Some thoughts on the future of Modal homotopy type theory

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For more on the applications see my talk in the Algebra session at the same meeting:
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1 Introduction

In [Euclid 300BC] it was shown that geometry could be formulated synthetically in first-order logic with a collection of axioms added that characterize points and lines. While this works well for the incidence geometry of the plane, Euclid's axioms are felt not to be useful for modern research-level geometry and physics.

In [Lawvere 67, Lawvere 97] it was suggested that differential geometry and the theory of differential equations, specifically those of relevance in continuum physics, could be formulated synthetically in the *internal logic*, in fact the internal type theory, of toposes that validate an extra axiom scheme, the *Kock-Lawvere axioms*. Relevant such toposes were eventually found [Dubuc 79] and basic differential geometry of manifolds was formulated in their internal logic [Kock 99, Kock 09]. While this works well as far as it goes, it remained unclear how modern research-level geometry and physics would benefit.

In [Lawvere 91, Lawvere 07] a rather different set of extra axioms on toposes was suggested to be relevant for the purpose of doing geometry formulated in their internal language: *axiomatic cohesion*. While the relation to the Kock-Lawvere axioms, or to any differential geometry, seems to have been left open, from the perspective of the logic it is noteworthy that cohesion on a topos is equivalent to the presence of a geometric modal operator [Goldblatt 81] acting on its internal intuitionistic type theory, such that this has two further (co-)modal operators left adjoint to it.

Might it be possible to lay useful *foundations for modern geometry and physics in the internal intuitionistic type theory of toposes with nothing but a system of modal operators added?*

Taken at face value this runs into the problem that intuitionistic type theory has, besides the problem of interpreting its identity types in categorical logic, no means to characterize its type universe as being that of a topos.

In [S 13, S 15] it was shown that implementing cohesion not on toposes but on ∞-toposes drastically increases the expressive power of the axiomatics and its relevance to modern geometry and physics. Moreover, the presence of synthetic infinitesimals otherwise encoded by the Kock-Lawvere axioms was shown to be captured by a second adjoint triple of modal operators, suitably compatible with the first: *differential cohesion*.

At the same time, while it might naively seem that passing from toposes to ∞-toposes makes the situation more complicated, the advent of homotopy type theory [UFP 13] showed that the opposite may be true: it is conjectured that homotopy type theory is accurately the internal language of (elementary) ∞-toposes:

Theorem 1.1.

- *HoTT has semantics in locally presentable locally Cartesian closed ∞-categories [Shulman 12];*
- *HoTT+UV$_{\mathrm{strict}}$ has semantics in the ∞-topos $\infty\mathrm{Grpd}$ [Kapulkin-Lumsdaine-Voevodsky 12];*
- *HoTT+UV$_{\mathrm{strict}}$ has semantics in a few infinite classes of ∞-presheaf ∞-toposes [Shulman 13, Shulman 15a];*

Remark 1.2.

- HoTT+UV$_{\mathrm{weak}}$ is argued to have semantics in all ∞-toposes [Shulman 14];
- HoTT+UV+Cohesion is developed in [S-Shulman 14, Licata-Shulman 15, Shulman 15b, Rijke-Shulman-Spitters 15].

synthetic geometry				
since		**add axioms to**	**with semantics in a**	**to obtain formalization of**:
BC 300	Euclid	first-order logic	hyperdoctrine	incidence geometry
1967	Lawvere	intuitionistic type theory	topos	differential geometry
1991	Lawvere	intuitionistic type theory	topos	$\sim$ differential topology
2014	Shulman-S	homotopy type theory	∞-topos	differential cohomology, differential geometry, PDE theory

The aim of the present note is to present *motivation* for the further development of modal homotopy type theory by surveying some statements in modern research-level geometry and physics that may neatly be obtained formally from just the structure of differentially cohesive ∞-toposes, hence that *ought* to be formalizable fairly directly within HoTT+UV+DifferentialCohesion.

Hence this note is not a note in type theory. It is instead the statement of an exercise for ambitious homotopy type theorists:

Exercise: *Formalize this!*

It should be within reach. For instance

- theorem 3.6 (existence of frame bundles) should be formalizable and provable with just one modality as taken from [UFP 13, section 7.7],
- theorem 6.8 (Stokes theorem) needs an adjoint pair of modalities and needs at least something close to the concept of stable objects;
- theorem 7.1 (Noether's theorem) involves all modalities of differential cohesion as well as the tower of truncation modalities [UFP 13, section 6.9].

but otherwise, the proofs of these theorems are simple (besides being elementary) and hence should lend themselves to formalization. For reference, below we spell out the proof of theorem 3.6 "informally", i.e. in the pseudocode formerly known as mathematics.

