of course. Now the inverse image $f^*$, preserving finite limits, preserves subobjects of 1 and therefore induces a function $f^- : \mathcal{O}(Y) \to \mathcal{O}(X)$. Since $f^*$ preserves colimits and finite limits, the function $f^-$ preserves the top element ($f^-(Y) = X$), finite intersections and arbitrary unions (in particular, $f^-(\emptyset) = \emptyset$).

Define a relation $R$ from $X$ to $Y$ as follows: $R(x, y)$ holds if and only if $x \in f^-(V)$ for every open neighbourhood $V$ of $y$. We shall show that $R$ is in fact a function $X \to Y$, leaving the remaining details as an exercise.

i) Assume $R(x, y)$ and $R(x, y')$ both hold, and $y \neq y'$. By the Hausdorff property, $y$ and $y'$ have disjoint open neighbourhoods $V_y$ and $V_{y'}$. By assumption and the preservation properties of $f^-$ we have:

$$x \in f^-(V_y) \cap f^-(V_{y'}) \;=\; f^-(V_y \cap V_{y'}) \;=\; f^-(\emptyset) \;=\; \emptyset$$

a clear contradiction. So the relation $R$ is single-valued.

ii) Suppose for $x \in X$ there is no $y \in Y$ satisfying $R(x, y)$. Then for every $y$ there is a neighbourhood $V_y$ such that $x \notin f^-(V_y)$. Then we have

$$x \notin \bigcup_{y \in Y} f^-(V_y) \;=\; f^-(\bigcup_{y \in Y} V_y) \;=\; f^-(Y) \;=\; X$$

also a clear contradiction. So the relation $R$ is total, and therefore a function.

**Exercise 9** Show that the function $R$ just constructed is continuous, and that it induces the given geometric morphism $f$.

2) Consider, for a group $G$, the category $\widehat{G}$ of right $G$-sets. Let $\Delta : \mathrm{Set} \to \widehat{G}$ be the functor which sends a set $X$ to the trivial $G$-set $X$ (i.e. the $G$-action is the identity). Note that $\Delta$ preserves finite limits. The functor $\Delta$ has a right adjoint $\Gamma$, which sends a $G$-set $X$ to its subset of *$G$-invariant elements*, i.e. to the set

$$\{x \in X \mid xg = x \text{ for all } g \in G\}$$

Note that $\widehat{G}(\Delta(Y), X)$ is naturally isomorphic to $\mathrm{Set}(Y, \Gamma(X))$, so we have a geometric morphism $\widehat{G} \to \mathrm{Set}$. Actually, this geometric morphism is essential, because $\Delta$ also has a left adjoint: we have that $\widehat{G}(X, \Delta(Y))$ is naturally isomorphic to $\mathrm{Set}(\mathrm{Orb}(X), Y)$, where $\mathrm{Orb}(X)$ denotes the set of orbits of $X$ under the $G$-action.

**Exercise 10** Prove that the functor Orb does not preserve equalizers (Hint: you can do this directly (think of two maps $G \to G$), or apply Theorem 1.22).

This example can be generalized in two directions, as the following items show.

3) Let $\mathcal{E}$ be a cocomplete topos. Then there is exactly one geometric morphism $\mathcal{E} \to \mathrm{Set}$, up to natural isomorphism. For, a geometric morphism is determined by its inverse image functor, which must preserve 1 and coproducts; and since, in Set, every object $X$ is the coproduct of $X$ copies of 1, for $f : \mathcal{E} \to \mathrm{Set}$ we must have $f^*(X) = \sum_{x\in X} 1$. For a function $\phi : X \to Y$ we have $[\mu_{\phi(x)}]_{x\in X} : \sum_{x\in X} 1 \to \sum_{y\in Y} 1$ (where $\mu_i$ sends 1 to the $i$'th cofactor of the coproduct $\sum_{y\in Y} 1$) which is $f^*(\phi) : f^*(X) \to f^*(Y)$. This defines $f^* : \mathrm{Set} \to \mathcal{E}$.

**Exercise 11** Show that the functor $f^*$ preserves finite limits.

The functor $f^*$ has a right adjoint: for a set $X$ and object $Y$ of $\mathcal{E}$ we have

$$\mathcal{E}(f^*(X), Y) \simeq \mathcal{E}(\sum_{x\in X} 1, Y) \simeq \prod_{x\in X} \mathcal{E}(1, Y) \simeq \mathrm{Set}(X, \mathcal{E}(1, Y))$$

so the functor which sends $Y$ to its set of *global sections* (arrows $1 \to Y$) is right adjoint to $f^*$. The "global sections functor" is usually denoted by the letter $\Gamma$; its left adjoint by $\Delta$.

4) Consider presheaf categories $\widehat{\mathcal{C}}, \widehat{\mathcal{D}}$, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We have a geometric morphism $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ constructed as follows. We have a functor $\widehat{F}^* : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ which sends a presheaf $X : \mathcal{D}^{\mathrm{op}} \to \mathrm{Set}$ to $X \circ F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. In other words,

$$\widehat{F}^*(X)(C) = X(F(C))$$

**Exercise 12** Prove that the functor $\widehat{F}^*$ preserves all small limits.

A right adjoint $\widehat{F}_*$ for $\widehat{F}^*$ may be constructed using the Yoneda Lemma. Indeed, for $\widehat{F}_*$ to exist, it should satisfy:

$$\widehat{F}_*(Y)(D) \simeq \widehat{\mathcal{D}}(y_D, \widehat{F}_*(Y)) \simeq \widehat{\mathcal{C}}(\widehat{F}^*(y_D), Y)$$

so we just *define* $\widehat{F}_*$ on objects by putting $\widehat{F}_*(Y)(D) = \widehat{\mathcal{C}}(\widehat{F}^*(y_D), Y)$.

**Exercise 13** Complete the definition of $\widehat{F}_*$ as a functor, and show that it is indeed a right adjoint for $\widehat{F}^*$.

The functor $\widehat{F}^* : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ has also a left adjoint (so the geometric morphism $\widehat{F}$ is essential). Recall that for a presheaf $X$ on $\mathcal{C}$ we have the *category of elements of* $X$, denoted $\mathrm{Elts}(X)$: objects are pairs $(x, C)$ with $x \in X(C)$, and arrows $(x, C) \to (x', C')$ are arrows $f : C \to C'$ in $\mathcal{C}$ satisfying $X(f)(x') = x$. We have the projection functor $\pi : \mathrm{Elts}(X) \to \mathcal{C}$. Define the functor $\widehat{F}_! : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ as follows: for $X \in \widehat{\mathcal{C}}$, $\widehat{F}_!(X)$ is the colimit in $\widehat{\mathcal{D}}$ of the diagram

$$\mathrm{Elts}(X) \xrightarrow{\pi} \mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{y} \widehat{\mathcal{D}}$$

We shall shortly see a more concrete presentation of functors of such "left Kan extensions".

5) In the course *Basic Category Theory and Topos Theory* we have seen that if Cov is a Grothendieck topology on a small category $\mathcal{C}$, then the category $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ of sheaves for Cov is a topos, and the inclusion functor $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov}) \to \widehat{\mathcal{C}}$ has a left adjoint (*sheafification* ) which preserves finite limits; so this is also an example of a geometric morphism. Henceforth we shall denote a Grothendieck topology by $J$ instead of Cov.

## 2.1 Points of $\widehat{\mathcal{C}}$

We recall from the course *Basic Category Theory and Topos Theory* that the functor $y : \mathcal{C} \to \widehat{\mathcal{C}}$ is the "free cocompletion of $\mathcal{C}$". That means the following: given an arbitrary functor $F$ from $\mathcal{C}$ to a cocomplete category $\mathcal{E}$ there is a unique (up to natural isomorphism) colimit-preserving functor $\widetilde{F} : \widehat{\mathcal{C}} \to \mathcal{E}$ such that the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "y"'] & \mathcal{E} \\ & \widehat{\mathcal{C}} \arrow[u, "\widetilde{F}"'] \end{tikzcd}$$

commutes up to isomorphism. The functor $\widehat{F}$ is called the "left Kan extension of $F$ along $y$".

Of course, $\widetilde{F}(X)$ can be defined as the colimit in $\mathcal{E}$ of the diagram $\mathrm{Elts}(X) \stackrel{\pi}{\to} \mathcal{C} \stackrel{F}{\to} \mathcal{E}$. We wish to present this colimit as a form of "tensor product". Let us review the definition from Commutative Algebra.

If $R$ is a commutative ring and $M, N$ are $R$-modules, the set $\mathrm{Hom}_R(M, N)$ of $R$-module homomorphisms from $M$ to $N$ is also an $R$-module (with pointwise operations), and the functor $\mathrm{Hom}_R(M, -) : R\text{-Mod} \to R\text{-Mod}$ has a left adjoint $(-) \otimes_R M$. For an $R$-module $L$ we define an equivalence relation $\sim$ on the set $L \times M$: it is the least equivalence relation satisfying

$$(x, y{\cdot}r) \sim (x{\cdot}r, y)$$

for all $x \in L, y \in M, r \in R$. The equivalence class of $(x, y)$ is denoted $x \otimes y$, and $L \otimes M$ is the $R$-module generated by all such elements $x \otimes y$, subject to the relations

$$((x + x') \otimes y) = x \otimes y + x' \otimes y \quad x \otimes (y + y') = x \otimes y + x \otimes y'$$

and with $R$-action $(x \otimes y)r = (xr \otimes y) = (x \otimes ry)$. In fact, one has a coequalizer diagram of abelian groups:

$$\begin{tikzcd} L \times R \times M \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & L \times M \arrow[r] & L \otimes M \end{tikzcd}$$

where $\phi(x, r, y) = (xr, y)$ and $\psi(x, r, y) = (x, ry)$. The $R$-module $M$ is called *flat* if the functor $(-) \otimes M$ preserves exact sequences; given that this functor is a left adjoint, this is equivalent to saying that it preserves finite limits.

Something similar happens if we have a functor $A : \mathcal{C} \to \mathrm{Set}$ and a presheaf $X$ on $\mathcal{C}$ and we wish to calculate the value of the left Kan extension $\widetilde{A}$ on $X$. Let $\mathcal{C}_1$ be the set of arrows of $\mathcal{C}$. On $\mathbb{A} = \sum_{C \in \mathcal{C}} A(C)$ there is a (partial) "left $\mathcal{C}_1$-action" $x \mapsto f{\cdot}x = A(f)(x)$, for $x \in A(C)$ and $f : C \to C'$. Similarly, on $\mathbb{X} = \sum_{C \in \mathcal{C}} X(C)$ there is a partial "right $\mathcal{C}_1$-action" $x \mapsto x{\cdot}f = X(f)(x)$, for $x \in X(C')$ and $f : C \to C'$. We can now represent the set $\widetilde{A}(X)$ as a coequalizer of sets

$$\textstyle\sum_{C,C' \in \mathcal{C}} X(C') \times \mathcal{C}(C, C') \times A(C) \overset{\phi}{\underset{\psi}{\rightrightarrows}} \sum_{C,C' \in \mathcal{C}} X(C) \times A(C) \longrightarrow \widetilde{A}(X)$$

where $\phi(x, f, a) = (x{\cdot}f, a)$ and $\psi(x, f, a) = (x, f{\cdot}a)$. Therefore we write, from now on, $X \otimes_{\mathcal{C}} A$ for $\widetilde{A}(X)$.

**Theorem 2.2 (MM VII.2.2)** *Let $A : \mathcal{C} \to \mathrm{Set}$ be a functor. Then we have an adjunction*

$$\mathrm{Set} \underset{R}{\overset{L}{\leftrightarrows}} \widehat{\mathcal{C}}$$

*with $L \dashv R$, $R(Y)(C) = \mathrm{Set}(A(C), Y)$ and $L(X) = X \otimes_{\mathcal{C}} A$.*

Now for geometric morphisms $\mathrm{Set} \to \widehat{\mathcal{C}}$ we need the left adjoint $(-) \otimes_{\mathcal{C}} A$ to preserve finite limits.

**Definition 2.3 (MM VII.5.1)** A functor $A : \mathcal{C} \to \mathrm{Set}$ is called *flat* if the functor $(-) \otimes_{\mathcal{C}} A$ preserves finite limits.

The following theorem summarizes our remarks so far.

**Theorem 2.4 (MM VII.5.2)** *Points of the presheaf topos $\widehat{\mathcal{C}}$ correspond to flat functors $\mathcal{C} \to \mathrm{Set}$.*

**Definition 2.5** A category $I$ is called *filtering* if the following conditions are satisfied:

i) $I$ is nonempty.

ii) For each pair of objects $(i, j)$ of $I$ there is a diagram $i \leftarrow k \to j$ in $I$.

iii) For each parallel pair $i \overset{a}{\underset{b}{\rightrightarrows}} j$ there is an arrow $k \xrightarrow{c} i$ which equalizes the pair.

Now let $A : \mathcal{C} \to \text{Set}$. We have the category $\text{Elts}(A)$: objects are pairs $(x, C)$ with $x \in A(C)$; an arrow $(x, C) \to (x', C')$ is a morphism $f : C \to C'$ in $\mathcal{C}$ such that $A(f)(x) = x'$.

**Definition 2.6** *A functor* $A : \mathcal{C} \to \text{Set}$ *is called filtering if the category* $\text{Elts}(A)$ *is filtering.*

**Exercise 14** Let $P$ be a poset and $A : P \to \text{Set}$ a filtering functor. Show that the category $\text{Elts}(A)$ is isomorphic to a *filter* in $P$, that is: a nonempty subset $F \subseteq P$ with the following properties:

i) The set $F$ is upwards closed: if $p \leq q$ and $p \in F$, then $q \in F$.

ii) Any two elements of $F$ have a common lower bound in $F$.

The following theorem provides a concrete handle on flat functors.

