

Getting to Grips with Sweedler Runners for Systems

Appendix C: Quantum runners – from QS monads to C^* -correspondences

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Aim. Part XII compared the prospective internal language of Sweedler runners with Linear Homotopy Type Theory and with Sati–Schreiber’s Quantum Monadology. There the comparison remained largely methodological: both programmes begin with a structured monoidal world, identify monads and comonads as computational effects or modalities, and seek an internal programming language in which those structures become usable syntax. In this appendix we push the comparison further. In the elementary finite-dimensional sector of Quantum Monadology, the principal quantum monads can be placed directly into the accessible-endofunctor setting used for Sweedler runners. We can then calculate what a runner between the QuantumEnvironment and QuantumState theories looks like. The calculation leads first to projective measurements, then through dagger and Stinespring/Naimark dilation to completely positive maps, and finally to C^* -correspondences. The outcome is not that the usual CPM construction simply commutes with **Run**(–). Rather, the operator-algebraic analogue of a hidden-state runner appears to be a pointed correspondence, with composition by relative tensor product and completely positive maps as the resulting extensional behaviour.

1 The elementary QS theories

Sati and Schreiber develop Quantum Monadology on categories of parameterised module spectra, with an elementary specialization to set-indexed complex vector spaces [2]. Their domain-specific language QS expresses the resulting monadic quantum effects in `do`-notation and is intended to embed into Linear Homotopy Type Theory. For the present calculation we restrict to the finite-dimensional linear sector and suppress the surrounding dependent structure.

Let W be a finite classical outcome set and write

$$QW := \bigoplus_{w \in W} \mathbb{C} \simeq \mathbb{C}^W.$$

The finite-dimensional QuantumEnvironment construction has underlying endofunctor

$$E_W(X) = X \otimes QW.$$

Likewise, for a finite-dimensional complex vector space H , the QuantumState construction has underlying endofunctor

$$S_H(X) = X \otimes H \otimes H^* \simeq X \otimes \text{End}(H).$$

Both carry Frobenius monad structure in the linear setting. Thus, if

$$\mathcal{D}_{\text{QS}} = [\mathbf{Vect}_{\mathbb{C}}, \mathbf{Vect}_{\mathbb{C}}]_{\text{acc}},$$

then E_W and S_H are accessible endofunctors and their monad structures make them monoid objects for composition. In the terminology of this tutorial, they are therefore theories:

$$E_W, S_H \in \text{Mon}(\mathcal{D}_{\text{QS}}).$$

This is the first point at which the comparison with QS becomes literal rather than metaphorical. The individual quantum monads play the role of our theories $T, R, \dots$, while QS itself is closer to the internal programming language generated by a structured family of such theories and their interactions.

The category $\mathbf{Vect}_{\mathbb{C}}$ is locally presentable, symmetric monoidal closed, and tensoring preserves colimits separately in each variable. McDermott, Rivas and Uustalu show that accessible monads in this setting admit the relevant monadic Sweedler theory: the final interacting comonad for a monad T and residual monad R is the Sweedler hom $M(T, R)$, and its coalgebras are precisely the corresponding stateful runners [1]. We may therefore ask concretely for

$$M(E_W, S_H).$$

2 What is an E_W -in- S_H runner?

Write

$$A := \mathbb{C}^W, \quad B := \text{End}(H).$$

Then

$$E_W(X) = X \otimes A, \quad S_H(X) = X \otimes B.$$

Let Y be a vector space of hidden implementation states. A runner

$$E_W \rightsquigarrow S_H$$

with state Y is a natural family

$$(X \otimes A) \otimes Y \longrightarrow (X \otimes Y) \otimes B$$

compatible with the unit and multiplication of the two monads. Naturality in X means that the family is determined by a single linear map

$$\rho : A \otimes Y \longrightarrow Y \otimes B.$$

The runner unit law is

$$\rho(1_A \otimes y) = y \otimes 1_B,$$

while multiplication compatibility says schematically

$$\rho(aa' \otimes y) = (1_Y \otimes \mu_B)(\rho \otimes 1_B)(1_A \otimes \rho)(a \otimes a' \otimes y),$$

with the evident associativity and symmetry maps suppressed.

