Cohomological quantization of boundary prequantum field theory

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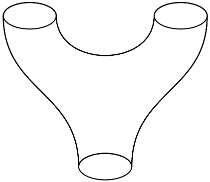
Functorial TQFT

A TQFT is a *local* assignment of linear propagators to cobordisms



$\mapsto$

$$\begin{array}{c} \mathrm{States}(S^1 \coprod S^1) \\ \big\downarrow \text{Propagate} \\ \mathrm{States}(S^1) \end{array}$$

More precisely: a functor

$$n\text{-category of cobordisms} \longrightarrow \text{some `linear' } n\text{-category}$$

Topological QFTs from quantization

Prequantum field theory:

- a field ϕ on Σ
- a local action functional $S[\phi] = \int_\Sigma \langle d\phi, \phi \rangle + \langle \phi, [\phi, \phi] \rangle + \ldots$

Can be described in (higher) differential geometry.

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Quantization:

- path integral $\int [D\phi] e^{iS[\phi]}$

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Quantization:

- path integral $\int [D\phi] e^{iS[\phi]}$

Idea: path integral = pushforward/fiber integration in cohomology

$$H^*_{dR}(M) \xrightarrow{\int_M} H^{*-\dim(M)}_{dR}(*) = \mathbb{C}$$

$$f(x) \cdot \mathrm{vol} \longmapsto \int_M f(x) \cdot \mathrm{vol}$$

Quantizing non-topological theories.

Idea: boundary to TFT quantizes to non-topological field theory.

Holographic principle:

partition function/correlator of boundary theory	$\leftrightarrow$	state of bulk TFT
boundary QFT evaluated on Σ	$\leftrightarrow$	*element* of bulk TFT evaluated on Σ

Example

- WZW-model at boundary of Chern-Simons theory.
- Poisson manifold at boundary of Poisson sigma model.

Contents

	Physics	Math
1.	Prequantum field theory	Corrrespondences of smooth spaces
2.	Linear space of quantum states	Twisted cohomology
3.	Propagator Boundary partition function Path integral	Linear map in cohomology Cocycle in cohomology Pushforward

4. Example: geometric quantization of Poisson manifold

pQFT: higher geometry

Main properties of fields:

1. Fields are smooth/geometric objects
2. Gauge principle:
   - different field configurations can be *gauge equivalent*
   - different gauge transformations can be *gauge equivalent*

Need for

$$\begin{array}{c}\text{smooth spaces}\\ +\ \text{gauge equivalences}\end{array} = \left\{\begin{array}{c}\text{smooth homotopy types}\\ \text{smooth } \infty\text{-groupoids}\\ \text{smooth } \infty\text{-stacks}\end{array}\right\}$$

These form an ∞-topos $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Gpd}$.

pQFT: field trajectories

A cobordism

$$\Sigma_{\text{left}} \hookrightarrow \Sigma \hookleftarrow \Sigma_{\text{right}}$$

gives a correspondence in **H**

$$\text{Fields}(\Sigma_{\text{left}}) \longleftarrow \text{Fields}(\Sigma) \longrightarrow \text{Fields}(\Sigma_{\text{right}})$$

pQFT: field trajectories

A cobordism between cobordisms

$$
\begin{array}{ccccc}
\Sigma_{\text{top,left}} & \hookrightarrow & \Sigma_{\text{top}} & \hookleftarrow & \Sigma_{\text{top,right}} \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma_{\text{left}} & \hookrightarrow & \Sigma & \hookleftarrow & \Sigma_{\text{right}} \\
\uparrow & & \uparrow & & \uparrow \\
\Sigma_{\text{bottom,left}} & \hookrightarrow & \Sigma_{\text{bottom}} & \hookleftarrow & \Sigma_{\text{bottom,right}}
\end{array}
$$

