Getting to Grips with Sweedler Runners for Systems

Appendix D: Higher analytic runners: from symmetric sequences to ∞-operads

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Aim. The basic construction of this tutorial has worked primarily with

$$\mathbf{Run}([\mathbf{Set}, \mathbf{Set}]_{\mathrm{acc}}),$$

where monoids for composition are monads, interpreted as theories, and loose arrows are stateful implementations between them. Several later examples suggest that this Set-based setting is only the 0-truncated part of a more homotopical picture. States may have symmetries and automorphisms; operations may form spaces rather than sets; and equations between operations may themselves carry coherent higher identifications. The natural higher replacement of **Set** is therefore the ∞-category $\mathcal{S}$ of ∞-groupoids or *anima*.

Rather than beginning immediately with the very large category

$$[\mathcal{S}, \mathcal{S}]_{\mathrm{acc}},$$

we concentrate on its analytic sector. This is particularly attractive because analytic monads on anima are equivalent to ∞-operads, and may also be presented by dendroidal Segal spaces. Thus a higher version of the runner construction in the analytic setting would have a concrete interpretation: its objects would be homotopy-coherent operadic theories, and its loose arrows should describe stateful, homotopy-coherent implementations between them.

The purpose of this appendix is to identify the categorical structure needed for such a construction. We begin with anima-valued symmetric sequences

$$\mathrm{SSeq}_\infty = \mathrm{Fun}(\mathrm{Fin}^{\simeq}, \mathcal{S}).$$

They carry the substitution product $\circ$, whose associative algebra objects are ∞-operads, and an arithmetic product $\boxtimes$, obtained by Day convolution from cartesian product of finite sets. We show that these fit together as the higher analogue of the normal duoidal structure on ordinary symmetric sequences:

$$(\mathrm{SSeq}_\infty, \boxtimes, \mathbb{X}, \circ, \mathbb{X}).$$

This is the structure which, in the ordinary case, supports universal measuring cooperads and the enrichment of operads in cooperads.

The remaining step is then sharply defined. One would like to construct, for ∞-operads $\mathcal{O}$ and $\mathcal{P}$, a universal measuring ∞-cooperad

$$\{\mathcal{O}, \mathcal{P}\}_\infty,$$

and thereby obtain a higher Sweedler theory of operadic implementations. Péroux's theory of universal measuring coalgebras in presentably symmetric monoidal ∞-categories provides a closely related higher-categorical result; what is required here is a duoidal extension in which the algebra structure is defined using substitution while the measuring interaction is supplied by the arithmetic product.

The larger systems-theoretic aim is therefore

> analytic monads and ∞-operads as higher theories,
> measuring ∞-cooperads as anima of implementations,
> and their coalgebras as homotopy-coherent stateful runners.

If this final measuring construction can be established, the analytic sector would provide a first concrete higher-groupoid version of the runner programme, connecting categorical systems theory directly with ∞-operads, dendroidal geometry and higher Lawvere theories.

1 A concrete motivation: symmetry and concurrency

One source of motivation for the passage from sets to anima[1] in the treatment of systems comes from physics with the development of a higher gauge theory ([13]). But a reason to take this step already appears in ordinary models of concurrent systems when symmetries of states and processes are retained rather than quotiented away, as we saw in Part VIII of the tutorial.

A simple example is furnished by Petri nets. If a marking contains two tokens of the same type A, collective-token semantics records only the multiplicity

$$2A.$$

At a more fine-grained level we may distinguish two occurrences a_1 and a_2, while recognizing that exchanging them does not change the physical marking. The state therefore carries an automorphism

$$a_1 \longleftrightarrow a_2.$$

It is naturally represented not merely by a point of a set but by an object of a groupoid with isotropy Σ_2.

This distinction is systematic in Kock's whole-grain treatment of Petri nets. States and processes retain their symmetries, and the process semantics forms a Segal space whose algebraic counterpart is a free coloured prop in groupoids. Collective-token semantics is recovered only after passing to connected components of the relevant groupoids. Thus moving from sets to groupoids does not add exotic structure to an otherwise elementary systems model; it prevents symmetry information already present in the model from being discarded.

Analytic functors provide a closely related mechanism. Classically an analytic functor has the form

$$F(X) = \coprod_{n\geq 0} F[n] \times_{\Sigma_n} X^n.$$

The orbit quotient identifies configurations which differ by a permutation of their n inputs. Over anima this becomes the homotopy quotient

$$F(X) \simeq \coprod_{n\geq 0} F[n] \times_{h\Sigma_n} X^n.$$

The symmetry is no longer forgotten after quotienting: it remains as homotopy data. An operation may therefore have interchangeable ports, nontrivial automorphisms, and coherent equivalences between different presentations.