Thus

$$M(E_W, S_H)\text{-}\mathbf{Coalg} \simeq \left\{ (Y, \rho) \mid \rho : \mathbb{C}^W \otimes Y \longrightarrow Y \otimes \text{End}(H) \text{ satisfies the runner laws} \right\}.$$

Already this is recognizable as a quantum implementation problem: a classical W -environment is to be realised through operations on a quantum state space, with an additional private implementation state Y .

3 Projector-valued implementations

The special form of $A = \mathbb{C}^W$ makes the runner equations particularly transparent. Let

$$e_w \in \mathbb{C}^W$$

be the characteristic idempotent at w . These satisfy

$$e_w e_v = \delta_{wv} e_w, \quad \sum_{w \in W} e_w = 1.$$

Using finite dimensionality of H , the map

$$\rho_w : Y \longrightarrow Y \otimes H \otimes H^*$$

corresponding to e_w may equivalently be regarded as an endomorphism

$$P_w : Y \otimes H \longrightarrow Y \otimes H.$$

The runner laws become

$$P_w P_v = \delta_{wv} P_w, \quad \sum_{w \in W} P_w = 1_{Y \otimes H}.$$

Hence an E_W -in- S_H runner is exactly a W -indexed complete family of algebraically orthogonal idempotents on the combined hidden-visible state space $Y \otimes H$.

When

$$Y = \mathbb{C},$$

we recover a stateless theory map

$$E_W \longrightarrow S_H,$$

equivalently an algebra map

$$\mathbb{C}^W \longrightarrow \text{End}(H).$$

Such a map sends the primitive idempotents e_w to a complete family

$$P_w : H \longrightarrow H, \quad P_w P_v = \delta_{wv} P_w, \quad \sum_w P_w = 1_H.$$

Thus the distinction between tight and loose arrows becomes concrete:

$E_W \longrightarrow S_H$ fixed decomposition of H	$E_W \rightsquigarrow S_H$ decomposition of $Y \otimes H$ with hidden state Y .
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4 The universal coefficient algebra

We can package all such runners into a single universal algebra. Choose a basis of H , with

$$n := \dim H,$$

and write

$$P_w(y \otimes h_j) = \sum_{i=1}^n p_{ij}^w(y) \otimes h_i$$

for operators

$$p_{ij}^w : Y \longrightarrow Y.$$

The projector equations become

$$\sum_{k=1}^n p_{ik}^w p_{kj}^v = \delta_{wv} p_{ij}^w, \quad \sum_{w \in W} p_{ij}^w = \delta_{ij} 1_Y.$$

Define $U_{W,H}$ to be the unital associative algebra generated by symbols

$$u_{ij}^w, \quad w \in W, \quad 1 \leq i, j \leq n,$$

subject to

$$\sum_{k=1}^n u_{ik}^w u_{kj}^v = \delta_{wv} u_{ij}^w, \quad \sum_{w \in W} u_{ij}^w = \delta_{ij}.$$

Then every runner state Y carries a representation

$$U_{W,H} \longrightarrow \text{End}(Y), \quad u_{ij}^w \longmapsto p_{ij}^w,$$

and conversely every such representation determines a runner. Morphisms of runners are exactly $U_{W,H}$ -linear maps. Hence

$$\boxed{M(E_W, S_H)\text{-}\mathbf{Coalg} \simeq U_{W,H}\text{-}\mathbf{Mod},}$$

compatibly with the forgetful functors to $\mathbf{Vect}_{\mathbb{C}}$.