(Notice that the issue of interpreting HoTT in ∞-toposes is, while of general interest, not strictly relevant for the purpose of the above exercise: the comparison to the semantics of ∞-toposes is just what helps to suggest that these theorems should indeed be provable in HoTT+UV+DifferentialCohesion.)

If anything on the semantics side is unclear, ask me. If you need help on the syntactic side, ask Mike Shulman.

2 A standard model

To get a quick feeling for the axioms of differential cohesion, it is helpful to first consider a standard model, and then abstract from it later.

Definition 2.1. Let

$$\mathrm{FormalSmoothCartSp} \hookrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\mathrm{op}}$$

be the full subcategory of formal duals of commutative $\mathbb{R}$-algebras, on those that are tensor products of the form

$$C^\infty(\mathbb{R}^n \times \mathbb{D}) := C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} (\mathbb{R} \oplus V)\,,$$

for $n \in \mathbb{N}$, where $C^\infty(\mathbb{R}^n)$ is the smooth functions in n variables, and V is finite dimensional and nilpotent.

Regard this as a site by equipping it with the coverage whose covers are of the form

$$\{U_i \times \mathbb{R} \stackrel{(\phi_i,\mathrm{id})}{\longrightarrow} X \times \mathbb{D}\}$$

for

$$\{U_i \stackrel{\phi_i}{\longrightarrow} X\}$$

an open cover of smooth manifolds.

Definition 2.2. Write

$$\mathbf{H}_0 := \mathrm{Sh}(\mathrm{FormalSmoothMfd})$$

for the sheaf topos over this site.

The topos $\mathbf{H}_0$ is a model for the Kock-Lawvere axioms of synthetic differential geometry. It was considered as such in [Dubuc 79] and has since been known as the *Cahiers topos*. This topos is moreover a *well-adapted model*, meaning that the ordinary category of smooth manifolds is a full subcategory

$$\mathrm{SmoothMfd} \hookrightarrow \mathbf{H}_0$$

such that the embedding preserves transversal pullbacks.

The idea of synthetic differential geometry was to place oneself inside a topos, assume that the Kock-Lawvere-axioms hold, and then (re-)do all of differential geometry using the internal logic of that topos [Lawvere 97, Kock 99, Kock 09]. In this respect it is noteworthy that the Cahiers topos also has the following abstract property:

Theorem 2.3 ([S 13])**.** *The topos $\mathbf{H}_0$ of def. 2.1 carries a system of idempotent (co-)monads of the form*

$$\begin{array}{ccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & \\
\vee & \bot & \vee & & & & \\
\Re & \dashv & \Im & \dashv & \& & & \\
 & & \vee & \bot & \vee & & \\
 & & \pi_0 & \dashv & \flat & \dashv & \sharp \\
 & & & & \vee & \bot & \vee \\
 & & & & \emptyset & \dashv & *
\end{array}$$

such that both adjoint triples are induced from adjoint quadruples to the base topos, where

1. *$\flat := \Delta \circ \Gamma$ is the comonad induced from the unique global section geometric morphism by which the Cahiers topos sits over the base topos;*

2. π_0 *sends manifolds to their set of connected components;*

3. *the quasi-topos of* $\sharp$*-separated objects* $(X \overset{\eta^\sharp_X}{\hookrightarrow} \sharp X)$ *is that of concrete sheaves, which here are the diffeological spaces;*

4. $\Re$ *sends formal smooth manifolds to their reduction,* $\Re(\mathbb{D}) \simeq *$*;*

5. $\Im := \mathrm{loc}_{\mathbb{D}}$ *is localization at the maps of the form* $\mathbb{D} \to *$*;*

and where

1. $L \dashv R$ *means that* L *is left adjoint to* R*;*

2. $\bigcirc_1 < \bigcirc_2$ *means that* $(\bigcirc_1 X \simeq X) \Rightarrow (\bigcirc_2 X \simeq X)$.

Definition 2.4 ([S 13])**.** Say that a topos which carries a system of idempotent (co-)modalities as in theorem 2.3 is *differentially cohesive*.

3 Manifolds and PDEs

A good bit of modern differential geometry is neatly formalized using axiomatic differential cohesion of def. 2.4. We now highlight two concepts: manifolds and partial differential equations.