[1] This term for ∞-groupoids is due to Peter Scholze. It is common to see people write *spaces* as well, but this risks confusing entities with cohesive geometric extent with those presenting homotopical identity structure.

For systems theory this matters whenever a component is assembled from repeated or indistinguishable resources. An abstract operation

$$A + A + B \longrightarrow C$$

has a symmetry exchanging its two A-inputs. A concrete implementation may route these resources through a process network in several equivalent ways. A Set-valued semantics records only the resulting equivalence class; a groupoid- or anima-valued semantics can retain the symmetry and require an implementation to respect it coherently.

Petri nets also show why the analytic setting should be regarded as a first step rather than a final graphical semantics. An operad describes many-input/one-output composition and hence tree-shaped processes, whereas a general Petri-net transition may have both many inputs and many outputs. Whole-grain Petri-net semantics consequently leads naturally to coloured PROPs in groupoids. The expected hierarchy is therefore

analytic functors and ∞-operads
tree-shaped operations with symmetry

$\downarrow$

graphical species, properads and PROPs
many-input/many-output processes

$\downarrow$

Petri-net and open-process semantics.

The analytic case is nevertheless the natural place to begin: it already contains nontrivial symmetry and higher coherence, while retaining the well-developed theory of analytic monads and ∞-operads.

There is a complementary higher phenomenon in models of true concurrency, cf Part VII. Two independent transitions a and b give interleavings ab and ba, but a higher-dimensional automaton records them as boundary paths of a common square. Cubical semantics therefore retains independence which an ordinary transition graph forgets, just as groupoid semantics retains symmetry which a set quotient forgets. These are different forms of higher structure, but they point in the same direction: categorical systems theory should avoid collapsing process geometry to equality before composition and implementation have been analysed.

At the opposite end of the spectrum, the same principle underlies the use of higher toposes in gauge theory. There the morphisms between configurations are gauge transformations, with higher gauge transformations between them. The move from sets to anima is therefore not tied to one physical application. From interchangeable tokens in a Petri net to gauge-equivalent fields, the common idea is to retain symmetries and equivalences as part of the state or process itself.

2 The boundary: Moore machines are not analytic monads

Moore machines remain an illuminating example of what the higher setting should eventually accommodate, but they are *not* a test case for analytic measuring. A Moore machine has transitions and observations

$$\delta : A \times S \to S, \qquad o : S \to O,$$

and is a comodel of the Moore theory in the cartesian Sweedler setting. Its continuation can be selected, duplicated, or discarded. That capability is absent from resource-exact operadic substitution.

The underlying endofunctor of the free Moore monad has a description by finite-leaf normal-form trees, but analyticity of an endofunctor does not imply analyticity of its monad structure. For finite O with $|O| \geq 2$, the reader multiplication contains

$$\mu_X : X^{O\times O} \longrightarrow X^O, \qquad (f_{o,o'}) \longmapsto (f_{o,o}).$$

Its naturality square along $X \to 1$ is not cartesian: for $|O| = |X| = 2$ the source has 16 elements while the required pullback has 4. In syntactic terms,

$$\mathrm{get}(\lambda o.\, \mathrm{get}(\lambda o'.x_{o,o'})) = \mathrm{get}(\lambda o.x_{o,o})$$

discards the off-diagonal variables. This violates the regular-linear equations characterising analytic monads in the sense of Szawiel–Zawadowski [2].

The step-only fragment $A^* \times (-)$, by contrast, is analytic: its operations are unary, and composition concatenates words. Yet even this fragment will not recover its ordinary comodels through measuring into the unit operad, as we see below. Thus the obstruction is not merely the reader equations: it is also the mismatch between operadic measuring and the cartesian store construction.

There remains a useful higher Moore story, with A, S, O groupoids or anima and transitions $A \times S \to S$, observations $S \to O$. But this belongs to the full accessible endofunctor setting, or to the actegorical viewpoint later in the appendix. Intrinsically groupoidal input positions present a further distinction between analytic and general polynomial functors; they are not the first obstruction in the Moore example.

3 Mathematical reasons not to stop at groupoids

A first enlargement of the Set-based theory would be

$$[\mathbf{Grpd}, \mathbf{Grpd}]_{\mathrm{acc}}.$$

This already records symmetries which are invisible at the level of sets. A state may have nontrivial automorphisms, while a polynomial operation may have a groupoid rather than a set of shapes. Groupoid-valued polynomial functors are therefore a natural home for species, quotient containers and other structures with symmetry.

