Collapsing Conformal Field Theories and quantum spaces with non-negative Ricci curvature

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Plan:

Collapse of CFT's and Mirror Symmetry. Conjectural precompactness of the moduli spaces of CFT's. Moduli space of CFTs': can it be defined as a functor?

Segal's axioms of CFT.

QFT on graphs as quantum Riemannian manifolds. Collapsing CFT's give rise to QFT on graphs.

Spectral triples. Spectral triples give rise to QFT on graphs. Sometimes the converse is true.

Bakry's CD(R,N) conditions. Invariant measure is involved. The case of spectral triples (not all of them,only those obtained by GNS construction to the algebra).

Generalization of spectral triples, so that in the commutative case they become Riemannian manifolds with the density. Quantum metric-measure space.

What is a "good" intrinsic definition of the dimension of metric-measure space?

Gromov-Hausdorff convergence and spectral convergence in the commutative case.

Quantum GH and spectral topologies. Collapsing CFTs via spectral topologies. Lattice model approximation of a CFT as an analog of the "finite set" approximation of a metric space.

Wasserstein metrics and Sturm-Lott-Villani precompactness results. Generalization to quantum length metric-measure spaces.

Quantum metric spaces a'la Rieffel-Hanfeng Li: generalization to NC metric-measure case.

d-dimensional QFT's (a'la Segal). QFTs on the metric space-time. Continuity w.r.t. measured Gromov-Hausdorff convergence of the space-time.

Relationship of GH to Zamolodchikov metric.

Applications: counting of string vacua. Non-archimedean Riemannian manifolds. Singular string background.

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1 Introduction

1.1

In this paper we discuss analogies between the moduli spaces of Conformal Field Theories (CFTs) and the space of isomorphism classes of compact metric-measure spaces equipped with the measured Gromov-Hausdorff topology. We hope that study some of these analogies will help to develop a mathematical model for various low energy limits in Quantum Field Theory (QFT). Since the underlying algebraic structures are often non-commutative, we need the notion of moduli space in the framework of non-commutative Riemannian geometry.

The concept of the moduli space as a space representing the functor of "isomorphism classes of families" is not very useful for Riemannian manifolds (same can be said about many other "functorial" concepts). On the other hand, the "moduli space" (i.e. the set of isometry classes) of compact Riemannian manifolds carries some natural Hausdorff topologies (e.g. Gromov-Hausdorff), so one can compactify it in a larger set consisting of isometry classes of compact metric spaces. It is well-known that some differential-geometric structures of Riemannian manifolds admit generalizations to the points of compactified moduli space. As an example we mention the notion of sectional curvature extended to Alexandrov spaces or, more recent, the property to have non-negative Ricci curvature extended to compact metric-measure spaces (see [LV], [St]).

One of the goals of present paper is to use this philosophy in the non-commutative case. We are motivated by Segal's axioms of the unitary Conformal Field Theory (see [Seg]) as well as by the approach to non-commutative Riemannian geometry developed by A. Connes in [Co1]. Many structures of Connes's approach (e.g. spectral triples, see [Co1], [CoMar]) are closely related to the structures considered in present paper.

1.2

One of our goals is to define a quantum Riemannian manifold (or, more generally, "Riemannian space") by a set of axioms similar to Segal's axioms of a unitary CFT. Set of "isometry classes" (i.e. the moduli space) should be treated similarly to the set of isometry classes of compact Riemannian manifolds with restrictions on the diameter and Ricci curvature. The central charge plays a role of the dimension of the space, and the "spectral gap" for the Virasoro operator $L_0 + \overline{L}_0$ plays a role of the (square root of the inverse to) diameter. In the spirit of Gromov, Cheeger, Colding and Fukaya we would like to compactify the "moduli space" of such objects by their "Gromov-Hausdorff degenerations". Notice that we treat the moduli space of CFTs (or,more generally, QFTs) as an object of metric geometry rather than the one of the algebraic geometry. We used this philosophy in [KoSo1], where the concept of collapsing family of unitary CFTs was introduced with the aim to explain Mirror Symmetry. Main idea of [KoSo1] is that the "moduli space of CFTs with the bounded central charge and the spectral gap (i.e. the minimal positive eigenvalue of the Virasoro operator $L_0 + \overline{L}_0$) bounded from below by a non-negative constant", is precompact in some natural topology. The topology itself was not specified in [KoSo1]. Since the notion of collapse depends on the spectral properties of the Virasoro operator $L_0 + \overline{L}_0$, one should use the topology which gives continuity of the spectral data, e.g. measured Gromov-Hausdorff topology. It was argued in loc. cit. that if the spectral gap approaches to zero, then the collapsing family of unitary CFTs gives rise to a topological space, which contains a dense open Riemannian manifold with non-negative Ricci curvature. The restriction on the Ricci curvature follows from the unitarity of the theory. This can be encoded into an informal statement:

Collapsing unitary two-dimensional CFTs= Riemannian manifolds (possibly singular) with non-negative Ricci curvature.

From this point of view the geometry of (possibly singular) Riemannian manifolds with non-negative Ricci curvature should be thought of as a limit of the (quantum) geometry of certain Quantum Field Theories (namely, two-dimensional unitary CFTs). Riemannian manifolds themselves appear as target spaces for non-linear sigma models. Hence non-linear sigma-models provide a partial compactification of the moduli space of two-dimensional unitary CFTs.

1.3

From the point of view of Segal's axioms, the geometry which underlies a collapsing family of CFTs is the geometry of 2-dimensional compact oriented Riemannian manifolds degenerating into metrized graphs. The algebra of the CFT is encoded into the operator product expansion (OPE). Its collapse gives rise to a commutative algebra A. The (rescaled) operator $L_0 + \overline{L}_0$ collapses into the second order differential operator on A. The rest of the conformal group does not survive. Hence the family of CFTs collapses into a QFT. The latter, according to [KoSo1], should be thought of as a Gromov-Hausdorff limit of the former.

The above considerations suggest the following working definitions in the framework of quantum geometry. A quantum compact Riemannian space (more precisely, 2-space, if we want to stress that surfaces are 2-dimensional) is defined by the following data: a separable complex Hilbert space H, an operator $S(\Sigma) : H^{\otimes n} \to H^{\otimes m}$ called the amplitude of Σ which is given for each compact Riemannian 2-dimensional oriented manifold Σ with n marked "input" circles and m marked "output" circles. The kernel of $S(\Sigma)$ is the tensor $K_\Sigma \in H^{\otimes m} \otimes (H^{\otimes n})^* = Hom(H^{\otimes n})^* \otimes H^{\otimes m}, \mathbf{C})$ called the correlator. In the same vein a quantum compact Riemannian 1-space is given by the similar data assigned to metrized graphs with marked input and output vertices. Natural gluing axioms should be satisfied in both cases, as well as continuity of the data with respect to some natural topology. In particular we allow degenerations of Riemannian 2-spaces into Riemannian 1-spaces. At the level of geometry this means that metrized graphs are limits of compact oriented Riemannian 2-dimensional manifolds with boundary. At the level of algebra all "algebraic data" (e.g. spaces of states, operators) associated with graphs are limits of the corresponding data for the surfaces. The notion of limit should have "geometric" and "algebraic" counterparts. Geometrically it can be the measured Gromov-Hausdorff limit, while for Hilbert spaces it can be any notion of limit which respects continuity of the spectral data of the associated positive self-adjoint operators. We are going to review several possibilities in the main body of the paper.

Also, we can (and should) relax the condition that H is a Hilbert space, since the limit of Hilbert spaces can be a locally convex vector space of more general type (e.g. a nuclear space). In order to obtain a "commutative" Riemannian geometry we require that the amplitude operator (or correlator) associated with a surface or a graph is invariant with respect to the (separate) permutations of inputs and outputs.

If we accept the above working definitions of quantum Riemannian spaces, then many natural questions arise, in particular:

a) What is the dimension of a quantum Riemannian space?

b) What is the diameter?

c) Which quantum Riemannian spaces should be called manifolds?

d) What are various curvature tensors, e.g. Ricci curvature?

When we speak about quantum Riemannian 1-spaces, Riemannian 2-spaces or, more generally, Riemannian d-spaces, the number d corresponds to the dimension of the *world-sheet*, not the *space-time*. In particular, one can associate a Riemannian 1-space with a compact Riemannian manifold of any dimension. Having in mind possible relationship with CFT we should allow the number d to be non-integer ("central charge").

By analogy with the commutative Riemannian geometry one can ask about the structure of the "moduli space" of quantum Riemannian manifolds rigidified by some geometric data. In particular, we can ask about the "space of isometry classes" of quantum compact Riemannian d-spaces equipped with a non-commutative version of the Gromov-Hausdorff topology. Then one can ask about analogs of classical precompactness and compactness theorems, e.g. those which claim precompactness of the "moduli space" of Riemannian manifolds having fixed dimension, diameter bounded from above and the Ricci curvature bounded from below (see e.g. [Gro1]). Introducing a non-commutative analog of measure, one can ask about non-commutative analogs of theorems due to

Cheeger, Colding, Fukaya and others for the class of *metric-measure spaces*, i.e. metric spaces equipped with a Borel probabilty a measure. To compare with the case of unitary CFTs we remark that the space of states of a unitary CFT plays a role of L_2-space (of the loop space of a manifold), with the vacuum expectation value playing a role of the measure. The unitarity condition ("reflection positivity" in the language of Euclidean Quantum Field Theory) turns out to be an analog of the non-negativeness of the Ricci curvature. Normalized boundary states should correspond to probability measures.

The above discussion motivates the idea to treat unitary CFTs and their degenerations as "quantum metric-measure spaces with bounded diameter and non-negative Ricci curvature". One expects that this moduli space is precompact and complete in the natural topology.[1] One hopes for a similar picture for QFTs which live on the space-time of a more general type than just a manifold. Leaving aside possible physical applications, one can ask (motivated by metric geometry) "what is a QFT with the space-time, which is a compact metric-measure space ?" Although the answer is not known, see Appendix for some ideas in this direction.

Defined in this way, quantum Riemannian spaces enjoy some functorial properties, well-known at the level of CFTs (e.g. one can take a tensor product of quantum Riemannian spaces).

1.4

The idea that QFTs should be studied by methods of non-commutative geometry was suggested by Alain Connes (see [Co1]). The idea to use Connes's approach for the description of CFTs and their degenerations goes back to [FG]. It was further developped in [RW] in an attempt to interpet the earlier approach of [KoSo1] from the point of view of Connes's spectral triples. The measure was not included in the list of data neither in [FG] nor in [RW], since in the framework of spectral triples the measure can be recovered from the rest of the data. Ricci curvature was not defined in the framework of spectral triples. In particular, it was not clear how to define a spectral triple with non-negative Ricci curvature. Present paper can be thought of as a step in this direction.

1.5

Recall (see e.g. [Co1]) that a spectral triple is given by a unital C^*-algebra of bounded operators in a Hilbert space and a 1-parameter semigroup continuously acting on the space. It is assumed that the semigroup has an infinitesimal generator D which is a positive unbounded self-adjoint operator with the compact resovent, and the commutator $[D, f]$ with any algebra element f is bounded. Similar structures appear in the theory of random walks and Markov semigroups

[1] More precisely, non-commutativity arises from CFTs with boundary conditions. Our point of view differs from [RW] where a unitary CFT already gives rise to a non-commutative space. We prefer to axiomatize the structure arising from the full space of states rather than from the subspace of invariants with respect to a W-algebra.

on singular spaces (see e.g. [Ba], [BaEm], [LV], [Led], [St]). In that case one also has a probability measure which is invariant with respect to the semigroup. This similarity makes plausible the idea that the "abstract calculus" of Markov semigroups developed by Bakry and Emery (see [BaEm], [Ba]) can be used for the description of the topological space obtained from a collapsing family of unitary CFTs. This idea was proposed by Kontsevich in a series of talks in 2003. In those talks Kontsevich introduced the notion of "singular Calabi-Yau manifold" defined in terms of what he called Graph Field Theory (and what we call commutative Riemannian 1-geometry below). One hopes that the "moduli space of singular Calabi-Yau manifolds" with bounded dimension and fixed diameter, being equipped with a (version of) Gromov-Hausdorff (or measured Gromov-Haudorff) topology, is compact. More generally one can expect a similar result for "quantum Riemannian 1-spaces" which have bounded dimension, diameter bounded from above and Ricci curvature bounded from below. [2] To our knowledge there is no precompactness theorem for the space of "abstract Bakry-Emery data".[3] In a similar vein we mention precompactness theorems for a class of metric-measure spaces which generalizes the class of Riemannian manifolds with non-negative Ricci curvature (see [LV], [St]). The authors introduced in [LV], [St] the notion of N-Ricci curvature, $N \in [1, \infty]$. The notion of N-Ricci curvature is defined in terms of geodesics in the space of probability measures equipped with the Wasserstein L_2-metric (see Section 6). We hope that there is a non-commutative generalization of the notion of N-Ricci curvature as well as of the Wasserstein metric, so that the class of quantum metric-measure spaces with non-negative N-Ricci curvature and bounded diameter is compact with respect to the non-commutative generalization of the measured Gromov-Hausdorff topology or non-commutative generalization of the metric introduced in [St].

1.6

Let us make few additional remarks about the relationship of our approach with the one of Connes (see [Co1]). In the notion of spectral triple he axiomatized the triple (A, H, Δ) where A is the algebra of smooth functions on a compact closed Riemannian manifold M (considered as a complex algebra with an anti-linear involution), $H = L_2(M, vol_M)$ is the Hilbert space of functions, which are square-integrable with respect to the volume form associated with the Riemannian metric, and Δ is the Laplace operator associated with the metric. [4] From a

[2] In the recent preprint [En] the precompactness of the moduli space of commutative measured Riemannian 1-spaces and the usual bounds on the diameter and Ricci curvature was proved. Methods of [En] are based on explicit estimates of the heat kernel as well as classical results by Cheeger and Colding [ChC3]. We do not see how to extend them to the non-commutative case.

[3] Bakry-Emery data naturally lead to the metric on the space of states of a C^*-algebra coinciding with the metric introduced by Connes's and generalized later by Rieffel (see [Rie]) in his notion of "quantum metric space". Although the precompactness theorem was formulated and proved by Rieffel for compact quantum metric spaces, it is not clear how to extend his approach to the case of quantum Riemannian 1-spaces discussed in this paper.

[4] In fact Connes considered the case of spin manifolds, so he used the Dirac operator D instead of $\Delta = D^2$, and H was the space of square-integrable sections of the spinor bundle.

slightly different perspective, the data are: involutive algebra A, a positive linear functional $\tau(f) = \int_M f\, vol_M$ which defines the completion H of A with respect to the scalar product $\tau(fg^*)$, the $*$-representation $A \to End(H)$, and the 1-parameter semigroup $exp(-t\Delta), t \geq 0$ acting on H by means of trace-class operators. The generator of the semigroup is a non-negative self-adjoint unbounded operator Δ with discrete spectrum, and the algebra A being naturally embedded to H belongs to the domain of Δ. Thus A encodes the topology of M, while τ encodes the measure, and Δ encodes the Riemannian structure. Let $B_1(f,g)$ be a bilinear form $A \otimes A \to A$ given by $2B_1(f,g) = \Delta(fg) - f\Delta(g) - \Delta(f)g$. The formula $d(\phi,\psi) = sup_{B_1(f,f)\leq 1}|\varphi(f) - \psi(f)|$ defines the distance function on the space of states of the C^*-completion of A in terms of the spectral triple data (the C^*-completion can be spelled out intrinsically in terms of the operator norm derived from the $*$-representation $A \to End(H)$). Every point $x \in M$ gives rise to a state (delta-function δ_x). One sees that the above formula recovers the Riemannian distance function on M without use of the language of points, so it can be generalized to the case of non-commutative algebra A. There are many non-trivial examples of spectral triples which do not correspond to commutative Riemannian manifolds (see e.g. [Co1], [CoMar]). Let us observe that in the case of Riemannian manifolds the 1-parameter semigroup $exp(-t\Delta), t \geq 0$ assigns a trace-class operator $exp(-l\Delta)$ to every segment $[0,l]$, which we can view as a very simple metrized graph with one input and one output. Moreover, the multiplication $m_A : A \otimes A \to A$ gives rise to the family of operators S_{l_1,l_2,l_3} $H^{\otimes 2} \to H$ such that

$$x_1 \otimes x_2 \mapsto exp(-l_3\Delta)(m_A(exp(-l_1\Delta)(x_1) \otimes exp(-l_2\Delta)(x_2))),$$

for any $l_1, l_2, l_3 > 0$.

In a bit more symmetric way, one has a family of trilinear forms $H^{\otimes 3} \to \mathbf{C}$ such that $(x_1, x_2, x_3) \mapsto \tau(m_A(m_A \otimes id)(exp(-l_1\Delta)(x_1) \otimes exp(-l_2\Delta)(x_2) \otimes exp(-l_3\Delta)(x_3)))$. Hence, starting with a commutative spectral triple, we can produce trace-class operators associated with two types of metrized graphs:

a) to a segment $I_l := [0,l]$ we associate an operator $exp(-l\Delta) := S_{I_l} : H \to H$, assuming that for $l = 0$ we have the identity operator;

b) to the Y-shape graph Γ_{l_1,l_2,l_3} with different positive lengths of the three edges we associate an operator $S_{l_1,l_2,l_3} := S(\Gamma_{l_1,l_2,l_3}) : H^{\otimes 2} \to H$. Notice that $m_A(f_1 \otimes f_2) = lim_{l_1+l_2+l_3 \to 0} S(\Gamma_{l_1,l_2,l_3})(f_1 \otimes f_2), f_i \in A, i = 1,2$, hence the multiplication on A can be recovered from operators associated to metrized graphs as long as we assume continuity of the operators with respect to the length of an edge of the tree.

