Getting to Grips with Sweedler Runners for Systems

Appendix C: Quantum runners – from QS monads to C^*-correspondences

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Aim. Part XII compared Sweedler runners with Sati–Schreiber's Quantum Monadology and Linear Homotopy Type Theory. Here we make that comparison concrete. A classical environment can be implemented through a quantum state monad with a private runner state; its commands become projections on the combined system. The key fact, valid beyond quantum mechanics, is that *writer runners are precisely freely presented bimodules*. Their composition is relative tensor product. This places the projector calculation, the universal coefficient algebra, and the passage to Hilbert C^*-correspondences in a single framework. With a dagger and an initialized ancilla, hiding recovers every finite POVM. The analytic question remaining is how to build a Sweedler hom whose *coalgebras*, rather than whose underlying object, recover general correspondences. The final sections develop a complementary dependent perspective: measurement instruments, typed classical feedback, and matrices of correspondences indexed by classical worlds.

1 The elementary QS theories

Sati and Schreiber develop Quantum Monadology on categories of parameterised module spectra, with an elementary specialization to set-indexed complex vector spaces [2]. Their domain-specific language QS expresses the resulting monadic quantum effects in do-notation and is intended to embed into Linear Homotopy Type Theory. For the present calculation we restrict to the finite-dimensional linear sector and suppress the surrounding dependent structure.

Let W be a finite classical outcome set and write

$$QW := \bigoplus_{w \in W} \mathbb{C} \simeq \mathbb{C}^W.$$

The finite-dimensional QuantumEnvironment construction has underlying endofunctor

$$E_W(X) = X \otimes QW.$$

Likewise, for a finite-dimensional complex vector space H, the QuantumState construction has underlying endofunctor

$$S_H(X) = X \otimes H \otimes H^* \simeq X \otimes \mathrm{End}(H).$$

Both carry Frobenius monad structure in the linear setting. Thus, if

$$\mathcal{D}_{\mathrm{QS}} = [\mathbf{Vect}_{\mathbb{C}}, \mathbf{Vect}_{\mathbb{C}}]_{\mathrm{acc}},$$

then E_W and S_H are accessible endofunctors and their monad structures make them monoid objects for composition. In the terminology of this tutorial, they are therefore theories:

$$E_W,\ S_H \in \mathrm{Mon}(\mathcal{D}_{\mathrm{QS}}).$$

This is the first point at which the comparison with QS becomes literal rather than metaphorical. The individual quantum monads play the role of our theories $T, R, \ldots$, while QS itself is closer to

the internal programming language generated by a structured family of such theories and their interactions.

The category $\mathbf{Vect}_{\mathbb{C}}$ is locally presentable, symmetric monoidal closed, and tensoring preserves colimits separately in each variable. McDermott, Rivas and Uustalu develop the final interacting-comonad interpretation of Sweedler homs and runner coalgebras [1]. We use its tensorially strong writer-instance here, constructing the relevant comonad explicitly rather than assuming all their general hypotheses carry over unchanged. Our target is

$$M(E_W, S_H).$$

Tensor convention. Throughout, implementation states combine by $\otimes$, not by the categorical product $\oplus$ of vector spaces. The two writer endofunctors have their canonical tensorial strengths, and the runner calculations below take place in that strong, $\mathbb{C}$-linear setting. We do not claim that arbitrary ordinary accessible endofunctors on $\mathbf{Vect}_{\mathbb{C}}$ have this strength, or that a general enriched accessible-endofunctor Sweedler existence theorem follows merely from local presentability. Independently of that general question, the explicit cofree comonad in § 5 exists and classifies exactly the writer runners considered here.

2 What is an E_W-in-S_H runner?

Write

$$A := \mathbb{C}^W, \qquad B := \mathrm{End}(H).$$

Then

$$E_W(X) = X \otimes A, \qquad S_H(X) = X \otimes B.$$

Let Y be a vector space of hidden implementation states. A runner

$$E_W \rightsquigarrow S_H$$

with state Y is a natural family

$$(X \otimes A) \otimes Y \longrightarrow (X \otimes Y) \otimes B$$

compatible with the unit and multiplication of the two monads. Naturality in X means that the family is determined by a single linear map

$$\rho : A \otimes Y \longrightarrow Y \otimes B.$$

The runner unit law is

$$\rho(1_A \otimes y) = y \otimes 1_B,$$

while multiplication compatibility says schematically

$$\rho(aa' \otimes y) = (1_Y \otimes \mu_B)(\rho \otimes 1_B)(1_A \otimes \rho)(a \otimes a' \otimes y),$$

with the evident associativity and symmetry maps suppressed.

Thus

$$\boxed{M(E_W, S_H)\text{-}\mathbf{Coalg} \simeq \left\{ (Y, \rho) \;\middle|\; \rho : \mathbb{C}^W \otimes Y \longrightarrow Y \otimes \mathrm{End}(H) \text{ satisfies the runner laws} \right\}.}$$

Already this is recognizable as a quantum implementation problem: a classical W-environment is to be realised through operations on a quantum state space, with an additional private implementation state Y.