For a self-contained higher theory, however, it is more natural to pass all the way to anima:

$$[\mathcal{S}, \mathcal{S}]_{\mathrm{acc}}.$$

The reason is that homotopy-coherent constructions do not stabilize at the 1-groupoid level. Passing from groupoids to anima closes the hierarchy of higher equivalences and allows polynomial and analytic constructions to be performed intrinsically.

Within the very large category of accessible endofunctors lies the more rigid category of analytic functors. Gepner, Haugseng and Kock show that analytic functors on anima are polynomial in the higher-categorical sense, and that analytic monads are equivalent to ∞-operads, or equivalently to dendroidal Segal spaces [7]. Thus

analytic monad = homotopy-coherent operadic theory.

4 Anima-valued symmetric sequences

The Moore-machine example shows what is gained by allowing commands, states and observations to carry intrinsic symmetry, but it still describes one particular theory. We now want a setting in which whole families of such higher operations can be organized uniformly, composed, and eventually equipped with Sweedler measuring objects. The analytic sector is the first setting in which this can be done while remaining close to familiar systems examples.

The key observation is that an analytic theory is built from operations of finite arity whose anima of operations may themselves carry nontrivial symmetry and higher homotopy. Thus the basic datum is no longer a set of n-ary operations, but an anima of n-ary operations, coherently acted on by permutations of the n input slots. Symmetric sequences package precisely this information.

Let

$$\mathbb{B} = \mathrm{Fin}^{\simeq}$$

be the ∞-groupoid of finite sets and bijections, and define

$$\mathrm{SSeq}_{\infty} = \mathrm{Fun}(\mathbb{B}, \mathcal{S}).$$

An object F assigns to each finite set I an anima $F(I)$ of I-ary operations, with coherent transport along bijections. In particular, $F[n]$ is no longer merely a set carrying a Σ_n-action: it is an anima with a homotopy-coherent Σ_n-action.

The relevance to the Moore example is immediate when the output object O is finite discrete. The normal-form trees for $T_{\mathrm{Moore}}(A, O)$ have finite sets of leaves, while symmetries may occur in the anima of operation shapes, for instance through groupoidal structure in A. Such theories therefore sit naturally inside the analytic world described by symmetric sequences. The purpose of passing to SSeq_{∞} is to study this entire analytic sector at once rather than one polynomial monad at a time.

Two monoidal structures on symmetric sequences will then play the roles already familiar from Sweedler theory. Substitution composes operations by inserting operations into input slots and has ∞-operads as its monoids. A second, arithmetic product combines operation shapes in the manner needed for measuring and enrichment. The central question is whether these two products form the higher analogue of the ordinary duoidal structure from which measuring cooperads arise.

There are two monoidal structures relevant to the runner programme. The first is substitution,

$$F \circ G,$$

whose associative algebras are ∞-operads. Haugseng constructs this product using Day convolution for double ∞-categories and shows that the resulting associative algebra objects are precisely ∞-operads [8].

The second is the arithmetic product. Cartesian product of finite sets gives

$$\times : \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}.$$

Day convolution along this functor defines

$$F \boxtimes G = \mathrm{Lan}_{\times}(F \times G),$$

or explicitly

$$(F \boxtimes G)(I) \simeq \operatorname*{colim}_{A \times B \simeq I} F(A) \times G(B).$$

Its unit is the singleton species

$$\mathbb{X} = y[1].$$

This is the anima-valued lift of the arithmetic product of ordinary symmetric sequences.

5 The proposed higher duoidal structure: a coherence problem

Arithmetic Day convolution $\boxtimes$ is defined using cartesian product of finite sets; substitution $\circ$ is governed instead by partitions, equivalently maps $A \to U$ with possibly empty fibres (nullary operations matter). Rectangular refinement suggests an interchange

$$(G_1 \circ F_1) \boxtimes (G_2 \circ F_2) \longrightarrow (G_1 \boxtimes G_2) \circ (F_1 \boxtimes F_2).$$

This is a candidate, not yet a proved normal duoidal structure on anima-valued symmetric sequences. Substitution is not ordinary Day convolution along a monoidal product on $\mathrm{Fin}^{\simeq}$. The necessary interchange and all higher coherences must be constructed using the double ∞-categorical machinery of Haugseng and checked in a framework such as Torii's duoidal ∞-categories [3, 4].