gives a higher correspondence in **H**

$$
\begin{array}{ccccc}
\text{Fields}(\Sigma_{\text{top,left}}) & \longleftarrow & \text{Fields}(\Sigma_{\text{top}}) & \longrightarrow & \text{Fields}(\Sigma_{\text{top,right}}) \\
\uparrow & & \uparrow & & \uparrow \\
\text{Fields}(\Sigma_{\text{left}}) & \longleftarrow & \text{Fields}(\Sigma) & \longrightarrow & \text{Fields}(\Sigma_{\text{right}}) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Fields}(\Sigma_{\text{bottom,left}}) & \longleftarrow & \text{Fields}(\Sigma_{\text{bottom}}) & \longrightarrow & \text{Fields}(\Sigma_{\text{bottom,right}})
\end{array}
$$

pQFT: locality

Consider the (∞, n)-categories:

- Bord_n (framed) cobordisms of dimension $\leq n$.
- $\mathrm{Corr}_n(\mathbf{H})$ n-fold corresponces of smooth stacks.

Definition

A n-dimensional prequantum field is a monoidal (∞, n)-functor

$$\mathrm{Bord}_n \xrightarrow{\text{Fields}} \mathrm{Corr}_n(\mathbf{H})$$

Functoriality $=$ *locality* of the field.

pQFT: fields

For *topological* field theories:

Proposition

Any prequantum field is defined by a classifying stack **Fields** *as*

$$\Sigma \mapsto \mathrm{Fields}(\Sigma) = \mathrm{Maps}\Big(\Pi(\Sigma), \mathbf{Fields}\Big)$$

These form the phase spaces of the pQFT.

Example

- sigma model: **Fields** = spacetime X.
- gauge theory: **Fields** = $\mathbf{B}G_{\mathrm{conn}}$ stack of G-principal connections

$$\Big\{G\text{-bundles} + \text{connection over } \Sigma\Big\} \simeq \Big\{\text{maps } \Sigma \to \mathbf{B}G_{\mathrm{conn}}\Big\}$$

Then $\mathrm{Fields}(\Sigma) = \mathrm{Flat}\,G\mathrm{Bund}(\Sigma)$ is the phase space of CS theory.

pQFT: local action functional

- Σ closed, n-dimensional, then

$$\begin{aligned} \mathrm{Fields}(\Sigma) &\longrightarrow U(1) \\ \phi &\longmapsto \exp(iS[\phi]) \end{aligned}$$

- Locality: $\exp(iS[\phi])$ by integrating 'higher phases' over Σ.

Such higher phases sit in higher circle groups $\mathbf{B}^n U(1)$.

Definition

An (exponentiated) *local action functional/Lagrangian* is a map

$$\mathbf{Fields} \xrightarrow{\chi} \mathbf{B}^n U(1)$$

Example

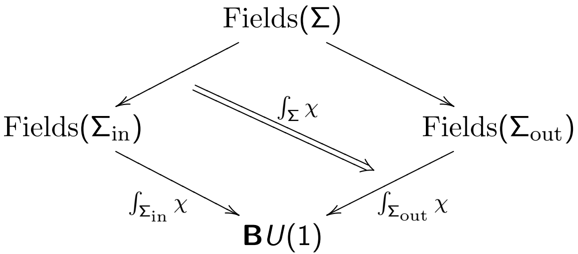
3D Chern-Simons theory: $\mathbf{B} G_{\mathrm{conn}} \xrightarrow{\mathbf{c}_2} \mathbf{B}^3 U(1)$

pQFT: local action functional

For Σ closed, n-dimensional (oriented):

$$\mathrm{Maps}\Big(\Pi(\Sigma), \mathbf{Fields}\Big) \xrightarrow{\chi} \mathrm{Maps}\Big(\Pi(\Sigma), \mathbf{B}^n U(1)\Big) \xrightarrow{\int_\Sigma} U(1)$$

More general: for a cobordism



a gauge equivalence between *prequantum circle bundles*.

