A NOTE ON SEMISIMPLICIAL SETS

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Abstract. This short note, written during the author's stay at the Institute for Advanced Study, investigates the possibility of imposing different kinds of model structure on the category of semisimplicial sets.

1. A model structure using simplicial boundary maps

In the following the will write Δ for the simplicial category and Δ_i for the semisimplicial category, which is the lluf subcategory of Δ containing only the injective maps. Moreover, we will write $s\mathcal{S}ets$ for the category of simplicial sets and $ss\mathcal{S}ets$ for the category of presheaves on Δ_i, *aka* semisimplicial sets.

The inclusion $i\colon \Delta_i \to \Delta$ determines a forgetful functor $i^*\colon s\mathcal{S}ets \to ss\mathcal{S}ets$, which has both adjoints. Its right adjoint i_* allows transfer of model structures, enabling one to carry over the Kan model structure on simplicial sets to one on semisimplicial sets. As a result there is a model structure on $ss\mathcal{S}ets$ in which:

(1) A map f is a weak equivalence iff i_*f is.
(2) A map f is a fibration iff i_*f is.
(3) The simplicial horn inclusions are the generating trivial cofibrations.
(4) The simplicial boundary inclusions are the generating cofibrations.

In addition, this should turn (i^*, i_*) into a Quillen equivalence. Item 4 implies that not every monomorphism in semisimplicial sets is a cofibration. In fact, the cofibrant objects are precisely the simplicial sets (more precisely: the objects in the image of i^*). In addition, we have:

Proposition 1.1. *It is not true in this model structure that the Π of a fibration along a fibration is again a fibration.*

Proof. It suffices to find a trivial cofibration whose pullback along a fibration is no longer a trivial cofibration. As a trivial cofibration we take the simplicial horn inclusion $\Lambda^0[1] \to \Delta[1]$, mapping the simplicial point to the endpoint of a simplicial interval, and as our fibration we take the map $\Delta_i[0] \to \Delta[1]$, mapping the semisimplicial point to the starting point of a simplicial interval. The latter is indeed a fibration for the trivial reason that, because there are no degeneracies in $\Delta_i[0]$, there can be

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no map from a simplicial horn into it. But the pullback is $0 \to \Delta_i[0]$, which is not a cofibration as $\Delta_i[0]$ is not cofibrant. □

2. No model structure using semisimplicial boundary maps

In order to circumvent the obstruction from the previous paragraph one might try to define a model structure on $ss\mathcal{S}ets$ based on semisimplicial boundary maps and semisimplicial horn inclusions. We will now show that this is impossible.

But first we recall some facts about Quillen model categories. Let A be any object in a model category. We can factor the codiagonal $A + A \to A$ as a cofibration followed by a trivial fibration:

$$A + A \xrightarrow{[\partial_A^0, \partial_A^1]} \mathrm{Cyl}(A) \xrightarrow{\sigma_A} A.$$

The object in the middle is called a *cylinder object* on A and determines a notion of *left homotopy* on maps from A to any other object X, the definition being that $f_0 \colon A \to X$ is left homotopic to $f_1 \colon A \to X$ whenever there is a map $H \colon \mathrm{Cyl}(A) \to X$ with $f_e = H\partial_A^e$ for $e \in \{0, 1\}$. If A is cofibrant and X is fibrant, then this defines an equivalence relation and we will write $[A, X]$ for the quotient.

Now consider the following non-triviality requirements for a model structure on the category of semisimplicial sets:

(1) There is a fibrant object P with $|P_0| \geq 2$ and $[A, P] = 1$ for any cofibrant object A (existence of a non-trivial contractible space).
(2) There is a fibrant object Q with $|[A, Q]| \geq 2$ whenever $|A_0| > 0$ and A is cofibrant (existence of a space with more than one connected component).

Remark 2.1. Note that in both requirements it sufficient to consider only *fibrant-cofibrant* A, because of the existence of fibrant replacements and the following fact: if $f \colon A \to B$ is a trivial cofibration and X is fibrant, then f determines a map $[B, X] \to [A, X]$ which is a bijection.

Proposition 2.2. *There is no model structure on the category of semisimplicial sets in which the semisimplicial boundary map* $0 = \partial\Delta_i[0] \to \Delta_i[0]$ *is a cofibration and which satisfies both non-triviality requirements.*

Proof. Suppose there would be. Then $D := \Delta_i[0]$, the representable semisimplicial set at 0, would be cofibrant. Consider a cylinder object on it:

$$D + D \xrightarrow{[\partial_D^0, \partial_D^1]} \mathrm{Cyl}(D) \xrightarrow{\sigma_D} D.$$

Note that because there is a map σ_D as shown, the object $\mathrm{Cyl}(D)$ can only have 0-simplices. Now we make a case distinction as to whether ∂_D^0 and ∂_D^1 map to the same 0-simplex in $\mathrm{Cyl}(D)$ or not.

Case 1: ∂_D^0 and ∂_D^1 map to the same 0-simplex. Then find a fibrant object P with $|P_0| \geq 2$ and $[A, P] = 1$ for any cofibrant A. The former means that there are two different maps $D \to P$, while the latter means that they have to be homotopic. But this homotopy cannot be witnessed by a suitable map $\mathrm{Cyl}(D) \to P$.

Case 2: ∂^0_D and ∂^1_D map to different 0-simplices. Then let X be an arbitrary fibrant object and $f, g \colon D \to X$ be two maps. Then one can define a left homotopy between them by sending the 0-simplex in the image of ∂^0_D to $f_0(\mathsf{id})$ and the one in the image of ∂^1_D to $f_1(\mathsf{id})$; the other 0-simplices can be sent to anything you like. (There are no 1-simplices to take care of: this is where the analogy with simplicial sets breaks.) So we have that $|[D, X]| \leq 1$ for any fibrant object X. But that contradicts the second non-triviality requirement. $\square$

So, in particular, there can be no non-trivial model structure on semisimplicial sets in which every object is cofibrant (as in Cisinski-style model structures), or in which the semisimplicial boundary maps are the generating cofibrations (in analogy with simplicial sets).