3 The key calculation: writer runners are free bimodules

Before specializing to quantum observables, take any unital $\mathbb{C}$-algebras A, B and their writer monads $X \mapsto X \otimes A$ and $X \mapsto X \otimes B$. A right B-linear endomorphism of $E = Y \otimes B$ is determined by its values on $Y \otimes 1_B$. Explicitly, a map $f : Y \to Y \otimes B$ extends uniquely to

$$\widehat{f}(y \otimes b) = (1_Y \otimes \mu_B)(f(y) \otimes b).$$

Moreover, $\widehat{f}\widehat{g}$ restricts to $(1_Y \otimes \mu_B)(f \otimes 1_B)g$. This is just the multiplication appearing in the runner law.

Theorem (writer runners as freely presented bimodules). For unital algebras A, B, a tensorial writer runner with state Y is equivalently a left A-action on the *chosen free right B-module* $Y \otimes B$, commuting with the right action. The correspondence sends ρ to

$$a \cdot (y \otimes b) = (1_Y \otimes \mu_B)(\rho(a \otimes y) \otimes b).$$

A runner morphism $f : Y \to Y'$ corresponds exactly to a bimodule map $f \otimes 1_B$. Thus this identifies the runner category with the category of freely *presented* A–B bimodules and presentation-preserving maps; forgetting presentations gives a faithful, generally non-full functor into all bimodules.

Proof. The runner unit law gives $1_A \cdot e = e$, and its multiplication law gives $(aa') \cdot e = a \cdot (a' \cdot e)$. Conversely, restricting any left action to $Y \otimes 1_B$ yields the runner laws. The displayed extension formula gives the inverse constructions and the assertion about morphisms. □

Composition is equally concrete. For a second runner $B \rightsquigarrow C$ on Z, its free bimodule is $F = Z \otimes C$. The bimodule composite has the canonical presentation

$$(Y \otimes B) \otimes_B (Z \otimes C) \;\cong\; (Y \otimes Z) \otimes C,$$

where the left-hand balanced tensor product uses the B-action on $Z \otimes C$. In these coordinates its left A-action is precisely the composite runner law, with state $Y \otimes Z$. (The isomorphism uses the runner's B-action, so it is not simply a cancellation that discards that action.) Hence horizontal runner composition *is* relative tensor product in the free, presented part of the bimodule bicategory. This will explain, rather than merely resemble, the correspondence calculus below.

4 Projector-valued implementations

The special form of $A = \mathbb{C}^W$ makes the runner equations particularly transparent. Let

$$e_w \in \mathbb{C}^W$$

be the characteristic idempotent at w. These satisfy

$$e_w e_v = \delta_{wv} e_w, \qquad \sum_{w \in W} e_w = 1.$$

Using finite dimensionality of H, the map

$$\rho_w : Y \longrightarrow Y \otimes H \otimes H^*$$

corresponding to e_w may equivalently be regarded as an endomorphism

$$P_w : Y \otimes H \longrightarrow Y \otimes H.$$

The runner laws become

$$P_w P_v = \delta_{wv} P_w, \qquad \sum_{w \in W} P_w = 1_{Y \otimes H}.$$

Hence an E_W-in-S_H runner is exactly a W-indexed complete family of algebraically orthogonal idempotents on the combined hidden-visible state space $Y \otimes H$.

When

$$Y = \mathbb{C},$$

we recover a stateless theory map

$$E_W \longrightarrow S_H,$$

equivalently an algebra map

$$\mathbb{C}^W \longrightarrow \mathrm{End}(H).$$

Such a map sends the primitive idempotents e_w to a complete family

$$P_w : H \longrightarrow H, \qquad P_w P_v = \delta_{wv} P_w, \qquad \sum_w P_w = 1_H.$$

Thus the distinction between tight and loose arrows becomes concrete:

$E_W \longrightarrow S_H$	$E_W \rightsquigarrow S_H$
fixed decomposition of H	decomposition of $Y \otimes H$ with hidden state Y.

5 The universal coefficient algebra

We can package all such runners into a single universal algebra. Choose a basis of H, with

$$n := \dim H,$$

and write

$$P_w(y \otimes h_j) = \sum_{i=1}^{n} p_{ij}^w(y) \otimes h_i$$

for operators

$$p_{ij}^w : Y \longrightarrow Y.$$

The projector equations become

$$\sum_{k=1}^{n} p_{ik}^w p_{kj}^v = \delta_{wv} p_{ij}^w, \qquad \sum_{w \in W} p_{ij}^w = \delta_{ij} 1_Y.$$

Define $U_{W,H}$ to be the unital associative algebra generated by symbols

$$u_{ij}^w, \qquad w \in W, \quad 1 \le i, j \le n,$$

subject to

$$\sum_{k=1}^{n} u_{ik}^w u_{kj}^v = \delta_{wv} u_{ij}^w, \qquad \sum_{w \in W} u_{ij}^w = \delta_{ij}.$$

Then every runner state Y carries a representation

$$U_{W,H} \longrightarrow \mathrm{End}(Y), \qquad u_{ij}^{w} \longmapsto p_{ij}^{w},$$

and conversely every such representation determines a runner. Morphisms of runners are exactly $U_{W,H}$-linear maps. Hence

$$\boxed{M(E_W, S_H)\text{-}\mathbf{Coalg} \simeq U_{W,H}\text{-}\mathbf{Mod},}$$

compatibly with the forgetful functors to $\mathbf{Vect}_{\mathbb{C}}$.