From the point of view of Quantum Field Theory it is natural to consider more general graphs. This leads to the notion of quantum Riemannian 1-geometry (see below). It turns out that this language is suitable for spelling out various differential-geometric properties of Riemannian manifolds (non-commutative and singular in general), in particular, the property to have the non-negative Ricci curvature.

1.7

One can speculate about possible applications of the ideas of this paper. The Gromov-Hausdorff (or measured Gromov-Hausdorff) topology is coarser than topologies typically used in physics. The question is: are these "Gromov-Hausdorff type" topologies "physical enough" to derive interesting properties of the moduli spaces? For example, it is interesting whether measured Gromov-Hausdorff topology can be useful in the study of the so-called "string landscape" and the problem of finiteness of the volume of the corresponding moduli spaces of QFTs (see e.g. [Dou 1], [Dou 2], [Dou L], [Va], [OVa]). For example, the problem of statistics of the string vacua leads to the counting of the number of critical points of a certain function (prepotential) on the moduli space of certain CFTs (see e.g. [Dou 1], [Z]). Finiteness of the volume of the moduli space of CFTs (the volume gives the first term of the asymptotic expansion of the number of critical points) is crucial. According to the previous discussion, the "Gromov-Hausdorff type" moduli space of unitary CFTs with the bounded central charge and bounded from below spectral gap is expected to be compact. Probably the requirement that the spectrum is discrete leads to the finiteness of the volume of the moduli space with respect to the measure derived from Zamolodchikov metric (cf. [Va]). This would give a bound for the number of string vacua.

1.8

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2 Reminder on degenerating Conformal Field Theories

This Section contains the material borrowed from [KoSo1].

Unitary Conformal Field Theory is well-defined mathematically thanks to Segal's axiomatic approach (see [Seg]). We are going to recall Segal's axioms later. In the case of the complex line **C** the data defining a unitary CFT can be summarized such as follows:

1) A real number $c \geq 0$ called central charge.

2) A bi-graded pre-Hilbert *space of states* $H = \oplus_{p,q\in \mathbf{R}_{\geq 0}} H^{p,q}, p-q \in \mathbf{Z}$ such that $dim(\oplus_{p+q\leq E} H^{p,q})$ is finite for every $E \in \mathbf{R}_{\geq 0}$. Equivalently, there is an action of the Lie group $\mathbf{C}^*$ on H, so that $z \in \mathbf{C}^*$ acts on $H^{p,q}$ as $z^p \bar{z}^q := (z\bar{z})^p \bar{z}^{q-p}$.

3) An action of the product of Virasoro and anti-Virasoro Lie algebras $Vir \times \overline{Vir}$ (with the same central charge c) on H, so that the space $H^{p,q}$ is an eigenspace for the generator L_0 (resp. $\overline{L}_0$) with the eigenvalue p (resp. q).

4) The space H carries some additional structures derived from the operator product expansion (OPE). The OPE is described by a linear map $H \otimes H \to H \widehat{\otimes} \mathbf{C}\{z,\bar{z}\}$. Here $\mathbf{C}\{z,\bar{z}\}$ is the topological ring of formal power series $f = \sum_{p,q} c_{p,q} z^p \bar{z}^q$ where $c_{p,q} \in \mathbf{C}$, $p,q \to +\infty$, $p,q \in \mathbf{R}$, $p-q \in \mathbf{Z}$. The OPE satisfies some axioms which do not recall here (see e.g. [Gaw]). One of the axioms is a sort of associativity of the OPE.

Let $\phi \in H^{p,q}$. Then the number $p+q$ is called the *conformal dimension* of ϕ (or the *energy*), and $p-q$ is called the *spin* of ϕ. Notice that, since the spin of ϕ is an integer number, the condition $p+q<1$ implies $p=q$.

The central charge c can be described intrinsically by the formula $dim(\oplus_{p+q\leq E} H^{p,q}) = exp(\sqrt{4/3\pi^2 cE(1+o(1))}$ as $E \to +\infty$. It is expected that all possible central charges form a countable well-ordered subset of $\mathbf{Q}_{\geq 0} \subset \mathbf{R}_{\geq 0}$. If $H^{0,0}$ is a one-dimensional vector space, the corresponding CFT is called irreducible. A general CFT is a sum of irreducible ones. The *trivial* CFT has $H = H^{0,0} = \mathbf{C}$ and it is the unique irreducible unitary CFT with $c = 0$.

Remark 2.0.1 *There is a version of the above data and axioms for Superconformal Field Theory (SCFT). In that case each $H^{p,q}$ is a hermitian super vector space. There is an action of the super extension of the product of Virasoro and anti-Virasoro algebra on H. In the discussion of the moduli spaces below we will speak about CFTs, not SCFTs. Segal's axiomatics for SCFT is not available from the published literature.*

2.1 Moduli space of Conformal Field Theories

For a given CFT one can consider its group of symmetries (i.e. automorphisms of the space $H = \oplus_{p,q} H^{p,q}$ preserving all the structures).

Conjecture 2.1.1 *The group of symmetries is a compact Lie group of dimension less or equal than $\dim H^{1,0}$.*

Let us fix $c_0 \geq 0$ and $E_{min} > 0$, and consider the moduli space $\mathcal{M}_{c\leq c_0}^{E_{min}}$ of all irreducible CFTs with the central charge $c \leq c_0$ and

$$min\{p+q>0 | H^{p,q} \neq 0\} \geq E_{min}.$$

Conjecture 2.1.2 *$\mathcal{M}_{c\leq c_0}^{E_{min}}$ is a compact real analytic stack of finite local dimension. The dimension of the base of the minimal versal deformation of a given CFT is less or equal than $\dim H^{1,1}$.*

We define $\mathcal{M}_{c\leq c_0} = \cup_{E_{min}>0}\mathcal{M}_{c\leq c_0}^{E_{min}}$. We would like to compactify this stack by adding boundary components corresponding to certain degenerations of the theories as $E_{min} \to 0$. The compactified space is expected to be a compact stack $\overline{\mathcal{M}}_{c\leq c_0}$. In what follows we will loosely use the word "moduli space" instead of the word "stack".

Remark 2.1.3 *There are basically two classes of rigorously defined CFTs: the rational theories (RCFT) and the lattice CFTs. They are defined algebraically (e.g. in terms of braided monoidal categories or vertex algebras). Physicists often consider so-called sigma models defined in terms of maps of two-dimensional Riemannian surfaces (world-sheets) to a Riemannian manifold (world-volume, or target space). Such a theory depends on a choice of the Lagrangian, which is a functional on the space of such maps. Descriptions of sigma models as path integrals does not have precise mathematical meaning. Segal's axioms arose from an attempt to treat the path integral categorically. As we will explain below, there is an alternative way to speak about sigma models in terms of degenerations of CFTs. Roughly speaking, sigma models "live" near the boudary of the compactified moduli space* $\overline{\mathcal{M}}_{c\leq c_0}$.

2.2 Physical picture of a simple collapse

In order to compactify $\mathcal{M}_{c\leq c_0}$ we consider degenerations of CFTs as $E_{min} \to 0$. A degeneration is given by a one-parameter (discrete or continuous) family $H_\varepsilon, \varepsilon \to 0$ of bi-graded spaces as above, where $(p,q) = (p(\varepsilon), q(\varepsilon))$, equipped with OPEs, and such that $E_{min} \to 0$. The subspace of fields with conformal dimensions vanishing as $\varepsilon \to 0$ gives rise to a commutative algebra $H^{small} = \oplus_{p(\varepsilon)\ll 1} H_\varepsilon^{p(\varepsilon),p(\varepsilon)}$ (the algebra structure is given by the leading terms in OPEs). The spectrum X of H^{small} is expected to be a compact space ("manifold with singularities") such that $dim\, X \leq c_0$. It follows from the conformal invariance and the OPE, that the grading of H^{small} (rescaled as $\varepsilon \to 0$) is given by the eigenvalues of a second order differential operator defined on the smooth part of X. The operator has positive eigenvalues and is determined up to multiplication by a scalar. This implies that the smooth part of X carries a metric g_X, which is also defined up to multiplication by a scalar. Other terms in OPEs give rise to additional differential-geometric structures on X.

Thus, as a first approximation to the real picture, we assume the following description of a "simple collapse" of a family of CFTs. The degeneration of the family is described by the point of the boundary of $\overline{\mathcal{M}}_{c\leq c_0}$ which is a triple $(X, \mathbf{R}_+^* \cdot g_X, \phi_X)$, where the metric g_X is defined up to a positive scalar factor, and $\phi_X : X \to \mathcal{M}_{c\leq c_0 - dim\, X}$ is a map. One can have some extra conditions on the data. For example, the metric g_X can satisfy the Einstein equation.

Although the scalar factor for the metric is arbitrary, one should imagine that the curvature of g_X is "small", and the injectivity radius of g_X is "large". The map ϕ_X appears naturally from the point of view of the simple collapse of CFTs described above. Indeed, in the limit $\varepsilon \to 0$, the space H_ε becomes an H^{small}-module. It can be thought of as a space of sections of an infinite-dimensional vector bundle $W \to X$. One can argue that fibers of W generically are spaces of states of CFTs with central charges less or equal than $c_0 - dim\, X$. This is encoded in the map ϕ_X. In the case when CFTs from $\phi_X(X)$ have non-trivial symmetry groups, one expects a kind of a gauge theory on X as well.

Purely bosonic sigma-models correspond to the case when $c_0 = c(\varepsilon) = dim\, X$ and the residual theories (CFTs in the image of ϕ_X) are all trivial. The target space X in this case should carry a Ricci flat metric. In the $N = 2$ supersymmetric case the target space X is a Calabi-Yau manifold, and the residual bundle of CFTs is a bundle of free fermion theories.

Remark 2.2.1 *It was conjectured in [KoSo1] that all compact Ricci flat manifolds (with the metric defined up to a constant scalar factor) appear as target spaces of degenerating CFTs. Thus, the construction of the compactification of the moduli space of CFTs should include a compactification of the moduli space of Einstein manifolds. As we already mentioned in the Introduction, there is a deep relationship between the compactification of the moduli space of CFTs and Gromov's compactification. Moreover, as we will discuss below, all target spaces appearing as limits of CFTs have in some sense non-negative Ricci curvature. More precisely, the limit of the rescaled Virasoro operator* $L_0 + \overline{L}_0$ *satisfies Bakry curvature-dimension inequality* $CD(0, \infty)$. *In the case of compact Riemannian manifolds the latter is equivalent to non-negativeness of Ricci curvature.*

2.3 Multiple collapse and the structure of the boundary

In terms of the Virasoro operator $L_0 + \overline{L}_0$ the degeneration of CFT (" collapse") is described by a subset (cluster) S_1 in the set of eigenvalues of $L_0 + \overline{L}_0$ which approach to zero "with the same speed" provided $E_{min} \to 0$. The next level of the collapse is described by another subset S_2 of eigenvalues of $L_0 + \overline{L}_0$. Elements of S_2 approach to zero "modulo the first collapse" (i.e. at the same speed, but "much slower" than elements of S_1). One can continue to build a tower of degenerations. It leads to an hierarchy of boundary strata. Namely, if there are further degenerations of CFTs parametrized by X, one gets a fiber bundle over the space of triples $(X, \mathbf{R}_+^* \cdot g_X, \phi_X)$ with the fiber which is the space of triples of similar sort. Finally, we obtain the following qualitative geometric picture of the boundary $\partial \overline{\mathcal{M}}_{c \le c_0}$.

A boundary point is given by the following data:

1) A finite tower of maps of compact topological spaces $p_i : \overline{X}_i \to \overline{X}_{i-1}, 0 \le i \le k$, $\overline{X}_0 = \{pt\}$.

2) A sequence of smooth manifolds $(X_i, g_{X_i}), 0 \le i \le k$, such that X_i is a dense subspace of $\overline{X}_i$, and $dim\, X_i > dim\, X_{i-1}$, and p_i defines a fiber bundle $p_i : X_i \to X_{i-1}$.

3) Riemannian metrics on the fibers of the restrictions of p_i to X_i, such that the diameter of each fiber is finite. In particular the diameter of X_1 is finite, because it is the only fiber of the map $p_1 : X_1 \to \{pt\}$.

4) A map $X_k \to \mathcal{M}_{c \leq c_0 - dim\, X_k}$.

The data above are considered up to the natural action of the group $(\mathbf{R}_+^*)^k$ (it rescales the metrics on fibers).

There are some additional data, like non-linear connections on the bundles $p_i : X_i \to X_{i-1}$. The set of data should satisfy some conditions, like differential equations on the metrics. It is an open problem to describe these conditions in general case. In the case of $N = 2$ SCFTs corresponding to sigma models with Calabi-Yau target spaces these geometric conditions were formulated as a conjecture in [KoSo1].

2.4 Example: Toroidal models

Non-supersymmetric toroidal model is described by the so-called Narain lattice, endowed with some additional data. More precisely, let us fix the central charge $c = n$ which is a positive integer number. What physicists call the Narain lattice $\Gamma^{n,n}$ is a unique unimodular lattice of rank $2n$ and the signature (n, n). It can be described as $\mathbf{Z}^{2n}$ equipped with the quadratic form $Q(x_1, ..., x_n, y_1, ..., y_n) = \sum_i x_i y_i$. The moduli space of toroidal CFTs is given by

$$\mathcal{M}_{c=n}^{tor} = O(n, n, \mathbf{Z}) \backslash O(n, n, \mathbf{R}) / O(n, \mathbf{R}) \times O(n, \mathbf{R}).$$

Equivalently, it is a quotient of the open part of the Grassmannian $\{V_+ \subset \mathbf{R}^{n,n} |\, dim\, V_+ = n, Q_{|V} > 0\}$ by the action of $O(n, n, \mathbf{Z}) = Aut(\Gamma^{n,n}, Q)$. Let V_- be the orthogonal complement to V_+. Then every vector of $\Gamma^{n,n}$ can be uniquely written as $\gamma = \gamma_+ + \gamma_-$, where $\gamma_\pm \in V_\pm$. For the corresponding CFT one has

$$\sum_{p,q} dim(\, H^{p,q}) z^p \bar{z}^q = \big|\prod_{k\geq 1}(1 - z^k)\big|^{-2n} \sum_{\gamma \in \Gamma^{n,n}} z^{Q(\gamma_+)} \bar{z}^{\,-Q(\gamma_-)}$$

Let us describe the (partial) compactification of the moduli space $\mathcal{M}_{c=n}^{tor}$ by collapsing toroidal CFTs. Suppose that we have a one-parameter family of toroidal theories such that $E_{min}(\varepsilon)$ approaches zero. Then for corresponding vectors in H_ε one gets $p(\varepsilon) = q(\varepsilon) \to 0$. It implies that $Q(\gamma(\varepsilon)) = 0, Q(\gamma_+(\varepsilon)) \ll 1$. It is easy to see that the vectors $\gamma(\varepsilon)$ form a semigroup with respect to addition. Thus one obtains a (part of) lattice of the rank less or equal than n. In the case of "maximal" simple collapse the rank will be equal to n. One can see that the corresponding points of the boundary give rise to the following data: $(X, \mathbf{R}_+^* \cdot g_X, \phi_X^{triv}; B)$, where (X, g_X) is a flat n-dimensional torus, $B \in H^2(X, i\mathbf{R}/\mathbf{Z})$ and ϕ_X^{triv} is the constant map form X to the trivial theory point in the moduli space of CFTs. These data in turn give rise to a toroidal CFT, which can be realized as a sigma model with the target space (X, g_X) and given B-field B. The residual bundle of CFTs on X is trivial.

Let us consider a 1-parameter family of CFTs given by $(X, \lambda g_X, \phi_X^{triv}; B = 0)$, where $\lambda \in (0, +\infty)$. There are two degenerations of this family, which define two points of the boundary $\partial\overline{\mathcal{M}}_{c=n}^{tor}$. As $\lambda \to +\infty$, we get a toroidal CFT defined by $(X, \mathbf{R}_+^* \cdot g_X, \phi_X^{triv}; B = 0)$. As $\lambda \to 0$ we get $(X^\vee, \mathbf{R}_+^* \cdot g_{X^\vee}, \phi_X^{triv}; B = 0)$, where $(X^\vee, g_{X^\vee})$ is the dual flat torus.

There might be further degenerations of the lattice. Thus one obtains a stratification of the compactified moduli space of lattices (and hence CFTs). Points of the compactification are described by flags of vector spaces $0 = V_0 \subset V_1 \subset V_2 \subset ... \subset V_k \subset \mathbf{R}^n$. In addition one has a lattice $\Gamma_{i+1} \subset V_{i+1}/V_i$, considered up to a scalar factor. These data give rise to a tower of torus bundles $X_k \to X_{k-1} \to ... \to X_1 \to \{pt\}$ over tori with fibers $(V_{i+1}/V_i)/\Gamma_{i+1}$. If $V_k \simeq \mathbf{R}^{n-l}, l \geq 1$, then one has also a map from the total space X_k of the last torus bundle to the point $[H_k]$ in the moduli space of toroidal theories of smaller central charge: $\phi_n : X_k \to \mathcal{M}_{c=l}^{tor}$, $\phi_k(X_k) = [H_k]$.

2.5 Example: WZW model for $SU(2)$

In this case we have a discrete family with $c = \frac{3k}{k+2}$, where $k \geq 1$ is an integer number called *level*. In the limit $k \to +\infty$ one gets $X = SU(2) = S^3$ equipped with the standard metric. The corresponding bundle is the trivial bundle of trivial CFTs (with $c = 0$ and $H = H^{0,0} = \mathbf{C}$). Analogous picture holds for an arbitrary compact simply connected simple group G.