For an ordinary algebra U , the forgetful functor

$$U\text{-}\mathbf{Mod} \longrightarrow \mathbf{Vect}_{\mathbb{C}}$$

has right adjoint

$$V \longmapsto \text{Hom}_{\mathbb{C}}(U, V).$$

The associated comonad has counit

$$\varepsilon_V(f) = f(1)$$

and comultiplication

$$\delta_V(f)(a)(b) = f(ba).$$

Its coalgebras are precisely U -modules. It follows that in this finite-dimensional QS example the monadic Sweedler hom may be described explicitly as

$$\boxed{M(E_W, S_H)(Y) \cong \text{Hom}_{\mathbb{C}}(U_{W,H}, Y).}$$

This formula is worth pausing over. The Sweedler hom is not merely a set or vector space of quantum observables. It is a comonad whose coalgebras are arbitrary hidden-state implementations of the classical theory E_W inside the quantum theory S_H .

5 Classical measuring coalgebras and the rational part

It is tempting to expect the previous result to reduce simply to classical Sweedler theory of algebras. For algebras A and B , let

$$P(A, B)$$

denote the usual measuring coalgebra. Since $A = \mathbb{C}^W$ and $B = \text{End}(H)$, one might guess

$$M(E_W, S_H) \stackrel{?}{\cong} (-) \otimes P(A, B).$$

This is generally too small.

The algebra $U_{W,H}$ above is the corresponding universal coefficient or comeasuring algebra: there is a universal algebra map

$$\mathbb{C}^W \longrightarrow \text{End}(H) \otimes U_{W,H}.$$

Classical Sweedler duality identifies the measuring coalgebra with the finite dual,

$$P(A, B) \cong U_{W,H}^\circ.$$

Thus tensoring with the measuring coalgebra gives

$$Y \longmapsto Y \otimes U_{W,H}^\circ,$$

whereas the full monadic Sweedler hom is

$$Y \longmapsto \text{Hom}_{\mathbb{C}}(U_{W,H}, Y).$$

There is a canonical comparison

$$Y \otimes U_{W,H}^\circ \longrightarrow \text{Hom}_{\mathbb{C}}(U_{W,H}, Y),$$

but it need not be an isomorphism. Coalgebras for $U_{W,H}^\circ$ correspond to rational or locally finite $U_{W,H}$ -modules, while the monadic Sweedler hom classifies all $U_{W,H}$ -modules.

This difference is already visible when

$$W = \{0, 1\}, \quad H = \mathbb{C}^2.$$

The universal data reduce to an idempotent 2×2 matrix. Over the polynomial algebra $\mathbb{C}[t]$,

$$P = \begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix}$$

satisfies

$$P^2 = P.$$

Hence the universal property produces a quotient

$$U_{\{0,1\}, \mathbb{C}^2} \twoheadrightarrow \mathbb{C}[t],$$

so $U_{\{0,1\}, \mathbb{C}^2}$ is already infinite-dimensional. Therefore the finite dual and the full linear dual differ, and the monadic Sweedler hom is genuinely larger than the tensor comonad arising from the classical measuring coalgebra.

Thus the passage from algebraic Sweedler theory to Sweedler theory of monads adds something essential:

classical measuring coalgebra $\subseteq$ full stateful implementation comonad.

6 Adding the dagger: projective quantum measurement

So far the projector equations have been purely algebraic. Actual projective quantum measurement requires

$$P_w^\dagger = P_w.$$

At the level of the universal coefficient algebra this amounts to adjoining an involution satisfying

$$(u_{ij}^w)^* = u_{ji}^w.$$

Let $U_{W,H}^\dagger$ denote the resulting universal unital $*$ -algebra. Its defining matrix

$$P_w = (u_{ij}^w)$$

satisfies

$$P_w^* = P_w, \quad P_w P_v = \delta_{wv} P_w, \quad \sum_w P_w = I.$$

A $*$ -representation

$$\pi : U_{W,H}^\dagger \longrightarrow B(Y)$$

therefore determines genuine mutually orthogonal projections

$$P_w^\pi : Y \otimes H \longrightarrow Y \otimes H.$$

Thus dagger-compatible runners correspond to projective measurements on the combined hidden and visible system:

$$\boxed{E_W \rightsquigarrow S_H \rightsquigarrow W\text{-valued PVM on } Y \otimes H.}$$