For an ordinary algebra U, the forgetful functor

$$U\text{-}\mathbf{Mod} \longrightarrow \mathbf{Vect}_{\mathbb{C}}$$

has right adjoint

$$V \longmapsto \mathrm{Hom}_{\mathbb{C}}(U, V).$$

The associated comonad has counit

$$\varepsilon_V(f) = f(1)$$

and comultiplication

$$\delta_V(f)(a)(b) = f(ba).$$

Its coalgebras are precisely U-modules. It follows that in this finite-dimensional QS example the monadic Sweedler hom may be described explicitly as

$$\boxed{M(E_W, S_H)(Y) \cong \mathrm{Hom}_{\mathbb{C}}(U_{W,H}, Y).}$$

This formula is worth pausing over. The Sweedler hom is not merely a set or vector space of quantum observables. It is a comonad whose coalgebras are arbitrary hidden-state implementations of the classical theory E_W inside the quantum theory S_H.

Earlier universal constructions. The algebra $U_{W,H}$ represents $R \mapsto \mathrm{Alg}(\mathbb{C}^W, M_n(R))$; it is a matrix-reduction or universal comeasuring algebra, a special case of Tambara's adjoint construction $a(M_n(\mathbb{C}), \mathbb{C}^W)$ [5, 6]. When the source and target of Tambara's construction coincide one obtains his universal *coacting bialgebra*; the present mixed pair is the more general universal coefficient algebra, not automatically a bialgebra. The classical matrix-reduction description of $U_{W,H}$ is also connected to the free product $\mathbb{C}^W * M_n(\mathbb{C})$: more precisely,

$$\mathbb{C}^W * M_n(\mathbb{C}) \cong M_n(U_{W,H}),$$

with the embedding of $M_n(\mathbb{C})$ specifying the matrix units and the coefficient algebra as their centralizer. Accordingly Morita equivalence identifies runner representations with modules for this free product. The two actions on the combined space $Y \otimes H$ are the $\mathbb{C}^W$ action supplied by the runner and the visible matrix-algebra action $1_Y \otimes M_n(\mathbb{C})$; there is no requirement that they commute.

6 Classical measuring coalgebras and the rational part

It is tempting to expect the previous result to reduce simply to classical Sweedler theory of algebras. For algebras A and B, let

$$P(A, B)$$

denote the usual measuring coalgebra. Since $A = \mathbb{C}^W$ and $B = \mathrm{End}(H)$, one might guess

$$M(E_W, S_H) \overset{?}{\cong} (-) \otimes P(A, B).$$

This is generally too small.

The algebra $U_{W,H}$ above is the corresponding universal coefficient or comeasuring algebra: there is a universal algebra map

$$\mathbb{C}^W \longrightarrow \mathrm{End}(H) \otimes U_{W,H}.$$

Classical Sweedler duality identifies the measuring coalgebra with the finite dual,

$$P(A, B) \cong U_{W,H}^{\circ}.$$

Thus tensoring with the measuring coalgebra gives

$$Y \longmapsto Y \otimes U_{W,H}^{\circ},$$

whereas the full monadic Sweedler hom is

$$Y \longmapsto \mathrm{Hom}_{\mathbb{C}}(U_{W,H}, Y).$$

There is a canonical comparison

$$Y \otimes U_{W,H}^{\circ} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(U_{W,H}, Y),$$

but it need not be an isomorphism. Coalgebras for $U_{W,H}^{\circ}$ correspond to rational or locally finite $U_{W,H}$-modules, while the monadic Sweedler hom classifies all $U_{W,H}$-modules.

This difference is already visible when

$$W = \{0, 1\}, \qquad H = \mathbb{C}^2.$$

The universal data reduce to an idempotent 2×2 matrix. Over the polynomial algebra $\mathbb{C}[t]$,

$$P = \begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix}$$

satisfies

$$P^2 = P.$$

Hence the universal property produces a quotient

$$U_{\{0,1\},\mathbb{C}^2} \twoheadrightarrow \mathbb{C}[t],$$

so $U_{\{0,1\},\mathbb{C}^2}$ is already infinite-dimensional. To see strictness, choose a functional $\lambda \in \mathbb{C}[t]^*$ that vanishes on no cofinite ideal (for example one whose coefficient sequence satisfies no eventual linear recurrence). Let $q : U_{W,H} \twoheadrightarrow \mathbb{C}[t]$. If λq vanished on a cofinite ideal $I \subseteq U_{W,H}$, then $q(I)$ would be a cofinite ideal of $\mathbb{C}[t]$ annihilated by λ, a contradiction. Thus $U_{W,H}^{\circ} \neq U_{W,H}^{*}$, and the monadic Sweedler hom is genuinely larger than the tensor comonad arising from the classical measuring coalgebra. The distinction is between *locally finite (rational)* implementations and arbitrary modules, not merely between two notations for duals.

In the two-outcome case this also has an operator-theoretic reading. The runner projector P_0 and the distinguished matrix projection $1_Y \otimes e_{11}$ are two generically noncommuting idempotents. Their free algebraic interaction is already infinite-dimensional, foreshadowing the two-projection/two-subspace analysis in the self-adjoint Hilbert-space setting [7].

