‘And’

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Conjunction is the connective of choice when philosophers look to give an account of the meaning of logical connectives.

A common view is that the meaning of '*and*' is determined by its introduction and elimination rules.

Conjunction via natural deduction

Rules for Conjunction

- Introduction

$$\frac{A \qquad\qquad B}{A \wedge B}$$

- Elimination

$$\frac{A \wedge B}{A} \qquad\qquad \frac{A \wedge B}{B}$$

Standard view of 'and'

> *...it's hard to see what else could constitute meaning conjunction by 'and' except being prepared to use it according to some rules and not others (most plausibly, the standard introduction and elimination rules for 'and'). (p. 488)*

> *This [an inference of Simon, a three-valued logician], then, is the basis for Williamson's confidence that not even something as seemingly safe as conjunction elimination is required for meaning* **and** *by 'and.' I am not persuaded. I don't believe that Simon presents us with an intelligible counterexample to the analyticity of conjunction elimination... (p. 489)*

Paul Boghossian (2011), 'Truth in Virtue of Meaning', *Philosophy and Phenomenological Research* Vol. LXXXII No. 2, March 2011, pp. 488-497

Bede Rundle - there's much more to 'and'

On the one hand, I am inclined to think that the standard philosophical treatment of conjunctions like 'and', 'but' and 'although' has been grossly inadequate, no concern being shown for anything more than a narrow aspect of their use, and no investigation of that use being conducted on the right principles. On the other hand, the preoccupation with truth-conditions which has resulted in this defective approach is one towards which it is easy, and proper, to be sympathetic, at least initially. I shall begin by indicating the considerations which might invite our sympathy, and then call upon the example of 'and' to show how the approach is defective. (1983, p. 386)

Common pattern

- Philosophical logicians represent some concept in some standard logical calculus.
- Ordinary language philosophers claim this treatment is (wholly) inadequate.
- This may lead to a general '*What has the predicate calculus done for philosophy?*' charge.
- One can agree with criticisms, and yet still think that some other calculus is much more appropriate.
- I'm looking at doing so for dependent type theory and HoTT in particular.

Hence, the prior treatment of 'the' in definite description, and of the modalities.

Today it is *and*'s turn. (I'm greatly indebted to the work of Aarne Ranta.).

- He used to lie in the sun and play cards.
- Jack fell down and broke his crown, and Jill came tumbling after.
- Pam took the key out of her bag and opened the door.
- It's raining now, and doing so heavily.

- He used to lie in the sun and play cards.

The introduction rule applied to ‘He used to lie in the sun’ and ‘He used to play cards’ is inappropriate here. We certainly mean that the cardplaying takes place *while* lying in the sun.

There’s also a possible asymmetry here:

- He used to play cards and lie in the sun.

The cardplaying corresponds to subintervals of sunbathing intervals.

Also, we don't conjoin just any propositions:

- Jack fell down and broke his crown, and Jill came tumbling after.
- Jack fell down and is twenty years old, and Jill is holidaying in Thailand.

There's often a relation between conjuncts which may be made explicit:

- He used to lie in the sun and play cards **then**
- Jack fell down and broke his crown, and Jill came tumbling after **him**.
- Pam took the key out of her bag and opened the door **with it**.

We'd be surprised if Pam had taken out the key, then not used it to open the door.

Some of these conjunctions could be conceived as answers to double questions (...and if so...?):

- Is it raining, and, if so, is it heavy?
- Yes, it's raining now, and doing so heavily.

There's a **dependency** of the second part of the question on the answer to the first part of the question. A negative answer to the first question and the second question makes no sense.

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What do you turn to when you want to capture dependency?

Dependent type theory.

Introduction and elimination done properly

	type theory	category theory
	syntax	semantics
	natural deduction	universal construction
	product type	**product**
type formation	$\dfrac{\vdash A : \mathrm{Type} \qquad \vdash B : \mathrm{Type}}{\vdash A \times B : \mathrm{Type}}$	$A, B \in \mathscr{C} \Rightarrow A \times B \in \mathscr{C}$
term introduction	$\dfrac{\vdash a : A \qquad \vdash b : B}{\vdash (a,b) : A \times B}$	$\begin{array}{ccccc} & & Q & & \\ & {}^{a}\swarrow & \downarrow_{(a,b)} & \searrow^{b} & \\ A & & A \times B & & B \end{array}$
term elimination	$\dfrac{\vdash t : A \times B}{\vdash p_1(t) : A} \qquad \dfrac{\vdash t : A \times B}{\vdash p_2(t) : B}$	$\begin{array}{ccccc} & & Q & & \\ & & \downarrow^{t} & & \\ A & \xleftarrow{p_1} & A \times B & \xrightarrow{p_2} & B \end{array}$
computation rule	$p_1(a, b) = a \quad p_2(a, b) = b$	$\begin{array}{ccccc} & & Q & & \\ & {}^{a}\swarrow & \downarrow_{(a,b)} & \searrow^{b} & \\ A & \xleftarrow{p_1} & A \times B & \xrightarrow{p_2} & B \end{array}$

In this type theory, these constructions hold whatever the kind of type: Propositions, sets,...