It is important not to overstate this result. The original Sweedler hom lives in

$$[\mathbf{Vect}_{\mathbb{C}}, \mathbf{Vect}_{\mathbb{C}}]_{\text{acc}},$$

which remembers neither inner products nor positivity. The dagger-compatible runners therefore form a distinguished refinement of the algebraic runner theory; they do not yet arise intrinsically as coalgebras for a dagger-Sweedler hom constructed inside a category of Hilbert spaces.

Nevertheless the interpretation is already informative. A tight map

$$E_W \longrightarrow S_H$$

gives a projective measurement directly on H , whereas a loose dagger runner gives a projective measurement on an enlarged space

$$Y \otimes H.$$

The hidden state has become an ancillary quantum system.

7 From projective measurements to POVMs

General quantum measurements are described not by projective measurements but by positive operator-valued measures. A finite POVM on H is a family

$$F_w \geq 0, \quad \sum_{w \in W} F_w = 1_H.$$

Equivalently, it is a unital positive map

$$\Phi : \mathbb{C}^W \longrightarrow B(H), \quad \Phi(e_w) = F_w.$$

Because the source $\mathbb{C}^W$ is commutative, positivity is already enough to imply complete positivity.

The passage from PVMs to POVMs is exactly Naimark dilation. There is an ancillary Hilbert space Y , an isometry

$$V : H \longrightarrow Y \otimes H,$$

and a PVM

$$\{P_w\}_{w \in W}$$

on $Y \otimes H$ such that

$$F_w = V^\dagger P_w V.$$

Equivalently,

$$\Phi(a) = V^\dagger \pi(a) V$$

for a $*$ -representation

$$\pi : \mathbb{C}^W \longrightarrow B(Y \otimes H).$$

This corrects a tempting but slightly too quick interpretation of the loose runner. The dagger runner by itself gives the PVM on the enlarged hidden-visible system. A general POVM is obtained after also specifying an initialization or embedding

$$V : H \longrightarrow Y \otimes H$$

and then hiding the ancillary system. Thus

$\text{POVM} = \text{initialized dagger runner} + \text{environmental hiding}.$

This is very close to the ordinary operational meaning of hidden runner state. The implementation contains an ancillary system which is prepared, allowed to interact, and then suppressed from the exposed behaviour.

8 Why CPM does not simply commute with Run

At this point it is natural to ask whether one should simply apply Selinger's CPM construction to the runner semantics. Selinger's construction sends a dagger compact category to a dagger compact category whose morphisms are completely positive maps [3]. One might therefore hope for something like

$$\text{CPM}(\mathbf{Run}(\mathcal{D})) \stackrel{?}{\simeq} \mathbf{Run}(\text{CPM}(\mathcal{D})).$$

In this form the equation is not even automatically well-typed: $\mathbf{Run}(\mathcal{D})$ is a double category rather than merely a dagger compact category, while $\mathbf{Run}(-)$ itself requires the duoidal and measuring structure used by Sweedler theory.

There is also a more substantive obstruction. If we first pass to a category whose morphisms are completely positive maps, the noncommutative multiplication underlying the QuantumState writer monad is no longer a morphism of the required kind. The multiplication

$$\mu : B(H) \otimes B(H) \longrightarrow B(H), \quad A \otimes B \longmapsto AB,$$

is not positive in general. For noncommuting positive operators P and Q , the product PQ need not even be self-adjoint. Thus the QuantumState theory itself cannot simply be reconstructed as the same tensor-by- $B(H)$ monad after CP completion.

The order of construction therefore matters:

form the quantum theories in the pure linear world first;
introduce complete positivity only after passing to implementations.

