Handout for
Joost Nuiten and Urs Schreiber: "Cohomological quantization"
String Geometry Network Meeting
Workshop at ESI Vienna
February 24-28, 2014
www.ingvet.kau.se/juerfuch/conf/esi14/esi14_34.html

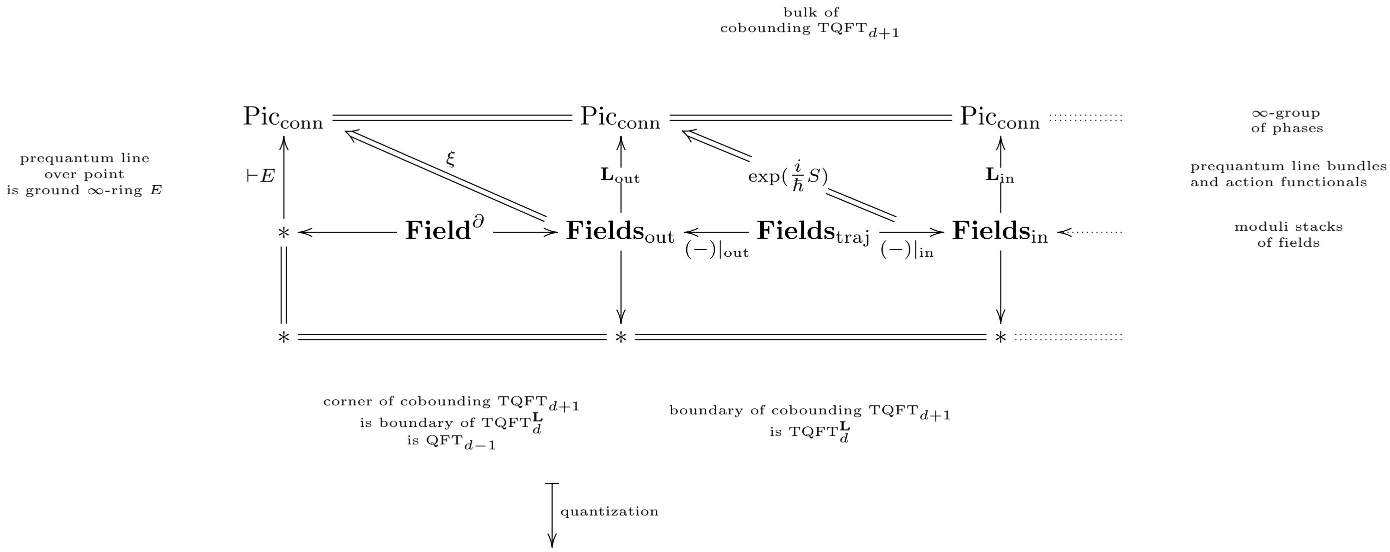
- Joost Nuiten, MSc Thesis, Utrecht (2013): ncatlab.org/schreiber/show/master+thesis+Nuiten
- talk notes: ncatlab.org/schreiber/show/Quantization+via+Linear+Homotopy+Types