**Theorem 2.7 (MM VII.6.3)** *A functor* $A : \mathcal{C} \to \text{Set}$ *is flat if and only if* $A$ *is filtering.*

**Proof**. Assume that $A : \mathcal{C} \to \text{Set}$ is flat. By definition, the following diagram commutes up to isomorphism:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "A"] \arrow[dr, "y_{\mathcal{C}}"'] & & \text{Set} \\
& \widehat{\mathcal{C}} \arrow[ur, "(-)\otimes_{\mathcal{C}} A"'] &
\end{tikzcd}$$

So, $y_C \otimes_{\mathcal{C}} A \simeq A(C)$, for objects $C$ of $\mathcal{C}$. We check the conditions for a filtering category.

i) Since $(-) \otimes_{\mathcal{C}} A$ preserves terminal objects, $1 \otimes_{\mathcal{C}} A$ is a one-point set. This shows that $A$ is nonempty.

ii) Since $(-) \otimes_{\mathcal{C}} A$ preserves binary products, we have that the map

$$(y_C \times y_D) \otimes_{\mathcal{C}} A \to A(C) \times A(D)((B \stackrel{u}{\to} C, B \stackrel{v}{\to} D), a) \mapsto (u{\cdot}a, v{\cdot}a)$$

must be an isomorphism; in particular it is surjective. That is condition ii) of the definition of a filtering functor.

iii) Finally, consider a parallel pair $C \overset{u}{\underset{v}{\rightrightarrows}} D$ in $\mathcal{C}$ and an element $a \in A(C)$ such that $u{\cdot}a = v{\cdot}a$ (that is, a parallel pair in $\mathrm{Elts}(A)$). Let

$$P \longrightarrow y_C \overset{y_u}{\underset{y_v}{\rightrightarrows}} y_D$$

be an equalizer diagram in $\widehat{\mathcal{C}}$. Since $(-) \otimes_{\mathcal{C}} A$ preserves equalizers, we have an equalizer diagram

$$P \otimes_{\mathcal{C}} A \xrightarrow{i} A(C) \overset{A(u)}{\underset{A(v)}{\rightrightarrows}} A(D)$$

in Set. Here, for $w \in P(B), b \in A(B)$, $i(w \otimes b) = w{\cdot}b \in A(C)$. Since $u{\cdot}a = v{\cdot}a$, there must be some pair $(w, b)$ for which $i(w \otimes b) = a$. This gives condition iii) of the definition of a filtering functor.

For the converse, only a sketch: suppose $A$ is filtering. Now for $R \in \widehat{\mathcal{C}}$, the set $R \otimes_{\mathcal{C}} A$ is a quotient of the sum $\sum_{C \in \mathcal{C}} R(C) \times A(C)$ by the equivalence relation $\sim$ generated by the set of equivalent pairs $((r{\cdot}g, a), (r, g{\cdot}a))$ for $r \in R(C), a \in A(C')$ and $g : C' \to C$. However, given that $A$ is filtering this can be simplified. We have: $(r, a) \in R(C) \times A(C)$ is equivalent to $(r', a') \in R(C') \times A(C')$ if and only if there is a diagram $C \xleftarrow{u} D \xrightarrow{v} C'$ in $\mathcal{C}$ and an element $b \in A(D)$ such that the equations

$$u{\cdot}b = a \quad v{\cdot}b = a' \quad r{\cdot}u = r{\cdot}v$$

hold. From this definition, it is straightforward to prove that $(-) \otimes_{\mathcal{C}} A$ preserves finite limits. ∎

**Corollary 2.8 (MM VII.6.4)** *Suppose $\mathcal{C}$ is a category with finite limits. Then a functor $A : \mathcal{C} \to \mathrm{Set}$ is flat if and only if it preserves finite limits.*

**Proof**. Again we use that the composite functor $((-) \otimes_{\mathcal{C}} A) \circ y : \mathcal{C} \to \mathrm{Set}$ is naturally isomorphic to $A$. If $A$ is flat, then $(-) \otimes_{\mathcal{C}} A$ preserves finite limits and $y$ always preserves existing finite limits, so then $A$ preserves all finite limits. Note, that this direction does not require $\mathcal{C}$ to have all finite limits.

Conversely, suppose $\mathcal{C}$ has finite limits and $A$ preserves them. Then $A$ is filtering:

i) $A(1) = 1$, so $A$ is nonempty.

ii) We have $A(C) \times A(D) \simeq A(C \times D)$ so in condition ii) of Definition 2.5 we can take the projections $C \xleftarrow{\pi_C} C \times D \xrightarrow{\pi_D} D$ and appropriate element of $A(C \times D)$.

iii) By a similar argument, now involving an equalizer in $\mathcal{C}$.

■

**Corollary 2.9 (MM VII.6.5)** *Let $\mathcal{D}$ be a small category. Then the colimit functor* $\mathrm{Set}^{\mathcal{D}} \to \mathrm{Set}$ *preserves finite limits if and only if $\mathcal{D}^{\mathrm{op}}$ is filtering.*

**Remark 2.10** In standard text books in category theory, for example MacLane, one finds a dual definition of "filtering" (i.e., a category is "filtering" in MacLane's sense if its opposite category is filtering in our sense). For this notion of filtering, part of Corollary 2.9 is contained in the slogan that "filtered colimits commute with finite limits in Set".

**Exercise 15** Deduce Corollary 2.9.

## 2.2 Geometric Morphisms $\mathcal{E} \to \widehat{\mathcal{C}}$ for cocomplete $\mathcal{E}$

The universal property of the Yoneda embedding $y : \mathcal{C} \to \widehat{\mathcal{C}}$ ($\widehat{\mathcal{C}}$ being the free cocompletion of $\mathcal{C}$) holds with respect to all cocomplete categories, not just Set. Therefore, every geometric morphism $f : \mathcal{E} \to \widehat{\mathcal{C}}$ is determined by the composite functor $f^* \circ y : \mathcal{C} \to \mathcal{E}$. Again, we have a suitably defined "tensor product" $X \otimes_{\mathcal{C}} A$ (when $A : \mathcal{C} \to \mathcal{E}$ is a functor and $X \in \widehat{\mathcal{C}}$), which is now defined as a colimit in $\mathcal{E}$ rather than in Set.

We cannot write down exactly the same formula for what will be the functor $(-) \otimes_{\mathcal{C}} A$ as we did for the case of Set, as something like "$X(C') \times \mathcal{C}(C, C') \times A(C)$" is not meaningful: $X(C')$ and $\mathcal{C}(C, C')$ are sets but $A(C)$ is an object of $\mathcal{E}$. However, using the cocompleteness of $\mathcal{E}$ we have the expression $\sum_{x \in X(C'), f : C \to C'} A(C')$ which, in the case of $\mathcal{E} = \mathrm{Set}$, is the same thing. Let, for a coproduct $\sum_{i \in I} X_i$, $\mu_i : X_i \to \sum_{i \in I} X_i$ denote the $i$'th coprojection. Then we define $X \otimes_{\mathcal{C}} A$ as the coequalizer

$$\textstyle\sum_{C \in \mathcal{C}, x \in X(C), f : C' \to C} A(C') \underset{\tau}{\overset{\theta}{\rightrightarrows}} \sum_{C \in \mathcal{C}, x \in X(C)} A(C) \longrightarrow X \otimes_{\mathcal{C}} A$$

where $\theta = [\theta_{C,x,f}]_{C \in \mathcal{C}, x \in X(C), f : C' \to C}$; and $\theta_{C,x,f}$ is defined to be the composite

$$A(C') \xrightarrow{A(f)} A(C) \xrightarrow{\mu_{C,x}} \sum_{C \in \mathcal{C}, x \in X(C)} A(C).$$

Likewise, $\tau = [\tau_{C,x,f}]_{C\in\mathcal{C},x\in X(C),f:C'\to C}$ where $\tau_{C,x,f}$ is the map

$$A(C) \stackrel{\mu_{C',x\cdot f}}{\longrightarrow} \sum_{C\in\mathcal{C},x\in X(C)} A(C).$$

Again, we define the functor $A : \mathcal{C} \to \mathcal{E}$ to be *flat* if the functor $(-) \otimes_{\mathcal{C}} A : \widehat{\mathcal{C}} \to \mathcal{E}$ preserves finite limits. And we have a similar notion of filtering as in 2.6:

**Definition 2.11 (MM VII.8.1)** A functor $A : \mathcal{C} \to \mathcal{E}$ is *filtering* if the following conditions hold:

i) The family of all maps $A(C) \to 1$ is epimorphic.

ii) For objects $C, D$ of $\mathcal{C}$, the family of maps

$$\{\langle A(u), A(v)\rangle : A(B) \to A(C) \times A(D) \,|\, u : B \to C, v : B \to D\}$$

is epimorphic.

iii) For any parallel pair of arrows $u, v : C \to D$ in $\mathcal{C}$ and equalizer diagram

$$E_{u,v} \xrightarrow{e} A(C) \overset{A(u)}{\underset{A(v)}{\rightrightarrows}} A(D)$$

in $\mathcal{E}$, the family of all arrows

$$\{A(B) \xrightarrow{f} E_{u,v} \,|\, \text{for some } w : B \to C \text{ in } \mathcal{C} \text{ with } uw = vw,\ ef = A(w)\}$$

is epimorphic.

Without proof, we record:

**Theorem 2.12 (MM VII.9.1)** *Let $\mathcal{E}$ be a cocomplete topos, and $\mathcal{C}$ a small category. Then a functor $A : \mathcal{C} \to \mathcal{E}$ is flat if and only if it is filtering.*

We see that geometric morphisms $\mathcal{E} \to \widehat{\mathcal{C}}$ correspond to filtering functors $\mathcal{C} \to \mathcal{E}$, for cocomplete $\mathcal{E}$.

## 2.3 Geometric morphisms to $\mathcal{E} \to \mathrm{Sh}(\mathcal{C}, J)$ for cocomplete $\mathcal{E}$

Recall that we use the letter $J$ to denote a general Grothendieck topology; so $J(C)$ is a collection of covering sieves on $C$ (where $C$ is an object of $\mathcal{C}$). Also recall that a sieve on $C$ can be regarded as a subobject of the representable presheaf $y_C$. Finally, we established in the basic course that an object $X$ of $\widehat{\mathcal{C}}$ is a sheaf for $J$, if and only if for every object $C$ of $\mathcal{C}$ and every $J$-covering sieve $R$ on $C$, any diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & y_C \\ X & \end{tikzcd}$$

has a *unique* filler: an arrow $y_C \to X$ making the triangle commute.
For the remainder of this section, $\mathcal{E}$ will always be a *cocomplete* topos.

**Exercise 16** Let $i : \mathrm{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ the geometric morphism where $i_*$ is the inclusion and $i^*$ is sheafification. Suppose $p : \mathcal{E} \to \widehat{\mathcal{C}}$ is a geometric morphism such that the direct image $p_*$ factors through $i_*$ by a functor $q : \mathcal{E} \to \mathrm{Sh}(\mathcal{C}, J)$. Show that the composite $p^*i_*$ is left adjoint to $q$ and conclude that the inverse image $p^*$ is isomorphic to a functor which factors through $\mathrm{Sh}(\mathcal{C}, J)$.

Exercise 16 tells us that a geometric morphism $p : \mathcal{E} \to \widehat{\mathcal{C}}$ factors through $\mathrm{Sh}(\mathcal{C}, J)$ if and only if every object $p_*(E)$ is a sheaf for $J$. The following exercise gives us a criterion for when this is the case.

**Exercise 17** Let $p : \mathcal{E} \to \widehat{\mathcal{C}}$ be a geometric morphism, and let $J$ be a Grothendieck topology on $\mathcal{C}$. Then the following two statements are equivalent:

i) For every object $E$ of $\mathcal{E}$, $p_*E$ is a sheaf for $J$.

ii) For every $J$-covering sieve $R$ on $C$, $p^*$ sends the inclusion $R \to y_C$ to an isomorphism in $\mathcal{E}$.

Now we characterized geometric morphisms $\mathcal{E} \to \widehat{\mathcal{C}}$ by flat functors $\mathcal{C} \to \mathcal{E}$; so we would like to characterize also geometric morphisms $p : \mathcal{E} \to \mathrm{Sh}(\mathcal{C}, J)$ in terms of such functors. Every such geometric morphism determines a geometric morphism into $\widehat{\mathcal{C}}$, hence a flat functor $A : \mathcal{C} \to \mathcal{E}$; we need to see which flat functors give rise to geometric morphisms which factor through $\mathrm{Sh}(\mathcal{C}, J)$. It should not be a surprise that we can characterize these functors by their behaviour on covering sieves, now seen as diagrams in $\mathcal{C}$: every sieve on $C$ is a diagram of arrows with codomain $C$.

**Lemma 2.13 (MM VII.7.3)** *Let $J$ be a Grothendieck topology on a small category $\mathcal{C}$, and let $f : \mathcal{E} \to \widehat{\mathcal{C}}$ be a geometric morphism. Then the following statements are equivalent:*

*i)* *The geometric morphism $f$ factors through* $\mathrm{Sh}(\mathcal{C}, J)$.

*ii)* *The composite $f^* \circ y : \mathcal{C} \to \mathcal{E}$ sends $J$-covering sieves to colimiting cocones in $\mathcal{E}$.*

*iii)* *The composite $f^* \circ y$ sends $J$-covering sieves to epimorphic families in $\mathcal{E}$.*

**Definition 2.14** A functor $A : \mathcal{C} \to \mathcal{E}$ is called *continuous* if it has the properties of the composite $f^* \circ y$ in Lemma 2.13.

We can now state:

**Theorem 2.15 (MM, Corollary VII.7.4)** *There is an equivalence of categories between*

$$\mathcal{T}op(\mathcal{E}, \mathrm{Sh}(\mathcal{C}, J))$$

*and the category of flat and continuous functors $\mathcal{C} \to \mathcal{E}$.*

## 2.4 Surjections and the Topos of Coalgebras

**Theorem 2.16 (MM V.8.4; PTJ 2.32)** *Let $(G, \delta, \varepsilon)$ be a comonad on a topos $\mathcal{E}$ such that the functor $G$ preserves finite limits. Then the category $\mathcal{E}_G$ of $G$-coalgebras is a topos, and there is a geometric morphism*

$$\mathcal{E} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{E}_G$$

*where $f^*$ is the forgetful functor and $f_*$ the cofree coalgebra functor.*

**Proof**. Finite limits are created by $V$ the forgetful functor $\mathcal{E}_G \to \mathcal{E}$, since $G$ preserves finite limits; so $\mathcal{E}_G$ has finite limits.