This is also compatible with the methodological structure of Quantum Monadology, where the Frobenius monads live in the linear semantic world and mixed-state quantum processes are recovered using the additional dagger and positivity structure [2].

9 Pointed C^* -correspondences

The correct operator-algebraic replacement for a plain hidden vector space turns out to be a C^* -correspondence. Let A and B be unital C^* -algebras. An A - B correspondence consists of a right Hilbert B -module E together with a nondegenerate $*$ -representation

$$\pi_E : A \longrightarrow \mathcal{L}_B(E).$$

Correspondences compose by the interior tensor product:

$${}_A E_B, \quad {}_B F_C \quad \longmapsto \quad {}_A (E \otimes_B F)_C.$$

With unitary intertwining maps as 2-cells, C^* -algebras and correspondences form a bicategory.

Now choose a distinguished vector

$$\xi \in E.$$

The pointed correspondence (E, ξ) determines a completely positive map

$$\Phi_{E, \xi} : A \longrightarrow B$$

by

$$\Phi_{E, \xi}(a) = \langle \xi, \pi_E(a) \xi \rangle_B.$$

If

$$\langle \xi, \xi \rangle_B = 1_B,$$

then the map is unital. Conversely, Paschke's GNS construction associates to every completely positive map

$$\Phi : A \longrightarrow B$$

a cyclic pointed correspondence (E_Φ, ξ_Φ) satisfying

$$\Phi(a) = \langle \xi_\Phi, \pi_\Phi(a) \xi_\Phi \rangle_B.$$

The cyclic correspondence is unique up to the appropriate pointed unitary equivalence [4].

The decisive calculation is composition. Let

$$(E, \xi) : A \rightsquigarrow B, \quad (F, \eta) : B \rightsquigarrow C.$$

Their composite implementation is

$$(E, \xi) \odot (F, \eta) := (E \otimes_B F, \xi \otimes \eta).$$

Then

$$\begin{aligned} \Phi_{E \otimes_B F, \xi \otimes \eta}(a) &= \langle \xi \otimes \eta, a(\xi \otimes \eta) \rangle_C \\ &= \langle \eta, \langle \xi, a\xi \rangle_B \eta \rangle_C \\ &= \Phi_{F, \eta}(\Phi_{E, \xi}(a)). \end{aligned}$$

Thus

$$\boxed{\Phi_{(E, \xi) \odot (F, \eta)} = \Phi_{F, \eta} \circ \Phi_{E, \xi}.}$$

This is precisely the composition law we wanted from an implementation semantics. Hidden implementation data compose by tensoring over the intermediate interface, while the exposed completely positive maps compose ordinarily.

10 Implementation versus extensional behaviour

Pointed correspondences contain more information than their associated CP maps. A correspondence may contain extra summands or hidden degrees of freedom which the chosen point never probes. These disappear under

$$(E, \xi) \longmapsto \Phi_{E, \xi}.$$

Thus completely positive maps arise naturally as an extensional quotient of a richer implementation world.

Minimal or cyclic GNS correspondences give canonical representatives of that quotient. For a CP map $\Phi : A \rightarrow B$, the associated correspondence is generated by

$$\overline{\text{span}}\{a\xi_\Phi b \mid a \in A, b \in B\}.$$

However, minimal representatives are not in general preserved by raw interior tensor product. If

$$(E_\Phi, \xi_\Phi)$$

and

$$(E_\Psi, \xi_\Psi)$$

are minimal GNS correspondences, then

$$(E_\Phi \otimes_B E_\Psi, \xi_\Phi \otimes \xi_\Psi)$$

represents

$$\Psi \circ \Phi,$$

but it may contain more hidden space than the minimal GNS representation of the composite. The correct minimal representative is obtained by taking the cyclic subcorrespondence generated by the composite point:

$$E_{\Psi \circ \Phi} \simeq \text{Cyc}(E_\Phi \otimes_B E_\Psi, \xi_\Phi \otimes \xi_\Psi).$$

This gives two complementary levels:

implementation composition : $E \otimes_B F$, minimal extensional composition : $\text{Cyc}(E \otimes_B F)$.
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The first retains all implementation structure. The second forgets implementation redundancy and recovers the canonical behaviour object associated with the composite CP map.