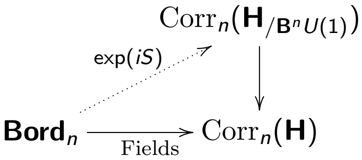
More general: higher gauge equivalence between circle n-bundles.

pQFT: local action functional

$\mathrm{Corr}_n(\mathbf{H}_{/\mathbf{B}^n U(1)})$ = n-fold correspondences in slice $\mathbf{H}_{/\mathbf{B}^n U(1)}$.

Definition

A functor



defines an n-dimensional *topological* prequantum field theory.

Proposition

Any such functor is obtained from a local action functional $\mathbf{Fields} \xrightarrow{\chi} \mathbf{B}^n U(1)$ *via*

$$\exp(iS[\phi]) = \int_\Sigma \chi(\phi).$$

pQFT: boundary theories

$\mathbf{Bord}_n^{\partial}$ = cobordisms with constrained boundary

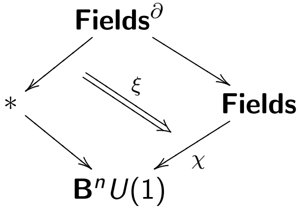
Definition

An n-dimensional boundary pQFT is a monoidal (∞, n)-functor

$$\mathbf{Bord}_n^{\partial} \to \mathrm{Corr}_n\left(\mathbf{H}_{/\mathbf{B}^n U(1)}\right)$$

Proposition (Fiorenza-Valentino)

A boundary pQFT is classified by a diagram



Summarizing:

Two sources of correspondences in $\mathbf{H}_{/\mathbf{B}^n U(1)}$:

1. As trajectories:

$$
\begin{array}{ccc}
 & \mathrm{Fields}(\Sigma) & \\
\swarrow & & \searrow \\
\mathrm{Fields}(\Sigma_{\mathrm{in}}) & \Longrightarrow & \mathrm{Fields}(\Sigma_{\mathrm{out}}) \\
\searrow & & \swarrow \\
 & \mathbf{B}^n U(1) &
\end{array}
$$

2. Classifying boundary theories:

$$
\begin{array}{ccc}
 & \mathbf{Fields}^{\partial} & \\
\swarrow & & \searrow \\
* & \Longrightarrow & \mathbf{Fields} \\
\searrow & & \swarrow \\
 & \mathbf{B}^n U(1) &
\end{array}
$$

Path integral quantization

Idea in 1d:

- map $U(1) \to GL_1(\mathbb{C})$.
- **Fields** $\to \mathbf{B}U(1)$ determines line bundle L.
- quantum state $=$ section of L
- propagators by *addition* of phases in $\mathbb{C}$.

For higher dimensions: replace $\mathbb{C}$ by higher (smooth) ring.

Linearization: rings and cohomology

'Higher ring' = (smooth) E_∞ ring spectrum

Cohomology

- X a smooth stack
- R a smooth E_∞ ring

The R-cohomology of X is

$$R^*(X) := \mathrm{Maps}(X, R)$$

Example

For X a manifold, R an ordinary geometrically discrete ring

$$R^*(X) = \left\{ R\text{-cochains in } X \right\}$$

Linearization: twisted cohomology

Let

- R a (smooth) E_∞ ring spectrum.
- $GL_1(R)$ its group of units in $\mathbf{H}$.

A map

$$X \xrightarrow{\alpha} \mathbf{B}GL_1(R)$$

classifies a (smooth) bundle

$$L \longrightarrow X$$

with fiber R.

Definition (Ando-Blumberg-Gepner-Hopkins-Rezk)

The α-twisted R-cohomology spectrum of X is

$$R^{*+\alpha}(X) := \mathrm{Maps}_{\mathrm{R-lin}}(L, R) = \Gamma(X, L^\vee)$$

Linearization: quantum states

A group homomorphism

$$\mathbf{B}^{n-1}U(1) \to GL_1(R)$$

gives a *universal* twist of R-cohomology

$$\mathbf{B}^n U(1) \to \mathbf{B}GL_1(R).$$

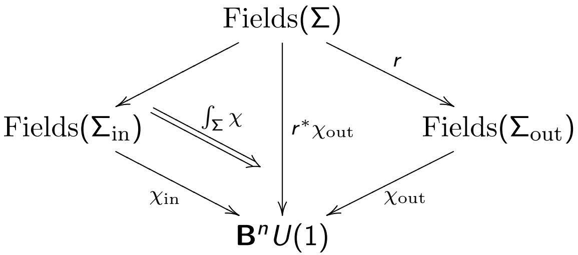
Then **Fields** $\xrightarrow{\chi}$ $\mathbf{B}^n U(1)$ gives