Let $R : \mathcal{E} \to \mathcal{E}_G$ be the cofree coalgebra functor: $RX = GX \overset{\delta_X}{\to} G^2X$. For coalgebras $(A, s), (B, t), (C, u)$ we have:

$$\mathcal{E}(A \times B, C) \simeq \mathcal{E}(A, C^B) \simeq \mathcal{E}_G((A, s), R(C^B))$$

where $f : A \times B \to C$ corresponds to $\tilde{f} : A \to C^B$ and to $f' = G(\tilde{f}) \circ s : A \to G(C^B)$. Note that $f = \mathrm{ev} \circ (\tilde{f} \times \mathrm{id})$.

Now $f : A \times B \to C$ is a coalgebra map if and only if the following diagram commutes:

$$\begin{tikzcd}
A \times B \arrow[r, "s\times t"] \arrow[d, "f"'] & GA \times GB \arrow[r, "\sim"] & G(A \times B) \arrow[d, "G(f)"] \\
C \arrow[rr, "u"'] & & GC
\end{tikzcd}$$

We consider the exponential transposes of both compositions in this diagram. The clockwise composition transposes to

$$(*) \quad A \xrightarrow{f'} G(C^B) \xrightarrow{\rho} GC^{GB} \xrightarrow{GC^t} GC^B$$

where $\rho$ is the transpose of the map $G(C^B) \times GB \overset{\sim}{\to} G(C^B \times B) \overset{G(\mathrm{ev})}{\to} GC$.

The counterclockwise composition transposes to

$$(**) \quad A \xrightarrow{\tilde{f}} C^B \xrightarrow{u^B} GC^B$$

We wish to describe those maps $f : A \times B \to C$ which make these two transposes equal. Let $V : \mathcal{E}_G \to \mathcal{E}$ be the forgetful functor and $C$ the cofree coalgebra functor; we have $V \dashv C$ and $VC = G$. Under this adjunction, the map $(*)$ corresponds to the compositie

$$A \xrightarrow{f'} G(C^B) \xrightarrow{\delta} G^2(C^B) \xrightarrow{G\rho} G(GC^{GB}) \xrightarrow{G(GC^t)} G(GC^B)$$

and the map $(**)$ corresponds to the composite

$$A \xrightarrow{f'} G(C^B) \xrightarrow{G(u^B)} G(GC^B)$$

Note that both these composites are maps of coalgebras. So, the maps $f : A \times B \to C$ we are looking for, correspond to maps $\bar{f} : A \to E$, where

$$\begin{tikzcd}
 & & G^2(C^B) \arrow[r, "G\rho"] & G(GC^{GB}) \arrow[dr, "G(GC^t)"] & \\
E \arrow[r] & G(C^B) \arrow[ur, "\delta"] \arrow[rrr, "G(u^B)"'] & & & G(GC^b)
\end{tikzcd}$$

is an equalizer in $\mathcal{E}_G$ (equalizer of two maps between cofree coalgebras). So $E$ is the exponent $(C, u)^{(B,t)}$ in $\mathcal{E}_G$.

It remains to show that $\mathcal{E}_G$ has a subobject classifier. To this end we have a look at subobjects of $(A, s)$ in $\mathcal{E}_G$. Our first remark is that if $m : D \to A$ is a subobject of $A$ in $\mathcal{E}$, there is at most one coalgebra structure $d : D \to GD$ on $D$ such that $m$ is a coalgebra map. Indeed, for $m$ to be a coalgebra map we should have $G(m)d = sm$; now $G(m)$ is mono, so there is at most one such $d$.

On the other hand, if $m : D \to A$ is a subobject and $d : D \to GD$ is any map such that $G(m)d = sm$, then $(D, d)$ is a $G$-coalgebra and the square

$$\begin{tikzcd} D \arrow[r, "d"] \arrow[d, "m"'] & GD \arrow[d, "Gm"] \\ A \arrow[r, "s"'] & GA \end{tikzcd}$$

is a pullback in $\mathcal{E}$. To see this, consider

$$\begin{tikzcd} D \arrow[rrr, "d"] \arrow[dr, "m"] \arrow[ddd, "d"'] & & & GD \arrow[dl, "Gm"'] \arrow[ddd, "Gd"] \\ & A \arrow[r, "s"] \arrow[d, "s"'] & GA \arrow[d, "Gs"] & \\ & GA \arrow[r, "\delta_A"'] & G^2A & \\ GD \arrow[ur, "Gm"] \arrow[rrr, "\delta_D"'] & & & G^2D \arrow[ul, "G^2m"'] \end{tikzcd}$$

The inner square commutes since $(A, s)$ is a coalgebra. The three upper squares commute because of the assumption $G(m)d = sm$, and the lower square is a naturality square for $\delta$. Hence the outer square commutes, which says that the map $d$ is coassociative. To see that $d$ is also counitary, consider the diagram

$$\begin{tikzcd} D \arrow[r, "d"] \arrow[d, "m"'] & GD \arrow[r, "\varepsilon_D"] \arrow[d, "Gm"'] & D \arrow[d, "m"] \\ A \arrow[r, "s"'] & GA \arrow[r, "\varepsilon_A"'] & A \end{tikzcd}$$

Since $m(\varepsilon_D d) = m$ and $m$ is mono, $\varepsilon_D d = \mathrm{id}_D$. Moreover, one sees that the left hand square is a pullback.

Now suppose $m : (D, d) \to (A, s)$ is the inclusion of a subobject in $\mathcal{E}_G$. Let $\tau : G(\Omega) \to \Omega$ be the classifying map of the mono $1 \simeq G(1) \stackrel{G(t)}{\to} G(\Omega)$.

Let $h : A \to \Omega$ be the classifying map of $m$. In the diagram

$$\begin{tikzcd}
D \arrow[r, "d"] \arrow[d, "m"'] & GD \arrow[r] \arrow[d, "Gm"'] & 1 \arrow[r] \arrow[d, "G(t)"'] & 1 \arrow[d, "t"] \\
A \arrow[r, "s"'] & GA \arrow[r, "G(h)"'] & G(\Omega) \arrow[r, "\tau"'] & \Omega
\end{tikzcd}$$

all three squares are pullbacks (check!), and therefore $\tau G(h)s = h$ by uniqueness of the classifying map. Moreover, since $(A, s)$ is a coalgebra we have $G(h)s = G(\tau)\delta_\Omega G(h)s$, so if we form an equalizer

$$\Omega_G \xrightarrow{e} G(\Omega) \underset{\mathrm{id}}{\overset{G(\tau)\delta_\Omega}{\rightrightarrows}} G(\Omega)$$

(equalizer taken in $\mathcal{E}_G$, the two maps seen as maps between cofree coalgebras), then we see that the map $G(h)s$ factors through $\Omega_G$. Also the map $G(t) : 1 \to G(\Omega)$ factors through this equalizer by a map $e : 1 \to \Omega_G$, which is the subobject classifier of $\mathcal{E}_G$. ∎

**Corollary 2.17 (MM V.7.7)** *If $(T, \eta, \mu)$ is a monad on a topos $\mathcal{E}$ and the functor $T$ has a right adjoint, then the category of $T$-algebras is again a topos.*

**Proof**. Combine Theorems 0.15 and 2.16. ∎

To give an example, consider a monoid $M$: a set with an associative multiplication, for which it has a two-sided unit element. The functor $(-) \times M$ : Set $\to$ Set has the structure of a monad (using the multiplication and the unit element of $M$). The category of algebras for this monad is the category of right $M$-sets, i.e. the category $\widehat{M}$. Note that the functor $(-) \times M$ has a right adjoint $(-)^M$, so we have another proof that $\widehat{M}$ is a topos.

The construction of the topos $\mathcal{E}_G$ and its accompanying geometric morphism $\mathcal{E} \to \mathcal{E}_G$ (the inverse image part of which is the forgetful functor, which is faithful) motivates the following definition.

**Definition 2.18** A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is called a *surjection* if the inverse image functor $f^*$ is faithful.

**Lemma 2.19 (MM Vii.4.3)** *For a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ the following are equivalent:*

*i)* *The inverse image $f^*$ is faithful.*

*ii)* *Every component of the unit $\eta$ of the adjunction $f^* \dashv f_*$ is a monomorphism.*

*iii)* *The functor $f^*$ reflects isomorphisms.*

*iv)* *The functor $f^*$ induces an injective homomorphism of lattices $\mathrm{Sub}_{\mathcal{E}}(E) \to \mathrm{Sub}_{\mathcal{F}}(f^*E)$.*

*v)* *The functor $f^*$ reflects the order on subobjects: for $A, B \in \mathrm{Sub}_{\mathcal{E}}(E)$, $f^*A \leq f^*B$ if and only if $A \leq B$.*

**Proof**. The equivalence (i)⇔(ii) is basic Category Theory.

For (i)⇒(iii): a faithful functor reflects monos and epis, and a topos is balanced (1.3).

For (iii)⇒(iv): Since $f^*$ preserves monos, it induces a map on subobjects. Furthermore $f^*$ preserves images and coproducts, hence unions of subobjects; also, $f^*$ preserves intersections. So $f^*$ induces a lattice homomorphism. Since $f^*$ reflects isomorphisms, it is injective.

For (iv)⇒(v): If $f^*A \leq f^*B$ then $f^*A = f^*A \cap f^*B = f^*(A \cap B)$ because $f^*$ is a lattice homomorphism. Hence $A = A \cap B$ since $f^*$ is injective; so $A \leq B$.

For (v)⇒(i): if $X \underset{v}{\overset{u}{\rightrightarrows}} Y$ is a parallel pair with equalizer $E \stackrel{e}{\to} X$, then $f^*(u) = f^*(v)$ entails (since $f^*$ preserves equalizers) that $f^*(E)$ is the maximal subobject of $f^*X$. By (v), this entails that $E$ is the maximal subobject of $X$; in other words, $u = v$. So $f^*$ is faithful. ∎

**Proposition 2.20 (MM VII.4.4)** *A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is a surjection if and only if $\mathcal{E}$ is equivalent to the topos of coalgebras for a finite limit preserving comonad on $\mathcal{F}$ and $f$ is, modulo this equivalence, the cofree-forgetful geometric morphism.*

**Proof**. One direction is clear, since the forgetful functor is always faithful. For the other, suppose $f$ is a surjection and consider the comonad $f^*f_*$ on $\mathcal{F}$. Let us spell out the dual version of Beck's Crude Tripleability Theorem (0.10):

CTT$^{\mathrm{op}}$ Let $A \underset{U}{\overset{F}{\leftrightarrows}} C$ be an adjunction with $F \dashv U$. Suppose $C$ has equalizers of coreflexive pairs, $F$ preserves them and $F$ reflects isomorphisms. Then the functor $F$ is comonadic.

It is clear that for a surjection $f$, the conditions are satisfied. The conclusion follows. ∎

**Examples 2.21** 1) For a continuous map $f$ of topological $T_1$-spaces, the induced geometric morphism is a surjection if and only if the map $f$ is surjective (**MM**, start of §VII.4).

2) For a morphism $f : A \to B$ in a topos $\mathcal{E}$, the induced geometric morphism $\mathcal{E}/A \to \mathcal{E}/B$ is a surjection if and only if $f$ is an epimorphism.

3) For a functor $F : \mathcal{C} \to \mathcal{D}$ between small categories, the induced geometric morphism $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ of Example 2.1 4) is a surjection if and only if every object of $\mathcal{D}$ is a retract of an object in the image of $F$ (**Elephant**, A4.2.7).

## 2.5 Embeddings and Sheaf Subtoposes

In this section we work again in an arbitrary (not necessarily cocomplete) topos $\mathcal{E}$. First we establish an internalization of the intersection ($\cap$) operation on subobjects.

**Proposition 2.22** *Let $1 \xrightarrow{t} \Omega$ be a subobject classifier and denote by $\wedge : \Omega \times \Omega \to \Omega$ the classifying map of the monomorphism $1 \xrightarrow{\langle t,t \rangle} \Omega \times \Omega$. Then for subobjects $M, N$ of $X$ we have: if $M$ is classified by $\phi : X \to \Omega$ and $N$ by $\psi : X \times \Omega$ then the intersection $M \cap N$ is classified by the composite*

$$X \xrightarrow{\langle \phi,\psi \rangle} \Omega \times \Omega \xrightarrow{\wedge} \Omega.$$

**Proof**. Consider maps $f : Y \to X$. If $\langle \phi, \psi \rangle \circ f : Y \to \Omega \times \Omega$ is equal to $\langle t \circ !, t \circ ! \rangle : Y \to \Omega \times \Omega$, then $\phi f = t!$ and $\psi f = t!$, so $f$ factors both through $M$ and through $N$, hence $f$ factors through the intersection $M \cap N$. We conclude that the diagram

$$\begin{tikzcd} M \wedge N \arrow[r] \arrow[d] & X \arrow[d, "\langle \phi,\psi \rangle"] \\ 1 \arrow[r, "\langle t,t \rangle"'] & \Omega \times \Omega \end{tikzcd}$$

is a pullback, and the statement follows. ∎

**Definition 2.23** A *Lawvere-Tierney topology* (MM) or simply *topology* (PTJ) in a topos $\mathcal{E}$ is an arrow $j : \Omega \to \Omega$ with the following properties:

i) $jt = t$ :

$$\begin{tikzcd} 1 \arrow[r, "t"] \arrow[dr, "t"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd}$$

ii) $jj = j$ :

$$\begin{tikzcd} \Omega \arrow[r, "j"] \arrow[dr, "j"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd}$$

iii) $j{\circ}\wedge = \wedge{\circ}(j \times j)$ :

$$\begin{tikzcd} \Omega \times \Omega \arrow[r, "\wedge"] \arrow[d, "j\times j"'] & \Omega \arrow[d, "j"] \\ \Omega \times \Omega \arrow[r, "\wedge"'] & \Omega \end{tikzcd}$$

In the course *Basic Category Theory and Topos Theory* we have seen that for $\mathcal{E} = \widehat{\mathcal{C}}$, Lawvere-Tierney topologies correspond to Grothendieck topologies on $\mathcal{C}$.

