Quantization In Linear Homotopy-Type theory

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We will be considering a formal system that refines traditional quantum logic such as to allow it to speak about quantization of field theories. Since the proverb has it that quantization is a mystery, such a formalization inevitably tends to raise the question of whether it helps with "interpreting" the theory (in the physicist's vague sense, not the logician's precise one). That is not our actual concern, but let it here serve as a lead-in.

Ever since Galileo, mathematical formalization alone serves to demystify (see also J. Butterfield's talk at the meeting). The ancients found the daily rising of the sun a mystery, some found it non-evident enough to sacrifice their own kin in the hope to propel the process. Later people were still mystified by the epicyclic intransparency of heavenly motions. That we are not mystified by any of this anymore is not because we now have some deep "interpretation" of the concept of moving point masses. Instead, we just found a formal system (Newton's equations of motion, to start with) that naturally allows to deduce these processes; and after staring at that for a while and finding all the useful facts it implies, it began to look very much self-evident.

The same ought to be true for a working formal quantum physics, a "quantum logic". A working quantum logic should be a formalism that is more than the "QM 101 made difficult" as which traditional quantum logic must appear; it should instead be a formalism that empowers us to think useful thoughts that were previously hard to think and that inform us about the genuine deep aspects of quantum physics. To recall some of these:

Deep structural aspects of quantum physics.

1. the path integral;
2. quantum anomaly cancellation;
3. non-perturbative effects;
4. holography;
5. motivic structures.

This points to what is a well-kep secret in much of the literature on quantum logic and related issues: Quantum mechanics is not actually our most fundamental theory of nature. Instead, modern physics says that reality is fundamentally governed by *quantum field theory*. More specifically, modern physics is based on *local Lagrangian gauge quantum field theory*:

Characteristics of fundamental quantum physics.

1. **fields** – types of configurations depending on n-dimensional spaces;
2. **gauge** – types of fields are really *moduli stacks*, geometric homotopy types;
3. **Lagrangian** – obtained via quantization from Lagrangian n-cocycle data;
4. **local** – n-dimensional theory is an n-categorical construction.

Quantum mechanics itself is just one limiting case of that. But if there is interesting formal structure in the foundations of physics at all, then it seems plausible that these are most purely exhibited by the very foundations, and not so much by some special limiting case.

The other proverb, the one referring to the effectiveness of mathematics in the physical sciences, suggests that a foundations of fundamental physics should go along well with the very foundations of mathematics and logic themselves. These have seen some considerable advances since, say, Hilbert, and we are going to take these into full account.

Logic sits inside type theory. A deep insight (attributed to Brouwer-Heyting-Kolmogorov, also to Howard) says that propositional or first order logic may naturally be regarded as a subsystem of what more fundamentally is *type theory* or *dependent type theory*, respectively. Under this *BHK correspondence* a proposition ϕ about terms x of type X is identified as the sub-type

$$\begin{array}{c} \sum\limits_{x \in X} \phi(x) = \{x \in X | \phi(x)\, \mathrm{true}\} \\ \big\downarrow \\ X \end{array} \quad \in \mathbf{H}_{/X}$$

of X of all those terms x which validate ϕ. Here $\mathbf{H}$ denotes the *category* of all types – and that is the insight of *categorical logic*, that the syntax of type theories has semantics in suitable categories. Moreover, $\mathbf{H}_{/X}$ denotes the slice category over the type X, and this is what interprets X-*dependent* types.

It should be plausible that using types instead of (just) propositons as the fundamental logical substrate suits formalization of fundamental physics, for here we are clearly concerned with talking not just about propositions, but about "things", notably when talking about some *type of fields*, which we generically write

- **Fields** $\in \mathbf{H}$ – a moduli space of fields.

And indeed these should depend on other types, such as on spacetimes $X \in \mathbf{H}$, for instance in order to form the *field bundle*

- $\begin{array}{c} \mathbf{Fields}_X \\ \downarrow \\ X \end{array} \in \mathbf{H}_{/X}$ – a bundle of moduli of fields parameterized over X

of which the actual field configurations of a field theory on X would be sections.