Definition 3.1. Say that a morphism $f : X \longrightarrow Y$ in $\mathbf{H}_0$ is *formally étale* if the naturality square of its $\Im$-unit is a pullback:

$$\begin{tikzcd} X \arrow[r, "\eta^\Im_X"] \arrow[d, "\mathrm{et}"'] \arrow[dr, phantom, "(\mathrm{pb})"] & \Im X \arrow[d, "\Im f"] \\ Y \arrow[r, "\eta^\Im_X"'] & \Im Y \end{tikzcd} .$$

Say that f is *formally smooth* if the comparison map $X \to Y \underset{\Im Y}{\times} \Im X$ is an epimorphism, and that f is *fomally unramified* if the comparison map is a monomorphism.

Proposition 3.2. *For* $X, Y \in \mathrm{SmoothMfd} \hookrightarrow \mathbf{H}_0$ *, then* $f : X \to Y$ *is formally étale/smooth/unramified in the abstract sense of def. 3.1 precisely if it is a local diffeomorphism/submersion/immersion of smooth manifolds, respectively, in the traditional sense.*

Definition 3.3. For $V \in \mathbf{H}_0$ a group object, say that an object $X \in \mathbf{H}_0$ is a V*-manifold* if it admits a V*-cover*, namely an object $U \in \mathbf{H}_0$ and a diagram of the form

$$\begin{tikzcd} & U \arrow[dl, "\mathrm{et}"'] \arrow[dr, two heads, "\mathrm{et}"] & \\ V & & X \end{tikzcd}$$

(both morphisms are formally étale, def. 3.1, the right one is in addition an epimorphism).

Definition 3.4. For $\Sigma \in \mathbf{H}_0$ any object, consider the base change adjoint triple along its $\Im$-unit:

$$\begin{tikzcd} (\mathbf{H}_0)_{/\Sigma} \arrow[r, shift left=2ex, "\sum_{\eta^\Im_\Sigma}"] \arrow[r, shift right=2ex, "\prod_{\eta^\Im_\Sigma}"'] & (\mathbf{H}_0)_{/\Im\Sigma} \arrow[l, "(\eta^\Im_\Sigma)^*"'] \end{tikzcd} .$$

Write

$$(T^\infty_\Sigma \dashv J^\infty_\Sigma) \;:=\; (\lozenge_{\eta^\Im_\Sigma} \dashv \Box_{\eta^\Im_\Sigma}) := \left((\eta^\Im_\Sigma)^* \circ \sum_{\eta^\Im_\Sigma} \dashv (\eta^\Im_\Sigma) \circ \prod_{\eta^\Im_\Sigma} \right)$$

for the induced adjoint pair of (co-)monads on $(\mathbf{H}_0)_{/\Sigma}$.

Proposition 3.5. *1. For $\Sigma \in \mathrm{SmoothMfd} \hookrightarrow \mathbf{H_0}$, then $T^\infty_\Sigma \Sigma \to \Sigma$ is the formal disk bundle over Σ, whose fiber over any point $\sigma \in \Sigma$ is the formal neighbourhood $\mathbb{D}_\sigma$ of that point.*

2. For $E \to \Sigma$ a bundle of manifolds, then $J^\infty_\Sigma E \to \Sigma$ is the jet bundle of E.

$$\left\{ \begin{tikzcd} & J^\infty_\Sigma E \arrow[d] \\ * \arrow[ur, dashed] \arrow[r, "\sigma"'] & \Sigma \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} & E \arrow[d] \\ \mathbb{D}_\sigma \arrow[ur, dashed] \arrow[r, hook] & \Sigma \end{tikzcd} \right\}$$

Theorem 3.6 ([S 15])**.** *For $V \in \mathbf{H}_0$ a group object in a differentially cohesive topos, then*

1. *the formal disk bundle $T^\infty_V V \simeq V \times \mathbb{D}^V$ is trivialized by left translation ("every group is canonically framed");*
2. *for X a V-manifold, def. 3.3, then $T^\infty_X X$ is a $\mathbb{D}^V$-fiber bundle which is associated to a $\mathrm{GL}(V) := \mathbf{Aut}(\mathbb{D}^V)$-principal bundle $\mathrm{Fr}(X)$ (the jet frame bundle).*

Proof. For the first statement, first observe that since $\Im$, being left and right adjoint, preserves group structure, so that the defining homotopy pullback of the infinitesimal disk bundle of V

$$\begin{tikzcd} T^\infty_V V \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & V \arrow[d] \\ V \arrow[r] & \Im V \end{tikzcd}$$

is a homotopy fiber product over a group object. This implies that a Mayer-Vietoris argument applies by which there is equivalently a pasting composite of homotopy pullbacks of the form

$$\begin{tikzcd} T^\infty_V V \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \mathbb{D}^V_e \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & * \arrow[d] \\ V \times V \arrow[r, "(-)\cdot(-)^{-1}"'] & V \arrow[r] & \Im V \end{tikzcd} .$$