2.6 Example: minimal models

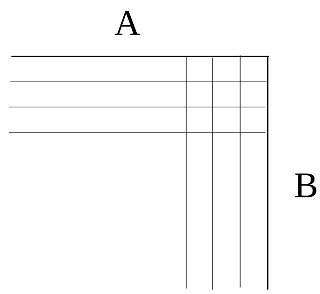
This example has been worked out in [RW], Section 4. One has a sequence of unitary CFTs H_m with the central charge $c_m = 1 - \frac{6}{m(m+1)}, m \to +\infty$. In this case $c_m \to 1$, and the limiting space is the interval $[0, \pi]$. The metric is given by $g(x) = \frac{4}{\pi^2} sin^4 x, x \in [0, \pi]$. The corresponding volume form is $vol_g = \sqrt{g(x)}dx = \frac{2}{\pi} sin^2 x dx$. In all above examples the volume form on the limiting space is the one associated with the Riemannian metric.

2.7 A-model and B-model of $N = 2$ SCFT as boundary strata

In the case of $N = 2$ Superconformal Field Theory one can modify the above considerations in the natural way. As a result one arrives to the following picture of the simple collapse.

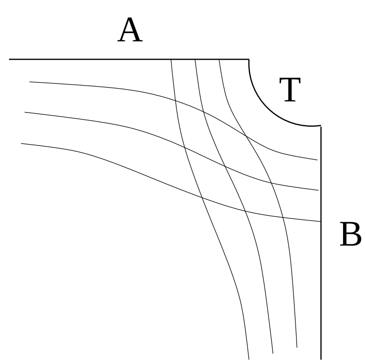
The boundary of the compactified moduli space $\overline{\mathcal{M}}^{N=2}$ of $N = 2$ SCFTs with a given central charge contains an open stratum given by sigma models with Calabi-Yau targets. Each stratum is parametrized by the classes of equivalence of quadruples $(X, J_X, \mathbf{R}_+^* \cdot g_X, B)$ where X is a compact real manifold, J_X a complex structure, g_X is a Calabi-Yau metric, and $B \in H^2(X, i\mathbf{R}/\mathbf{Z})$ is a B-field. The residual bundle of CFTs is a bundle of free fermion theories.

As a consequence of supersymmetry, the moduli space $\mathcal{M}^{N=2}$ of superconformal field theories is a complex manifold which is locally isomorphic to the product of two complex manifolds. [5] It is believed that this decomposition (up to certain corrections) is global. Also, there are two types of sigma models with Calabi-Yau targets: A-models and B-models. Hence, the traditional picture of the compactified moduli space looks as follows:



Here the boundary consists of two open strata (A-stratum and B-stratum) and a mysterious meeting point. This point corresponds, in general, to a submanifold of codimension one in the closure of A-stratum and of B-stratum.

As we explained in [KoSo1] this picture should be modified. Namely, there is another open stratum of $\partial\overline{\mathcal{M}}^{N=2}$ (called T-stratum in [KoSo1]). It consists of toroidal models (i.e. CFTs associated with Narain lattices), parametrized by a manifold Y with a Riemannian metric defined up to a scalar factor. This T-stratum meets both A and B strata along the codimension one stratum corresponding to the double collapse. Therefore the "true" picture is obtained from the traditional one by the real blow-up at the corner:



2.8 Mirror symmetry and the collapse

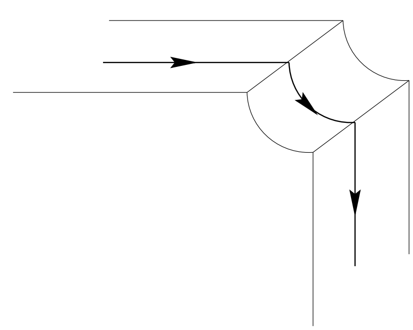
Mirror symmetry is related to the existence of two different strata of the boundary $\partial\overline{\mathcal{M}}^{N=2}$ which we called A-stratum and B-stratum. As a corollary, same quantities admit different geometric descriptions near different strata. In the traditional picture, one can introduce natural coordinates in a small neighbor-

[5] Strictly speaking, one should exclude models with chiral fields of conformal dimension $(2,0)$, e.g. sigma models on hyperkähler manifolds.

hood of a boundary point corresponding to $(X, J_X, \mathbf{R}_+^* \cdot g_X, B)$. Skipping X from the notation, one can say that the coordinates are (J, g, B) (complex structure, Calabi-Yau metric and the B-field). Geometrically, the pairs (g, B) belong to the preimage of the Kähler cone under the natural map $Re : H^2(X, \mathbf{C}) \to H^2(X, \mathbf{R})$ (more precisely, one should consider B as an element of $H^2(X, i\mathbf{R}/\mathbf{Z})$). It is usually said, that one considers an open domain in the complexified Kähler cone with the property that it contains together with the class of metric $[g]$ also the ray $t[g], t \gg 1$. The mirror symmetry gives rise to an identification of neighborhoods of $(X, J_X, \mathbf{R}_+^* \cdot g_X, B_X)$ and $(X^\vee, J_{X^\vee}, \mathbf{R}_+^* \cdot g_{X^\vee}, B_{X^\vee})$ such that J_X is interchanged with $[g_{X^\vee}] + iB_{X^\vee})$ and vice versa.

We can describe this picture in a different way. Using the identification of complex and Kähler moduli, one can choose $([g_X], B_X, [g_{X^\vee}], B_{X^\vee})$ as local coordinates near the meeting point of A-stratum and B-stratum. There is an action of the additive semigroup $\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}$ in this neighborhood. It is given explicitly by the formula $([g_X], B_X, [g_{X^\vee}], B_{X^\vee}) \mapsto (e^{t_1}[g_X], B_X, e^{t_2}[g_{X^\vee}], B_{X^\vee})$ where $(t_1, t_2) \in \mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}$. As $t_1 \to +\infty$, a point of the moduli space approaches the B-stratum, where the metric is defined up to a positive scalar only. The action of the second semigroup $\mathbf{R}_{\geq 0}$ extends by continuity to the non-trivial action on the B-stratum. Similarly, in the limit $t_2 \to +\infty$ the flow retracts the point to the A-stratum.

This picture should be modified, if one makes a real blow-up at the corner, as we discussed before. Again, the action of the semigroup $\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}$ extends continuously to the boundary. Contractions to the A-stratum and B-stratum carry non-trivial actions of the corresponding semigroups isomorphic to $\mathbf{R}_{\geq 0}$. Now, let us choose a point in, say, A-stratum. Then the semigroup flow takes it along the boundary to the new stratum, corresponding to the double collapse. The semigroup $\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}$ acts trivially on this stratum. A point of the double collapse is also a limiting point of a 1-dimensional orbit of $\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}$ acting on the T-stratum. Explicitly, the element (t_1, t_2) changes the size of the tori defined by the Narain lattices, rescaling them with the coefficient $e^{t_1 - t_2}$. This flow carries the point of T-stratum to another point of the double collapse, which can be moved then inside of the B-stratum. The whole path, which is the intersection of $\partial \overline{\mathcal{M}}^{N=2}$ and the $\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}$-orbit, connects an A-model with the corresponding B-model through the stratum of toroidal models. We can depict it as follows:



The conclusion of the above discussion is the following: in order to explain the mirror symmetry phenomenon it is not necessary to build full SCFTs. It is sufficient to work with simple toroidal models on the boundary of the compactified moduli space $\overline{\mathcal{M}}^{N=2}$.

3 Segal's axioms and collapse

3.1 Segal's axioms

Let us recall Segal's axioms of a 2-dimensional unitary CFT (see e.g. [FFRS], [Gaw], [Ru], [Seg]). Similarly to [FFRS] and [Ru] we use surfaces with a metric rather than surfaces with complex structure. To simplify the description we use oriented surfaces only. The data of a unitary CFT are described such as follows:

1) Real number c called *central charge*.

2) 2-dimensional *world-sheet*. The latter is defined by:

2a) an oriented 2-dimensional manifold Σ, possibly disconnected, with finitely many boundary components $(C_i)_{i\in I}$ labeled by elements of the disjoint union of finite sets: $I = I_- \sqcup I_+$. Components labeled by the elements of I_- (resp. I_+) are called *incoming* (resp. *outcoming*). The orientation of Σ induces orientations of all C_i as boundary components. This orientation is called canonical;

2b) a Riemannian metric g_Σ on Σ;

2c) for some $\varepsilon > 0$ (which depends on $i \in I$) a real-analytic conformal embedding to (Σ, g_Σ) of either the flat annulus $A_\varepsilon^- = \{z \in \mathbf{C} | 1-\varepsilon < |z| \leq 1\}$ (for $i \in I_-$) or the flat annulus $A_\varepsilon^+ = \{z \in \mathbf{C} | 1 \leq |z| < 1+\varepsilon\}$ (for $i \in I_+$) (the annuli are equipped with the standard flat metric $\frac{dz\,d\overline{z}}{z\overline{z}}$) such that: $f_i^{\pm}(|z| = 1) = C_i$ and f_i^- (resp f_i^+) is orientation preserving (resp. reversing) with respect to the canonical orientation. The map f_i^+ (resp.f_i^-) is called a *parametrization* of C_i.

3) Complex separable Hilbert space H [6] equipped with an antilinear involution $x \mapsto \sigma(x)$.

4) A trace-class operator (*amplitude*) $S(\Sigma, (f_i)_{i\in I}, g_\Sigma) : \otimes_{i\in I_-} H \to \otimes_{i\in I_+} H$ (by convention the empty tensor product is equal to $\mathbf{C}$). We will sometimes denote by H_i the tensor factor corresponding to $i \in I$.

These data are required to satisfy the following axioms:

CFT 1) If $(\Sigma, (f_i), g_\Sigma) = \sqcup_\alpha (\Sigma_\alpha, (f_i^\alpha), g_{\Sigma_\alpha})$ then $S(\Sigma, (f_i), g_\Sigma) = \otimes_\alpha S(\Sigma_\alpha, (f_i^\alpha), g_{\Sigma_\alpha})$.

CFT 2) Let $i_0 \in I$, and $\overline{f}_{i_0}(z, \overline{z}) = f_{i_0}(\frac{1}{z}, \frac{1}{\overline{z}})$ (i.e. $\overline{f}_{i_0}$ induces the opposite orientation on C_{i_0}). For the new world-sheet we require that if i_0 was in I_- (resp. I_+) then it is now in I_+ (resp. I_-). The condition says: $\langle S(\Sigma, \overline{f}_{i_0}, (f_i)_{i\in I\setminus i_0}, g_\Sigma) x_{i_0} \otimes x, y\rangle = \langle S(\Sigma, (f_i)_{i\in I}, g_\Sigma)(\sigma(x_{i_0}) \otimes x, y\rangle$, where $\langle , \rangle$ denotes the hermitian scalar product on H.

CFT 3) Let $i_0 \in I_-$, $j_0 \in I_+$, and f_{i_0} and f_{j_0} be the corresponding parametrizations. Let us change (by a real-analytic change of coordinates in the annulus A_ε^+) the parametrization f_{j_0} in such a way that the pull-backs of the metric g_Σ under f_{i_0} and f_{j_0} coincide at the corresponding points of the circle $|z| = 1 \subset A_\varepsilon^\pm$.

[6] This condition can be relaxed so one can assume that H is a locally compact vector space, e.g. a nuclear vector space.

Let us keep the same notation f_{j_0} for the new parametrization. Identifying points $f_{i_0}(z,\overline{z})$ and $f_{j_0}(z,\overline{z})$, for $|z|=1$ we obtain a new 2-dimensional oriented surface Σ_{i_0,j_0} such that $C_{i_0}, i_0 \in I_-$ and $C_{j_0}, j_0 \in I_+$ are isometrically identfied. By construction, the surface Σ_{i_0,j_0} carries a smooth metric induced by g_Σ. In order to formulate the next condition we need to introduce the notion of partial trace. Let $A : V \otimes H_{i_0} \to W \otimes H_{j_0}$ be a linear map. Using an antilinear involution σ let us identify $H_{j_0} = H$ with the dual space $H^*_{i_0} = H^*$ such as follows: $y \mapsto l_y = \langle \bullet, \sigma(y)\rangle$. Then we define $Tr_{i_0,j_0}(A) : V \to W$ by the formula $Tr_{i_0,j_0}(A)(v \otimes x_{i_0}) = \sum_m l_{e_m}(A(v \otimes x_{i_0}))$, where the sum is taken over elements e_m of an orthonormal basis of H_{j_0}. The condition says: $S(\Sigma_{i_0,j_0}, (f_i)_{i\in I\setminus\{i_0,j_0\}}, g_{\Sigma_{i_0,j_0}}) = Tr_{i_0,j_0} S(\Sigma, (f_i)_{i\in I}, g_\Sigma)$.

CFT 4) Let $\overline{\Sigma}$ denotes the same 2-dimensional manifold Σ, but with the orientation changed to the opposite one, and with I_- being interchanged with I_+, but all parametrizations remained the same. Using the involution σ we identify each dual space H^*_i with $H_i, i \in I$ as above. The condition says that the operator corresponding to $\overline{\Sigma}$ coincides with the dual operator $S(\Sigma, (f_i), g_\Sigma)^* : \otimes_{i\in I_+} H \to \otimes_{i\in I_-} H$.

CFT 5) If the metric g_Σ is replaced by $e^h g_\Sigma$ where h is a real-valued smooth function, then

$$S(\Sigma, (f_i), e^h g_\Sigma) = exp(\frac{c}{96\pi} D(h)) S(\Sigma, (f_i), g_\Sigma),$$

where

$$D(h) = \int_\Sigma (|\nabla h|^2 + 4Rh) d\mu_\Sigma,$$

R is the scalar curvature of the metric g_Σ, $d\mu_\Sigma$ is the measure corresponding to g_Σ, and ∇h is the gradient of h with respect to g_Σ.

CFT 6) The operator $S(\Sigma, (f_i), g_\Sigma)$ is invariant with respect to isometries of world-sheets which respect labelings and parametrizations of the boundary components.

3.2 Collapse of CFTs as a double-scaling limit

Collapse of a family of unitary CFTs admits a description in the language of Segal's axioms. Let us consider the set $\mathcal{W}$ of isomorphism classes of wordlsheets defined in the previous subsection. An isomorphism of wordlsheets is an isometry of two-dimensional manifolds with the boundary, which respects separately labelings of incoming and outcoming circles as well as parametrizations.

For a fixed a non-negative integer number $g \geq 0$ let us consider a subset $\mathcal{P}_g \subset \mathcal{W}$ consisting of worldsheets Σ which are surfaces of genus $g \geq 0$ glued from a collection of spheres with three holes (a.k.a. pants) joined by flat cylinders. The number of cylinders depend on the genus g only. Every worldsheet is conformally equivalent to the one like this.

Without giving a definition of the topology on the moduli space of unitary CFTs we would like to describe "a path to infinity" in the moduli space.

Namely, let us consider the amplitudes for a family of unitary CFTs with the same central charge c, depending on the parameter $\varepsilon \to 0$, which satisfy the following properties:

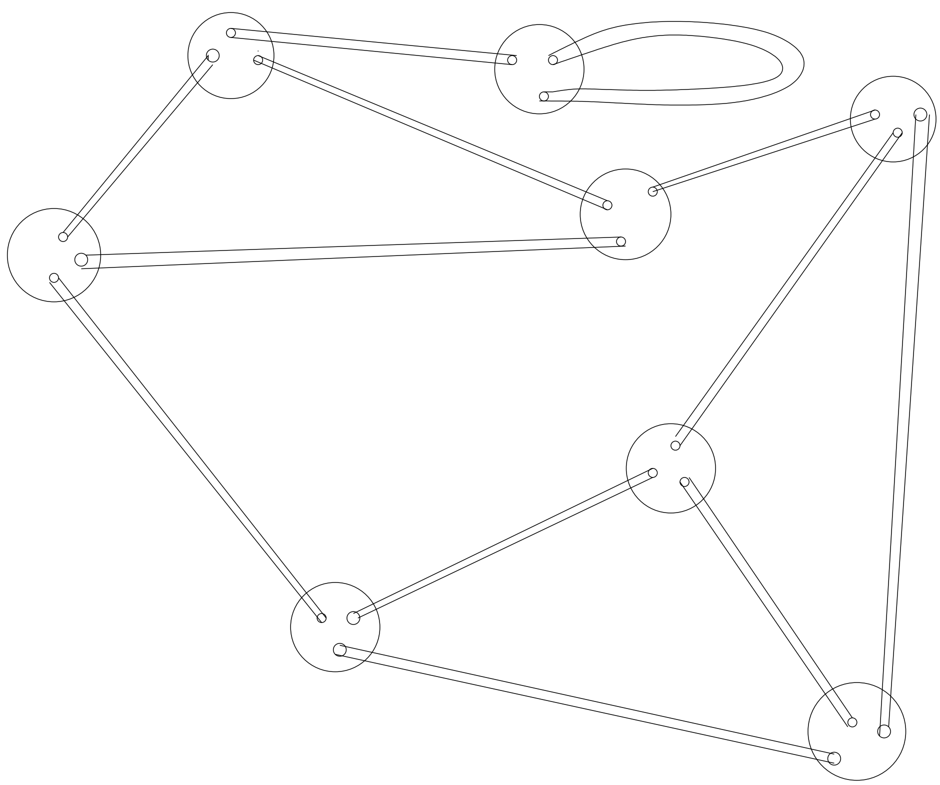
1) The minimal eigenvalue $\lambda_{min}(\varepsilon)$ of the non-negative operator $L_0 + \overline{L}_0$ corresponding to the standard cylinder $S^1_{R=1} \otimes [0,1]$ satisfies the property $\lambda_{min}(\varepsilon) = \lambda_0 \varepsilon \to 0 + o(\varepsilon)$.