Dependent types and Existence. The only basic operations on (dependent) types are these: for any morphism $f \; : \; X \longrightarrow Y$ in $\mathbf{H}$ (hence a function sending terms of type X to terms of type Y) there is an adjoint triple

$$(\sum_f \dashv f^* \dashv \prod_f) \; : \; \mathbf{H}_{/X} \underset{\prod_f}{\overset{\sum_f}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{f^*}}}}} \mathbf{H}_{/Y}$$

whose operations are called, in type theory and in order of appearance, the dependent sum, the context extension and the dependent product along f. (In geometry and topos theory this is instead known as base change, or similar.) Here sum and product are to be understood fiberwise over the fibers of f, and so if we think of a bundle E over X as a collection of types $E(x)$ for $x \in X$, then the dependent sum reads

$$(\sum_f E)(y) = \sum_{x \in f^{-1}(y)} E(x)$$

and is hence manifestly a form of fiber integration along f. This is going to play arole in the path integral below.

We already used the $\sum$-notation above for indicating how propositions appear as types, and indeed the combined restriction and co-restriction of $\sum\limits_X : \mathbf{H}_{/X} \to X$ to propositions is exponential quantification $\underset{x \in f^{-1}(y)}{\exists} \phi(x)$. So where propositional logic has the proposition "There exists an x such that $\phi(x)$ is true." its embedding into type theory replaces that with "The collection of all x such that $\phi(x)$ is true." Something to keep in mind when we get to the path integral below.

Constructive $\leftrightarrow$ Physically realizable. Closely related is the fact that type theory embodies *constructive mathematics*, where nothing is regarded as true unless its proof may be constructed in a way that yields an algorithm. For instance in type theory to prove that there exists x such that $\phi(x)$ is true is to actually construct a term $t \in \sum\limits_{x \in X} \phi(X)$ from the deductive rules of the theory. This explains the fundamental relevance of type theory in computer science, where these proofs are the very programs – which shows that the constructive concept of existence is in some way closely related to the physical concept of existence.

Taking this constructivism fully seriously lead to a breakthrough convergence of formerly disparate concepts:

Constructive identity types $\leftrightarrow$ Gauge principle. Constructivism demands that given a (dependent) type $\mathbf{Fields}_X$ (of fields, for instance) and given two terms ϕ_1, ϕ_2 of that type (so: two field configurations of given type over a spacetime X) then it is misguided to ask whether these are equal or not, instead we have to construct a witness α exhibiting their equivalence, hence produce a *gauge transformation* making them gauge equivalent. It is in turn wrong to assert that two such gauge equivalences α_1 and α_2 are equal, instead we have to exhibit a gauge-of-gauge transformation between those:

$$
\begin{tikzcd}
\phi_1 \arrow[rr, bend left=50, "\overset{\alpha_1}{\simeq}"] \arrow[rr, bend right=50, "\underset{\alpha_2}{\simeq}"'] \arrow[rr, phantom, "\Downarrow \simeq"] & & \phi_2
\end{tikzcd} .
$$

And so on. This is really the *gauge principle* in physics. At the same time, this is how constructive type theory is *automatically* a theory of homotopy types (of ∞-groupoids) [UFP13].

Example [Sch13a]. For the purpose of the following exposition, a running example for $\mathbf{H}$ to keep in mind is

$$
\mathbf{H} = \mathrm{SynthDiff}\infty\mathrm{Grpd} := \mathrm{Sh}_\infty(\mathrm{FormMfd})
$$

the homotopy topos of sheaves of homotopy types on formal smooth manifolds. This is a homotopy-theoretic version of a topos that interprets synthetic differential geometry.

Noteworthy geometric homotopy types that we encounter below are

$$
\mathbf{B}^n U(1) \in \mathbf{H}
$$

which are obtained by deloopoing the abelian Lie group $U(1) = \mathbb{R}/\mathbb{Z} \in \mathrm{Grp}(\mathbf{H})$ n times. These are the moduli for instanton sectors of n-form $U(1)$-gauge fields.

