Synthetic geometry of differential equations: I. Jets and comonad structure.

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Abstract

We give an abstract formulation of the formal theory partial differential equations (PDEs) in synthetic differential geometry, one that would seamlessly generalize the traditional theory to a range of enhanced contexts, such as super-geometry, higher (stacky) differential geometry, or even a combination of both. A motivation for such a level of generality is the eventual goal of solving the open problem of covariant geometric pre-quantization of locally variational field theories, which may include fermions and (higher) gauge fields.

A remarkable observation of Marvan [23] is that the jet bundle construction in ordinary differential geometry has the structure of a comonad, whose (Eilenberg-Moore) category of coalgebras is equivalent to Vinogradov's category of PDEs. We give a synthetic generalization of the jet bundle construction and exhibit it as the base change comonad along the unit of the "infinitesimal shape" functor, the differential geometric analog of Simpson's "de Rham shape" operation in algebraic geometry. This comonad structure coincides with Marvan's on ordinary manifolds. This suggests to consider PDE theory in the more general context of any topos equipped with an "infinitesimal shape" monad (a "differentially cohesive" topos). We give a new natural definition of a category of formally integrable PDEs at this level of generality and prove that it is always equivalent to the Eilenberg-Moore category over the synthetic jet comonad. When restricted to ordinary manifolds, Marvan's result shows that our definition of the category of PDEs coincides with Vinogradov's, meaning that it is a sensible generalization in the synthetic context.

Finally we observe that whenever the base space Σ is formally smooth, then the category of formally integrable PDEs with independent variables ranging in Σ is also equivalent simply to the slice category over $\Im\Sigma$. This yields in particular a convenient site presentation of the categories of PDEs in general contexts.

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1 Introduction and Summary

Local variational calculus—equivalently, local Lagrangian classical field theory—concerns the analysis of variational partial differential equations (PDEs), their symmetries and other properties. It is most usefully formulated in the language of jet bundles (e.g., [26, 1]). Much of the time, in this formalism, it is sufficient to work in local coordinates, which leads to very useful but quite intricate formulas. Unfortunately, this means that often certain technical aspects go ignored or unnoticed, which includes the consequences of the global topology of the spaces of either dependent or independent variables, as well as aspects of analysis on infinite dimensional jet bundles. Of course, in order to consider some generalizations of this framework (to, for instance, supermanifolds, differentiable spaces more singular than manifolds, or even the more general spaces of higher geometry), it is desirable to have a comprehensive discussion of these issues based on a precise and sufficiently flexible technical foundation.

In this work, we achieve this goal by giving a succinct and transparent formalization of jet bundles and PDEs (eventually also laying the ground for a formalization of local variational calculus) within a "convenient category" for differential geometry. A key example of such a category is Dubuc's "Cahiers topos" [6] (reviewed below in section 2.1), which was originally introduced as a well-adapted model of the Kock-Lawvere axioms for *synthetic differential geometry* [15].

Here "synthetic" means that the axioms formulate natural properties of objects (for instance that to every object X there is naturally associated an object TX that behaves like the tangent bundle of X), instead of prescribing the objects themselves. Synthetic axioms instead prescribe what the *category* of objects is to be like (for instance that it carries an endomorphism T with a natural transformation $T \to \mathrm{id}$). While our formulation of differential geometry is "synthetic" in this sense, we do not actually use the Kock-Lawvere axiom scheme, but another axiom called "differential cohesion" in [28], see proposition 2.14 and definition 2.15 below.

The Cahiers topos is the category of sheaves on formal Cartesian spaces (meaning $\mathbb{R}^n$'s, possibly "thickened" by infinitesimal directions). From the perspective of the foundations of what is called the *formal theory* of PDEs, having access to spaces with actual infinitesimal directions turns out to be of great practical and conceptual help. Traditionally, the formal (meaning infinitesimal) aspects of PDE theory have only been treated informally (meaning heuristically). But in the Cahiers topos, such heuristic arguments can actually be made precise. In a technical sense, this category is "convenient" because it fully-faithfully includes ordinary manifolds[1] but also ensures the existence of objects resulting from constructions with intersections, quotients and limits, which may be too singular or too infinite dimensional to correspond to ordinary manifolds. In particular the categories of infinite-dimensional Fréchet manifolds and of smooth locally pro-manifolds, LocProMfd (introduced in section 2.2 below), embed fully faithfully into the Cahiers topos, this we discuss in section 2.2. Locally pro-manifolds constitute a slight extension of the category of smooth manifolds that admits spaces locally modeled on projective limits of smooth manifolds. This class of spaces has been used extensively, though only semi-explicitly, as a minimalistic setting for discussing analytical aspects of infinite jet bundles [31, 25, 27, 23, 11]. By giving it a precise definition and by embedding it fully faithfully in the Cahiers topos, we finally provide a precise and flexible categorical framework for locally pro-manifolds.

Another "convenience," which will be of more use in a followup is the existence of moduli spaces (or more loosely *classifying spaces*) of differential form data as bona fide objects in this category.

In 3.2 we discuss how a differentially cohesive topos $\mathbf{H}$, such as the Cahiers topos, comes equipped with a natural and intrinsic notion of formal infinitesimal neighborhoods, which are formalized with the help of a monad functor $\Im\colon \mathbf{H} \to \mathbf{H}$ with the property that the following pullback diagram in $\mathbf{H}$ defines the formal

[1] This statement may require some interpretation for those not familiar with the use of sheaves in category theory. A manifold X may be characterized by its coordinate atlas, which technically forms a sheaf on the category of all Cartesian spaces $\mathbb{R}^n$, with $n = \dim X$, and local diffeomorphisms as arrows. The Cahier topos generalizes this correspondence by dropping the dimensionality condition on $\mathbb{R}^n$ and the smooth invertibility condition on arrows. In addition the Cahiers topos contains infinitesimal spaces.

infinitesimal neighborhood bundle $T^\infty\Sigma \to \Sigma$

$$\begin{tikzcd} T^\infty\Sigma \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \Sigma \arrow[d, "\eta_\Sigma"] \\ \Sigma \arrow[r, "\eta_\Sigma"'] & \Im\Sigma \end{tikzcd},$$

where $\eta_\Sigma \colon \Sigma \to \Im\Sigma$ the unit (natural transformation) of the monad. We may call $\Im\Sigma$ the "de Rham shape" (or "infinitesimal shape") of Σ, by analogy with a well-known construction in algebraic geometry [29]. We recall basics of (co-)monads in section B.3.

Since infinite-order jets are essentially functions on formal infinitesimal neighborhoods, we may synthetically define the construction

$$J^\infty_\Sigma \;:\; \mathbf{H}_{/\Sigma} \longrightarrow \mathbf{H}_{/\Sigma}$$

of jet bundles over some base manifold Σ as nothing but the *base change comonad* (definition B.25 below) along the unit of $\Im$, thus abstractly exhibiting its comonad structure. This is the content of section 3.3. (Here $\mathbf{H}_{/\Sigma}$ denotes the slice topos, recalled as definition B.12 below.)

Aspects of the theory of jet bundles have been formalized in synthetic differential geometry before, in [14], but only at finite jet order and without the comonad structure. It is important to note that for us the J^∞_Σ functor is then defined on any synthetic category $\mathbf{H}$, such as the Cahiers topos, which when suitably restricted to ordinary smooth bundles over Σ (spaces of dependent variables) yields the usual infinite jet bundles, with their canonical comonad structure coinciding with the explicit comonad structure previously observed in the work of Marvan [23] (presented in expanded form in [24]). This we prove as theorem 3.22 below.

The central result in [23] is that key aspects of the theory of differential operators and PDEs over Σ can be reinterpreted as certain well-known categorical constructions. For instance, given a comonad like J^∞_Σ acting on the category of bundles over Σ, we can consider the corresponding category of cofree coalgebras over it (the *Kleisli category* over the comonad, definition B.35 below), with the result being precisely the category of (possibly non-linear) partial differential operators over Σ:

$$\mathrm{DiffOp}_{\downarrow\Sigma}(\mathrm{LocProMfd}) \simeq \mathrm{Kl}(J^\infty_\Sigma|_{\mathrm{LocProMfd}_{\downarrow\Sigma}})\,.$$

This we discuss in section 3.4.

Moreover, dropping the cofreeness condition, the category of coalgebras over the jet comonad (the *Eilenberg-Moore category* over the comonad, definition B.30 below) is equivalent to Vinogradov's category of formally integrable partial differential equations (PDEs) [32, 33] with free variables ranging in Σ:

$$\mathrm{PDE}_{\downarrow\Sigma}(\mathrm{LocProMfd}) \simeq \mathrm{EM}(J^\infty_\Sigma|_{\mathrm{LocProMfd}_{\downarrow\Sigma}})\,.$$

In section 3.5 we prove a synthetic generalization of this result of [23]. First we consider a general definition of the category $\mathrm{PDE}_{/\Sigma}(\mathbf{H})$ of formally integrable PDEs in any topos $\mathbf{H}$ equipped with an infinitesimal shape operation $\Im$. This is definition 3.47 below. Then we show that generally there is an equivalence of categories

$$\mathrm{PDE}_{/\Sigma}(\mathbf{H}) \simeq \mathrm{EM}(J^\infty_\Sigma)\,.$$

Moreover for $\mathcal{C} \hookrightarrow \mathbf{H}_{/\Sigma}$ any full subcategory stable under forming jet bundles via J^∞_Σ, then the above restricts to an equivalence

$$\mathrm{PDE}(\mathcal{C}) \simeq \mathrm{EM}(J^\infty_\Sigma|_{\mathcal{C}})\,.$$

This is theorem 3.52 below. In the special case that $\mathcal{C} = \mathrm{LocProMfd}_{\downarrow\Sigma}$, $\mathrm{PDE}_{\downarrow\Sigma}(\mathrm{LocProMfd}) := \mathrm{PDE}(\mathcal{C})$ is the category of fibered locally pro-manifolds over Σ (definition 2.30 below). This reduces via [23] to an equivalence between our synthetic PDEs and Vinogradov's classical definition of PDEs (this is corollary 3.55 below, see remark 3.54).

An advantage of working with a synthetic formalization of PDEs that is worth mentioning is the ability to form products and equalizers (theorem 3.57), hence also arbitrary finite limits (corollary 3.58), in the resulting category. From a "big picture" point of view, having worked out the notion of a category of PDEs at this abstract level, this notion can now be extended in a mechanical way to the supergeometric version of the Cahiers topos [35], or to an ∞-topos of ∞-stacks, or even a combination of both [28].

Finally we observe in section 3.7 that whenever the unit $\Sigma \to \Im\Sigma$ of the infinitesimal shape operation on Σ is epimorphic, hence if Σ is formally smooth (Definition 3.1), which is true for all objects in the Cahiers topos (Proposition 2.18), then there is a further equivalence of categories

$$\mathrm{PDE}_{/\Sigma}(\mathbf{H}) \simeq \mathbf{H}_{/\Im\Sigma}$$

which identifies the category of formally integrable PDEs in $\mathbf{H}$ over Σ simply with the slice topos over the infinitesimal shape $\Im\Sigma$. This is theorem 3.60 below.

In algebraic geometry the linear version of this equivalence is familiar: linear differential equations over a scheme Σ incarnated as $\mathcal{D}$-modules over Σ are equivalently quasicoherent sheaves over the de Rham shape $\Im\Sigma$ (e.g. [20, above theorem 0.4], [7, sections 2.1.1 and 5.5]). Under this identification the algebro-geometric analog of the category $\mathrm{PDE}_{\downarrow\Sigma}(\mathrm{LocProMfd})$ of general (non-linear) PDEs in manifolds are the *$\mathcal{D}$-schemes* of [2, chapter 2.3].

The above equivalence of categories has various interesting implications. One is that it allows to exhibit PDEs in the generalized sense as sheaves over (and hence as colimits of) PDEs in the ordinary sense: There is an equivalence of categories

$$\mathrm{PDE}_{\Sigma}(\mathbf{H}) \;\simeq\; \mathrm{Sh}(\mathrm{PDE}_{\Sigma}(\mathrm{FormalLocProMfd}))\,.$$

This is theorem 3.63 below.

Particularly interesting sheaves on the ordinary category of PDEs are the variational bicomplex and the Euler-Lagrange complex of variational calculus [1, 26, 33]. With the above equivalence this means that the collection of all Euler-Lagrange complexes is secretly a single complex of generalized PDEs, in fact it turns out to be the *moduli stack* of variational calculus. This we will discuss in a followup.

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2 Spaces

We work in a "convenient category" for differential geometry, which faithfully contains smooth manifolds, but also contains more general smooth spaces. The idea is that even when we are interested only in smooth manifolds, then for some constructions it is convenient to be able to pass through this larger category that contains them.

The actual model we use is the category $\mathbf{H}$ of sheaves on the category of all formal manifolds [6, 28], containing the category $\mathbf{H}_{\Re}$ of sheaves on the category of ordinary manifolds, recalled below in section 2.1. This model serves to exhibit our constructions in section 3.3 as subsuming and generalizing traditional constructions in differential geometry.

But the only property of $\mathbf{H}$ that we actually use for the formal development of variational calculus below is that the inclusion $\mathbf{H}_{\Re} \hookrightarrow \mathbf{H}$ is exhibited by an idempotent monad $\Re$ ("reduction") which has two further right adjoint functors:

$$(\Re \dashv \Im \dashv \&) \;:\; \mathbf{H} \longrightarrow \mathbf{H}\,.$$

See proposition 2.14 and definition 2.15 below.

For technical aspects of the discussion of jet bundles in section 3.3, it is important that $\mathbf{H}_{\Re}$ (hence also $\mathbf{H}$) not only contains finite dimensional manifolds fully faithfully, but also infinite-dimensional Fréchet manifolds, in particular manifolds locally modeled on $\mathbb{R}^n$, for $n \in \mathbb{N} \cup \{\infty\}$. This we recall below in section 2.2.

2.1 Formal smooth sets

Here we discuss the category of "formal smooth sets," which is the category in which all our developments of variational calculus in section 3.3 take place. A "formal smooth set" is a sheaf on the category of Cartesian spaces (which can be interpreted as a generalized differentiable space in the sense of footnote 1) that may be equipped with an infinitesimal thickening.

The most immediate way to improve smooth manifolds to a "convenient category" (a topos) is to pass to sheaves over the category of all smooth manifolds. These sheaves we may think of as very general "smooth sets."

Definition 2.1. (a) Write

- SmthMfd for the category of finite-dimensional paracompact smooth (C^∞) manifolds with a countable set of connected components;
- CartSp $\hookrightarrow$ SmthMfd for its full subcategory on the Cartesian spaces $\mathbb{R}^n$, $n \in \mathbb{N}$;

We agree to say "manifold" for the objects of SmthMfd from now on.

(b) Regard SmthMfd and CartSp as sites (definition B.17), equipped with the Grothendieck pre-topology of (good) open covers, where "good" as usual means locally finite with contractible multiple intersections. Write

$$\mathrm{SmoothSet} := \mathrm{Sh}(\mathrm{SmthMfd}) \simeq \mathrm{Sh}(\mathrm{CartSp})$$

for the corresponding category of sheaves on smooth manifolds ("smooth sets").

We are to think of an object X in $\mathbf{H}_{\Re}$ as a generalized smooth space, defined not in terms of an underlying set of points, but defined entirely by a consistent declaration of the sets $X(U)$ of smooth maps "$U \to X$" out of smooth manifolds into the would-be space X.

Example 2.2. The Yoneda embedding $M \mapsto \mathrm{Hom}(-, M)$ constitutes a fully faithful inclusion

$$\mathrm{SmthMfd} \hookrightarrow \mathrm{SmoothSet}$$

of smooth manifolds into smooth sets.

Example 2.3. A *diffeological space* (or *Chen smooth space*) is a "concrete" smooth set, in the sense of concrete sheaves, i.e. a sheaf X on SmthMfd with the property that there exists a set X_s such that for each $U \in \mathrm{SmthMfd}$ there is a natural inclusion

$$X(U) \hookrightarrow \mathrm{Hom}_{\mathrm{Set}}(U_s, X_s)$$

of the set $X(U)$ (of functions declared to be smooth) into the set of all functions from the underlying set U_s of U into X_s.

A homomorphism of diffeological spaces is a morphism of the corresponding sheaves. Smooth manifolds form a full subcategory of diffeological spaces, which in turn form a full subcategory of smooth sets:

$$\mathrm{SmthMfd} \hookrightarrow \mathrm{DiffSp} \hookrightarrow \mathrm{SmoothSet}\,.$$

The intuition about the objects in SmoothSet as being smooth spaces defined by a rule for how to *locally* probe them by smoothly parameterized ways of mapping points into them is further supported by the following fact:

Proposition 2.4 ([28]). *For each $n \in \mathbb{N}$ consider the operation of forming the stalk of some sheaf $X \in$ $\mathrm{SmoothSet} := \mathrm{Sh}(\mathrm{CartSp})$ at the origin of $\mathbb{R}^n$*

$$n^* X := \varinjlim_{\delta > 0} X(B^n_\delta)\,.$$

Here $\delta \in \mathbb{R}_{>0}$ and $B^n_\delta \hookrightarrow \mathbb{R}^n$ denotes the ball of radius δ around the origin in $\mathbb{R}^n$, and the colimit is over the diagram of inclusions $B^n_{\delta_1} \hookrightarrow B^n_{\delta_2}$ for all $\delta_1 < \delta_2$. (Here we are implicitly using diffeomorphisms $B^n_\delta \simeq \mathbb{R}^n$ in order to evaluate X on B^n_δ.)

Then:

1. *the functor n^* is a* point of the topos SmoothSet *in the sense of definition B.21;*
2. *the set $\{n^*\}_{n\in\mathbb{N}}$ constitutes* enough points *for* SmoothSet *in the sense of definition B.21.*

In order to appreciate the following constructions, it is useful to recall the following fact:

Proposition 2.5 ([17, §35.8–10]). *The functor that sends a smooth manifold X to the $\mathbb{R}$-algebra of its smooth functions is fully faithful, hence exhibits a full subcategory inclusion*

$$C^\infty(-) \;:\; \mathrm{SmthMfd} \hookrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\mathrm{op}}$$

of smooth manifolds into the opposite of commutative $\mathbb{R}$-algebras.

(Notice that this statement is *not* restricted to compact manifolds.) We may hence generalize the category of smooth manifolds by expanding it inside the opposite of the category of commutative $\mathbb{R}$-algebras.

Definition 2.6. (a) Write

$$\mathrm{InfThPoints} \hookrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\mathrm{op}}$$

for category of "infinitesimally thickened points", being the full subcategory of the opposite of that of commutative $\mathbb{R}$-algebras, on those whose underlying $\mathbb{R}$-vector space is of the form

$$C^\infty(\mathbb{D}) := \mathbb{R} \oplus V\,,$$

where V is a finite dimensional nilpotent ideal, i.e. such that there exists $n \in \mathbb{N}$ with $V^n = 0$. Hence these are *local Artin $\mathbb{R}$-algebras*; in the literature on synthetic differential geometry they are called *Weil algebras*.

(b) The formal dual of such a Weil algebra we generically denote by $\mathbb{D}$ and think of it as an "infinitesimally thickened point". Write

$$\mathrm{Sh}(\mathrm{InfPoint})$$

for the category of presheaves on this category— these are sheaves with respect to the induced Grothendieck topology on InfPoint from that on FormalCartSp from Definition 2.11 since there are no non-trivial open covers of a point.

The key fact for discussion of these infinitesimally thickened points is this:

Proposition 2.7 (Hadamard's lemma). *For $f : \mathbb{R} \to \mathbb{R}$ a smooth function, then there exists a smooth function $g : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = f(0) + x \cdot g(x)\,.$$

It follows that $g(0) = f'(0)$ is the derivative of f at 0, and more generally that for every $k \in \mathbb{N}$ then the remainder of the partial Taylor expansion of f at 0 to order k is a smooth function $h : \mathbb{R} \to \mathbb{R}$:

$$f(x) = f(0) + f'(0)\cdot x + \tfrac{1}{2} f''(0)\cdot x^2 + \cdots + x^{k+1}\cdot h(x)\,.$$

Still more generally, this implies that for $f : \mathbb{R}^n \to \mathbb{R}$ a smooth function, then the remainder of any partial Taylor expansion in partial derivatives are smooth functions $h_{i_1\cdots i_{k+1}} : \mathbb{R}^n \to \mathbb{R}$:

$$f(\vec{x}) = f(0) + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(0)\cdot x^i + \tfrac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x^i \partial x^j}(0)\cdot x^i \cdot x^j + \cdots + \sum_{i_1,\cdots,i_{k+1}=1}^n x^{i_1}\cdots x^{i_{k+1}}\cdot h_{i_1\cdots i_{k+1}}(\vec{x})\,.$$

Definition 2.8. For $n \in \mathbb{N}$ write

$$\mathbb{D}^n(k) \in \mathrm{InfPoint}$$

for the infinitesimally thickened point whose corresponding algebra is a *jet algebra*, namely a quotient algebra of the form

$$C^\infty(\mathbb{R}^n)/(x^1, \cdots, x^n)^{k+1},$$

where $\{x^i\}_{i=1}^n$ are the canonical coordinates on $\mathbb{R}^n$. We call this the *standard infinitesimal n-disk of order k*.

The archetypical example is the following:

Example 2.9. Write $\mathbb{D}^1(1)$ for the infinitesimally thickened point (definition 2.6) corresponding to the "ring of dual numbers" over $\mathbb{R}$, i.e.

$$C^\infty(\mathbb{D}^1(1)) := \mathbb{R} \oplus \varepsilon\mathbb{R}$$

with $\varepsilon^2 = 0$. By Hadamard's lemma (proposition 2.7), this is equivalently the quotient of the $\mathbb{R}$-algebra of smooth functions on $\mathbb{R}$ by the ideal generated by x^2 (for $x \colon \mathbb{R} \to \mathbb{R}$ the canonical coordinate function):

$$C^\infty(\mathbb{D}^1(1)) \simeq C^\infty(\mathbb{R}^1)/(x^2).$$

Proposition 2.10 (e.g. [4, proposition 4.43])**.** *Every infinitesimally thickened point $\mathbb{D}$ (definition 2.6) embeds into an infinitesimal n-disk of some order k (example 2.8)*

$$\mathbb{D} \hookrightarrow \mathbb{D}^n(k).$$

Dually, every Weil algebra $\mathbb{R} \oplus V$ is a quotient of a jet algebra

$$\mathbb{R} \oplus V \simeq C^\infty(\mathbb{R}^n)/(x_1, \cdots x_n)^{k+1}$$

Definition 2.11. Write

$$\mathrm{FormalCartSp} \hookrightarrow \mathrm{CAlg}_{\mathbb{R}}^{\mathrm{op}}$$

for the full subcategory of the opposite of commutative $\mathbb{R}$-algebras which are Cartesian products $\mathbb{R}^n \times \mathbb{D}$ of a Cartesian space (via the embedding of proposition 2.5) with an infinitesimally thickened point (via the defining embedding of definition 2.6).

Explicitly, since the coproduct in $\mathrm{CAlg}_{\mathbb{R}}$ is the tensor product over $\mathbb{R}$, this is the full subcategory on those commutative $\mathbb{R}$-algebras of the form

$$C^\infty(\mathbb{R}^n \times \mathbb{D}) := C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} (\mathbb{R} \oplus V),$$

for $n \in \mathbb{N}$ and where V is a finite-dimensional nilpotent ideal.

We regard this as a site by taking the covering families those sets of morphisms

$$\left\{ U_i \times \mathbb{D} \xrightarrow{\phi_i \times \mathrm{id}_{\mathbb{D}}} \mathbb{R}^n \times \mathbb{D} \right\}_{i \in I} \quad \text{such that} \quad \left\{ U_i \xrightarrow{\phi_i} \mathbb{R}^n \right\}_{i \in I}$$

is a covering family in CartSp (i.e., a good open cover).

We write

$$\mathrm{FormalSmoothSet} := \mathrm{Sh}(\mathrm{FormalSmoothCartSp})$$

for the category of sheaves over this site ("formal smooth sets").

The category FormalSmoothSet in definition 2.11 was introduced in [6] as a well-adapted model of the Kock-Lawvere axioms [15, I.12] [16, 1.3] for synthetic differential geometry. It is sometimes referred to as the "Cahiers topos." While we do discuss a kind of "synthetic" axiomatization of differential geometry in FormalSmoothSet in section 3, we do not use the Kock-Lawvere axioms, but instead the fact that FormalSmoothSet satisfies the axioms of "differential cohesion" [28], a property recalled as proposition 2.14 below.

Example 2.12. By Hadamard's lemma (proposition 2.7), morphisms in FormalCartSp (definition 2.11) out of $\mathbb{D}^1(1)$ into any Cartesian space

$$\mathbb{D}^1(1) \longrightarrow \mathbb{R}^n$$

are in bijection with tangent vectors

$$\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i} \in T_x \mathbb{R}^n \, .$$

the corresponding algebra homomorphism

$$C^\infty(\mathbb{R}^n) \longrightarrow (\mathbb{R} \oplus \varepsilon \mathbb{R})$$

is given by

$$f \mapsto f(x) + \varepsilon \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(x) \, .$$

More generally, a morphism

$$\mathbb{D}^n(k) \longrightarrow \mathbb{R}^n$$

out of the order-k infinitesimal n-disk (definition 2.8) is equivalently the equivalence class of a smooth function

$$\mathbb{R}^n \longrightarrow \mathbb{R}^n \, ,$$

where two such functions are regarded as equivalent if they have the same partial derivatives at 0 up to order k. Such an equivalence class is called a "k-jet" of a smooth function.

The corresponding algebra homomorphism

$$C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)/(x^1, \cdots, x^n)^{k+1}$$

is given by restricting smooth functions to their partial derivatives up to order k around a given point. In particular there is an open neighbourhood $U \subset \mathbb{R}^n$ around that point such that functions supported on $\mathbb{R}^n \setminus U$ are sent to zero by this algebra homomorphism.

Proposition 2.13. *For $X \in$ SmthMfd $\hookrightarrow$ FormalSmoothSet a smooth manifold of dimension d, regarded as a formal smooth set (definition 2.11) and $\mathbb{D} \in$ FormalCartSp $\hookrightarrow$ FormalSmoothSet an infinitesimally thickened point (definition 2.6), then every morphism*

$$\mathbb{D} \longrightarrow X$$

in FormalSmoothSet *factors through an infinitesimal d-disk of order k (definition 2.8) as*

$$\mathbb{D} \longrightarrow \mathbb{D}^d(k) \longrightarrow X$$

for some k.

Proof. There exists a smooth embedding $X \hookrightarrow \mathbb{R}^n$, for some $n \in \mathbb{N}$. By proposition 2.10 the composite

$$\mathbb{D} \longrightarrow X \hookrightarrow \mathbb{R}^n$$

factors as

$$\mathbb{D} \longrightarrow \mathbb{D}^n(k) \longrightarrow \mathbb{R}^n \, .$$

Let $U \subset \mathbb{R}^n$ an open neighbourhood around the corresponding point in $\mathbb{R}^n$, and let U_X be its pre-image in X. We may find an open d-ball $\simeq \mathbb{R}^d \subset U_X$ around the given point. By example 2.12 the original morphism factors now as

$$\mathbb{D} \longrightarrow \mathbb{R}^d \hookrightarrow X \, .$$

Using again proposition 2.10, this now factors as

$$\mathbb{D} \longrightarrow \mathbb{D}^d(k) \longrightarrow X \, . \quad \square$$

Proposition 2.14 (differential cohesion of formal smooth sets)**.** *The canonical embedding of Cartesian spaces into formal Cartesian spaces (definition 2.11) is coreflective, i.e., the embedding functor i has a right adjoint functor p*

$$\begin{tikzcd} \mathrm{CartSp} \arrow[r, hook, shift left=2, "i"] \arrow[r, phantom, "\perp"] & \mathrm{FormalCartSp} \arrow[l, shift left=2, "p"] \end{tikzcd} .$$

Moreover, Kan extension (proposition B.9) along these functors induces a quadruple of adjoint functors (definition B.4) of the form

$$\begin{tikzcd} \mathrm{SmoothSet} \arrow[r, hook, shift left=4.5, "i_!"] \arrow[r, hook, shift right=1.5, "i_* \simeq p^*"] & \mathrm{FormalSmoothSet} \arrow[l, shift right=1.5, "i^* \simeq p_!"'] \arrow[l, shift left=4.5, "p_*"] \end{tikzcd} .$$

(where each functor on top is left adjoint to the functor below) between smooth sets (definition 2.1) and formal smooth sets (definition 2.11).

Proof. For the first statement, observe that in FormalCartSp every morphism out of a finite manifold into an infinitesimally thickened point

$$\mathbb{R}^n \longrightarrow \mathbb{D}$$

necessarily factors through the actual point

$$\mathbb{R}^n \xrightarrow{\exists !} * \xrightarrow{\exists !} \mathbb{D} .$$

This is because, dually, an algebra homomorphism of the form

$$C^\infty(\mathbb{R}^n) \longleftarrow (\mathbb{R} \oplus V)$$

has to vanish on the nilpotent ideal V (since respect for the product implies that nilpotent algebra elements are sent to nilpotent algebra elements, but there are no non-zero nilpotent elements in $C^\infty(\mathbb{R}^n)$). This means that there is a natural bijection between morphisms in FormalCartSp of the form

$$\mathbb{R}^{n_1} \longrightarrow \mathbb{R}^{n_2} \times \mathbb{D}$$

and morphisms in CartSp $\hookrightarrow$ FormalCartSp of the form

$$\mathbb{R}^{n_1} \longrightarrow \mathbb{R}^{n_2} .$$

This in turn means that if a right adjoint p to the inclusion exists then it is given by projection on the non-thickened part (the reduced part)

$$p(\mathbb{R}^n \times \mathbb{D}) \simeq \mathbb{R}^n .$$

It only remains to observe that this is indeed functorial.

With this, the adjoint quadruple between categories of sheaves follows directly from the corresponding system of adjunctions on presheaves (proposition B.9) because the site FormalCartSp is by definition such that the Grothendieck topology along the infinitesimal directions is trivial. □

The synthetic theory that we develop below will be based entirely on the existence of an adjoint quadruple as in proposition 2.14. Therefore, while Dubuc's *Cahiers topos* of formal smooth sets is the archetypical example of an ambient category that we consider, we consider this structure in more generality:

Definition 2.15 (differential cohesion [28])**.** We say that a full inclusion of toposes

$$i_! : \mathbf{H}_{\Re} \hookrightarrow \mathbf{H}$$

exhibits $\mathbf{H}$ as being *differentially cohesive* over $\mathbf{H}_{\Re}$ (and exhibits $\mathbf{H}_{\Re}$ as being the full subcategory of *reduced objects* of $\mathbf{H}$) if the inclusion has three adjoint functors to the right

$$\begin{tikzcd} \mathbf{H}_{\Re} \arrow[r, hook, shift left=3, "i_!"] \arrow[r, hook, shift right=1, "i_*"] & \mathbf{H} \arrow[l, shift right=1, "i^*"'] \arrow[l, shift left=3, "i_!"] \end{tikzcd} .$$

We then write

$$(\Re \dashv \Im \dashv \text{Et}) \;:=\; (i_! \circ i^* \;\dashv i_* \circ i^* \;\dashv i_* \circ i_!) \;:\; \mathbf{H} \longrightarrow \mathbf{H}$$

for the triple of adjoint idempotent (co-)monads on formal smooth set induced (via example B.33) by this adjoint quadruple. We pronounce $\Re$ "reduction" (due to proposition 2.16 below) and $\Im$ "de Rham shape" [29] or "infinitesimal shape" [28].

So in this terminology proposition 2.14 says that the topos $\mathbf{H} = \text{FormalSmoothSet}$ is differentially cohesive over the full subtopos $\mathbf{H}_{\Re} = \text{SmoothSet}$.