Inspection of the left square (`http://ncatlab.org/nlab/show/Mayer-Vietoris+sequence#HTTArgumentForPullback`) reveals the triviality of the formal disk bundle

$$T^\infty_V V \simeq V \times \mathbb{D}^V_e \,.$$

Now to see how this impacts on the formal disk bundle of V-manifolds, first observe that for $\iota : U \xrightarrow{\mathrm{et}} X$ a formally étale morphism, then pullback along it preserves formal disk bundles:

$$\iota^* T^\infty_X X \simeq T^\infty_U U \,.$$

This follows by the naturality of the $\Im$-unit and using the pasting law to see that there is an equivalence of homotopy pullback diagrams of the following form:

$$\begin{tikzcd} \iota^* T^\infty_X X \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & T^\infty_X X \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & X \arrow[d] \\ U \arrow[r] & X \arrow[r] & \Im X \end{tikzcd} \simeq \begin{tikzcd} T^\infty_U U \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & U \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & X \arrow[d] \\ U \arrow[r] & \Im U \arrow[r] & \Im X \end{tikzcd} .$$

Together with the previous statement this implies that the formal disk bundle of a V-manifold trivializes when pulled back to any V-cover $V \xleftarrow{\text{et}} U \xrightarrow{\text{et}} X$.

To conclude it is now sufficient to observe that every locally trivial F-fiber bundle is associated to an $\mathbf{Aut}(F)$-principal bundle, this is prop. 5.7 below. ☐

Remark 3.7. The axiomatization of frame bundles in theorem 3.6 opens the door to axiomatization of all kinds of flavors of geometry: complex, symplectic, Riemannian, conformal, etc. All these are naturally encoded via reduction of the structure group of the frame bundle.

Proposition 3.8. *The Eilenberg-Moore category of J^∞_Σ-coalgebras is equivalent to the slice topos $(\mathbf{H}_0)_{/\Im\Sigma}$. The subcategory of those coalgebra objects whose underlying object is a smooth bundle of manifolds is equivalent to the category* PDE_Σ*, whose objects are partial differential equations on sections of bundles over* Σ*, and whose morphisms are differential operators preserving solutions of PDEs:*

$$
\begin{tikzcd}
\mathrm{SmoothMfd}_{/\Sigma} \arrow[d, hook] \arrow[r, shift right, "F"'] & \mathrm{PDE}_\Sigma \arrow[l, shift right, "U"'] \arrow[d, hook] \\
(\mathbf{H}_0)_{/\Sigma} \arrow[d, equal] \arrow[r, shift left, "U"] \arrow[r, shift right, "F"'] & \mathrm{EM}(J^\infty_\Sigma) \arrow[d, "\simeq"] & =: & \mathrm{PDE}_\Sigma(\mathbf{H}) \\
(\mathbf{H}_0)_{/\Sigma} \arrow[r, shift right, "\Pi_{\eta^{\Im}_\Sigma}"'] & (\mathbf{H}_0)_{/\Im\Sigma} \arrow[l, shift right, "(\eta^{\Im}_\Sigma)^*"']
\end{tikzcd}
$$

Axiomatizing ordinary differential equations and in particular equations of motion in physics via the internal language of toposes has been one of the motivations [Lawvere 97] for the Kock-Lawvere-axioms of synthetic differential geometry. We saw so far that with the alternative axioms of differential cohesion one gets the state-of-the-art formulation of partial differential equations.

We next see that we also get the state-of-the-art formulation of equations of motion in physical field theory, after passing to ∞-toposes.

4 More on the standard model

Definition 4.1. Write

$$
\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{FormalSmoothMfd})
$$

for the ∞-topos, according to [L-Topos], over the site from def. 2.1.

(The topological localization of $\mathrm{Sh}_\infty(\mathrm{FormalSmoothMfd})$ is already hypercomplete, hence we don't have to make a distinction.)