Thus the passage from algebraic Sweedler theory to Sweedler theory of monads adds something essential:

classical measuring coalgebra	$\subseteq$	full stateful implementation comonad.

7 Adding the dagger: projective quantum measurement

So far the projector equations have been purely algebraic. Actual projective quantum measurement requires

$$P_w^\dagger = P_w.$$

At the level of the universal coefficient algebra this amounts to adjoining an involution satisfying

$$(u_{ij}^w)^* = u_{ji}^w.$$

Let $U_{W,H}^\dagger$ denote the resulting universal unital $*$-algebra. Its defining matrix

$$P_w = (u_{ij}^w)$$

satisfies

$$P_w^* = P_w, \qquad P_w P_v = \delta_{wv} P_w, \qquad \sum_w P_w = I.$$

A $*$-representation

$$\pi : U_{W,H}^\dagger \longrightarrow B(Y)$$

therefore determines genuine mutually orthogonal projections

$$P_w^\pi : Y \otimes H \longrightarrow Y \otimes H.$$

Thus dagger-compatible runners correspond to projective measurements on the combined hidden and visible system:

$$\boxed{E_W \rightsquigarrow S_H \quad \rightsquigarrow \quad W\text{-valued PVM on } Y \otimes H.}$$

It is important not to overstate this result. The original Sweedler hom lives in

$$[\mathbf{Vect}_{\mathbb{C}}, \mathbf{Vect}_{\mathbb{C}}]_{\mathrm{acc}},$$

which remembers neither inner products nor positivity. The dagger-compatible runners therefore form a distinguished refinement of the algebraic runner theory; they do not yet arise intrinsically as coalgebras for a dagger-Sweedler hom constructed inside a category of Hilbert spaces.

Nevertheless the interpretation is already informative. A tight map

$$E_W \longrightarrow S_H$$

gives a projective measurement directly on H, whereas a loose dagger runner gives a projective measurement on an enlarged space

$$Y \otimes H.$$

The hidden state has become an ancillary quantum system.

8 From projective measurements to POVMs

A finite POVM on H is a family $F_w \geq 0$ with $\sum_w F_w = 1_H$, equivalently a unital positive map $\Phi : \mathbb{C}^W \to B(H)$. Since the source is commutative, this map is automatically completely positive.

Initialized-runner realization. Every such POVM comes from a dagger writer runner on a finite-dimensional ancilla Y, together with one initial unit vector $y_0 \in Y$:

$$F_w = (\langle y_0| \otimes 1_H) P_w (|y_0\rangle \otimes 1_H).$$

Indeed let $Y = \mathbb{C}^W$ and define the Naimark isometry

$$Vh = \sum_{w \in W} |w\rangle \otimes \sqrt{F_w} h.$$

It is an isometry because $\sum_w F_w = 1$. Fix a unit $y_0 \in Y$. In finite dimensions an isometry from $y_0 \otimes H$ onto $V(H)$ extends to a unitary U on $Y \otimes H$, with $U(y_0 \otimes h) = Vh$. Put

$$P_w = U^*(|w\rangle\langle w| \otimes 1_H)U.$$

These are orthogonal projections summing to 1, and compressing against y_0 yields F_w. By the writer-runner theorem they define a dagger runner. The same formula can be understood as applying a joint unitary before a sharp measurement; the joint unitary is absorbed into the runner PVM.

Thus the runner's own initial state suffices: no arbitrary isometry is needed as extra primitive data. An *individual* Naimark embedding $H \to Y \otimes H$ need not have product form, but an equivalent dilation can be chosen with the fixed product initialization above.

$$\boxed{\text{POVM} = \text{initialized dagger runner} + \text{hiding.}}$$

The runner retains the private mechanism; compression forgets which internal implementation produced the exposed effects.

9 What changes upon passing to CPM?

Selinger's construction maps pure processes into completely positive processes by a strong monoidal embedding $J : \mathbf{FHilb} \to \mathrm{CPM}(\mathbf{FHilb})$ [3]. In particular, it transports monoid objects: the contraction multiplication of $H \otimes H^*$ is a pure linear map, whose image under J is a CP morphism. Therefore the observation that ordinary multiplication $B(H) \otimes B(H) \to B(H)$ is not a positive map does *not* obstruct the transported QuantumState monad.

The distinction is between an object in a pure category and its *doubled* CPM image. States of the latter are positive operators on the Hilbert space $H \otimes H^*$, not simply elements of $B(H)$ regarded as a naive tensor factor. Ordinary multiplication of observables, considered directly as a linear map of operator algebras, is indeed not generally positive, but that is a different map from $J(\mu)$.

A formula such as $\mathrm{CPM}(\mathbf{Run}(\mathcal{D})) \simeq \mathbf{Run}(\mathrm{CPM}(\mathcal{D}))$ is moreover not automatically well-typed: the left input is a double category, while the right construction needs a specified duoidal endofunctor setting, appropriate strengths, and measuring objects. Whether a suitable comparison exists remains open. A first test is to compute runners between the *transported doubled monads*, not to infer their failure from positivity of naive matrix multiplication.