2) Consider a family of worldsheets $\Sigma = \Sigma(l_1, ..., l_m, R_1, ..., R_m)$ of genus g such that $min_i\, l_i \to +\infty, max_i R_i/l_i \to 0$ and which are glued from 2-dimensional spheres with three holes each, connected by m "long tubes", which are flat cylinders each of which is isometric to $S^1_{R_i} \times [0, l_i]$ (see Figure below), where $S^1_{R_i}$ is the circle of radius R_i, $1 \le i \le m$. Then the following holds

$$l_i \lambda_{min}(\varepsilon) \to l_i^0 < \infty$$

as $l_i \to \infty, \varepsilon \to 0$ uniformly for $1 \le i \le m$.

FIGURE 0 (surface degenerating into a graph)



Let us normalize the partition function Z of the standard sphere S^2 in such a way that $Z(S^2) = 1$.

Conjecture 3.2.1 *For the family of closed surfaces Σ as above we have*

$$Z(\Sigma)exp(-\frac{c}{6}\cdot l_i\lambda_{min})\leq const,$$

where the constant does not depend on Σ.

In order to formulate the next conjecture we need to use the terminology of subsequent sections.

Conjecture 3.2.2 *After rescaling* $L_0+\overline{L}_0$ *by the factor* λ_{min}^{-1}, *there is a limit in the sense of Section 4.2 of the unitary CFTs, which is a quantum Riemannian* 1*-space in the sence of Section 5.2, such that the limiting for* $(L_0+\overline{L}_0)/\lambda_{min}$ *operator* L *satisfies the curvature-dimension inequlity* $CD(0,\infty)$ *from Section 6.1.*

In this sense CFTs collapse to QFTs with the space-time having non-negative Ricci curvature.

4 Quantum Riemannian d-geometry

4.1 2-dimensional case and CFT

Here we follow [Ru]. For any $\varepsilon>0$ we denote by A_ε the annulus $\{z\in \mathbf{C}|1-\varepsilon<|z|<1+\varepsilon\}=A_\varepsilon^-\cup A_\varepsilon^+$. Let consider the (non-unital) category $Riem_2$ whose objects are $(k+1)$-tuples $X=(\varepsilon;f_1,...,f_k), k\geq 0$, where $\varepsilon>0$ and $f_i:A_\varepsilon\to \mathbf{R}_{>0}, 1\leq i\leq k$ is a smooth function. Notice that each function f_i defines a metric $f_i(x,y)(dx^2+dy^2)$ on A_ε. Let $Y=(\eta;g_1,...,g_l)$ be another object of $Riem_2$. A morphism $\phi:X\to Y$ is by definition a triple (M,g_M,j_-,j_+), where (M,g_M) is a 2-dimensional compact oriented Riemannian manifold with the boundary, and $j_-:\sqcup_{1\leq i\leq k}A_\varepsilon^-\to M$ and $j_+:\sqcup_{1\leq i\leq l}A_\eta^-\to M$ orientation preserving (reversing for j_+) isometric embeddings such that $Im(j_-)\cap Im(j_+)=\emptyset$, and ∂M is the union of the images of $S^1=\{z\in A_\varepsilon||z|=1\}$ under $j_\pm$. Embeddings $j_\pm$ are called parametrizations. Boundary components of M parametrized by j_- (resp. j_+) are called *incoming* (resp. *outcoming*). Composition of morphisms $\phi\circ\psi$ is defined in the natural way: one identifies the point $j_+(z), z\in S^1$ on the outcoming boundary component of the surface defined by ψ with the point $j_-(z), z\in S^1$ on the corresponding incoming boundary component of the surface defined by ϕ. One makes $Riem_2$ into a rigid symmetric monoidal category via disjoint union operation on objects: $(\varepsilon_1,f_1)\otimes(\varepsilon_2,f_2)=(\varepsilon,\underline{f_1,f_2})$, where $\varepsilon=min\{\varepsilon_1,\varepsilon_2\}$. The duality functor is given by $(\varepsilon,f(z))^*=(\varepsilon,\overline{f(\overline{z})})$, where the bar means complex conjugation. On morphisms the tensor product is given by the disjoint union of surfaces together with taking the minimal ε for all incoming and outcoming boundary components. Finally, for a morphism $\phi:X\otimes Y\to Y\otimes Z$ one has a trace map $tr_Y(\phi):X\to Z$ obtained by identification of points $j_+(z), z\in S^1$ and $j_-(z), z\in S^1$ of outcoming and incoming boundary components of Y.

Definition 4.1.1 *A (commutative) Riemannian* 2*-space is a monoidal functor* $F : Riem_2 \to Hilb_{\mathbf{C}}$ *to the rigid monoidal category of separable complex Hilbert spaces*

As usual, the tensor product of Hilbert spaces is defined as the completed algebraic tensor product. We will denote it simply by $\otimes$, changing sometimes the notation to $\widehat{\otimes}$ in order to avoid a confusion.

A commutative Riemannian 2-space defines a unitary CFT with the central charge $c \in \mathbf{R}$ if F is a functor to the category of complex Hilbert spaces satisfying the following properties:

a) F depends on the isometry class of a surface defined by a morphism, not by the surface itself. It assigns the same Hilbert space H to all objects $(\varepsilon; f_1, ..., f_k)$;

b) F preserves trace maps;

c) if $\phi : X \to Y$ is a morphism and $\overline{\phi} : Y \to X$ is obtained by reversing the orientation of the corresponidng surface then $F(\overline{\phi})$ is the Hermitian conjugate to $F(\phi)$;

d) change of the metric $g_M \mapsto e^h g_M$ on the surface (recall that h is a smooth real function) defined by a morphism $\phi = (M, g_M, j_-, j_+)$ leads to a new morphism ϕ_h (between the same objects according to a)) such that $F(\phi_h) = exp(\frac{c}{96\pi} D(h))F(\phi)$, and $D(h)$ is the Liouville action described in CFT 5) in the previous Section.

In order to define a *quantum Riemannian* 2*-space*, we start with the category $Riem_2^{NC}$ with the objects which are $(k+1)$-tuples $(\varepsilon; f_1, ..., f_k)$, as before. A morphism is defined as a quadruple (M, g_M, j_+, j_-, p) where (M, g_M, j_+, j_-) is a Riemannian 2-dimensional surface with parametrized neighborhood of the boundary, as before, and p is a *marking* of the incoming (resp. outcoming) boundary components of ∂M by finite sets $\{1, ..., k\}$ (resp. $\{1, ..., l\}$). On the set of objects we now introduce a rigid *monoidal* structure, which is not symmetric. The tensor product $(\varepsilon_1, f_1) \otimes (\varepsilon_2, f_2)$ is defined as $(min\{\varepsilon_1, \varepsilon_2\}, f_1, f_2)$, as before. But now there is no commutativity isomorphism $X_1 \otimes X_2 \to X_2 \otimes X_1$ if f_1 is not equal to f_2, since isomorphisms in $Riem_2^{NC}$ must respect markings. A quantum Riemannian 2-space is defined by a monoidal functor $F : Riem_2^{NC} \to Hilb_{\mathbf{C}}$. In general F does not commute with the natural action of the product of symmetric groups $S_k \times S_l$ on the markings by the sets $\{1, ..., k\}$ (incoming circles) and $\{1, ..., l\}$ (outcoming circles).

4.2 General case and spaces with measure

Let us fix a non-negative integer d. A *quantum Riemannian d-space* is defined as a monoidal functor $F : Riem_d^{NC} \to Hilb_{\mathbf{C}}$, where $Riem_d^{NC}$ is a rigid monoidal category of marked Riemannian manifolds described below (cf. [Seg], comments to Section 4):

a) An object of $Riem_d^{NC}$ is an isometry class of a neighborhood of a compact oriented Riemannian $(d-1)$-manifold in an oriented d-manifold (called

Riemannian d-germ) together with a bijection between the set of its connected components and a finite set $\{1, ..., n\}$ for some $n \geq 1$.

b) Morphisms are oriented compact Riemannian d-dimensional bordisms. The Riemannian metric is trivialized near the boundary as the product of the given metric on the boundary and the standard flat metric on the interval (this what we called parametrization in the two-dimensional case). Composition of morphisms is defined via pasting and gluing operation which respects to the metric, similarly to the case $d = 2$ considered above.

c) Tensor product $X_1 \otimes X_2$ is defined as the disjoint union of Riemannian d-germs $X_1 \sqcup X_2$ equipped with the marking such that connected components of X_1 are marked first. In this way $X_1 \otimes X_2$ is not necessarily isomorphic to $X_2 \otimes X_1$. Duality corresponds to the reversing of the orientation.

Similarly to the two-dimensional case we have an operation $tr_Y(\phi) : X \to Z$ associated with a morphism $\phi : X \otimes Y \to Y \otimes Z$.

The Hilbert space $H_X := F(X)$ is called the *space of states* assigned to X, for a morphism $f : X \to Y$ the linear map $S(X, Y) := F(f)$ is called the *amplitude* assigned to f.

Next, we would like to introduce a topology on the space of quantum Riemannian d-spaces specified by the definition of the convergence. The latter depends on the notion of convergence of Hilbert vector spaces which carry self-adjoint positive operators (each operator is a generator of the semigroup corresponding to the intervals $[0, t]$). Let us discuss this notion of convergence. Suppose that we have a sequence of Hilbert spaces H_α parametrized by $\alpha \in \mathbf{Z}_+ \cup \{\infty\}$, where $H_\infty := H$. Assume that every H_α carries a self-adjoint non-negative (in general unbounded) operator L_α where $L_\infty := L$. We would like to define a topology on the union of H_α. We will do that following [KS].

Definition 4.2.1 *1) We say that a sequence H_α converges to H as $\alpha \to \infty$, if there exist an open dense subspace $D \subset H$, and for any α there is a continuous linear map $R_\alpha : D \to H_\alpha$, such that $lim_{\alpha\to\infty} ||R_\alpha(x)||_{H_\alpha} = ||x||_H$ for any $x \in D$.*

2) Suppose that H_α converges to H in the above sense. We say that a sequence $x_\alpha \in H_\alpha$ strongly converges to $x \in H$ if there is a sequence $y_\beta \in D$ converging to x such that

$$lim_{\beta\to\infty} lim_{\alpha\to\infty} ||R_\alpha(y_\beta) - x_\alpha||_{H_\alpha}.$$

3) We say that a sequence (H_α, L_α) strongly converges to (H, L) if D contains the domain of L, the sequence H_α converges to H is the sense of 1), and for any sequence $x_\alpha \in H_\alpha$ which strongly converges to $x \in H$ and any continuous function $\varphi : [0, \infty) \to \mathbf{R}$ with compact support the sequence $\varphi(L_\alpha)(x_\alpha)$ strongly converges to $\varphi(L)(x)$.

4) We say that a sequence of quantum Riemannian d-spaces $\{M_\alpha\}_{\alpha\in\mathbf{Z}_+}$ converges to a quantum Riemannian d-space M as $\alpha \to \infty$, if for any morphism $f : X \to Y$ in $Riem_d^{NC}$ the correspondning sequence $(H^*_{X,\alpha} \otimes H_{Y,\alpha}, S(X, Y, \alpha))$ strongly converges to $(H^*_X \otimes H_Y, S(X, Y))$.

Remark 4.2.2 *One can define the notion of quantum locally convex d-geometry over a complete normed field K by considering monoidal functors to the monoidal category $Vect_K^{lc}$ of locally convex topological spaces over K. Probably the case of locally compact K (e.g. the field of p-adic numbers) is of some interest.*

A natural generalization of the above definitions gives rise to the notion of *quantum Riemannian d-space with a measure*. It is an analog of the notion of Riemannian manifold M equipped with a probability measure, which is absolutely continuous with respect to the measure $dvol_M/vol(M)$ defined by the Riemannian metric. Namely, we assume that an additional datum is given: a continuous linear functional $\tau : H \to \mathbf{C}$ such that $\tau(e^{-tL}x) = \tau(x)$ for any $x \in H, t \geq 0$, and which satisfies the "trace" property described below. Let $\Gamma_{l_1,l_2,l}^{in_1,in_2,out}$ be a Y-shape graph with two inputs (marked by in_1, in_2), one output (marked by out) and three edges with the lengths l_i (outcoming from $in_i, i = 1, 2$) and l (incoming to out). Then the trace property says:

$$\tau \circ S(\Gamma_{l_1,l_2,l}^{in_1,in_2,out}) = \tau \circ S(\Gamma_{l_2,l_1,l}^{in_2,in_1,out}).$$

This is an analog of the property: $\tau(fg) = \tau(gf)$ in case when H is obtained by the Gelfand-Naimark-Segal construction from a C^*-algebra and a tracial state τ on it.

We define the category $Riem_d^{NC,mes}$ of *measured* $(d-1)$*-dimensional Riemannian germs* with the objects which are $(d-1)$-Riemannian germs as before, equipped with probability measures which are absolutely continuous with respect to the volume measure associated with the germ of the Riemannian metric. Morphisms between measured $(d-1)$-Riemannian germs are measured compact Riemannian d-dimensional manifolds, such that in the neighborhood of a boundary both metric and measure are trivialized as products of the given metric and measure on the boundary and the standard metric and measure on the interval.

Then we modify the above Property 4) by the following requirement:

if $x_\alpha \to x, x_\alpha \in H_\alpha, x \in H$, then $\tau_\alpha(x_\alpha) \to \tau(x)$ for the corresponding linear functionals. In this way we obtain a topology on quantum measured Riemannian d-spaces.

In the above discussion of convergence we spoke about convergence of quantum Riemannian spaces, not underlying germs of Riemannian manifolds. In other words, having a sequence of quantum Riemannian d-spaces given by functors $F_\alpha : Riem_d^{NC} \to Hilb_{\mathbf{C}}, \alpha \in \mathbf{Z}_+$ we say that the sequence converges to a quantum Riemannian d-space given by a functor $F : Riem_d^{NC} \to Hilb_{\mathbf{C}}$ if for any morphims $f : X \to Y$ in $Riem_d^{NC}$ we have $F_\alpha(f) \to F(f)$ as $\alpha \to \infty$. The latter convergence is defined in the Definition 4.2.1. On the other hand, we have Gromov-Hausdorff (or measured Gromov-Hausdorff) topology on the objects of $Riem_d^{NC}$ (we leave as an exercise to the reader to work out the modification of either topology which takes into account labelings of the boundary components). Moreover we have those topologies on the spaces of morphisms as well. This can be used to define the notion of *continuous* functor.

Definition 4.2.3 *We say that a functor $F : Riem_d^{NC} \to Hilb_{\mathbf{C}}$ is continuous if for any sequence of morphisms $f_\alpha : X_\alpha \to Y_\alpha$ in $Riem_d^{NC}$ which converges in the Gromov-Hausdorff sense to $f : X \to Y$ we have: $F(f_\alpha)$ converges to $F(f)$ in the sense of Definition 4.2.1.*

In particular, if F is continuous then $X_\alpha \to X, \alpha \to \infty$ implies $F(X_\alpha) \to F(X), \alpha \to \infty$.

Continuity is too strong property, because it allows to contract e.g. loops in the $d = 1$ case. A weaker property is the *non-collapsing continuity.* It is the same as the one given in the above definition with the restriction that for the limit $f_\alpha \to f$ we require $vol(f_\alpha)(B(0, r)) \geq const > 0$ for any ball $B(0, r) \subset f_\alpha$ of the Riemannian d-bordism f_α. In particular, for $d = 1$ non-collapsing continuity prohibits contraction not only loops of a graph but edges as well.

There is a natural modification of the above definitions to the case of quantum measured Riemannian d-spaces, which we leave as an exercise to the reader. In this case we have that measured Gromov-Hausdorff convergence of objects in $Riem_d^{NC,mes}$ implies the above-mentioned convergence of Hilbert spaces and linear functionals. More realistic continuity property corresponds to the limit $\varepsilon \to 0$ for an ε-neighborhood of a submanifolds.

5 Graphs and quantum Riemannian 1-geometry

The content of this section is heavily influenced by conversations with Maxim Kontsevich. The notion of commutative Riemannian 1-space is basically what he called Graph Field Theory.

5.1 Quantum Riemannian 1-spaces

A quantum Riemannian 1-space is defined by the following data:

1) A class G of *metrizable labeled graphs* (Γ, I_-, I_+, l, p) described in 1a), 1b) below:

1a) $\Gamma \in G$ is a finite graph with external vertices having the valency one and labeled by elements of the disjoint union of finite sets: $I = I_- \sqcup I_+$ (the letter p above denotes the labeling). Vertices parametrized by the elements of I_- (resp. I_+) are called *incoming* (resp. *outcoming*), or, simply, *in* (resp. *out*) vertices. We denote by $V_{in}(\Gamma)$ the set of inner vertices of Γ, by $E(\Gamma)$ the set of edges of Γ, etc.

1b) A length function $l : E(\Gamma) \to \mathbf{R}_{>0}$, where $E(\Gamma)$ is the set of edges of Γ.

2) Separable real Hilbert space H (or a complex Hilbert space with real structure).[7]

3) To each (Γ, l, I_-, I_+) a trace-class operator $S(\Gamma, l, I_-, I_+) : \otimes_{I_-} H \to \otimes_{I_+} H$, which we will often denote simply by $S(\Gamma)$.