...	...
2	2-groupoid
1	groupoid
0	set
-1	mere proposition
-2	contractible type

So product for propositions is conjunction, and for sets is (cartesian) product.

But we need a **dependent** version of this type formation.

Dependent sum/pair via natural deduction

	type theory	**category theory**
	syntax	semantics
	natural deduction	universal construction
	dependent sum type	**dependent sum**
type formation	$\dfrac{\vdash X\colon \mathrm{Type} \qquad x\colon X \vdash A(x)\colon \mathrm{Type}}{\vdash (\sum_{x:X} A(x))\colon \mathrm{Type}}$	$\left(X \in \mathscr{C},\ \begin{array}{c} A \\ \downarrow^{p_1} \\ X \end{array} \in \mathscr{C}/X \right) \Rightarrow (A \in \mathscr{C})$
term introduction	$\dfrac{x\colon X \vdash a\colon A(x)}{\vdash (x,a)\colon \sum_{x':X} A(x')}$	$\begin{array}{ccc} Q & \xrightarrow{(x,a)} & A \\ & {}_x\searrow & \\ & & X \end{array}$
term elimination	$\dfrac{\vdash t\colon (\sum_{x:X} A(x))}{\vdash p_1(t)\colon X \qquad \vdash p_2(t)\colon A(p_1(t))}$	$\begin{array}{ccc} Q & \xrightarrow{t} & A \\ & & \downarrow^{p_1} \\ & & X \end{array}$
computation rule	$p_1(x,a) = x \quad p_2(x,a) = a$	$\begin{array}{ccc} Q & \xrightarrow{(x,a)} & A \\ & {}_x\searrow & \downarrow^{p_1} \\ & & X \end{array}$

Dependent sum	Dependent product
$\sum_{x:A} B(x)$ is the collection of pairs (a, b) with $a : A$ and $b : B(a)$	$\prod_{x:A} B(x)$, is the collection of functions, f, such that $f(a) : B(a)$
When A is a set and $B(x)$ is a constant set B: The product of the sets.	When A is a set and $B(x)$ is a constant set B: The set of functions from A to B.
When A is a proposition and $B(x)$ is a constant proposition, B: The conjunction of A and B.	When A is a proposition and $B(x)$ is a constant proposition, B: The implication $A \to B$.

It is easy to think of cases where the sets $B(x)$ genuinely depend on $x : A$, and also where the propositions $B(x)$ genuinely depend on $x : A$.

- Consider the case where A is a set, and $B(a)$ is a proposition for each a in A.
- Perhaps A is the set of animals, and $B(a)$ states that a particular animal, a, is bilateral.
- Then an element of the dependent sum is an element a of A and a proof of $B(a)$, so something witnessing a bilateral animal.
- Meanwhile an element of the dependent product is a mapping from each $a : A$ to a proof of $B(a)$.
- There will only be such a mapping if $B(a)$ is true for each a.
- If this were the case, we would have a proof of the universal statement 'for all x in A, $B(x)$', in our example, 'All animals are bilateral.'

- The dependent sum is almost expressing the existential quantifier 'there exists x in A such that $B(x)$', except that it's gathering all such a for which $B(a)$ holds, or, in our case, gathering all bilateral animals.
- To treat this dependent sum as a proposition, there needs to be a 'truncation' from set to proposition, so that we ask merely whether this set is inhabited, in our case 'Does there exist a bilateral animal?'
- Any two bilateral animals are treated as the same. An element of this truncated existential proposition is something like 'the fact that some bilateral animal exists'.

Now what happens when A is a proposition, and B depends upon A?

On the face of it, if A is a subsingleton, so a type which may be empty (false) or inhabited by a single element (true), there's no scope for variation over A.

- The only type that depends on the empty type is the empty type.
- If proposition A is true, then there's just one proposition to consider depending on it. How can that not be constant, and in some sense then not dependent?

However, see how options for the B type are being constrained by whether A is inhabited. This isn't just conjunction of independent propositions. If A is false, there is no type B.

The key is to think of the formation of $B(x)$ itself. Natural language versions may even have a word indicating this dependency.

Consider the song '*If you're happy and you know it, clap your hands*'. What does the 'it' refer to in the antecedent?

- You're happy and you know it

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Following Vendler, we might say this is

- You're happy and you know the fact that you're happy.