Proposition 2.16. *For $\mathbb{R}^n \times \mathbb{D} \in \text{FormalCartSp} \hookrightarrow \text{FormalSmoothSet}$ we have*

$$\Re(\mathbb{R}^n \times \mathbb{D}) \simeq \mathbb{R}^n$$

and the counit (definition B.28) of the comonad $\Re$ (definition 2.15) is the canonical inclusion

$$\epsilon_{\mathbb{R}^n \times \mathbb{D}} : \Re(\mathbb{R}^n \times \mathbb{D}) \simeq \mathbb{R}^n \overset{(\text{id},0)}{\longrightarrow} \mathbb{R}^n \times \mathbb{D}\,.$$

It follows that for $X \in \mathbf{H} = \text{Sh}(\text{FormalCartSp})$ a sheaf, then $\Im X \in \text{Sh}(\text{FormalCartSp})$ is the sheaf given by

$$\Im X \;:\; \mathbb{R}^n \times \mathbb{D} \mapsto X(\mathbb{R}^n)\,.$$

and that the unit (definition B.28)

$$\eta_X \;:\; X \longrightarrow \Im X$$

of the monad $\Im$ (definition 2.15) is the morphism of sheaves which over any $\mathbb{R}^n \times \mathbb{D} \in \text{FormalCartSp}$ is given by the function

$$\text{Hom}_{\mathbf{H}}(\mathbb{R}^n \times \mathbb{D}, X) \longrightarrow \text{Hom}_{\mathbf{H}}(\mathbb{R}^n \times \mathbb{D}, \Im X) \simeq \text{Hom}_{\mathbf{H}}(\Re(\mathbb{R}^n \times \mathbb{D}), X)\,,$$

given by precomposition with ϵ,

$$(\mathbb{R}^n \times \mathbb{D} \overset{\phi}{\longrightarrow} X) \;\mapsto\; \left(\Re(\mathbb{R}^n \times \mathbb{D}) \overset{\epsilon}{\longrightarrow} \mathbb{R}^n \times \mathbb{D} \overset{\phi}{\longrightarrow} X\right)\,.$$

Proof. Using that left Kan extension $i_!$ along a functor i is on representables given by that functor (proposition B.9), we have, for all $\mathbb{R}^{n_1} \times \mathbb{D} \in \text{FormalCartSp}$ and all $\mathbb{R}^{n_2} \in \text{CartSp} \hookrightarrow \text{FormalCartSp}$, the natural isomorphisms

$$\begin{aligned} \text{Hom}_{\mathbf{H}_{\Re}}(\mathbb{R}^{n_2}, i^*(\mathbb{R}^{n_1} \times \mathbb{D})) &\simeq \text{Hom}_{\mathbf{H}}(i_!(\mathbb{R}^{n_2}), \mathbb{R}^{n_1} \times \mathbb{D}) \\ &\simeq \text{Hom}_{\text{FormalCartSp}}(\mathbb{R}^{n_2}, \mathbb{R}^{n_1} \times \mathbb{D}) \\ &\simeq \text{Hom}_{\text{FormalCartSp}}(\mathbb{R}^{n_2}, \mathbb{R}^{n_1})\,, \end{aligned}$$

where in the last line we used proposition 2.14. This shows that

$$i^*(\mathbb{R}^{n_1} \times \mathbb{D}) \simeq \mathbb{R}^{n_1} \qquad \in \mathbf{H}_{\Re}\,.$$

Use again that left Kan extension $i_!$ preserves representables to find that

$$\Re(\mathbb{R}^{n_1} \times \mathbb{D}) := i_! i^*(\mathbb{R}^{n_1} \times \mathbb{D}) \simeq i_!(\mathbb{R}^{n_1}) \simeq \mathbb{R}^{n_1}\,.$$

Now, the counit $\epsilon = \epsilon_{\mathbb{R}^{n_2}\times\mathbb{D}} : \Re(\mathbb{R}^{n_2} \times \mathbb{D}) \to \mathbb{R}^{n_2} \times \mathbb{D}$ is characterized by the identity $g = \epsilon_Y \circ i_!(f)$ with respect to an arbitrary morphism $f\colon X \to i^*(Y)$ and its adjoint $g\colon i_!(X) \to Y$. Setting $Y = \mathbb{R}^{n_2} \times \mathbb{D}$ and $X = i^*(\mathbb{R}^{n_1} \times \mathbb{D}')$, and recalling also that $i_!(f) = f$, this means that the formula $g = \epsilon \circ f$ must give a bijection between morphisms of the form

$$\mathbb{R}^{n_1} \simeq i^*(\mathbb{R}^{n_1} \times \mathbb{D}') \xrightarrow{f} i^*(\mathbb{R}^{n_2} \times \mathbb{D}) \simeq \mathbb{R}^{n_2} \quad \text{and} \quad \mathbb{R}^{n_1} \simeq i_! i^*(\mathbb{R}^{n_1} \times \mathbb{D}') \simeq \Re(\mathbb{R}^{n_1} \times \mathbb{D}') \xrightarrow{g} \mathbb{R}^{n_2} \times \mathbb{D}\,.$$

From the proof of proposition 2.14, we know that g uniquely factors as

$$g\colon \mathbb{R}^{n_1} \xrightarrow{g'} \mathbb{R}^{n_2} \xrightarrow{(\mathrm{id},0)} \mathbb{R}^{n_2} \times \mathbb{D}\,.$$

Hence, under the hypothesis that $g' = f$, we conclude that we can identify $\epsilon = (\mathrm{id}, 0)$. It remains only to verify this hypothesis. First, note that when $\mathbb{D} = \{*\}$, we simply have $g \simeq f$ (the adjunction is unique, if it exists, and this satisfies all the desired properties). Second, recalling the naturality of the adjunction, we find that, with $p_2\colon \mathbb{R}^{n_2} \times \mathbb{D} \to \mathbb{R}^{n_2}$ the projection onto the second factor, the adjunct of $i^*(p_2) \circ f = f$ is $p_2 \circ g = g'$, from which we can conclude that $g' = f$.

Finally, since $\Re \dashv \Im$ and the unit η_X is correspondingly adjunct to the counit ϵ_X, it is straight forward to see that the sheaf morphism representing η_X is given by the formulas in the statement of the proposition. □

Example 2.17. The reduction counit on $\mathbb{R}^n \times \mathbb{D}$ (proposition 2.16) has in fact a retraction:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "(\mathrm{id},0)"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathbb{R}^n \times \mathbb{D} \arrow[r] & \mathbb{R}^n \end{tikzcd}$$

given by the product with $\mathbb{R}^n$ applied to the basic retraction

$$\begin{tikzcd} * \arrow[r, "\exists!"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathbb{D} \arrow[r, "\exists!"] & * \end{tikzcd}\,.$$

Dually this is the retraction of commutative $\mathbb{R}$-algebras

$$\begin{tikzcd} C^\infty(\mathbb{R}^n) & C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} C^\infty(\mathbb{D}) \arrow[l, "\text{reduction}"'] & C^\infty(\mathbb{R}^n) \arrow[l, "f \mapsto (f,0)"'] \arrow[ll, bend left, "\mathrm{id}"] \end{tikzcd}\,,$$

where the left morphism "reduction" sends every nilpotent algebra element to 0 and is the identity on the remaining elements.

From this the statements about $\Im$ follow by the adjunction $(\Re \dashv \Im)$.

Proposition 2.18. *For every $X \in$ FormalSmoothSet (definition 2.11), the unit $\eta_X : X \longrightarrow \Im X$ of the infinitesimal shape monad from definition 2.15 is an epimorphism in* FormalSmoothSet*, i.e. all objects of* FormalSmoothSet *are formally smooth in the sense of Definition 3.1.*

Proof. By proposition B.22 it is sufficient to see that for $\mathbb{R}^n \times \mathbb{D} \in \mathrm{FormalCartSp} \hookrightarrow \mathbf{H}$ any representable, then the induced function on hom-sets

$$\mathrm{Hom}_{\mathbf{H}}(\mathbb{R}^n \times \mathbb{D}, X) \longrightarrow \mathrm{Hom}_{\mathbf{H}}(\mathbb{R}^n \times \mathbb{D}, \Im X)$$

is a surjection. By proposition 2.16 this is the case precisely if every morphism in $\mathbf{H}$ of the form

$$\mathbb{R}^n \longrightarrow X$$

factors as

$$\mathbb{R}^n \xrightarrow{(\mathrm{id},0)} \mathbb{R}^n \times \mathbb{D} \longrightarrow X\,.$$

This follows from example 2.17. □

2.2 Locally pro-manifolds

We consider here a category of "locally pro-manifolds" (definition 2.30 below), by which we mean projective limits of finite-dimensional smooth manifolds, formed inside the category of Fréchet manifolds.

Fréchet manifolds are smooth manifolds of possibly infinite-dimension, which are however only "mildly infinite-dimensional" in that they arise as projective limits of Banach manifolds. They still embed fully faithfully into the category SmoothSet of smooth sets (proposition 2.22 below), hence also into the category FormalSmoothSet of formal smooth sets from the previous section. The key point is that projective limits of finite dimensional smooth manifolds do exist in the category of Fréchet manifolds (proposition 2.26 below). Accordingly, infinite jet bundles of smooth manifolds (definition 3.20 below) may naturally be regarded as Fréchet manifolds (a point of view advocated in [27, 25]), in fact they are in the more restrictive class of $\mathbb{R}^\infty$-manifolds (definition 2.24 below).

Beware that sometimes jet bundles are instead regarded as pro-objects in the category of finite dimensional smooth manifolds [11], hence as purely formal projective limits. This perspective is not equivalent: While a smooth function on a pro-manifold is, by definition, globally a function on one of the finite stages in the projective system, a smooth function on the corresponding projective limit formed in Fréchet manifolds is in general only *locally* a function on one of the finite stages (proposition 2.29 below). This is why we speak of "locally pro-manifolds". Notice that while the smooth functions are not the same in both cases, the cohomology of the complexes of the corresponding differential forms may still agree, see [8] or [5, appendix to Chapter 2].

The calculus of smooth functions on Fréchet manifolds is a special case of a calculus that may be defined more generally on manifolds locally modeled on locally convex topological vector spaces (see [25] for details, in particular Chapter 8 for the precise definition of smoothness).

Definition 2.19 ([25, §8.1–3], [27, Definition 7.1.5])**.** Let $f\colon X \to Y$ be a continuous map (not necessarily linear) between two Fréchet spaces X and Y. The *Gâteaux derivative* $Df\colon U \times X \to Y$ of f on an open $U \subset X$ is defined as the directional derivative

$$Df(x,w) := D_w f(x) = \lim_{t\to 0} \frac{f(x+tw)-f(x)}{t}\,,$$

with the limit taken in the topology of Y. The map f is smooth at $x \in X$ when there exists an open neighborhood $U \ni x$ such that the iterated Gâteaux derivatives $(u, v_1, \ldots, v_k) \mapsto D^k_{v_1,\ldots,v_k} f(u)$, defined in the obvious way, exist and are jointly continuous

$$D^k f\colon U \times \underbrace{X \times \cdots \times X}_{k \text{ times}} \to Y$$

for each $k \geq 0$.

This notion of smoothness is at the very least compatible with the fundamental theorem of calculus.

Proposition 2.20 ([25, §8.4])**.** *Let $f\colon X \to Y$ be a map between Fréchet spaces, with f smooth on an open convex subset $U \subset X$. Then, for any $u, v \in U$,*

$$f(v) - f(u) = \int_0^1 \mathrm{d}t\, D_{v-u} f(u + t(v-u)),$$

using the Bochner integral (Riemann sums convergent in the topology of Y).

Usually the Bochner integral is defined on a measure space and is valued in a Banach space. Since we are only considering integration over the interval $[0,1]$ and only of continuous functions, there is no difference between the Riemann and Lebesgue integrals. On a Banach space the Bochner integral is defined by convergence with respect to a norm, which is equivalent to convergence in the topology of the Banach space (as opposed to to a weak topology, for example). Though Fréchet spaces are not normed, we can still define the Bochner integral by convergence in the topology of the Fréchet space.

Definition 2.21. Let FrMfd be the category of topological spaces endowed with atlases of charts onto open subsets of Fréchet vector spaces, with continuous maps that are smooth in local charts in the above sense as morphisms.

There is the canonical inclusion

$$\mathrm{SmthMfd} \hookrightarrow \mathrm{FrMfd}$$

of finite dimensional smooth manifolds (definition 2.1) into Fréchet manifolds (definition 2.21).

Proposition 2.22. *Let $i : \mathrm{SmthMfd} \hookrightarrow \mathrm{FrMfd}$ be the canonical inclusion of finite-dimensional smooth manifolds (definition 2.1) into Fréchet manifolds (definition 2.21). Then the functor*

$$i_{\mathrm{Sh}} \;:\; \mathrm{FrMfd} \xrightarrow{y_{\mathrm{FrMfd}}} \mathrm{PSh}(\mathrm{FrMfd}) \xrightarrow{i^*} \mathrm{PSh}(\mathrm{SmthMfd}) \xrightarrow{L_{\mathrm{SmthMfd}}} \mathrm{Sh}(\mathrm{SmthMfd}) = \mathrm{SmoothSet}\;,$$
$$i_{\mathrm{Sh}}(X) : U \mapsto \mathrm{Hom}_{\mathrm{FrMfd}}(i(U), X)\,,$$

where $y_{(-)} : (-) \to \mathrm{PSh}(-)$ is the Yoneda embedding and $L_{(-)} : \mathrm{PSh}(-) \to \mathrm{Sh}(-)$ is the sheafification functor, is fully faithful, hence exhibits Fréchet manifolds as a full subcategory of smooth sets (definition 2.1).

Proof. The functor factors evidently through the full subcategory of concrete sheaves on smooth manifolds, which is the category of diffeological spaces from example 2.3. It is hence sufficient to see that the factorization

$$\mathrm{FrMfd} \longrightarrow \mathrm{DiffSp}$$

is fully faithful. This is the content of [19, theorem 3.1.1]. □

Proposition 2.23. *There exists a subcanonical (definition B.20) Grothendieck topology on* FrMfd *such that its sheaf topos agrees with that over* SmthMfd*:*

$$\mathrm{Sh}(\mathrm{FrMfd}) \simeq \mathrm{Sh}(\mathrm{SmthMfd})\,.$$

Proof. This is the result of applying proposition 2.22 in proposition B.27. □

Definition 2.24. Write

$$\mathbb{R}^\infty \in \mathrm{FrMfd}$$

for the Fréchet manifold whose underlying topological vector space is the *projective* limit over the sequence of the standard projections $\cdots \to \mathbb{R}^2 \to \mathbb{R}^1 \to \mathbb{R}^0$, equipped with the family of seminorms given for each $n \in \mathbb{N}$ by the composites

$$\| - \|_n \;:\; \mathbb{R}^\infty \xrightarrow{p_n} \mathbb{R}^n \xrightarrow{\|-\|} \mathbb{R}$$

of the standard projection to $\mathbb{R}^n$ followed by the standard norm on $\mathbb{R}^n$.

Remark 2.25. The definition 2.24 of $\mathbb{R}^\infty$ says that for each $x \in \mathbb{R}^n$ the open balls

$$B^n_\delta(x) := \{ y \in \mathbb{R}^\infty \mid \|y - x\|_n < \delta \} \qquad \text{for } \delta > 0, n \in \mathbb{N}$$

form a base of neighborhoods of x for the topology on $\mathbb{R}^\infty$. For fixed n these are of course the preimages under $p_n : \mathbb{R}^\infty \to \mathbb{R}^n$ of the opens of the standard base of neighborhoods of $\mathbb{R}^n$, hence the open balls $B^n_\delta(x)$ induce the coarsest topology on $\mathbb{R}^\infty$ such that all the p_n are continuous functions. This exhibits the underlying topological space of $\mathbb{R}^\infty$ as the projective limit in topological spaces over the system of topological spaces $(\cdots \to \mathbb{R}^2 \to \mathbb{R}^1 \to \mathbb{R}^0)$.

Proposition 2.26. *The Fréchet manifold $\mathbb{R}^\infty$ from definition 2.24 is the projective limit, formed in the category* FrMfd *of Fréchet manifolds, of the projective system $(\cdots \to \mathbb{R}^2 \to \mathbb{R}^1 \to \mathbb{R}^0)$ of finite dimensional manifolds, regarded as a system in* FrMfd*.*

Proof. We need to check that for any $X \in \mathrm{FrMfd}$ and a sequence $\{X \xrightarrow{f_n} \mathbb{R}^n\}$ of compatible morphisms, there is a unique morphism $f\colon X \to \mathbb{R}^\infty$ that commutes with the f_n and the projections $X \to \mathbb{R}^n$. By remark 2.25 we already know that the underlying topological space of $\mathbb{R}^\infty$ is the projective limit of the projective sequence of topological spaces spaces $(\cdots \to \mathbb{R}^2 \to \mathbb{R}^1 \to \mathbb{R}^0)$. So all of our conditions are already satisfied, though at this point we can only conclude that f is continuous. Hence what remains to be shown is that our f is smooth precisely when all the corresponding f_n are. This is easily done by repeating the above argument for the iterated derivatives $D^k f_n$. The details can be found in [27, Lemma 7.1.8]. □

Example 2.27. It follows from proposition 2.26 that for $y(\mathbb{R})^n \simeq i_{\mathrm{Sh}}(\mathbb{R}^n) \in \mathrm{Sh}(\mathrm{SmthMfd})$ the Cartesian spaces regarded as smooth sets (via Yoneda embedding, hence equivalently via the embedding of proposition 2.22) then their projective limit as smooth sets is represented (via proposition 2.22) by their projective limit as Fréchet manifolds

$$\underset{\longleftarrow n}{\lim}\, i_{\mathrm{Sh}}(\mathbb{R}^n) \simeq i_{\mathrm{Sh}}(\mathbb{R}^\infty)\,.$$

Moreover, the full faithfulness of the embedding due to proposition 2.22 says that morphisms of smooth sets of the form

$$\underset{\longleftarrow n}{\lim}\, i_{\mathrm{Sh}}(\mathbb{R}^n) \simeq i_{\mathrm{Sh}}(\mathbb{R}^\infty) \longrightarrow i_{\mathrm{Sh}}(\mathbb{R})$$

are in natural bijection with morphisms of Fréchet manifolds of the form

$$\mathbb{R}^\infty \longrightarrow \mathbb{R}\,.$$

Definition 2.28 ([31])**.** A function (of underlying sets) on $\mathbb{R}^\infty$ (definition 2.24)

$$f : \mathbb{R}^\infty \longrightarrow \mathbb{R}$$

we call *smooth and locally of finite order* if for every point $x \in \mathbb{R}^\infty$ there is an open neighbourhood $U_k \subset \mathbb{R}^k$ of $p_k(x)$ and a smooth function $f_k : U_k \to \mathbb{R}$ which determines the restriction of f to the pre-image $U = p_k^{-1}(U)$:

$$f|_U = f_k \circ p_k\,.$$

Proposition 2.29 ([25, §9.5.9])**.** *Morphisms of Fréchet manifolds of the form*

$$f : \mathbb{R}^\infty \longrightarrow \mathbb{R}$$

are precisely the functions of underlying sets that are smooth and locally of finite order *in the sense of definition 2.28.*

Proof. If f is smooth and locally of finite order, then it is straightforward to check that it is smooth also in the sense of definition 2.19. On the other hand, if we already know that f is locally of finite order, Fréchet smoothness directly implies the smoothness of each of its finite order restrictions. Thus, it remains only to show that a Fréchet smooth function is locally of finite order.

This result eventually follows from the joint continuity of the derivative $(u, v) \mapsto D_v f(u)$ as a map $Df\colon U \times \mathbb{R}^\infty \to \mathbb{R}$, which turns out to be a rather strong requirement.

For any point $x \in \mathbb{R}^\infty$, by linearity in the second argument, $D_0 f(x) = 0$. By joint continuity, the preimage $(Df)^{-1}(-1, 1)$ is open and contains every point $(x, 0) \in \mathbb{R}^\infty \times \mathbb{R}^\infty$ with $x \in X$. The product topology implies the existence of a product neighborhood $U_x \times V_x \ni (x, 0)$ for any $x \in \mathbb{R}^\infty$, such that $U_x \times V_x \subset Df^{-1}(-1, 1)$. The projective limit topology implies that there exists a positive integer k such that we can choose $U_x = U_x^k \times W$ and $V = V_x^k \times W$, where $U_x^k, V_x^k \subset \mathbb{R}^k$ are open and convex and $W = \ker(\mathbb{R}^\infty \to \mathbb{R}^k)$. The order k could be different for U_x and V_x, but we can just take the maximum of the two. And, of course, k depends both on x and on the choice of the product neighborhood $U_x \times V_x$.

With $w \in W$, $tw \in W$ for any $t \in \mathbb{R}$. But then for any $u \in U_x$, by construction and linearity,

$$|D_{tw} f(u)| = |t||D_w f(u)| \leq 1.$$

Since $|t|$ could be arbitrarily large, this means that $D_w f(u) = 0$ for any $w \in W$. Thus, k gives us a local uniform upper bound on the number of independent non-vanishing partial derivatives of f in U_x. To conclude the proof, we use the fundamental theorem of calculus (Proposition 2.20) to show that k is also an upper bound on the number of coordinates on which f depends in a non-trivial way in U_x. Namely, any $u \in U_x = U_x^k \times W$ has the form $u = (u_0, w)$, where $u_0 \in U_x^k$ and $w \in W$, so that

$$f(u_0, w) - f(u_0, 0) = \int_0^1 \mathrm{d}t \, D_w f(u_0, tw) = 0.$$

Hence $f|_{U_x}$ is constant along the fibers of $U_x \to U_x^k$ and thus factors through that projection. Since x was arbitrary, f is everywhere locally of finite order, which completes the proof. □

We will be considering the following full subcategory of "locally pro-manifolds" (see remark 2.32 below) inside Fréchet manifolds (this is essentially the unnamed definition in [23, section 1]):

Definition 2.30. Write

$$\mathrm{LocProMfd} \hookrightarrow \mathrm{FrMfd}$$

for the full subcategory of that of Fréchet manifolds (definition 2.21) on those Fréchet manifolds which are n-dimensional for $n \in \mathbb{N}$ (hence "$\mathbb{R}^n$-manifolds") or which are $\mathbb{R}^\infty$-manifolds (definition 2.24).

Remark 2.31. Hence by proposition 2.22 there is a sequence of full inclusion

$$\mathrm{SmthMfd} \hookrightarrow \mathrm{LocProMfd} \hookrightarrow \mathrm{FrMfd} \hookrightarrow \mathrm{SmoothSet} \hookrightarrow \mathrm{FormalSmoothSet}\,.$$

Since full inclusions reflect limits, it is still true that $\mathbb{R}^\infty$ is the projective limit over the $\mathbb{R}^n$-s when formed in LocProMfd.

Remark 2.32. The terminology in definition 2.30 is motivated as follows: A pro-object in finite dimensional manifolds ("pro-manifold" for short) is a *formal* projective limit of finite dimensional manifolds. This implies that a smooth function out of a pro-manifold is represented by a smooth function on a finite stage of the corresponding projective system, hence is *globally of finite order*. Now by proposition 2.26 also the objects in LocProMfd are projective limits of finite dimensional manifolds, but, by proposition 2.29, smooth functions on them are in general only locally of finite order. In this sense objects of LocProMfd locally look like pro-manifolds, while globally they are a little more flexible.

We will need to combine the concept of locally pro-manifolds with that of formal thickening, yielding the following concept of formal locally pro-manifolds:

Definition 2.33. Write

$$\mathrm{FormalLocProMfd} \hookrightarrow \mathrm{FormalSmoothSet}$$

for the full subcategory of the Cahiers topos (definition 2.11) on objects X such that

1. the reduction (definition 2.15) of X is a locally pro-manifold (definition 2.30)

$$\Re X \in \mathrm{LocProMfd} \hookrightarrow \mathrm{FormalSmoothSet}\,,$$

2. there exists an order k-infinitesimal disk $\mathbb{D} \in \mathrm{InfPoint} \in \mathrm{FormalSmoothSet}$ (definition 2.8, definition 3.5) or a formal disk $\mathbb{D} \in \mathbf{H}$ (proposition 3.7) such that X is locally the Cartesian product of a locally pro-manifold with $\mathbb{D}$, hence such that there exists an open cover $\{U_i \xrightarrow{\phi_i} \Re X\}_{i \in I}$ and for each $i \in I$ a commuting diagram of the form

$$\begin{tikzcd} U_i \arrow[r, "\epsilon_{U_i \times \mathbb{D}}"] \arrow[d, "\mathrm{et}"', "\phi_i"] & U_i \times \mathbb{D} \arrow[d, "\mathrm{et}"] \\ \Re X \arrow[r, "\epsilon_X"'] & X \end{tikzcd} ,$$

where the horizontal morphisms are the units of the $\Re$-monad, and where in addition to the left vertical morphisms (which is formally étale by proposition 3.2) also the right morphisms is formally étale (definition 3.1).

Following [14] we call these objects *formal locally pro-manifolds*, or just *formal manifolds*, for short.

Definition 2.34 ([23, Sec.1.2])**.** We say that a morphism $f\colon X \longrightarrow Y$ between locally pro-manifolds (definition 2.30) is a *submersion* if for every $x \in X$, there exist open neighborhoods U of x and V of $y = f(x)$ of the form $U \simeq \mathbb{R}^m \times \mathbb{R}^n$ and $V \simeq \mathbb{R}^n$, with $n = \dim Y$ and m either a finite non-negative integer or ∞, such that $U \xrightarrow{f} V$ is equivalent to the projection $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ onto the second factor.

3 Jets and PDEs

We discuss here how purely formal constructions induced by the presence of the infinitesimal shape operation $\Im$ in a differentially cohesive topos $\mathbf{H}$ (definition 2.15) yields a synthetic theory of jet bundles, and of partial differential equations.

We frequently illustrate the general discussion with the archetypical example the topos

$$\mathbf{H} = \mathrm{FormalSmoothSet}$$

from definition 2.11 (Dubuc's "Cahiers topos"), which is differentially cohesive by proposition 2.14, showing that and how it reproduces and generalizes traditional constructions.

3.1 V-Manifolds

We consider now an axiomatization of manifolds in the generality where the local model space may be any group object V in a differentially cohesive topos such as that of formal smooth sets, or rather the formal neighbourhood of its neutral element (see above definition 3.3 below). Ordinary n-dimensional manifolds constitute the special case where $V = \mathbb{R}^n \in \mathrm{FormalSmoothset}$, equipped with its canonical translation group structure.

For the present purpose of studying PDEs, the main point of the V-manifolds is that their formal disk bundles have good "microlinear" structure, as shown by proposition 3.13 below, highlighted by corollary 3.16. This plays a key role in the analysis of generalized PDEs with free variables ranging in V-manifolds, below in section 3.5.

Beyond that, the full force of the abstract definition of V-manifolds is obtained when passing to the formal smooth ∞-groupoids, in which case we obtain a concept of étale ∞-stacks locally modeled on some ∞-group V. This we discuss in a followup.

Throughout, let $\mathbf{H}$ be a differentially cohesive topos (definition 2.15) such as Dubuc's Cahiers topos FormalSmoothSet (definition 2.11).

Definition 3.1. For $f : X \longrightarrow Y$ a morphism in $\mathbf{H}$, we say that it is *formally étale* precisely if the naturality square of the unit of the $\Im$-monad (definition 2.15) is Cartesian (a pullback square):

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{(pb)}"] & \Im X \arrow[d, "\Im f"] \\ Y \arrow[r, "\eta_Y"'] & \Im Y \end{tikzcd} .$$

Moreover, we say that the *formal étalification* of f is the pullback $(\Im_Y X \overset{\Im_Y f}{\to} Y) := \eta_Y^* \Im f$ in

$$\begin{tikzcd} \Im_Y X \arrow[r] \arrow[d, "\Im_Y f"'] \arrow[dr, phantom, "\text{(pb)}"] & \Im X \arrow[d, "\Im f"] \\ Y \arrow[r, "\eta_Y"'] & \Im Y \end{tikzcd} .$$

If the canonical comparison morphism

$$X \longrightarrow \Im_Y X$$

is

- an epimorphism, then f is called *formally smooth*,
- a monomorphism, then f is called *formally unramified*.

Finally, we say that an object $X \in \mathbf{H}$ is formally smooth or formally unramified if the terminal morphism $X \to *$ is so, respectively. This is the case precisely if

$$\eta_X : X \to \Im X$$

is an epimorphism or monomorphism, respectively.

Proposition 3.2. *For $X, Y \in \mathrm{SmthMfd} \hookrightarrow \mathrm{SmoothSet} \overset{i_!}{\hookrightarrow} \mathrm{FormalSmoothSet}$ two smooth manifolds, regarded as formal smooth sets, then a morphism $f : X \to Y$ between them is formally étale in the sense of definition 3.1 precisely if it is a local diffeomorphism in the traditional sense.*

Proof. A function f between smooth manifolds is a local diffeomorphism in the traditional sense precisely if the induced square

$$\begin{tikzcd} TX \arrow[r] \arrow[d, "df"'] \arrow[dr, phantom, "\text{(pb)}"] & X \arrow[d, "f"] \\ TY \arrow[r] & Y \end{tikzcd}$$

of tangent bundles is Cartesian (is a pullback) in SmthMfd. This equivalently means that for all $U \in \mathrm{SmthMfd}$ then the image

$$\mathrm{Hom}_{\mathrm{SmthMfd}}\left(U, \begin{tikzcd} TX \arrow[r] \arrow[d, "df"'] \arrow[dr, phantom, "\text{(pb)}"] & X \arrow[d, "f"] \\ TY \arrow[r] & Y \end{tikzcd} \right)$$

of this square under $\mathrm{Hom}_{\mathrm{SmthMfd}}(U, -) : \mathrm{SmthMfd} \to \mathrm{Set}$ is a Cartesian square in sets.

This we may equivalently rewrite as

$$\mathrm{Hom}_{\mathrm{SmthMfd}}\left(U, \begin{tikzcd} X^{\mathbb{D}^1(1)} \arrow[r, "X^{(* \to \mathbb{D}^1(1))}"] \arrow[d, "f^{\mathbb{D}^1(1)}"'] \arrow[dr, phantom, "\text{(pb)}"] & X \arrow[d, "f"] \\ Y^{\mathbb{D}^1(1)} \arrow[r, "Y^{(* \to \mathbb{D}^1(1))}"'] & Y \end{tikzcd} \right),$$

where $(-)^{\mathbb{D}^1(1)}$ denotes the internal hom[2] out of the standard first order infinitesimal 1-disk (definition 2.8). (Hence $X^{(* \to \mathbb{D}^1(1))}$ denotes the morphism on internal homs into X out of $*$ (which is X) and out of $\mathbb{D}^1(1)$, respectively, which is induced by the unique point inclusion $* \to \mathbb{D}^1(1)$.) By the $(\Re \dashv \Im)$-adjunction (definition 2.15) and since $\Re(\mathbb{D}^1(1)) \simeq *$ (proposition 2.16), this in turn is a pullback of sets precisely if the following is:

$$\mathrm{Hom}\left(U \times \mathbb{D}^1(1), \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \Im X \arrow[d, "\Im f"] \\ X \arrow[r, "\eta_Y"'] & \Im Y \end{tikzcd} \right). \qquad (\star)$$

[2]The *internal hom* A^B is an object satisfying the identity $\mathrm{Hom}(-, A^B) \simeq \mathrm{Hom}(- \times B, A)$ as a natural bijection of sets.

Now the square on the right is Cartesian precisely if the stronger condition holds that

$$\mathrm{Hom}\left(U \times \mathbb{D}, \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \Im X \arrow[d, "\Im f"] \\ X \arrow[r, "\eta_Y"'] & \Im Y \end{tikzcd} \right) \tag{$\star\star$}$$

is a pullback, for all $\mathbb{D} \in \mathrm{InfPoint}$.

So this shows that f being formally étale implies that it is a locally diffeomorphism. To see that there is no further condition from using more general $\mathbb{D}$ than $\mathbb{D}^1(1)$ observe that a local diffeomorphism between smooth manifolds is in fact an isomorphism on every sufficiently small open neighbourhood around every point $x \in X$. Since all infinitesimal points $\mathbb{D}$ necessarily factor through any such open neighbourhood, this shows that ($\star\star$) is a pullback already if ($\star$) is.

□

The following gives a generalized concept of manifolds locally modeled on some model space V. In this article the only explicit examples we consider are $V = \mathbb{R}^n$ for $n \in \mathbb{N} \cup \{\infty\}$ (example 3.4 below), but for later development more generality is needed. The idea is that all we need of a local model space V is that it has some kind of differentiable structure and that it has some kind of additive structure that allows to translate along it, even if it is not commutative. The first point is satisfied by taking V to be an object of our category $\mathbf{H}$. The second point means that it is a group object in $\mathbf{H}$ (definition B.2).

Definition 3.3. Let $V \in \mathrm{Grp}(\mathbf{H})$ be a group object in $\mathbf{H}$ (definition B.2). We say that an object $X \in \mathbf{H}$ is a *V-manifold* if there exists $U \in \mathbf{H}$ and a diagram of the form

$$\begin{tikzcd} & U \arrow[dl, "\mathrm{et}"'] \arrow[dr, two heads, "\mathrm{et}"] & \\ V & & X \end{tikzcd},$$

i.e., a span from V to X such that both legs are formally étale according to definition 3.1 and such that in addition the right leg is an epimorphism in $\mathbf{H}$. Any such U we call a *V-atlas* of X.

Example 3.4. An ordinary smooth manifold of dimension n becomes a V-manifold in the sense of definition 3.3 by taking $V = \mathbb{R}^n$ equipped with its canonical additive group structure. A V-atlas is obtained from any ordinary atlas with charts $\{\mathbb{R}^n \stackrel{\phi_i}{\to} X\}_{i \in I}$ by setting $U := \coprod_{i \in I} \mathbb{R}^n$ and taking the two morphisms to be the canonical ones

$$\begin{tikzcd} & U = \coprod\limits_{i \in I} \mathbb{R}^n \arrow[dl, "(\mathrm{id})_{i \in I}"'] \arrow[dr, two heads, "(\phi_i)_{i \in I}"] & \\ \mathbb{R}^n & & X \end{tikzcd}.$$

Since $U \to X$ is an open cover, every germ of a function on an open n-ball $B^n_\delta \to X$ factors through $\mathbb{R}^n \simeq B^n_\delta \to U \to X$. Thinking of U and X as sheaves in the sheaf topos $\mathrm{SmoothTopos} = \mathrm{Sh}(\mathrm{CartSp})$, this means that $U \to X$ induces an epimorphism from stalks of U to the stalks of X over enough points of the topos SmoothSet (proposition 2.4), which in turn implies that $U \to X$ is itself an epimorphism (proposition B.22). Since both morphisms are local diffeomorphisms, by construction, they are formally étale morphisms in SmoothSet by proposition 3.2.