Notice a basic fact of homotopy theory, a first little hint of holography: Giving a homotopy as on the left of

$$
\begin{tikzcd}
 & \prod\limits_{\Sigma} \mathbf{Fields}_\Sigma \arrow[dl] \arrow[dr] & \\
* \arrow[dr] & \arrow[l, phantom, "\swarrow"] & * \arrow[dl] \\
 & \mathbf{B}^{n+1}U(1) &
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
 & \prod\limits_{\Sigma} \mathbf{Fields}_\Sigma \arrow[ddl] \arrow[ddr] \arrow[dd, dashed, "\exp(\frac{i}{\hbar}S)"] & \\
 & & \\
* \arrow[ddr] & \mathbf{B}^n U(1) \arrow[l] \arrow[r] \arrow[d, phantom, "\swarrow"] & * \arrow[ddl] \\
 & & \\
 & \mathbf{B}^{n+1}U(1) &
\end{tikzcd}
$$

is equivalently a dashed map as shown on the right. If here we think of the tip as a type of fields over a closed manifold Σ, and if we furthermore restrict to $n = 0$ in which case $\mathbf{B}^0 U(1) \simeq U(1)$, then on the right the map denoted $\exp(\frac{i}{\hbar}S)$ may be regarded as an *action functional* on these fields, as indicated. More generally Σ may have boundaries, in which case the situation is more interesting, we come to this below.

Infinitesimal identity types $\leftrightarrow$ homotopy Lie algebroids / BRST complexes. Homotopy types of gauge equivalences are best known in physics in the approximation of perturbation theory, where they appear as homotopy Lie algebroids known as BV-BRST complexes. For instance if $\mathbf{Fields} \in \mathbf{H}$ is a type of fields and G is a gauge group acting on that, then there is the homotopy quotient $\mathbf{Fields}//G$. Infinitesimally this is the *BRST-complex*.

2 Categories of being and Prequantum geometry

To make formal sense of what we mean by "infinitesimally" here, and generally by "differential" etc. we need to equip the types with *geometric quality*. In logic and type theory such "quality" is called "modality". For propositions a modality is "a way of being true" – for instance: "possibly being true" or "necessarily being true". But for types a modality is simply "a way of being".

A modality is formalized as *monad* or *comonad* on the type system $\mathbf{H}$. Lawvere observed [Law91] that adding an *adjoint idempotent modality*

$$\begin{array}{ccc} \emptyset & \subset & \flat \\ \bot & & \bot \\ * & \subset & \sharp \end{array}$$

naturally has the following meaning:

- $\flat$ is the modality of "being geometrically discrete";
- $\sharp$ is "being geometrically codiscrete".

Adding one more makes it a *category of cohesion*

$$\begin{array}{ccc} & & \int \\ & & \bot \\ \emptyset & \subset & \flat \\ \bot & & \bot \\ * & \subset & \sharp \end{array}$$

In homotopy-type theory we have that

- $\int$ is the modality of "being homotopy invariant".

In [Sch13a] we add three more to obtain what we called a *category of differential cohesion*

$$\begin{array}{ccccc} & & & & \Re \\ & & & & \bot \\ & & \int & \subset & \& \\ & & \bot & & \bot \\ \emptyset & \subset & \flat & \subset & \Im \\ \bot & & \bot & & \\ * & \subset & \sharp & & \end{array}$$

We find that

- $\&$ is the modality of "being formally étale", orthogonal to "being infinitesimal".

Claim (Theorem) a) ([Sch13a]): In cohesive homotopy-type theory there is a natural construction of differential cohomology = higher gauge theory fields.

For instance there is the type $\mathbf{B}U(1)_{\mathrm{conn}} \in \mathbf{H}$ characterized by the fact that for X a smooth manifold, then functions

$$\nabla : \; X \longrightarrow \mathbf{B}U(1)_{\mathrm{conn}}$$

are equivalently $U(1)$-principal connections on X – for instance electromagnetic field configurations.

Improved Claim a) ([BNV13]) In cohesive homotopy-type theory *every* stable homotopy-type represents a generalized differential cohomology theory (such as differential K-theory, differential elliptic cohomology, etc.) hence a higher gauge theory.