**Definition 2.24 (PTJ 3.13)** A *universal closure operation* on a topos $\mathcal{E}$ is given by, for each object $X$, a map $c_X : \mathrm{Sub}(X) \to \mathrm{Sub}(X)$, which system has the following properties:

i) $M \leq c_X(M)$ for every subobject $M$ of $X$ (the operation is *inflationary*).

ii) $M \leq N$ implies $c_X(M) \leq c_X(N)$ for $M, N \in \mathrm{Sub}(X)$ (the operation is order-preserving).

iii) $c_X(c_X(M)) = c_X(M)$ for each $M \in \mathrm{Sub}(X)$ (the operation is idempotent).

iv) For every arrow $f : Y \to X$ and every $M \in \mathrm{Sub}(X)$ we have

$$c_Y(f^*(M)) = f^*(c_X(M))$$

(the operation is *stable*).

Instead of $c_X(M)$ we shall also sometimes write $\overline{M}$, if the subobject lattice in which we work is clear.

**Exercise 18** Use the stability (requirement iv) of 2.24) to deduce that a closure operation commutes with finite intersections: $\overline{M \cap N} = \overline{M} \cap \overline{N}$.

Note that the result of Exercise 18 means that a universal closure operation is different from "closure" in Topology, where closure commutes with *union*, not with intersection of subsets.

**Proposition 2.25 (MM V.1.1; PTJ 3.14)** *There is a bijection between universal closure operations and Lawvere-Tierney topologies.*

**Proof**. If $j$ is a Lawvere-Tierney topology, define for $M \in \mathrm{Sub}(X)$, classified by $\phi : X \to \Omega$, $\overline{M}$ as the subobject of $X$ classified by $j\phi$. We use the letter $J$ to denote the subobject of $\Omega$ classified by $j$:

$$\begin{tikzcd} J \arrow[r] \arrow[d] & \Omega \arrow[d, "j"] \\ 1 \arrow[r, "t"'] & \Omega \end{tikzcd}$$

We see that $J$ is the closure of the subobject $(1 \xrightarrow{t} \Omega)$. We have: $\overline{M}$ is the vertex of the pullback

$$\begin{tikzcd} \overline{M} \arrow[r] \arrow[d] & X \arrow[d, "\phi"] \\ J \arrow[r] & \Omega \end{tikzcd}$$

and we conclude that $M \leq \overline{M}$. The other properties of the universal closure operation are straightforward and left to you.

In the other direction, given a universal closure operation $c_X(-)$, let $j$ be the classifying map of $c_\Omega(1 \xrightarrow{t} \Omega)$. The verification of the properties of a Lawvere-Tierney topology, as well as that the two described operations are inverse to each other, is again left to you. ∎

**Definition 2.26** Given a Lawvere-Tierney topology $j$ with associated closure operation $c_X(-)$ (or $\overline{(-)}$), we call a subobject $M$ of $X$:

> *dense* if $\overline{M} = X$
>
> *closed* if $\overline{M} = M$.

**Definition 2.27** Consider, for an object $X$, partial maps into $X$ with domain a dense subobject:

$$\begin{tikzcd} M' \arrow[r, "m"] \arrow[dr] & M \\ & X \end{tikzcd}$$

with $m : M' \to M$ a dense mono (i.e., the subobject represented by the mono $m$ is dense).

> The object $X$ is called *separated for* $j$ if any such partial map has *at most one* extension to a map $M \to X$.
>
> The object $X$ is called a *sheaf* for $j$ (or a $j$-sheaf) if any such partial map has *exactly one* extension to a map $M \to X$.

We write $\mathrm{Sh}_j(\mathcal{E})$ for the full subcategory of $\mathcal{E}$ on the sheaves for $j$.

**Theorem 2.28 (MM V.2.5; PTJ §3.2)** *For any topos $\mathcal{E}$ with Lawvere-Tierney topology $j$, the category $\mathrm{Sh}_j(\mathcal{E})$ is a topos. The inclusion functor $\mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$ preserves finite limits and exponentials, and $\mathrm{Sh}_j(\mathcal{E})$ is closed under finite limits in $\mathcal{E}$.*

**Proof**. Suppose $\mathcal{I}$ is a finite category and $X : \mathcal{I} \to \mathrm{Sh}_j(\mathcal{E})$ a functor with limiting cone $(N, \mu)$ in $\mathcal{E}$. Given a diagram

$$\begin{tikzcd} M' \arrow[r, "m"] \arrow[dr] & M \\ & N \end{tikzcd}$$

we have partial maps $M \rightharpoonup X(i)$ for all objects $i$ of $\mathcal{I}$, and these partial maps have unique extensions $M \to X(i)$ since the $X(i)$ are sheaves. Therefore we have a cone for $X$ with vertex $M$ and hence a unique map of cones $M \to N$, which is also the unique extension of the given partial map. Therefore, $N$ is a sheaf and we see that $\mathrm{Sh}_j(\mathcal{E})$ is closed under the finite limits of $\mathcal{E}$, that it has finite limits and that the inclusion preserves them.

Secondly, if $F$ is a sheaf, then the exponential $F^Y$ is a sheaf, for any object $Y$. For, given a partial map

$$\begin{tikzcd} M' \arrow[r, "m"] \arrow[dr] & M \\ & F^Y \end{tikzcd}$$

with $m$ dense, this diagram transposes under the exponential adjunction to a partial map

$$\begin{tikzcd} M' \times Y \arrow[r, "m \times \mathrm{id}"] \arrow[dr] & M \times Y \\ & F \end{tikzcd}$$

Now by the stability of the closure operation, the subobject $M' \times Y \stackrel{m\times\mathrm{id}}{\longrightarrow} M \times Y$ is dense. Sine $F$ is a sheaf we have a unique extension $M \times Y \to F$, which transposes back to give a unique extension for the original diagram. We conclude that $\mathrm{Sh}_j(\mathcal{E})$ is cartesian closed and that the inlusion into $\mathcal{E}$ preserves exponentials.

For the subobject classifier of $\mathrm{Sh}_j(\mathcal{E})$ we need an intermediate result, which we have already seen in the case $\mathcal{E} = \widehat{\mathcal{C}}$.

**Lemma 2.29** *Let $M$ be a sheaf and $M'$ a subobject of $M$. Then $M'$ is a sheaf if and only if $M'$ is closed in $M$.*

**Proof**. Suppose $M'$ is closed in $M$ and $M' \xleftarrow{f} N' \longrightarrow N$ is a partial map with $N'$ dense in $N$. Let $i : M' \to M$ be the inclusion. Now $i \circ f$ has a unique extension $g : N \to M$. Let

$$\begin{tikzcd} L \arrow[r] \arrow[d] & N \arrow[d, "g"] \\ M' \arrow[r] & M \end{tikzcd}$$

be a pullback. Then $f : N' \to M'$ factors through $L \to M'$, so $N' \leq L$ as subobjects of $N$, but $L$ is closed (since it is a pullback of $M' \to M$) and $N'$ is dense. We see that $N = \overline{N'} \leq \overline{L} = L$, so $L \to N$ is an isomorphism and we have $g : N \to M'$. So $M'$ is a sheaf.

Conversely if $M' \in \mathrm{Sub}(M)$ is a sheaf, consider the partial map

$$\begin{tikzcd} M' \arrow[r] \arrow[d, "\mathrm{id}"'] & \overline{M'} \\ M' & \end{tikzcd}$$

Since $M' \to \overline{M'}$ is dense, there is a unique extension $\overline{M'} \to M'$. It follows that $M' = \overline{M'}$, so $M'$ is closed in $M$. ∎

Returning to the proof of 2.28: closed subobjects of $X$ are classified by maps of the form $j\phi$, hence their classifying maps land in the image of $j$, which is (by the idempotence of $j$) the equalizer

$$\Omega_j \longrightarrow \Omega \underset{j}{\overset{\mathrm{id}}{\rightrightarrows}} \Omega$$

Hence, $\Omega_j$ is a subobject classifier for $\mathrm{Sh}_j(\mathcal{E})$ provided we can show that it is a sheaf.

Now partial maps $\Omega_j \longleftarrow M' \longrightarrow M$ correspond to closed subobjects of $M'$. But given that $M'$ is dense in $M$, there is an order-preserving bijection between the closed subobjects of $M'$ and of $M$, given as follows: for $A$ closed in $M$, we have $A \cap M'$ closed in $M'$ and for $B$ closed in $M'$ we have $c_M(B)$ closed in $M$. To see that these operations are each other's inverse, observe that for $A$ closed in $M$:

$$c_M(A \cap M') = c_M(A) \cap c_M(M') = c_M(A) = A$$

and for $B$ closed in $M'$ we have

$$c_M(B) \cap M' = c_{M'}(B) = B$$

The given partial map has therefore a unique extension $M \to \Omega_j$ (the classifier of the closed subobject of $M$ corresponding to the closed subobject of $M'$ classified by the partial map); and $\Omega_j$ is a sheaf, as desired. ∎

**Proposition 2.30** *For an object $X$ of $\mathcal{E}$ the following are equivalent:*

*i)* *$X$ is $j$-separated.*

*ii)* *$X$ is a subobject of a $j$-sheaf.*

*iii)* *$X$ is a subobject of a sheaf of the form $\Omega_j^E$.*

*iv)* *The diagonal $\delta : X \to X \times X$ is a $j$-closed subobject of $X \times X$.*

**Proof**. We prove i)⇒iv)⇒iii)⇒ii)⇒i).

For i)⇒iv): let $X$ be separated and let $\overline{\delta}$ be the closure of $\delta$ as subobject of $X \times X$. Consider the partial map

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\mathrm{id}"'] & \overline{\delta} \\ X & \end{tikzcd}$$

If $i : \overline{\delta} \to X \times X$ is the inclusion and $p_1, p_2 : X \times X \to X$ are the projections, then both $p_1 i$ and $p_2 i$ are fillers for this diagram, so since $X$ is separated, $p_1 i = p_2 i$. This means that $i : \overline{\delta} \to X \times X$ factors through the equalizer of $p_1$ and $p_2$, which is $\delta$. So $\overline{\delta} = \delta$ as subobjects of $X \times X$.

For iv)⇒iii): Let $\Delta : X \times X \to \Omega$ classify the diagonal $\delta$, and $\{\cdot\} : X \to \Omega^X$ its exponential transpose, which is a monomorphism. Since $\delta$ is closed in $X \times X$, $\Delta$ factors through $\Omega_j$, and therefore $\{\cdot\}$ factors through $\Omega_j^X$. So $X$ is a subobject of $\Omega_j^X$, which is a $j$-sheaf.

The implication iii)⇒ii) is trivial.

For ii)⇒i): Let $X \xrightarrow{i} F$ be mono, with $F$ a $j$-sheaf. Suppose that

$$\begin{tikzcd} M' \arrow[r, "m"] \arrow[d, "k"'] & M \\ X & \end{tikzcd}$$

is a partial map with $m$ a dense mono. If both of $f, g : M \to X$ are fillers for this diagram then $if = ig$ since $F$ is a sheaf; hence $f = g$ since $i$ is mono. So $X$ is $j$-separated. ∎

**Lemma 2.31** *Let $j$ be a Lawvere-Tierney topology in a topos $\mathcal{E}$, and let $X$ be an object of $\mathcal{E}$. As usual, we denote the diagonal subobject of $X \times X$ by $\delta$ and its closure by $\overline{\delta}$.*

*a) If $f, g : Z \to X$ is a parallel pair of arrows into $X$, then the morphism $\langle f, g\rangle : Z \to X \times X$ factors through $\overline{\delta}$ if and only if the equalizer of $f$ and $g$ is a $j$-dense subobject of $Z$.*

*b) The subobject $\overline{\delta}$ of $X \times X$ is an equivalence relation on $X$.*

*c) Let $X \to MX$ be the coequalizer of the pair $\overline{\delta} \rightrightarrows X$. Then any map $X \to L$, for a $j$-separated object $L$ of $\mathcal{E}$, factors uniquely through $X \to MX$. Hence the assignment $X \mapsto MX$ induces a functor which is left adjoint to the inclusion $\mathrm{sep}_j(\mathcal{E}) \to \mathcal{E}$, where $\mathrm{sep}_j(\mathcal{E})$ denotes the full subcategory of $\mathcal{E}$ on the $j$-separated objects.*

**Proof**. a) Let $E_{fg} \to Z$ denote the equalizer of $f, g$. Consider the diagram:

$$\begin{tikzcd} & E' \arrow[dr] \arrow[dd] & \\ E_{fg} \arrow[ur] \arrow[rr] \arrow[dd] & & Z \arrow[dd, "\langle f,g\rangle"] \\ & \overline{\delta} \arrow[dr] & \\ X \arrow[ur] \arrow[rr, "\delta"'] & & X \times X \end{tikzcd}$$

where all the squares are pullbacks. We see that $E'$ is the closure of $E_{fg}$, and we see that the map $\langle f, g\rangle$ factors through $\overline{\delta}$ if and only if $E' \to Z$ is an isomorphism, which holds if and only if $E_{fg}$ is a dense subobject of $Z$.

b) We prove that for an arbitrary object $Z$ of $\mathcal{E}$, the set of ordered pairs

$$\{(f,g) \in \mathcal{E}(Z,X)^2 \mid \langle f,g\rangle \text{ factors through } \overline{\delta}\}$$

is an equivalence relation on $\mathcal{E}(Z,X)$. Now reflexivity and symmetry are obvious, and using the notation above for equalizers we easily see that $E_{fg} \wedge E_{gh} \le E_{fh}$. Since the meet of two dense subobjects is dense, we see that the relation is transitive.