[7]As in in the case of CFT we can relax this condition assuming that H is a locally compact, in particular, a nuclear space. In case if H is a complex vector space the formulas below have to be modified in order to include complex conjugation.

These data are required to satisfy the following axioms:

QFT 1) If Γ is obtained by gluing Γ_1 and Γ_2 (with an obvious definition of the sets $I_\pm$ and the length function) then $S(\Gamma) = S(\Gamma_1) \circ S(\Gamma_2)$ (composition of operators).

QFT 2) Operators $S(\Gamma)$ are invariant with respect to the isometries.

QFT 3) If $\Gamma^\vee$ is obtained from Γ by relabling so that all incoming vertices are declared outcoming and vice versa then $S(\Gamma^\vee) = S(\Gamma)^*$ (conjugate operator). Here we use the scalar product in order to identify H and H^*.

We will add two more axioms below. They will allow us to compactify various "moduli spaces" of quantum Riemannian 1-spaces.

QFT 4) The operators $S(\Gamma)$ enjoy the non-collapsing continuity property. In other words, a small deformation of a metrized graph Γ leads to a small change of $S(\Gamma)$ in the *normed* operator topology. In addition, the operator $S(\Gamma)$ is a continuous function (in the *strong* operator topology) with respect to the length of an edge which is not a loop. Moreover if $F \subset \Gamma$ is a subforest (i.e. a collection of internal edges without loops), and $(\Gamma/F, I_-, I_+, l)$ is the metrized graph obtained by contracting all of the edges from F, then $S(\Gamma/F)$ coincides with the limiting operator $lim_{l(F)\to 0} S(\Gamma)$, which is the limit of $S(\Gamma)$ as lengths of all edges belonging to F simultanuously approach 0.

QFT 5) If Γ^0_ε is a graph, which is a segment $[0,\varepsilon] \subset \mathbf{R}$ such that $I_- = \{0\}$ and $I_+ = \{\varepsilon\}$ then $lim_{\varepsilon\to 0} S(\Gamma^0_\varepsilon) = id_H$, where the limit is taken in the strong operator topology. Similarly, if Γ^1_ε is a graph, which is the same segment but with $I_- = \{0,\varepsilon\}$ and $I_+ = \{\emptyset\}$, then $lim_{\varepsilon\to 0} S(\Gamma^1_\varepsilon) : H \otimes H \to \mathbf{R}$ is the scalar product on H.

Example 5.1.1 *Let M be a compact Riemannian manifold with the metric g_M. We can normalize the Riemannian measure $d\mu = \sqrt{det\, g_M}\, dx$ so that $\int_M d\mu(M) = 1$. Let $H = L^2(M, d\mu)$. Then with each $t > 0$ we associate a trace-class operator $P_t = exp(-t\Delta)$, where Δ is the Laplace operator on M. One has an integral representation $(P_t f)(x) = \int_M G_t(x,y) f(y) d\mu(y)$, where $G_t(x,y)$ is the heat kernel. To a graph $(\Gamma, l) \in G$ we assign the following function on $M^{I_- \cup I_+}$:*

$$K_{(\Gamma,l)}((x_i)_{i\in I_-}, (y_j)_{j\in I_+}) = \int_{M^{V_{in}(\Gamma)}} d\mu^{V_{in}(\Gamma)} \prod_{e\in E(\Gamma)} G_{l(e)}(x,y),$$

where $d\mu^{V_{in}(\Gamma)} = \prod_{v\in V_{in}(\Gamma)} d\mu$ is the product measure on $M^{V_{in}(\Gamma)}$, associated with $d\mu$. In other words, we assign a measure $d\mu$ to every internal vertex, the kernel $G_{l(e)}(x,y)$ (propagator) to every edge and then integrate over all internal vertices. We obtain a function whose variables are parametrized by input and output vertices. Finally, we define $S(\Gamma, l) : \otimes_{I_-} H \to \otimes_{I_+} H$ as the integral operator with the kernel $K(\Gamma, l)$.

We see that every Riemannian compact manifold defines a quantum Riemannian 1-space. In fact it is a commutative Riemannian 1-space, since the operators $S(\Gamma, l)$ are invariant with respect to the action of the product of symmetric groups $S_{I_-} \times S_{I_+}$ on the set $I_- \times I_+$ (and hence the product on the algebra of smooth functions $A \subset H$ defined by the Y-shape graph is commutative).

Let us restate the above example in terms more suitable for non-commutative generalization. In order to define a collection of operators $S(\Gamma, l) : \otimes_{I_-} H \to \otimes_{I_+} H$ it suffices to have tensors $K(\Gamma, l) \in (\otimes_{I_-} H^*) \otimes (\otimes_{I_+} H)$ satisfying the composition property: $K(\Gamma_1 \circ \Gamma_2, l_1 \circ l_2) = ev^{\otimes_{I_+} H}(K(\Gamma_1, l_1) \otimes K(\Gamma_1, l_2))$, where $ev : H^* \otimes H \to \mathbf{C}$ is the natural pairing, $\Gamma_1 \circ \Gamma_2$ is the gluing operation (see QFT 1)), and $l_1 \circ l_2$ is the length function obtained by the natural extension of l_1 and l_2 to $\Gamma_1 \circ \Gamma_2$.

Suppose that to each vertex $v \in V(\Gamma)$ we assigned a tensor $T_v \in \otimes_{i \in Star(v)} H$, where $Star(v)$ is the set of adjacent vertices (i.e. vertices $w \in V(\Gamma)$ such that (w, v) is an edge), and to every edge $e \in E(\Gamma)$ with the endpoints v_1, v_2 we assigned a linear functional $\tau_e : H \otimes H \to \mathbf{C}$. We define $T(\Gamma) := \otimes_{v \in V(\Gamma)} T_v$. Let τ_{I_+} be the tensor product $\otimes_e \tau_e$ taken over all edges which are not of the form $e = (v, w)$, where $w \in I_+$ (i.e. e does not have an endpoint which is an *out* vertex). Then the element $\tau_{I_+}(T(\Gamma))$ belongs to $\otimes_{i \in I_+} H$.

In the above example we thought of $G_{l(e)}(x, y)$ as of an element of the space $H \otimes H$, where tensor factors are assigned to the enpoints of e. To such an edge we assigned the pairing $\tau_e(f, g) = \int d\mu_x d\mu_y G_{l(e)}(x, y) f(x) g(y)$, where $d\mu_x = d\mu_y = d\mu$. We remark, that in order to define $T(\Gamma)$ it suffices to have vectors $z_i \in H, i \in I_- \cup I_+$ and tensors $T_e \in H \otimes H$ for every edge $e = (v_1, v_2) \in E(\Gamma)$.

5.2 Spectral triples and quantum Riemannian 1-geometry

As we explained in the Introduction, a spectral triple in the sense of Connes gives rise to a quantum Riemannian 1-geometry. Let us make this point more precise. We will use a slightly different definition than in [Co1], since we would like to use an abstract version of the Laplace operator rather than the Dirac operator. More precisely, let us consider a triple (A, H, L) which consists of an unital complex $*$-algebra A, a separable Hilbert left A-module H (i.e. the $*$-algebra A acts on H by bounded operators), a self-adjoint non-negative unbounded operator L on H, such that the operator $P_t = exp(-tL) : H \to H$ has finite trace for all $t > 0$. The latter implies that that spectrum of L consists of eigenvalues only, and the operator $(1 + L)^{-1}$ is compact. To formulate the second assumption, for any $a \in A$ let us consider the function $\phi_a(t) = e^{ta} L e^{-ta}, t \geq 0$. It takes value in unbounded operators in H, and the domains of all $\phi_a(t)$ belong to the domain of L for all $a \in A, t \geq 0$. We say that $\phi_a(t)$ has k-th derivative at $t = 0$ if it can be represented as a sum $\phi_a(t) = C_0(a) + C_1(a)t + \frac{C_2(a)}{2!}t^2 + ... + \frac{C_k(t,a)}{k!}t^k$ such that $C_i(a)$ are some (possibly unbounded, but with a non-empty common domain) operators in H and $C_k(0, a) := C_k(a)$ is a bounded operator in H. We will denote it by $ad_a^k(L)$. Now the second assumption says that $ad_a^2(L)$ exists for all $a \in A$. Basically this means that the double commutator $[[L, a], a]$ exists and bounded for all $a \in A$.

Let us assume that the spectral triple has finite metric dimension n (a.k.a.

n-summable, see [Co1], [CoMar]) and satisfies the regularity conditions (see loc cit.) such that the volume functional

$$\tau(f) = Tr_{\omega}(fL^{-n/2})$$

is finite for all $f \in exp(-tL)(H)$ (here Tr_{ω} denotes the Dixmier trace, see loc. cit.). For simplicity we will also assume that H is obtained from A by a GNS completion with respect to the scalar product $\tau(fg^*)$ (this assumption can be relaxed). Then it looks plausible that one can define a quantum Riemannian 1-geometry similarly to the last paragraph of the previous subsection.

6 Ricci curvature, diameter and dimension: probabilistic and spectral approaches to precompactness

In this Section we review some results presented in [BBG], [Ba], [Led], [LV], [St], [KS], [KMS], [KaKu1-2] (see also [U]). Recall that there are basically three approaches to precompactness of metric (and metric-measure) spaces with the diameter bounded from above and the Ricci curvature bounded from below. "Geometric" approach (which goes back to Gromov, see e.g. [Gro1], and which was developed in a deep and non-trivial way by Cheeger, Colding, Fukaya and many others, see e.g. [Fu], [ChC1-3]) deals with Gromov-Hausdorff (or mesured Gromov-Hausdorff) topology, and (very roughly speaking) embeds a compact metric space (X, d) into the Banach space $C(X)$ of continuous functions via $x \mapsto d(x, \bullet)$. Then precompacness follows from a version of Arzela-Ascoli theorem, since the space X is approximated by a finite metric space. By the nature of this approach one needs the notion of Ricci curvature to be defined "locally", in terms of points of X.

In the "spectral" approach (see e.g. [BBG], [KaKu1-2]) one embeds the metric-measure space $(X, d, d\mu)$ into $L_2(X, d\mu)$ via $x \mapsto K_t(x, \bullet)$ where $K_t(x, y)$ is the "heat kernel" (which needs to be defined if (X, d) is not a Riemannian manifold). Then precompactness follows from a version of Rellich's theorem, since the assumptions on the diameter and curvature imply that the image of X belongs to a Sobolev space, which is compactly embedded in $L_2(X, d\mu)$. Precompactness relies on the estimates for eigenvectors and eigenvalues of the "generalized Laplacian". The former are still local while the latter depends on the global geometry of X. The spectral topology in general does not coincide with the measured Gromov-Hausdorff topology (restrictions on the diameter and Ricci curvature can make these topologies equivalent, see e.g. [KS], Remark 5.1).

In the "probabilistic" approach (see a good review in [L], or original proofs in [LV], [St]) one uses the ideas of optimal transport (different point of view is presented in [AGS]). It is a mixture of the previous two approaches, since one studies the "heat flow" on the space $P(X)$ of probabilistic Borel measures on X, but proves the precompactness theorem in the measured Gromov-Hausdorff topology via a kind of Arzela-Ascoli arguments. The point is that the heat flow on measures can be interpreted either as a gradient line of a functional (entropy) or as a geodesic for some metric on $P(X)$. More precisely the space $P(X)$ carries a family $W_p, p \geq 1$ of the so-called Wasserstein metrics. In the case $p = 1$ such a metric (called also Monge-Kantorovich metric) being restricted to delta-functions $\delta_x, x \in X$ reproduces the original distance on X. The distance $W_1(\phi, \psi)$ coinsides with the "non-commutative distance" on the space of states on $C(X)$ discussed in the Introduction, thus making a connection with Connes's approach to non-commutative Riemannian geometry. In order to use the above ideas of optimal transport on needs a non-commutative analog of the Wasserstein metric W_2. This is an interesting open problem. [8]

6.1 Semigroups and curvature-dimension inequalities

We follow closely [Ba], [Led]. Let $(X, d\mu)$ be a space with a measure $d\mu$ (which is assumed to be a Borel probability measure), and $P_t, t \geq 0$ a semigroup of bounded operators acting continuously in the operator norm topology on the Hilbert space of real-valued functions $L_2(X, d\mu)$, and such that

$$(P_t f)(x) = \int_X G_t(x,y) f(y) d\mu,$$

where the kernels $G_t(x,y)d\mu$ are non-negative for all $t \geq 0$. It is also assumed that $P_t(1) = 1$, which is true for semigroups arising from Markov processes (main application of loc.cit).

Example 6.1.1 *For the Brownian motion in $\mathbf{R}^n$ starting from the origin one has*

$$G_t(x,y)d\mu = \frac{1}{(2\pi t)^{n/2}} e^{-|x-y|^2/2t} dy.$$

One defines the generator of the semigroup P_t as

$$Lf = lim_{t \to 0} \frac{(P_t f - f)}{t}.$$

Then on the domain of the non-negative operator L one has $\frac{\partial}{\partial t} P_t(f) = LP_t(f)$. For the Brownian motion the operator L is just the standard Laplace operator on $\mathbf{R}^n$. We will assume that L is symmetric on its domain (this corresponds to the so-called time reversible measures). This also implies that the finite measure $d\mu$ is invariant with respect to the semigroup $P_t, t \geq 0$. We will also assume that the domain of L contains a dense unital subalgebra A (typically,

[8] I thank to Dima Shlyakhtenko who pointed me out the paper [BiVo] where the non-commutative analogs of the Wasserstein metrics W_p were introduced in the framework of the free probability theory. Since we define the tensor product of quantum spaces by means of the tensor product of algebras, rather than their free product, it is not clear how to use that definition for the purposes of quantum Riemannian 1-geometry.

the algebra of real-valued smooth functions on a manifold, or the algebra of real-valued Lipschitz functions on a metric space). Following Bakry we introduce a sequence of bilinear forms $A \otimes A \to A$:

0) $B_0(f,g) = fg$,

1) $2B_1(f,g) = LB_0(f,g) - fL(g) - L(f)g = L(fg) - fL(g) - L(f)g$,

2) $2B_n(f,g) = LB_{n-1}(f,g) - B_{n-1}(f,Lg) - B_{n-1}(Lf,g), n \geq 2$.

In the case when $L = \Delta$ is the Laplace operator in $\mathbf{R}^n$ (Brownian motion case) one has $B_2(f,f)(x) = |Hess(f)|^2(x) := \sum_{1 \leq i,j \leq n} (\partial^2 f / \partial x_i \partial x_j)^2(x)$. If L is the Laplace operator on the Riemannian manifold M then

$$B_1(f,f) = g_M(\nabla f, \nabla f),$$

$$B_2(f,f) = Ric(\nabla f, \nabla f) + |Hess(f)|^2,$$

where Ric denotes the Ricci tensor. In the case of general Riemannian manifold the Hessian matrix $Hess(f) := \nabla^2(f)$ can be defined as a second derivative of f in the Riemannian structure, so the above equality for $B_2(f,f)$ still holds. All that can be axiomatized such as follows.

Suppose that we are given a triple (A, H, L) such that

1) H is a separable real Hilbert space;

2) L is a (possibly unbounded) operator on H, which is symmetric on its domain;

3) $A \subset H$ is a unital real algebra, dense in the domain of L, such that $L(1) = 0$.

Definition 6.1.2 *a) An operator L satisfies a curvature-dimension condition $CD(R,N)$, where $R \in \mathbf{R}, N \geq 1$ if for all $f \in A$ one has*

$$B_2(f,f) \geq RB_1(f,f) + \frac{(Lf)^2}{N}.$$

b) An operator L has the Ricci curvature greater or equal than R if it satisfies $CD(R,\infty)$, i.e.

$$B_2(f,f) \geq R\,B_1(f,f),$$

for any $f \in A$.

Remark 6.1.3 *The above considerations can be generalized to the case of complex Hilbert spaces with the real structure defined by an involution $x \mapsto x^*, x \in H$, which is compatible with the involution on $A \subset H$. In that case we define a sequence of bilinear forms $B_n(f, g^*)$. In what follows we will discuss for simplicity real Hilbert spaces.*

Suppose that a triple (A, H, L) satisfies the above conditions 1)-3).

Definition 6.1.4 *We say that (A, H, L) has Ricci curvature greater or equal than R if the operator L satisfies $CD(R,\infty)$.*

Example 6.1.5 *a) If $d\mu = e^h dvol_M$, where $g = g_M$ is a Riemannian metric on the n-dimensional manifold M, $dvol_M$ is the corresponding volume form, and h is a smooth real function then $L = \Delta + \nabla(h)\nabla$, where Δ is the Laplace operator associated with the metric g. It was shown by Bakry and Emery (see [BaEm]) that $CD(R, N), N \geq n$ for L is equivalent to the following inequality of symmetric tensors:*

$$Ric \geq Hess(h) + Rg_M + \frac{dh \otimes dh}{(N - n)}.$$

In particular, $CD(R, n)$ implies $h = const$, and hence $Ric \geq Rg_M$ everywhere.

b) If $L = (d/dx)^2 - q(x)d/dx$ then $CD(R, N)$ is equivalent to

$$q' \geq R + \frac{q^2}{N - 1}.$$

Definition 6.1.6 *A spectral triple (A, L, H) is called measured spectral triple if we are given also Gelfand-Naimark-Segal state γ on A which is invariant with respect to the semigroup $exp(-tL), t > 0$.*

6.2 Wasserstein metric and N-curvature tensor

Here we recall definitions and results of [LV], [St].