Claim (Theorem) b) ([FRS13], [Sch13a]): Cohesive homotopy-type theory naturally encodes local prequantum geometry, hence Lagrangian cocycle data on higher moduli stacks of fields.

We continue to provide some examples of this.

running Example a) – The particle at the boundary of 2d Poisson-Chern-Simons theory. Consider a Poisson manifold (X, π), the phase spaces of a folitation by mechanical systems. It is encoded in its Poisson Lie algebroid $\mathfrak{P}$, which is equipped with a 2-plectic cocycle π

$$\mathfrak{P} \xrightarrow{\pi} \mathbf{B}^2\mathbb{R}$$

Theorem. [FSS12][Bongers13] Higher Lie integration $\exp(-)$ of π yields the Lagrangian $\mathbf{L}_{2d\mathrm{CS}}$ for non-perturbative 2-d Poisson-Chern-Simons theory

$$\mathbf{L}_{2d\mathrm{CS}} = \exp(\pi) \;:\; \mathrm{SymplGrpd}(\mathfrak{P}, \pi)_{\mathrm{conn}} \longrightarrow \mathbf{B}^2 U(1)_{\mathrm{conn}}$$

whose moduli stack of fields is differential cohomology refinement of the "symplectic groupoid". Moreover, the original Poisson manifold is a boundary condition of this 2-d theory exhibited by a correspondence diagram in the slice $\mathbf{H}_{\mathbf{B}(\mathbf{B}U(1)_{\mathrm{conn}})}$

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\swarrow"] & \\
{*} \arrow[dr] & & \mathrm{SymplGrpd}(X, \pi)_{\mathrm{conn}} \arrow[dl, "\mathbf{L}"] \\
 & \mathbf{B}\mathbf{B}U(1)_{\mathrm{conn}} &
\end{tikzcd}$$

which describes the points of X as "trajectories" along which the fields of the 2d theory may approach the boundary.

running Example b) – The string at the boundary of 3d Chern-Simons theory.
Theorem [FSS12], exposition in [FSS13]. The Lie integration of the canonical Lie algebra 3-cocycle

$$\mathfrak{so} \xrightarrow{\langle -,[-,-]\rangle} \mathbf{B}^2 2\mathbb{R}$$

is a differential cohomology refinement of the first fractional Pontryagin class $\frac{1}{2}p_1$

$$\mathbf{L}_{3d\mathrm{CS}} = \tfrac{1}{2}\mathbf{p} \;:\; \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

and this is the local Lagrangian for 3d Spin Chern-Simons theory.

Moreover, the *universal* boundary condition for this is the delooped String 2-group $\mathbf{B}$String, and hence a manifold X via a Spin-structure ∇_{Spin} yields a boundary for the 3d Chern-Simons theory here precisely if it lifts to a String-structure ∇_{String}

$$\begin{tikzcd}
 & X \arrow[dl, bend right] \arrow[d, dashed, "\nabla_{\mathrm{String}}"] \arrow[ddr, "\nabla_{\mathrm{Spin}}"] & \\
 & \mathbf{B}\mathrm{String}_{\mathrm{conn}} \arrow[dl] \arrow[dr] & \\
 * \arrow[dr] & \Swarrow & \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \arrow[dl, "\frac{1}{2}\mathbf{p}_1"] \\
 & \mathbf{B}^3 U(1)_{\mathrm{conn}} &
\end{tikzcd}$$

Below we see the holographic quantization of these two examples. In general, a space of field trajectories $\mathbf{Fields}_{\mathrm{traj}}$ equipped with action functional data for an n-dimensional field theory is a correspondence in $\mathbf{B}_{/\mathbf{B}^n U(1)}$ of the form

$$\begin{tikzcd}
 & \mathbf{Fields}_{\mathrm{traj}} \arrow[dl, "(-)|_{\mathrm{out}}"'] \arrow[dr, "(-)|_{\mathrm{in}}"] & \\
\mathbf{Fields}_{\mathrm{out}} \arrow[dr, "\mathbf{L}_{\mathrm{out}}"'] & \Swarrow \exp(\frac{i}{\hbar}S) & \mathbf{Fields}_{\mathrm{in}} \arrow[dl, "\mathbf{L}_{\mathrm{in}}"] \\
 & \mathbf{B}^n U(1) &
\end{tikzcd}$$

Now to quantize all this higher prequantum geometry.