Similarly, a locally pro-manifold in the sense of definition 2.30 becomes a V-manifold in the sense of definition 3.3, with $V = \mathbb{R}^n$ and $n \in \mathbb{N} \cup \{\infty\}$, under the embedding of Fréchet manifolds (proposition 2.22).

3.2 Infinitesimal disks

Jets (discussed below in section 3.3) are functions out of infinitesimal and formal disks (balls). Therefore here we first discuss infinitesimal and formal disks.

Definition 3.5. For $X \in \mathrm{SmthMfd} \hookrightarrow \mathrm{SmoothSet} \overset{i_!}{\hookrightarrow} \mathrm{FormalSmoothSet}$ a smooth manifold of dimension n, for $x : * \to X$ a point and for $k \in \mathbb{N}$, then the *infinitesimal disk of order k around x in X* is the subobject

$$\mathbb{D}_x(k) \hookrightarrow X$$

in FormalSmoothSet given, up to isomorphism, as the composite

$$\mathbb{D}^n(k) \hookrightarrow \mathbb{R}^n \xrightarrow{\phi} X \ ,$$

where the first inclusion is the defining one of the standard infinitesimal d-disk of order k into $\mathbb{R}^d$ (definition 2.8) and where ϕ is any smooth embedding sending the origin of $\mathbb{R}^n$ to x.

More abstractly, we may speak of the formal disk around any point in any object of a differentially cohesive topos $\mathbf{H}$ as follows.

Definition 3.6. Let $X \in \mathbf{H}$ and $x : * \to X$ a point. Then the *formal disk* in X around x is the fiber product

$$\mathbb{D}_x := X \times_{\Im X} \{x\} \, ,$$

i.e., the object sitting in the pullback diagram of the form

$$\begin{tikzcd} \mathbb{D}_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & * \arrow[d, "x"] \\ X \arrow[r, "\eta_X"'] & \Im X \end{tikzcd} \ ,$$

where $\Im$ is the monad from definition 2.15, and $\eta_X : X \to \Im X$ its unit at X.

The following proposition unwinds this abstract definition for the case that X is a manifold.

Proposition 3.7. *Let $X \in \mathrm{SmthMfd} \hookrightarrow \mathrm{FormalSmoothSet}$ be a smooth manifold and $x\colon * \to X$ a point. Then the formal disk $\mathbb{D}_x \in \mathbf{H} \simeq \mathrm{Sh}(\mathrm{FormalSmoothSet})$ according to definition 3.6 is given by the sheaf which is the filtered colimit over the infinitesimal disks $\mathbb{D}_x(n) \hookrightarrow X$ in X at x (from definition 3.5)*

$$\mathbb{D}_x \simeq \varinjlim_k \mathbb{D}_x(k) \, .$$

Proof. First of all we observe that the diagram

$$\begin{tikzcd} \varinjlim_k \mathbb{D}_x(k) \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r, "x"'] & \Im X \end{tikzcd}$$

indeed commutes, where the top morphism is given component-wise by the defining inclusions $\mathbb{D}_x(k) \hookrightarrow X$. By the $(\Re \dashv \Im)$ adjunction (definition 2.15) this corresponds to the diagram

$$\begin{tikzcd} \Re(\varinjlim_k \mathbb{D}_x(k)) \arrow[r] \arrow[d] & \Re X \arrow[d] \\ \Re * \arrow[r] & X \end{tikzcd} \ .$$

Since X is an ordinary manifold and hence already reduced by assumption, the morphism on the right is an isomorphism, and the bottom morphism is just the inclusion of the point x. Moreover, since $\Re$ is left adjoint it preserves the colimit in the top left, and since $\Re(\mathbb{D}) \simeq *$ for all infinitesimally thickened points $\mathbb{D}$ (proposition 2.16) also the top morphism is just the inclusion of the point x. Hence the diagram commutes.

Observe that the top morphism in the first diagram is a monomorphism

$$\varinjlim_k \mathbb{D}_x(k) \hookrightarrow X \,.$$

(Since each $\mathbb{D}_x(k) \to X$ is a monomorphism and monomorphisms as well as colimits of sheaves are detected on stalks.)

Therefore to check that the first square above is indeed Cartesian, it is sufficient to check that for every $\mathbb{R}^n \times \mathbb{D} \in \mathrm{FormalCartSp}$ (definition 2.11) then morphisms $f : \mathbb{R}^n \times \mathbb{D} \longrightarrow X$ such that $\eta_X \circ f$ is constant on x factor through $\varinjlim_k \mathbb{D}_x(k) \to X$.

Let a given $\mathbb{D}$ be of infinitesimal order $k_{\mathbb{D}}$ (meaning that in $C^\infty(\mathbb{D}) = \mathbb{R} \oplus V$ then $V^{k_{\mathbb{D}}+1} = 0$). We will show that then in fact we have factorization through $\mathbb{D}^n(k_{\mathbb{D}}) \to \varinjlim_k \mathbb{D}_x(k) \to X$

To that end, observe that for every point $y : * \to \mathbb{R}^n$ we have the restriction $f|_y$ of f to that point

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "{(y,\mathrm{id})}"'] \arrow[dr, "f|_y"] & \\
\mathbb{R}^n \times \mathbb{D} \arrow[r, "f"'] & X
\end{tikzcd}$$

and the bottom morphism f is fully determined once all its restrictions $f|_x$ are known (since, dually, smooth functions in the image of $C^\infty(X) \xrightarrow{f^*} C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} C^\infty(\mathbb{D})$ are uniquely determined by their restrictions along $(-)|_y : C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} C^\infty(\mathbb{D}) \to C^\infty(\mathbb{D})$, for all $y \in \mathbb{R}^n$).

Therefore it is sufficient to check the factorization property for morphisms of the form $\mathbb{D} \to X$, with $\mathbb{D} \in \mathrm{InfPoint}$ (definition 2.6), and through a fixed stage k of the colimit: because if we have factorizations of the form shown by the dashed morphism here

$$\begin{tikzcd}
\mathbb{D} \arrow[dr, "{(y,\mathrm{id})}"] \arrow[ddrr, dashed, bend right] & & & \\
& \mathbb{R}^n \times \mathbb{D} \arrow[drr, bend left] \arrow[dr, dotted] \arrow[ddr, bend right] & & \\
& & \mathbb{D}_x(k_{\mathbb{D}}) \arrow[r, hook] \arrow[d] & X \arrow[d] \\
& & * \arrow[r, hook] & \Im X
\end{tikzcd}$$

for all $y \in \mathbb{R}^n$, then there there is an induced factorization as shown by the dotted arrow.

Observe that the composite morphism $\mathbb{D} \longrightarrow X$ in the above diagram must take the global point $* \to \mathbb{D}$ to $* \xrightarrow{x} X$, by the same kind of argument as for the commutativity of the square at the beginning of the proof.

Now if $\mathbb{D}$ is of order k (i.e., if the maximal ideal V of $C^\infty(\mathbb{D})$ satisfies $V^{k+1} = 0$), this means that morphisms $f : \mathbb{D} \to X$ are equivalently $\mathbb{R}$-algebra homomorphisms of the form $(\mathrm{ev}_x, \rho) : C^\infty(X) \to \mathbb{R} \oplus V$ where ρ is a function of the partial derivatives of f at x of order at most k. By proposition 2.13 such an algebra homomorphism factors through $C^\infty(\mathbb{D}_x(k))$. Hence we have a factorization

$$\begin{tikzcd}
f : \mathbb{D} \arrow[r] \arrow[rr, bend right] & \mathbb{D}_x(k) \arrow[r] & \varinjlim_k \mathbb{D}_x(k) \arrow[r] & X
\end{tikzcd} .$$

$\square$

We collect all the infinitesimal disks (definition 3.6) of a formal smooth set as follows:

Definition 3.8. Let $X \in \mathbf{H}$ be any object of a differentially cohesive topos. Then its *formal disk bundle*

$$\begin{tikzcd} T^\infty X \arrow[d] \\ X \end{tikzcd}$$

is the pullback in

$$\begin{tikzcd} T^\infty X \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "(\mathrm{pb})"] & X \arrow[d, "\eta_X"] \\ X \arrow[r, "\eta_X"'] & \Im X \end{tikzcd} .$$

More generally, for $E \xrightarrow{p} X$ a morphism in $\mathbf{H}$ (e.g., a bundle over X), then $T^\infty_X E$ is the left vertical composite $\pi \circ \mathrm{ev}^* p$ in the pasting diagram

$$\begin{tikzcd} T^\infty_X E \arrow[r, "\mathrm{ev}_E"] \arrow[d, "\mathrm{ev}^* p"'] \arrow[dr, phantom, "(\mathrm{pb})"] & E \arrow[d, "p"] \\ T^\infty X \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "(\mathrm{pb})"] & X \arrow[d, "\eta_X"] \\ X \arrow[r, "\eta_X"'] & \Im X \end{tikzcd} .$$

(This definition is essentially that in [14], just preceding its Proposition 2.2).

More succinctly, according to proposition B.25, T^∞_X is the monad on the slice of $\mathbf{H}$ over X given by left base change along the unit of the $\Im$-monad (definition 2.15):

$$T^\infty_X \;:=\; (\eta_X)^* \circ (\eta_X)_! \;:\; \mathbf{H}_{/X} \longrightarrow \mathbf{H}_{/X} \,.$$

Remark 3.9. The object $T^\infty X$ in definition 3.8 is also called the *formal neighbourhood of the diagonal* (cf. for instance below Proposition 2.1 in [14]). As such one wants to think of the two canonical maps out of it on a more symmetric footing, and write $\mathrm{pr}_1 := \pi$, $\mathrm{pr}_2 = \mathrm{ev}$. By the universal property of the pullback, a generalized point in $T^\infty X$ is an ordinary point of X together with two infinitesimal neighbours

$$\begin{tikzcd} & \circ_2 \arrow[d, squiggly, dash] \\ \bullet \arrow[ur, squiggly, dash] \arrow[r, squiggly, dash] & \circ_1 \end{tikzcd} .$$

The projection pr_i takes such a triple to $\circ_i$. Definition 3.8 above takes the non-symmetric perspective that $\mathrm{pr}_1 = \pi$ is to be regarded as a bundle projection. Viewed this way then the above picture illustrates a point $\circ_2$ in the fiber of $T^\infty X$ over $\circ_1 \in X$.

Assume that $X \in \mathbf{H}$ is formally smooth (Definition 3.1), hence that $\eta_X : X \longrightarrow \Im X$ is an epimorphism in the topos $\mathbf{H}$ (which is the case generally for $\mathbf{H} = \mathrm{FormalSmoothSet}$, according to Proposition 2.18). All epimorphisms in a sheaf topos are regular epimorphisms, meaning that they are the coequalizers of their kernel pair (prop. B.23). Here this means that the diagram

$$\begin{tikzcd} T^\infty X \arrow[r, shift left, "\mathrm{pr}_1 = \pi"] \arrow[r, shift right, "\mathrm{pr}_2 = \mathrm{ev}"'] & X \arrow[r, "\eta_X", "\mathrm{coeq}"'] & \Im X \end{tikzcd}$$

is a coequalizer diagram, manifestly exhibiting $\Im X$ as the quotient of the equivalence relation

$$(x \sim y) \Leftrightarrow (x \text{ is an infinitesimal neighbour of } y) \,.$$

Example 3.10. For $X \in \mathbf{H}$ any object and for $x : * \to X$ a point, regarded as an object of $\mathbf{H}_{/X}$, then its formal disk bundle according to definition 3.8 is just the formal disk $\mathbb{D}_x$ in X (according to definition 3.6) at that point:

$$T^\infty_X \{x\} \simeq \mathbb{D}_x \,.$$

Proof. Def. 3.8 gives that

$$T^\infty_X \{x\} \simeq T^\infty X \times_X \{x\} \,.$$

The statement thus follows with the pasting law (proposition B.1) and definition 3.6:

$$\begin{tikzcd}
\mathbb{D}_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & * \arrow[d, "s"] \\
T^\infty X \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\text{(pb)}"] & X \arrow[d, "\eta_X"] \\
X \arrow[r, "\eta_X"'] & \Im X
\end{tikzcd} \,.$$

∎

Proposition 3.11. *For $X \in \mathbf{H}$, the formal disk bundle monad $T^\infty_X : \mathbf{H}_{/X} \to \mathbf{H}_{/X}$ from definition 3.8 has the following structure morphisms:*

1. *The monad unit at an object $[E \xrightarrow{p} X]$ is $\eta_p = (p, \mathrm{id}) : E \to X \times_{\Im X} E = T^\infty_X E$;*
2. *the monad product is given by $\Delta_X = T^\infty_X \mathrm{ev}_E : T^\infty_X T^\infty_X E \to T^\infty_X E$,*

where in the second item we regard ev_E as a morphism in $\mathbf{H}_{/X}$ via

$$\begin{tikzcd}
T^\infty_X E \arrow[rr, "\mathrm{ev}_E"] \arrow[dr] & & E \arrow[dl] \\
& X &
\end{tikzcd} \,.$$

Proof. This follows by direct unwinding of the definition of the monad structure induced by an adjunction (example B.29). For the first item observe that this is the unit of the adjunction $(\eta_X)_! \dashv (\eta_X)^*$:

$$\begin{tikzcd}
E \arrow[dr, dashed, "\eta_E"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddd, bend right, "p"'] & & \\
& T^\infty_X E \arrow[r, "\mathrm{ev}_E"] \arrow[d, "\pi_E"'] \arrow[dr, phantom, "\text{(pb)}"] & E \arrow[d, "p"] \\
& T^\infty X \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\text{(pb)}"] & X \arrow[d, "\eta_X"] \\
& X \arrow[r, "\eta_X"'] & \Im X
\end{tikzcd} \,.$$

For the second item, observe that the counit of the $(\eta_X)_! \dashv (\eta_X)^*$-adjunction on any $Q \xrightarrow{p} \Im X$ is the top horizontal morphism in

$$\begin{tikzcd}
(\eta_X)^* Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & Q \arrow[d, "p"] \\
X \arrow[r, "\eta_X"'] & \Im X
\end{tikzcd} \,,$$

regarded as a morphism in $\mathbf{H}_{/\Im X}$. For $(Q \to \Im X) = (\eta_X)_! E = (E \to X \overset{\eta_X}{\to} \Im X)$ this square becomes the defining rectangle from definition 3.8:

$$
\begin{tikzcd}
T^\infty_X E \arrow[r, "\mathrm{ev}_E"] \arrow[d, "\pi_E"'] & E \arrow[d, "p"] \\
T^\infty X \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"'] & X \arrow[d, "\eta_X"] \\
X \arrow[r, "\eta_X"'] & \Im X
\end{tikzcd}
.
$$

Hence the monad product is the image of the top morphism ev_E here, regarded as a morphism over $\Im X$, under base change via pullback along $\eta_X : X \to \Im X$. This yields the claim. ☐

We now establish some basic properties inherited by $T^\infty X$ in the case that X is a V-manifold.

Proposition 3.12. *If a morphism $f : X \longrightarrow Y$ in $\mathbf{H}$ is formally étale (definition 3.1), then pullback along f preserves infinitesimal disk bundles (definition 3.8), in that*

$$
f^*(T^\infty Y) \simeq T^\infty X \quad \in \mathbf{H}_{/X} .
$$

Proof. Observe that we have a commuting cube as follows

$$
\begin{tikzcd}
f^* T^\infty Y \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & T^\infty Y \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & Y \arrow[d, "\eta_Y"] \\
X \arrow[r, "f"'] & Y \arrow[r, "\eta_Y"'] & \Im Y
\end{tikzcd}
\simeq
\begin{tikzcd}
T^\infty X \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & Y \arrow[d, "\eta_Y"] \\
X \arrow[r, "\eta_X"'] & \Im X \arrow[r, "\Im f"'] & \Im Y
\end{tikzcd}
.
$$

Here the bottom composites agree by the naturality of η, and from this the squares are identified via the pasting law for pullbacks (proposition B.1) and using the definition of the formal disk bundle (definition 3.8) and of the property that f is formally étale (definition 3.1). The resulting isomorphism in the top left is the statement to be proven. ☐

Proposition 3.13. *Let $V \in \mathrm{Grp}(\mathbf{H})$ be a group object if a differentially cohesive topos.*

1. *The infinitesimal disk bundle $T^\infty V$ over V (definition 3.8) trivializes, with typical fiber the formal disk $\mathbb{D}_e$ (definition3.6) of V at the neutral element $e \in V$, i.e., there is an equivalence of the form*

$$
\begin{tikzcd}
T^\infty V \arrow[rr, "\simeq"] \arrow[dr, "\pi"'] & & V \times \mathbb{D}_e \arrow[dl, "\mathrm{pr}_1"] \\
& V &
\end{tikzcd}
.
$$

Moreover, under this equivalence the morphism $\mathrm{ev} : T^\infty V \to V$ from definition 3.8 is identified with the operation of right translation of elements in V by elements in $\mathbb{D}_e \overset{i}{\to} V$ in that the following diagram commutes

$$
\begin{tikzcd}
V \times \mathbb{D}_e \arrow[r, "\mathrm{id} \times i"] \arrow[d, "\simeq"'] & V \times V \arrow[r, "(-)\cdot(-)^{-1}"] & V \arrow[d, "="] \\
T^\infty V \arrow[rr, "\mathrm{ev}"'] & & V
\end{tikzcd}
.
$$

2. *For X a V-manifold (definition 3.3), its infinitesimal disk bundle $T^\infty X$ (definition 3.8) is a locally trivial $\mathbb{D}_e$-fiber bundle. Specifically, for $U \xrightarrow{\mathrm{et}} X$ any V-atlas of X, there is a Cartesian diagram*

$$\begin{tikzcd} U \times \mathbb{D}_e \arrow[r] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "(pb)"] & T^\infty X \arrow[d] \\ U \arrow[r, two heads, "\mathrm{et}"'] & X \end{tikzcd} .$$

Proof. Since $\Im$ is a right adjoint (by proposition 2.14, definition 2.15) it preserves limits and hence in particular it preserves group objects. Therefore the defining Cartesian square

$$\begin{tikzcd} T^\infty V \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "(\mathrm{pb})"] & V \arrow[d, "\eta_V"] \\ V \arrow[r, "\eta_V"'] & \Im V \end{tikzcd}$$

(from definition 3.5) is a fiber product over a group object $\Im V$. Hence the "nonabelian Mayer-Vietoris argument" applies (proposition B.3) and shows that there is equivalently a Cartesian square of the form

$$\begin{tikzcd} T^\infty V \arrow[r] \arrow[d, "(\pi{,}\mathrm{ev})"'] \arrow[dr, phantom, "(\mathrm{pb})"] & * \arrow[d, "e"] \\ V \times V \arrow[r, "(\eta_V)\cdot(\eta_V)^{-1}"'] & \Im V \end{tikzcd} .$$

Since η_V is a group homomorphism, the bottom morphism here is equivalently $\eta_V \circ ((-) \cdot (-)^{-1})$. Hence by the pasting law (proposition B.1) and using the definition of the formal disk $\mathbb{D}_e$ (definition 3.6), we obtain a factorization of the above diagram as

$$\begin{tikzcd} T^\infty V \arrow[r] \arrow[d, "(\pi{,}\mathrm{ev})"'] \arrow[dr, phantom, "(\mathrm{pb})"] & \mathbb{D}_e \arrow[r] \arrow[d, "i"] \arrow[dr, phantom, "(\mathrm{pb})"] & * \arrow[d, "e"] \\ V \times V \arrow[r, "(-)\cdot(-)^{-1}"'] & V \arrow[r, "\eta_V"'] & \Im V \end{tikzcd} .$$

From the universal property of the pullback, it follows that the square on the left here is equivalently

$$\begin{tikzcd} V \times \mathbb{D}_e \arrow[r, "\mathrm{pr}_2"] \arrow[d, "(\mathrm{pr}_1{,}\,(-)\cdot(i(-))^{-1})"'] \arrow[dr, phantom, "(\mathrm{pb})"] & \mathbb{D} \arrow[d, "i"] \\ V \times V \arrow[r, "(-)\cdot(-)^{-1}"'] & V \end{tikzcd} .$$

The equivalence of this square with the left square above is the first statement to be shown.

For the second statement, observe, by proposition 3.12, that pullback along the two legs of any V-atlas (definition 3.3)

$$\begin{tikzcd} & U \arrow[dl, "\mathrm{et}"'] \arrow[dr, two heads, "\mathrm{et}"] & \\ V & & X \end{tikzcd}$$

preserves formal disk bundles. Hence the previous statement, given that $T^\infty V$ is trivial, implies that $T^\infty U \simeq U \times \mathbb{D}_e$ is also trivial, and that so is the pullback of $T^\infty X$ to the V-atlas U. Hence $T^\infty X$ is locally trivial. □

Remark 3.14. The proof of proposition 3.13 is "purely formal," and works in every differentially cohesive topos (definition 2.15). As such it has been formalized in the formal language of modal homotopy type theory [34]. This means that the proof also works in the ∞-topos version of $\mathbf{H}$. There we may furthermore formally conclude that for X a V-manifold, then the formal disk bundle $T^\infty X$ is associated to an $\mathbf{Aut}(\mathbb{D})_e$-principal bundle. This is the *jet frame bundle* of the V-manifold X. The frame bundle (at any jet order) is the starting point for Cartan geometry (i.e., the theory of torsion-free G-structures on manifolds). This is a rich part of the theory, but here we will not further dwell on it.

In order to highlight the content of proposition 3.13 it will be convenient to introduce the following notation:

Definition 3.15. Let X be a V-manifold (definition 3.3). Denote the local evaluation action from item 1 of proposition 3.13 by

$$+ : V \times \mathbb{D}_e \longrightarrow V\,.$$

For $E \in \mathbf{H}$ an object, we may regard any morphism $p : \cdots \to E$ into it as a generalized element of E. Say that a generalized element of the total space of a bundle $E \to X$ is *local* if its projection $x : \cdots \to E \to X$ factors through some V-cover $U \to X$. By item 2 of proposition 3.13 a local generalized element of the infinitesimal disk bundle $T^\infty_X E$ is equivalently a pair $p = (x, a)$ consisting of a generalized base point $x : \cdots \to X$ and of a generalized element $a : \cdots \to \mathbb{D}_e \times E$, such that the evaluation morphism

$$\mathrm{ev}_E : T^\infty_X E \longrightarrow E$$

is given by

$$(x, a) \mapsto x + a\,.$$

Corollary 3.16. *In the notation of definition 3.15 then the structure morphism (definition B.28) of the T^∞_X-monad from definition 3.8 are given on local generalized elements as follows:*

1. *the unit morphism*

$$\eta_E \;:\; E \to T^\infty_X E$$

   is given by

$$x \mapsto (x, 0)$$

   (where $0 : \cdots \to * \to \mathbb{D}_e$ is the generalized element constant on the unique global base point of $\mathbb{D}_e$)

2. *the product morphism*

$$\nabla_E \;:\; T^\infty_X T^\infty_X E \longrightarrow T^\infty_X E$$

   is given by

$$(b, a, b) \mapsto (b, a + b)\,.$$

Proof. In view of proposition 3.13 this is the statement of proposition 3.11. $\square$

The following example shows what the abstract phenomena of proposition 3.13 look like when realized in a basic concrete special case.

Example 3.17. Let $X = \mathbb{R} \in \mathrm{CartSp} \hookrightarrow \mathrm{FormalSmoothSet}$ be the real line and write

$$\mathbb{D}_0 = \varinjlim_k \mathbb{D}_0(k) \hookrightarrow \mathbb{R}$$

for the formal infinitesimal disk in $\mathbb{R}$ around the origin (definition 3.5). Then the product morphism of the infinitesimal disk bundle monad according to proposition 3.11 is of the form

$$\begin{tikzcd} T^\infty_X T^\infty_X X \arrow[r] \arrow[d, "\simeq"'] & T^\infty_X X \arrow[d, "\simeq"] \\ \mathbb{R} \times \mathbb{D}_0 \times \mathbb{D}_0 \arrow[r] & \mathbb{R} \times \mathbb{D}_0 \end{tikzcd}$$

and is given at infinitesimal order $k \in \mathbb{N}$ by the morphism

$$\mathbb{R} \times \mathbb{D}_0(k) \times \mathbb{D}_0(k) \longrightarrow \mathbb{R} \times \mathbb{D}_0(k)$$

in FormalCartSp $\hookrightarrow$ FormalSmoothSet which is dual to the algebra homomorphism

$$C^\infty(\mathbb{R}) \otimes_{\mathbb{R}} \langle(\mathbb{R} \oplus \langle\varepsilon_1\rangle)/(\varepsilon_1^{k+1}) \otimes_{\mathbb{R}} (\mathbb{R} \oplus \langle\varepsilon_2\rangle)/(\varepsilon_2^{k+1}) \longleftarrow C^\infty(\mathbb{R}) \otimes_{\mathbb{R}\oplus\langle\varepsilon\rangle} /(\varepsilon^{k+1})$$

that takes

$$f(x,\varepsilon) \mapsto f(x, \varepsilon_1 + \varepsilon_2)\,.$$

Proof. By proposition 3.13 and proposition 3.7 the morphism

$$\mathrm{ev} \;:\; T^\infty_{\mathbb{R}}\mathbb{R} \longrightarrow \mathbb{R}$$

is the composite

$$\mathbb{R} \times \mathbb{D}_0 \hookrightarrow \mathbb{R} \times \mathbb{R} \xrightarrow{(-)+(-)} \mathbb{R}$$

in FormalCartSp $\hookrightarrow$ FormalSmoothSet (definition 2.11). Dually, at infinitesimal order $k \in \mathbb{N}$, this is the algebra homomorphism

$$C^\infty(\mathbb{R}) \otimes_{\mathbb{R}} (\mathbb{R} \oplus \langle\varepsilon_1\rangle)/(\varepsilon_1^{k+1}) \longleftarrow C^\infty(\mathbb{R}) \;\; : \exp(\varepsilon\partial_x)$$

which takes

$$f(x) \mapsto \exp(\varepsilon\partial_x) f(x) := \sum_{n_1=0}^{k} \tfrac{1}{n!} f^{(n_1)}(x)\varepsilon_1^{n_1}\,.$$

By proposition 3.11 the morphism that we are after is the image $T^\infty_X \mathrm{ev}$ of this morphism under T^∞_X, hence its base change, as a morphism over $\Im\mathbb{R}$, along $\eta_{\mathbb{R}} : \mathbb{R} \to \Im\mathbb{R}$. To analyse what this does on fibers, we check what it does over any point, say $0 \in \mathbb{R}$, by base changing further along $\{0\} \to \mathbb{R} \to \Im\mathbb{R}$. This is given by the front top morphism in the following diagram

$$\begin{tikzcd}
 & \mathbb{R} \times \mathbb{D}_0 \arrow[rr, "\mathrm{ev}"] \arrow[dd] & & \mathbb{R} \arrow[dd] \\
\mathbb{D}_0 \times \mathbb{D}_0 \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbb{D}_0 \arrow[ur] \arrow[dd] & \\
 & \mathbb{R} \arrow[rr] & & \Im\mathbb{R} \\
\mathbb{D}_0 \arrow[ur] \arrow[rr] & & * \arrow[ur, "0"'] &
\end{tikzcd},$$

where all squares are Cartesian (are pullback squares). Consider the top square. Since colimits are universal in $\mathbf{H}$ (proposition B.24) we may analyze this at any finite infinitesimal order $\mathbb{D}_0(k) \hookrightarrow \mathbb{D}_0$, where it becomes a pullback in the site FormalCartSp $\overset{y}{\hookrightarrow}$ FormalSmoothSet. Since the Yoneda embedding preserves limits, we may compute this pullback square in FormalCartSp. Since by definition there is a full inclusion $\mathrm{FormalCartSp}^{\mathrm{op}} \hookrightarrow \mathrm{CAlg}_{\mathbb{R}}$, for this it is sufficient to find a pushout square in $\mathrm{CAlg}_{\mathbb{R}}$. As such this is the following:

$$
\begin{tikzcd}
C^\infty(\mathbb{R}) \otimes_{\mathbb{R}} (\mathbb{R} \oplus \langle \varepsilon_1 \rangle)/(\varepsilon_1^{k+1}) \arrow[d, "{\substack{\sum_{n_1=0}^{k} \frac{1}{n_1!} g_{n_1}(x) \varepsilon_1^{n_1} \\ \mapsto \sum_{n_1,n_2=0}^{k} \frac{1}{n_1! n_2!} g_{n_1}^{(n_2)}(0) \varepsilon_1^{n_1} \varepsilon_2^{n_2}}}"'] & & C^\infty(\mathbb{R}) \arrow[ll, "f(x) \mapsto \sum_{n_1=0}^{k} \frac{1}{n!} f^{(n_1)}(x) \varepsilon_1^{n_1}"'] \arrow[d, "f(x) \mapsto \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(0) \varepsilon^n"] \arrow[dll, "f(x) \mapsto \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(\varepsilon_1 + \varepsilon_2)^n" description] \\
(\mathbb{R} \oplus \langle \varepsilon_1, \varepsilon_2 \rangle)/(\varepsilon_1^{k+1}, \varepsilon_2^{k+1}) & & (\mathbb{R} \oplus \langle \varepsilon \rangle)/(\varepsilon^{k+1}) \arrow[ll, "\varepsilon \mapsto (\varepsilon_1 + \varepsilon_2)"]
\end{tikzcd}
$$

Here the top homomorphism is from the previous discussion, while the right morphism is the one that defines the inclusion $\mathbb{D}_0(k) \hookrightarrow \mathbb{R}$ by definition 2.8. Hence the pushout of commutative $\mathbb{R}$-algebras simply identifies the tensor copy of $C^\infty(\mathbb{R})$ in the algebra at the top left with the algebra in the bottom right. This shows that the left morphism has to be as shown. To see that the composite of the top and the left morphism is the diagonal morphism as shown, use the exponential expression from before, in terms of which this identity is

$$\exp(\varepsilon_2 \partial_x) \exp(\varepsilon_1 \partial_x) f(x) = \exp((\varepsilon_1 + \varepsilon_2)\partial_x) f(x) \,.$$

This implies that the bottom morphism is as shown, which is the claim to be proven. □

3.3 Jet bundles

We recall the traditional definition of jet bundles, and then show that this is reproduced by abstract constructions in the Cahiers topos FormalSmoothSet and finally use this to give a general synthetic discussion of infinite-order jet bundles in any differentially cohesive topos, via the abstract construction right adjoint to that of formal disk bundles in the previous section 3.2.

A traditional arena for defining jets and jet bundles is the following slight generalization of fiber bundles (this is essentially the unnamed definition in [23, section 1], see also [1, sec.I.A]):

Definition 3.18 (fibered manifold)**.** We say a morphism $E \longrightarrow \Sigma$ of locally pro-manifolds (definition 2.30) is a *fibered manifold* if the morphism is a surjective submersion according to definition 2.34. We write

$$\mathrm{LocProMfd}_{\downarrow \Sigma} \hookrightarrow \mathrm{LocProMfd}_{/\Sigma}$$

for the full subcategory of the slice category (definition B.12) of locally pro-manifolds over Σ on those objects which are fibered manifolds in this sense.

In generalization of example 2.12 one says:

Definition 3.19 (e.g. [25, §§1.2,1.12])**.** Let $E \xrightarrow{p} \Sigma$ be a morphism in FormalSmoothSet (thought of as a bundle over Σ, example B.16). For $s \colon * \to \Sigma$ a point, and $k \in \mathbb{N}$, then a *k-jet* of p at s is a morphism $\phi : \mathbb{D}_s(k) \longrightarrow E$ out of the infinitesimal disk of order k around that point (definition 3.5) making the following diagram commute

$$
\begin{tikzcd}
\mathbb{D}_s(k) \arrow[rr, "\phi"] \arrow[dr, hook] & & E \arrow[dl, "p"] \\
& \Sigma &
\end{tikzcd}
,
$$

where on the left we have the defining inclusion (from definition 3.5).

The collection of all k-jets of E naturally forms a smooth manifold $J^k_\Sigma E$, called the k-th *jet bundle* of E. When E is a fibered manifold (definition 3.18), then so is $J^k_\Sigma E$. As k-varies, these naturally form a projective system of smooth manifolds

$$\cdots \longrightarrow J^2_\Sigma E \longrightarrow J^1_\Sigma E \longrightarrow J^0_\Sigma E = E \,.$$

The following definition is equivalent to the standard definition in the literature, see remark 3.21 below.

Definition 3.20. For $E \to \Sigma$ a fibered manifold (definition 3.18) then its *infinite jet bundle* $J^\infty_\Sigma E$ is the limit over the projective system of finite jet bundles, according to definition 3.19, taken in the category of locally pro-manifolds (definition 2.30, remark 2.31):

$$J^\infty_\Sigma E := \varprojlim_k J^k_\Sigma E \quad \in \mathrm{LocProMfd}\,.$$

Remark 3.21. In [27] the infinite jet bundle $J^\infty_\Sigma E$ is considered more explicitly as the set of infinite jets equipped with the Fréchet manifold structure locally modeled on $\mathbb{R}^\infty$, with $\mathbb{R}^\infty$ regarded as a Fréchet manifold via the evident seminorms $\| - \|_n : \mathbb{R}^\infty \to \mathbb{R}^n \overset{\|-\|}{\to} \mathbb{R}$ (definition 2.24). But by proposition 2.26 this is equivalently the projective limit of the finite dimensional $\mathbb{R}^n$s formed in Fréchet manifolds, hence (by remark 2.31) in locally pro-manifolds:

$$\mathbb{R}^\infty \simeq \varprojlim_n \mathbb{R}^n \quad \in \mathrm{LocProMfd} \hookrightarrow \mathrm{FrMfd}\,.$$

Hence for the case that $E \to \Sigma$ is a fibered manifold (definition 3.18), then definition 3.20 reduces to what is considered in [27].