There is a particularly instructive arity-zero warning. The product functor on finite-set groupoids loses automorphisms of $(\varnothing, B)$; consequently $\boxtimes$ at arity zero can involve homotopy orbits $G(n)_{h\Sigma_n}$, and even classifying spaces $B\Sigma_n$. It is not obtained by simply lifting the set-level quotient formula to discrete spaces. Unit maps, normality, and interchange must all be checked at arity zero.

Structural programme (not a theorem). Construct the rectangular interchange within Haugseng's double ∞-categorical Day convolution and establish the normal duoidal axioms in Torii's sense, including nullary operations and arity-zero homotopy orbits. Only then use the resulting structure as input to a measuring construction.

6 Analytic monads and ∞-operads

The significance of the previous result is that the monoids for the substitution structure are already familiar objects. In the one-colour setting relevant here, associative algebra objects for substitution give the corresponding one-coloured ∞-operads; the varying-colour theory is naturally formulated in Haugseng's double ∞-category of symmetric collections. Schematically,

$$\mathrm{Alg}_{\circ}(\mathrm{SSeq}_{\infty}) \simeq \infty\text{-}\mathbf{Operad}_{1\mathrm{col}}.$$

With the colour data treated in the appropriate varying-object setting, Gepner–Haugseng–Kock identify the same analytic structures as analytic monads. Thus, schematically,

$$\infty\text{-}\mathbf{Operad} \simeq \mathbf{AnMnd},$$

where the notation suppresses the corresponding colour or pinning data. Moreover these analytic monads admit a dendroidal nerve:

$$\mathbf{AnMnd} \simeq \mathcal{P}_{\mathrm{Seg}}(\Omega).$$

Thus the same theory may be read in three ways:

analytic monad	$\longleftrightarrow$	∞-operad
	$\longleftrightarrow$	dendroidal Segal space.

Haugseng gives a sharper fourth description: the Lawvere theories of analytic monads are precisely pinned ∞-operads, using a variant of the operadic definition based on spans of finite sets rather than finite pointed sets [9]. This is especially relevant for the present tutorial, where monads have throughout been interpreted simultaneously as computational effects and as theories.

7 Measuring into the unit: why ordinary comodels disappear

In ordinary symmetric sequences, the López Franco–Vasilakopoulou measuring construction uses arithmetic product to define a universal measuring cooperad. Its behaviour at the unit operad $\mathbb{X}$ is decisive for the proposed systems interpretation.

Unit-target obstruction (ordinary species; conditional higher analogue). Suppose a measuring is represented by a map $C \boxtimes \mathcal{O} \to \mathbb{X}$, where $\mathbb{X}$ is supported only in arity one. If $\mathcal{O}$ has an operation of arity $m \neq 1$, the arity-support condition forces $C(1)$ to be empty, because $\mathcal{O}(1)$ contains the identity. It likewise forces $C(n)$ empty for $n \neq 1$. Thus no nonempty measuring object of this form can detect an ordinary comodel. For a unary operad, the only possible support is arity one; in the standard cooperadic measuring construction the resulting measuring object is the unit cooperad.

Indeed, a summand indexed by $A \times B \cong K$ in $(C \boxtimes \mathcal{O})(K)$ cannot map to the empty space $\mathbb{X}(K)$ when $|K| \neq 1$. Nonempty arities therefore give the support obstruction immediately. This calculation should be repeated carefully in the chosen higher category, including existence and conventions for the initial cooperad; the argument does not by itself establish the universal higher measuring object.

Even for the unary step operad $A^* \cdot \mathbb{X}$, measuring into $\mathbb{X}$ does not classify A^*-actions on a state space. In the full endofunctor setting, the runner uses the store comonoid $S \times (-)^S$. Its continuation-copying map changes arity from $|S|$ to $|S|^2$ and is not a map of species. This is the precise boundary between cartesian comodels and resource-exact operadic implementations.

8 A first resource-exact runner: deterministic tree transducers

A useful replacement for the Moore test is a state-threading translation of operadic operations. For discrete operads $\mathcal{O}, \mathcal{P}$ and a state set S, consider equivariant maps

$$\rho_n : S \times \mathcal{O}(n) \longrightarrow \mathcal{P}(n) \times S^n.$$

On receiving an n-ary source operation in state s, the transducer emits one n-ary target operation and assigns a successor state to each of its n input branches. Equivariance permutes the branches and successor states together. Compatibility with operadic units and substitution says that translating a composite tree agrees with translating its nodes recursively.

Equivalently, in the discrete one-colour setting, this has the shape of a distributive translation

$$(S \cdot \mathbb{X}) \boxtimes \mathcal{O} \longrightarrow \mathcal{P} \circ (S \cdot \mathbb{X}),$$

with the requisite coherence laws. This is a *one-node-output* deterministic top-down tree transducer, not a classification of all linear runners. Allowing target trees, colours, or homotopy-coherent state spaces requires further definitions and proofs.