The analog of theorem 2.3 remains true:

Theorem 4.2 ([S 13])**.** *The ∞-topos $\mathbf{H}$ carries a system of idempotent ∞-(co-)-monads of the form*

$$
\begin{array}{ccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & \\
\vee & \llcorner & \vee & & & & \\
\Re & \dashv & \Im & \dashv & \& & & \\
 & & \vee & \llcorner & \vee & & \\
 & & \pi_\infty & \dashv & \flat & \dashv & \sharp \\
 & & & & \vee & \llcorner & \vee \\
 & & & & \emptyset & \dashv & *
\end{array}
$$

whose nature is verbatim as in theorem 2.3, only that π_0 is promoted to

$$\pi_\infty \simeq \mathrm{loc}_{\mathbb{R}^1}$$

which acts as follows:

1. *For $X \in \mathbf{H}$ a smooth manifold, then $\pi_\infty X$ is its fundamental ∞-groupoid whose objects are the points of X, morphisms are the smooth paths in X, 2-morphisms are the smooth paths-of-paths, etc.*

2. *more generally for $X_\bullet \in \mathbf{H}$ a simplicial smooth manifold, then $\pi_\infty X_\bullet$ is the homotopy type of the fat geometric realization of $X_\bullet$.*

Definition 4.3 ([S 13])**.** Call an ∞-topos with the properties as in theorem 4.2 *differentially cohesive.*

Example 4.4. Write

$$S^1 \in \mathrm{SmoothMfd} \hookrightarrow \mathbf{H}$$

for the circle in its incarnation as a smooth manifold, and write

$$\mathbf{B}\mathbb{Z} \in \infty\mathrm{Grpd} \hookrightarrow \mathbf{H}$$

for the circle in its incarnation as a homotopy type. Then

$$\pi_\infty S^1 \simeq \mathbf{B}\mathbb{Z}\,.$$

5 Groups, actions, fiber bundles

Theorem 5.1 ([L-Alg])**.** *For $\mathbf{H}$ an ∞-topos, then forming loop space objects constitutes an equivalence between the ∞-categories of pointed connected objects and of group objects:*

$$\mathrm{Grp}(\mathbf{H}) \underset{\mathbf{B}}{\overset{\Omega}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \mathbf{H}^{*/}_{\geq 1}$$

Example 5.2. For $\mathbf{H}$ from def. 4.1, let $G \in \mathrm{Grp}(\mathrm{SmoothMfd}) \hookrightarrow \mathrm{Grp}(\mathbf{H})$ a Lie group, write $BG, K(G,1) \in \infty\mathrm{Grpd} \hookrightarrow \mathbf{H}$ for the homotopy types that go by these symbols in algebraic topology, the classifying space BG of G, and the Eilenberg-MacLane space $K(G,1)$ with fundamental group the discrete group underlying G. Then

$$\begin{aligned} \pi_\infty(\mathbf{B}G) &\simeq BG \\ \flat(\mathbf{B}G) &\simeq K(G,1) \end{aligned}\,.$$

Theorem 5.3 ([Nikolaus-S-Stevenson 14],[S 15])**.** *Let $\mathbf{H}$ be an ∞-sheaf ∞-topos. For $G \in \mathrm{Grp}(\mathbf{H})$, the G-principal bundles over any X are equivalent to maps $X \to \mathbf{B}G$, via the construction that sends such a map to its homotopy fiber.*

More generally, the ∞-category of G-actions is equivalent to the slice over $\mathbf{B}G$

$$G\mathrm{Act}(\mathbf{H}) \simeq \mathbf{H}_{/\mathbf{B}G}\,.$$

Given a G-action ρ on an object V here, then

$$\prod_{\mathbf{B}G} \rho \simeq H_{\mathrm{Grp}}(G,\rho)$$

is the group cohomology of G with coefficients in ρ and

$$\sum_{\mathbf{B}G} \rho \simeq V/G$$

is the homotopy quotient. More generally for $f : \mathbf{B}G \longrightarrow \mathbf{B}H$ a group homomorphism, then $\sum_f$ and $\prod_f$ are the constructions and induced and coinduced representations, respectively.

Remark 5.4. Theorems 5.1 and 5.3 imply in particular that in ∞-toposes the concepts of ∞-groups, ∞-actions, ∞-fiber bundles, ∞-group cohomology and ∞-representation theory all have an *elementary* axiomatization.