The similarity with our general systems picture is striking:

Sweedler systems	operator-algebraic systems
hidden runner state	C^* -correspondence E
implementation law	left representation $A \rightarrow \mathcal{L}_B(E)$
initial state	distinguished vector ξ
runner composition	$E \otimes_B F$
hiding internal realization	$a \mapsto \langle \xi, a\xi \rangle$
extensional behaviour	completely positive map
behavioural minimization	cyclic GNS reduction.

11 Back to the QS example

For the QuantumEnvironment–QuantumState pair we take

$$A = \mathbb{C}^W, \quad B = B(H).$$

A pointed correspondence

$$(\mathbb{C}^W E_{B(H)}, \xi)$$

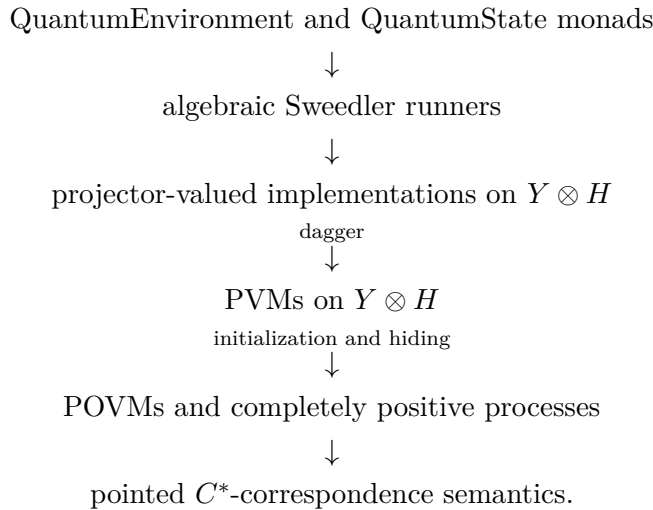
therefore yields a completely positive map

$$\Phi : \mathbb{C}^W \longrightarrow B(H), \quad \Phi(f) = \langle \xi, f\xi \rangle.$$

If the point is normalized, Φ is unital and hence determines a POVM. In the finite-dimensional matrix-algebra case the correspondence may be presented by adjoining an ancillary Hilbert space, while the left action becomes a projective representation on the enlarged system. The general correspondence formula therefore reduces to the Naimark expression

$$F_w = V^\dagger P_w V.$$

This places the stages of the calculation in a single sequence:



The original **Vect** _{$\mathbb{C}$} -based Sweedler calculation should therefore be regarded as the algebraic shadow of a richer analytic implementation theory.

12 A correspondence-valued quantum Sweedler hom

We can now formulate the conceptual endpoint. In ordinary Sweedler theory the measuring object

$$\{T, R\}$$

is universal for stateful implementations of T in R , and its coalgebras are concrete runner states. Composition is encoded by

$$\{R, P\} \star \{T, R\} \longrightarrow \{T, P\}.$$

The operator-algebraic calculation suggests that the quantum analogue should not necessarily be an ordinary comonoid whose coalgebras are vector spaces. The natural implementation objects between C^* -algebras A and B form instead a category of correspondences, and composition is the relative tensor product

$${}_A E_B, \quad {}_B F_C \quad \longmapsto \quad {}_A (E \otimes_B F)_C.$$

This suggests a categorified replacement

$$\boxed{\{A, B\}_{\text{quant}} \quad \text{“=”} \quad \text{the category of } A\text{-}B \text{ correspondences,}}$$

with a composition operation

$$\{B, C\}_{\text{quant}} \times \{A, B\}_{\text{quant}} \longrightarrow \{A, C\}_{\text{quant}}$$

implemented by the interior tensor product.