Theorem 3.22. *Let $E \overset{p}{\to} \Sigma$ be a fibered manifold (definition 3.18). Then under the embedding (proposition 2.22)*

$$\mathrm{LocProMfd} \hookrightarrow \mathbf{H}_{\Re} \hookrightarrow \mathbf{H}\,,$$

the infinite-jet bundle $J^\infty_\Sigma E$ in the sense of definition 3.20 is isomorphic in $\mathbf{H}_{/\Sigma}$ to the right base change of E (according to proposition B.25) along the unit $\eta_\Sigma : \Sigma \to \Im\Sigma$ of the $\Im$-monad (definition 2.15):

$$J^\infty_\Sigma E \;\simeq\; (\eta_\Sigma)^*(\eta_\Sigma)_* E \quad \in \mathrm{FormalSmoothSet}\,.$$

Proof. By applying proposition B.26 to definition 2.11, we have an equivalence of categories

$$\mathrm{FormalSmoothSet}_{/\Sigma} \simeq \mathrm{Sh}(\mathrm{FormalCartSp}/\Sigma)\,.$$

Therefore it is sufficient to show that for every morphism $U \to \Sigma$ out of a formal Cartesian space $U = \mathbb{R}^d \times \mathbb{D}$, there is a natural bijection

$$\mathrm{Hom}_{\mathbf{H}_{/\Sigma}}(U, (\eta_\Sigma)^*(\eta_\Sigma)_* E) \;\simeq\; \mathrm{Hom}_{\mathbf{H}_{/\Sigma}}(U\,, J^\infty_\Sigma E)$$

(where we leave the maps to Σ notationally implicit). Now by the base change adjoint triple $(\eta_\Sigma)_! \dashv (\eta_\Sigma)^* \dashv (\eta_\Sigma)_*$ (proposition B.25) the left hand side is equivalently

$$\begin{aligned}\mathrm{Hom}_{\mathbf{H}_{/\Sigma}}(U, (\eta_\Sigma)^*(\eta_\Sigma)_* E) &\simeq \mathrm{Hom}_{\mathbf{H}_{/\Sigma}}((\eta_\Sigma)^*(\eta_\Sigma)_! U, E) \\ &= \mathrm{Hom}_{\mathbf{H}_{/\Sigma}}(T^\infty_\Sigma U, E)\end{aligned}\,,$$

where we used definition 3.8 to identify the formal disk bundle $T^\infty_\Sigma U = (\eta_\Sigma)^*(\eta_\Sigma)_! U = T^\infty \Sigma \times_\Sigma U$.

Since the statement to be proven is local over Σ, we may assume without restriction of generality that $\Sigma \simeq \mathbb{R}^n$ (otherwise restrict to the charts of any atlas for the manifold Σ). By proposition 3.13 then $T^\infty\Sigma \simeq \Sigma \times \mathbb{D}^n$ and hence (using definition 3.8)

$$\begin{aligned}T^\infty_\Sigma U &\simeq U \times \mathbb{D}^n \\ &\simeq U \times \varinjlim_k \mathbb{D}^n(k) \\ &\simeq \varinjlim_k U \times \mathbb{D}^n(k)\end{aligned}\,,$$

where in the second step we used proposition 3.7 and in the last step proposition B.24. Hence there is a natural bijection

$$
\begin{aligned}
\mathrm{Hom}_{\mathbf{H}_{/\Sigma}}\left(U,\, (\eta_\Sigma)^*(\eta_\Sigma)_* E\right) &\simeq \mathrm{Hom}_{\mathbf{H}_{/\Sigma}}\left(\varinjlim_k U \times \mathbb{D}^n(k), E\right) \\
&\simeq \varprojlim_k \mathrm{Hom}_{\mathbf{H}_{/\Sigma}}\left(U \times \mathbb{D}^n(k), E\right) \\
&\simeq \varprojlim_k \mathrm{Hom}_{\mathbf{H}_{/\Sigma}}\left(U, J^k_\Sigma E\right)
\end{aligned}
\quad ,
$$

where in the last step we observe that a morphism $U \times \mathbb{D}^n(k) \to E$ over Σ is precisely a smooth U-parameterized family of k-jets in E, according to definition 3.19. Now with the fact that locally pro-manifolds embed fully faithfully $\mathrm{LocProMfd} \hookrightarrow \mathbf{H}_{\Re} \hookrightarrow \mathbf{H}$ (proposition 2.22, remark 2.31) we conclude:

$$
\begin{aligned}
\varprojlim_k \mathrm{Hom}_{\mathbf{H}_{/\Sigma}}(U, J^k_\Sigma E) &\simeq \varprojlim_k \mathrm{Hom}_{\mathrm{LocProMfd}_{/\Sigma}}(U, J^k_\Sigma E) \\
&\simeq \mathrm{Hom}_{\mathrm{LocProMfd}_{/\Sigma}}(U, \varprojlim_k J^k_\Sigma E) \\
&= \mathrm{Hom}_{\mathrm{LocProMfd}_{/\Sigma}}(U, J^\infty_\Sigma E) \\
&\simeq \mathrm{Hom}_{\mathbf{H}_{/\Sigma}}(U, J^\infty_\Sigma E)\,.
\end{aligned}
$$

□

We now see some of the power of the synthetic formalism. Classically, jets are defined for fibered manifolds over Σ (definition 3.21), where locally adapted coordinate charts enter the definition in an essential way. Above, we have found a synthetic formalization of that construction. But now, this construction of J^∞_Σ can be applied to the entire slice topos $\mathbf{H}_{/\Sigma}$, which admits objects $E \to \Sigma$ where E is a much more general space than a locally pro-manifold and the map to Σ is no longer needs to be a surjective submersion. That is, theorem 3.22 says that the following abstract terminology makes good sense:

Definition 3.23. For $\mathbf{H}$ any differentially cohesive topos, and for $\Sigma \in \mathbf{H}$, then we say that the base change comonad (from proposition B.25) along the unit $\eta_\Sigma : \mathbf{H}_{/\Sigma} \to \mathbf{H}_{/\Im\Sigma}$ of the $\Im$-monad (definition 2.15) is the *infinite jet bundle* operation J^∞_Σ:

$$J^\infty_\Sigma := (\eta_\Sigma)^* \circ (\eta_\Sigma)_*\,.$$

We denote its coproduct and counit natural transformations, respectively, by $\Delta\colon J^\infty_\Sigma \to J^\infty_\Sigma J^\infty_\Sigma$ and $\epsilon\colon J^\infty_\Sigma \to \mathrm{id}$. In other words (by proposition B.25) this is the right adjoint to the construction of formal disk bundles (according to definition 3.8):

$$T^\infty_\Sigma \;\dashv\; J^\infty_\Sigma \;:\; \mathbf{H}_{/\Sigma} \longrightarrow \mathbf{H}_{/\Sigma}\,.$$

If $\sigma \in \mathrm{Hom}(T^\infty_\Sigma X, Y)$ and $\tau \in \mathrm{Hom}(X, J^\infty_\Sigma Y)$ are adjunct morphisms, in the sequel we will use the notation $\tau = \widetilde{\sigma}$ and $\sigma = \overline{\tau}$.

That forming jets is right adjoint to forming infinitesimal disk bundles was observed before in [14, Proposition 2.2] (at least for jets of finite order).

Proposition 3.24. *For $\Sigma \in \mathrm{SmthMfd}$, the jet bundle functor from proposition 3.23 restricts to an endofunctor on the category of fibered manifolds $\mathrm{LocProMfd}_{\downarrow\Sigma}$ (definition 3.18):*

$$J^\infty_\Sigma \;:\; \mathrm{LocProMfd}_{\downarrow\Sigma} \longrightarrow \mathrm{LocProMfd}_{\downarrow\Sigma}\,.$$

Remark 3.25. As a direct corollary, theorem 3.22 implies that the infinite jet bundle functor J^∞_Σ of definition 3.23 naturally carries the structure of a comonad on $\mathrm{FormalSmoothSet}_{/\Sigma}$, and by proposition 3.24 this restricts to a comonad structure on the traditional jet bundle construction on (infinite-dimensional) smooth manifolds.

That there is a comonad structure on the traditional infinite jet bundle construction has been observed before in [23] (and, apparently, only there). We now unwind the structure morphisms in the abstract comonad structure on J^∞_Σ and show that, restricted to locally pro-manifolds, they indeed coincide with the traditional definitions.

Example 3.26. For any $\Sigma \in \mathbf{H}$, regarded trivially as a bundle over itself

$$[\Sigma \xrightarrow{\mathrm{id}} \Sigma] \quad \in \mathbf{H}_{/\Sigma}\,,$$

then the corresponding infinite jet bundle (according to definition 3.23) coincides with Σ, in that the unique morphism

$$J^\infty_\Sigma \Sigma \xrightarrow{\simeq} \Sigma$$

is an isomorphism.

Proof. In full generality, this follows from the fact that $\Sigma \in \mathbf{H}_{/\Sigma}$ (i.e., id_Σ) is a terminal object, and the fact that J^∞_Σ is a right adjoint, by definition 3.23, hence preserves terminal objects.

Explicitly, if $\mathbf{H} = \mathrm{FormalSmoothSet}$ and $\Sigma \in \mathrm{SmthMfd} \hookrightarrow \mathrm{FormalSmoothSet}$ happens to be a smooth manifold, then definition 3.19 says that the jets of id_Σ all have to be constant. □

Definition 3.27. For $E \xrightarrow{p} \Sigma$ any morphism in $\mathbf{H}$, regarded as an object of $\mathbf{H}_{/\Sigma}$, and with $J^\infty_\Sigma E \in \mathbf{H}_{/\Sigma}$ denoting the corresponding infinite jet bundle according to definition 3.23, then *jet prolongation* (or *jet extension*) is the function

$$j^\infty \;:\; \Gamma_\Sigma(E) \longrightarrow \Gamma_\Sigma(J^\infty_\Sigma E)$$

from sections of E (definition B.16) to sections of $J^\infty_\Sigma E$ which is given by

$$(\Sigma \xrightarrow{\sigma} E) \;\mapsto\; (j^\infty\sigma \colon \Sigma \xrightarrow{\simeq} J^\infty_\Sigma \Sigma \xrightarrow{J^\infty_\Sigma \sigma} J^\infty_\Sigma E)\,,$$

where the equivalence on the right is that of example 3.26.

Notice that by the $(T^\infty_\Sigma \dashv J^\infty_\Sigma)$-adjunction, E'-parameterized sections of a jet bundle $J^\infty_X E$ (definition 3.23), namely morphisms of the form

$$\sigma : E' \longrightarrow J^\infty_X E$$

in $\mathbf{H}_{/X}$ (see example B.16) are in natural bijection with morphisms of the form

$$\overline{\sigma} : T^\infty_X E' \longrightarrow E$$

in $\mathbf{H}_{/X}$. Now:

Proposition 3.28. *Let $X \in \mathbf{H}$ be any object, and let $E', E \in \mathbf{H}_{/X}$ be two bundles over X. Then the double $(T^\infty_\Sigma \dashv J^\infty_\Sigma)$-adjunct of a morphism of the form*

$$E' \longrightarrow J^\infty_X E \xrightarrow{\Delta_E} J^\infty_X J^\infty_X E\,,$$

hence of the image of an E'-parameterized section of the jet bundle under the jet coproduct operation, is the image of $\overline{\sigma}$ under the product $\nabla_{E'}$ of the infinitesimal disk bundle monad, namely the morphism

$$T^\infty_X T^\infty_X E' \xrightarrow{\nabla_{E'}} T^\infty_X E \xrightarrow{\overline{\sigma}} E\,.$$

Proof. By proposition B.34. □

The following two propositions show that the abstract concept of jet prolongation in definition 3.27 reduces to the traditional concept of jet prolongation on jets of sections of smooth manifolds.

Proposition 3.29. *For $[E \longrightarrow \Sigma] \in \mathbf{H}_{/\Sigma}$ any slice in $\mathbf{H}$ (definition 2.15), the operation of jet prolongation from definition 3.27*

$$j^\infty : \Gamma_\Sigma(E) \longrightarrow \Gamma_\Sigma(J^\infty_\Sigma E)$$

takes a smooth section $\sigma : \Sigma \to E$ (definition B.16) to the section of the infinite jet bundle whose value over any point $s \in \Sigma$ is the $(T^\infty_\Sigma \dashv J^\infty_\Sigma)$-adjunct infinite jet of σ at that point (definition 3.19):

$$j^\infty \sigma(s) = \widetilde{\sigma|_{\mathbb{D}_s}} : \mathbb{D}_s \longrightarrow \Sigma \xrightarrow{\sigma} E\,.$$

Proof. By definition 3.23 the value of $j^\infty\sigma$ over s is

$$j^\infty\sigma(s) : \{s\} \longrightarrow \Sigma \simeq J^\infty_\Sigma \Sigma \xrightarrow{J^\infty_\Sigma \sigma} J^\infty_\Sigma E\,.$$

By the adjunction $(T^\infty_\Sigma \dashv J^\infty_\Sigma)$ of definition 3.23, this corresponds to

$$T^\infty_\Sigma\{s\} \longrightarrow \Sigma \xrightarrow{\sigma} E\,.$$

By example 3.10 this is

$$\mathbb{D}_s \longrightarrow \Sigma \xrightarrow{\sigma} E$$

as claimed. ∎

But this statement holds generally:

Proposition 3.30. *Let $\Sigma \in \mathbf{H}$ and $[E \to \Sigma] \in \mathbf{H}_{/\Sigma}$ be any objects, and let $\sigma : \Sigma \to E$ be any section. Then the $(T^\infty_\Sigma \dashv J^\infty_\Sigma)$-adjunct $\overline{j^\infty\sigma}$ (definition B.4) of the jet prolongation $j^\infty\sigma : \Sigma \to J^\infty_\Sigma E$ of σ (definition 3.29) is the composite*

$$\overline{j^\infty\sigma} \;:\; T^\infty\Sigma \xrightarrow{\mathrm{ev}} \Sigma \xrightarrow{\sigma} E\,,$$

where ev *is as in definition 3.6.*

Proof. We are to find the adjunct for the composite

$$\Sigma \underset{\simeq}{\xrightarrow{\phi}} J^\infty_\Sigma \Sigma \xrightarrow{J^\infty_\Sigma \sigma} J^\infty_\Sigma E\;\;.$$

By the naturality of forming adjuncts, this is

$$T^\infty_\Sigma \Sigma \xrightarrow{\overline{\phi}} \Sigma \xrightarrow{\sigma} E\,,$$

where $\overline{\phi}$ is the adjunct of ϕ. By the formula for adjuncts (proposition B.5), the latter is

$$\overline{\phi} :\; T^\infty_\Sigma \Sigma \underset{\simeq}{\xrightarrow{T^\infty_\Sigma \phi}} T^\infty_\Sigma J^\infty_\Sigma \Sigma \longrightarrow \Sigma\;\;,$$

where the first morphism is the canonical isomorphism (since ϕ is) and where the second morphism is the counit of the $(T^\infty_\Sigma \dashv J^\infty_\Sigma)$-adjunction, which is given by the units of the adjoint triple $((\eta_\Sigma)_! \dashv (\eta_\Sigma)^* \dashv (\eta_\Sigma)_*)$ as the composite

$$(\eta_\Sigma)^*(\eta_\Sigma)_!(\eta_\Sigma)^*(\eta_\Sigma)_*\Sigma \xrightarrow{\tau} (\eta_\Sigma)^*(\eta_\Sigma)_*\Sigma \xrightarrow{\simeq} \Sigma\,.$$

The second morphism here is the isomorphism $J^\infty_\Sigma\Sigma \simeq \Sigma$ already used. So we are reduced to seeing that the first morphism here is $\tau = \mathrm{ev}$.

To see that this is indeed the case, first notice that $(\eta_\Sigma)_*\Sigma \simeq \Im\Sigma$, since the right adjoint $(\eta_\Sigma)_*$ preserves terminal objects (which is also the origin of the very isomorphism just mentioned). Hence $(\eta_\Sigma)_!(\eta_\Sigma)^*(\eta_\Sigma)_*\Sigma \to (\eta_\Sigma)_*\Sigma$ is the top morphism in the pullback square

$$\begin{tikzcd}
\Sigma \arrow[r] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\text{(pb)}"] & \Im\Sigma \arrow[d, "="] \\
\Sigma \arrow[r, "\eta_\Sigma"'] & \Im\Sigma
\end{tikzcd}$$

which is again η_Σ. Hence the full morphism in question is the pullback of η_Σ along itself. This is either π or ev (definition 3.8), depending on the order of the arguments. Checking the commutativity of the pullback diagram, one finds that it is ev. $\square$

The following proposition shows that the abstractly induced comonad structure on J^∞_Σ (remark 3.25) reduces on jets of bundles of smooth manifolds to the comonad structure considered in [23].

Proposition 3.31. *Consider $[E \longrightarrow \Sigma]$ in $\mathbf{H}_{/\Sigma}$. Then the comultiplication of the abstractly defined jet comonad $\Delta : J^\infty_\Sigma \to J^\infty_\Sigma J^\infty_\Sigma$ from definition 3.23 takes a jet $j^\infty_\Sigma\sigma(s)$, written as a jet prolongation of some section σ, via definition 3.27, to the jet prolongation of that jet prolongation*

$$\Delta_E(j^\infty\sigma(s)) = (j^\infty j^\infty\sigma)(s)\,.$$

Hence if $\mathbf{H} = \mathrm{FormalSmoothSet}$ and $[E \xrightarrow{p} \Sigma] \in \mathrm{LocProMfd}_{\downarrow\Sigma} \hookrightarrow \mathbf{H}_{/\Sigma}$ is a fibered manifold (definition 3.18) then this reduces to the jet coproduct considered in [23, p.3].

Proof. Under the adjunction $T^\infty_\Sigma \dashv J^\infty_\Sigma$ (definition 3.23)

$$j^\infty\sigma(s) \colon \{s\} \longrightarrow J^\infty_\Sigma E$$

is represented by

$$\overline{j^\infty\sigma}(s) \;:\; T^\infty_\Sigma\{s\} \simeq \mathbb{D}_s \hookrightarrow \Sigma \xrightarrow{\sigma} E\,,$$

where the isomorphism on the left is from example 3.10. By proposition 3.28, its image

$$\{s\} \longrightarrow J^\infty_\Sigma E \xrightarrow{\Delta_E} J^\infty_\Sigma J^\infty_\Sigma E$$

under the jet coproduct Δ_E corresponds dually to the pre-composition of this morphism with the product operation in the formal disk bundle monad (definition 3.8):

$$T^\infty_\Sigma T^\infty_\Sigma\{s\} \xrightarrow{\nabla_{\{s\}}} T^\infty_\Sigma\{s\} \simeq \mathbb{D}_s \hookrightarrow \Sigma \xrightarrow{\sigma} E\,.$$

The $(T^\infty_\Sigma \dashv J^\infty_\Sigma)$-adjunct of this morphism is (by proposition B.8) the composite

$$\begin{tikzcd} \mathbb{D}_s \simeq T^\infty_\Sigma\{s\} \arrow[r] \arrow[rr, bend right, "\simeq"'] & J^\infty_\Sigma T^\infty_\Sigma T^\infty_\Sigma\{s\} \arrow[r] & J^\infty_\Sigma T^\infty_\Sigma\{s\} \arrow[r, "J^\infty_\Sigma(\overline{j^\infty\sigma(s)})"] & J^\infty_\Sigma E \end{tikzcd}\;,$$

where the left composition has to be an isomorphism, as shown, because $J^\infty_\Sigma T^\infty_\Sigma\{s\} \simeq J^\infty_\Sigma\mathbb{D}_s \simeq \mathbb{D}_s \simeq T^\infty_\Sigma\{s\}$ (by example 3.10 and example 3.26) and since in $\mathbf{H}_{/\Sigma}$ the object $\mathbb{D}_s$ is sub-terminal so that its only endomorphism is the identity. Hence this morphism is identified with $(j^\infty j^\infty\sigma)(s)$ by example 3.27. $\square$

3.4 Differential operators

We recall the traditional concept of differential operators, formulated in terms of jet bundles. Then we show that the category whose objects are bundles over some Σ and whose morphisms are differential operators between their sections is equivalently the "Kleisli category" of the jet comonad J^∞_Σ from definition 3.23. (This is a special case of the more general identification of PDEs with coalgebras over the jet comonad, that we turn to below in sections 3.5 and 3.6.)

Notice that throughout by "differential operator" we mean in general non-linear differential operators.

Definition 3.32 (e.g. [27, definition 6.2.22])**.** Consider two objects $E \to \Sigma$ and $F \to \Sigma$ in $\mathbf{H}_{/\Sigma}$.

(a) A morphism $D \colon J^\infty_\Sigma E \to F$ induces, by composition with jet prolongation j^∞ (definition 3.27), a map $\hat{D}(\sigma) := D \circ j^\infty\sigma$ between spaces of sections (definition B.16),

$$\hat{D} \colon \Gamma_\Sigma(E) \to \Gamma_\Sigma(F),$$

whose effect is illustrated by the following commutative diagram:

$$
\begin{tikzcd}
 & E \arrow[d] \\
\Sigma \arrow[ur, "\sigma"] \arrow[r, equal] & \Sigma
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
 & & J^\infty_\Sigma E \arrow[r, "D"] \arrow[d] & F \arrow[d] \arrow[dl] \\
\Sigma \arrow[r, "\simeq"] & J^\infty_\Sigma \Sigma \arrow[ur, "j^\infty\sigma \simeq J^\infty_\Sigma \sigma"] \arrow[r, equal] & J^\infty_\Sigma \Sigma & \Sigma \arrow[l, "\simeq"'] \arrow[u, bend right, "\hat{D}(\sigma)"']
\end{tikzcd}
.
$$

(The equivalences on the right are those from example 3.26.)

(b) Conversely, a map between spaces of sections $B \colon \Gamma_\Sigma(E) \to \Gamma_\Sigma(F)$ is called a *differential operator* if it factors through the jet bundle this way. That is $B = \hat{D}$ for some morphism $D \colon J^\infty_\Sigma E \to F$. Naturally, the morphism D is also referred to as a differential operator, or as a *formal differential operator* when more precision is needed.

The formal differential operator D corresponding to a differential operator $B = \hat{D}$ is clearly unique, which we will denote by $D = \underline{B}$.

Remark 3.33. Definition 3.32 is the standard formalization of the informal notion of a differential operator that usually goes along with the notation $\hat{D}[\sigma](x) = D(x, \sigma(x), \partial\sigma(x), \partial\partial\sigma(x), \ldots)$.

Definition 3.34. Given a formal differential operator (definition 3.32)

$$
D : J^\infty_\Sigma E \longrightarrow F\,.
$$

the composite

$$
p^\infty D := J^\infty_\Sigma E \overset{\Delta_E}{\longrightarrow} J^\infty_\Sigma J^\infty_\Sigma E \overset{J^\infty_\Sigma D}{\longrightarrow} J^\infty_\Sigma F
$$

is called its *infinite prolongation*. It is characterized by the relation

$$
\widehat{p^\infty D}[\sigma] = j^\infty \hat{D}[\sigma]
$$

for any section $\sigma \in \Gamma_\Sigma(E)$.

Proposition 3.35. *For $D_1 \colon J^\infty_\Sigma E_1 \longrightarrow E_2$ and $D_2 \colon J^\infty_\Sigma E_2 \longrightarrow E_3$ two formal differential operators according to definition 3.32, then the formal differential operator corresponding to the direct composite of the corresponding maps on spaces of sections (definition B.16)*

$$
\hat{D}_2 \circ \hat{D}_1 \colon\; \Gamma_\Sigma(E_1) \overset{\hat{D}_1}{\longrightarrow} \Gamma_\Sigma(E_2) \overset{\hat{D}_2}{\longrightarrow} \Gamma_\Sigma(E_3)
$$

is the composite

$$
\underline{\hat{D}_2 \circ \hat{D}_1} = D_2 \circ p^\infty D_1 \colon\;\; J^\infty_\Sigma E_1 \overset{p^\infty D_1}{\longrightarrow} J^\infty_\Sigma E_2 \overset{D_2}{\longrightarrow} E_3\,,
$$

(with $p^\infty D_1$ the infinite prolongation of D_1 from definition 3.34) of D_2 with the image under J^∞_Σ of D_1 and with the coproduct Δ of the jet comonad.

In terms of infinite prolongations this is

$$
p^\infty \left(\underline{\hat{D}_2 \circ \hat{D}_1} \right) = (p^\infty D_2) \circ (p^\infty D_1)
$$

Proof. Regarding the first statement: Applying the formula in definition 3.32 twice gives that for $\sigma \in \Gamma_\Sigma(E_1)$ any section, then $(\hat{D}_2 \circ \hat{D}_1)(\sigma)$ is the section given by the total top horizontal composite in the following diagram:

$$
\begin{tikzcd}
\Sigma \arrow[r, "\simeq"] \arrow[dr, "\simeq"'] & J^\infty_\Sigma J^\infty_\Sigma \Sigma \arrow[r, "J^\infty_\Sigma J^\infty_\Sigma \sigma"] & J^\infty_\Sigma J^\infty_\Sigma E_1 \arrow[r, "J^\infty_\Sigma D_1"] & J^\infty_\Sigma E_2 \arrow[r, "D_2"] & E_3 \\
 & J^\infty_\Sigma \Sigma \arrow[u, "\simeq"', "\Delta_\Sigma"] \arrow[r, "J^\infty_\Sigma \sigma"'] & J^\infty_\Sigma E_1 \arrow[u, "\Delta_{J^\infty_\Sigma E_1}"] \arrow[ur, "p^\infty D_1"] \arrow[urr, bend right, "D_2 \circ p^\infty D_1"'] & &
\end{tikzcd}
.
$$

Here the square commutes by the naturality of the coproduct Δ of the jet comonad, and the equivalences on the left are the unique ones from example 3.26. Hence the total top composite is equivalent to the total bottom composite, which is the formula to be proven.

Regarding the second statement, consider the diagram

$$
\begin{tikzcd}
J^\infty_\Sigma E_1 \arrow[rr, bend left, "p^\infty D_1"] \arrow[r, "\Delta_{E_1}"] \arrow[d, "\Delta_{E_1}"'] & J^\infty_\Sigma J^\infty_\Sigma E_1 \arrow[r, "J^\infty D_1"] \arrow[d, "\Delta_{J^\infty_\Sigma E_1}"'] & J^\infty_\Sigma E_2 \arrow[dr, bend left, "p^\infty D_2"] \arrow[d, "\Delta_{E_2}"'] & \\
J^\infty_\Sigma J^\infty_\Sigma E_1 \arrow[r, "J^\infty_\Sigma(\Delta_{E_1})"'] \arrow[rr, bend right, "J^\infty_\Sigma p^\infty D_1"] \arrow[rrr, bend right=40, "J^\infty_\Sigma(\underline{\hat{D}_2 \circ \hat{D}_1})"'] & J^\infty_\Sigma J^\infty_\Sigma J^\infty_\Sigma E_1 \arrow[r, "J^\infty_\Sigma J^\infty_\Sigma D_1"'] & J^\infty_\Sigma J^\infty_\Sigma E_2 \arrow[r, "J^\infty_\Sigma D_2"'] & J^\infty E_3
\end{tikzcd}
.
$$

Here both squares are naturality squares of the coproduct Δ. The total top composite is $p^\infty D_2 \circ p^\infty D_1$, while the total left and bottom composite is $p^\infty(\underline{\hat{D}_2 \circ \hat{D}_1})$. □

Definition 3.36. For Σ a smooth manifold, we write $\mathrm{DiffOp}_{\downarrow\Sigma}(\mathrm{LocProMfd})$ for the category whose objects are fibered manifolds over Σ (definition 3.18) and whose morphisms are differential operators between their spaces of sections, according to definition 3.32.

Proposition 3.37. *There is an equivalence of categories*

$$\mathrm{DiffOp}_{\downarrow\Sigma}(\mathrm{LocProMfd}) \simeq \mathrm{Kl}(J^\infty_\Sigma|_{\mathrm{LocProMfd}_{\downarrow\Sigma}})$$

between the category of differential operators over Σ (definition 3.36) and the co-Kleisli category $\mathrm{Kl}(J^\infty_\Sigma)$ of the jet comonad J^∞_Σ restricted to fibered manifolds (according to proposition 3.24).

Proof. This is a direct consequence of definition B.35 and remark B.37, together with definition 3.32 and proposition 3.35. □

Remark 3.38. Proposition 3.37 implies that it makes sense to regard the full Kleisli category of the J^∞_Σ comonad on all of $\mathbf{H}_{/\Sigma}$ as the category of differential operators on sections of all objects of $\mathbf{H}_{/\Sigma}$, which we will denote by $\mathrm{DiffOp}_{/\Sigma}(\mathbf{H}) := \mathrm{Kl}(J^\infty_\Sigma)$. Hence a general morphism of the form

$$D : J^\infty_\Sigma E \longrightarrow F$$

in $\mathbf{H}_{/\Sigma}$ may be thought of as a *generalized differential operator* (the generalization being that E and F may both be far from having the structure of fibered manifolds).

3.5 Partial differential equations

We now give a general synthetic definition of partial differential equations (PDEs) in a differentially cohesive topos $\mathbf{H}$ (definition 2.15 below). Throughout this section we make detailed comments on how to connect this synthetic definition to traditional concepts.

We are concerned with the geometric perspective on partial differential equations, from the point of view of jet bundles, sometimes known as the *formal theory of PDEs*. This has a long history going back to the works of Riquier, E. Cartan, Janet and many others, with the influential ideas of Spencer [30] and Vinogradov [33] (eventually developed by them and many others) being essential for its modern formulation. The idea here is that, just as with ordinary (algebraic) equations, which we can put in correspondence with the loci of solutions that they carve out inside the domain of definition of their variables, so we may think of partial differential equations as being embodied by the sub-loci which they carve out inside spaces of partial derivatives, hence inside jet bundles:

Definition 3.39 (generalized PDEs)**.** Let $\Sigma \in \mathbf{H}$ be a V-manifold (definition 3.3), and let $Y \in \mathbf{H}_{/\Sigma}$ be an object in the slice over Σ (thought of as a bundle, example B.16). Then a *generalized partial differential equation* (PDE) on the space of sections of Y is an object $\mathcal{E} \in \mathbf{H}_{/\Sigma}$ together with a monomorphism into the infinite jet bundle of Y (definition 3.23),

$$\mathcal{E} \hookrightarrow J^\infty_\Sigma Y \,.$$

Note that we are not excluding the possibility that $Y = \mathcal{E}$. We omit the *generalized* attribute when Σ is a smooth finite-dimensional manifold and $\mathcal{E} \in \mathrm{LocProMfd}_{\downarrow\Sigma} \hookrightarrow \mathrm{FormalSmoothSet}_{/\Sigma} = \mathbf{H}_{/\Sigma}$ is a fibered manifold (definition 3.18).

We say that a section $\sigma \in \Gamma_\Sigma(Y)$ (definition B.16) is a *solution* to this partial differential equation if its jet prolongation $j^\infty_\Sigma \sigma$ (definition 3.27) factors through $\mathcal{E}$, i.e., if there exists the dashed morphism s making the following diagram commute:

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook] & J^\infty_\Sigma Y \arrow[d] \\ \Sigma \arrow[u, dashed, "s"] \arrow[ur, "j^\infty\sigma"] \arrow[r, equal] & \Sigma \end{tikzcd} .$$

We write

$$\mathrm{Sol}_\Sigma(\mathcal{E}) \hookrightarrow \Gamma_\Sigma(Y)$$

for the subset of solutions to the PDE, inside the set of sections (example B.16).

Next we pass attention to *formal theory of PDEs*. The reason that it is referred to as *formal* is that in it center stage is taken not by true solutions (in the sense of definition 3.39 above), but *formal solutions* (definition 3.40 below), essentially formal Taylor series that satisfy the PDE order by order. See remark 3.41 below for how to interpret the abstract definition 3.40. In order to make this work in full generality it invokes parameterization of families of formal solutions by infinitesimal disk bundles as above in proposition 3.28. Below in example 3.42 we show that this general definition reduces pointwise to the expected one.

The concept of formal solutions is natural since jets do not see past the "formal power series horizon." While formal solutions have been used as a guiding heuristic in the traditional literature on PDEs, their treatment has traditionally been only informal (meaning heuristic). What here allows us to work with formal solutions directly is the flexibility of synthetic differential geometry in the guise of differential cohesion, which makes formal infinitesimal neighbourhoods in a manifold a reality. As the central role of families of formal solutions becomes apparent below, the ability to work with them directly clearly becomes an advantage of the synthetic formalization of PDEs.