Let $f : \mathbf{R}_{\geq 0} \to \mathbf{R}$ be a continuous convex function, such that $f(0) = 0$. If $x^N f(x^{-N})$ is convex on $(0, \infty)$, we will say that the function f is N-convex, where $1 \leq N < \infty$. If $e^x f(e^{-x})$ is convex on $(-\infty, \infty)$ we will say that the function f is ∞-convex.

Example 6.2.1 *a) The function $f_N(x) = Nx(1 - x^{-1/N})$ is N-convex for $1 < N < \infty$.*

b) The function $f_\infty(x) = x log\, x$ is ∞-convex.

Let $(X, d\mu)$ be a compact Hausdorff space equipped with a finite probability measure, and f be an arbitrary continuous convex function as above. For any probability measure $d\nu = \rho\, d\mu$, which is absolutely continuous with respect to $d\mu$ we define the f-relative entropy of $d\nu$ with respect to $d\mu$ by the formula

$$E^f_{d\mu}(d\nu) = \int_X f(\rho(x))d\mu$$

(one can slightly modify this formula in order to include measures with a non-trivial singular part in the Lebesgue decomposition).

Let now (X, d) be a compact metric space. We define the L_2-Wasserstein metric (or, simply, the Wasserstein metric, since we will not consider other Wasserstein metrics) on the space $P(X)$ of all Borel probability measures on X by the formula

$$W_2(d\nu_1, d\nu_2)^2 = inf\{\int_{X \times X} d^2(x_1, x_2)d\xi\},$$

where infimum is taken over all probability measures $d\xi \in P(X \times X)$ such that $(\pi_i)_* d\xi = d\nu_i, i = 1, 2$, where π_i are the natural projections of $X \times X$ to the factors. Then $(P(X), W_2)$ becomes a compact metric space (in fact a length space, if X is a length space), and the corresponding metric topology coincides with the weak $*$-topology on measures.

Definition 6.2.2 *([LV], [St])*

a) We say that the compact metric-measure length space $(X, d, d\mu)$ has a non-negative N-Ricci curvature, $1 \leq N < \infty$ if for all $d\nu_0, d\nu_1 \in P(X)$ which have supports belonging to $supp(d\mu)$ there is a Wasserstein geodesic $d\nu_t, 0 \leq t \leq 1$ joining $d\nu_0$ and $d\nu_1$ such that for all N-convex functions f and all $0 \leq t \leq 1$ one has

$$E^f_{d\mu}(d\nu_t) \leq tE^f_{d\mu}(d\nu_1) + (1-t)E^f_{d\mu}(d\nu_0).$$

In other words, the f-relative entropy $E^f_{d\mu}$ is convex along a geodesic joining $d\nu_0$ and $d\nu_1$.

b) Given $R \in \mathbf{R}$ we say that $(X, d, d\mu)$ has ∞-Ricci curvature bounded below by R if for all $d\mu_0, d\mu_1$ as in part a) there is a Wasserstein geodesic $d\nu_t$ joining $d\nu_0$ and $d\nu_1$ such that for all ∞-convex functions f and all $t \in [0, 1]$ one has:

$$E^f_{d\mu}(d\nu_t) \leq tE^f_{d\mu}(d\nu_1) + (1-t)E^f_{d\mu}(d\nu_0) - \frac{1}{2}\lambda(f)t(1-t)W_2(d\nu_0, d\nu_1)^2,$$

and $\lambda = \lambda_R$ is a certain map from ∞-convex functions to $\mathbf{R} \cup \{-\infty\}$ defined in [LV], Section 5. For $f = x\log x$ one can take $\lambda(f) = R$.

Let now $N \in [1, \infty]$, and $d\mu = e^h\, dvol_g$ be a probability measure on a smooth compact connected Riemannian manifold $(M, g), dim\, M = n$, associated with an arbitary smooth real function h. One defines the Ricci N-curvature Ric_N such as follows (see [LV]):

a) $Ric_N = Ric - Hess(h)$, if $N = \infty$,

b) $Ric_N = Ric - Hess(h) - \frac{dh \otimes dh}{N-n}$, if $n < N < \infty$,

c) $Ric_N = Ric - Hess(h) - \infty \cdot (dh \otimes dh)$, if $N = n$. Here by convention $0 \cdot \infty = 0$.

d) $Ric_N = -\infty$, if $N < n$.

The following theorem was proved in [LV], [St].

Theorem 6.2.3 *1) For $N \in [1, \infty)$ the measured length space $(M, g, d\mu)$ has non-negative N-Ricci curvature iff $Ric_N \geq 0$ as a symmetric tensor.*

2) It has ∞-Ricci curvature bounded below by R iff $Ric_\infty \geq Rg$.

Notice that in the above assumptions the condition $Ric_N \geq Rg$ is equivalent to Bakry's $CD(R, N)$ condition for the operator $\Delta + \nabla(h)\nabla$.

One defines the measured Gromov-Hausdorff topology on the set of metric-measure spaces in the usual way: a sequence $(X_i, d_i, d\mu_i)$ converges to $(X, d, d\mu)$ if there are ε_i-approximations $f_i : (X_i, d_i) \to (X, d)$ such that $\varepsilon_i \to 0$ as $i \to \infty$, which satisfy the condition that the direct images $(f_i)_* d\mu_i$ converge (in the weak topology) to $d\mu$.

Theorem 6.2.4 *([LV])*

a) For all $1 \leq N < \infty$ the set of length metric-measure spaces with non-negative N-Ricci curvature is precompact and complete in the measured Gromov-Hausdorff topology.

b) For $N = \infty$ the same is true for the set of length metric-mesure spaces with the ∞-Ricci curvature greater or equal than fixed R.

Similar result was proved in [St], where the precompactness theorem was established with respect to the following distance function on metric-measure spaces:

$$\mathbf{D}^2((X_1, d_1, d\mu_1), (X_2, d_2, d\mu_2)) = inf\{\int_{X_1 \times X_2} d^2(x_1, x_2) d\chi(x_1, x_2)\},$$

where infimum is taken over all $d\chi \in P(X_1 \times X_2)$ which projects onto $d\mu_1$ and $d\mu_2$ respectively under the natural projections, and all metrics d on $X_1 \sqcup X_2$ which coincide with the given metrics $d_i, i = 1, 2$ on $X_i, i = 1, 2$.

6.3 Remark about the Laplacian

As we have seen, the notion of Laplacian (maybe generalized one) plays an important role in the description of the collapsing CFTs. In the case of metric-measure spaces one can use the following approach to the notion of Laplacian (see [Kok]). Let $(X, d, d\mu)$ be a metric-measure space. For any point $x \in X$ and the open ball $B(x, r)$ with the center at x we define the operator "mean value" $f \mapsto \langle f \rangle_{B(x,r)}$, where

$$\langle f \rangle_{B(x,r)} = \frac{1}{\mu(B(x, r))} \int_{B(x,r)} f(y) d\mu(y).$$

Let us assume that the measure of every ball is positive. Then one defines the Laplacian $\Delta_\mu(f)$ by the formula

$$\Delta_\mu(f)(x) := lim_{r \to 0} sup_{r>0} \frac{2}{r^2} \langle (f - f(x)) \rangle_{B(x,r)} =$$

$$lim_{r \to 0} inf_{r>0} \frac{2}{r^2} \langle (f - f(x)) \rangle_{B(x,r)},$$

provided the last two limits exist and coincide. It was observed in [Kok] that this *measured* Laplacian coinsides with $\frac{1}{n+2}\Delta_g$ on a Riemannian n-dimensional manifold (M, g). Moreover, for a large class of metric-measure spaces it is symmetric non-negative operator on certain classes of functions built out from Lipshitz functions on $(X, d, d\mu)$. In a different framework of spaces with diffusion and codiffusion the notion of the heat operator and the Laplacian was in introduced in [Gro2], Sect.3.3. The Laplacian introduced in [Gro2] is in fact a vector field, which is the gradient of the energy function. Probably in all cases when the Laplacian or the heat operator can be defined as a symmetric non-negative operator, there is an interesting quantum Riemannian 1-geometry of spaces with non-negative Ricci curvature.

6.4 Spectral metrics

Here we follow [BBG].

Let (M, g_M) be a compact closed Riemannian manifold, $d\mu = dvol_M$ denotes the associated Riemannian measure. Let us choose an orthonormal basis $\psi_j, j \geq 0$ of eigenvectors of the Laplacian $\Delta = \Delta_{g_M}$, which is unbounded non-negative self-adjoint operator on the Hilbert space $H = L_2(M, d\mu)$. Let λ_j be the eigenvalue corresponding to ψ_j. Then for every $t > 0$ one defines a map $\Phi_t : M \to l_2(\mathbf{Z}_+)$ by the formula

$$\Phi_t(x) = \sqrt{Vol(M)}(e^{-\lambda_j t/2}\psi_j(x))_{j\geq 0}.$$

In this way one obtains an embedding of M into $l_2(\mathbf{Z}_+)$. For any two Riemannian manifolds (M_1, g_{M_1}) and (M_2, g_{M_2}) as above, and any two choices of the orthonormal bases one can compute the Hausdorff distance d_H between compact sets $\Phi_t(M_1, g_{M_1})$ and $\Phi_t(M_2, g_{M_2})$ inside of the metric space $l_2(\mathbf{Z}_+)$. This number depends on the choices of orthonormal bases for the Laplace operators on M_1 and M_2, but one can remedy the problem by taking $sup\, inf\, d_H$ over all possible pairs of choices. This gives the distance $d_t((M_1, g_{M_1}), (M_2, g_{M_2}))$ defined in [BBG]. It was proved there that the distance (for a fixed $t > 0$) is equal to zero if and only if the Riemannian manifolds are isometric. More invariant way to spell out this embedding is via the heat kernel. Namely, one considers the map $M \to L_2(M, d\mu)$ such that $x \mapsto K_M(t/2, x, \bullet)$, where $K_M(t, x, y) = \sum_{j\geq 0} e^{-\lambda_j t}\psi_j(x)\psi_j(y)$ is the heat kernel. Subsequently one can identify $L_2(M, d\mu)$ with $l_2(\mathbf{Z}_+)$ since any two separable Hilbert spaces are isometric. Thus we obtain the above embedding.

Let $\mathcal{M}(n, R, D)$ be the set of compact closed Riemannian manifolds such that $dim\, M = n, Ric(M) \geq R, diam(M) \leq D$. It was proved in [BBG] that the Φ_t-image of $\mathcal{M}(n, R, D)$ belongs to a bounded subset of the Sobolev space $h^1(\mathbf{Z}_+)$ (the latter consists of sequences $(a_0, a_1, ...) \in l_2(\mathbf{Z}_+)$ such that $\sum_{j\geq 0}(1+j^{2/n})a_j^2 < \infty$). By Rellich's theorem the embedding $h^1(\mathbf{Z}_+) \to l_2(\mathbf{Z}_+)$ is a compact operator. This implies that the image of the embedding of $\mathcal{M}(n, R, D)$ via Φ_t is precompact in $l_2(\mathbf{Z}_+)$. Moreover, the eigenvalues of the Laplacian are continuous with respect to the spectral distance d_t. Since only smooth manifolds were considered in [BBG] the measure $d\mu$ was always the one associated with the Riemannian metric. Approach of [BBG] was further developed and generalized in [KaKu1-2]. In the loc. cit the authors discussed the compactification of $\mathcal{M}(n, R, D)$ with respect to their version of the spectral distance, which is different from the one in [BBG].

6.5 Spectral structures and measured Gromov-Hausdorff topology

Here we briefly recall the approach suggested in [KS].

As we have seen above, the Laplacian and the heat kernel can be defined for more general spaces than just Riemannian manifolds (see e.g. [KMS], [S] for the case of Alexandrov spaces). The notion of spectral structure was introduced in [KS] with the purpose to study the behavior of eigenvalues of the Laplacian with respect to perturbations of the metric and topology of not necessarily compact Riemannian manifolds. A *spectral structure* is a tuple of compatible data: $(L, Q, E, U_t, R(z), H)$ which consists of a separable Hilbert space H (complex or real), self-adjoint non-negative linear operator $L : H \to H$, densely defined quadratic form Q generated by $\sqrt{L}$, spectral measure $E = E_\lambda(L)$, pointwise continuous contraction semigroup U_t with the infinitesimal generator L, pointwise continuous resolvent $(z - L)^{-1}$. One can associate a spectral structure with a pointed locally compact metric space equipped with a Radon measure (or with a not necessarily pointed compact metric-measure space). Two types of topologies on the set of spectral triples were introduced in [KS]: strong topology and compact topology. For both topologies the natural forgetful map from "geometric" spectral structures to the corresponding metric-measure spaces is continuous. Main results of [KS] concern convergence of spectral structures under the condition that the underlying metric-measure spaces converge. In particular, one has such a convergence for the class of Riemannian complete (possibly non-compact) pointed n-dimensional manifolds with the Ricci curvature bounded from below ([KS], Theorem 1.3). The results of [KS] can be considered as a generalization of the results of [Fu], [ChC1] about continuity of eigenvalues of the Laplacian with respect to measured Gromov-Hausdorff topology. Every commutative Riemannian 1-space associated with a complete pointed Riemannian manifold (or compact closed non-pointed Riemannian manifold) gives rise to a spectral structure. Therefore, applying results of [KS] one can deduce precompactness of the moduli space of such Riemannian 1-spaces. Unfortunately, [KS] does not contain any precompactness results about the moduli space of "abstract" spectral structures, i.e. those which are not associated with meatric-measure spaces. Same is true for [BBG].

6.6 Non-negative Ricci curvature for quantum 1-geometry

Suppose that we are given a quantum Riemannian 1-space which satisfies the following property: there is a dense pre-Hilbert subspace $A \subset H$, such that for any graph $\Gamma \in G$ the tensor product $A^{\otimes I_-}$ belongs to the domain of the operator $S(\Gamma)$. Then, taking a Y-shape graph, as in the Introduction, we recover from the axoms QFT 1)-QFT 4) an associative product on A. Following Kontsevich we impose the following three conditions:

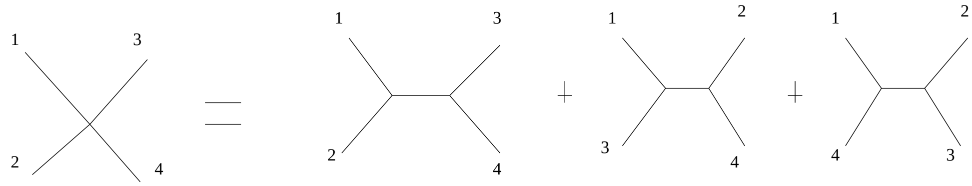
QFT 6) (spectral gap, or boundness of the diameter) Let L be a generator of the semigroup $S(\Gamma^0_\varepsilon)$ (see QFT 5)). Then the spectrum of L belongs to the set $\{0\} \cup \{a\} \cup [b, +\infty)$, where $b > a > 0$ are some numbers.

QFT 7) (7-term relation) For a graph $\Gamma(l_1, l_2, l_3, l_4)$ with one internal vertex and four attached edges of length $l_i, 1 \le i \le 4$ one has the following identity:

$$\sum_{1 \le i \le 4} \frac{\partial}{\partial l_i} S_{\Gamma(l_1,l_2,l_3,l_4)} = \sum_{1 \le j \le 3} \frac{\partial}{\partial \varepsilon_j}|_{\varepsilon_j = 0} S_{\Gamma(l_1,l_2,l_3,l_4,\varepsilon_j)},$$

where $\Gamma(l_1, l_2, l_3, l_4, \varepsilon_j)$ is a graph obtained from $\Gamma(l_1, l_2, l_3, l_4)$ by inserting one internal edge of the length ε_j (three summands in the RHS correspond to three different ways of pairing of four external vertices of the graph $\Gamma(l_1, l_2, l_3, l_4)$, see the Figure below).

FIGURE 1 (corolla with 4 external vertices and 3 graphs obtained from it by inserting a new internal edge)



QFT 8) (non-negativness of Ricci curvature) If $\Gamma(l_1, l_1, l_3, l_3, \varepsilon)$ is the graph $\Gamma(l_1, l_2, l_3, l_4, l)$ from QFT 7) with $l_2 = l_1, l_4 = l_3, \varepsilon = l$ then the following identity holds:

$$(\partial/\partial l_1 - \partial/\partial l)^2 S_{\Gamma(l_1,l_1,l_3,l_3,l)} \geq 0.$$

This condition is equivalent to the following one:

$$(\frac{d}{dt})^2|_{t=0}(e^{tL}(e^{-tL}(f) \cdot e^{-tL}(f))) \geq 0,$$

for all $f \in A$. The LHS of this inequality is equal to $B_2(f, f)$. Another way to state the above inequality is to say that the amplitude $S_{\Gamma(l_1+t,l_1+t,l_3,l_3,l-t)}$ is convex at $t = 0$. The continuity of the amplitudes $S_{\Gamma,l}$ (see QFT 4)) implies that the condition $B_2(f, f) \geq 0$ is preserved if we contract a subforset of Γ (i.e. we allow to contract a disjoint union of trees, while the contraction of loops is prohibited).

Remark 6.6.1 *a) The condition QFT 6) is motivated by the property that for any $t > 0$ we have $Tr(e^{-tL}) < \infty$. This implies that there is an interval $(0, \lambda), \lambda > 0$ which contains finitely many points of the spectrum of L. If $L = \Delta$, the Laplace operator on a compact Riemannian manifold M, then the spectral gap λ is of the magnitude $1/(diamM)^2$.*

b) The condition QFT 7) is motivated by the case $L = \Delta$. In this case

$$\Delta(f_1 f_2 f_3) - \Delta(f_1 f_2) f_3 + ... + \Delta(f_1) f_2 f_3 - \Delta(1) f_1 f_2 f_3 = 0$$

(7-term relation for the second order differential operator Δ).

c) The condition QFT 8) is motivated by the $CD(0, \infty)$ inequalities of Bakry.