From free bimodules to Hilbert modules. When $B = B(H)$ and H is finite-dimensional, the analytic completion of the standard free module may be written $E = B(H, Y \otimes H)$, for a Hilbert space Y, with $\mathcal{L}_{B(H)}(E) \cong B(Y \otimes H)$. Thus a dagger runner gives a $\mathbb{C}^W$–$B(H)$ correspondence via its PVM representation on $Y \otimes H$. A general correspondence over the matrix algebra is $B(H, K)$ for a Hilbert space K carrying the left $\mathbb{C}^W$ representation. Such a K is abstractly of the form $Y \otimes H$ after choosing a multiplicity space Y (for finite-dimensional H). Consequently, in this *fixed matrix-algebra* example, dropping a *chosen* free presentation does not itself enlarge the class of correspondence objects up to unitary equivalence; it chiefly changes the allowed intertwiners and the canonicality of the presentation. For general C^*-algebras, not every Hilbert module is

free or stably free. That is where genuinely non-free implementations can arise. The algebraic tensor $Y \otimes B(H)$ should not be confused with its completed Hilbert-module version when Y is infinite-dimensional.

10 Pointed C^*-correspondences

The correct operator-algebraic replacement for a plain hidden vector space turns out to be a C^*-correspondence. Let A and B be unital C^*-algebras. An A-B correspondence consists of a right Hilbert B-module E together with a nondegenerate $*$-representation

$$\pi_E : A \longrightarrow \mathcal{L}_B(E).$$

Correspondences compose by the interior tensor product:

$$_AE_B, \qquad _BF_C \quad \longmapsto \quad _A(E \otimes_B F)_C.$$

With unitary intertwining maps as 2-cells, C^*-algebras and correspondences form a bicategory.

Now choose a distinguished vector

$$\xi \in E.$$

The pointed correspondence (E, ξ) determines a completely positive map

$$\Phi_{E,\xi} : A \longrightarrow B$$

by

$$\boxed{\Phi_{E,\xi}(a) = \langle \xi, \pi_E(a)\xi \rangle_B.}$$

If

$$\langle \xi, \xi \rangle_B = 1_B,$$

then the map is unital. Conversely, Paschke's GNS construction associates to every completely positive map

$$\Phi : A \longrightarrow B$$

a cyclic pointed correspondence (E_Φ, ξ_Φ) satisfying

$$\Phi(a) = \langle \xi_\Phi, \pi_\Phi(a)\xi_\Phi \rangle_B.$$

The cyclic correspondence is unique up to the appropriate pointed unitary equivalence [4].

The decisive calculation is composition. Let

$$(E, \xi) : A \rightsquigarrow B, \qquad (F, \eta) : B \rightsquigarrow C.$$

Their composite implementation is

$$(E, \xi) \odot (F, \eta) := (E \otimes_B F, \xi \otimes \eta).$$

Then

$$\begin{aligned} \Phi_{E\otimes_B F, \xi\otimes\eta}(a) &= \langle \xi \otimes \eta, a(\xi \otimes \eta) \rangle_C \\ &= \langle \eta, \langle \xi, a\xi \rangle_B \eta \rangle_C \\ &= \Phi_{F,\eta}(\Phi_{E,\xi}(a)). \end{aligned}$$

Thus

$$\boxed{\Phi_{(E,\xi)\odot(F,\eta)} = \Phi_{F,\eta} \circ \Phi_{E,\xi}.}$$

This is precisely the composition law we wanted from an implementation semantics. Hidden implementation data compose by tensoring over the intermediate interface, while the exposed completely positive maps compose ordinarily.

11 Implementation versus extensional behaviour

Pointed correspondences contain more information than their associated CP maps. A correspondence may contain extra summands or hidden degrees of freedom which the chosen point never probes. These disappear under

$$(E, \xi) \longmapsto \Phi_{E,\xi}.$$

Thus completely positive maps arise naturally as an extensional quotient of a richer implementation world.

Minimal or cyclic GNS correspondences give canonical representatives of that quotient. For a CP map $\Phi : A \to B$, the associated correspondence is generated by

$$\overline{\text{span}}\{a\xi_\Phi b \mid a \in A,\ b \in B\}.$$

However, minimal representatives are not in general preserved by raw interior tensor product. If

$$(E_\Phi, \xi_\Phi)$$

and

$$(E_\Psi, \xi_\Psi)$$

are minimal GNS correspondences, then

$$(E_\Phi \otimes_B E_\Psi, \xi_\Phi \otimes \xi_\Psi)$$

represents

$$\Psi \circ \Phi,$$

but it may contain more hidden space than the minimal GNS representation of the composite. The correct minimal representative is obtained by taking the cyclic subcorrespondence generated by the composite point:

$$E_{\Psi\circ\Phi} \simeq \text{Cyc}(E_\Phi \otimes_B E_\Psi, \xi_\Phi \otimes \xi_\Psi).$$

This gives two complementary levels:

implementation composition	:	$E \otimes_B F$,
minimal extensional composition	:	$\text{Cyc}(E \otimes_B F)$.

The first retains all implementation structure. The second forgets implementation redundancy and recovers the canonical behaviour object associated with the composite CP map.