0.0.1 Translation between linear homotopy-type theory, generalized cohomology and quantization

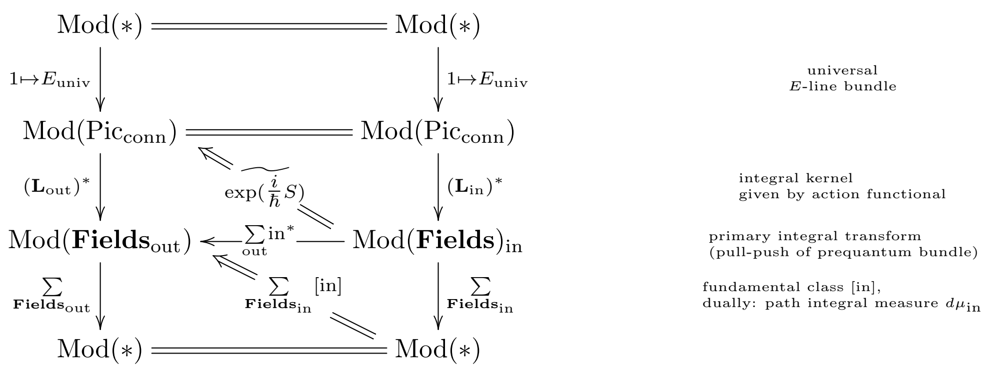
linear homotopy-type theory	**twisted generalized cohomology**	**quantum theory**
linear homotopy-type	(module-)spectrum	state space
multiplicative conjunction	smash product of spectra	composite system
dependent linear type	module spectrum bundle	
Frobenius reciprocity	six operation yoga in Wirthmüller context	linearity of integrals
dual type (linear negation)	Spanier-Whitehead duality	dual state space
invertible type	twist	prequantum line bundle, quantum anomaly
dependent sum	generalized homology spectrum	space of compactly supported quantum states "bra"
dual of dependent sum	generalized cohomology spectrum	space of quantum states "ket"
linear implication	bivariant cohomology	quantum operators
exponential modality	Goodwillie exponential	Fock space
dependent sum over finite homotopy type	Thom spectrum	
dualizable dependent sum over finite homotopy type	Atiyah duality between Thom spectrum and suspension spectrum	
(twisted) self-dual type	Poincaré duality	inner product (Hilbert) space
dependent sum coinciding with dependent product	ambidexterity, semiadditivity	system of inner product state spaces
dependent sum coinciding with dependent product up to invertible type	Wirthmüller isomorphism (twisted ambidexterity)	anomalous system of inner product state spaces
$(\sum_f \dashv f^*)$-counit	pushforward in generalized homology	
(twisted-)self-duality-induced dagger of this counit	(twisted-)Umkehr map, fiber integration	quantum superposition and interference
linear polynomial functor	primary integral transform	propagator in cobounding TQFT_{d+1}
correspondence with linear implication	motive	prequantized Lagrangian correspondence, action functional
composite of this linear implication with unit and daggered counit	secondary integral transform	cohomological path integral, motivic transfer
trace	Euler characteristic	partition function

0.0.2 The quantization process.

Prequantum $d+1$-dimensional field theory in $\mathrm{Corr}_2(\mathbf{H})$.



Quantum $d+1$-dimensional field theory in Mod_2.



encodes

Quantum d-dimensional field theory

as unit-component in $\mathrm{Mod}(*)$ of the above transformation: $\mathbb{D}\int_{\mathbf{Fields}_{\mathrm{traj}}} \exp(\frac{i}{\hbar}S)d\mu :=$

$$\sum_{\mathbf{Fields}_{\mathrm{out}}} \mathbf{L}_{\mathrm{out}} \xleftarrow{\sum_{\mathbf{Fields}_{\mathrm{out}}} \epsilon_{\mathbf{L}_{\mathrm{out}}}} \sum_{\mathbf{Fields}_{\mathrm{out}}} \mathrm{out}_! \mathrm{out}^* \mathbf{L}_{\mathrm{out}} \xleftarrow{\simeq} \sum_{\mathbf{Fields}_{\mathrm{traj}}} \mathrm{out}^* \mathbf{L}_{\mathrm{out}} \xleftarrow{\sum_{\mathbf{Fields}_{\mathrm{traj}}} \exp(\frac{i}{\hbar}S)} \sum_{\mathbf{Fields}_{\mathrm{traj}}} \mathrm{in}^* \mathbf{L}_{\mathrm{in}} \xleftarrow{\simeq} \sum_{\mathbf{Fields}_{\mathrm{in}}} \mathrm{in}_! \mathrm{in}^* \mathbf{L}_{\mathrm{in}} \xleftarrow{\sum_{\mathbf{Fields}_{\mathrm{in}}} [\mathrm{in}]} \sum_{\mathbf{Fields}_{\mathrm{in}}} \mathbf{L}_{\mathrm{in}} \otimes \tau$$

(secondary integral transform: pull-push of states).