$$R^{*+\chi}(\mathbf{Fields}) = \Gamma(\mathbf{Fields}, L^\vee)$$

the space of 'higher wave functions/quantum states'.

Linearized trajectories

A trajectory



gives rise to

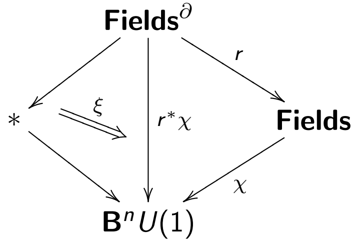
$$R^{*+\chi_{\mathrm{in}}}(\mathrm{Fields}_{\mathrm{in}}) \xrightarrow{\int_\Sigma \chi} R^{*+r^*\chi_{\mathrm{out}}}\left(\mathrm{Fields}(\Sigma)\right) \xleftarrow{r^*} R^{*+\chi_{\mathrm{out}}}\left(\mathrm{Fields}(\Sigma_{\mathrm{out}})\right)$$

Quantization: turn this into a propagator

$$R^{*+\chi_{\mathrm{in}}}(\mathrm{Fields}_{\mathrm{in}}) \xrightarrow{r_! \circ \int_\Sigma \chi} R^{*+\chi_{\mathrm{out}}}\left(\mathrm{Fields}(\Sigma_{\mathrm{out}})\right)$$

Linearized boundaries

A boundary



gives rise to

$$R \xrightarrow{\xi} R^{*+r^*\chi}(\mathbf{Fields}^{\partial}) \xleftarrow{r^*} R^{*+\chi}(\mathbf{Fields})$$

Quantization: turn this into a state

$$R \xrightarrow{r_!(\xi)} R^{*+\chi}(\mathbf{Fields}).$$

This is the holographic quantization of the boundary theory.

Quantization

Idea: fiber integration by duality

For M a closed manifold:

$$\begin{array}{ccc} H^*(M) & \xrightarrow{f_!} & H^{*-\mathrm{dim(M)}}(*) \\ {\scriptstyle P.D.}\downarrow\simeq & & \simeq\uparrow \\ H_{*-\mathrm{dim}(M)}(M) & \xrightarrow[f_*]{} & H_{*-\mathrm{dim}(M)}(*) \end{array}$$

In general:

- identify $R^{*+\chi}(X) \xrightarrow{\sim} R^{*+\chi}(X)^\vee$ with its dual (*orientation*).
- use the dual map to form the pushforward.
- do this fiberwise.

Constraints: compactness + orientability

Example: quantization of Poisson manifolds

Math	Physics
symplectic manifold	mechanical system.
Poisson manifold	foliation of mechanical systems.

Both describe a *non-topological* particle.

Holographic quantization: quantize them as the *boundary* of a 2d *topological* pQFT.

Analogue in *geometric* quantization of

$$\begin{array}{c}\text{deformation quantization}\\ \text{of Poisson manifold}\end{array} = \begin{array}{c}\text{perturbative quantization}\\ \text{of Poisson sigma model}\end{array}$$

by Kontsevich and Cattaneo-Felder.

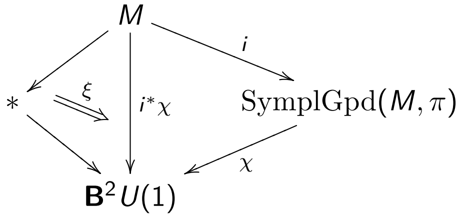
Example: Poisson sigma model

$$\text{Poisson manifold } (M, \pi) \xrightarrow{\text{exponentiate}} \mathrm{SymplGpd}(M, \pi) \in \mathbf{H}$$

Under suitable conditions:

- $\mathrm{SymplGpd}(M, \pi)$ a Lie groupoid.
- with multiplicative prequantum line bundle on space of morphisms.
- $\chi : \mathrm{SymplGpd}(M, \pi) \to \mathbf{B}^2 U(1)$ describes Poisson sigma-model.