c) We have to prove that any map $f : X \to L$ with $L$ separated, coequalizes the parallel pair $r_0, r_1 : \overline{\delta} \to X$ which is the equivalence relation from part b). Now clearly for $f \times f : X \times X \to L \times L$, the composite $(f \times f)\circ\delta$ factors through the diagonal subobject $L \overset{\delta_L}{\to} L \times L$, so the composite $(f \times f)\circ\langle r_0, r_1\rangle$ factors through the closure of $\delta_L$. But $\delta_L$ is closed by Proposition 2.30iv), so $fr_0 = fr_1$ and $f$ factors uniquely through $X \to MX$. The adjointness is also clear, provided we can show that $MX$ is separated. Now $\delta$ is classified by $\Delta : X \times X \to \Omega$, which has as exponential transpose the map $\{\cdot\} : X \to \Omega^X$. So, $\delta$ is the kernel pair of $\{\cdot\}$. Now $\overline{\delta}$ is classified by $j\circ\Delta$, the exponential transpose of which is $j^X\circ\{\cdot\} : X \to \Omega_j^X$. And $\overline{\delta}$ is the kernel pair of $j^X\circ\{\cdot\}$. We see that, by the construction of epi-mono factorizations in a regular category, $X \to MX \to \Omega_j^X$ is an epi-mono factorization. So $MX$ is a subobject of a sheaf, and therefore separated by 2.30. ∎

**Lemma 2.32** *Suppose we have an operation which, to any object $X$ of $\mathcal{E}$, assigns a sheaf $LX$ and a dense inclusion $MX \overset{i_X}{\to} LX$. Then this extends to a unique functor $L : \mathcal{E} \to \mathcal{E}$. Moreover, this functor has the property that for every $X$, every map from $X$ to a sheaf factors uniquely through the composite $X \to MX \overset{i_X}{\to} LX$, so $L : \mathcal{E} \to \mathrm{Sh}_j(\mathcal{E})$ is left adjoint to the inclusion of sheaves.*

**Proof**. For $f : X \to X'$, define $Lf : LX \to LX'$ as the unique filler for the partial map

$$\begin{array}{ccc} MX & \xrightarrow{i_X} & LX \\ {\scriptstyle i_{X'}\circ Mf}\downarrow & & \\ LX' & & . \end{array}$$

The functoriality and the adjointness follow at once. ∎

**Theorem 2.33** *The inclusion functor $\mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$ has a left adjoint which preserves finite limits. Hence, we have a geometric morphism $i : \mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$.*

**Proof**. Let, as before, $\Delta : X \times X \to \Omega$ classify the diagonal $\delta : X \to X \times X$. Then $j \circ \Delta : X \times X \to \Omega_j$ classifies the closure $\overline{\delta}$; let $\overline{\{\cdot\}} : X \to \Omega_j^X$ be its exponential transpose. One can easily verify that the kernel pair of $\overline{\{\cdot\}}$ is $\overline{\delta}$, so $\overline{\{\cdot\}}$ factors as $X \to MX \to \Omega_j^X$ which, since a topos is regular, is the epi-mono factorization of $\overline{\{\cdot\}}$. Let $LX$ be the closure of the subobject $MX$ of $\Omega_j^X$. Then we have the assumptions of Lemma 2.32 verified, so $L$ is a functor left adjoint to the inclusion $\mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$. We need to prove that $L$ preserves finite limits. The following proof is taken from **Elephant**, A4.4.7.

First of all, we have seen in the proof of Theorem 2.28 that $\mathrm{sh}_j(\mathcal{E})$ is an exponential ideal in $\mathcal{E}$ (for a sheaf $F$ and an arbitrary $X$, $F^X$ is a sheaf). From this, it follows easily that $L$ preserves finite products: for objects $A$ and $B$ of $\mathcal{E}$ and a sheaf $F$, we have the following natural bijections:

$$\begin{aligned}&\mathcal{E}(L(A \times B), F) \simeq \mathcal{E}(A \times B, F) \simeq \mathcal{E}(A, F^B) \simeq \mathcal{E}(LA, F^B) \simeq \\ &\mathcal{E}(B, F^{LA}) \simeq \mathcal{E}(LB, F^{LA}) \simeq \mathcal{E}(LA \times LB, F)\end{aligned}$$

so $L(A \times B) \simeq LA \times LB$.

Furthermore, by Exercise 6, an object in $\mathrm{sh}_j(\mathcal{E})$ is injective if and only if it is a retract of some $\Omega_j^X$; since the inclusion $\mathrm{sh}_j(\mathcal{E}) \to \mathcal{E}$ preserves exponentials and since $\Omega_j$ is a retract of $\Omega$ (hence $\Omega_j^X$ is a retract of $\Omega^X$), we see that the inclusion preserves injective objects. Given that $\mathrm{sh}_j(\mathcal{E})$ has enough injectives, by the same exercise we have that $L$ preserves monos.

Now we wish to show that $L$ preserves "coreflexive equalizers". A *coreflexive pair* is a parallel pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ with common retraction $Y \xrightarrow{h} X$: $hf = hg = \mathrm{id}_X$. A coreflexive equalizer is an equalizer of a coreflexive pair.

In a category with finite products, every equalizer appears also as coreflexive equalizer: the arrow $E \xrightarrow{e} X$ is an equalizer of $f, g : X \to Y$ if and only if $e$ is an equalizer of the coreflexive pair $\langle \mathrm{id}_X, f\rangle, \langle \mathrm{id}_X, g\rangle : X \to X \times Y$ (which has as common retraction the projection $X \times Y \to X$). Therefore, if coreflexive equalizers exist, all equalizers exist and if coreflexive equalizers are preserved (by a product-preserving functor) then all equalizers are preserved.

Let $f, g : X \to Y$ be a coreflexive pair. You should check that $e : E \to X$ is an equalizer of $f, g$ if and only if the square

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, "e"'] & X \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

is a pullback. Therefore, if $e$ is an equalizer of $f, g$ then $E \to X$ is the meet (intersection) in $\mathrm{Sub}(Y)$ of the subobjects represented by $f$ and $g$. We wish therefore to show that $L$ preserves meets of subobjects.

To this end, let $M \stackrel{m}{\to} X, N \stackrel{n}{\to} X$ be monos representing subobjects $M$ and $N$, and let $M \cap N$, $M \cup N$ be their intersection and union. The square

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r] & M \cup N \end{tikzcd}$$

is a pushout in $\mathcal{E}$ by Proposition 1.29. Since $L$ is a left adjoint, the square

$$\begin{tikzcd} L(M \cap N) \arrow[r] \arrow[d] & LM \arrow[d] \\ LN \arrow[r] & L(M \cup N) \end{tikzcd}$$

is a pushout in $\mathrm{sh}_j(\mathcal{E})$. We know that $L$ preserves monos, so $L(M \cap N) \to LN$ is mono; so Corollary 1.9 applies and the square is also a pullback. Since also $L(M \cup N) \to LX$ is mono, also the square

$$\begin{tikzcd} L(M \cap N) \arrow[r] \arrow[d] & LM \arrow[d] \\ LN \arrow[r] & X \end{tikzcd}$$

is a pullback. We conclude that $L(M \cap N) = LM \cap LN$ so $L$ indeed preserves meets of subobjects. ∎

**Definition 2.34** A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is called an *embedding* if the direct image functor $f_*$ is full and faithful.

The geometric morphism of Theorem 2.33 is an embedding. Moreover we shall see that every embedding is of this form (Proposition 2.37).

**Examples 2.35** For our usual examples of geometric morphisms, we have:

1) Given a continuous map of topological spaces $f : X \to Y$, the associated geometric morphism $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is an embedding if and only if $f$ is an embedding of topological spaces (i.e. $X$ is a subspace of $Y$).

2) For a morphism $u : X \to Y$ in a topos $\mathcal{E}$, the geometric morphism $\mathcal{E}/X \to \mathcal{E}/Y$ is an embedding if and only if $u$ is mono.

3) For a functor $F : \mathcal{C} \to \mathcal{D}$ between small categories, the geometric morphism $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ is an embedding if and only if $F$ is full and faithful.

## 2.6 The Factorization Theorem

**Theorem 2.36 (MM VII.4.6)** *Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. There exists a Lawvere-Tierney topology $j$ in $\mathcal{E}$ such that $f$ factors as*

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{E} \\
& \mathrm{Sh}_j(\mathcal{E}) \arrow[ur, "i"'] &
\end{tikzcd}$$

*where $p$ is a surjection and $i$ is the geometric morphism from Theorem 2.33. Moreover, given another factorization $\mathcal{F} \xrightarrow{q} \mathcal{G} \xrightarrow{k} \mathcal{E}$ of $f$ with $q$ a surjection and $k$ an embedding, there is an equivalence $\mathcal{G} \to \mathrm{Sh}_j(\mathcal{E})$ which makes the following diagram commute:*

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "f"] \arrow[dr, "p"] \arrow[ddr, "q"'] & & \mathcal{E} \\
& \mathrm{Sh}_j(\mathcal{E}) \arrow[ur, "i"'] & \\
& \mathcal{G} \arrow[u] \arrow[uur, "k"'] &
\end{tikzcd}.$$

**Proof**. Consider the closure operation $c_{(-)}$ on $\mathcal{E}$ defined as follows: for a subobject $U \xrightarrow{u} X$, $c_X(u)$ is the subobject of $X$ given by the following pullback:

$$\begin{tikzcd}
c_X(u) \arrow[r] \arrow[d] & f_*f^*U \arrow[d, "f_*f^*u"] \\
X \arrow[r, "\eta_X"'] & f_*f^*X
\end{tikzcd}$$

where $\eta$ is the unit of the adjunction $f^* \dashv f_*$.

**Exercise 19** Check yourself that this defines a universal closure operation.

We claim that for arbitrary subobjects $U, V$ of $X$ the following holds: $V \leq c_X(U)$ if and only if $f^*V \leq f^*U$. Indeed, consider the commuting diagram:

$$\begin{tikzcd}
V \arrow[r, "\eta"] \arrow[d] & f_*f^*V \arrow[d] \\
X \arrow[r, "\eta"] & f_*f^*X \\
c_X(U) \arrow[r] \arrow[u] & f_*f^*U \arrow[u]
\end{tikzcd}$$

where $\eta$ is the unit of the adjunction $f^* \dashv f_*$. If $f^*V \leq f^*U$ then $f_*f^*V \leq f_*f^*U$ so, since the lower square is a pullback, the arrow $V \to X$ factors through $c_X(U)$; i.e., $V \leq c_X(U)$.

Conversely, if $V \leq c_X(U)$, we obtain an arrow $V \xrightarrow{\mu} f_*f^*U$ such that the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "\mu"] \arrow[dr] & f_*f^*U \arrow[r] & f_*f^*X \\
& X \arrow[ur, "\eta"'] &
\end{tikzcd}$$

Transposing along $f^* \dashv f_*$ we get

$$\begin{tikzcd}
f^*V \arrow[r, "\hat{\mu}"] \arrow[dr] & f^*U \arrow[r] & f^*X \\
& f^*X \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

and, since $f^*V \to f^*X$ is mono, also $\hat{\mu}$ is mono, and $f^*V \leq f^*U$.

The following exercise is very similar to Exercise 17b):

**Exercise 20** Suppose $\mathcal{F} \xrightarrow{f} \mathcal{E}$ is a geometric morphism and $j$ is a Lawvere-Tierney topology in $\mathcal{E}$. Then $f_*$ factors through the inclusion $\mathrm{sh}_j(\mathcal{E}) \to \mathcal{E}$ if and only if $f^*$ maps $j$-dense monos to isomorphisms in $\mathcal{F}$.

Now if $U \xrightarrow{u} X$ is a mono which is dense for (the topology associated to) the closure operator $c_X$, then $X \leq c_X(U)$, so $f^*X \leq f^*U$ and $f * u$ is an isomorphism. By the exercise, we conclude that $f_*$ factors through $\mathrm{sh}_j(\mathcal{E})$. And by reasoning as in Exercise 16, we obtain a factorization of geometric morphisms:

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{E} \\ & \mathrm{sh}_j(\mathcal{E} \arrow[ur, "i"'] & \end{tikzcd} .$$

Remains to see that $p$ is a surjection. Consider subobjects $U \leq V$ of $X$ in $\mathrm{sh}_j(\mathcal{E})$; suppose $p^*U \simeq p^*V$. Then $f^*i^*U \simeq f^*i^*V$ so, since $U$ and $V$ are *closed* subobjects of $X$, we have $i_*U \simeq i_*V$. Since $i_*$ is full and faithful, $U \simeq V$ follows. We conclude that $p^*$ reflects isomorphisms of subobjects; by Lemma 2.19, $p$ is a surjection as claimed.