How is ‘X knows P’ formed?

$$X : Person, P : Proposition, x : P \vdash know(X, P, x) : Type$$

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Once we have the judgements:

- $\vdash you : Person,$
- $\vdash you're\ happy : Prop,$
- $\vdash h : you're\ happy$
- then, $\vdash know(you, you're\ happy, h) : Prop$

How is 'X knows P' formed?

$$X : Person, P : Proposition, x : P \vdash know(X, P, x) : Type$$

You might take this to be a proposition if there's one fact of X knowing a true P without concern for different ways X may be said to know P.

Once we have the judgements:

- $\vdash you : Person$,
- $\vdash you're\ happy : Prop$,
- $\vdash h : you're\ happy$
- then, $\vdash know(you, you're\ happy, h) : Prop$
- and we may have $\vdash k : know(you, you're\ happy, h)$,
- and so $\vdash (h, k) : \sum_{x : You're\ happy} know(you, you're\ happy, x)$

The 'it' of '*You're happy and you know it*' is the pair $(you're\ happy, h)$.

Vendler claims that we know facts but believe propositions. So perhaps we have

$$X : Person, P : Proposition, x : P \vdash know(X, P, x) : Type$$

but

$$X : Person, P : Proposition \vdash believe(X, P) : Type$$

So we don't sing 'If you're happy and you believe it, clap your hands.'

Of course, 'I believe it (her claim)' is fine, and the claim may in fact be false.

- He used to lie in the sun and play cards **then**
- Jack fell down and broke his crown, and Jill came tumbling after **him**.
- Pam took the key out of her bag and opened the door **with it**.

In each case there is a context, and some dependency on that context.

Recall what is meant by a context in type theory.

> *A man walks into a bar. He's whistling a tune. A woman sits at a table in the bar. She's nursing a drink. On hearing the tune, she jumps up, knocking over the drink. She hurls the glass at him. "Is that any way to greet your husband", he says.*

One long context, with dependency structure

$x_1 : Man, x_2 : Bar, x_3 : WalksInto(x_1, x_2), x_4 : Tune, x_5 : Whistle(x_1, x_4), x_6 : Woman, x_7 : Table, x_8 : Locate(x_7, x_2), x_9 : SitsAt(x_6, x_7), x_{10} : Drink, x_{11} : Nurse(x_6, x_{10}), x_{12} : Hear(x_6, x_5), \ldots$

You could easily populate this with plenty of '*and*'s, especially where there is dependency.

[Note that there is a huge amount of tacit lexical knowledge deployed.]

- He used to lie in the sun and play cards.
- Jack fell down and broke his crown, and Jill came tumbling after.
- Pam took the key out of her bag and opened the door.

These can be treated by dependency relations to a common context. E.g., we must have introduced Jack as a person to form 'Jack fell down'.

Dependent type theory and its contexts will allow a much better approach to event structures:

> *'There are no straightforward rules for translating ordinary event-recording sentences into the canonical notation in advance of displaying and analysing their logical structure, not in the forms of the predicate calculus, but in terms of the verbs (and their specific meanings), the qualifying adverbs (and their specific significance, and hence effect upon the overall meaning of the expression or expressions they qualify), the application of the nominalizing operation to different types of adverbially qualified verbs, etc.' (Peter Hacker)*

For another time...

What of *but*, *while*, *although*, *whereas*? All of these are forms of *and*, but with yet more stringent conditions:

- Jay likes beer, whereas Kay prefers wine.
- Jane fell over, but she still won the race.

Background (lexical) knowledge is key to making sense of this usage in terms of contrasts and expectations.

What of Winston Churchill's

- Give us the tools, and we will finish the job.

Rundle had noted that we use 'and' to connect propositions, commands and questions, and mixtures.

- Where do you come from and what do you do?
- Put out the dog and bring in the cat.
- He was decent enough to apologize, and make sure you do too.

This raised the suspicion for him:

> *...given that the conjunctive role of 'and' is quite indifferent to mood, we should surely be suspicious of any account of its use that makes reference to truth essential.*

Pressing on with our use of dependent type theory, here we're probably seeing in the Churchill case:

- There is a job. If you give us the tools for this job, we will finish the job with these tools.

Then the original 'and' here is picking up now a dependent **product**, a map from instances of tools to the truth of the claim that we will be successful with them.

This provides a way to think about what disturbs beginning logic students that $P \to Q$ is constructed without there being any relevance between the components. Just as logical conjunction is a degenerate (non-dependent) form of dependent sum, so material implication is a degenerate form of dependent product. [For a proof of the implication, when P is false, there is the trivial map from the empty type, and when Q is true, there is the constant map to the proof of Q.]

The 'and' in Churchill's remark is picking up genuine dependency.

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There can be genuine dependence in $\prod_{x:A} B(x)$ for A and B propositions. 'If you're happy, then you'll know it'.

Thank you.