3 Categories of linear homotopy-types and Quantization

There is a simple idea: quantization is *linearization* of the above pre-quantum geometry, analogous to how motivic geometry is a linearization of algebraic geometry. To formalize this, think of "linear" as being "affine with basepoint".

quantum $\leftrightarrow$ linear = base-point + affine. The "modality of being pointed" is the *maybe monad*

$$*/ \;:\; X \mapsto X \coprod * \,.$$

Indeed, the $*/$-modal types are equivalently the pointed types canonically equipped with the smash tensor product as "linear conjunction". Notice that this is a non-cartesian tensor product, the hallmark of quantum theory in category theory.

Moreover, according to deformation theory a pointed space is affine if it is infinitesimally extended, which by the above means it is orthogonal to $\oint$-modal types.
Definition: For $X \in \mathbf{H}$ then $\mathrm{Mod}(X)$ are the $*/$-modal types in $\mathbf{H}_{/X}$ which are left orthogonal to $\oint$-modal types.
Proposition: this forms a *linear* homotopy-type theory

$$\begin{tikzcd}
\mathrm{Mod}(X) \arrow[r, shift left=2, "\sum_f"] \arrow[r, shift right=2, "\prod_f"'] & \mathrm{Mod}(Y) \arrow[l, "f^*" description]
\end{tikzcd}$$

Under this extension the meaning of the existential quantifier $\sum$ changes drastically:

For the following assume for simplicity of notation that $\mathrm{Line}(*)$ has a reflection as $\mathrm{Line} \in \mathbf{H}$ that classifies invertible linear types.

Definition (the path integral): First turn an action functional on trajectories as above into an *integral kernel* by associating linear coefficients

$$\begin{tikzcd}[column sep=small]
 & \mathbf{Fields}_{\mathrm{traj}} \arrow[dl, "(-)|_{\mathrm{out}}"'] \arrow[dr, "(-)|_{\mathrm{in}}"] & \\
\mathbf{Fields}_{\mathrm{out}} \arrow[dr, "\mathbf{L}_{\mathrm{out}}"'] & \arrow[l, phantom, "\Leftarrow \exp(\frac{i}{\hbar}S)"] & \mathbf{Fields}_{\mathrm{in}} \arrow[dl, "\mathbf{L}_{\mathrm{in}}"] \\
 & \mathbf{B}^n U(1) \arrow[d] & \\
 & \mathrm{Line}(*) &
\end{tikzcd}$$

The path integral is then effectively the linear sum over this: $\sum\limits_{\mathbf{Fields}_{\mathrm{traj}}} \exp(\frac{i}{\hbar}S)$.

The full expression involves pre-composing this with the $(\prod\limits_{\mathrm{in}} \dashv \mathrm{in}^*)$-unit followed by a "twisted ambidexterity" measure $d\mu$, and postcomposing with the $(\sum\limits_{\mathrm{out}} \dashv \mathrm{out}^*)$-counit:

$$\mathbb{D} \int\limits_{\mathbf{Fields}_{\mathrm{traj}}} \exp(\tfrac{i}{\hbar}S) d\mu =$$