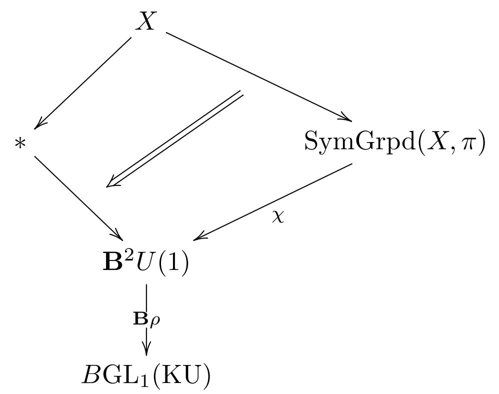
field theory	spaces of states	propagator
TQFT_{d+1}	$\mathrm{Mod}_2 \in \mathrm{Cat}_2$	integral transform
TQFT_d^{τ}	$\mathrm{Mod}(*) \in \mathrm{Mod}_2$	secondary integral transform, path integral
QFT_{d-1}	$\sum_X \mathbf{L}_X \in \mathrm{Mod}(*)$	equivariance under Hamiltonian group action

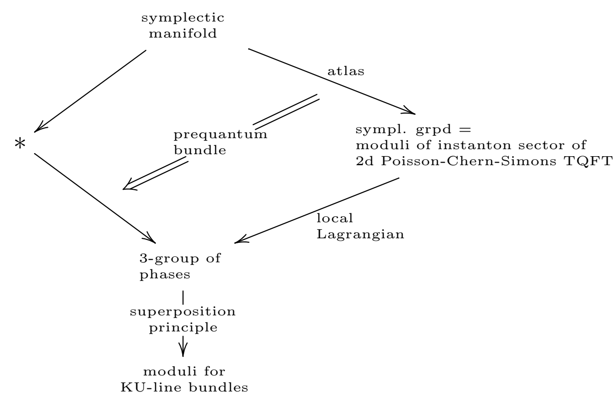
0.0.3 Translation between linear homotopy-type theory in $E\mathrm{Mod}$ and twisted E-cohomology.

special case	**linear homotopy-type theory**	**higher linear algebra** viz. **generalized cohomology theory**
	$E \in \mathrm{CRing}_\infty$	ground ring
	$X \in \infty\mathrm{Grpd}$	base homotopy type (base space)
	$\tau : X \longrightarrow \mathrm{Pic}(E)$	twist
	$\widehat{\tau} := \tau^* \widehat{\mathrm{Pic}}(E) \in \mathrm{Mod}(X)$	E-line bundle
canonical twist on moduli for stable vector bundles	$J^E :\ \mathbb{Z} \times BO \xrightarrow{J} \mathrm{Pic}(\mathbb{S}) \xrightarrow{\mathrm{Pic}(\mathbb{S}\to E)} \mathrm{Pic}(E)$	J-homomorphism
	$\sum_X \widehat{\tau} \simeq E_{\bullet+\tau}(X)$	spectrum of τ-twisted E-homology cycles
trivial twist	$\sum_X 1_X \simeq E_\bullet(X) = E \wedge \Sigma^\infty_+ X$	suspension spectrum
$X \xrightarrow{\xi} \mathbb{Z} \times BO$ modulating stable vector bundle	$\sum_X \widehat{J^E \circ \xi} = E \wedge X^\xi$	Thom spectrum
canonical twist on $X := BO\langle n \rangle$ $J^E_{BO\langle n \rangle} : BO\langle n \rangle \to BO \xrightarrow{J^E} \mathrm{Pic}(E)$	$\sum_{BO\langle n \rangle} \widehat{J^E_{BO\langle n \rangle}} \simeq MO\langle n \rangle$	universal Thom spectrum
low n	$n = 0$: MO $n = 1$: $M\mathrm{SO}$ $n = 2$: $M\mathrm{Spin}$ $n = 4$: $M\mathrm{String}$	Riemannian- oriented- spin- string- } cobordism spectrum
	$\mathbb{D}$	Spanier-Whitehead duality
	$\mathbb{D} \sum_X \widehat{\tau} = E^{\bullet+\tau}(X)$	spectrum of τ-twisted E-cohomology cocycles
X compact smooth manifold with tangent bundle TX and stable normal bundle $NX = -TX$	$\mathbb{D}(E \wedge \Sigma^\infty_+ X) \simeq E_{\bullet+NX}(X)$	Atiyah-Whitehead duality
	$\begin{array}{ccc} Z & \xrightarrow{\;f\;} & X \\ & \swarrow\!\!\!\swarrow_o & \\ {}_{\tau_Z}\searrow & & \swarrow_{\tau_X} \\ & \mathrm{Pic}(E) & \end{array}$	fiberwise E-orientation of τ_Z relative to τ_X
to the point	$\begin{array}{ccc} Z & \longrightarrow & * \\ & \swarrow\!\!\!\swarrow_o & \\ {}_{\tau_Z}\searrow & & \swarrow_{0} \\ & \mathrm{Pic}(E) & \end{array}$	τ_Z-twisted E-orientation of Z
vanishing twist on domain	$\begin{array}{ccc} Z & \xrightarrow{\;f\;} & X \\ & \swarrow\!\!\!\swarrow_o & \\ {}_{0}\searrow & & \swarrow_{\tau_X} \\ & \mathrm{Pic}(E) & \end{array}$	E-orientation of f
fiberwise fundamental class with twist τ	$f_! f^* \widehat{\tau_X} \simeq \mathbb{D} f_! f^* \mathbb{D}(\widehat{\tau_X} \otimes \widehat{\tau})$	fiberwise twisted Poincaré duality