For the essential uniqueness of the decomposition, I refer to **MM**, Theorem VII.4.8. ∎

We can now give the promised characterization of embeddings:

**Proposition 2.37** *For a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ the following statements are equivalent:*

*i)* *$f$ is an embedding (i.e., $f_*$ is full and faithful).*

*ii)* *The counit $\varepsilon : f^*f_* \Rightarrow \mathrm{id}_{\mathcal{F}}$ is an isomorphism.*

*iii)* *There is a Lawvere-Tierney topology $j$ in $\mathcal{E}$ and an equivalence $e : \mathcal{F} \to \mathrm{sh}_j(\mathcal{E})$ such that the diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f"] \arrow[dr, "e"'] & \mathcal{E} \\ & \mathrm{sh}_j(\mathcal{E}) \arrow[u] \end{tikzcd}$$

*commutes up to isomorphism.*

**Proof**. The equivalence between i) and ii) is standard Category Theory, and the implication iii)⇒i) is clear. For the converse, assume $f$ is an embedding. By Theorem 2.36, there is a factorization

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f"] \arrow[dr, "p"'] & \mathcal{E} \\ & \mathrm{sh}_j(\mathcal{E}) \arrow[u, "i"'] \end{tikzcd}$$

with $p$ a surjection. Since $i_*$ and $f_*$ are full and faithful, so is $p_*$ (check!). Therefore the counit $\varepsilon$ for $p^* \dashv p_*$ is an isomorphism. Consider the "triangular identity" from basic Category Theory for arbitrary $E \in \mathcal{E}$:

$$\begin{tikzcd}
p^*E \arrow[r, "p^*\eta_E"] \arrow[dr, "\mathrm{id}"'] & p^*p_*p^*E \arrow[d, "\varepsilon_{p^*E}"] \\
& p^*E
\end{tikzcd}$$

Since $\varepsilon$ is an isomorphism, we see that $p^*(\eta_E)$ is an isomorphism. But $p$ is a surjection, so $\eta_E$ is an isomorphism. We see that both $\varepsilon$ and $\eta$ are isomorphisms, so $p$ is an equivalence. ∎

**Examples 2.38** Let us see how standard geometric morphisms decompose:

1) Every continuous map $f : X \to Y$ of topological spaces factors as $X \to Z \to Y$, where $Z$ is the image of $X$, topologized as a subspace of $Y$. The map $X \to Z$ is surjective, the map $Z \to Y$ is an embedding. Hence the geometric morphism $\mathrm{Sh}(X) \to \mathrm{Sh}(Z)$ is a surjection and $\mathrm{Sh}(Z) \to \mathrm{Sh}(Y)$ is an embedding.

2) Every morphism in a topos has an epi-mono factorization, as we have seen. This gives at once a surjection-embedding factorization of the geometric morphism between the slice toposes.

3) For a functor $F : \mathcal{C} \to \mathcal{D}$ between small categories, let $\mathcal{B}$ be the full subcategory of $\mathcal{D}$ on objects in the image of $F$; and let $\mathcal{C} \xrightarrow{G} \mathcal{B} \xrightarrow{H} \mathcal{D}$ be the evident factorization. Then $G$ is surjective on objects and $H$ is full and faithful; so $\widehat{\mathcal{C}} \xrightarrow{\widehat{G}} \widehat{\mathcal{B}} \xrightarrow{\widehat{H}} \widehat{\mathcal{D}}$ is a surjection-embedding factorization of $\widehat{F}$.

# 3 Logic in Toposes

In 2.22 we have seen the map $\wedge : \Omega \times \Omega \to \Omega$, which classifies the subobject $1 \xrightarrow{\langle t,t \rangle} \Omega \times \Omega$; whenever subobjects $M, N$ of $X$ are classified by $\phi, \psi$ respectively, the intersection (meet) $M \cap N$ is classified by the composite

$$X \xrightarrow{\langle \phi,\psi \rangle} \Omega \times \Omega \xrightarrow{\wedge} \Omega.$$

We have also seen that $\mathrm{Sub}(X)$ has unions (joins): for subobjects $M \xrightarrow{m} X$, $N \xrightarrow{n} X$, $M \cup N \to X$ is the mono-part of the epi-mono factorization of $\begin{bmatrix} m \\ n \end{bmatrix} : M + N \to X$. Equivalently, by Proposition 1.29, $M \cup N$ may be constructed as the pushout:

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r] & M \cup N \end{tikzcd}$$

Let $\vee : \Omega \times \Omega \to \Omega$ classify the union of the subobjects $\Omega \xrightarrow{\langle \mathrm{id},t \rangle} \Omega \times \Omega$ and $\Omega \xrightarrow{\langle t,\mathrm{id} \rangle} \Omega \times \Omega$.

**Exercise 21** If the subobjects $M, N$ of $X$ are classified by $\phi, \psi : X \to \Omega$ respectively, then $M \cup N$ is classified by the composite

$$X \xrightarrow{\langle \phi,\psi \rangle} \Omega \times \Omega \xrightarrow{\vee} \Omega.$$

We define also the mono $\Omega_1 \to \Omega \times \Omega$ as the equalizer of $\pi_0, \wedge : \Omega \times \Omega \to \Omega$ (where $\pi_0$ is the first projection).

**Exercise 22** For subobjects $M, N$ of $X$, classified by $\phi, \psi$ we have: $M \leq N$ if and only if the map $\langle \phi, \psi : X \to \Omega \times \Omega$ factors through $\Omega_1$.

Furthermore, we have the arrow $f : 1 \to \Omega$, which classifies the mono $0 \to 1$.

**Definition 3.1** In a category $\mathcal{C}$ with finite limits, an *internal lattice* is an object $L$ together with morphisms $\top, \bot : 1 \to L$ and $\sqcup, \sqcap : L \times L \to L$ such that the diagrams expressing the following equations commute:

| | | |
|---|---|---|
| 1 | $x \sqcap \top = x$ | $x \sqcup \top = \top$ |
| 2 | $x \sqcap \bot = \bot$ | $x \sqcup \bot = x$ |
| 3 | $x \sqcap y = y \sqcap x$ | $x \sqcup y = y \sqcup x$ |
| 4 | $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$ | $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ |
| 5 | $x \sqcap (x \sqcup y) = x$ | $y \sqcup (x \sqcap y) = y$ |
| 6 | $x \sqcap x = x$ | $x \sqcup x = x$ |

Note that in a lattice $(L, \bot, \top, \sqcap, \sqcup)$ a partial ordering can be defined as the equalizer of $\sqcap$ and the first projection (equivalently, the equalizer of $\sqcup$ and the second projection). If, in the context of an internal lattice, we use the symbol $\leq$, it refers to this ordering.

Such a lattice is *distributive* if moreover $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$ holds.

**Exercise 23** Show that, with the operations $\wedge$ and $\vee$ and the elements $t, f : 1 \to \Omega$, $\Omega$ is a distributive internal lattice.

But $\Omega$ has more structure. A *Heyting algebra* is a lattice which is cartesian closed as a poset: for elements $x$ and $y$ there is the exponent $x \Rightarrow y$ which has the universal property that $z \leq (x \Rightarrow y)$ if and only if $z \sqcap x \leq y$. This property can also be given equationally, so we can define the notion of *internal Heyting algebra*:

**Definition 3.2** In a category $\mathcal{C}$ with finite limits, a *internal Heyting algebra* is an internal lattice $(L, \top, \bot, \sqcap, \sqcup)$ and an operation $\Rightarrow : L \times L \to L$ such that the diagram expressing the following equations commute:

$$\begin{array}{ll} (x \Rightarrow x) = \top & (x \Rightarrow (y \sqcap z)) = (x \Rightarrow y) \sqcap (x \Rightarrow z) \\ x \sqcap (x \Rightarrow y) = x \sqcap y & y \sqcap (x \Rightarrow y) = y \end{array}$$

For $\Omega$, we have a map $\Rightarrow : \Omega \times \Omega \to \Omega$ which classifies the mono $\Omega_1 \to \Omega \times \Omega$.

**Exercise 24** Prove that $(\Omega, t, f, \wedge, \vee, \Rightarrow)$ is an internal Heyting algebra. If $M, N$ are subobjects of $X$ classified by $\phi, \psi$ respectively, then the composite

$$X \xrightarrow{\langle \phi, \psi \rangle} \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$$

classifies the largest subobject $K$ of $X$ which satisfies $K \cap M \leq N$.

In any Heyting algebra, we have the operation $\neg$ (*pseudocomplement*): $\neg x = x \Rightarrow \bot$. A Heyting algebra is a *Boolean algebra* if $x \sqcup \neg x = \top$ for all $x$.

**Definition 3.3** A topos $\mathcal{E}$ is called *Boolean* if $\Omega$ is an internal Boolean algebra.

**Proposition 3.4** *For a topos $\mathcal{E}$, the following statements are equivalent:*

*i)* *$\mathcal{E}$ is Boolean.*

*ii)* *The map $\neg\neg$ is the identity map on $\Omega$.*

*iii)* *Every subobject $M$ of an object $X$ has a* complement*, that is: a subobject $N$ of $X$ satisfying $M \cap N = 0$ and $M \cup N = X$.*

*iv)* *The map* $\begin{bmatrix} t \\ f \end{bmatrix} : 1 + 1 \to \Omega$ *is an isomorphism.*

Obviously, if $f : \mathcal{F} \to \mathcal{E}$ is a logical functor and $\mathcal{F}$ is Boolean, then so is $\mathcal{E}$.

**Theorem 3.5 (PTJ 5.17)** *For every topos $\mathcal{E}$, $\neg\neg$ is a Lawvere-Tierney topology in $\mathcal{E}$, and $\mathrm{sh}_{\neg\neg}(\mathcal{E})$ is Boolean.*

**Proof**. The map $\neg$ is order-reversing, and for subobjects $M, M'$ of $X$ $M \leq \neg M'$ if and only if $M' \leq \neg M$. Hence (taking $\neg M$ for $M'$) $M \leq \neg\neg M$ (i.e., $\neg\neg$ is inflationary) and $\neg M = \neg\neg\neg M$; so $\neg\neg$ is idempotent. Also, $\neg\neg X = X$ and $\neg\neg$ commutes with meets: $\neg\neg M \wedge \neg\neg M' = \neg\neg(M \wedge M')$.

The subobject classifier of $\mathrm{sh}_{\neg\neg}(\mathcal{E})$ is $\Omega_{\neg\neg}$, which is an internal Boolean algebra in $\mathcal{E}$, hence also an internal Boolean algebra in $\mathrm{sh}_{\neg\neg}(\mathcal{E})$. ∎

For the following proposition, we need the pointwise ordering on maps into $\Omega$: for $f, g : X \to \Omega$ we set $f \leq g$ if and only if the map $\langle f, g \rangle : X \to \Omega \times \Omega$ factors through $\Omega_1$.

**Proposition 3.6** *The map $\neg\neg$ is the largest topology for which the inclusion $0 \to 1$ is closed (equivalently, for which the object $0$ is a sheaf).*

**Proof**. Clearly, $\neg\neg f = f$ (since $\neg f = t$), so $0 \to 1$ is closed.

Conversely, let $j$ be a topology for which $0 \to 1$ is closed. Let $X' \stackrel{\sigma}{\to} X$ be a $j$-dense mono. Let $X'' = \neg X'$ (in $\mathrm{Sub}(X)$). We have a pullback

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X' \arrow[d, "\sigma"] \\ X'' \arrow[r] & X \end{tikzcd}$$

so $0 \to X''$ is $j$-dense. But $0 \to X''$ is also $j$-closed, since also.the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ X'' \arrow[r] & 1 \end{tikzcd}$$

is a pullback. So, $0 \to X''$ is an isomorphism, and $X = \neg X'' = \neg\neg X'$; so $\sigma$ is $\neg\neg$-dense. It follows that $j \leq \neg\neg$. ∎

**Definition 3.7** Let $\mathcal{E}$ be a topos.

1) We say that *supports split* in $\mathcal{E}$ (or, that $\mathcal{E}$ *satisfies* SS) if, whenever

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & 1 \\ & \sigma_1(X) \arrow[ur] & \end{tikzcd}$$

is an epi-mono factorization, the epi $X \to \sigma_1(X)$ is split.

2) We say that $\mathcal{E}$ *satisfies the Axiom of Choice (AC)* if every epi in $\mathcal{E}$ is split.

**Exercise 25** Prove that $\mathcal{E}$ satisfies SS if and only if every subobject of 1 is projective in $\mathcal{E}$; and that $\mathcal{E}$ satisfies AC if and only if every object is projective in $\mathcal{E}$.

The following is a classical result, but we shall later see a stronger statement, so the proof is deferred.

**Theorem 3.8 (Diaconescu; PTJ 5.23)** *If a topos satisfies AC, it is Boolean.*

**Exercise 26** For a group $\mathcal{G}$, show that $\widehat{\mathcal{G}}$ is Boolean but does not satisfy AC.

**Definition 3.9** An object $X$ of a topos $\mathcal{E}$ is called *internally projective* if the functor $(-)^X$ preserves epimorphisms.

An epimorphism $f : X \to Y$ in $\mathcal{E}$ is called *locally split* if there is an object $V$ of $\mathcal{E}$ such that $V \to 1$ is epi and $V^*(f)$ is split epi in $\mathcal{E}/V$.

**Exercise 27** Show that an epi $f : X \to Y$ is locally split if and only if there is an epi $h : Z \to Y$ such that $h^*(f)$ is split epi in $\mathcal{E}$.

**Proposition 3.10 (PTJ 5.25)** *The following statements are equivalent for a topos $\mathcal{E}$:*

*i) Every object of $\mathcal{E}$ is internally projective.*

*ii) Every epi in $\mathcal{E}$ is locally split.*

*iii) If $X \xrightarrow{f} Y$ is epi, then $\prod_Y(f) \to 1$ is epi.*

**Proof**. Recall from the proof of Corollary 1.18 that

$$\prod_Y(f) \longrightarrow Y^X \underset{\ulcorner \mathrm{id} \urcorner \circ !}{\overset{f^X}{\rightrightarrows}} X^X$$

is an equalizer diagram, where $\ulcorner \mathrm{id} \urcorner : 1 \to X^X$ denotes the exponential transpose of the identity arrow on $X$. ∎

# 4 Classifying Toposes

## 4.1 Examples

**Example 4.1 (Torsors)** Let $G$ be a group and suppose $\gamma : \mathcal{E} \to \mathrm{Set}$ is a geometric morphism (we speak of a "topos over Set", i.e. a topos with a geometric morphism to Set). Then $\gamma^*(G)$ is a group object in $\mathcal{E}$. A *$G$-torsor* over $\mathcal{E}$ is an object $T$ of $\mathcal{E}$ equipped with a left group action

$$\mu : \gamma^*(G) \times T \to T$$

which, apart from the axioms for a group action, satisfies the following conditions:

i) $T \to 1$ is an epimorphism.

ii) The action $\mu$ induces an isomorphism

$$\langle \mu, p_1 \rangle : \gamma^*(G) \times T \to T \times T$$

(recall that $p_1$ denotes the projection on the *second* coordinate)

In the topos $\widehat{G}$ of right $G$-sets, we have a torsor whose underlying set is $G$ itself, with its canonical action on the left (note that the actions on the left and on the right commute with each other, so the left action is a map of $G$-sets). We call this $G$-torsor $U_G$.