$$\sum_{\mathbf{Fields}_{\mathrm{out}}} \mathbf{L}_{\mathrm{out}} \xleftarrow{\sum\limits_{\mathbf{Fields}_{\mathrm{out}}} \epsilon_{\mathbf{L}_{\mathrm{out}}}} \sum_{\mathbf{Fields}_{\mathrm{out}}} \mathrm{out}_! \mathrm{out}^* \mathbf{L}_{\mathrm{out}} \xleftarrow{\simeq} \sum_{\mathbf{Fields}_{\mathrm{traj}}} \mathrm{out}^* \mathbf{L}_{\mathrm{out}} \xleftarrow{\sum\limits_{\mathbf{Fields}_{\mathrm{traj}}} \exp(\frac{i}{\hbar}S)} \sum_{\mathbf{Fields}_{\mathrm{traj}}} \mathrm{in}^* \mathbf{L}_{\mathrm{in}} \xleftarrow{\simeq} \sum_{\mathbf{Fields}_{\mathrm{in}}} \mathrm{in}_! \mathrm{in}^* \mathbf{L}_{\mathrm{in}} \xleftarrow{\sum\limits_{\mathbf{Fields}_{\mathrm{in}}} [\mathrm{in}]} \sum_{\mathbf{Fields}_{\mathrm{in}}} \mathbf{L}_{\mathrm{in}} \otimes \tau$$

Notice how here what used to be existential quantification becomes the path integral with its superposition and quantum interference:
The quantum incarnation of existential quantification.

1. in logic: That there exists a path.
2. in type theory: The collection of all paths.
3. in homotopy-type theory: The collection of all paths with gauge equivalences between them.
4. in linear type theory: The linear addition (superposition) of amplitudes of all paths.
5. in linear homotopy-type theory: The linear addition (superposition) of amplitudes of all paths with gauge equivalences taken into account.

Theorem (wth. Nuiten): This is quantum anomaly free if there is cobounding theory.

running example a) – The particle at the boundary of 2d Poisson-Chern-Simons theory:
Proposition [Nuiten13]: The above path integral yields the geometric quantization of symplectic manifolds in its K-theoretic incarnation (following Bott). It generalizes it to a geometric quantization of Poisson manifolds and reproduces there for instance the universal orbit method of [FHT05].

running example b) – The string at the boundary of 3d Chern-Simons theory

Proposition: Quantization of the boundary of Spin Chern-Simons with coefficients in the universal elliptic cohomology ring tmf

$$\begin{tikzcd}
 & B\mathrm{String} \arrow[dl] \arrow[dr] \arrow[ddd, bend right, "J_{\mathrm{String}}"'] & \\
\ast \arrow[ddr] \arrow[dr, phantom, "\swarrow^{\sigma}"] & & B\mathrm{Spin} \arrow[dl, "\frac{1}{2}p_1"'] \arrow[ddl, "J_{\mathrm{Spin}}"] \\
 & B^3U(1) \arrow[d, "B\rho"] & \\
 & B\mathrm{GL}_1(\mathrm{tmf}) &
\end{tikzcd} .$$

yields the non-perturbative refinement of the Witten genus [Wi87], the partition function of the string, to the String-orientation of tmf [AHR10]

$$\mathrm{tmf} = \sum_{\ast} 1_{\ast} \longleftarrow \sum_{B\mathrm{String}} J_{\mathrm{String}} = M\mathrm{String} \wedge \mathrm{tmf} \ .$$

(This follows using section 8 of [ABG11].)

4 Outlook

We are seeing here the pattern of the holographic principle [Maldacena97, Wi98]. The next example of interest of this form is induced from a local Lagrangian for 7-dimensional Chern-Simons theory [FSS12].

field theory	**spaces of states**	**propagator**	particle/2dCS	string/3dCS	6dSCFT/7dCS
TQFT_{d+1}	$\mathrm{Mod}_2 \in \mathrm{Cat}_2$	integral transform	TQFT_3	TQFT_4	TQFT_8
TQFT_d^{τ}	$\mathrm{Mod}(\ast) \in \mathrm{Mod}_2$	secondary integral transform	CS_2	CS_3	CS_7
QFT_{d-1}	$\sum_X A_X \in \mathrm{Mod}(\ast)$		QM_1	WZW_2	WZW_6

See also [Freed12]. Notice that according to [Wi07] the Kaluza-Klein compactification of WZW_6 on a torus is 4-dimensional (super-)Yang-Mills theory. This way we find at least a sketch of a plausible path here for how to approach the quantization of Yang-Mills theory [JaWi] in linear homotopy-type theory. A central open question from this perspective is which brave new ring would serve as the right coefficient for quantization of the 7d Chern-Simons theory.