Definition 3.40 (formal solutions of PDEs)**.** Let Σ be a V-manifold (definition 3.28). Consider a generalized PDE $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ over Σ according to definition 3.39 and a further parameter object $E \in \mathbf{H}_{/\Sigma}$. Then:

(a) An *(E-parametrized) family of formally holonomic sections* of $J^\infty_\Sigma Y$ is a morphism

$$\sigma_E : T^\infty_\Sigma E \longrightarrow J^\infty_\Sigma Y$$

from the infinitesimal disk bundle of E (definition 3.8) such that its $(T^\infty_\Sigma \dashv J^\infty_\Sigma)$-adjunct $\widetilde{\sigma_E} : E \to J^\infty_\Sigma J^\infty_\Sigma E$ (definition B.4) makes the following diagram commute:

$$\begin{tikzcd} T^\infty_\Sigma E \arrow[r, "\sigma_E"] & J^\infty_\Sigma Y \arrow[d, "\Delta_Y"] \\ E \arrow[u, "\eta_E"] \arrow[r, "\widetilde{\sigma_E}"'] & J^\infty_\Sigma J^\infty_\Sigma Y \end{tikzcd} ,$$

or equivalently that its $(T^\infty_\Sigma \dashv J^\infty_\Sigma)$-adjunct "to the other side" $\overline{\sigma_E} : T^\infty_\Sigma T^\infty_\Sigma E \to E$ makes the following diagram commute:

$$\begin{tikzcd} T^\infty_\Sigma T^\infty_\Sigma E \arrow[r, "\overline{\sigma_E}"] \arrow[d, "\nabla_E"'] & Y \\ T^\infty_\Sigma E \arrow[r, "\sigma_E"'] & J^\infty_\Sigma Y \arrow[u, "\epsilon_Y"'] \end{tikzcd} .$$

(b) An *(E-parametrized) family of formal solutions* of $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ is a morphism

$$s_E : T^\infty_\Sigma E \longrightarrow \mathcal{E}$$

such that the composite

$$\sigma_E : T^\infty_\Sigma E \xrightarrow{s_E} \mathcal{E} \hookrightarrow J^\infty_\Sigma Y$$

is an E-parametrized family of formally holonomic sections of $J^\infty_\Sigma Y$.

Remark 3.41. By corollary 3.16 the two conditions in definition 3.40 for a morphism σ_E to be a formally holonomic section σ_E mean in terms of local generalized elements (definition 3.15) equivalently that $\overline{\sigma_E}(x, a, b)$ depends "symmetrically" on its infinitesimal arguments a and b: The first condition says that

$$\widetilde{\sigma}_E(x)(a, b) = \sigma_E(x, 0)(a + b)$$

hence equivalently

$$\sigma_E(x, a)(b) = \sigma_E(x, 0)(a + b) .$$

while the second similarly states that

$$\overline{\sigma_E}(x, a, b) = \sigma_E(x, a + b)(0)$$

hence equivalently

$$\sigma_E(x, a)(b) = \sigma_E(x, a + b)(0) .$$

This also shows that the two conditions are indeed equivalent: The first condition implies the second by replacing $a \mapsto a + b$ and $b \mapsto 0$, while the second implies the first by replacing $a \mapsto 0$ and $b \mapsto a + b$.

Example 3.42 (traditional formal solutions)**.** Let $\mathbf{H} = \text{FormalSmoothSet}$ be the Cahiers topos (definition 2.1) and let Σ be an ordinary smooth manifold, according to example 3.4. Let $E = *$ be the abstract point, with the slice morphism $x\colon * \to \Sigma$ picking some point in Σ. Recall, by example 3.10, that its formal neighborhood bundle $T^\infty_\Sigma\{*\} \to \Sigma$ is the inclusion $\mathbb{D}_x \to \Sigma$ of the formal neighborhood of the point x.

Then, a $*$-parametrized family of formal solutions of $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$, in the general sense of definition 3.40, is precisely what is usually known as a *formal solution at* $x \in \Sigma$, namely a formal power series $T^\infty_\Sigma\{*\} \simeq \mathbb{D}_x \xrightarrow{\sigma} Y$ at $x \in \Sigma$ valued in Y, such that its jet extension $T^\infty_\Sigma\{*\} \simeq \mathbb{D}_x \overset{j^\infty\sigma}{\hookrightarrow} J^\infty_\Sigma Y$ factors through the solution locus $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$.

Proof. Consider a $*$-parametrized family of formal solutions $T^\infty_\Sigma\{*\} \xrightarrow{\sigma^\infty} \mathcal{E}$ and the two commutative diagrams equivalently exhibiting this property (definition 3.40):

$$\begin{tikzcd} & & \mathcal{E} \arrow[d, hook] \\ T^\infty_\Sigma\{*\} \simeq \mathbb{D}_x \arrow[urr, "\sigma^\infty"] \arrow[rr, "j^\infty\sigma"] & & J^\infty_\Sigma Y \arrow[d, "\Delta_Y"] \\ * \arrow[u, "\eta_*"] \arrow[urr, "\tilde{\sigma}"] \arrow[rr, "\widetilde{j^\infty\sigma}"'] & & J^\infty_\Sigma J^\infty_\Sigma Y \end{tikzcd} \qquad \begin{tikzcd} T^\infty_\Sigma T^\infty_\Sigma\{*\} \arrow[r, "\overline{j^\infty\sigma}"] \arrow[d, "\nabla_*"'] & Y \\ T^\infty_\Sigma\{*\} \simeq \mathbb{D}_x \arrow[ur, "\sigma"] \arrow[r, "j^\infty\sigma"'] \arrow[dr, "\sigma^\infty"'] & J^\infty_\Sigma Y \arrow[u, "\epsilon_Y"'] \\ & \mathcal{E} \arrow[u, hook] \end{tikzcd} .$$

By remark 3.41, we can recover the formal solution σ^∞ from the knowledge of the diagonal morphism in the diagram on the right, which we have denoted by σ. By proposition 3.29 the top horizontal morphism on the right is the $(T^\infty_\Sigma \dashv J^\infty_\Sigma)$-adjunct $\overline{j^\infty \sigma}$ of the jet prolongation $j^\infty\sigma$ (definition 3.27). This identifies the bottom horizontal morphism on the right and hence the top horizontal morphism on the left with the jet prolongation $j^\infty\sigma$ itself, as shown. Now the commutativity of the triangle diagrams shows that $j^\infty\sigma$ factors as $\mathbb{D}_x \xrightarrow{\sigma^\infty} \mathcal{E} \to J^\infty_\Sigma Y$, which concludes the proof in one direction.

In the other direction, suppose that we have a section $\sigma\colon T^\infty_\Sigma\{*\} \to Y$, such that $j^\infty\sigma$ factors through $\mathcal{E}$. Then, by remark 3.41, we recover a commutative diagram like the one above on the right and hence a $*$-parametrized family of formal solutions. $\square$

Example 3.43. Given a section $\sigma\colon \Sigma \to Y$, it is a true solution in the sense of definition 3.39, precisely if the $(T^\infty_\Sigma \dashv J^\infty_\Sigma)$-adjunct $\sigma^\infty : T^\infty\Sigma \to J^\infty_\Sigma Y$ (definition B.4) of its double jet prolongation $j^\infty j^\infty\sigma\colon \Sigma \to J^\infty_\Sigma J^\infty_\Sigma Y$ (definition 3.27), is a Σ-parametrized family of formal solutions according to definition 3.40.

Proof. By proposition 3.30, an equivalent formula for σ^∞ is $\sigma^\infty\colon T^\infty_\Sigma\Sigma \xrightarrow{\mathrm{ev}} \Sigma \xrightarrow{j^\infty\sigma} J^\infty_\Sigma Y$.

Then, consider the diagram

$$\begin{tikzcd}
 & & \mathcal{E} \arrow[d, hook] \\
T^\infty_\Sigma\Sigma \arrow[urr, dashed] \arrow[rr, "\sigma^\infty"] & & J^\infty_\Sigma Y \arrow[d, "\Delta_Y"] \\
\Sigma \arrow[u, "\eta_\Sigma"] \arrow[urr, "j^\infty\sigma"] \arrow[rr, "j^\infty j^\infty\sigma"'] & & J^\infty_\Sigma J^\infty_\Sigma Y
\end{tikzcd},$$

where we do not a priori know whether the dashed morphism exists. By the definition of σ^∞ and the identity $j^\infty j^\infty\sigma = \Delta_Y \circ j^\infty\sigma$ (proposition 3.31), the solid arrows are known to commute. Hence, σ^∞ is a Σ-parametrized family of formally holonomic sections. If σ^∞ happens to be a family of formal solutions, then the dashed arrow exists and the whole diagram commutes, implying that $j^\infty\sigma$ factors through $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$, hence a true solution. On the other hand, if σ is a true solution, then $j^\infty\sigma$ factors through $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$. Precomposing this factorization with $T^\infty_\Sigma\Sigma \to \Sigma$ then shows that σ^∞ also factors through $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$, meaning that the dashed morphism exists and commutes with the rest of the diagram, or in other words that σ^∞ is a family of formal solutions. $\square$

We close this section with remarks on how to connect the above synthetic formalization to traditional concepts.

Remark 3.44. In the case that $\mathbf{H} = \mathrm{FormalSmoothSet}$ is the Cahiers topos (definition 2.1) and when $\mathcal{E}$ and solutions $\sigma\colon \Sigma \to \mathcal{E}$ belong to $\mathrm{LocProMfd}_{\downarrow\Sigma} \hookrightarrow \mathbf{H}_{/\Sigma}$ (definition 3.18), the above definitions coincide with the one given by Vinogradov [33, 23]. A slightly more traditional definition [9, 11] restricts Vinogradov's notion of a PDE on the sections of $Y \to \Sigma$ by the extra requirement that image of $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ is a closed submanifold; sometimes the restriction that the composition with the natural projection $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y \to Y$ is surjective is also invoked. It should be noted that, by ignoring the latter requirements, Vinogradov's notion of PDE admits also examples that have been called *partial differential relations* [10], which may be specified by inequalities (rather than equalities) between differential operators.

Remark 3.45. In a category of sheaves such as the slice topos $\mathbf{H}_{/\Sigma}$, the difference between equations and inequalities gets blurred. There, every monomorphism is a *regular monomorphism* (prop. B.23), which means that every subobject is characterized by an equation, in that every subobject inclusion like $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ is part of an equalizer diagram of the form

$$\begin{tikzcd}
\mathcal{E} \arrow[r, hook] & J^\infty_\Sigma Y \arrow[r, shift left, "D_1"] \arrow[r, shift right, "D_2"'] & Z
\end{tikzcd}$$

for some object Y and some morphisms D_1, D_2 in $\mathbf{H}_{/\Sigma}$. Since by remark 3.38 we may view the morphisms D_1 and D_2 above as generalized differential operators, this exhibits every PDE in the general sense of definition 3.39 as the locus carved out by an equation between differential operators. However, even if the inclusion $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ lives in $\mathrm{LocProMfd}_{\downarrow\Sigma} \hookrightarrow \mathbf{H}_{/\Sigma}$, the target object Z of the corresponding morphisms D_1, D_2 may be quite far from a standard smooth manifold. For instance, the inclusion $(0,1) \hookrightarrow \mathbb{R}$ is the equalizer of two maps $\mathbb{R} \mapsto \mathbb{R} \sqcup_{(0,1)} \mathbb{R}$ into the non-Hausdorff space constructed by glueing two copies of $\mathbb{R}$ along the $(0,1)$ subinterval.

Remark 3.46. It is worth dwelling a bit on the condition that a generalized PDE $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ be included in a jet bundle by a monomorphism. In our definition, this is a monomorphism in the slice category $\mathbf{H}_{/\Sigma}$. As will be seen below, no further conditions on this morphism will be necessary. On the other hand, although we have a full inclusion of the sub-category fibered manifolds, $\mathrm{LocProMfd}_{\downarrow\Sigma} \hookrightarrow \mathbf{H}_{/\Sigma}$, the notion of an $\mathbf{H}_{/\Sigma}$-monomorphism is strictly stronger than a monomorphism in fibered manifolds. Since Marvan was working in fibered manifolds [23, 24], he had to require an additional condition[3]: the inclusion monomorphism needed to remain a monomorphism under the V-functor (vertical tangent bundle functor). The main consequence of this extra hypothesis is that such monomorphisms are then also preserved by the J^∞_Σ functor. In our case, $\mathbf{H}_{/\Sigma}$-monomorphisms (and even more generally all limits) are preserved by J^∞_Σ because it is defined as a right adjoint (definition 3.23, proposition B.7).

3.6 Formally integrable PDEs

A PDE is to be called *integrable* if given a solution jet at one point, it may be extended to a jet prolongation of a local section on an open neighbourhood of that point. Accordingly, a PDE is to be called *formally integrable* if given a solution jet at one point, it may be extended to a jet prolongation of a section defined at least on a *formal infinitesimal neighbourhood* of that point. We now give an abstract synthetic definition (not actually referring to points) of this concept of formally integrable PDEs, this is definition 3.47 below.

Then we prove in this generality that the category formally integrable PDEs in a differentially cohesive topos is equivalent to the category of coalgebras over the jet-comonad. This is theorem 3.52 below. For the special case that $\mathbf{H} = \mathrm{FormalSmoothSet}$ is Dubuc's Cahiers topos (definition 2.11), and that $\mathcal{C} = \mathrm{LocProMfd}_{\downarrow\Sigma}$ is the category of locally pro-manifolds fibered over an ordinary manifold Σ (definition 3.18) we recover $\mathrm{PDE}(\mathcal{C}) \hookrightarrow \mathrm{PDE}_{/\Sigma}(\mathbf{H})$ as the image of the embedding of Vinogradov's category of PDEs over Σ, since by the main result of [23, 24] they are both equivalent to the category of colagebras over the jet-comonad in fibered manifolds (corollary 3.55).

Finally we use this equivalence in order to discuss finite limits in the category of formally integrable PDEs (theorem 3.57 and corollary 3.58 below). This will be crucial (in future work) for the discussion of variational (Euler-Lagrange) PDEs, which characterize the vanishing locus (hence a certain kernel) of certain PDE morphisms, more precisely given by variational derivatives of Lagrangian densities.

Definition 3.47 (formally integrable PDEs)**.** Consider $\mathcal{E}, Y \in \mathbf{H}_{/\Sigma}$ and a generalized PDE $e_Y \colon \mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ (definition 3.39).

(a) Its *(infinite) prolongation* is the generalized PDE denoted by the $\mathbf{H}_{/\Sigma}$-monomorphism

$$e^\infty_Y \colon \mathcal{E}^\infty \hookrightarrow J^\infty_\Sigma Y$$

[3] This condition was originally and erroneously omitted from Proposition 1.4 of [23], which was later corrected in Theorem 1.3 of [24]. The condition itself can be seen as a strengthened version of being an *immersion* or as a weakened version of being *transversal*.

and defined by the pullback square[4]

$$\begin{tikzcd} \mathcal{E}^\infty \arrow[r, "e_Y^\infty"] \arrow[d, "\rho_{\mathcal{E}}^\infty"'] \arrow[dr, phantom, "\text{(pb)}"] & J^\infty_\Sigma Y \arrow[d, "\Delta_Y"] \\ J^\infty_\Sigma \mathcal{E} \arrow[r, "J^\infty_\Sigma e_Y"'] & J^\infty_\Sigma J^\infty_\Sigma Y \end{tikzcd} .$$

We call the $\mathbf{H}_{/\Sigma}$-monomorphism $e_{\mathcal{E}}^\infty = \rho_{\mathcal{E}}^\infty \circ \epsilon_{\mathcal{E}} \colon \mathcal{E}^\infty \hookrightarrow \mathcal{E}$ the *canonical inclusion* of the prolongation in the original PDE.

(b) If the canonical inclusion is in fact an isomorphism, $e_{\mathcal{E}}^\infty \colon \mathcal{E}^\infty \xrightarrow{\simeq} \mathcal{E}$, we say that the generalized PDE $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ is *formally integrable*.

(c) If $e'_Y \colon \mathcal{E}' \hookrightarrow J^\infty_\Sigma Y'$ is another generalized PDE, then an $\mathbf{H}_{/\Sigma}$-morphism $\phi \colon \mathcal{E} \to \mathcal{E}'$ is said to *preserve formal solutions* if for any parametrized family of formal solutions $\sigma^\infty \colon T^\infty_\Sigma E \to \mathcal{E}$, the composition $\phi \circ \sigma^\infty \colon T^\infty_\Sigma E \to \mathcal{E}'$ is still a parametrized family of formal solutions.

(d) Denote by $\mathrm{PDE}_{/\Sigma}(\mathbf{H})$ the category whose objects are formally integrable generalized PDEs $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$, with $\mathcal{E}, Y \in \mathbf{H}_{/\Sigma}$, and whose morphisms are $\mathbf{H}_{/\Sigma}$-morphisms $\phi \colon \mathcal{E} \to \mathcal{E}'$ that preserve formal solutions.

Remark 3.48. (a) Definition 3.47(a) says that the prolongation of a PDE locus $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ (definition 3.39) yields another PDE locus $\mathcal{E}^\infty \hookrightarrow \mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ that is *smaller or equal* to the original one. In the sense of remark 3.45 this means that there are in general *more/stronger* equations characterizing $\mathcal{E}^\infty$ than there are equations characterizing $\mathcal{E}$, and it is in this sense that the differential equation is *prolonged* as we pass to $\mathcal{E}^\infty$.

(b) Combining the definition of the prolongation $\mathcal{E}^\infty$ of a PDE with example 3.43 shows that both $\mathcal{E}$ and its prolongation have the same true solutions, because by definition they have the same formal solutions. But it is important to note that dealing with formal solutions is indispensable in this context. There are examples of PDEs that have lots of formal solutions, but none that can be extended even to local sections [18] (see [36] for an insightful exposition). Thus, if we were to define the prolongation $\mathcal{E}^\infty$ of $\mathcal{E}$ as the largest sub-object that admits the same true solutions, it would be empty in those cases. However, under such a definition the prolongation would be much more difficult to compute, because jets do not see past the "horizon" of formal solutions.

(c) Finally, a comment on the term *formally integrable*. One way construct a formal solution $\sigma^\infty \colon T^\infty_\Sigma(*) \simeq \mathbb{D}_* \to \mathcal{E}$ is as a limit $\sigma^\infty = \varprojlim_k \sigma^k$, where $\mathbb{D}_* = \varinjlim_k \mathbb{D}_*(k)$ and $\sigma^k \colon \mathbb{D}_*(k) \to \mathcal{E}$ are formal solutions of finite orders. The construction would proceed inductively, by starting with a $\sigma^0 \colon * \to \mathcal{E}$ and successively constructing higher orders. However, if the σ^0 does not factor through the canonical inclusion $\mathcal{E}^\infty \hookrightarrow \mathcal{E}$, this inductive procedure will be obstructed at some higher order and it is said that such a finite order formal solution is *not integrable*. Thus, another way to interpret $\mathcal{E}^\infty$ is as the largest sub-object of $\mathcal{E}$ that consists of 0-th order formal solutions that are integrable (lemma A.2 makes this more precise by proving that $\mathcal{E}^\infty$ parametrizes a universal family of formal solutions). These ideas have lead to a finite order version of the notion of formally integrable (for instance, as expressed in Definition 7.1 of [9]), which when extended to infinite order coincides with our definition 3.47.

Remark 3.49. (a) Consider any full subcategory $\mathcal{C} \hookrightarrow \mathbf{H}_{/\Sigma}$ such that we can restrict $J^\infty_\Sigma \colon \mathcal{C} \to \mathcal{C}$. One example is the subcategory $\mathcal{C} = \mathrm{LocProMfd}_{\downarrow\Sigma}$ of locally pro-manifolds fibered over Σ (definition 3.18).

(b) We can then denote by $\mathrm{PDE}(\mathcal{C}) \hookrightarrow \mathrm{PDE}_{/\Sigma}(\mathbf{H}) =: \mathrm{PDE}(\mathbf{H}_{/\Sigma})$ the full subcategory where $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ and $\phi \colon \mathcal{E} \to \mathcal{E}'$ are all morphisms in $\mathcal{C}$. It is important to note that the definition of $\mathrm{PDE}(\mathcal{C})$ still refers to the larger category $\mathbf{H}_{/\Sigma}$, which is needed to check the $\mathbf{H}_{/\Sigma}$-monomorphism property of $\mathcal{E} \hookrightarrow J^\infty_\Sigma$ as well as to introduce families of formal solutions (for instance, $\mathcal{C}$ itself might not admit any manifolds with infinitesimal dimensions), to verify that they are preserved under morphisms and to check formal integrability.

[4]Note that, because e_Y^∞ and Δ_Y are monomorphisms, the rest of the morphisms in the diagram are also monomorphisms, since J^∞_Σ preserves monomorphisms (proposition B.7) and pullbacks of monomorphisms are monomorphisms.

(c) The example of $\mathrm{PDE}_{\downarrow\Sigma}(\mathrm{LocProMfd}) := \mathrm{PDE}(\mathrm{LocProMfd}_{\downarrow\Sigma})$ was explicitly considered in [23, 24] where it was identified with a subcategory of Vinogradov's category of PDEs [32, 33]. Strictly speaking, morphisms in Vinogradov's category are defined differently: instead of preserving formal solutions they preserve an alternative geometric structure (the *Cartan distribution*). On the other hand, informal comments in the literature (such as those at the top of [32, §8.6]) indicate that our definition is the intuitively preferred one. However, since infinitesimals are not strictly part of classical differential geometry, the difficulty of describing formal solutions in a precise and concise way has made the Cartan distribution a favored proxy in the existing literature. We will not go into the details here of how the Cartan distribution relates to formal solutions. It suffices to note that our definition of the PDE category ultimately coincides with Vinogradov's (remark 3.54 and corollary 3.55), at least in the $\mathrm{PDE}_{\downarrow\Sigma}(\mathrm{LocProMfd})$ case. Later, in remark 3.59, we will comment on how identifying the coalgebra structure underlying a PDE could be interpreted as putting the PDE in "canonical form."

Next we prove some lemmas containing the main technical results needed to establish our main theorem 3.52 below on the structure of the category of generalized PDEs.

Lemma 3.50. *Consider $E, E', \mathcal{E}, \in \mathbf{H}_{/\Sigma}$, a generalized PDE $e_Y\colon \mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ (definition 3.39) and an E-parametrized family of formal solutions $s_E\colon T^\infty_\Sigma E \to \mathcal{E}$. Then, for any morphism $\phi\colon E' \to E$, the composite $s_E \circ T^\infty_\Sigma \phi\colon T^\infty_\Sigma E' \to \mathcal{E}$ is also an E'-parametrized family of formal solutions.*

Proof. The only property that we need to check is that the composition $e_Y \circ s_E \circ T^\infty_\Sigma \phi\colon T^\infty_\Sigma E' \to J^\infty_\Sigma Y$ is an E'-parametrized family of formally holonomic sections. This property is precisely captured by the commutativity of the outer part of the following diagram:

$$\begin{tikzcd}
T^\infty_\Sigma E' \arrow[r, "T^\infty_\Sigma \phi"] \arrow[rr, bend left, dashed] & T^\infty_\Sigma E \arrow[r, "e_Y \circ s_E"] & J^\infty_\Sigma Y \arrow[d, "\Delta_Y"] \\
E' \arrow[u, "\eta_{E'}"] \arrow[r, "\phi"'] \arrow[rr, bend right, dashed, "\widetilde{(e_Y \circ s_E \circ T^\infty_\Sigma \phi)}"'] & E \arrow[u, "\eta_E"'] \arrow[r, "\widetilde{e_Y \circ s_E}"'] & J^\infty_\Sigma J^\infty_\Sigma Y
\end{tikzcd} .$$

Here the square on the right commutes by hypothesis; the left square commutes by naturality of the unit of the monad T^∞_Σ; while the top triangle commutes by definition. Finally, the bottom triangle commutes by the natural hom-isomorphism of the $(T^\infty_\Sigma \dashv J^\infty_\Sigma)$-adjunction, which provides us with a bijection between the following kinds of commutative triangles:

$$\begin{tikzcd}
E' \arrow[d, "\phi"'] \arrow[dr] & \\
E \arrow[r] & J^\infty_\Sigma Z
\end{tikzcd}
\quad \leftrightarrow \quad
\begin{tikzcd}
T^\infty_\Sigma E' \arrow[d, "T^\infty_\Sigma \phi"'] \arrow[dr] & \\
T^\infty_\Sigma E \arrow[r] & Z
\end{tikzcd} .$$

□

Proposition 3.51. *Consider a formally integrable generalized PDE $e_Y\colon \mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ (definition 3.47).*

(a) Using the canonical inclusion isomorphism, $e^\infty_{\mathcal{E}}\colon \mathcal{E}^\infty \xrightarrow{\simeq} \mathcal{E}$, a formally integrable PDE parametrizes its own universal family of formal solutions, $\overline{\rho_{\mathcal{E}}}\colon T^\infty_\Sigma \mathcal{E} \to \mathcal{E}$, where $\rho_{\mathcal{E}} = \rho^\infty_{\mathcal{E}} \circ e^\infty_{\mathcal{E}}$ (cf. lemma A.2).

(b) The adjunct morphism $\rho_{\mathcal{E}}\colon \mathcal{E} \hookrightarrow J^\infty_\Sigma \mathcal{E}$ to $\overline{\rho_{\mathcal{E}}}$ defines a coalgebra structure (definition B.30) over the jet comonad J^∞_Σ.

(c) Conversely, any coalgebra structure $\rho_{\mathcal{E}}\colon \mathcal{E} \hookrightarrow J^\infty_\Sigma \mathcal{E}$ is also a formally integrable generalized PDE.

(d) For $e'_Y \colon \mathcal{E}' \hookrightarrow J^\infty Y'$ another formally integrable generalized PDE, an $\mathbf{H}_{/\Sigma}$-morphism $\phi \colon \mathcal{E} \to \mathcal{E}'$ preserves formal solutions iff it is a morphism of corresponding coalgebras, meaning that it fits into the commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\phi"] \arrow[d, "\rho_{\mathcal{E}}"'] & \mathcal{E}' \arrow[d, "\rho_{\mathcal{E}'}"] \\ J^\infty_\Sigma \mathcal{E} \arrow[r, "J^\infty_\Sigma \phi"'] & J^\infty_\Sigma \mathcal{E}' \end{tikzcd} .$$

Proof. (a) This is just a restatement (placed here for convenience) of the result of lemma A.2 in light of the definition of formal integrability (definition 3.47(b)).

(b) First to see the counitality condition, let us identify $\mathcal{E}^\infty \simeq \mathcal{E}$ and $e_Y \simeq e_Y^\infty$ by formal integrability, and consider the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook, "e_Y"] \arrow[d, "\rho_{\mathcal{E}}"'] \arrow[dr, phantom, "\text{(pb)}"] & J^\infty_\Sigma Y \arrow[d, "\Delta_Y"] \arrow[dd, bend left=60, "\mathrm{id}"] \\ J^\infty_\Sigma \mathcal{E} \arrow[r, hook, "J^\infty_\Sigma e_Y"'] \arrow[d, "\epsilon_{\mathcal{E}}"'] & J^\infty_\Sigma J^\infty_\Sigma Y \arrow[d, "\epsilon_{J^\infty_\Sigma Y}"] \\ \mathcal{E} \arrow[r, hook, "e_Y"'] & J^\infty_\Sigma Y \end{tikzcd} .$$

Here the top square is the defining pullback square of the formally integrable PDE, while the bottom square is the naturality square for the counit of the jet comonad. The right vertical composite is the identity by counitality of the coproduct Δ. A consequence of the commutativity of the whole diagram is the factorization identity $e_Y \circ (\epsilon_{\mathcal{E}} \circ \rho_{\mathcal{E}}) = e_Y$. But, since e_Y is a monomorphism and $e_Y \circ \mathrm{id} = e_Y$ is another factorization, the two factorizing morphisms must be the same,

$$\epsilon_{\mathcal{E}} \circ \rho_{\mathcal{E}} = \mathrm{id} ,$$

which is the counitality condition for a coalgebra structure $\rho_{\mathcal{E}}$. It also means that $\rho_{\mathcal{E}}$ is a split monomorphism with retraction morphism $\epsilon_{\mathcal{E}}$.

Now, to see the coaction property, consider the diagram

$$\begin{tikzcd} J^\infty_\Sigma J^\infty_\Sigma \mathcal{E} \arrow[r, "J^\infty_\Sigma J^\infty_\Sigma e_Y"] & J^\infty_\Sigma J^\infty_\Sigma J^\infty_\Sigma Y & J^\infty_\Sigma J^\infty_\Sigma Y \arrow[l, "J^\infty_\Sigma \Delta_Y"'] \\ J^\infty_\Sigma \mathcal{E} \arrow[u, "\Delta_{\mathcal{E}}"] \arrow[r, "J^\infty_\Sigma e_Y"'] & J^\infty_\Sigma J^\infty_\Sigma Y \arrow[u, "\Delta_{J^\infty_\Sigma Y}"] & J^\infty_\Sigma Y \arrow[u, "\Delta_Y"'] \arrow[l, "\Delta_Y"] \end{tikzcd}$$

made from naturality squares of the coproduct Δ (it consists of two faces of the "commuting cube" diagram in [23, §§2.3–4]). Since J^∞_Σ is a right adjoint it preserves fiber products. By definition of a formally integrable PDE, the fiber product over the bottom arrows is $\mathcal{E}$, and since J^∞_Σ preserves limits the fiber product over the top morphisms is $J^\infty_\Sigma \mathcal{E}$. By the universal property of the latter fiber product, there is hence a unique dashed morphism e making the two rectangles in the following diagram commute:

$$\begin{tikzcd} J^\infty_\Sigma \mathcal{E} & J^\infty_\Sigma J^\infty_\Sigma \mathcal{E} \arrow[l, "J^\infty_\Sigma \epsilon_{\mathcal{E}}"'] & J^\infty_\Sigma \mathcal{E} \arrow[l, "J^\infty_\Sigma \rho_{\mathcal{E}}"] \arrow[r, "J^\infty_\Sigma e_Y"] \arrow[ll, bend right, "J^\infty_\Sigma \mathrm{id} = \mathrm{id}"'] & J^\infty_\Sigma J^\infty_\Sigma Y \\ & J^\infty_\Sigma \mathcal{E} \arrow[ul, "\mathrm{id}"] \arrow[u, "\Delta_{\mathcal{E}}"] & \mathcal{E} \arrow[l, "\rho_{\mathcal{E}}"] \arrow[u, dashed, "e"] \arrow[r, "e_Y"'] & J^\infty_\Sigma Y \arrow[u, "\Delta_Y"'] \end{tikzcd} ,$$

where we have also added the triangle on the left, which commutes by the counitality of Δ, and have also noted that the two left-most horizontal morphisms compose to the identity, by the counitality of $\rho_{\mathcal{E}}$ just established above. Hence, from commutativity of the left part of this diagram, we can conclude that

$$e = \rho_{\mathcal{E}} \,.$$

With this identification, the commutativity of the left square is exactly the coaction property of $\rho_{\mathcal{E}}$.

(c) Suppose that $\rho_{\mathcal{E}} \colon \mathcal{E} \to J^\infty_\Sigma \mathcal{E}$ gives $\mathcal{E}$ a coalgebra structure. Then, by the counitality condition, $\epsilon_{\mathcal{E}} \circ \rho_{\mathcal{E}} = \mathrm{id}$, meaning that $\rho_{\mathcal{E}}$ is a split monomorphism (in $\mathbf{H}_{/\Sigma}$ in either case), in the same way as we concluded in part (a). Now, the Beck equalizer theorem (proposition B.31) says that $\rho_{\mathcal{E}}$ is an equalizer of the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook, "\rho_{\mathcal{E}}"] & J^\infty_\Sigma \mathcal{E} \arrow[r, shift left, "\Delta_{\mathcal{E}}"] \arrow[r, shift right, "J^\infty_\Sigma \rho_{\mathcal{E}}"'] & J^\infty_\Sigma J^\infty_\Sigma \mathcal{E} \end{tikzcd} \,.$$

Recall the canonical the canonical inclusion $e^\infty_{\mathcal{E}} \colon \mathcal{E}^\infty \hookrightarrow \mathcal{E}$ (definition 3.47) of the prolongation $\mathcal{E}^\infty$ of $\mathcal{E}$. The pullback diagram defining $\mathcal{E}^\infty$ (lemma A.2) can be combined with the Beck equalizer diagram to give the following commutative diagram:

$$\begin{tikzcd} \mathcal{E} \arrow[drr, hook, bend left, "\rho_{\mathcal{E}}"] \arrow[dr, dashed] \arrow[ddr, hook, bend right, "\rho_{\mathcal{E}}"'] & & \\ & \mathcal{E}^\infty \arrow[ul, hook, "e^\infty_{\mathcal{E}}"'] \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & J^\infty_\Sigma \mathcal{E} \arrow[d, "\Delta_{\mathcal{E}}"] \\ & J^\infty_\Sigma \mathcal{E} \arrow[r, hook, "J^\infty_\Sigma \rho_{\mathcal{E}}"'] & J^\infty_\Sigma J^\infty_\Sigma \mathcal{E} \end{tikzcd} \,,$$

where we have illustrated by a dashed arrow the unique monomorphism that factors $\rho_{\mathcal{E}}$ through $\mathcal{E}^\infty$ (note that monomorphisms always factor through monomorphisms), which exists by the pullback property. Composing the morphisms $\mathcal{E}^\infty \hookrightarrow \mathcal{E} \hookrightarrow \mathcal{E}^\infty$ gives a factorization of the cone with vertex $\mathcal{E}^\infty$ through itself. By the pullback property, there is only one way for such a factorization to be done, which hence must coincide with the identity $\mathcal{E}^\infty \xrightarrow{\mathrm{id}} \mathcal{E}^\infty$. If the composition of two monomorphisms is the identity, then they both had to be isomorphisms to begin with. Thus, $\mathcal{E} \simeq \mathcal{E}^\infty$ and we have shown that $\rho_{\mathcal{E}} \colon \mathcal{E} \hookrightarrow J^\infty_\Sigma \mathcal{E}$ is a generalized formally integrable PDE.