0.0.4 Examples

a) Particle at the boundary of 2d Poisson-Chern-Simons TQFT.

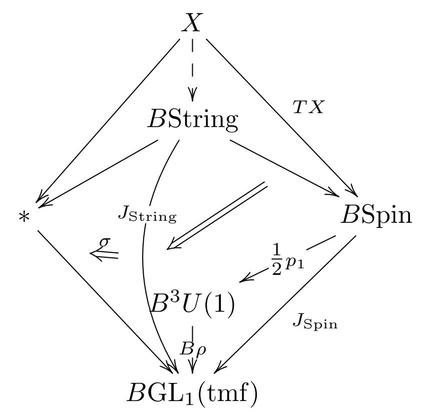


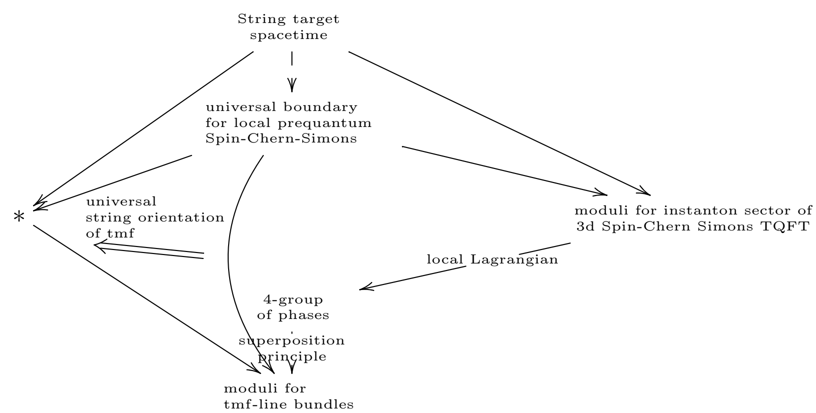


$$\mathrm{KU} \longleftarrow \mathrm{KU}_{\bullet+\chi}(\mathrm{SymGrpd}(X,\pi))$$

bundle of Hilbert spaces
of quantum states
from symplectic leaf-wise
geometric quantization

b) Superstring at boundary of 3d Spin**-Chern-Simons TQFT.**

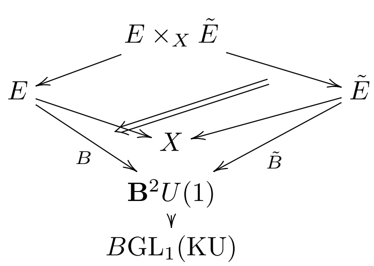


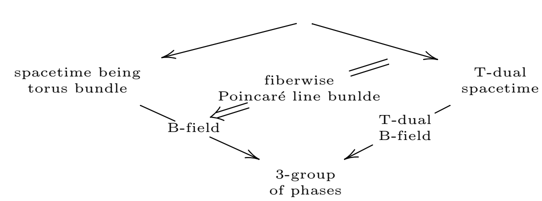


$$\mathrm{tmf} \longleftarrow X^{TX}$$

integral Witten genus =
non-perturbative string partition function

c) D-Brane Charge and T-Duality.





$$\mathrm{KU}_{\bullet+B}(E) \xleftarrow{\simeq} \mathrm{KU}_{\bullet-\mathrm{rk}(E)+\tilde{B}}(\tilde{E})$$

T-duality equivalence on
D-brane charges
in K-theory