Example 5.5. For any $F \in \mathbf{H}$, $\mathbf{Aut}(F)$ is the looping of the 1-image factorization of the morphism into the type universe that classifies F:

$$\begin{tikzcd} * \arrow[r, two heads] \arrow[rr, bend right, "\vdash F"'] & \mathbf{BAut}(F) \arrow[r, hook] & \mathrm{Type} \end{tikzcd} .$$

The canonical action of $\mathbf{Aut}(F)$ on F is exhibited, via theorem 5.3, by the pullback of the universal type fibration along the monomorphism on the right:

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & F/\mathbf{Aut}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \widetilde{\mathrm{Type}} \arrow[d] \\ * \arrow[r, two heads] \arrow[rr, bend right, "\vdash F"'] & \mathbf{BAut}(F) \arrow[r, hook] & \mathrm{Type} \end{tikzcd} .$$

Definition 5.6. For $F, X \in \mathbf{H}$ two objects, then an *F-fiber bundle over X* is a morphism $E \to X$ such that there is a 1-epimorphism $U \to X$ pulled back along which it trivializes:

$$\begin{tikzcd} U \times F \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & E \arrow[d] \\ U \arrow[r, two heads] & X \end{tikzcd} .$$

Proposition 5.7. *Every F-fiber bundle, def. 5.6, is associated to a unique, up to equivalence $\mathbf{Aut}(F)$-principal bundle.*

Proof. E and its local trivialization are classified by maps to the type universe as shown by the solid arrows the following diagram

$$\begin{tikzcd} & * \arrow[d] \\ U \arrow[ur] \arrow[r] \arrow[d] & \mathbf{BAut}(F) \arrow[d, hook] \\ X \arrow[ur, dashed] \arrow[r, "\vdash E"'] & \mathrm{Type} \end{tikzcd} ,$$

where on the right we show the 1-image factorization of example 5.5. By the (1-epi, 1-mono) factorization, this implies a unique, up to equivalence, dashed lift as show. ☐

6 Differential cohomology, differential forms

Theorem 6.1. *For $G \in \mathrm{Grp}(\mathrm{SmoothMfd}) \hookrightarrow \mathrm{Grp}(\mathbf{H})$, and for $P \to X$ a G-principal bundle, then a flat connection on P is equivalently a lift ∇ of the classifying map through the $\flat$-counit:*

$$\begin{tikzcd} & \flat \mathbf{B}G \arrow[d, "\epsilon^{\flat}_{\mathbf{B}G}"] \\ X \arrow[ur, dashed, "\nabla"] \arrow[r] & \mathbf{B}G \end{tikzcd}$$

Hence $\flat \mathbf{B}G$ classifies *flat* G-principal connections, whence the notation.

Recall the following classical fact:

Theorem 6.2 (Brown representability theorem)**.** *A generalized cohomology theory is equivalently a stable object in the base ∞-topos $\infty\mathrm{Grpd}$.*

We consider now a differential refinement of this state,ent.

Definition 6.3. For $\bigcirc$ an $\infty-$(co-)monad, write $\overline{\bigcirc}$ for the homotopy (co-)fiber of its (co-)unit.

Theorem 6.4 ([Bunke-Nikolaus-Völkl 13])**.** *For A a stable object in a cohesive ∞-topos, then the canonical hexagon*

$$\begin{tikzcd}
 & \overline{\pi_\infty} A \arrow[rr, "d"] \arrow[dr] & & \bar{\flat} A \arrow[dr] & \\
\overline{\pi_\infty}\flat A \arrow[ur] \arrow[dr] & & A \arrow[ur] \arrow[dr] & & \pi_\infty \bar{\flat} A \\
 & \flat A \arrow[rr] \arrow[ur] & & \pi_\infty A \arrow[ur] &
\end{tikzcd}$$

is exact, in that, in addition to the diagonals being homotopy fiber sequences,

1. *both squares are homotopy Cartesian;*
2. *the outer sequences are long homotopy fiber sequences.*

Remark 6.5. The proof of theorem 6.4 is elementary, both in the technical as well as in the ordinary sense: use that homotopy pullbacks of stable objects may be detected on homotopy fibers and use the idempotency of π_∞ and $\flat$.

Remark 6.6 (exegesis of theorem 6.4)**.** By theorem 6.2, $\pi_\infty A$ is a generalized cohomology theory. Hence A is a geometric enrichment of that. By theorem 6.1, the object $\flat A$ classifies flat A-connections, and hence by exactness and stability, the map $A \to \bar{\flat} A$ is the universal obstruction to flatness, hence $\bar{\flat} A$ classifies the curvature differential forms for A-connections. Similarly, by exactness $\overline{\pi_\infty} A \to A$ is the inclusion of those A-connections whose underlying bundle is trivial, hence these are the globally defined connection forms. Accordingly, the top morphism sends connection forms to their curvature, and hence plays the role of the de Rham differential.