The $G$-torsors in $\mathcal{E}$ form a category $\mathrm{Tor}(\mathcal{E}, G)$, whose objects are $G$-torsors over $\mathcal{E}$ and whose morphisms are morphisms of left $G$-sets in $\mathcal{E}$. Since for cocomplete toposes, the geometric morphism to Set is essentially unique, we have, for a geometric morphism $f : \mathcal{E} \to \widehat{G}$, a diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "f"] \arrow[dr, "\gamma"'] & & \widehat{G} \arrow[dl, "g"] \\ & \mathrm{Set} & \end{tikzcd}$$

which commutes up to isomorphism (where $g$ is the geometric morphism we have already seen).

Clearly, the structures of a $G$-torsor and of a map between $G$-torsors are preserved by inverse images of geometric morphisms, so any geometric morphism $f : \mathcal{F} \to \mathcal{E}$ gives rise to a functor $f^* : \mathrm{Tor}(\mathcal{E}, G) \to \mathrm{Tor}(\mathcal{F}, G)$.

For the following theorem we should state the 2-dimensional character of the category $\mathcal{T}op$: for two geometric morphisms $f, g : \mathcal{F} \to \mathcal{E}$ we can also consider natural transformations $f^* \to g^*$. In this way we have, for any two toposes $\mathcal{F}, \mathcal{G}$ a category $\mathcal{T}op(\mathcal{F}, \mathcal{E})$.

**Theorem 4.2 (MM VIII.2.7)** *For a topos $\mathcal{E}$ over* Set *there is an equivalence of categories*

$$\mathcal{T}op(\mathcal{E}, \widehat{G}) \;\simeq\; \mathrm{Tor}(\mathcal{E}, G).$$

*This equivalence is, on objects, induced by the operation which sends the geometric morphism $g : \mathcal{E} \to \widehat{G}$ to the $G$-torsor $g^*(U_G)$ and is therefore natural in $\mathcal{E}$.*

This example is an instance of a general phenomenon. We consider, for a topos $\mathcal{E}$, the category $\mathcal{E}_T$ of "structures of a type $T$" in $\mathcal{E}$. For the moment, let us not worry about what these structures are or what the morphisms could be, except that we suppose that when $M$ is such a structure in $\mathcal{E}$ and $f : \mathcal{F} \to \mathcal{E}$ is a geometric morphism, then $f^*M$ is such a structure in $\mathcal{F}$; and similarly, if we have an arrow $\mu : M \to N$ in $\mathcal{E}_T$ then $f^*(\mu)$ is an arrow $f^*M \to f^*N$ in $\mathcal{F}_T$, so that we have a functor $f^* : \mathcal{E}_T \to \mathcal{F}_T$.

**Definition 4.3** A *classifying topos* for structures of type $T$ is a topos $\mathcal{B}(T)$ over Set, for which there is a natural equivalence of categories

$$\mathcal{T}op(\mathcal{E}, \mathcal{B}(T)) \to \mathcal{E}_T$$

Applying the equivalence to the identity geometric morphism on $\mathcal{B}(T)$ and reasoning like in the Yoneda Lemma, we see that there is a structure $U_T$ of type $T$ in $\mathcal{B}(T)$ (the *universal* $T$-structure), such that the equivalence of Definition 4.3 is given by: $f \mapsto f^*(U_T)$.

We shall later specify what "structures of type $T$" will be (models of a certain logical theory); for now, we continue with some more examples.

**Example 4.4 (Objects)** The simplest "structure of type $T$" is: just an object. If $\mathcal{B}$ is a classifying topos for objects, we have an equivalence of categories

$$\mathcal{T}op(\mathcal{E}, \mathcal{B}) \to \mathcal{E}$$

given by $f \mapsto f^*(U)$ for some "universal object" $U$ of $\mathcal{B}$.

**Lemma 4.5 (MM VIII.4.1)** *Let* $\mathrm{Set}_f$ *be the category of finite sets. Then* $\mathrm{Set}_f$ *is the free category with finite colimits generated by one object.*

**Proof**. The statement of the lemma means: there is a finite set $X$ such that for every category $\mathcal{C}$ with finite colimits and every object $C$ of $\mathcal{C}$, there is an essentially unique functor $F_C : \mathrm{Set}_f \to \mathcal{C}$ which preserves finite colimits and sends $X$ to $C$. Indeed, let $X$ be a one-element set. For an arbitrary finite set $E$, let

$$F_C(E) \;=\; \sum_{e \in E} C$$

Clearly, $F_C(X) = C$. Moreover, $F_C$ preserves all finite colimits (see MM VIII.4.1 for details). ∎

Dual to Lemma 4.5 we have:

**Lemma 4.6 (MM VIII.4.2)** *The category* $\mathrm{Set}_f^{\mathrm{op}}$ *is the free category with finite limits, generated by one object.*

Now we have a chain of equivalences:

$$\begin{array}{rc}
\text{Geometric morphisms } \mathcal{E} \to \mathrm{Set}^{\mathrm{Set}_f} & \simeq \\
\text{Flat functors } \mathrm{Set}_f^{\mathrm{op}} \to \mathcal{E} & \simeq \\
\text{Finite limit preserving functors } \mathrm{Set}_f^{\mathrm{op}} \to \mathcal{E} & \simeq \\
\mathcal{E} &
\end{array}$$

So, the classifying topos for objects is $\mathrm{Set}^{\mathrm{Set}_f}$.

**Exercise 28** What is the "universal object" in $\mathrm{Set}^{\mathrm{Set}_f}$?

**Example 4.7 (Rings)** Our next example concerns commutative rings, here just called rings. In a category $\mathcal{C}$ with finite limits, a *ring object* is a diagram

$$1 \underset{1}{\overset{0}{\rightrightarrows}} R \underset{\cdot}{\overset{+}{\leftleftarrows}} R \times R$$

for which the axioms for rings (expressed by commuting diagrams) hold. We have an obvious definition of homomorphism of ring objects in $\mathcal{C}$, and hence a category $\mathrm{ring}(\mathcal{C})$. Any finite limit preserving functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor $\mathrm{ring}(\mathcal{C}) \to \mathrm{ring}(\mathcal{D})$.

**Definition 4.8** *A ring is* finitely presented *if it is isomorphic to*

$$\mathbb{Z}[X_1, \ldots, X_n]/I$$

*where* $\mathbb{Z}[X_1, \ldots, X_n]$ *is the ring of polynomials in* $n$ *variables with integer coefficients, and* $I$ *is an ideal. Since* $\mathbb{Z}[X_1, \ldots, X_n]$ *is Noetherian, the ideal* $I$ *can be written as* $(P_1, \ldots, P_k)$ *for elements* $P_1, \ldots, P_k$ *of* $\mathbb{Z}[X_1, \ldots, X_n]$.

Let **fp-rings** be the full subcategory of the category of rings on the finitely presented rings. A morphism

$$\alpha : \mathbb{Z}[X_1, \ldots, X_n]/(P_1, \ldots, P_k) \to \mathbb{Z}[Y_1, \ldots, Y_m]/(Q_1, \ldots, Q_l)$$

is given by an $n$-tuple $(\alpha(X_1), \ldots, \alpha(X_n))$ of polynomials in $Y_1, \ldots, Y_m$, such that the polynomials

$$P_j(\alpha(X_1), \ldots, \alpha(X_n))$$

are elements of the ideal $(Q_1, \ldots, Q_l)$.

The category **fp-rings** has finite coproducts: the initial object is $\mathbb{Z}$, and the sum

$$\mathbb{Z}[X_1, \ldots, X_n]/(P_1 \ldots, P_k) + \mathbb{Z}[Y_1, \ldots, Y_m]/(Q_1, \ldots, Q_l)$$

(where we assume that the strings of variables $\vec{X}$ and $\vec{Y}$ are disjoint) is the ring

$$\mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_m]/(P_1, \ldots, P_k, Q_1, \ldots, Q_l)$$

Moreover, the category **fp-rings** has coequalizers: given a parallel pair of arrows

$$\begin{tikzcd} \mathbb{Z}[\vec{X}]/(\vec{P}) \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \mathbb{Z}[\vec{Y}]/(\vec{Q}) \end{tikzcd}$$

its coequalizer is the quotient ring

$$\mathbb{Z}[\vec{Y}]/(\vec{Q}, \alpha(X_1) - \beta(X_1), \ldots, \alpha(X_n) - \beta(X_n))$$

with the evident quotient map.

Now, we consider $\mathbf{fp\text{-}rings}^{\mathrm{op}}$. This is a category with finite limits. Note that $\mathbb{Z}$ is terminal in $\mathbf{fp\text{-}rings}^{\mathrm{op}}$. A ring object in $\mathbf{fp\text{-}rings}^{\mathrm{op}}$ is a diagram

$$\begin{tikzcd} \mathbb{Z} & R \arrow[l, shift right, "0"'] \arrow[l, shift left, "1"] \arrow[r, shift left] \arrow[r, shift right] & R + R \end{tikzcd}$$

in **fp-rings**, subject to the duals of the axioms for rings. An example of such a structure in **fp-rings** is the ring $\mathbb{Z}[X]$ with maps $0, 1 : \mathbb{Z}[X] \to \mathbb{Z}$ sending a polynomial $P$ to $P(0)$ and to $P(1)$ respectively; and $+, \cdot : \mathbb{Z}[X] \to \mathbb{Z}[X, Y]$ (note that $\mathbb{Z}[X, Y] = \mathbb{Z}[X] + \mathbb{Z}[X]$ in fp-rings, sending $P(X)$ to $P(X + Y)$ and to $P(XY)$ respectively.

**Lemma 4.9 (MM VIII.5.1)** *The category* $\mathbf{fp\text{-}rings}^{\mathrm{op}}$*, together with the ring object* $\mathbb{Z}[X]$ *as just described, is the free category with finite limits and a ring object.*

The statement of the lemma means: for any category $\mathcal{C}$ with finite limits and ring object $R$, there is an essentially unique finite limit preserving functor from $\mathbf{fp\text{-}rings}^{\mathrm{op}}$ to $\mathcal{C}$ which sends $\mathbb{Z}[X]$ to $R$.

We can now argue in exactly the same way as in the two previous examples: ring objects in a topos $\mathcal{E}$ correspond to flat, that is: finite limit preserving, functors from $\mathbf{fp\text{-}rings}^{\mathrm{op}}$ to $\mathcal{E}$, which correspond to geometric morphisms from $\mathcal{E}$ to $\mathrm{Set}^{\mathbf{fp-rings}}$; the latter therefore being the "classsifying topos for rings".

In each of the three examples we have just seen, the classifying topos was a presheaf topos. That is because of the "algebraic character" of the type of structures we considered: the structure is given by a number of operations and the axioms are equations. Not every structure which admits a classifying topos is of such a simple kind. But let us now define what kind of structures we have in mind: structures for *geometric logic*.

## 4.2 Geometric Logic

We consider a *multi-sorted language*. That is: we have a set of *sorts*, a stock of variables for each sort (we write $x^S$ in order to indicate that the variable $x$ has sort $S$), and constants, function symbols and relation symbols with also specified sorts. We write:

> $c^S$ to indicate that the constant $c$ is of sort $S$;
>
> $f : S_1, \ldots, S_n \to T$ to indicate that the function symbol $f$ takes arguments of sorts $S_1, \ldots, S_n$, and then yields something of sort $T$;
>
> $R \subseteq S_1, \ldots, S_n$ to indicate that the relation symbol $R$ takes arguments of sorts $S_1, \ldots, S_n$.

All terms of the language have a specified sort: for a variable $x^S$ of sort $S$, $x^S$ is a term of sort $S$. Every constant of sort $S$ is a term of sort $S$. If $f : S_1, \ldots, S_n \to T$ is a function symbol and $t_1, \ldots, t_n$ are terms of sorts $S_1, \ldots, S_n$ respectively, then $f(t_1, \ldots, t_n)$ is a term of sort $T$.

An *atomic formula* is an expression of one of three forms: it is the symbol $\top$ (for "true"), it is an equation $t = s$ where $t$ and $s$ are terms of the same sort, or it is an expression $R(t_1, \ldots, t_n)$, where $R \subseteq S_1, \ldots, S_n$ is a relation symbol and $t_i$ is a term of sort $S_i$ for $i = 1, \ldots, n$.

The class of *geometric formulas* (for a given language) is defined as follows:

> Every atomic formula is a geometric formula;
>
> If $\phi$ and $\psi$ are geometric formulas, then $\phi \wedge \psi$ is a geometric formula;
>
> If $\phi$ is a geometric formula and $x^S$ is a variable, then $\exists x^S \phi$ is a geometric formula;
>
> If $X$ is a set of geometric formulas and $X$ contains only finitely many free variables, then $\bigvee X$ is a geometric formula.

If $\mathcal{E}$ is a cocomplete topos, then there is a straightforward definition of what a *structure* for a language in $\mathcal{E}$ should be: for every sort $S$, we have an object $[\![ S ]\!]$ of $\mathcal{E}$; for every function symbol $f : S_1, \ldots, S_n \to T$ we have a morphism $[\![ f ]\!] : [\![ S_1 ]\!] \times \cdots \times [\![ S_n ]\!] \to [\![ T ]\!]$ in $\mathcal{E}$; for every relation symbol $R \subseteq S_1, \ldots, S_n$ we have a subobject $[\![ R ]\!]$ of $[\![ S_1 ]\!] \times \cdots \times [\![ S_n ]\!]$.

Just as straightforwardly, one now obtains, for any formula $\phi$ with free variables $x_1^{S_1}, \ldots, x_n^{S_n}$, a subobject $[\![ \phi ]\!]$ of $[\![ S_1 ]\!] \times \cdots \times [\![ S_n ]\!]$. For the case when $\phi$ is of the form $\bigvee X$, we use of course the cocompleteness of $\mathcal{E}$, which implies that subobject lattices are complete (have arbitrary joins).