(d) In one direction, suppose that $\phi \colon \mathcal{E} \to \mathcal{E}'$ preserves formal solutions. Then, using formal integrability and lemma A.2, we get the commutative diagram

$$\begin{tikzcd} T^\infty_\Sigma \mathcal{E} \arrow[r, "T^\infty_\Sigma \phi"] \arrow[d, "\overline{\rho_{\mathcal{E}}}"'] & T^\infty_\Sigma \mathcal{E}' \arrow[d, "\overline{\rho_{\mathcal{E}'}}"] \\ \mathcal{E} \arrow[r, "\phi"'] & \mathcal{E}' \end{tikzcd} \,,$$

which expresses the fact that the composition of ϕ with the universal family of formal solutions of $\mathcal{E}$ factors through the universal family of solutions of $\mathcal{E}'$. By the $T^\infty_\Sigma \dashv J^\infty_\Sigma$ adjunction, we get precisely the commutative diagram showing that ϕ is a morphism of corresponding coalgebras.

In the other direction, reversing the preceding argument, applying the adjunction to the commutative square showing that ϕ is a morphism of coalgebras, we get the commutative square with the $T^\infty_\Sigma \phi$ morphism as illustrated above, implying the identity $\phi \circ \overline{\rho_{\mathcal{E}}} = \overline{\rho_{\mathcal{E}'}} \circ T^\infty_\Sigma \phi$. By lemma 3.50, this means that ϕ preserves the universal family of formal solutions $\overline{\rho_{\mathcal{E}}}$ and hence preserves all families of formal solutions. $\square$

Now the main theorem:

Theorem 3.52. *Let $\mathbf{H}$ be a differentially cohesive topos (definition 2.15) and let Σ be a V-manifold in $\mathbf{H}$ (definition 3.3). Then there is an equivalence of categories*

$$\mathrm{PDE}_{/\Sigma}(\mathbf{H}) \simeq \mathrm{EM}(J^\infty_\Sigma)$$

between the category of formally integrable generalized PDEs in $\mathbf{H}$ (definition 3.47(d)) and the Eilenberg-Moore category of coalgebras (definition B.30) over the jet comonad $J^\infty_\Sigma \colon \mathbf{H}_{/\Sigma} \to \mathbf{H}_{/\Sigma}$ (definition 3.23), all over Σ. Moreover, for any full subcategory $\mathcal{C} \hookrightarrow \mathbf{H}_{/\Sigma}$, the equivalence restricts to the corresponding subcategories $\mathrm{PDE}(\mathcal{C}) \simeq \mathrm{EM}(J^\infty_\Sigma|_\mathcal{C})$, meaning that the following diagram commutes:

$$\begin{tikzcd} \mathrm{PDE}(\mathcal{C}) \arrow[r, "\simeq"] \arrow[d, hook] & \mathrm{EM}(J^\infty_\Sigma|_\mathcal{C}) \arrow[d, hook] \\ \mathrm{PDE}_{/\Sigma}(\mathbf{H}) \arrow[r, "\simeq"] & \mathrm{EM}(J^\infty_\Sigma) \end{tikzcd} .$$

Proof. Proposition 3.51 shows that there are functors in both directions between $\mathrm{PDE}_{/\Sigma}(\mathbf{H})$ and $\mathrm{EM}(J^\infty_\Sigma)$. It remains only to check that they compose to identity, up to natural isomorphism, in both directions. It is obvious from part (c) that the coalgebra $\rho_\mathcal{E} \colon J^\infty_\Sigma \hookrightarrow \mathcal{E}$ gets sent back to itself under composition in one direction. In the other direction, a formally integrable PDE $e_Y \colon \mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ gets sent back to $\rho_\mathcal{E} \colon \mathcal{E} \hookrightarrow J^\infty_\Sigma \mathcal{E}$. However, the natural morphism $\mathrm{id}_\mathcal{E} \colon \mathcal{E} \to \mathcal{E}$ clearly preserves formal solutions and is an isomorphism. Hence, we have the desired equivalence of categories.

Finally, consider a full subcategory $\mathcal{C} \hookrightarrow \mathbf{H}_{/\Sigma}$ that is closed under J^∞_Σ. If $\mathcal{E} \in \mathcal{C}$, then all the morphisms needed to define a PDE or coalgebra structure on it are also $\mathcal{C}$, because it is a full subcategory. Similarly, any morphism in $\mathrm{PDE}_{/\Sigma}(\mathbf{H})$ or $\mathrm{EM}(J^\infty_\Sigma)$ breaks down to morphisms in $\mathcal{C}$ if its source and target have underlying objects in $\mathcal{C}$. Hence, clearly, $\mathrm{PDE}(\mathcal{C}) \hookrightarrow \mathrm{PDE}_{/\Sigma}(\mathbf{H})$ and $\mathrm{EM}(J^\infty_\Sigma|_\mathcal{C}) \hookrightarrow \mathrm{EM}(J^\infty_\Sigma)$ are full subcategories that are respected by the equivalence. □

Remark 3.53. We introduced the notion of a PDE (definition 3.39) by relying on the extrinsic structure of its inclusion $\mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ in a jet bundle, which also plays a crucial role in the corresponding notion of a solution. While this extrinsic structure seems natural from the traditional point of view on PDEs, a complete equivalence of the category of PDEs based on this definition (definition 3.47(d)) with coalgebras over the J^∞_Σ comonad shows that this extrinsic information is not actually necessary and the only inclusion that counts is that of a (formally integrable) PDE $\rho_\mathcal{E} \colon \mathcal{E} \hookrightarrow J^\infty_\Sigma \mathcal{E}$ in its own jet bundle, with $\rho_\mathcal{E}$ precisely giving it the structure of a coalgebra. This completely intrinsic structure can be still seen as a traditional PDE when written out in the following way:

$$j^\infty \phi = \rho_\mathcal{E}(\phi) ,$$

where $\phi \colon \Sigma \to \mathcal{E}$ is a section. In a sense, this form can be seen as the canonical form of a PDE achieved by "solving for the highest derivatives," in analogy with how it is commonly done for ordinary differential equations. Of course, any such equation may have integrability conditions, following from the differential consequences of the equation as written above. From this point of view, the coaction identity (definition B.30) satisfied by $\rho_\mathcal{E}$ is a necessary and sufficient condition for the absence of non-trivial integrability conditions. That is, plugging the above equation into the universal integrability condition $j^\infty j^\infty \phi = \Delta_\mathcal{E}(j^\infty \phi)$ implies the identity $p^\infty \rho_\mathcal{E} = \Delta_\mathcal{E} \circ \rho_\mathcal{E}$, but it does not yield any new conditions on the section ϕ, because $p^\infty \rho_\mathcal{E} = J^\infty_\Sigma \rho_\mathcal{E} \circ \Delta_\mathcal{E}$ by definition 3.34 and $\rho_\mathcal{E}$ already satisfies the identity $J^\infty_\Sigma \rho_\mathcal{E} \circ \Delta_\mathcal{E} = \Delta_\mathcal{E} \circ \rho_\mathcal{E}$ by virtue of being defining a coalgebra structure on $\mathcal{E}$.

Remark 3.54. As mentioned earlier in remark 3.49(c), while our definition for the category of generalized PDEs shares the same heuristic foundations with the one in Vinogradov's approach [32, 33], they are a priori distinct, though ours could be said to be intuitively closer to these heuristics. Both have an intrinsic description, without requiring an extrinsic embedding into a jet bundle, ours in terms of jet coalgebra structures and Vinogradov's in terms of the Cartan distribution. We are finally at a point where we can state a precise relation between these two categories. Instead of relating the two definitions directly, which would require a detailed synthetic formalization of the notion of the Cartan distribution, we will simply rely on Marvan's original and insightful observation [23, 24] of the equivalence of Vinogradov's category with the Eilenberg-Moore category of coalgebras over the jet comonad in fibered manifolds (definition 3.18), which by our main theorem 3.52 embeds fully and faithfully in our category of generalized PDEs in the Cahiers topos $\mathbf{H} = \mathrm{FormalSmoothSet}$.

Corollary 3.55. *Vinogradov's category of PDEs over an ordinary manifold Σ embeds fully and faithfully in* $\mathrm{PDE}_{/\Sigma}(\mathrm{FormalSmoothSet})$*, with image* $\mathrm{EM}(J^\infty_\Sigma|_{\mathrm{LocProMfd}_{\downarrow\Sigma}})$.

Example 3.56. There are several immediate sources of examples of morphisms in the PDE category.

(a) Recall that the co-Kleisli category of cofree coalgebras over a comonad (definition B.35) embeds as a full subcategory of the Eilenberg-Moore category of coalgebras over a comonad (definition B.30) and, in particular, we have $\mathrm{Kl}(J^\infty_\Sigma) \hookrightarrow \mathrm{EM}(J^\infty_\Sigma)$. Putting together the notation from definition 3.34 and the equivalences $\mathrm{DiffOp}_{/\Sigma}(\mathbf{H}) \simeq \mathrm{Kl}(J^\infty_\Sigma)$ (proposition 3.37) and $\mathrm{PDE}_{/\Sigma}(\mathbf{H}) \simeq \mathrm{EM}(J^\infty_\Sigma)$, we have the full embedding

$$\mathrm{DiffOp}_{/\Sigma}(\mathbf{H}) \xhookrightarrow{(J^\infty_\Sigma, p^\infty)} \mathrm{PDE}_{/\Sigma}(\mathbf{H}) \ ,$$

where the notation means that objects are embedded as *cofree PDEs*, $E \mapsto (J^\infty_\Sigma E \xrightarrow{\Delta_E} J^\infty_\Sigma J^\infty_\Sigma E)$, and morphisms as infinitely prolonged differential operators, $(J^\infty_\Sigma E_1 \xrightarrow{\phi} E_2) \mapsto (J^\infty_\Sigma E_1 \xrightarrow{p^\infty\phi} J^\infty_\Sigma E_2)$. Since the embedding is full (remark B.37), it means that the only allowed morphisms between cofree PDEs are prolonged differential operators.

(b) The trivial bundle $[\Sigma \xrightarrow{\mathrm{id}} \Sigma] \in \mathbf{H}_{/\Sigma}$ embeds in $\mathrm{DiffOp}_{/\Sigma}(\mathbf{H})$ and hence as a cofree object $\Sigma \in \mathrm{PDE}_{/\Sigma}(\mathbf{H})$. Recall that $\Sigma \simeq J^\infty_\Sigma \Sigma$ (example 3.26). Hence, every PDE morphism $\sigma^\infty \colon \Sigma \to J^\infty_\Sigma Y$ into a cofree object must be of the form $\sigma^\infty = p^\infty \sigma = j^\infty \sigma$, for some section $\sigma \colon \Sigma \to Y$.

(c) The inclusion morphism of a formally integrable generalized PDE in a jet bundle, $e_Y \colon \mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ is another example of a morphism in $\mathrm{PDE}_{/\Sigma}(\mathbf{H})$. Keeping in mind the isomorphism $\mathcal{E}^\infty \simeq \mathcal{E}$, this can be seen directly from the top square of the commutative diagram in lemma A.2, which precisely shows that e_Y is a morphism of coalgebras.

(d) For any full subcategory $\mathcal{C} \hookrightarrow \mathbf{H}_{/\Sigma}$, such as $\mathcal{C} = \mathrm{LocProMfd}_{\downarrow\Sigma}$, to which we can restrict the J^∞_Σ comonad, all of the following embeddings of categories are compatible:

$$\begin{tikzcd} \mathrm{DiffOp}(\mathcal{C}) \arrow[r, hook, "(J^\infty_\Sigma, p^\infty)"] \arrow[d, hook] & \mathrm{PDE}(\mathcal{C}) \arrow[d, hook] \\ \mathrm{DiffOp}_{/\Sigma}(\mathbf{H}) \arrow[r, hook, "(J^\infty_\Sigma, p^\infty)"'] & \mathrm{PDE}_{/\Sigma}(\mathbf{H}) \end{tikzcd} .$$

Theorem 3.57 (Products and equalizers)**.** *Consider two formally integrable PDEs $\mathcal{E}, \mathcal{F} \in \mathrm{PDE}_{/\Sigma}(\mathbf{H})$, presented as coalgebras over J^∞_Σ with coaction morphisms $\rho_\mathcal{E} \colon \mathcal{E} \to J^\infty_\Sigma \mathcal{E}$ and $\rho_\mathcal{F} \colon \mathcal{E} \to J^\infty_\Sigma \mathcal{F}$.*

(a) The $\mathbf{H}_{/\Sigma}$-product[5] *$\mathcal{E} \times \mathcal{F}$ has the coalgebra structure*

$$\rho_{\mathcal{E}\times\mathcal{F}} = (\rho_\mathcal{E}, \rho_\mathcal{F}) \colon \mathcal{E} \times \mathcal{F} \to J^\infty_\Sigma \mathcal{E} \times J^\infty_\Sigma \mathcal{F} \simeq J^\infty_\Sigma (\mathcal{E} \times \mathcal{F}) \, ,$$

which makes it into the $\mathrm{PDE}_{/\Sigma}(\mathbf{H})$-product of $\mathcal{E}$ and $\mathcal{F}$.

(b) For any pair of coalgebra morphisms $f, g \colon \mathcal{E} \to \mathcal{F}$, the $\mathbf{H}_{/\Sigma}$-equalizer

$$\begin{tikzcd} \mathcal{K} \arrow[r, hook, "e"] & \mathcal{E} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathcal{F} \end{tikzcd}$$

has a unique coalgebra structure, $\rho_\mathcal{K} \colon \mathcal{K} \to J^\infty_\Sigma \mathcal{K}$, that makes $e \colon \mathcal{K} \to \mathcal{E}$ into a morphism of coalgebras and moreover the $\mathrm{PDE}_{/\Sigma}(\mathbf{H})$-equalizer of f and g.

[5] Since $\Sigma \in \mathbf{H}_{/\Sigma}$ is the terminal object, the product in $\mathbf{H}_{/\Sigma}$ must be computed as the $\mathcal{E} \times_\Sigma \mathcal{F}$ product in $\mathbf{H}$.

(c) When, $\mathcal{E} = J^\infty_\Sigma Y$, $\mathcal{F} = J^\infty_\Sigma Z$ and $f = p^\infty f_0$, $g = p^\infty g_0$ for some differential operators $f, g\colon J^\infty_\Sigma Y \to Z$. The $\mathrm{PDE}_{/\Sigma}(\mathbf{H})$-equalizer $\mathcal{K}$ of $p^\infty f_0$ and $p^\infty g_0$ coincides with the prolongation of the $\mathbf{H}_{/\Sigma}$-equalizer

$$\begin{tikzcd} \mathcal{K}_0 \arrow[r, hook, "e_0"] & J^\infty_\Sigma Y \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] & Z \end{tikzcd} .$$

That is, $\mathcal{K} \simeq (\mathcal{K}_0)^\infty$ (lemma A.2), as formally integrable generalized PDEs.

Proof. (a) This part follows directly from the observation that J^∞_Σ preserves $\mathbf{H}_{/\Sigma}$-products.

(b) Given that $e = \ker_{\mathbf{H}_{/\Sigma}}(f, g)\colon \mathcal{K} \to \mathcal{E}$, it is a monomorphism. Since equalizers are preserved by J^∞_Σ (from definition 3.23 it is right adjoint and hence preserves all limits), $J^\infty_\Sigma e\colon J^\infty_\Sigma \mathcal{K} \to J^\infty_\Sigma \mathcal{E}$ is also a monomorphism. By the properties of morphisms of coalgebras, the solid arrows in the following diagram in $\mathbf{H}_{/\Sigma}$ commute:

$$\begin{tikzcd} \mathcal{K} \arrow[r, hook, "e"] \arrow[d, dashed, "\rho_{\mathcal{K}}"'] & \mathcal{E} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, hook, "\rho_{\mathcal{E}}"'] & \mathcal{F} \arrow[d, hook, "\rho_{\mathcal{F}}"'] \\ J^\infty_\Sigma \mathcal{K} \arrow[r, hook, "J^\infty_\Sigma e"] & J^\infty_\Sigma \mathcal{E} \arrow[r, shift left, "J^\infty_\Sigma f"] \arrow[r, shift right, "J^\infty_\Sigma g"'] & J^\infty_\Sigma \mathcal{F} \end{tikzcd} .$$

And, because $J^\infty_\Sigma e$ is a monomorphism, there exists the dashed arrow $\rho_{\mathcal{K}}$ that is unique in making the whole diagram commute. It follows that the left square in the diagram is a pullback square for $\mathcal{K}$. Pasting it together with the coaction square for $\rho_{\mathcal{E}}$ (which is also a pullback square, by Proposition 3.52(c)), we get the diagram

$$\begin{tikzcd} \mathcal{K} \arrow[r, "e"] \arrow[d, "\rho_{\mathcal{K}}"'] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{E} \arrow[r, "\mathcal{E}"] \arrow[d, "\rho_{\mathcal{E}}"'] \arrow[dr, phantom, "\text{(pb)}"] & J^\infty_\Sigma \mathcal{E} \arrow[d, "\Delta_{\mathcal{E}}"] \\ J^\infty_\Sigma \mathcal{K} \arrow[r, "J^\infty_\Sigma e"'] & J^\infty_\Sigma \mathcal{E} \arrow[r, "J^\infty_\Sigma \rho_{\mathcal{E}}"'] & J^\infty_\Sigma J^\infty_\Sigma \mathcal{E} \end{tikzcd} ,$$

whose outer square is hence also a pullback square for $\mathcal{K}$, by the pasting law (Proposition B.1). In other words $\rho_{\mathcal{E}} \circ e\colon \mathcal{K} \to J^\infty_\Sigma \mathcal{E}$ is a formally integrable PDE and hence, by proposition 3.52(b), the morphism $\rho_{\mathcal{K}}\colon \mathcal{K} \to J^\infty_\Sigma \mathcal{K}$ endows $\mathcal{K}$ it with the structure of a coalgebra. The commutativity of the left square of the above diagram then means that $e\colon \mathcal{K} \to \mathcal{E}$ is a morphism of coalgebras, with $\rho_{\mathcal{K}}$ the unique coalgebra structure making it true.

It remains to show that any other other coalgebra morphism $h\colon \mathcal{H} \to \mathcal{E}$ that equalizes f and g must uniquely factor through $\mathcal{K}$. Since both e and $J^\infty_\Sigma e$ are equalizers in $\mathbf{H}_{/\Sigma}$, there exists dashed arrow u that is unique in making the top loop in the following diagram commute, with the bottom loop also commuting since it is the J^∞_Σ image of the top one:

$$\begin{tikzcd} \mathcal{H} \arrow[r, dashed, "u"'] \arrow[rr, bend left, "h"] \arrow[d, "\rho_{\mathcal{H}}"'] & \mathcal{K} \arrow[r, "e"'] \arrow[d, "\rho_{\mathcal{K}}"'] & \mathcal{E} \arrow[d, "\rho_{\mathcal{E}}"'] \\ J^\infty_\Sigma \mathcal{H} \arrow[r, dashed, "J^\infty_\Sigma u"] \arrow[rr, bend right, "J^\infty_\Sigma h"'] & J^\infty_\Sigma \mathcal{K} \arrow[r, "J^\infty_\Sigma e"] & J^\infty_\Sigma \mathcal{E} \end{tikzcd} .$$

We already know that the right square and the outer square commute. Hence, we can conclude that

$$J^\infty_\Sigma e \circ (\rho_{\mathcal{K}} \circ u) = J^\infty_\Sigma e \circ (J^\infty_\Sigma u \circ \rho_{\mathcal{H}}) .$$

But, since $J^\infty_\Sigma e$ is a monomorphism, it must be true that $\rho_{\mathcal{K}} \circ u = J^\infty_\Sigma u \circ \rho_{\mathcal{K}}$ and therefore that the left square commutes as well. This shows that $u\colon \mathcal{H} \to \mathcal{K}$ is a morphism of coalgebras. Therefore, we have shown that any h as above must uniquely factor through e in $\mathrm{PDE}_{/\Sigma}(\mathbf{H})$, we can conclude that $e = \ker_{\mathrm{PDE}_{/\Sigma}(\mathbf{H})}(f, g)$.

(c) Consider the following diagram:

$$
\begin{tikzcd}
\mathcal{K} \arrow[drr, hook, bend left, "e"] \arrow[dr, dashed, hook, shift left] \arrow[ddr, dotted, hook, bend right] & & \\
& (\mathcal{K}_0)^\infty \arrow[r, "(e_0)^\infty"] \arrow[d] \arrow[ul, dashed, hook, shift left] \arrow[dr, phantom, "\text{(pb)}"] & J^\infty_\Sigma Y \arrow[d, "\Delta_Y"] \\
& J^\infty_\Sigma \mathcal{K}_0 \arrow[r, hook, "J^\infty_\Sigma e_0"'] & J^\infty_\Sigma J^\infty_\Sigma Y \arrow[r, shift left, "J^\infty_\Sigma f_0"] \arrow[r, shift right, "J^\infty_\Sigma g_0"'] & J^\infty_\Sigma Z
\end{tikzcd} .
$$

The bottom line is an equalizer, since J^∞_Σ preserves equalizers, while the inner square is the defining pullback square of the prolongation $(\mathcal{K}_0)^\infty$. Since e is the equalizer of $p^\infty f_0 = J^\infty_\Sigma f_0 \circ \Delta_Y$ and $p^\infty g_0 = J^\infty_\Sigma g_0 \circ \Delta_Y$, $\Delta_Y \circ e$ equalizes $J^\infty_\Sigma f_0$ and $J^\infty_\Sigma g_0$. Hence $\Delta_Y \circ e$ uniquely factors through $J^\infty_\Sigma \mathcal{K}_0$, illustrated by the dotted monomorphism above (recall that monomorphisms always factor through monomorphisms), which hence commutes with all the solid arrows. This commutativity and the pullback property of $(\mathcal{K}_0)^\infty$ implies the unique commuting factorization $\mathcal{K} \hookrightarrow (\mathcal{K}_0)^\infty$, illustrated by one of the dashed morphisms above. On the other hand, the commutativity of the pullback square implies that $(e_0)^\infty$ equalizes $p^\infty f_0$ and $p^\infty g_0$, which implies the unique commuting factorization $(\mathcal{K}_0)^\infty \hookrightarrow \mathcal{K}$, illustrated by the other dashed morphism above.

Now, composing the two dashed morphisms as $(\mathcal{K}_0)^\infty \hookrightarrow \mathcal{K} \hookrightarrow (\mathcal{K}_0)^\infty$ gives a commuting factorization of $(\mathcal{K}_0)^\infty$ through itself, which by uniqueness of commuting factorizations through a pullback must be equal to the identity $(\mathcal{K}_0)^\infty \xrightarrow{\text{id}} (\mathcal{K}_0)^\infty$. But if the composition of two monomorphisms is the identity, they must have both been isomorphisms to begin with. Thus, $e \simeq (e_0)^\infty$ and $\mathcal{K}, (\mathcal{K}_0)^\infty \hookrightarrow J^\infty_\Sigma Y$ are equivalent subobjects. Since each of them can be uniquely endowed with the structure of a coalgebra such that the monomorphisms e and $(e_0)^\infty$ are morphisms of coalgebras, $\mathcal{K}$ and $(\mathcal{K}_0)^\infty$ are also isomorphic as objects in $\mathrm{PDE}_{/\Sigma}(\mathbf{H})$. □

If all pairwise products and equalizers exist, then all finite products and equalizers exist. Moreover, it is well-known that any category with a terminal object, finite products and finite equalizers has all finite limits [21, cor.V.2.1]. Hence, from parts (a) and (b) of the preceding theorem 3.57 we have the

Corollary 3.58. *The category* $\mathrm{PDE}_{/\Sigma}(\mathbf{H})$ *has all finite limits and they can be computed as* $\mathbf{H}_{/\Sigma}$ *limits.*

We conclude this section by underscoring one of the central advantages of the equivalence $\mathrm{PDE}_{/\Sigma}(\mathbf{H}) \simeq \mathrm{EM}(J^\infty_\Sigma)$ that we have established in theorem 3.52. Namely with this equivalence at hand, we may precisely characterize solutions of a PDE using only its intrinsic coalgebra structure, this is proposition 3.59 below.

Proposition 3.59. *Given a formally integrable generalized PDE* $e_Y : \mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ *over* Σ *(definition 3.47), the induced* J^∞_Σ*-coalgebra* $\rho_{\mathcal{E}} : \mathcal{E} \hookrightarrow J^\infty_\Sigma \mathcal{E}$ *(proposition 3.52) has the property that coalgebra morphisms from* Σ *(regarded as a jet coalgebra via example 3.56(b)) into it are in natural bijection with solutions to the PDE according to definition 3.39:*

$$\mathrm{Sol}_\Sigma(\mathcal{E}) \simeq \mathrm{Hom}_{\mathrm{EM}(J^\infty_\Sigma)}(\Sigma, \mathcal{E})\,.$$

Proof. Suppose that $\sigma^\infty_{\mathcal{E}} : \Sigma \to \mathcal{E}$ is a morphism in $\mathrm{PDE}_{/\Sigma}(\mathbf{H})$. By example 3.56(c), so is $e_Y : \mathcal{E} \to J^\infty_\Sigma Y$. Hence, so is $\sigma^\infty_Y = e_Y \circ \sigma^\infty_{\mathcal{E}} : \Sigma \to J^\infty_\Sigma Y$. But now, by example 3.56(b), any such morphism must be of the form $\sigma^\infty_Y = j^\infty \sigma_Y$ for some section $\sigma_Y : \Sigma \to Y$. Noticing now that $j^\infty \sigma_Y = e_Y \circ \sigma^\infty_{\mathcal{E}}$ immediately implies that $\sigma_Y \in \mathrm{Sol}_\Sigma(\mathcal{E})$.

In the other direction, suppose that $\sigma_Y : \Sigma \to Y$ has a jet extension that factors through $\mathcal{E}$, that is $j^\infty \sigma_Y = e_Y \circ \sigma^\infty_{\mathcal{E}}$ for some $\sigma^\infty_{\mathcal{E}} : \Sigma \to \mathcal{E}$. Hence, it is easy to see that the following is a pullback diagram in $\mathbf{H}_{/\Sigma}$:

$$
\begin{tikzcd}
\Sigma \arrow[r, "\sigma^\infty_{\mathcal{E}}"] \arrow[d, "\text{id}"'] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{E} \arrow[d, "e_Y"] \\
\Sigma \arrow[r, "j^\infty \sigma_Y"'] & J^\infty_\Sigma Y
\end{tikzcd} .
$$

By corollary 3.58, the same diagram is also a pullback diagram in $\mathrm{PDE}_{/\Sigma}(\mathbf{H}) \simeq \mathrm{EM}(J^\infty_\Sigma)$, meaning that $\sigma^\infty_{\mathcal{E}} \in \mathrm{Hom}_{\mathrm{EM}(J^\infty_\Sigma)}(\Sigma, \mathcal{E})$. ☐

3.7 The topos of synthetic PDEs

We have seen above in section 3.5 (theorem 3.52) that for Σ a V-manifold, the Eilenberg-Moore category (definition B.30) of coalgebras for the jet comonad (definition 3.23), acting on any object of a differentially cohesive topos $\mathbf{H}$ (definition 2.15) is equivalently the category of formally integrable generalized PDEs, generalized in the sense that both their underlying bundles as well as their solution loci may again be arbitrary objects of $\mathbf{H}$:

$$\mathrm{PDE}_{/\Sigma}(\mathbf{H}) \simeq \mathrm{EM}(J^\infty_\Sigma)\,.$$

It is a general fact that for J a comonad, such that

1. J acts on an elementary topos,
2. J is a right adjoint,

the Eilenberg-Moore category $\mathrm{EM}(J)$ is itself an elementary topos [22, V8, cor. 7]. Here we show (theorem 3.63 below) that at least for $\mathbf{H} = \mathrm{FormalSmoothSet}$ then $\mathrm{PDE}_\Sigma(\mathbf{H})$ is in fact a category of sheaves (definition B.18) over the category of "ordinary" PDEs, i.e., of PDEs in the category of (infinitesimally thickened) fibered manifolds.

This result shows that the generalized formally integrable PDEs of definition 3.47 are related to ordinary PDEs in the same way that the (formal) smooth sets of definition 2.1 and definition 2.11 are related to (formal) smooth manifolds. In particular this implies that every PDE in the generalized sense is but a colimit of PDEs in the ordinary sense. In a followup we will show that this may be used in order to identify a synthetic formulation for the variational bicomplex [1] construction (which lies at the foundation of variational calculus and Lagrangian field theory) inside $\mathrm{PDE}_{/\Sigma}(\mathbf{H})$.

Let throughout this section $\mathbf{H} := \mathrm{FormalSmoothSet} := \mathrm{Sh}(\mathrm{FormalCartSp})$ be the Cahiers topos (definition 2.11) with $\Im\colon \mathbf{H} \to \mathbf{H}$ denoting the infinitesimal shape monad from proposition 2.15, and the natural transformation $\eta\colon \mathrm{id} \to \Im$ its unit.

Theorem 3.60. *Let $\Sigma \in \mathrm{FormalSmoothSet}$ be any object. Then the functor*

$$(\eta_\Sigma)^* \;:\; \mathrm{FormalSmoothSet}_{/\Im\Sigma} \xrightarrow{\simeq} \mathrm{EM}(J^\infty_\Sigma) \simeq \mathrm{PDE}_{/\Sigma}(\mathrm{FormalSmoothSet})$$

(from the slice over $\Im\Sigma$ to the Eilenberg-Moore category of theorem 3.52) which equips the pullback $(\eta_\Sigma)^*E$ with the coalgebra structure given by

$$(\eta_\Sigma)^*E \xrightarrow{(\eta_\Sigma)(\epsilon_E)} (\eta_\Sigma)^*(\eta_\Sigma)_*(\eta_\Sigma)^*E = J^\infty_\Sigma\left((\eta_\Sigma)^*E\right)$$

(where $\epsilon\colon \mathrm{id} \to (\eta_\Sigma)_*(\eta_\Sigma)^*$ is the unit of the adjunction $(\eta_\Sigma)^* \dashv (\eta_\Sigma)_*$) is an equivalence of categories.

Proof. By proposition B.42 it is sufficient that $(\eta_\Sigma)\colon \Sigma \longrightarrow \Im\Sigma$ is an epimorphism. This is the case by proposition 2.18. ☐

Proposition 3.61. *For $\Sigma \in \mathrm{FormalSmoothSet}$ any object, there is an equivalence of categories*

$$\mathrm{PDE}_{/\Sigma}(\mathrm{FormalSmoothSet}) \simeq \mathrm{Sh}(\mathrm{FormalCartSp}/\Im\Sigma)$$

which identifies the category of generalized PDEs (definition 3.47(d)) with the category of sheaves (definition B.18) over the slice site (proposition B.26) of that of formal Cartesian spaces (definition 2.11) over Σ.