It was shown by Kontsevich that Segal's axioms of the unitary CFT imply that the collapsing sequence of unitary CFTs gives rise to a commutative Riemannian 1-space, which satisfies axioms QFT1)-QFT 8). The underlying Riemannian manifold M is the smooth part of $X = Spec(H^{small})$ (see Section 2.2) with L derived from the limit of the rescaled Virasoro operators $(L_0 + \overline{L}_0)/\varepsilon$, where ε is the minimal eigenvalue of $(L_0 + \overline{L}_0)$, $\varepsilon \to 0$.

Recall that for a non-negative self-adjoint operator L acting in a Hilbert space H a *spectral gap* $\lambda_1(L)$ is the smallest positive eigenvalue of L. Let us consider quantum Riemannian 1-spaces with measure, for which the Hilbert space H is obtained by the Gelfand-Naimark-Segal construction from a unital involutive algebra A and a state $\tau : A \to \mathbf{C}$. To every such a quantum space one can associate a spectral triple (A, H, L).

Conjecture 6.6.2 *The space of isomorphism classes of quantum Riemannian 1-spaces with a measure, which have non-negative Ricci curvature, spectral gap bounded below by a given number C, and such that the corresponding spectral triples have dimensional spectrum belonging to a given interval $[a, b]$ (see [Co-Mar] for the definition of the dimension spectrum) is precompact in the topology defined in Section 4.2.*

Finally, we are going to discuss an example in which a version of the above conjecture was verified. Let $M_j, j \geq 1$ be a sequence of compact Riemannian manifolds of the same dimension n with the diameter equal to 1 (recall that rescaling of the metric does not chence the Ricci curvature). Then, following Example 5.1.1 we can associate with this sequence a sequence of non-commutaive Riemannian 1-spaces $V(M_j), j \geq 1$. Suppose that N is a measured Gromov-Hausdorff limit of M_j as $j \to \infty$. Then it follows from [ChC3] that N carries a generalized Laplacian, and the measure, hence the same formulas as in Example 5.1.1 allows us to associate with N a quantum Riemannian 1-space $V(N)$. Let us say that $V(M_j)$ *weakly converges* to $V(N)$ if for any sequence of Lipshitz functions $f_j : M_j \to \mathbf{R}, f : N \to \mathbf{R}$ such that $|f_j \circ \psi_j - f|_{L_\infty(N)}$ as $j \to \infty$, and for any metrized graph Γ, we have: $S_\Gamma(f_j) \to S_\Gamma(f)$ as $j \to \infty$. Here $\psi_j : N \to M_j$ is any sequence of ε_j-approximations such that $\varepsilon_j \to 0$ as $j \to \infty$. Weak convergence gives rise to the topology on the space of equivalence classes of quantum Riemannian 1-spaces of *geometric origin* (i.e. those which correspond to metric-measured spaces). Then the following result holds (see [En] for the proof).

Theorem 6.6.3 *The subspace of the space of the above quantum Riemannian 1-spaces corresponding to manifolds with non-negative Ricci curvature is precompact in the weak topology.*

Having in mind results of [LV] and [St] we expect that it is in fact compact.

7 Appendix: Deformations of Quantum Field Theories and QFTs on metric spaces

Here we reproduce a portion of the unfinished paper with Maxim Kontsevich with the title "Deformations of Quantum Field Theories" started in December 2000. Among other things it contains the definition of a QFT on an arbitrary metric space-time. Hopefully in the case of Riemannian manifolds, our definition can be formulated in the language similar to the language of Segal axioms for

CFTs. Then this kind of "generalized QFT" give an example of a quantum metric-measure space. From the point of view of present paper it is natural to ask how such generalized QFTs behave with respect to the Gromov-Hausdorff topology on the "moduli space" of space-times which have non-negative Ricci curvature.

7.1 Moduli space of translation-invariant QFTs: what to expect?

Here we will briefly explain part of the structures we would like to have in the deformation theory of a general translation-invariant QFT on $\mathbf{R}^d$. For such a theory $\mathcal{C}$ we have a space of local fields H, which is filtered by dimensions $\Delta \geq 0$. It is convenient to introduce the space $\Omega^\bullet(H) = H \otimes \wedge^\bullet(\mathbf{R}^{*d})$ of H-valued differential forms. We introduce the grading in this space, so that the degree of $dx_i, 1 \leq i \leq d$ is -1. Since translations act on H we have the action of the corresponding vector fields $\partial_i, 1 \leq i \leq d$ on H. Therefore $\Omega^\bullet(H)$ carries a differential $m_1 = d$ (Koszul differential). Intuitively the tangent space to formal deformations of the QFT with the space of fields H is given by $H/\sum_i \partial_i H$. This corresponds to a part

$$H \otimes \wedge^1(\mathbf{R}^{*d}) \to H$$

of the complex $(\Omega^\bullet(H), m_1)$. We will explain why the deformation theory of $\mathcal{C}$ is controlled by a certain L_∞-algebra structure on $\Omega^\bullet(H)_{\leq 0}$ (here ≤ 0 stands for the total degree). This structure is related to the action of $H \otimes \wedge^{d-1}(\mathbf{R}^{*d})$ on $\Omega^\bullet(H)$, and we are going to explain this action.

Since H is filtered, we can consider the corresponding graded space. It gives rise to a CFT $gr(\mathcal{C})$. The renormalization group flow contracts $\mathcal{C}$ to $gr(\mathcal{C})$.

When discussing foundational questions of Quantum Field Theory one faces the problem of basic definitions. For example, what is the moduli space of QFTs (on a given space-time)? Without answering this question, one meets difficulties in defining deformations of a given QFT. In practice physicists speak about "deforming the lagrangian of a theory". This is not satisfactory, because lagrangians are not fundamental objects. One can have a theory (CFT, for example) which is a priori defined without a lagrangian. Obviously, one wants to have definitions which work in all cases.

We suggest the point of view which can be explained by analogy with Morse theory: one can reconstruct a compact manifold from a general gradient field on it. Let us explain this approach in a "simplified picture of the world". Namely, imagine that there is a smooth moduli space $\mathcal{M}$ of translation-invariant QFTs on a given Euclidean space-time $\mathbf{R}^d$. This moduli space carries a vector field (called β-function by physicists). This vector field has the following origin. Let us consider the group of dilations of the metric g on $\mathbf{R}^d$, namely $g \mapsto \lambda g$. We assume that there is a canonical lifting of this action to $\mathcal{M}$. The corresponding group is called the *renormalizarion group* (RG for short), and by definition it acts on quantum field theories. We prefer to call it RG vector field (rather than β-function) and denote by β. Locally it is a gradient vector field.

Critical points of the RG vector field β are CFTs. We assume that for each Morse critical point x we have: $\mathcal{M} = \mathcal{M}_x^{in} \bigcup \mathcal{M}_x^{out}$ (union of points which are attracted to x and repelled from x as $t \to +\infty$).

Definition 7.1.1 *Renormalizable (with respect to x) QFTs correspond to points of $\mathcal{M}_x^{out}$. Unrenormalizable QFTs correspond to points of $\mathcal{M}_x^{in}$.*

Let us assume that β is a gradient vector field: $\beta = grad(c)$ locally near $x_0 \in \mathcal{M}, \beta(x_0) = 0$. Here $c(x)$ is a smooth function on $\mathcal{M}$ (Zamolodchikov c-function), which (near the Morse critical point x_0) can be written as $c(x) = \sum_i \lambda_i x_i^2$ (possibly infinite sum). We assume (?) that $Hess(c) := d^2(c)_{|T_{x_0}\mathcal{M}}$ has finitely many negative eigenvalues, $\lambda_i = \Delta_i - d$. It follows that $\mathcal{M}_{x_0}^{out}$ is a finite-dimensional manifold. Since β is a gradient vector field, all numbers Δ_i (called dimensions of local fields) are real. The deformation theory we described at the very beginning of this section is the deformation theory of the CFT associated with the point $x_0 \in \mathcal{M}$ in the direction of $\mathcal{M}_{x_0}^{out}$. All deformed QFTs can be described by means of the operator product expansion (OPE). There is an L_∞-algebra controlling these deformations. It gives rise to the finite-dimensional moduli space $\mathcal{M}_{x_0}^{out}$. More precisely, the L_∞-algebra gives rise to a formal pointed dg-manifold (see [KoSo2], [KoSo3]) with x_0 be the marked point. It is foliated (in the sense of dg-manifolds) by the leaves of the odd vector field. Equivalent theories belong to the same leaf. "Moduli space" of leaves is isomorphic to $\mathcal{M}_{x_0}^{out}$.

Remark 7.1.2 *The full picture is more complicated. Critical points of β are not necessarily Morse. Linearization of β is no longer acting on the tangent space to the critical point without kernel. As a result one has "marginal" local fields such that $\lambda_i = \Delta_i - d = 0$. The OPE can contain fractional powers.*

7.2 Physics

The deformation theory we are going to develop deals rather with OPE than with the quantum field theory itself. This means that a QFT is determined by correlators between fields. The latter depend not only on OPE but also on the boundary conditions at infinity. In this section we will ignore the behavior at infinity.

7.2.1 Renormalization

Renormalization have been discussed in a series of papers by Connes and Kreimer (see e.g. [CoKr]). Here we present a slightly different point of view, motivated by our approach to deformation theory of QFTs.

The OPE is determined by the behavior of the correlators if some of points collide. This means that the Fulton-Macpherson operad $FM(\mathbf{R}^d)$ will be of use. For example, let us consider the space $H \otimes \Omega_c^d(\mathbf{R}^d)$ of H-valued compactly supported differential forms of the top degree. For a choice $\phi_1, ..., \phi_n \in H \otimes \Omega^d_c(\mathbf{R}^d)$ we can define the universal partition function Z such that

$$Z(\varepsilon \sum_{1\le i\le n} \phi_i) = \sum_{n\ge 0} \frac{\varepsilon^n}{n!} \sum_{i_1,...,i_n} \int'_{(\mathbf{R}^d)^n} \langle \phi_{i_1} ... \phi_{i_n} \rangle,$$

where $\int'$ denotes a regularization of the integral at $x_i = x_j$, and $\langle .... \rangle$ denotes the correlator. Notice that since we take compactly supported forms on $\mathbf{R}^d$ the regularized integral converges.

From this formula we see that

$$\langle \phi_{i_1} ... \phi_{i_n} \rangle = \frac{\delta^n Z}{\delta\phi_{i_1} ... \delta\phi_{i_n}}.$$

Morally the formulas above should correspond to the deformation of a QFT by adding $\sum_i \phi_i$ to the lagrangian. The choice of the regularization is not fixed. Normally people take as the correlators for the deformed theory the coefficients for ε^0. We can take any linear functional $R : C^\infty(0, r) \to \mathbf{R}$ such that if f is continuous at zero then $R(f) = f(0)$. We call such a functional a *regularization operator*.

Assume that we have fixed a regularization operator. Then for a choice of fields $\phi \in \varepsilon\Omega^\bullet(H)[[\varepsilon]], \phi_i \in \Omega^\bullet(H), 1 \le i \le n$, choice of points $x_i \in \mathbf{R}^d, 1 \le i \le n$ we can introduce new correlators $\langle \phi_1(x_1)...\phi_n(x_n) \rangle_{\phi,R} \in C^\infty((\mathbf{R}^d)^n \backslash diag)[[\varepsilon]]$ such that

$$\langle \phi_1(x_1)...\phi_n(x_n) \rangle_{\phi,R} = \sum_{k\ge 0} \frac{1}{k!} R(\int d^k y \langle \prod_{1\le j\le k} \phi(y_j) \prod_{1\le i\le n} \phi_i(x_i) \rangle),$$

where the integration is taken over the subspace $|y_i - y_j| > \delta, |y_i - x_j| > \delta, |y_i| < \frac{1}{\delta}$, and $\delta > 0$ is some number.

Definition 7.2.1 *1) We say that the correlators $\langle ... \rangle_{\phi,R}$ define the deformed theory.*

2) Two deformed theories given by (ϕ, R) and (ϕ', R') are called equivalent if there exists $L \in End(H)[[\varepsilon]]$ that $L = id + o(1)$ and $\langle \prod_i \phi_i(x_i) \rangle_{\phi,R} = \langle \prod_i (L\phi_i)(x_i) \rangle_{\phi',R'}$ for any choice of fields ϕ_i and pairwise different points x_i.

Remark 7.2.2 *a) The field ϕ gives a tangent vector to the space of deformations of the given QFT. It can be taken from $H \otimes \wedge^d((\mathbf{R}^d)^*)$.*

b) The above definitions should be modified because $\mathbf{R}^d$ is non-compact. We can either assume that the space-time is compact or introduce a cut-off at infinity. For pedagogical reasons the reader can assume that instead of $\wedge^\bullet((\mathbf{R}^d)^*)$ we take the compactly supported differential forms on $\mathbf{R}^d$ (in this case we lose translation invariance).

c) The definition above corresponds to the intuitively clear picture: in order to deform a QFT we consider the space of all lagrangians $\mathcal{L}$ modulo $\partial_i\mathcal{L}, 1 \le$

$i \leq d$ plus a regularization procedure. A choice of an element in this space gives a QFT. (Morally a choice of $\mathcal{L}$ gives an affine structure on the space of QFTs). Then we need to identify all gauge equivalent theories. The moduli space of QFTs consists of the classes of gauge equivalent theories.

7.2.2 The operad responsible for the OPE

In order to explain the conjecture about L_∞-structure we need to discuss the role of Fulton-Macpherson operad $FM(\mathbf{R}^d)$ in OPE. Traditionally, when talking about OPE, people consider two colliding points in the space-time. More generally one can consider several points approaching to the same one with the same speed. It gives only one stratum in the Fulton-Macpherson compactification of the space-time. There are more strata, corresponding to groups of points colliding to the same one "with the same speed" (see [FM]). Hence the "true" picture for OPE should describe the asymptotic behavior of correlators near all the strata. Each operadic space $FM_n(\mathbf{R}^d)$ is a real manifold with corners. We assume that asymptotic expansion of a correlator at each corner is given by a series $r^\beta log^n r f(s), \beta \in \mathbf{Q}, n \in \mathbf{Z}_+$, where r is the distance to the corner, and $f(s)$ is a smooth function in all other variables. Such functions behave nicely with respect to the operadic composition in $FM(\mathbf{R}^d)$. The space of such functions on $FM_n(\mathbf{R}^d)$ will be denoted by V_n. We should also take care about two things:

a) filtration of H by dimensions;

b) action of the group of translations.

Let $\mathbf{C}_\varepsilon$ be the space of formal series $\sum_i c_i \varepsilon^{\beta_i}$ such that $c_i \in \mathbf{C}$ and $\beta_i \in \mathbf{R}, lim_{i \to +\infty} \beta_i = +\infty$. We define the cooperad $\mathcal{A} = (\mathcal{A}_n)_{n \geq 1}$ such that $\mathcal{A}_n^* = \mathbf{C}_\varepsilon \widehat{\otimes} V_{n+1} \widehat{\otimes} \mathbf{C}[[x]]$, where $\widehat{\otimes}$ means the completed tensor product and the formal series $\mathbf{C}[[x]]$ is the space dual to the algebra of differential operators with constant coefficients on $\mathbf{R}^d$ (it is responsible for the action of the group of translations on H). Notice that the spaces $\mathcal{A}_n^*$ are filtered by the degrees β_i and the powers of r at the corners. The operadic structure on $\mathcal{A}$ comes naturally from the operadic structure on $FM(\mathbf{R}^d)$.

Conjecture 7.2.3 *Let g be an L_∞-algebra which controls formal deformations of H as an $\mathcal{A}$-algebra. Then:*

a) There is a structure of L_∞-algebra on $\Omega^d(H) = H \otimes \wedge^d(\mathbf{R}^d)$.

b) There exists an L_∞-morphism $\Omega^d(H) \to g$.

c) There is a structure of d-algebra (see [KoSo2]) on $\Omega^\bullet(H)$.

In particular, it follows from c) that the cooperad of differential forms on the configuration space of d-dimensional discs in $(\mathbf{R}^d)^*$ acts on $\Omega^\bullet(H)$.

7.3 Some metric geometry

In this section we will introduce useful constructions and language for what follows.

Let $X = \{x_1, ..., x_n\}$ be a finite set, T be a planar tree with the tails parametrized by X, and $\{\lambda_v\}, 0 < \lambda_v < 1$ be the set of numbers parametrized by all but tail vertices of T.

Proposition 7.3.1 *There exists a metric ρ on X such that for any two different points x_i, x_j one has*

$$\rho(x_i, x_j) = C \prod_{v<i,j} \lambda_v,$$

where $C > 0$ depends on the set X only, and the notation $v < i$ means that the vertex v is closer to the root vertex with respect to the natural order on the vertices of T (so that the root vertex is the smallest element and tail vertices are maximal elements).

7.3.1 Clusters

Let X be a finite set, $|X| \geq 2$, equipped with a metric ρ. We will assign a tree $T = T_X$ to these data.

Definition 7.3.2 *Subset $Y \subset X$ is called cluster if Y contains at least two elements and for any points $a, b \in Y$ and $c \in X \setminus Y$ one has $\rho(a, b) < \rho(b, c)$.*

Definition 7.3.3 *Let us fix a positive number $\varepsilon < 1$. We say that Y is an ε-cluster if for any $c \in X \setminus Y$ one has $diam(Y) < \varepsilon\rho(c, Y)$, where $diam(Y)$ is the diameter of the set Y.*

Clearly any ε-cluster is a cluster. Clusters enjoy the following "non-archimedean" property.