The free-bimodule theorem makes the first part of this dictionary an identity (at the algebraic, presented level); pointing and CP behaviour add analytic structure:

Sweedler systems	operator-algebraic systems
runner state Y	free module $Y \otimes B$ (then completion)
implementation law	left representation $A \to \mathcal{L}_B(E)$
initial state	distinguished vector ξ
runner composition	$E \otimes_B F$
hiding internal realization	$a \mapsto \langle \xi, a\xi \rangle$
extensional behaviour	completely positive map
behavioural minimization	cyclic GNS reduction.

12 Back to the QS example

For the QuantumEnvironment–QuantumState pair we take

$$A = \mathbb{C}^W, \qquad B = B(H).$$

A pointed correspondence

$$({}_{\mathbb{C}^W}E_{B(H)}, \xi)$$

therefore yields a completely positive map

$$\Phi : \mathbb{C}^W \longrightarrow B(H), \qquad \Phi(f) = \langle \xi, f\xi \rangle.$$

If the point is normalized, Φ is unital and hence determines a POVM. In the finite-dimensional matrix-algebra case the correspondence may be presented by adjoining an ancillary Hilbert space, while the left action becomes a projective representation on the enlarged system. The general correspondence formula therefore reduces to the Naimark expression

$$F_w = V^\dagger P_w V.$$

This places the stages of the calculation in a single sequence:

QuantumEnvironment and QuantumState monads
$\downarrow$
algebraic Sweedler runners
$\downarrow$
projector-valued implementations on $Y \otimes H$
dagger
$\downarrow$
PVMs on $Y \otimes H$
initialization and hiding
$\downarrow$
POVMs and completely positive processes
$\downarrow$
pointed C^*-correspondence semantics.

The original $\mathbf{Vect}_{\mathbb{C}}$-based Sweedler calculation should therefore be regarded as the algebraic shadow of a richer analytic implementation theory.

13 Towards an analytic Sweedler hom

There is an important distinction of levels. The measuring comonad $M(T, R)$ is an *object* of the ambient endofunctor category; its coalgebras form the *category of implementations*. Hence a correspondence category, if it arises by an analytic measuring construction, belongs at the level of coalgebras, not as a replacement for the comonad itself. Already in the algebraic writer setting we have the concrete theorem

$$M(- \otimes A, - \otimes B)\text{-}\mathbf{Coalg} \simeq \mathbf{Mod}(U_{A,B}),$$

with its freely presented A–B bimodule interpretation. In the matrix example the comonad is explicitly

$$M(E_W, S_H)(Y) \cong \mathrm{Hom}_{\mathbb{C}}(U_{W,H}, Y).$$

An *analytic Sweedler hom* should consequently be an appropriate measuring object or comonad whose coalgebras recover Hilbert-module implementations, possibly general C^*-correspondences, with their intertwiners. Its compositional structure should induce the interior tensor product, and a normalized chosen point ξ should expose the CP map

$$a \longmapsto \langle \xi, \pi(a)\xi \rangle_B.$$

This is a precise construction problem, not a proposed identification $\{A, B\}_{\mathrm{quant}} = \mathbf{Corr}(A, B)$.

Established and open. Algebraically, tensorial writer runners *are* freely presented bimodules and compose by relative tensor product. Analytically, the standard Hilbert-module completion gives free correspondences, and initialized dagger runners recover every finite POVM. What remains open is an intrinsic dagger/C^*-analytic Sweedler construction producing general correspondence coalgebras and accounting for CP behaviour, completion and intertwiners.

The link with profunctors can now be stated more strongly. Bimodules are **Vect**-enriched profunctors between one-object enriched categories (with the appropriate opposite conventions), and their coend composition *is* tensor product over the intermediate algebra. The previously established behaviour construction $\mathbf{Run}(\mathcal{D}) \to \mathbf{CofProf}$ therefore belongs to the same pattern. The interior tensor product is its analytic Hilbert-module counterpart. A unified enriched or double-categorical comparison beyond the algebraic case is still to be constructed.

The remaining Morita question is particularly sharp: for a finite matrix target, every correspondence has an abstract free standard realization, but not a preferred presentation. For general coefficient C^*-algebras, can non-free Hilbert modules be absorbed by changing the state/interface, or do they constitute new forms of implementation? This is the useful analytic frontier.

14 Why retain the classical outcome as a world?

The preceding calculations are made in a single linear fibre: both the outcome algebra $\mathbb{C}^W$ and the ancillary system Y are encoded inside $\mathbf{Vect}_{\mathbb{C}}$. The ambient tangent ∞-topos considered by Sati–Schreiber invites a different organization. One can put classical control data in a base of worlds and linear quantum data in fibres. This does not invalidate the writer calculation: it explains which information its one-fibre presentation has compressed.

There are two separate ingredients in a quantum measurement. An *outcome protocol* supplies a finite set W of possible answers and a continuation that may depend on $w \in W$. A *quantum instrument* specifies the probabilities and conditional state updates. An ordinary dependent polynomial command with a continuation in each outcome world describes the first ingredient, but does not alone provide the second. In particular, a polynomial continuation position does not itself sample an outcome according to the Born rule.