A pointed object

$$(E, \xi) \in \{A, B\}_{\text{quant}}$$

then represents an initialized implementation, while

$$(E, \xi) \longmapsto (a \mapsto \langle \xi, a\xi \rangle)$$

passes from hidden implementation to exposed CP behaviour.

This leads to the following prospective principle.

Quantum runner programme. A genuinely quantum extension of Sweedler runners should retain the pure Frobenius monads or theories in a dagger-linear semantic world, but replace plain state objects at the implementation level by analytic correspondence objects. Their composition should be relative tensor product over the intermediate theory or interface. Pointing or initializing a correspondence and then hiding its internal degrees of freedom should recover completely positive behaviour, with cyclic GNS reduction playing the role of extensional minimization.

This is not yet a construction of a dagger or C^* -enriched $\mathbf{Run}(\mathcal{D})$. In particular, we have not exhibited a duoidal category whose Sweedler homs are literally correspondence categories, nor proved a universal property equating the bicategory of C^* -correspondences with a measuring construction. What we have established is more modest but already substantial: the finite-dimensional QS monads fit directly into the ordinary Sweedler theory of accessible monads; their runner coalgebras admit an explicit projector interpretation; dagger refinement produces projective measurements; environmental hiding produces CP maps; and the canonical implementation objects for general CP maps are pointed C^* -correspondences whose tensor product reproduces composition.

The comparison with the Set-based behaviour story is suggestive. There, loose systems acquire extensional semantics through profunctorial behaviour:

$$\mathbf{Run}(\mathcal{D}) \longrightarrow \mathbf{CofProf}.$$

In the quantum case, the corresponding analytic intermediary may be some bicategory of Hilbert or C^* -correspondences:

$$\text{quantum runners} \longrightarrow \mathbf{Corr}_{C^*} \longrightarrow \mathbf{CP}.$$

Profunctor composition uses a coend; correspondence composition uses an interior tensor product. Both constructions compose mediating realization data over an intermediate interface. Whether this parallel can be promoted to a common enriched or double-categorical construction is an open question, but it now has a concrete mathematical form.

13 Conclusion

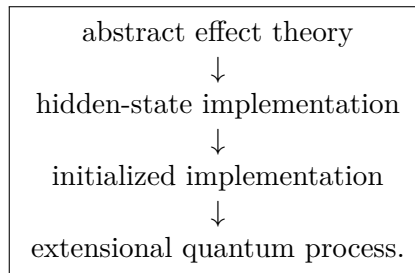
The point of this appendix was to ask whether the quantum monads appearing in QS can be recognized inside the Sweedler systems architecture. In the elementary finite-dimensional sector the answer is yes, but the comparison does not stop with the observation that both theories contain monads.

The QuantumEnvironment and QuantumState constructions provide genuine monoid objects in an accessible-endofunctor setting. Their Sweedler hom

$$M(E_W, S_H)$$

classifies hidden-state implementations of a classical W -environment in quantum-state dynamics. These implementations are controlled by a universal coefficient algebra $U_{W,H}$, and their dagger-compatible part is described by projective measurements on an enlarged state space. Initialization and hiding then recover general POVMs, while the general operator-algebraic form of the same construction is Paschke–GNS: a completely positive map is represented by a pointed correspondence and composition is implemented by interior tensor product.

The resulting picture is



This is precisely the hierarchy which motivated runners in the first place. The quantum case therefore does not merely supply another example of a monad. It suggests that the Sweedler idea of a universal space of implementations may admit an analytic, bicategorical refinement in which the “space” between two theories is itself a category of correspondences.

The open problem is consequently sharper than the naive question of whether CPM commutes with **Run**:

construct a correspondence-enriched or dagger-analytic version of **Run**
whose loose arrows recover Stinespring/GNS composition.

If such a construction exists, QS would not merely be analogous to the internal language proposed in Part XII. Its quantum effect theories would provide a first substantial test case for a genuinely quantum Sweedler runner calculus.

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