References

[AHR10] M. Ando, M. Hopkins, C. Rezk, *Multiplicative orientations of* KI *and of the spectrum of topological modular forms*, 2010 www.math.uiuc.edu/mando/papers/koandtmf.pdf

[ABG11] M. Ando, A. Blumberg, D. Gepner, *Parametrized spectra, multiplicative Thom spectra, and the twisted Umkehr map* arXiv:1112.2203

[Bongers13] S. Bongers, *Geometric quantization of symplectic and Poisson manifolds*, MSc thesis, Utrecht (2013), ncatlab.org/schreiber/show/master+thesis+Bongers

[BNV13] U. Bunke, T. Nikolaus, M. Völkl, *Differential cohomology theories as sheaves of spectra* arXiv:1311.3188

[FSS12] D. Fiorenza, H. Sati, U. Schreiber, *Multiple M5-branes, String 2-connections, and 7d nonabelian Chern-Simons theory* to appear in Advances in Theoretical and Mathematical Physics arXiv:1201.5277

[FSS13] D. Fiorenza, H. Sati, U. Schreiber, *A higher stacky perspective on Chern-Simons theory*, in D. Calaque et al. (eds.) *Mathematical Aspects of Quantum Field Theories*, Spring 2014, arXiv:1301.2580

[FSS12] D. Fiorenza, U. Schreiber, J. Stasheff, *Cech cocycles for differential characteristic classes*, Advances in Theoretical and Mathematical Physics, Volume 16 Issue 1 (2012), pages 149-250 arXiv:1011.4735

[FRS13] D. Fiorenza, C. Rogers, U. Schreiber, *Higher geometric prequantum theory*, arXiv:1304.0236

[Freed12] D. Freed, *4-3-2 8-7-6*, talk at *ASPECTS of Topology*, Oxford, Dec 2012, `https://people.maths.ox.ac.uk/tillmann/ASPECTSfreed.pdf`

[FHT05] D. Freed, M. Hopkins, C. Teleman, *Loop Groups and Twisted K-Theory II*, J. Amer. Math. Soc. 26 (2013), 595-644 arXiv:math/0511232

[JaWi] A. Jaffe, E. Witten, *Quantum Yang-Mills theory* www.arthurjaffe.com/Assets/pdf/QuantumYangMillsWebRevised.pdf

[Law91] W. Lawvere, *Some Thoughts on the Future of Category Theory* in A. Carboni et. al. (eds.) *Category Theory*, Proceedings of the International Conference held in Como, Lecture Notes in Mathematics 1488, Springer (1991),
ncatlab.org/nlab/show/Some+Thoughts+on+the+Future+of+Category+Theory

[Maldacena97] J. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2:231 (1998), arXiv:hep-th/9711200

[Nuiten13] J. Nuiten, *Cohomological quantization of boundary field theory* , MSc thesis, Utrecht, 2013,
ncatlab.org/schreiber/show/master+thesis+Nuiten

[Sch13a] U. Schreiber, *Differential cohomology in a cohesive topos.* arXiv:1310.7930

[UFP13] Univalent Foundations Project, *Homotopy Type Theory Univalent Foundations of Mathematics*, Institute for Advaced Study, Princeton 2013, homotopytypetheory.org/book/

[Wi87] E. Witten, Elliptic Genera And Quantum Field Theory, Commun.Math.Phys. 109 525 (1987), projecteuclid.org/euclid.cmp/1104117076

[Wi98] E. Witten, *Anti-de Sitter space and holography*, Advances in Theoretical and Mathematical Physics 2: 253291 (1998) arXiv:hep-th/9802150

[Wi07] E. Witten, *Conformal field theory in four and six dimensions*, in U. Tillmann (ed.) *Topology, geometry and quantum field theory* LMS Lecture Note Series (2004) arXiv:0712.0157
Geometric Langlands From Six Dimensions, in P. Kotiuga (ed.) *A Celebration of the Mathematical Legacy of Raoul Bott*, AMS 2010 arXiv:0905.2720