$$\begin{tikzcd}
 & \boxed{\text{differential forms}} \arrow[rr, "\text{de Rham differential}"] \arrow[dr, "\text{regard as}"] & & \boxed{\text{curvature forms}} \arrow[dr, "\text{de Rham theorem}"] & \\
\boxed{\text{closed differential forms}} \arrow[ur, "\text{regard as}"] \arrow[dr, "\text{regard as}"'] & & \boxed{\text{connections}} \arrow[ur, "\text{curvature}"] \arrow[dr, "\text{topol. class}"] & & \boxed{\text{rationalized class}} \\
 & \boxed{\text{flat connections}} \arrow[rr] \arrow[ur, "\text{regard as}"] & & \boxed{\text{integral class}} \arrow[ur, "\text{Chern character}"] &
\end{tikzcd}$$

It was long known that every generalized differential cohomology theory sits in such a hexagon (see e.g. [Bunke 12]). In [Simons-Sullivan 07] it was observed that exactness of the hexagon already characterizes ordinary differential cohomology and it was suggested that this may be true more generally. Theorem 6.4 resolves this:

Just as a generalized cohomology theory is equivalently a spectrum object in an ∞-topos, so a generalized differential cohomology theory is a spectrum object in a cohesive ∞-topos.

Example 6.7. In $\mathbf{H}$ from def. 4.1 there is a hexagon as in theorem 6.4

$$\begin{tikzcd}
 & \mathbf{\Omega}^{\bullet \leq p+1} \arrow[rr, "d_{\mathrm{dR}}"] \arrow[dr] & & \mathbf{\Omega}^{p+2}_{\mathrm{cl}} \arrow[dr] & \\
\flat \mathbf{B}^{p+1}\mathbb{R} \arrow[ur] \arrow[dr] & & \mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}} \arrow[ur] \arrow[dr] & & \flat \mathbf{B}^{p+2}\mathbb{R} \\
 & \flat \mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z}) \arrow[rr] \arrow[ur] & & \mathbf{B}^{p+2}\mathbb{Z} \arrow[ur] &
\end{tikzcd}$$

whose

- top sequence exhibits the filtration on the $\flat \mathbf{B}^k\mathbb{R}$s induced by the Poincaré lemma;
- the bottom sequence is part of the long homotopy fiber sequence induced by the exponential sequence

$$\mathbb{Z} \xhookrightarrow{2\pi\hbar} \mathbb{R} \twoheadrightarrow \mathbb{R}/_{\hbar}\mathbb{Z}$$

A morphism

$$\nabla : X \longrightarrow \mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}}$$

to the object at the heart of this hexagon is equivalently known as

- a p-gerbe with connection and band $(\mathbb{R}/_{\hbar}\mathbb{Z})$;
- a $\mathbf{B}^p(\mathbb{R}/_{\hbar})$-principal connection;
- a Deligne cocycle of degree $(p+2)$.

Theorem 6.8 (∞-Stokes' theorem [Bunke-Nikolaus-Völkl 13])**.** *If $\mathbf{H}$ is a cohesive ∞-topos such that there is an object $(0,1) : * \coprod * \to \mathbb{R}$ with $\pi_\infty \simeq \mathrm{loc}_{\mathbb{R}}$, then there is canonically an integration map*

$$\int_0^1 : [\mathbb{R}, \bar{\flat} A] \longrightarrow \overline{\pi_\infty} A$$

such that

$$\int_0^1 \circ\, d \;\; \simeq \;\; 1^* - 0^* \,.$$

Moreover, applied to example 6.7 this reproduces the traditional fiber integration of differential forms.

Remark 6.9. The proof of the first part of theorem 6.8 is elementary, in the technical sense. It consists entirely of forming suitable pasting composites of canonically given squares filled by homotopies.

7 Field theory and equations of motion

The following is discussed in [Khavkine-S].

Write

$$(-)_\Sigma : \mathbf{H} \xrightarrow{\Sigma^*} \mathbf{H}_{/\Sigma} \xrightarrow{F} \mathrm{PDE}_\Sigma(\mathbf{H})$$

for the context-extension of type in $\mathbf{H}$ to types in $\mathbf{H}_{/\Im\Sigma}$.