Lemma 7.3.4 *If two clusters intersect non-trivially then one of them contains the other one.*

Proof. Let Y_1 and Y_2 be the clusters which intersect non-trivially. Then we can choose a common element c. Let $a_1 \in Y_1 \setminus Y_2$ and $a_2 \in Y_2 \setminus Y_1$. Then $\rho(a_1, c) < \rho(a_2, c)$, because $c \in Y_1$. Since $c \in Y_2$ we have $\rho(a_2, c) < \rho(a_1, c)$. Contradiction. The lemma is proved. ■

Having a set X with a metric ρ, as above, one can construct a tree T in the following way. Tails of T are parametrized by X. Internal vertices are clusters. Two internal vertices are connected by an edge if one cluster belongs to the other one. A tail vertex corresponding to $x \in X$ is connected to an internal vertex corresponding to a cluster Y if $x \in Y$.

Let us fix a set X, and consider all metrics on X such that $diam(X) = 1$. To every such a metric ρ we can assign the tree $T = T(X, \rho)$, as above.

The following proposition is easy to prove.

Proposition 7.3.5 *Let ρ be a metric on X, as above. One can assign numbers $\lambda_v, 0 < \lambda_v < 1$ to all internal vertices v of $T(X,\rho)$ in such a way that for any two points $x_i, x_j \in X$ one has*

$$C_1 < \frac{\rho(x_i, x_j)}{\prod_{v<i,j} \lambda_v} < C_2.$$

Here $C_i, i = 1, 2$ are positive numbers depending on X (not on the metric), and the notation $v < i$ means that there is a path along the edges of T which starts at the root vertex, ends at the tail vertex i and contains v (i.e. v is "closer" to the root than i).

Let us fix the tree T corresponding to a finite set X. By definition T has $|X| := n$ tail vertices. Let us fix numbers $\lambda_v, v \in V_i(T)$ such that $0 < \lambda_v < 1$, the metric ρ as in the Proposition 1, and real numbers $\Delta, \Delta_1, ..., \Delta_n$. For a given subset $S \subset \{1, ..., n\}$ we denote by B_S the subset $x_i, i \in S$.

Proposition 7.3.6 *The following formula holds:*

$$(diam\, X)^{-\Delta} \prod_{v \in V_i(T)} \lambda_v^{\sum_{v<j} \Delta_j} = C\, (diam\, X)^{-\Delta} \prod_{S \subset \{1,...,n\}, |S| \geq 2} (diam\, B_S)^{(-1)^{|S|+1} (\sum_{j \in S} \Delta_j)},$$

where C is a positive number depending on the set X (not on the metric).

Proof. Straightforward. ■

We will denote the LHS of the formula from the previous proposition by $R(x_1, ..., x_n, y)$ or by $R(x_1, ..., x_n, y; \Delta_1, ..., \Delta_n, \Delta)$. In this notation $x_1, ..., x_n \in X$ corresponds to the tail vertices of T and y corresponds to the root vertex. Although the point y is not an element of the set X, one can imagine that all points $x_1, ..., x_n, y$ belong to a metric space $\widehat{X} = X \cup \{y\}$ such that X is a metric subspace of $\widehat{X}$, and $\rho(y, x_i) = diam\, X$ for $1 \leq i \leq n$. The meaning of this notation will become clear later, when we discuss the operator product expansion (OPE). At this time we would like to make few comments about the meaning of the function R.

The function $R(x_1, ..., x_n, y)$ will play the following role in our considerations. Suppose we have local fields $\phi_1, ..., \phi_n$ sitting in $x_1, ..., x_n$ and having dimensions $\Delta_1, ..., \Delta_n$. When all points x_i approach to the same point y, we can write the OPE for the given fields:

$$\phi_1(x_1)...\phi_n(x_n) = \sum_\lambda C_\lambda(x_1, ..., x_n, y)\phi_\lambda(y),$$

where the coefficients $C_j(x_1, ..., x_n, y)$ depend on the configuration $\{x_1, ..., x_n, y\}$ as well as on the dimensions Δ_λ of the fields $\phi_\lambda(y)$.

Let us take a local field $\phi = \phi_\lambda$ of dimension Δ sitting at y such that ϕ appears in the OPE. The function $R(x_1, ..., x_n, y; \Delta_1, ..., \Delta_n, \Delta)$ will be responsible for the singular part of the OPE. The tree T appears because the points

$x_i, 1 \le i \le n$ can approach y with different speeds. Geometrically we have a structure of cluster on the set $\{x_1, ..., x_n\}$. The point y corresponds to the root of T.

We can describe the same picture in a slightly different way. The correlator $\langle \phi_1(x_1)...\phi_n(x_n)\rangle$ has different behavior near different strata of the Fulton-Macpherson compactification of $Conf_n(X)$, where X is the space-time. If the stratum D_T corresponds to a tree T, then there are scaling factors $\lambda_v, v \in V_i(T)$ such that the leading asymptotic term of the correlator behaves near D_T as the LHS of the formula from the proposition (up to a positive scalar). This will be one of the axioms (which can be checked in all known examples). Therefore the function R represents the leading singular asymptotic term of the correlator $\langle \phi_1(x_1)...\phi_n(x_n)\phi(y)\rangle$. The constant factor C depends on the dimensions $\Delta, \Delta_i, 1 \le i \le n$ as well as on the tree T. One can think of it as of function which is bounded on any compact subset of the stratum D_T.

Proposition 7.3.7 *The function R satisfies the following properties:*

1) If $n = 0, 1$ then $R = 1$.

2) If $n \ge 2$ then

$$R = C_1\,(diam\,X)^{-\Delta} \prod_{1\le i\le n} (min_{j\ne i, 1\le j\le n}\rho(x_i, x_j))^{-\Delta_i}$$
$$= C_2\,(diam\,X)^{-\Delta} min_{r_1,...,r_n} \prod_{1\le i\le n} r_i^{-\Delta_i}.$$

Here $C_i, i = 1, 2$ are positive constants, and the second product is taken over all disjoint union of balls $\sqcup_{1\le i\le n} B(x_i, r_i) \subset X$.

Proof. Notice that the original formula for the function R can be written as $R = (diam\,X)^{-\Delta} \prod_{1\le i\le n}(diam B_i)^{-\Delta_i}$, where B_i is the diameter of the minimal cluster containing the point x_i. This leads to a proof of the first equality. In order to prove the second one, one can take $r_i = \frac{1}{2}\rho_i, 1 \le i \le n$, where ρ_i is the distance to a point closest to x_i. ■

7.4 Operator product expansion

Let (X, ρ) be a metric space of finite diameter. It will play a role of the space-time. We assume that we are given a family $H_x, x \in X$ of vector spaces, each space is equipped with an increasing discrete filtration:

$$H_x = \bigcup_{\Delta\ge 0} H_x^{\le\Delta}.$$

We assume that each $H_x^{\le\Delta}$ is a finite-dimensional vector space, which carries a norm $|\bullet|_x$. We will call elements of $H_x^{\le\Delta}$ local fields at x of dimension less or equal than Δ. In what follows we will assume that the filtrations are locally constant (although this condition as well as many others can be relaxed).

Let $Conf_n(X), n \ge 0$ be the configuration space of X. By definition $Conf_n(X)$ consists of sequences of n pairwise distinct points of X, and $Conf_0(X) = \emptyset$.

Definition 7.4.1 *Let us fix $(x_1, ..., x_n) \in Conf_n(X), y \in X$ and non-negative numbers $\Delta_1, ..., \Delta_n, \Delta$.*

Operator product expansion (OPE) associated to these data is a class of equivalence of linear maps $m_n = m_n(x_1, ...x_n, y; \Delta_1, ..., \Delta_n, \Delta)$ such that

$$m_n : \otimes_{1 \leq i \leq n} H_{x_i}^{\leq \Delta_i} \to H_y^{\leq \Delta},$$

and $m_n, n \geq 1$ satify the axioms listed below. For $n = 0$ it is the equivalence class of linear maps $m_0 : \mathbf{C} \to H_y^{\leq \Delta}$.

The equivalence relation is the following one.

Definition 7.4.2 *Two maps m_n and m'_n as above are said to be equivalent if there exists $\varepsilon > 0$ such that one has the following inequality for the norm of the linear map:*

$||m_n - m'_n|| \leq C\, min\{r^{\Delta+\varepsilon} \prod_{1 \leq i \leq n} r_i^{-\Delta_i}\}$,

where $C > 0$ depends on the points $x_1, ..., x_n$ (not on the metric or dimensions), and the minimum is taken over all disjoint unions of balls $\sqcup_{i=1}^{i=n} B(x_i, r_i) \subset B(y, r)$.

Now we are going to list the axioms for $m_n, n \geq 0$ (or rather properties of these maps).

A1 (the norm inequality). One has:

$||m_n|| \leq C(\Delta_1, ..., \Delta_n, \Delta) R(x_1, ..., x_n, y)$, where in the LHS we take the norm of any representative from an equivalence class, and R is the function introduced in the previous section.

A2. All maps m_n from a given equivalence class are S_n-equivariant.

A3. Let $\Delta'_i \leq \Delta_i, 1 \leq i \leq n$ and $\Delta'' \geq \Delta$ be fixed. Then the linear maps m_n and $j_2 m'_n j_1$, where $j_l, l = 1, 2$ are natural embeddings associated with the filtrations, are equivalent in the sense of the definition above (they are considered as maps $\otimes_{1 \leq i \leq n} H_{x_i}^{\leq \Delta'_i} \to H_y^{\leq \Delta''}$).

A4. Let $n = 1$, $\Delta \geq \Delta_1$ and $x_1 = y$. Then the well-defined map $m_1 : H_y^{\leq \Delta_1} \to H_y^{\leq \Delta}$ coincides with the natural embedding induced by the filtration.

A5. (operadic composition) For every planar tree T with n tails, and any choice of representatives of m_k assigned to internal vertices v of T in such a way that k is the incoming valency of v, the corresponding composition map is equivalent to m_n (for any choice of a point in the configuration space and any choice of dimensions).

Remark 7.4.3 *1) It is easy to see that differences $m_n - m'_n$ of equivalent maps form a vector space. Hence the equivalence classes form a quotient vector space.*

2) Our definition of equivalent maps means that we do not consider the input to OPE of local fields of dimensions greater than Δ.

Question 7.4.4 *1) Is it true that m_1 automatically gives a D-module structure on the space of local fields H?*

2) Can one treat fractional powers in the asymptotic series for correlators in terms of decompositions at zero of certain curves $f(t)$ in the moduli space of QFTs?

3) What is the deformation theory for m_1? First order deformations are given by the elements of $H \otimes \Omega^d$.

8 Quantum spaces over metric-measure spaces

Let us consider the following category $\mathcal{C}$. Objects of $\mathcal{C}$ are sequences $M := ((\varepsilon_1, Y_1), ..., (\varepsilon_n, Y_n))$ where $\varepsilon_i, 1 \leq i \leq n$ are postive numbers and Y_i are compact metric spaces (as before, we will often denote by X the metric-measure space (X, d_X)) . Let $M' = ((\varepsilon_1', Y_1'), ..., (\varepsilon_n', Y_m'))$ be another object of $\mathcal{C}$. A morphism $f : M \to M'$ is defined by:

1) a sequence $(X, X_1^{\varepsilon_1}, ...X_n^{\varepsilon_n}, X_{n+1}^{\varepsilon_1'}, ..., X_{n+m}^{\varepsilon_m'})$, where X is a compact metric space, $X_k^{\varepsilon_k}, X_l^{\varepsilon_l'}$ are non-intersecting compact metric subspaces;

2) isometry embeddings $i_k : Y_k \to X, i_k' : Y_k' \to X$ such that

the closure of the ε_k-neighborhood of $i_k(Y_k)$ is equal to $X_k^{\varepsilon_k}$, and similarly for $i_k(Y_k')$ and $X_{k+n}^{\varepsilon_k'}, 1 \leq k \leq m$.

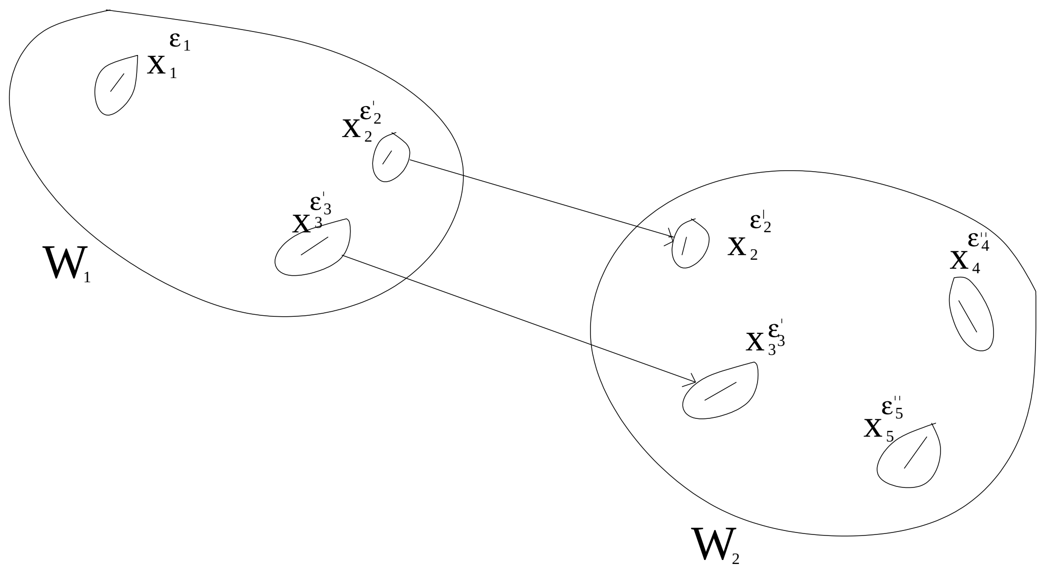
Let us call such a morphism a *metric bordism between M and M'*. Let us formally add the empty metric space to $\mathcal{C}$. Then the metric bordism between M and the empty space is a sequence of isometric embeddings i_k as above. Let us call the corresponding W the *metric collar* of M. Composition of morphisms is defined in the following way. Let W_1 and W_2 be two metric bordisms representing morphisms $f_1 : M \to M'$ and $f_2 : M' \to M''$. Then we can construct a metric bordism W between M and M' representing the composition $f_2 \circ f_1$ such as follows. As a topological space W is obtained from W_1 and W_2 by the gluing along canonically isometrically identified compact subsets $X_{k+n}^{\varepsilon_k'}, 1 \leq k \leq m$ which belong to both W_1 and W_2. Let j_k denotes this isometric identification. The distance between $w_1 \in W_1$ and $w_2 \in W_2$ is defined in the following way:

a) if both w_1 and w_2 belong to one of $X_{k+n}^{\varepsilon_k'}$ then $d_W(w_1, w_2)$ is the distance inside of $X_{k+n}^{\varepsilon_k'}$;

b) otherwise we define $d_W(w_1, w_2)$ as a minimum of the numbers $d_{W_1}(w_1, y) + d_{W_2}(j_k(y), w_2)$ where the minimum is taken over all points y belonging to the union of the subsets $X_{k+n}^{\varepsilon_k'}$.

One checks that the distance function d_W is symmetric and satisfies the triangle inequality.

FIGURE 2 (composition of metric bordisms)



The category $\mathcal{C}$ carries a symmetric monoidal structure with the tensor product given by the disjoint union of ordered sequences, e.g. $(\varepsilon_1, Y_1) \otimes (\varepsilon_2, Y_2) = ((\varepsilon_1, Y_1), (\varepsilon_2, Y_2))$.

Definition 8.0.5 *A quantum metric space is a monoidal functor $F : \mathcal{C}_X \to Hilb_{\mathbf{C}}$.*

Let W be a metric collar for an object M as above. Then W is also a metric collar for all $M_{\delta_1,...,\delta_n}$ where $\delta_j < \varepsilon_j, 1 \leq j \leq n$ and the metric spaces $Y_j, 1 \leq j \leq n$ are the same. We say that M and $M_{\delta_1,...,\delta_n}$ are equivalent in W. Suppose that all metric spaces above are in fact metric-measure spaces. Then we have the following version of the above category. For a fixed metric collar W of M we fix $\varepsilon_2, \varepsilon_3, ..., \varepsilon_n$ and let $\varepsilon := \varepsilon_1 \to 0$. Suppose that the measure $d\mu_X$ being restricted to X_1^ε admits an asymptoric expansion $d\mu_X = \varepsilon^{l_1} d\mu_{Y_1} + o(\varepsilon^{l_1})$, where $l_1 \geq 0$ and similarly for other Y_k. We obtain a sequence $(l_1, ..., l_n)$ of non-negative real numbers which we call exponents of M with respect to W. Then we define the category $\mathcal{C}^{mes}$ with objects which are equivalence classes as above, and in addition we assume that all Y_k, Y_j', X are metric-measure spaces. In the definition of a morphism we will require that the bordism between M and M' satisfies the above-mentioned property for the measures. One checks that in this way $\mathcal{C}^{mes}$ becomes a symmetric monoidal category. There is a natural monoidal functor $F : \mathcal{C}^{mes} \to Hilb_{\mathbf{C}}$ such that $F((\varepsilon_1, Y_1), ..., (\varepsilon_n, Y_n)) = \otimes_{1 \leq i \leq n} L_2(Y_i, d\mu_i)$.

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