For finite-dimensional Hilbert spaces H and $(K_w)_{w \in W}$, an instrument in the Schrödinger picture is a family of completely positive, trace-nonincreasing maps

$$\mathcal{I}_w : \mathcal{T}(H) \longrightarrow \mathcal{T}(K_w)$$

whose sum is trace-preserving:

$$\sum_w \operatorname{tr} \mathcal{I}_w(\rho) = \operatorname{tr}(\rho), \qquad \rho \geq 0.$$

Equivalently it is one CPTP map to the finite classical–quantum sum $\bigoplus_w \mathcal{T}(K_w)$. In a Kraus presentation,

$$\mathcal{I}_w(\rho) = \sum_\alpha K_{w\alpha} \rho K_{w\alpha}^\dagger, \qquad \sum_{w,\alpha} K_{w\alpha}^\dagger K_{w\alpha} = 1_H, \quad K_{w\alpha} : H \to K_w. \tag{1}$$

The outcome probability is $p_w = \operatorname{tr} \mathcal{I}_w(\rho)$; when $p_w > 0$ the conditional state is $\rho_w = \mathcal{I}_w(\rho)/p_w$. The unnormalized block $\mathcal{I}_w(\rho)$, rather than the normalized state, is the compositional datum.

Two different claims. A dependent runner can type which continuations and world transitions are allowed. To obtain a physical instrument one must additionally specify CP branch maps satisfying (1), or construct such maps from a suitable dagger/analytic refinement. No equivalence between unrestricted dependent Sweedler coalgebras and quantum instruments is asserted here.

14.1 The two-outcome qubit calculation

Let $W = \{0, 1\}$ and $H = K_0 = K_1 = \mathbb{C}^2$. A computational-basis measurement has the Lüders instrument

$$P_0 = |0\rangle\langle 0|, \qquad P_1 = |1\rangle\langle 1|, \qquad \mathcal{I}_w(\rho) = P_w \rho P_w.$$

For $\rho = \left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right)$ with $a + d = 1$, the instrument gives

$$\mathcal{I}_0(\rho) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{I}_1(\rho) = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}.$$

It remembers a and d as outcome probabilities and retains the corresponding output state in the appropriate outcome fibre. By contrast, the associated POVM remembers only the effects $F_w = \mathcal{I}_w^*(1_{K_w}) = P_w$. Different instruments can have the same POVM: postcompose each branch with an arbitrary CPTP channel Λ_w on K_w. The effects remain F_w because $\Lambda_w^*(1) = 1$, but the output states can change. Consequently the POVM classification earlier in this appendix must not be silently upgraded to an instrument classification.

There is a canonical CP preparation of the classical record together with the unnormalized conditional quantum outputs:

$$\rho \longmapsto (P_0 \rho P_0,\ P_1 \rho P_1).$$

As with the previous Naimark construction, a dilation may implement this channel, but the dilation and its choice of postmeasurement system are extra data beyond the POVM itself.

15 World-typed classical feedforward

The programming-language value of retaining outcomes is immediate. Let a command at a premeasurement world p have continuations in the worlds $w \in W$. A subsequent program in world w may apply a channel $\Lambda_w : \mathcal{T}(K_w) \to \mathcal{T}(L_w)$. Its compound instrument has branches

$$\mathcal{J}_w = \Lambda_w \circ \mathcal{I}_w. \tag{2}$$

If Λ_w itself produces a classical outcome v, with branch maps $\Lambda_{v|w}$, sequential composition is the finer family

$$\mathcal{J}_{w,v} = \Lambda_{v|w} \circ \mathcal{I}_w.$$

One may subsequently forget w by summing the relevant CP blocks; this is a separate classical-data operation, not a physical selection rule supplied by a bare dependent sum.

For the qubit example take $U_0 = 1$, $U_1 = X$, with X the Pauli bit flip, and put $\Lambda_w(\sigma) = U_w \sigma U_w^\dagger$. Then

$$\mathcal{J}_0(\rho) = a|0\rangle\langle 0|, \qquad \mathcal{J}_1(\rho) = d|0\rangle\langle 0|, \qquad \sum_w \mathcal{J}_w(\rho) = |0\rangle\langle 0|$$

for every density matrix ρ as above. The controller has used a classical measurement result to choose a quantum gate. If the record is retained, the two output blocks remain distinguishable even though their normalized quantum states coincide.

This example separates three operations: world-typed branching, branchwise quantum evolution, and forgetting a classical record. It is a useful operational test for a dependent measuring construction: its composition laws should recover (2), not merely the multiplication of outcome algebras within one linear fibre.

16 Many-object correspondences and spans of worlds

The existing writer theorem says that an A-in-B runner amounts to a left A-action on a freely presented right B-module. Over finite classical world sets we can calculate an unambiguous many-object analogue *at the level of correspondences*. Let

$$A = \bigoplus_{w \in W} A_w, \qquad B = \bigoplus_{v \in V} B_v,$$

where the summands are unital C^*-algebras and both sets are finite. An A–B correspondence E splits into components

$$E_{wv} = \pi_E(e_w) E f_v, \qquad E \cong \bigoplus_{w,v} E_{wv},$$

where e_w and f_v denote the canonical central projections and Ef_v denotes the right-module summand. The component E_{wv} is an A_w–B_v correspondence (possibly the zero correspondence). This decomposition uses central projections in the *coefficient algebras*, not a canonical splitting of an arbitrary Hilbert space unrelated to them.