For $E \in \mathrm{SmoothMfd}_{/\Sigma} \xrightarrow{F} \mathrm{PDE}_\Sigma(\mathbf{H})$, the joint coimage of

$$[E, (\mathbf{\Omega}^{\bullet \leq p+1})_\Sigma] \xrightarrow{\phi^*} [\Sigma, (\mathbf{\Omega}^{\bullet \leq p+1})_\Sigma]$$

over all $\phi : \Sigma \to E$ is representable by an object $\mathbf{\Omega}_H^{\bullet \leq p+1}$. This induces a projection map

$$H : (\mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}})_\Sigma \longrightarrow \mathbf{B}_H^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}}$$

via the induced morphism in the $(-)_\Sigma$-image of the hexagon in example 6.7:

$$\begin{tikzcd}
 & (\mathbf{\Omega}^{\bullet \leq p+1})_\Sigma \arrow[rr, "d_{\mathrm{dR}}"] \arrow[dr] \arrow[dl] \arrow[ddr] & & (\mathbf{\Omega}^{p+2}_{\mathrm{cl}})_\Sigma \arrow[dr] \arrow[ddr] & & \\
 & & \mathbf{\Omega}_H^{\bullet \leq p+1} \arrow[rr, "\delta_V"] \arrow[ddr] & & \mathbf{\Omega}_S^{p+1,1} \arrow[ddr] & \\
(\flat \mathbf{B}^{p+1}\mathbb{R})_\Sigma \arrow[dr, equal] \arrow[ddr] & & (\mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}})_\Sigma \arrow[uur, "\mathrm{curv}"] \arrow[dr, "H"] \arrow[ddr] & & (\flat \mathbf{B}^{p+2}\mathbb{R})_\Sigma \arrow[dr, equal] & \\
 & (\flat \mathbf{B}^{p+1}\mathbb{R})_\Sigma \arrow[uur] \arrow[ddr] & & \mathbf{B}_H^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}} \arrow[uur, "\mathrm{curv}"] \arrow[ddr] & & (\flat \mathbf{B}^{p+1}\mathbb{R})_\Sigma \\
 & (\flat \mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z}))_\Sigma \arrow[uur] \arrow[rr, "\beta_\Sigma"'] \arrow[dr, equal] & & (\mathbf{B}^{p+2}\mathbb{Z})_\Sigma \arrow[uur] \arrow[dr, equal] & & \\
 & & (\flat \mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z}))_\Sigma \arrow[uur] \arrow[rr, "\beta_\Sigma"'] & & (\mathbf{B}^{p+2}\mathbb{Z})_\Sigma \arrow[uur] &
\end{tikzcd}$$

For $E \in \mathbf{H}_{/\Sigma} \xrightarrow{F} \mathrm{PDE}_\Sigma(\mathbf{H})$ this formalizes the principle of extremal action in field theory/variational calculus:

$$\begin{tikzcd}
 & & \underset{\mathcal{E}}{\text{Euler−Lagrange equation}} \arrow[d, hook, "\ker(\mathrm{EL})"] & \mathbf{B}_H^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}} \arrow[dr, "\delta_V", "\text{variational differential}"'] & \\
\text{spacetime/worldvolume} & \Sigma \arrow[ur, dashed, "\text{solution}"] \arrow[r, "\phi", "\text{field configuration}"'] & E \arrow[ur, "\mathbf{L}", "\text{local Lagrangian}"'] \arrow[rr, "\mathrm{EL}"'] & & \mathbf{\Omega}_S^{p+1,1}
\end{tikzcd}$$

Theorem 7.1 (Noether's theorem [Fiorenza-Rogers-S 13a, Sati-S 15, Khavkine-S]). *There is homotopy fiber sequence of group stacks like so:*

$$\left\{ \begin{tikzcd} E \arrow[d, bend right, "\mathbf{L}"'] \arrow[d, bend left, "\mathbf{L}"] \arrow[d, phantom, "\Downarrow \text{topological current}"] \\ \mathbf{B}_H^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}} \end{tikzcd} \right\} \longrightarrow \left\{ \begin{tikzcd} E \arrow[rr, "\text{symmetry}", "\simeq"'] \arrow[dr, "\mathbf{L}"'] & \arrow[d, phantom, "\Downarrow \text{Noether current}"] & E \arrow[dl, "\mathbf{L}"] \\ & \mathbf{B}_H^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}} & \end{tikzcd} \right\} \longrightarrow \left\{ E \underset{\text{symmetry}}{\overset{\text{variational}}{\xrightarrow{\simeq}}} E \right\}$$

$$\Big\downarrow$$

$$\mathbf{B} \left\{ \begin{tikzcd} E \arrow[d, bend right, "\mathbf{L}"'] \arrow[d, bend left, "\mathbf{L}"] \arrow[d, phantom, "\Downarrow \text{topological current}"] \\ \mathbf{B}_H^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}} \end{tikzcd} \right\}$$

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