A *geometric sequent* is an expression of the form $\phi \vdash_{\vec{x}} \psi$, where $\phi$ and $\psi$ are geometric formulas, and $\vec{x}$ is a finite list of variables which contains every variable which appears freely in $\phi$ or $\psi$.

If a structure for the language is given, let us write $[\![ \vec{x} ]\!]$ for the product $\prod_{i=1}^n [\![ S_i ]\!]$ if $\vec{x} = (x_1^{S_1}, \ldots, x_n^{S_n})$. If $\vec{y}_\phi$ is the list of variables appearing freely in $\phi$ and $\vec{y}_\psi$ the list of those in $\psi$, then we have evident projections $p_\phi : [\![ \vec{x} ]\!] \to [\![ \vec{y}_\phi ]\!]$ and $p_\psi : [\![ \vec{x} ]\!] \to [\![ \vec{y}_\psi ]\!]$, and hence subobjects $[\![ \phi ]\!]_{\vec{x}} = p_\phi^*([\![ \phi ]\!])$ and $[\![ \psi ]\!]_{\vec{x}} = p_\psi^*([\![ \psi ]\!])$ of $[\![ \vec{x} ]\!]$.

We say that the sequent $\phi \vdash_{\vec{x}} \psi$ is *true* in the given structure, if $[\![ \phi ]\!]_{\vec{x}} \leq [\![ \psi ]\!]_{\vec{x}}$ in $\mathrm{Sub}([\![ \vec{x} ]\!])$. We think of the sequent $\phi \vdash_{\vec{x}} \psi$ as of the "formula"

$$\forall \vec{x}(\phi \Rightarrow \psi)$$

For instance, if for one of the variables $x^S$ in $\vec{x}$ we have that the object $[\![ S ]\!]$ is initial, then the sequent $\phi \vdash_{\vec{x}} \psi$ is always true.

Let us denote a structure for a given language by $\mathcal{M}$. So we have the interpretation $[\![ \cdot ]\!]^{\mathcal{M}}$ of the sorts, function symbols, constants and relation symbols in some topos $\mathcal{E}$. If $f : \mathcal{F} \to \mathcal{E}$ is a geometric morphism, we have a structure $f^*\mathcal{M}$ in $\mathcal{F}$ by applying the inverse image functor $f^*$ to all the data of $\mathcal{M}$. We now have interpretations $[\![ \phi ]\!]^{\mathcal{M}}$ in $\mathcal{E}$ and $[\![ \phi ]\!]^{f^*\mathcal{M}}$ in $\mathcal{F}$.

**Proposition 4.10** *Let $\mathcal{M}$ be a structure for a language in a topos $\mathcal{E}$, and suppose $f : \mathcal{F} \to \mathcal{E}$ is a geometric morphism. Then we have:*

*a) For any formula $\phi$ of the language, $[\![ \phi ]\!]^{f^*\mathcal{M}} = f^*([\![ \phi ]\!]^{\mathcal{M}})$.*

*b) If the sequent $\phi \vdash_{\vec{x}} \psi$ is true with respect to the structure $\mathcal{M}$, then it is also true with respect to $f^*\mathcal{M}$.*

*c) If the geometric morphism $f$ is a surjection, then the converse of b) holds: if $\phi \vdash_{\vec{x}} \psi$ is true with respect to the structure $f^*\mathcal{M}$ then it is true with respect to $\mathcal{M}$.*

A *geometric theory* in a given language is a set of geometric sequents in that language. If $\mathcal{M}$ is a structure in which every sequent of a theory is true, then $\mathcal{M}$ is called a *model* of the theory.

Now we can be more precise about the "structures of a type $T$" mentioned in Definition 4.3: they are, in fact, models of a geometric theory. One advantage of making this notion precise is, that we can investigate geometric theories also syntactically, and, much as in classical Model Theory, study relations between syntactic properties of theories and topos-theoretic properties of their classifying toposes.

For example, in the examples we have discussed so far, the classifying toposes were presheaf toposes (as we already remarked). This is connected to the fact that the respective theories are all *universal*: no $\bigvee$ and no existential quantifier (you might object by saying that in the theory of rings we need to express that every element has an additive inverse, and that we need an existential quantifier for this; however, since the additive inverse is unique this existential quantifier is not essential and we could expand the language with an extra function symbol).

**Example 4.11 (Flat functors)** Let us now consider a theory where the use of existential quantifiers and (possibly infinite) disjunctions is necessary: the *theory of flat functors* form a small category $\mathcal{C}$.

Given a small category $\mathcal{C}$, let $\mathcal{L}_{\mathcal{C}}$ be the language which has:

for every object $C$ of $\mathcal{C}$ a sort $C$;

for every arrow $f : C \to D$ in $\mathcal{C}$, a function symbol $f : C \to D$.

The geometric theory $\mathrm{Flat}(\mathcal{C})$ has the following sequents:

1) For every commutative triangle

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "h"'] & D \arrow[d, "g"] \\ & E \end{tikzcd}$$

a sequent $\top \vdash_{x^C} h(x) = g(f(x))$.

2) A sequent

$$\top \vdash \bigvee_{C \in \mathcal{C}_0} \exists x^C (x = x).$$

3) A sequent

$$\top \vdash_{x^C, y^D} \bigvee_{f:E\to C, g:E\to D} \exists z^E (f(z) = x \wedge g(z) = y)$$

4) A sequent

$$f(x) = g(x) \vdash_{x^C} \bigvee_{h:D\to C, fh=gh} \exists y^D (h(y) = x)$$

**Exercise 29** Show that for a topos $\mathcal{E}$, a model of $\mathrm{Flat}(\mathcal{C})$ in $\mathcal{E}$ is nothing but a flat functor $\mathcal{C} \to \mathcal{E}$; and hence, that the topos $\widehat{\mathcal{C}}$ classifies models of $\mathrm{Flat}(\mathcal{C})$.

Admittedly, in this example the classifying topos is still a presheaf topos. However, this changes if we extend the theory $\mathrm{Flat}(\mathcal{C})$ according to section 2.3.

**Definition 4.12** Let $(\mathcal{C}, J)$ be a site. The theory $\mathrm{FlatCont}(\mathcal{C}, J)$ of flat and $J$-continuous functors from $\mathcal{C}$, is an extension of the theory $\mathrm{Flat}(\mathcal{C})$ by the following axioms: for every object $C$ of $\mathcal{C}$ and every covering sieve $R \in J(C)$ we have the axiom

$$\top \vdash_{x^C} \bigvee_{f:D\to C, f\in R} \exists y^D (f(y) = x)$$

Theorem 2.15 now implies:

**Proposition 4.13** *A model of* $\mathrm{FlatCont}(\mathcal{C}, J)$ *is a topos* $\mathcal{E}$ *is nothing but a flat and* $J$*-continuous functor from* $\mathcal{C}$ *to* $\mathcal{E}$*. Therefore, the topos* $\mathrm{Sh}(\mathcal{C}, J)$ *classifies models of* $\mathrm{FlatCont}(\mathcal{C}, J)$*.*

And we conclude:

**Theorem 4.14 (Classifying Topos Theorem, part I)** *Every Grothendieck topos is the classifying topos of some geometric theory.*

The geometric theory which a Grothendieck topos classifies is by no means unique, as the following example shows.

**Example 4.15 (MM, §VIII.8)** Let $\Delta$ be the category of nonempty finite ordinals and order-preserving (i.e., $\leq$-preserving) functions. The presheaf category $\widehat{\Delta}$ is of paramount importance in algebraic topology and higher category theory; it is the *category of simplicial sets*. In the indicated section of their book, MacLane and Moerdijk give a detailed proof of the fact that $\widehat{\Delta}$ classifies the theory of linear orders with distinct top and bottom elements, and order-preserving maps which also preserve top and bottom.

This looks rather different from the category $\mathrm{Flat}(\Delta)$!

If geometric theories $T$ and $T'$ have equivalent classifying toposes, we call them *Morita equivalent*. In a picture strongly advocated by Olivia Caramello, the classifying topos forms a "bridge" between the theories $T$ and $T'$.

## 4.3 Syntactic categories

In section 4.2 we have already seen (in the notations $\phi \vdash_{\vec{x}} \psi$ and $[\![ \phi ]\!]_{\vec{x}}$) that it is useful to consider so-called *formulas in context*: a formula in context is a pair $[\vec{x}.\phi]$ where $\phi$ is a geometric formula and $\vec{x}$ a finite list of variables which contains all variables which appear freely in $\phi$.

Given a geometric theory $T$ and a geometric sequent $\phi \vdash_{\vec{x}} \psi$, we write $T \models (\phi \vdash_{\vec{x}} \psi)$ to mean that $\phi \vdash_{\vec{x}} \psi$ is true in every model of $T$ in every topos.

There is a deduction system for geometric logic, giving a notion $T \vdash (\phi \vdash_{\vec{x}} \psi)$, which is described in **Elephant**, §D1.3. We have a *Completeness Theorem*, which says that the notions $T \models (\phi \vdash_{\vec{x}} \psi)$ and $T \vdash (\phi \vdash_{\vec{x}} \psi)$ are equivalent; this theorem is outside the scope of these lecture notes. We shall only use the $\models$-notion.

We construct for any geometric theory $T$ a so-called *syntactic category* $\mathrm{Syn}(T)$, as follows.

Call two geometric formulas in context $[\vec{x}.\phi]$ and $[\vec{y}.\psi]$ equivalent if $[\vec{y}.\psi]$ is obtained from $[\vec{x}.\phi]$ by a renaming of variables (both free and bound). An object of $\mathrm{Syn}(T)$ is an equivalence class of such formulas in context. We shall just write $[\vec{x}.\phi]$ for its equivalence class.

When discussing arrows from $[\vec{x}.\phi]$ to $\vec{y}.\psi]$ we may, by our convention on equivalence, assume that the contexts $\vec{x}$ and $\vec{y}$ are disjoint.

A morphism $[\vec{x}.\phi] \to [\vec{y}.\psi]$ in $\mathrm{Syn}(T)$ is an equivalence class of formulas in context $[\vec{x}, \vec{y}.\theta]$ which satisfy:

i) $T \models (\theta(\vec{x}, \vec{y}) \vdash_{\vec{x},\vec{y}} \phi(\vec{x}) \wedge \psi(\vec{y}))$.

ii) $T \models (\phi(\vec{x}) \vdash_{\vec{x}} \exists \vec{y} \theta(\vec{x}, \vec{y}))$.

iii) $T \models (\theta(\vec{x}, \vec{y}) \wedge \theta(\vec{x}, \vec{y'}) \vdash_{\vec{x},\vec{y},\vec{y'}} \vec{y} = \vec{y'})$.

where in the last clause, for $\vec{y} = y_1, \ldots, y_n$ and $\vec{y'} = y'_1, \ldots, y'_n$, $\vec{y} = \vec{y'}$ abbreviates the formula $y_1 = y'_1 \wedge \cdots \wedge y_n = y'_n$.

Two such $\theta(\vec{x}, \vec{y})$ and $\theta'(\vec{x}, \vec{y})$ represent the same morphism if they are equivalent modulo $T$.

Given morphisms $\theta(\vec{x}, \vec{y}) : [\vec{x}.\phi] \to [\vec{y}.\psi]$ and $\xi : [\vec{y}.\psi] \to [\vec{z}.\chi]$, the composition $\xi \circ \theta : [\vec{x}.\phi] \to [\vec{z}.\chi]$ is represented by the formula $\exists \vec{y}(\theta(\vec{x}, \vec{y}) \wedge \xi(\vec{y}, \vec{z}))$. For any object $[\vec{x}.\phi]$, the identity arrow $[\vec{x}.\phi] \to [\vec{y}.\phi]$ (recall our convention about equivalent formulas in context) is the formula $x_1 = y_1 \wedge \cdots \wedge x_n = y_n$.

**Exercise 30** Prove that $\mathrm{Syn}(T)$ is a category.

**Definition 4.16** A *geometric category* is a regular category in which subobject lattices have arbitrary joins, and these joins are stable under pullback.

**Exercise 31** i) Characterize the monomorphisms in the category $\mathrm{Syn}(T)$.

ii) Show that $\mathrm{Syn}(T)$ is a regular category.

iii) Show that $\mathrm{Syn}(T)$ is a geometric category.

The category $\mathrm{Syn}(T)$ has a tautological model of $T$: for any sort $S$, $[\![ S ]\!]$ is the formula in context $[x^S.x = x]$; for any function symbol $f : S_1, \ldots, S_n \to T$, the arrow $[\![ f ]\!]$ is the formula

$$f(x_1^{S_1}, \ldots, x_n^{S_n}) = y^T$$

and for any relation symbol $R \subseteq S_1, \ldots, S_n$, the subobject $[\![ R ]\!]$ is represented by the evident monomorphism with domain $R(x_1^{S_1}, \ldots, x_n^{S_n})$.

For every geometric category $\mathcal{C}$, there is a *geometric topology* on $\mathcal{C}$: the covering sieves are those families$\{f_i : D_i \to C\}_{i \in I}$ for which the subobject

$$\bigvee_{i \in I} \mathrm{im}(f_i)$$

is the maximal subobject of $C$ (here $\mathrm{im}(f_i)$ denotes the image of $f_i$ as subobject of $C$).

Without proof, we state:

**Theorem 4.17** *Let $T$ be a geometric theory. For any cocomplete topos $\mathcal{E}$ (or, for any geometric category $\mathcal{E}$), the category of models of $T$ in $\mathcal{E}$ is equivalent to the category of flat and continuous functors from* $\mathrm{Syn}(T)$ *to $\mathcal{E}$.*

Therefore we have:

**Theorem 4.18 (Classifying Topos Theorem, part II)** *The category* $\mathrm{Sh}(\mathrm{Syn}(T), J)$, *where* $J$ *is the geometric topology on* $\mathrm{Syn}(T)$*, is a classifying topos for* $T$*.*
*Hence every geometric theory has a classifying topos.*

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