Proof. This follows via theorem 3.60 by proposition B.26. ☐

By theorem 3.60 we may identify the objects in the slice site FormalCartSp$/\Im\Sigma$ of proposition 3.61 as generalized PDEs. But in fact these turn out to be just very mildly generalized, in that their solution locus is a locally pro-manifold that may admit a formal infinitesimal thickening:

Proposition 3.62. *For $\Sigma \in \mathrm{SmthMfd} \hookrightarrow \mathbf{H} := \mathrm{FormalSmoothSet}$ a smooth manifold, the equivalence of theorem 3.60 restricted to the objects in the image under the Yoneda embedding of the slice site of proposition 3.61 factors through jet coalgebra structures (definition B.30) in formal locally pro-manifolds (definition 2.33). That is, there exists a dashed arrow that makes the following diagram commute:*

$$\begin{tikzcd}
\mathrm{FormalCartSp}_{/\Im\Sigma} \arrow[r, hook, dashed] \arrow[d, "y"'] & \mathrm{EM}(J^\infty_\Sigma|_{\mathrm{FormalLocProMfd}_{\downarrow\Sigma}}) \arrow[r, "\simeq"] \arrow[d, hook] & \mathrm{PDE}_{\downarrow\Sigma}(\mathrm{FormalLocProMfd}) \arrow[d, hook] \\
\mathbf{H}_{/\Im\Sigma} \arrow[r, "\simeq"'] & \mathrm{EM}(J^\infty_\Sigma) \arrow[r, "\simeq"'] & \mathrm{PDE}_{/\Sigma}(\mathbf{H})
\end{tikzcd} .$$

Proof. By theorem 3.60 the bottom left equivalence in the above diagram sends an object

$$X = \mathbb{R}^k \times \mathbb{D} \in \mathrm{FormalCartSp}$$

in the slice over $\Im\Sigma$

$$[X \xrightarrow{f} \Im\Sigma] \in \mathrm{FormalCartSp}_{/\Im\Sigma}$$

to its pullback along $\Sigma \overset{\eta_\Sigma}{\to} \Im\Sigma$ into the slice over Σ

$$[\eta_\Sigma^* X \longrightarrow \Sigma] \in \mathrm{FormalSmoothSet}_{/\Sigma}$$

and equipped there with some coalgebra structure. This coalgebra structure is of no concern for the proof, all we need to check is that this pullback actually lands in the image of the inclusion

$$\mathrm{FormalLocProMfd}_{\downarrow\Sigma} \hookrightarrow \mathrm{FormalSmoothSet}_{/\Sigma} .$$

To this end, we claim that:

Claim: Every morphism of the form

$$f \colon X \longrightarrow \Im\Sigma$$

for $X = \mathbb{R}^k \times \mathbb{D} \in \mathrm{FormalCartSp}$ factors as

$$X \xrightarrow{f'} \Sigma \xrightarrow{\eta_\Sigma} \Im\Sigma .$$

With this claim the desired statement follows: By the pasting law (proposition B.1) the pullback of any f that is so factored is given by the pasting of two pullbacks as in the following diagram

$$\begin{tikzcd}
T^\infty_\Sigma X \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & X \arrow[d] \arrow[dd, bend left, "f"] \\
T^\infty\Sigma \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "(\mathrm{pb})"] & \Sigma \arrow[d, "\eta_\Sigma"] \\
\Sigma \arrow[r, "\eta_\Sigma"'] & \Im\Sigma
\end{tikzcd} .$$

This identifies the pullback with the infinitesimal disk bundle from definition 3.8

$$\eta_\Sigma^* X \simeq T^\infty_\Sigma X$$

as shown. Moreover, by example 3.4 and proposition 3.13, the object $T^\infty\Sigma$ is locally a Cartesian product of a chart

$$\mathbb{R}^n \hookrightarrow \Sigma$$

with the formal neighbourhood $\mathbb{D}^n$ of any point in $\mathbb{R}^n$, and according to corollary 3.16 the morphism ev is given on generalized elements (definition 3.15) by $(s,d) \mapsto s-d$. Therefore $T^\infty_\Sigma X$ is locally the pullback in the following diagram:

$$\begin{tikzcd}
X|_{\mathbb{R}^n} \times \mathbb{D}^n \arrow[r, "(x,d)\mapsto x"] \arrow[d, "(x,d)\mapsto f'(x)+d"'] \arrow[dr, phantom, "\text{(pb)}"] & X|_{\mathbb{R}^n} \arrow[d, "x\mapsto f'(x)"] \\
\mathbb{R}^n \times \mathbb{D}^n \arrow[r, "(s,d)\mapsto s-d"'] & \mathbb{R}^n
\end{tikzcd}$$

Now since $X|_{\mathbb{R}^n} \times \mathbb{D}^n$ is the Cartesian product of a formal manifold with an infinitesimally thickened point, it is itself a formal manifold and so in particular a formal locally pro-manifold according to definition 2.33. This is the desired conclusion.

It remains to prove the above claim that every f factors through f' in this way. To that end, recall the adjoint quadruple

$$i_! \dashv i^* \dashv i_* \dashv i_! \ : \ \mathrm{SmoothSet} \leftrightarrow \mathrm{FormalSmoothSet}$$

from proposition 2.14 and the identification $\Im = i_* i^*$ from definition 2.15. Using this and forming adjuncts, (definition B.4) the morphisms in FormalSmoothSet of the form $f : X \to \Im\Sigma$ are in natural bijection with morphisms in SmoothSet of the form

$$\phi : i^* X \longrightarrow i^*\Sigma\,.$$

Here $i^*X = \mathbb{R}^k$ is just the ordinary Cartesian space underlying the formal Cartesian space $X = \mathbb{R}^k \times \mathbb{D}$, and $i^*\Sigma$ is just the smooth manifold Σ itself, regarded in SmoothSet. Now if we denote by $\tilde f$ the composite

$$\tilde f : X \to \Re X = i_! i^* X \xrightarrow{i_!\phi} i_! i^* \Sigma = \Sigma$$

then

$$\phi = i^* \tilde f$$

by the fact that $i_!$ is fully faithful, and using proposition B.6.

Now by the formula for adjuncts (proposition B.8) the fact that ϕ is, by definition, the $(i^* \dashv i_*)$-adjunct of f means that the left triangle in the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "\tilde f"] \arrow[d, "\eta_X"'] \arrow[dr, "f"] & \Sigma \arrow[d, "\eta_\Sigma"] \\
i_* i^* X \arrow[r, "{\substack{i_*\phi \\ = \\ i_* i^* \tilde f}}"'] & i_* i^* \Sigma
\end{tikzcd} .$$

But the equality which we just established, shown on the bottom edge of the diagram, now exhibits this triangle as being one half of the naturality square of the unit $\eta : \mathrm{id} \to i_* i^*$ on $\tilde f$. The other half is the right triangle shown in the above diagram. Therefore this right triangle also commutes, and this is the factorization to be shown. □

This now allows us to use the category of ordinary PDEs as a site that presents the category of generalized PDEs:

Theorem 3.63. *For $\Sigma \in \mathrm{SmthMfd} \hookrightarrow \mathrm{FormalSmoothSet}$ a smooth manifold, then there is a subcanonical coverage (definition B.20) on the category $\mathrm{PDE}_{\downarrow\Sigma}(\mathrm{FormalLocProMfd})$ of formally integrable PDEs in formal manifolds (definition 2.33), making it a site (definition B.17), such that there is an equivalence of categories*

$$\mathrm{PDE}_{/\Sigma}(\mathrm{FormalSmoothSet}) \simeq \mathrm{Sh}(\mathrm{PDE}_{/\Sigma}(\mathrm{FormalLocProMfd}))$$

which identifies the category of generalized formally integrable PDEs $\mathrm{PDE}_{/\Sigma}(\mathbf{H}) \simeq \mathrm{EM}(J^\infty_\Sigma)$ *from theorem 3.52 with the category of sheaves (definition B.18) over the category of PDEs in the ordinary sense.*

Proof. By proposition 3.61 there is an equivalence of categories

$$\mathrm{PDE}_{/\Sigma}(\mathrm{FormalSmoothSet}) \simeq \mathrm{Sh}(\mathrm{FormalCartSp}/\Im\Sigma)$$

with sheaves on the slice site of formal Cartesian spaces over $\Im\Sigma$. By proposition 3.62 there is a full inclusion

$$\mathrm{FormalCartSp}/\Im\Sigma \hookrightarrow \mathrm{PDE}_{\downarrow\Sigma}(\mathrm{FormalLocProMfd})$$

of the slice site into the category of ordinary PDEs. Moreover, since $\mathrm{FormalCartSp}/\Im\Sigma$ is a site of definition for PDEs, the objects in $\mathrm{PDE}_{\downarrow\Sigma}(\mathrm{FormalLocProMfd})$ are faithfully tested on $\mathrm{FormalCartSp}/\Im\Sigma$. Thus proposition B.27 implies the claim. ☐

A Formal solutions of PDEs

We state and prove here two technical lemmas regarding the general concept of formal solutions of PDEs (definition 3.40).

In order to prevent an explosion of notation, we will use the following device in the sequel. Since the adjunction $T^\infty_\Sigma \dashv J^\infty_\Sigma$ (definition 3.23) provides us with a natural bijection $\mathrm{Hom}(T^\infty_\Sigma(-), -) \simeq \mathrm{Hom}(-, J^\infty_\Sigma(-))$ (definition B.4), we may freely use one kind of morphism to label the other kind. For instance, recalling the notation for adjunct morphisms introduced in definition 3.23, we might use $\overline{\alpha}$ to refer to an arbitrary morphism $T^\infty_\Sigma E \to Y$, which is labelled by the corresponding morphism $\alpha\colon E \to J^\infty_\Sigma Y$.

Lemma A.1. *Consider $E, \mathcal{E}, Y \in \mathbf{H}_{/\Sigma}$, together with a monomorphism $e_Y\colon \mathcal{E} \hookrightarrow J^\infty_\Sigma Y$.*

(a) For an E-parametrized family of formal solutions $\overline{\rho^E}\colon T^\infty_\Sigma E \to \mathcal{E}$, the morphism $\rho^E\colon E \to J^\infty_\Sigma \mathcal{E}$ fits into a commutative diagram of the following form

$$\begin{tikzcd}
E \arrow[r, "e^E_Y"] \arrow[d, "\rho^E"] \arrow[dd, bend right=60, "e^E_{\mathcal{E}}"'] & J^\infty_\Sigma Y \arrow[d, "\Delta_Y"'] \arrow[dd, bend left=60, "\mathrm{id}"] \\
J^\infty_\Sigma \mathcal{E} \arrow[r, "J^\infty_\Sigma e_Y"'] \arrow[d, "\epsilon_{\mathcal{E}}"] & J^\infty_\Sigma J^\infty_\Sigma Y \arrow[d, "\epsilon_{J^\infty_\Sigma Y}"'] \\
\mathcal{E} \arrow[r, "e_Y"'] & J^\infty_\Sigma Y
\end{tikzcd} ,$$

where

1. $e^E_{\mathcal{E}} := \epsilon_{\mathcal{E}} \circ \rho^E$ *as shown;*

2. $e^E_Y := e_Y \circ \overline{\rho^E} \circ \eta_E$.

(b) For a pair of morphisms $\rho^E\colon E \to J^\infty_\Sigma \mathcal{E}$ and $e^E_Y\colon E \to J^\infty_\Sigma Y$ that fit into a commutative diagram of the form

$$\begin{tikzcd}
E \arrow[r, "e^E_Y"] \arrow[d, "\rho^E"'] & J^\infty_\Sigma Y \arrow[d, "\Delta_Y"] \\
J^\infty_\Sigma \mathcal{E} \arrow[r, "J^\infty_\Sigma e_Y"'] & J^\infty_\Sigma J^\infty_\Sigma Y
\end{tikzcd} ,$$

the adjunct morphism $\overline{\rho^E}\colon T^\infty_\Sigma E \to \mathcal{E}$ is an E-parametrized family of formal solutions and the above commutative square coincides with the top square of the commutative diagram associated to $\overline{\rho^E}$ in part (a).

Proof. (a) Let us start with the defining property of $\overline{\rho^E}$ as a family of formal solutions. Namely, $\overline{\rho^E}$ and its adjunct ρ^E together fit into the following commutative diagram:

$$
\begin{tikzcd}
& \mathcal{E} \arrow[d, "e_Y"] \\
T^\infty_\Sigma E \arrow[ur, "\overline{\rho^E}"] \arrow[r] & J^\infty_\Sigma Y \arrow[d, "\Delta_Y"] \\
E \arrow[u, "\eta_E"] \arrow[ur, dashed, "e^E_Y"] \arrow[r] \arrow[dr, "\rho^E"'] & J^\infty_\Sigma J^\infty_\Sigma Y \\
& J^\infty_\Sigma \mathcal{E} \arrow[u, "J^\infty_\Sigma e_Y"']
\end{tikzcd}
\quad .
$$

The middle square commutes because by hypothesis the composition $e_Y \circ \overline{\rho^E}$ is a family of formally holonomic sections of $J^\infty_\Sigma Y$, which factors through $e_Y \colon \mathcal{E} \hookrightarrow J^\infty_\Sigma Y$ by virtue of consisting of formal solutions. The top and bottom triangle are related by the $T^\infty_\Sigma \dashv J^\infty_\Sigma$ adjunction, which explains the appearance of the morphism $J^\infty_\Sigma e_Y$. The dashed diagonal arrow, which we have chosen to denote by e^E_Y, is the unique one that commutes with the rest of the diagram, namely $e^E_Y = e_Y \circ \overline{\rho^E} \circ \eta_E$.

This shows that the subdiagram consisting of the morphisms ρ^E, e^E_Y, $J^\infty_\Sigma e_Y$ and Δ_Y commutes and is identical to the top square of the desired diagram in part (a) of the theorem. We can now conclude that this square commutes.

The rest of the desired diagram is constructed by pasting to it the naturality square of the ϵ counit (definition B.28) of J^∞_Σ (bottom), the counit-coproduct commutative triangle (definition B.30.2) of J^∞_Σ (right) and the composite morphism $e^E_{\mathcal{E}} = \epsilon_{\mathcal{E}} \circ \rho^E$ (left). Since all the pasted subdiagrams commute, the whole diagram commutes as well, as was desired.

(b) Let us split the commutative square from the hypothesis in part (b) of the theorem into two triangles, with τ denoting the common morphism between them:

$$
\begin{tikzcd}
E \arrow[d, "\rho^E"'] \arrow[dr, "\tau"] & \\
J^\infty_\Sigma \mathcal{E} \arrow[r, "J^\infty_\Sigma e_Y"'] & J^\infty_\Sigma J^\infty_\Sigma Y
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "e^E_Y"] \arrow[dr, "\tau"'] & J^\infty_\Sigma Y \arrow[d, "\Delta_Y"] \\
& J^\infty_\Sigma J^\infty_\Sigma Y
\end{tikzcd}
\quad .
$$

Now observe that the $(T^\infty_\Sigma \dashv J^\infty_\Sigma)$-adjuncts of these diagrams are the following triangles shown with solid arrows (for the left triangle this is just the naturality of forming adjuncts, for the right triangle this is by proposition B.34):

$$
\begin{tikzcd}
& \mathcal{E} \arrow[d, "e_Y"] \\
T^\infty_\Sigma E \arrow[ur, "\overline{\rho^E}"] \arrow[r, "\overline{\tau}"'] & J^\infty_\Sigma Y
\end{tikzcd}
\qquad
\begin{tikzcd}
T^\infty_\Sigma T^\infty_\Sigma E \arrow[r, "\overline{\overline{\tau}}"] \arrow[d, "\nabla_E"'] & Y \\
T^\infty_\Sigma E \arrow[ur, "\overline{e^E_Y}"] \arrow[r, dashed, "\overline{\tau}"] & J^\infty_\Sigma Y \arrow[u, dashed, "\epsilon_Y"'] \arrow[d, dashed, "\Delta_Y"] \\
E \arrow[u, dashed, "\eta_E"] \arrow[r, dashed, "\tau"] & J^\infty_\Sigma J^\infty_\Sigma Y
\end{tikzcd}
\quad .
$$

Now we argue with the method of local generalized elements as in remark 3.41 (using that Σ is assumed to be a V-manifold) that the diagrams shown with dashed morphisms exist and commute: Since τ is the image of composing with the jet coproduct over a V-manifold, it depends on its infinitesimal arguments symmetrically, $\tau(x)(a,b) = e_Y^E(x)(a+b)$. Hence,

$$\begin{aligned}\epsilon_Y \circ \overline{\tau} \circ \nabla_E(x,a,b) &= \epsilon_Y \circ \overline{\tau}(x, a+b) = \overline{\tau}(x,a+b)(0) = \tau(x)(a+b,0) \\ &= e_Y^E(x)(a+b) = \tau(x)(a,b) = \overline{\overline{\tau}}(x,a,b)\,.\end{aligned}$$

This shows that the square on the top right commutes. On the other hand, the commutative triangle on the left shows that $\overline{\tau}$ factors through $\overline{\rho^E}\colon T^\infty_\Sigma E \to \mathcal{E}$. In other words, as was desired, we have shown that $\overline{\rho^E}$ is a family of formal solutions.

Finally, it is obvious that applying to $\overline{\rho^E}$ the argument of part (a) we recover the same commutative square that we have started with. $\square$

Lemma A.2. *Consider $\mathcal{E}, Y \in \mathbf{H}_{/\Sigma}$ and a generalized PDE $e_Y\colon \mathcal{E} \hookrightarrow J^\infty_\Sigma Y$.*

Every parametrized family of formal solutions of this PDE factors uniquely through a universal one, $\overline{\rho^\infty_{\mathcal{E}}}\colon T^\infty_\Sigma \mathcal{E}^\infty \to \mathcal{E}$, which is defined by the pullback square in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{E}^\infty \arrow[r, "e_Y^\infty"] \arrow[d, "\rho^\infty_{\mathcal{E}}"] \arrow[dd, bend right=60, "e^\infty_{\mathcal{E}}"'] \arrow[dr, phantom, "(pb)"] & J^\infty_\Sigma Y \arrow[d, "\Delta_Y"'] \arrow[dd, bend left=60, "\mathrm{id}"] \\
J^\infty_\Sigma \mathcal{E} \arrow[r, "J^\infty_\Sigma e_Y"'] \arrow[d, "\epsilon_{\mathcal{E}}"] & J^\infty_\Sigma J^\infty_\Sigma Y \arrow[d, "\epsilon_{J^\infty_\Sigma Y}"'] \\
\mathcal{E} \arrow[r, "e_Y"'] & J^\infty_\Sigma Y
\end{tikzcd} ,$$

which defines the $e^\infty_{\mathcal{E}}\colon \mathcal{E}^\infty \to \mathcal{E}$ morphism and where each of e^∞_Y, $\rho^\infty_{\mathcal{E}}$ and $e^\infty_{\mathcal{E}}$ is a monomorphism.

If $\overline{\rho^E}\colon T^\infty_\Sigma E \to \mathcal{E}$, with $E \in \mathbf{H}_{/\Sigma}$ is a parametrized family of formal solutions, then it factors through the universal family $\overline{\rho^\infty_{\mathcal{E}}}$ as illustrated in the commutative diagram

$$\begin{tikzcd}
T^\infty_\Sigma E \arrow[r, dashed, "T^\infty_\Sigma \phi"'] \arrow[rr, bend left, "\overline{\rho^E}"] & T^\infty_\Sigma \mathcal{E}^\infty \arrow[r, "\overline{\rho^\infty_{\mathcal{E}}}"'] & \mathcal{E}
\end{tikzcd} ,$$

where $\phi\colon E \to \mathcal{E}^\infty$ is the unique morphism such that the above diagram is commutative.

Proof. As demonstrated in lemma A.1, an equivalent way of presenting an E-parametrized family of formal solutions $\overline{\rho^E}\colon T^\infty_\Sigma E \to J^\infty_\Sigma Y$ is by the pair of morphisms $\rho^\infty_{\mathcal{E}}$ and $e^E_Y = \epsilon_Y \circ (J^\infty_\Sigma e_Y) \circ \rho^E$ that fit into the commutative diagram illustrated in part (b) of that proposition. But by the definition of $\mathcal{E}^\infty$ through the pullback diagram above and by the universality property of pullbacks, there exists a unique morphism ϕ (dashed below) that makes the following diagram commute:

$$\begin{tikzcd}
E \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "e^E_Y"] \arrow[ddr, bend right, "\rho^E"'] & & \\
& \mathcal{E}^\infty \arrow[r, "e^\infty_Y"] \arrow[d, "\rho^\infty_{\mathcal{E}}"'] & J^\infty_\Sigma Y \arrow[d, "\Delta_Y"] \\
& J^\infty_\Sigma \mathcal{E} \arrow[r, "J^\infty_\Sigma e_Y"'] & J^\infty_\Sigma J^\infty_\Sigma Y
\end{tikzcd} .$$

By the $T^\infty_\Sigma \dashv J^\infty_\Sigma$ adjunction, it then follows that the morphism $T^\infty_\Sigma \phi$ (dashed below) making the following diagram commute:

$$\begin{tikzcd}
T^\infty_\Sigma E \arrow[r, dashed, "T^\infty_\Sigma \phi"'] \arrow[rr, bend left, "\overline{\rho^E}"] & T^\infty_\Sigma \mathcal{E}^\infty \arrow[r, "\overline{\rho^\infty_{\mathcal{E}}}"'] & \mathcal{E}
\end{tikzcd} .$$

Finally, the commutativity of the pullback square defining $\mathcal{E}^\infty$ and lemma A.1(b) imply that $\overline{\rho^\infty_{\mathcal{E}}} : T^\infty_\Sigma \mathcal{E}^\infty \to \mathcal{E}$ is itself a parametrized family of formal solutions. In other words, $\overline{\rho^\infty_{\mathcal{E}}}$ is the desired universal family of formal solutions through which every other one factors uniquely, in the manner indicated above.

Now we conclude by checking the monomorphism conditions. The commutative diagram in the hypothesis is obtained by taking the pullback square defining $\mathcal{E}^\infty$ and pasting to it the naturality square of the ϵ counit (definition B.28) of J^∞_Σ (bottom), the counit-coproduct commutative triangle (definition B.30.2) of J^∞_Σ (right) and the composite morphism $e^\infty_{\mathcal{E}} = \epsilon_{\mathcal{E}} \circ \rho^\infty_{\mathcal{E}}$ (left). Since all the pasted subdiagrams commute, the whole diagram commutes as well, as was desired. Recall that both Δ_Y and $J^\infty_\Sigma e_Y$ are monomorphisms, hence their pullbacks $\rho^\infty_{\mathcal{E}}$ and e^∞_Y are also monomorphisms. The commutativity of the diagram in the hypothesis implies the identity $e^\infty_Y = e_Y \circ e^\infty_{\mathcal{E}}$. Hence, since e^∞_Y is a monomorphism, its factorization map through e_Y, that is $e^\infty_{\mathcal{E}}$, must also be a monomorphism. □

B Category theoretic background

For reference, we collect here some standard facts from category theory that are referred to in the main text. Unless indicated otherwise, proof of these facts may be found in [21, 3].

B.1 Universal constructions

We list some basic statements about (co-)limits and Kan extensions, see for instance [3, vol 1, section 2]

Proposition B.1 (pasting law, e.g. [3, vol 1, prop. 2.5.9])**.** *Consider in any category a commuting diagram of the from*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "(pb)"] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

such that the right square is Cartesian (is a pullback square). Then the left square is Cartesian precisely if the total rectangle is.

Definition B.2 (e.g.[21, III.6])**.** For $\mathcal{C}$ a category with finite products (hence with a terminal object $*$ and with binary Cartesian products $(-)\times(-)$), then a *monoid object* in $\mathcal{C}$ is an object $G \in \mathcal{C}$ equipped with

1. (unit) a morphism $e : * \longrightarrow G$
2. (binary product) a morphism $(-)\cdot(-) : G \times G \longrightarrow G$;

such that

- (associativity) the following diagram commutes:

$$\begin{tikzcd} G \times G \times G \arrow[r, "{(\mathrm{id},(-)\cdot(-))}"] \arrow[d, "{((-)\cdot(-),\mathrm{id})}"'] & G \times G \arrow[d, "(-)\cdot(-)"] \\ G \times G \arrow[r, "(-)\cdot(-)"'] & G \end{tikzcd}$$

- (unitality) the following diagram commutes:

$$\begin{tikzcd} * \times G \arrow[rr, "e\times\mathrm{id}"] \arrow[dr, "\simeq"'] & & G \times G \arrow[dl, "(-)\cdot(-)"] \\ & G & \end{tikzcd}$$

Such a monoid object is called *commutative* if

- the following diagram commutes:

$$\begin{tikzcd} G \times G \arrow[rr, "\tau"] \arrow[dr, "(-)\cdot(-)"'] & & G \times G \arrow[dl, "(-)\cdot(-)"] \\ & G & \end{tikzcd}$$

(where the top morphism exchanges the two factors $(g_1, g_2) \mapsto (g_2, g_1)$).

A monoid object is a *group object* if

- (inverses) there exists a morphism $(-)^{-1} : G \longrightarrow G$

such that

- (invertibility) the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\mathrm{id}\times\mathrm{id}"] \arrow[d, equal] & G \times G \arrow[d, "\mathrm{id}\times(-)^{-1}"] \\ G & G \times G \arrow[l, "(-)\cdot(-)"] \end{tikzcd}$$

A monoid object that is both commutative as well as a group object is also called an *abelian group object.*

Proposition B.3 (nonabelian Mayer-Vietoris lemma)**.** *Let $\mathcal{C}$ be a category with finite products, and let $G \in \mathcal{C}$ be equipped with the structure of a group object (definition B.2). Then for $f : X \to G$ and $g : Y \to G$ two morphisms in $\mathcal{C}$, their fiber product also makes the following square Cartesian:*

$$\begin{tikzcd} X \times_G Y \arrow[rr] \arrow[d, "(\mathrm{pr}_1,\mathrm{pr}_2)"'] \arrow[drr, phantom, "(pb)"] & & * \arrow[d, "e"] \\ X \times Y \arrow[r, "f\times g"'] & G \times G \arrow[r, "(-)\cdot(-)^{-1}"'] & G \end{tikzcd} .$$

Definition B.4 (e.g. [3, vol 1, section 3])**.** A pair of *adjoint functors* denoted $(L \dashv R)$ (or an *adjunction* between two functors), is a pair of functors of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift right, "R"'] & \mathcal{D} \arrow[l, shift right, "L"'] \end{tikzcd}$$

equipped with a natural isomorphism ("forming adjuncts") between their hom-functors of the form

$$\mathrm{Hom}_{\mathcal{C}}(L(-), -) \simeq \mathrm{Hom}_{\mathcal{D}}(-, R(-)) \,.$$

Here L is called *left adjoint to* R and R is called *right adjoint to* L. The image η_d of id_{Ld} under this isomorphism is called the *unit* of the adjunction at $d \in \mathcal{D}$

$$\eta_d : d \longrightarrow R(L(d)) \,,$$

while, conversely, the image ϵ_d of id_{Rc} is called the *counit*

$$\epsilon_d : L(R(d)) \longrightarrow d \,.$$

(Unit and counit are themselves natural transformations $\eta : \mathrm{id}_{\mathcal{D}} \longrightarrow R \circ L$ and $\epsilon : L \circ R \longrightarrow \mathrm{id}_{\mathcal{C}}$.)

One also writes horizontal lines for indicating these bijections between sets of adjunct morphisms:

$$\frac{d \longrightarrow Rc}{Ld \longrightarrow c}$$

There are various equivalent definitions of adjoint functors:

Proposition B.5. *An adjunction $L \dashv R$ between two functors (definition B.4) is equivalent to two natural transformations, the unit*

$$\eta : \mathrm{id} \longrightarrow R \circ L$$

and the counit

$$\epsilon : L \circ R \longrightarrow \mathrm{id},$$

such that the following "zig-zag"-identities hold:

$$\begin{tikzcd} L \arrow[r, "L(\eta)"] \arrow[rr, bend right, "\mathrm{id}"'] & L \circ R \circ L \arrow[r, "\epsilon_L"] & L \end{tikzcd}$$

and

$$\begin{tikzcd} R \arrow[r, "\eta_R"] \arrow[rr, bend right, "\mathrm{id}"'] & R \circ L \circ R \arrow[r, "R(\epsilon)"] & R \end{tikzcd} .$$

Here is a list of basic properties of adjoint functors:

Proposition B.6 (e.g. [3, vol 1, prop. 3.4.1])**.** *For a pair of adjoint functors $(L \dashv R)$ (definition B.4) the following are equivalent*

1. *the right adjoint R is a fully faithful functor;*
2. *the adjunction counit $L \circ R \longrightarrow \mathrm{id}$ is a natural isomorphism*

and similarly the following are equivalent:

1. *the left adjoint L is a fully faithful functor;*
2. *the adjunction unit $\mathrm{id} \longrightarrow R \circ L$ is a natural isomorphism.*

Proposition B.7 (e.g [21, thm.V.5.1][3, vol 1, prop. 3.2.2])**.** *A right adjoint functor (definition B.4) preserves all small limits. Dually, a left adjoint functor preserves all small colimits.*

Proposition B.8. *Given an adjunction $(L \dashv R)$ as in definition B.4, then*

- *the adjunct of a morphism of the form $f : d \longrightarrow Rc$ is equivalently the composite*

$$Ld \xrightarrow{L(f)} LRc \xrightarrow{\epsilon_c} c \;;$$

- *the adjunct of a morphism of the form $g : Lc \longrightarrow d$ is equivalently the composite*

$$c \xrightarrow{\eta_c} RLc \xrightarrow{R(g)} Rd \;.$$

Key examples of adjoint pairs and adjoint triples are Kan extensions:

Proposition B.9 (Kan extension, e.g. [3, vol 1, section 3.7]). *Given a functor $f : C \longrightarrow D$ between small categories, then the induced functor on categories of presheaves $f^* : \mathrm{PSh}(D) \longrightarrow \mathrm{PSh}(C)$ (given by precomposing a presheaf with f) has both a left and a right adjoint (definition B.4), denoted $f_!$ and f_* respectively, and called the operations of left and right Kan extension along f.*

$$
(f_! \dashv f^* \dashv f_*) : \ \mathrm{PSh}(C) \begin{array}{c} \xrightarrow{f_!} \\ \xleftarrow{f^*} \\ \xrightarrow[f_*]{} \end{array} \mathrm{PSh}(D) \ .
$$

Moreover, the left Kan extension of a presheaf $A \in \mathrm{PSh}(C)$ is equivalently the presheaf which to any object $d \in D$ assigns the set expressed by the coend formula

$$
(f_! A)(d) \simeq \int^{c \in C} \mathrm{Hom}_D(d, f(c)) \times \mathrm{Hom}_{\mathrm{PSh}(C)}(c, A) \,,
$$

where on the right we are identifying c with the presheaf that it represents. Explicitly, this coend gives the set of equivalence classes of pairs of morphisms

$$
(d \to f(c), c \to A)
$$

where two such pairs are regarded as equivalent if there is a morphism $\phi : c_1 \to c_2$ in C such that the following two triangles commute

$$
\begin{tikzcd}
 & d \arrow[dl] \arrow[dr] & \\
f(c_1) \arrow[rr, "f(\phi)"] & & f(c_2) \\
c_1 \arrow[rr, "\phi"] \arrow[dr] & & c_2 \arrow[dl] \\
 & A &
\end{tikzcd} \ .
$$

In particular the left Kan extension of a representable presheaf $y(c)$ is the presheaf represented by the image under the given functor f of the representing object c:

$$
f_!(y(c)) \simeq y(f(x)) \,.
$$

Remark B.10. For a presheaf A on $\mathcal{C}$, the presheaf $f_* (A)$ can also be described explicitly. Namely, for D an object of $\mathcal{D}$, one has $f_* (A) (D) \cong \mathrm{Hom}_{\mathrm{Psh}(\mathcal{C})} (f^* y (D) , A)$. This follows easily from the Yoneda lemma.

Proposition B.11. *If $f : C \hookrightarrow D$ is a fully faithful functor, then so is its left Kan extension (proposition B.9) $f_! : \mathrm{PSh}(C) \hookrightarrow \mathrm{PSh}(D)$, hence (by proposition B.6 then the adjunction unit $\mathrm{id} \xrightarrow{\simeq} f^* f_!$ is a natural isomorphism.*

Definition B.12. Given a category $\mathcal{C}$ and an object $c \in \mathcal{C}$, then the *slice category* $\mathcal{C}_{/c}$ has as objects the morphisms of $\mathcal{C}$ into c, and as morphisms between these the commuting triangles in $\mathcal{C}$ of the form

$$
\begin{tikzcd}
a_1 \arrow[rr] \arrow[dr, "f_1"'] & & a_2 \arrow[dl, "f_2"] \\
 & c &
\end{tikzcd} \ .
$$

Proposition B.13. *The hom-spaces in a slice category $\mathcal{C}_{/c}$, definition B.12 are equivalently given by the fiber product:*

$$\mathcal{C}_{/c}(f_1, f_2) \simeq \mathcal{C}(a_1, a_2) \underset{\mathcal{C}(a_2,c)}{\times} \{f_2\}$$

of hom-spaces in $\mathcal{C}$:

$$\begin{tikzcd}
\mathcal{C}_{/c}(f_1, f_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "(pb)"] & \mathcal{C}(a_1, a_2) \arrow[d] \\
* \arrow[r, "\tilde{f}_2"'] & \mathcal{C}(a_1, c)
\end{tikzcd} .$$

where $\tilde{f}_2$ picks the element f_2 in $\mathcal{C}(a_2, c)$.

Example B.14. If $* \in \mathcal{C}$ is a terminal object, then there is an equivalence of categories

$$\mathcal{C}_{/*} \simeq \mathcal{C}$$

between the slice category over $*$ (definition B.12) and the original category.

Example B.15. If $\mathcal{C}$ is a category with finite limits, then for every object $c \in \mathcal{C}$ the slice category $\mathcal{C}_{/c}$ (definition B.12) has terminal object given by

$$[c \xrightarrow{\mathrm{id}} c]$$

and with Cartesian product given by the fiber product over c in $\mathcal{C}$:

$$[a \to c] \times [b \to c] \;\simeq\; [a \times_c b \to c] .$$

Example B.16. If $\mathcal{C}$ is a category thought of as a category of spaces (as in section 2) then for $\Sigma \in \mathcal{C}$ any object, thought of as a base space, we may think of the slice category $\mathcal{C}_{/\Sigma}$ (definition B.12) as the *category of bundles* over Σ, in the generality where bundles are not required to be fiber bundles and in particular may have empty fibers. For

$$[E \xrightarrow{p} \Sigma] \in \mathcal{C}_{/\Sigma}$$

any such bundle, then bundle morphisms from the terminal bundle

$$[\Sigma \xrightarrow{\mathrm{id}} \Sigma] \in \mathcal{C}_{/\Sigma}$$

are equivalently sections of the bundle $E \overset{p}{\to} \Sigma$. We write

$$\Gamma_\Sigma(E) := \mathrm{Hom}_{\mathcal{C}_{/\Sigma}}\left([\Sigma \xrightarrow{\mathrm{id}} \Sigma]\,,\; [E \xrightarrow{p} \Sigma]\right)$$

B.2 Categories of sheaves

For reference, we collect here some facts about categories of sheaves (Grothendieck toposes) that we use in the main text, see for instance [3, vol 3]

Definition B.17 (site)**.** For $\mathcal{C}$ a small category, then a *coverage* or *Grothendieck pre-topology* on $\mathcal{C}$ is for each object $X \in \mathcal{C}$ a set of families of morphisms $\{U_i \xrightarrow{\phi_i} X\}_{i \in I}$ into X, called the *covering families*, which are such that for each morphism $Y \longrightarrow X$ there exists a covering family $\{V_i \xrightarrow{\psi_j} Y\}_{j \in J}$ such that for each $i \in I$ there is a commuting square of the form

$$\begin{tikzcd}
V_{j(i)} \arrow[r] \arrow[d, "\psi_{j(i)}"'] & U_i \arrow[d, "\phi_i"] \\
Y \arrow[r] & X
\end{tikzcd} .$$

A small category $\mathcal{C}$ equipped with a coverage is called a *site*.