Suppose also $C = \bigoplus_{u \in U} C_u$ and F is a B–C correspondence. There is a canonical componentwise unitary isomorphism

$$(E \otimes_B F)_{wu} \;\cong\; \bigoplus_{v \in V} E_{wv} \otimes_{B_v} F_{vu}. \tag{3}$$

To see this, decompose an elementary balanced tensor by the orthogonal central idempotents f_v. Mixed terms with different v vanish under balancing over B, and the surviving terms complete to the finite Hilbert direct sum on the right. Thus correspondence composition is matrix multiplication, with $\oplus$ as addition and interior tensor product as multiplication. This statement is an established fact about finite sums of C^*-algebras; it is *not* yet a new theorem about dependent Sweedler coalgebras.

A span $W \xleftarrow{s} D \xrightarrow{t} V$ gives one possible syntax for the allowed classical transitions; a family of correspondences E_d can decorate its arrows. Different elements d may have the same endpoints, so a span remembers multiplicity that an undecorated binary relation forgets. The direct sum of

those decorations yields a matrix entry E_{wv}, at least when the stated finite-direct-sum hypotheses apply. This connects the quantum example to our earlier span-of-worlds runner problem, but does not prove that *all* such decorated spans arise from a particular dependent measuring comonad.

Proposed many-object Sweedler theorem (open). Choose a dependent writer-like theory over each finite world set and a precise notion of world-changing runner, including its strength and base-change laws. Determine whether its coalgebras are equivalent to an appropriate class of correspondence matrices, and whether composition of runners is carried to (3). One must state which matrices admit the requisite free presentations. The one-world algebraic writer theorem established above is the essential consistency test, not a proof of this generalization.

The correspondence matrix is not by itself a channel or instrument: a pointed correspondence determines a CP map in the *Heisenberg* direction $A \to B$, while an instrument's unnormalized state updates are in the Schrödinger direction. One must use the appropriate adjoint and specify normalized points or other Stinespring data before identifying physical processes. In particular, the existence of an abstract bimodule matrix does not impose a trace-preservation law.

17 Beyond discrete worlds: the tangent ∞-topos

Finite sets already explain world-typed classical control. What is specifically gained by the tangent ∞-topos is the possibility of a base with nontrivial paths and higher coherences. For example, a local system of complex vector spaces over BG, for a discrete group G, is equivalently a complex representation of G; with a compatible Hermitian metric it is a unitary representation. Loop transport then acts on fibres by the representation operators. This is a model for geometric or topological gates, not yet an account of measurement or of complete positivity.

An extension of (3) from finite discrete bases to ∞-groupoids would require a homotopy-coherent category of correspondences over the base, together with appropriate pullback, pushforward and relative tensor products. The comparison between base change and runner composition must be checked rather than inferred from the discrete case. Likewise, although finite dependent sums and products can be ambidextrous in suitable linear contexts, the Frobenius comonad structure on a quantum monad should not be identified automatically with the *Sweedler measuring* comonad. Relating these two structures is another separate question.

Finally, the tangent picture does not manufacture a dagger, positivity, or C^*-completion. Such extra structure is necessary for the instrument semantics used above. The analytic open problem of the previous section and the dependent/homotopical generalization are thus complementary, rather than alternative solutions.

Research sequence. (i) Specify the dependent command and calculate its world-changing runner laws; (ii) characterize the CP/trace-preserving refinement and verify the two-outcome calculation and feedforward composition; (iii) extend the free-bimodule theorem to world-indexed runner matrices with the correct base-change laws; (iv) only then formulate the homotopy-coherent version over non-discrete bases. The explicit instrument and the correspondence matrix calculation above are checks for these proposed constructions, not substitutes for them.

18 Conclusion

The QS writer theories yield a concrete and unexpectedly general theorem. An environment command interpreted with hidden state is a left algebra action on a freely presented right module. The familiar projection family on $Y \otimes H$ is one example. The universal coefficient algebra $U_{W,H}$ controls *all* such algebraic implementations, rather than only their rational finite-dual part. Dagger structure turns the idempotents into a PVM; initialization by one ancilla vector and hiding then recover every finite POVM.

Thus the relation between runners and correspondences is no longer a similarity. *At the algebraic level it is the free-bimodule theorem, and composition agrees exactly with balanced tensor product.* Hilbert correspondence theory adds analytic completion and the possibility of non-free modules; Paschke–GNS adds the pointed realization of CP maps and cyclic minimization. These extensions preserve the implementation-versus-extensional-behaviour distinction that motivates the tutorial.

The question left for the general programme is to construct an analytic Sweedler hom *whose coalgebras are suitable correspondences*, extending the free writer case without confusing the measuring object with its implementation category. This would connect the QS example, the enriched arities programme and the profunctorial behaviour semantics of earlier Parts in one architecture. The dependent extension adds a second, independent challenge: explaining classical branching and quantum state update simultaneously, and deriving the composition of correspondence matrices from actual world-changing runner laws. The qubit instrument and feedforward calculations give concrete tests for that proposal.

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