M describes a boundary of the 2d Poisson sigma model.



K-theory for differentiable stacks

To quantize: map higher group $\mathbf{B}U(1)$ to units of smooth ring.

Expected good choice: smooth K-theory $\mathbf{KU}$.

K-theory for differentiable stacks

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Topological approximation:

Theorem (Landsman, Joachim-Stolz, Tu e. a., ...)

There is a lax monoidal functor

$$\mathrm{DiffStack}^{\mathrm{prop}}_{/\mathbf{B}^2 U(1)} \xrightarrow{C^*(-)} \mathrm{KK} \longrightarrow ho\big(KU\mathrm{Mod}\big)$$

taking a differentiable stack to the K-theory spectrum of its twisted convolution algebra.

This gives:

- twisted topological K-theory.
- twisted G-equivariant K-theory for compact G.

Example: Poisson sigma model

If $M \xrightarrow{i} \mathrm{SymplGpd}(M,\pi)$ is K-oriented, we obtain

$$i_!(\xi) \in K^{*+\chi}\big(\mathrm{SymplGpd}(M,\pi)\big)$$

Interpret this as

- twisted vector bundle over leaf space $\mathrm{SymplGpd}(M,\pi)$
- with fibers the quantizations of the symplectic leaves.

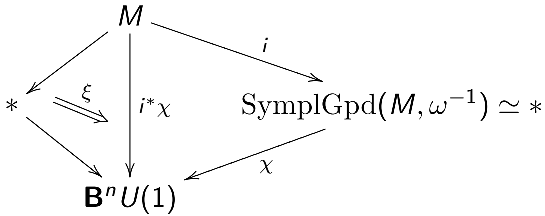
This combines the

- K-theoretic quantization of symplectic manifolds
- quantization of symplectic groupoids (Hawkins)

to complete Weinstein's quantization programme for Poisson manifolds using their symplectic groupoid.

Example: symplectic manifold

If $(M, \pi) = (M, \omega^{-1})$ symplectic, then



describes the prequantum circle bundle L over M.

This produces the traditional geometric quantization of (M, ω):

- A spin^c-structure on M defines an orientation.
- $i_!(\xi)$ as index of the spin^c Dirac operator, coupled to L.
- $i_!(\xi) \in K^0(*)$ gives the virtual space of states.

Example: Lie-Poisson manifold

- G a compact, simply connected Lie group.
- $\mathfrak{g}^*$ carries a linear Poisson structure π_{Lie}.
- $\mathrm{SymplGpd}(\mathfrak{g}^*, \pi_{\text{Lie}}) \simeq \mathfrak{g}^*//G$ under the coadjoint action.
- The map $\mathfrak{g}^* \to \mathfrak{g}^*//G$ has natural K-orientation.

When restricted to suitable defects (given by coadjoint orbits), this produces Kirillov's orbit method.

Interpretation the 'inverse orbit method theorem' of Freed-Hopkins-Teleman as defects of 2d Poisson sigma model.

Outlook

More examples of holographic quantization:

- *D*-brane charges in string theory (Brodzki e.a.).
- Witten genus quantizing the heterotic string.
- '*M*-brane charge' quantizing string at end of 2-brane.

Examples of cohomological quantization of TFTs:

- string topology operations (Chas-Sullivan, Godin, ...).
- CS-theory as '2-1'-theory (Freed-Hopkins-Teleman).

Requires functoriality + a *consistent choice of orientation*.

Next step: use pull-push quantization to produce a TQFT

$$\mathrm{Bord}_n \to (R\mathrm{Mod})^{\square^n}$$