Definition B.18. For $\mathcal{S}$ a small site (definition B.17) then a presheaf $F\colon \mathcal{S}^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is called a *sheaf* if for all covering families $\{U_i \xrightarrow{\phi_i} U\}_{i\in I}$ in $\mathcal{S}$ and for all tuples of elements $(x_i \in F(U_i))_{i\in I}$ and for all pairs of morphisms $g\colon V \to U_{i_1}$ and $h\colon V \to U_{i_2}$ in $\mathcal{S}$ such that $\phi_{i_1} \circ g = \phi_{i_2} \circ h$ and such that $F(g)(x_{i_1}) = F(h)(x_{i_2})$ then there exists a unique $x \in F(U)$ such that $x_i = F(\phi_i)(x)$ for all $i \in I$.

We write

$$\mathrm{Sh}(\mathcal{S}) \hookrightarrow \mathrm{PSh}(\mathcal{S})$$

for the full inclusion of the sheaves into the category of presheaves. We call this the *category of sheaves* or the *sheaf topos* or the *Grothendieck topos* over $\mathcal{S}$.

The following is an important structure theorem for categories of sheaves:

Proposition B.19 (e.g. [3, vol. 3, cor. 3.5.5 with vol 1, def. 3.5.5])**.** *Given a small category $\mathcal{S}$ with the structure of a site (definition B.17) then the full inclusion of the sheaves into the presheaves (definition B.18) has a left adjoint functor $L_{\mathcal{S}}$ ('sheafification'") which preserves finite limits*

$$\begin{tikzcd} \mathrm{Sh}(\mathcal{S}) \arrow[r, hook, "i"', shift right] & \mathrm{PSh}(\mathcal{S}) \arrow[l, dashed, "L_{\mathcal{S}}"', shift right] \arrow[l, phantom, "\perp"] \end{tikzcd} .$$

Conversely, every full inclusion of this form into a category of presheaves on some small category $\mathcal{C}$ is the inclusion of a sheaves for some site structure on $\mathcal{C}$.

Here is a list of some basic extra conditions on sites:

Definition B.20. A site $\mathcal{C}$ (definition B.17) is called *sub-canonical* if every representable presheaf is a sheaf (definition B.18), i.e. if the Yoneda embedding factors through the full inclusion of sheaves (proposition B.19)

$$\begin{tikzcd} \mathcal{C} \arrow[r, dashed] \arrow[rr, bend right, "y"'] & \mathrm{Sh}(\mathcal{C}) \arrow[r, hook] & \mathrm{PSh}(\mathcal{C}) \end{tikzcd} .$$

Definition B.21. Let $\mathcal{C}$ be a small site (definition B.17).

1. A *point of the site* is a geometric morphism from the base topos to its sheaf topos (definition B.18), hence a pair of adjoint functors $(x^* \dashv x_*)$ (definition B.4) of the form

$$\begin{tikzcd} \mathrm{Set} \arrow[r, "x_*"', shift right] & \mathrm{Sh}(\mathcal{C}) \arrow[l, "x^*"', shift right] \arrow[l, phantom, "\perp"] \end{tikzcd}$$

   and such that x^* preserves finite limits. In this case left adjoint x^* is called *forming the stalk at* x, hence for $X \in \mathrm{Sh}(\mathcal{C})$ any sheaf, then $x^*X \in Set$ is called the *stalk* of X at x.

2. The site $\mathcal{C}$ is said to have *enough points* if there exists a set $\{x_i\}_{i\in I}$ of points, in the above sense, such that a morphism $f : X \to Y$ in $\mathrm{Sh}(\mathcal{C})$ is an isomorphism precisely if all its stalks $x_i^* f : x_i^* X \to x_i^* Y$ are bijections of sets.

The following is a list of some basic properties of categories of sheaves that we need in the main text:

Proposition B.22. *Let S be a small site (definition B.17), and let $f : X \to Y$ be a morphism in the category of sheaves $\mathrm{Sh}(S)$ over it (definition B.18). Then in generality:*

1. *f is a monomorphism or isomorphism precisely it is so globally, hence if for all object $U \in S$ its component $f_U : X(U) \to Y(U)$ is an injection or bijection of sets, respectively;*

2. *f is an epimorphism precisely if it is so locally, hence if for each object $U \in S$ there exists a cover $\{U_i \stackrel{\phi_i}{\to} U\}_i$ such that for each element $y_U \in Y(U)$ its restriction $y_{U_i} := f(\phi_i)(y_U)$ is in the image of $f_{U_i} : X(U_i) \to Y(U_i)$ for all i.*

But if S has enough points *$\{x_i\}_{i\in I}$ in the sense of definition B.21, then f is an epi-/iso-/mono-morphisms precisely if it is so stalkwise, hence precisely if for each $i \in I$ then $x_i^* f : x_i^* X \to x_i^* Y$ is a surjection/bijection/injection of sets, respectively.*

Proposition B.23 (e.g. [3, vol 3, prop. 3.4.11, 3.4.13, 3.4.15])**.** *In a category of sheaves $\mathrm{Sh}(\mathcal{C})$ (definition B.18)*

1. *every epimorphism $f\colon X \to Y$ is regular and indeed effective, meaning that it is the coequalizer of its kernel pair, hence that $X \underset{Y}{\sqcup} X \rightrightarrows X \xrightarrow{f} Y$ is a colimiting co-cone;*
2. *every monomorphism $f\colon X \to Y$ is regular, meaning that it is the equalizer of some pair of parallel morphisms.*

Proposition B.24 (universal colimits, e.g. [3, vol 3, prop. 3.4.4])**.** *In a category of sheaves $\mathrm{Sh}(\mathcal{C})$ (definition B.18), colimits are compatible with fiber products : If $X : I \longrightarrow \mathrm{Sh}(\mathcal{C})$ is a diagram and $\varinjlim_i X_i \longrightarrow B$ a morphism out of its colimit, then for every morphism $f : A \longrightarrow B$ the following square is Cartesian (is a pullback square)*

$$\begin{tikzcd} \varinjlim_i f^* X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "(pb)"] & \varinjlim_i X_i \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd} .$$

Proposition B.25 (base change)**.** *Let $\mathbf{H} = \mathrm{Sh}(\mathcal{C})$ be a category of sheaves (definition B.18). Then for*

$$f : X \longrightarrow Y$$

any morphism in $\mathbf{H}$, there is an adjoint triple of functors (definition B.4) between the slice categories (definition B.12)

$$(f_! \dashv f^* \dashv f_*) \ : \ \begin{tikzcd} \mathbf{H}_{/X} \arrow[r, shift left=2ex, "f_!"] \arrow[r, shift right=2ex, "f_!"'] & \mathbf{H}_{/Y} \arrow[l, "f^*" description] \end{tikzcd} ,$$

where

1. *$f_!$ (left push-forward) is given by post-composition with f in $\mathbf{H}$;*
2. *f^* is given by pullback along f in $\mathbf{H}$.*

Accordingly, by example B.29 composition of left and right pushforward with pullback, respectively, yields an adjoint pair of a monad L_f and a comonad R_f (definition B.28)

$$(L_f \dashv R_f) \ := \ ((\eta_f)^* \circ (\eta_f)_! \dashv (\eta_f)^* \circ (\eta_f)_*) \ : \ \mathbf{H}_{/X} \longrightarrow \mathbf{H}_{/X} .$$

Proposition B.26. *Let $\mathcal{C}$ be a small category equipped with a subcanonical coverage. Let $X \in \mathrm{Sh}(\mathcal{C})$ be an object in the category of sheaves over $\mathcal{C}$. Write $\mathcal{C}/X$ for the category whose objects are morphisms $y(c) \to X$ in $\mathrm{Sh}(\mathcal{C})$, with $c \in \mathcal{C}$ (and y denoting the Yoneda embedding) and whose morphisms are commuting triangles*

$$\begin{tikzcd} y(c_1) \arrow[rr] \arrow[dr] & & y(c_2) \arrow[dl] \\ & X & \end{tikzcd} .$$

Say that a set of morphism in $\mathcal{C}/X$ is a covering family if the underlying horizontal morphisms in $\mathcal{C}$ form a covering family. Then sheaves on $\mathcal{C}/X$ are equivalently sheaves on $\mathcal{C}$ sliced over X:

$$\mathrm{Sh}(\mathcal{C}/X) \simeq \mathrm{Sh}(\mathcal{C})/X .$$

Proposition B.27. *Let $\mathcal{S}$ be a small site (definition B.17) and let $i : \mathcal{S} \hookrightarrow \mathcal{C}$ be a fully faithful inclusion of the underlying category into a small category $\mathcal{C}$. Then there exists a Grothendieck topology on $\mathcal{C}$ and an equivalence of categories of sheaves (definition B.18) of the form*

$$\mathrm{Sh}(\mathcal{S}) \xrightarrow{\simeq} \mathrm{Sh}(\mathcal{C}) \ .$$

If moreover the functor

$$\mathcal{C} \xrightarrow{y_{\mathcal{C}}} \mathrm{PSh}(\mathcal{C}) \xrightarrow{i^*} \mathrm{PSh}(\mathcal{S}) \xrightarrow{L_{\mathcal{S}}} \mathrm{Sh}(\mathcal{S})$$

is fully faithful (y denotes Yoneda embedding, i^* denotes restriction of presheaves, proposition B.25, and L denotes sheafification, definition B.19), then this Grothendieck topology is subcanonical.

Proof. Consider[6] the composition of pairs of adjoint functors

$$\begin{tikzcd} \mathrm{Sh}(\mathcal{S}) \arrow[r, hook, shift right, "i_{\mathcal{S}}"'] \arrow[r, phantom, "\perp"] & \mathrm{PSh}(\mathcal{S}) \arrow[l, shift right, "L_{\mathcal{S}}"'] \arrow[r, hook, shift right, "i_*"'] \arrow[r, phantom, "\perp"] & \mathrm{PSh}(\mathcal{C}) \arrow[l, shift right, "i^*"'] \end{tikzcd} \ ,$$

where on the left we have the sheafification adjunction over $\mathcal{S}$ (proposition B.19), while on the right we have the pullback / right Kan extension adjunction along i (proposition B.9). Here L preserves finite limits (by proposition B.19) and i^* preserves all limits (since it has a further left adjoint given by left Kan extension). Hence the composite adjunction is a geometric embedding

$$\begin{tikzcd} \mathrm{Sh}(\mathcal{S}) \simeq \mathrm{Sh}(\mathcal{C}) \arrow[r, hook, shift right, "i_{\mathcal{C}}"'] \arrow[r, phantom, "\perp"] & \mathrm{PSh}(\mathcal{C}) \arrow[l, shift right, "L_{\mathcal{C}}"'] \end{tikzcd}$$

of the topos $\mathrm{Sh}(\mathcal{S})$ into the presheaf topos $\mathrm{PSh}(\mathcal{C})$. By proposition B.19 every such corresponds to the sheafification adjunction for some Grothendieck topology on $\mathcal{C}$, identifying the former with the category of sheaves over the latter.

In particular

$$L_{\mathcal{S}} \circ i^* \circ y_{\mathcal{C}} \simeq L_{\mathcal{C}} \circ y_{\mathcal{C}} \ ,$$

and hence if $L_{\mathcal{S}} \circ i^* \circ y_{\mathcal{C}}$ is fully faithful then so is $L_{\mathcal{C}} \circ y_{\mathcal{C}}$ and hence also $i_{\mathcal{C}} \circ L_{\mathcal{C}} \circ y_{\mathcal{C}}$. This means that for $C \in \mathcal{C}$, then the sheafification of $y(\mathcal{C})$ is the presheaf given on $D \in \mathcal{C}$ by

$$\begin{aligned} (L_{\mathcal{C}} \circ y_{\mathcal{C}}(C))(D) &\simeq (i_{\mathcal{C}} \circ L_{\mathcal{C}} \circ y_{\mathcal{C}}(C))(D) \\ &\simeq \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(y_{\mathcal{C}}(D), i_{\mathcal{C}} \circ L_{\mathcal{C}} \circ y_{\mathcal{C}}(C)) \\ &\simeq \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(L_{\mathcal{C}} \circ y_{\mathcal{C}}(D), L_{\mathcal{C}} \circ y_{\mathcal{C}}(C)) \\ &= \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(i_{\mathcal{C}} \circ L_{\mathcal{C}} \circ y_{\mathcal{C}}(D), i_{\mathcal{C}} \circ L_{\mathcal{C}} \circ y_{\mathcal{C}}(C)) \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(C, D) \\ &\simeq y_{\mathcal{C}}(D) \,. \end{aligned}$$

This says that $y_{\mathcal{C}}(C)$ coincides with its sheafification, hence that the Grothendieck topology on $\mathcal{C}$ is subcanonical. ∎

B.3 Monads

We collect some basic facts on (co-)monads and (co-)monadic descent, see for instance [3, vol 2, section 4].

Definition B.28. For $\mathcal{C}$ a category, then a *monad* on $\mathcal{C}$ is an endofunctor $T \colon \mathcal{C} \to \mathcal{C}$ equipped with natural transformations

[6] We are grateful to Dave Carchedi for providing this argument.

* (product) $\nabla : T \circ T \longrightarrow T$;
* (unit) $\eta : \mathrm{id}_{\mathcal{C}} \longrightarrow T$

such that these satisfy the following associativity and unitality properties:

$$
\begin{tikzcd}
 & T & \\
T \arrow[ur, "\mathrm{id}_{\mathcal{C}}"] \arrow[r, "\eta(T)"'] & TT \arrow[u, "\nabla"] & T \arrow[ul, "\mathrm{id}_{\mathcal{C}}"'] \arrow[l, "T(\eta)"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
T & TT \arrow[l, "\nabla"'] \\
TT \arrow[u, "\nabla"] & TTT \arrow[l, "T(\nabla)"] \arrow[u, "\nabla(T)"']
\end{tikzcd} .
$$

Dually, a *comonad* on $\mathcal{C}$ is an endofunctor $J\colon \mathcal{C} \to \mathcal{C}$ equipped with natural transformations

* (coproduct) $\Delta : J \longrightarrow J \circ J$
* (counit) $\epsilon : J \longrightarrow \mathrm{id}_{\mathcal{C}}$

such that these satisfy the following coassociativity and counitality properties:

$$
\begin{tikzcd}
 & J \arrow[dl, "\mathrm{id}_{\mathcal{C}}"'] \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}_{\mathcal{C}}"] & \\
J & JJ \arrow[l, "\epsilon(J)"] \arrow[r, "J(\epsilon)"'] & J
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
J \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & JJ \arrow[d, "\Delta(J)"] \\
JJ \arrow[r, "J(\Delta)"'] & JJJ
\end{tikzcd} .
$$

Example B.29. Given a pair of adjoint functors $(L \dashv R)$ (definition B.4) then

1. $R \circ L$ becomes a monad (definition B.28) by taking the monad unit to be the adjunction unit, and taking the monad product to be the image of the adjunction counit on L under R:

$$(R \circ L) \circ (R \circ L) \xrightarrow{R\epsilon_L} R \circ L$$

2. $L \circ R$ becomes a comonad (definition B.28) by taking the comonad counit to be the adjunction counit, and taking the comonad coproduct to be the image of the adjunction unit on R under L:

$$L \circ R \xrightarrow{L\eta_R} (L \circ R) \circ (L \circ R) \ .$$

Definition B.30. Given a comonad (J, ϵ, Δ) on $\mathcal{C}$, definition B.28, then a *coalgebra* over the comonad is an object $E \in \mathcal{C}$ equipped with a morphism

$$\rho\colon E \longrightarrow JE$$

such that

1. (coaction property) the following diagram commutes:

$$
\begin{tikzcd}
E \arrow[r, "\rho"] \arrow[d, "\rho"'] & JE \arrow[d, "J\rho"] \\
JE \arrow[r, "\Delta_E"'] & JJE
\end{tikzcd}
$$

2. (counitality) the following diagram commutes:

$$
\begin{tikzcd}
 & E \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\rho"] & \\
E & & JE \arrow[ll, "\epsilon_E"']
\end{tikzcd}
$$

A homomorphism of coalgebras $f : (E_1, \rho_1) \longrightarrow (E_2, \rho_2)$ is a morphism $f : E_1 \longrightarrow E_2$ in $\mathcal{C}$ which respects these coaction morphisms in that the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "\rho_1"'] & E_2 \arrow[d, "\rho_2"] \\ JE_1 \arrow[r, "Jf"'] & JE_2 \end{tikzcd} .$$

The resulting category of coalgebras is denoted $\mathrm{EM}(J)$ (for "Eilenberg-Moore category").

Proposition B.31 (Beck equalizer)**.** *For $J \colon \mathcal{C} \longrightarrow \mathcal{C}$ any comonad (definition B.28) and $\rho : E \longrightarrow JE$ a coalgebra over J (definition B.30) then the diagram*

$$E \xrightarrow{\ \rho\ } JE \underset{J\rho}{\overset{\Delta_E}{\rightrightarrows}} JJE$$

is an equalizer diagram, in fact it is an absolute equalizer (meaning that it is preserved by every functor $F : \mathcal{C} \to \mathcal{D}$). In particular therefore ρ is a monomorphism.

Proposition B.32. *For $(L \dashv R) : \mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ an adjunction, definition B.4, then the endofunctor*

$$T := L \circ R \; : \; \mathcal{C} \longrightarrow \mathcal{C}$$

becomes a comonad on $\mathcal{C}$ (definition B.28) with counit the adjunction counit $L \circ R \to \mathrm{id}_{\mathcal{C}}$ (definition B.4), and with coproduct induced from the unit of the adjunction by

$$\Delta_T := L(\eta_{R(-)}) .$$

Dually, $R \circ L$ is canonically equipped with the structure of a monad.

Example B.33. Given an adjoint triple $(L \dashv C \dashv R)$ of functors (definition B.4) then the monad $T := C \circ L$ and the comonad $J := C \circ R$ (induced via proposition B.32) themselves form an adjoint pair:

$$(T \dashv J) : \mathcal{C} \longrightarrow \mathcal{C} .$$

Proposition B.34. *In the situation of example B.33 then the double $(T \dashv J)$-adjunct $\tilde{\tilde{f}}$ (definition B.4) of a morphism $f : X \to JJY$ of the form*

$$\begin{tikzcd} & JY \arrow[d, "\Delta_Y"] \\ X \arrow[ur, "g"] \arrow[r, "f"'] & JJY \end{tikzcd}$$

is given by the $(T \dashv J)$-adjunct $\tilde{g}$ of g via

$$\begin{tikzcd} TTX \arrow[r, "\tilde{\tilde{f}}"] \arrow[d, "\nabla_X"'] & Y \\ TX \arrow[ur, "\tilde{g}"'] & \end{tikzcd} .$$

In fact both these kinds of morphisms are in natural bijection with those of the form

$$TX \simeq CLX \xrightarrow{\ C\overline{g}\ } CRX \simeq JX \ ,$$

where $\overline{g}$ denotes the $(C \dashv R)$-adjunct of g.

Proof. By definition of the monad unit (definition B.32), f is of the form

$$X \xrightarrow{\quad g \quad} CRY \xrightarrow{\quad C(\eta_{RY}) \quad} CRCRY \;.$$

By composition of adjunctions, the $(CL \dashv CR)$-adjunct of this morphism is the $(L \dashv C)$-adjunct of its $(C \dashv R)$-adjunct. Since the morphism on the right is in the image of C, the naturality of the adjunction isomorphism (definition B.4) implies that f is in natural bijection to

$$LX \xrightarrow{\quad \overline{g} \quad} RY \xrightarrow{\quad \eta_{RY} \quad} RCRY \,.$$

Now by the formula for adjuncts from proposition B.8, the further $(C \dashv R)$-adjunct of this morphism is

$$\begin{tikzcd} CLX \arrow[r, "C\overline{g}"] & CRY \arrow[r, "C\eta_{RY}"] \arrow[rr, bend right, "id"'] & CRCRY \arrow[r, "\epsilon_{CRY}"] & CRY \end{tikzcd} \;.$$

Here the composite on the right is the identity morphism, as shown, by proposition B.5. This shows that the single $(T \dashv J)$-adjunct of f is $C\overline{g}$. In complete formal duality one finds that also the single $(T \dashv J)$-adjunct (in the other direction) of $TTX \overset{\nabla_X}{\longrightarrow} TX \overset{\tilde{g}}{\longrightarrow} Y$ is $C\overline{g}$. Hence the statement follows. ☐

Definition B.35. The full subcategory of $\mathrm{EM}(J)$ on the cofree coalgebras, i.e., on the objects in the image of F, proposition B.36, is denoted $\mathrm{Kl}(J)$ (from "Kleisli category", or more appropriately "co-Kleisli category" given that J is a comonad).

Proposition B.36. *The category of coalgebras over a comonad on a category $\mathcal{C}$, definition B.30, is related to $\mathcal{C}$ by a pair of adjoint functors, definition B.4, of the form*

$$\begin{tikzcd} (\mathrm{U} \dashv \mathrm{F}) : \mathcal{C} \arrow[rr, shift right, "\mathrm{F}"'] \arrow[dr] & & \mathrm{EM}(J) \arrow[ll, shift right, "\mathrm{U}"'] \\ & \mathrm{Kl}(J) \arrow[ur, hook] & \end{tikzcd} \;,$$

where the left adjoint U ("underlying") forgets the coalgebra structure, $U : (E, \rho) \mapsto E$, while the right adjoint F ("cofree") sends an object $c \in \mathcal{C}$ to to the object Jc with coaction given by the coproduct Δ_J. The comonad induced from this adjunction via proposition B.32 coincides with J:

$$J \simeq \mathrm{U} \circ \mathrm{F} \,.$$

Remark B.37. In the situation of proposition B.36, given objects $c_1, c_2 \in \mathcal{C}$, then by adjunction we have a bijection of morphisms of the form

$$\frac{\qquad Fc_1 \overset{f}{\longrightarrow} Fc_2}{Jc_1 \simeq UFc_1 \overset{\tilde{f}}{\longrightarrow} c_2}$$

Hence morphisms f in the co-Kleisli category $\mathrm{Kl}(J)$, definition B.35, are equivalently morphisms in $\mathcal{C}$ of the form $\tilde{f} : Jc_1 \longrightarrow c_2$. According to proposition B.8, the adjunct morphisms are related by $\tilde{f} = \epsilon_{c_2} \circ f$, where we have identified $f \simeq Uf$, and $f = J(\tilde{f}) \circ \Delta_{c_1}$, where we have identified $J(\tilde{f}) \simeq F(\tilde{f})$.

Under identification between the adjunct morphisms f and $\tilde{f}$, the composition of morphisms $g \circ f$ in $\mathrm{Kl}(J)$ is given by the "co-Kleisli composite"

$$\widetilde{g \circ f} : Jc_1 \overset{\Delta_{c_1}}{\longrightarrow} JJc_1 \overset{J(\tilde{f})}{\longrightarrow} Jc_2 \overset{\tilde{g}}{\longrightarrow} c_3 \,.$$

The fact that a morphism between free coalgebras $f\colon Fc_1 \to Fc_2$ must be of the form $f = J(\tilde{f}) \circ \Delta_{c_1}$, as observed above, follows from the general proposition B.36. But there is also an elementary way to see it. Consider the following diagram:

$$
\begin{tikzcd}
Jc_1 \arrow[r, "f"] \arrow[d, "\Delta_{c_1}"'] \arrow[rr, bend left, "\tilde{f}=\epsilon_{c_2}\circ f"] & Jc_2 \arrow[r, "\epsilon_{c_2}"] \arrow[d, "\Delta_{c_2}"] \arrow[dr, "\mathrm{id}"] & c_2 \\
JJc_1 \arrow[r, "Jf"'] \arrow[rr, bend right, "J(\tilde{f})"'] & JJc_2 \arrow[r, "\epsilon_{Jc_2}\simeq J\epsilon_{c_2}"'] & Jc_2
\end{tikzcd}
,
$$

which commutes because f is a morphism of free coalgebras and because of the counit-coproduct identities of a comonad. From the commutativity, as was desired, it follows that $f = J(\tilde{f}) \circ \Delta_{c_1}$, with $\tilde{f} = \epsilon_{c_2} \circ f$.

Definition B.38. A functor $F : \mathcal{D} \longrightarrow \mathcal{C}$ is called *conservative* if it reflects equivalences, hence if for a morphism f in $\mathcal{D}$ we have that if $F(f)$ is an equivalence then already f was an equivalence.

Theorem B.39 (Beck monadicity theorem, e.g. [3, vol. 4 sect. 2])**.** *Sufficient conditions for an adjunction* $(L \dashv R)$*, definition B.4, to be equivalent to a comonadic adjunction* $(\mathrm{U} \dashv \mathrm{F})$ *as in proposition B.36 is that*

1. *L is conservative, definition B.38;*
2. *L preserves certain limits called* equalizers of L-split pairs*.*

Hence it is useful to record some facts about conservative functors:

Proposition B.40 (e.g. [13, lemma 1.3.2])**.** *For $\mathbf{H}$ a category of sheaves (Definition B.18) and $f : X \longrightarrow Y$ an epimorphism in $\mathbf{H}$, then the pullback functor $f^* : \mathbf{H}_{/Y} \longrightarrow \mathbf{H}_{/X}$ (proposition B.25) is conservative, definition B.38.*

Proposition B.41. *A conservative functor reflects all the limits and colimits which it preserves.*

Proposition B.42 (comonadic descent, e.g. [12, 2.4])**.** *Given an epimorphism $X \overset{f}{\twoheadrightarrow} Y$ in a category of sheaves $\mathbf{H}$, with induced base change comonad*

$$J := f^* f_* : \mathbf{H}_{/X} \to \mathbf{H}_{/X}$$

(via proposition B.25 and proposition B.32), then there is an equivalence of categories

$$\mathbf{H}_{/Y} \xrightarrow{\simeq} \mathrm{EM}(J)$$

between the slice category $\mathbf{H}_{/Y}$ (definition B.12) and the Eilenberg-Moore category of J-coalgebras in $\mathbf{H}_{/X}$, definition B.30.

Moreover, under this identification the comonadic adjunction $(U_J \dashv F_J)$ from proposition B.36 coincides with the base change adjunction $(f^* \dashv f_*)$ of proposition B.25:

$$(U_J \dashv F_J) \simeq (f^* \dashv f_*)\,.$$

Proof. Since f is assumed to be epi, proposition B.40 says that f^* is conservative. Moreover, since f^* is right adjoint to $f_!$ by proposition B.25, it preserves all small limits (proposition B.7). Therefore the conditions in the monadicity theorem B.39 are satisfied:

$$\begin{tikzcd}
\mathbf{H}_{/X} \arrow[r, shift right, "f_*"'] \arrow[d, equal] & \mathbf{H}_{/Y} \arrow[l, shift right, "f^*"'] \arrow[d, "\simeq"] \\
\mathbf{H}_{/X} \arrow[r, shift right, "F_J"'] & \mathrm{EM}(J) \arrow[l, shift right, "U_J"']
\end{tikzcd} .$$

□

References

[1] I. M. Anderson, "The variational bicomplex." Unpublished draft, 1989.

[2] A. Beilinson and V. Drinfeld, *Chiral Algebras*, vol. 51 of *Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[3] F. Borceux, *Categories and Structures*, vol. 2 of *Handbook of Categorical Algebra*. Cambridge University Press, Cambridge, UK, 1994.

[4] D. Carchedi and D. Roytenberg, "On theories of superalgebras of differentiable functions," 2013. arXiv:1211.6134.

[5] P. Deligne and D. S. Freed, "Classical field theory," in *Quantum fields and strings: a course for mathematicians*, P. Deligne, D. Kazhdan, P. Etingof, J. W. Morgan, D. S. Freed, D. R. Morrison, L. C. Jeffrey, and E. Witten, eds., vol. 1, pp. 137–225. AMS, Providence, RI, 1999.

[6] E. J. Dubuc, "Sur les modèles de la géométrie différentielle synthétique," *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **20** (1979) 231–279. http://eudml.org/doc/91216.

[7] D. Gaitsgory and N. Rozenblyum, "Crystals and D-modules," *Pure and Applied Mathematics Quarterly* **10** (2014) 57–154, arXiv:1111.2087.

[8] G. Giachetta, L. Mangiarotti, and G. Sardanashvily, "Cohomology of the infinite-order jet space and the inverse problem," *Journal of Mathematical Physics* **42** (2001) 4272–4282, arXiv:math/0006074.

[9] H. Goldschmidt, "Integrability criteria for systems of nonlinear partial differential equations," *Journal of Differential Geometry* **1** (1967) 269–307. http://projecteuclid.org/euclid.jdg/1214428094.

[10] M. Gromov, *Partial Differential Relations*, vol. 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, Berlin, 1986.

[11] B. Güneysu and M. J. Pflaum, "The profinite dimensional manifold structure of formal solution spaces of formally integrable PDEs," *SIGMA* **13** (2017) 003, arXiv:1308.1005. http://dx.doi.org/10.3842/SIGMA.2017.003.

[12] G. Janelidze and W. Tholen, "Facets of descent, I," *Applied Categorical Structures* **2** (1994) 245–281.

[13] P. T. Johnstone, *Sketches of an Elephant: A Topos Theory Compendium*, vol. 43,44 of *Oxford Logic Guides*. Clarendon Press, 2002.

[14] A. Kock, "Formal manifolds and synthetic theory of jet bundles," *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **21** (1980) 227–246. http://eudml.org/doc/91230.

[15] A. Kock, *Synthetic differential geometry*, vol. 333 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, UK, 2nd ed., 2006.

[16] A. Kock, *Synthetic Geometry of Manifolds*, vol. 180 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, UK, 2009. http://home.math.au.dk/kock/SGM-final.pdf.

[17] I. Kolař, P. W. Michor, and J. Slovák, *Natural Operations in Differential Geometry*. Springer, Berlin, 1993.

[18] H. Lewy, "An example of a smooth linear partial differential equation without solution," *The Annals of Mathematics* **66** (1957) 155–158.

[19] M. V. Losik, "Categorical differential geometry," *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **35** (1994) 274–290. http://eudml.org/doc/91549.

[20] J. Lurie, "Notes on crystals and algebraic $\mathcal{D}$-modules," 2009. http://www.math.harvard.edu/~gaitsgde/grad_2009/SeminarNotes/Nov17-19(Crystals).pdf.

[21] S. Mac Lane, *Categories for the Working Mathematician*, vol. 5 of *Graduate Texts in Mathematics*. Springer, New York, NY, 2 ed., 1978.

[22] S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Springer, Berlin, corrected ed., 1992.

[23] M. Marvan, "A note on the category of partial differential equations," in *Differential geometry and its applications, Proceedings of the Conference (Aug 24–30, Brno, 1986)*, D. Krupka and A. Švec, eds., pp. 235–244. Univ. J. E. Purkyné, Brno, 1987. https://ncatlab.org/nlab/files/MarvanJetComonadPDE.pdf.

[24] M. Marvan, *Horizontal cohomology with general coefficients*. PhD thesis, Moscow State University, 1989. http://www.slu.cz/math/cz/lide/marvan-michal/docs/mar89-ocr.pdf.

[25] P. W. Michor, *Manifolds of differentiable mappings*. Shiva Publishing, Orpington, 1980.

[26] P. J. Olver, *Applications of Lie groups to differential equations*, vol. 107 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second ed., 1993.

[27] D. J. Saunders, *The Geometry of Jet Bundles*, vol. 142 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1989.

[28] U. Schreiber, "Differential cohomology in a cohesive infinity-topos," 2013. arXiv:1310.7930.

[29] C. Simpson, "Homotopy over the complex numbers and generalized de Rham cohomology," in *Moduli of Vector Bundles*, M. Maruyama, ed., vol. 179 of *Lecture Notes in Pure and Applied Mathematics*, pp. 229–264. Marcel Dekker, New York, NY, 1996.

[30] D. C. Spencer, "Overdetermined systems of linear partial differential equations," *Bulletin of the American Mathematical Society* **75** (1969) 179–240.

[31] F. Takens, "A global version of the inverse problem of the calculus of variations," *Journal of Differential Geometry* **14** (1979) 543–562. http://projecteuclid.org/euclid.jdg/1214435235.

[32] A. M. Vinogradov, "Geometry of nonlinear differential equations," *Journal of Soviet Mathematics* **17** (1981) 1624–1649.

[33] A. M. Vinogradov and I. S. Krasilshchik, eds., *Symmetries and conservation laws for differential equations of mathematical physics*, vol. 182 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1999.

[34] F. Wellen, *Formalizing Higher Cartan Geometry in Modal Homotopy Type Theory*. PhD thesis, KIT, Karlsruhe, 2017. https://github.com/felixwellen/DCHoTT-Agda.

[35] D. N. Yetter, "Models for synthetic supergeometry," *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **29** (1988) 87–108. http://eudml.org/doc/91418.

[36] M. Zworski, "Numerical linear algebra and solvability of partial differential equations," *Communications in Mathematical Physics* **229** (2002) 293–307.