For $\mathcal{P} = \mathbb{X}$, non-unary source operations have no one-node output of matching arity, consistently with the measuring obstruction. The important operational feature is that every input branch is passed along exactly once. This is the same resource discipline that makes whole-grain token accounts of Petri-net firing attractive.

9 Higher measuring: questions rather than an existence claim

Péroux's measuring theorem [12] provides a valuable model, but it cannot simply be applied to substitution. Its hypotheses include colimit preservation in each tensor variable; substitution preserves sifted colimits in its right argument but not arbitrary colimits. Accessibility of substitution-comonoids, their identification with an appropriate notion of higher cooperad, and the distinction between homotopy orbits and fixed points require separate work.

The immediate programme is therefore: construct and verify the duoidal interchange; establish an appropriate category of cooperads and universal measuring objects; determine whether a classifying object for resource-exact stateful transducers exists, perhaps using coloured operads; and compare the result at Set level with the measuring comonoids in $[\mathbf{Set}, \mathbf{Set}]_{\mathrm{acc}}$. No inclusion of analytic runner double categories into the full Sweedler runner double category is claimed: tight maps and loose maps require separate comparisons.

10 Potential significance for categorical systems theory

This construction would enlarge the systems programme in several directions at once.

First, symmetries cease to be quotiented away. An implementation may have automorphisms, and the theory remembers them as genuine homotopy data.

Second, equations become coherent rather than strict. Theories can contain paths between operations and higher homotopies between those paths. A runner must respect those identifications coherently.

Third, multi-input composition is made explicit at the level of operation spaces. General monads may of course encode operations of arbitrary arity, but operadic presentation exposes their arities, permutation symmetries and tree-shaped substitution directly.

Fourth, the tree geometry of composition becomes part of the theory itself. This brings the runner programme into direct contact with dendroidal methods and gives a natural operadic sector of the graphical constructions considered elsewhere in the tutorial.

Fifth, analytic monads provide a direct bridge between the systems-theoretic and logical readings of a theory:

analytic monads $\longleftrightarrow$ ∞-operads $\longleftrightarrow$ dendroidal Segal models, with a corresponding higher Lawvere-theory presentation.

A higher runner may therefore be viewed equally as a stateful implementation between homotopy-coherent effect theories, between ∞-operads, or between higher Lawvere theories.

Finally, this suggests that the broad construction

$$\mathbf{Run}([\mathcal{S}, \mathcal{S}]_{\mathrm{acc}})$$

should contain a distinguished analytic sector

$$\mathbf{Run}_{\mathrm{an}}$$

whose objects are ∞-operads and whose loose arrows are homotopy-coherent stateful implementations. This analytic sector would stand between the elementary Set-based runner theory and the more general graphical and homotopical systems theories required for higher-dimensional automata, Petri nets and wheeled structures.

The proposed hierarchy is therefore

$$\begin{array}{c} [\mathbf{Set},\mathbf{Set}]_{\mathrm{acc}} \quad \text{ordinary theories and runners} \\ \downarrow \\ [\mathcal{S},\mathcal{S}]_{\mathrm{acc}} \quad \text{higher theories and runners} \\ \supset \\ \mathbf{Analytic}(\mathcal{S}) \\ \text{operadic higher theories} \\ \downarrow \ \mathrm{Mnd} \\ \infty\text{-}\mathbf{Operad}. \end{array}$$

The next step is no longer to guess what the relevant higher theories should be. Their identity is established. The problem is to construct the universal measuring ∞-cooperads which turn their maps of implementation into a genuine higher Sweedler theory.

There is one further direction which should be kept distinct from the passage to anima. Part X varied the geometric nature of the state space, whereas the present appendix varies its intrinsic homotopy level. To combine these two directions one should eventually work not merely in $\mathcal{S}$ but in a cohesive ∞-topos $\mathbf{H}$:

$$\boxed{\mathrm{Fun}(\mathrm{Fin}^{\simeq},\mathbf{H}).}$$

This would allow operation objects to possess both cohesive geometric variation and intrinsic higher symmetries. The distinction is important: moving from sets to anima adds higher identity structure, whereas passing to a cohesive ∞-topos adds geometry.

11 From groupoidal states to actegorical automata

The preceding discussion replaced a set of states by a groupoid, or more generally by a space. There is another nearby categorification which is worth separating from this homotopical one. For an ordinary Moore machine with input alphabet A, the transition structure is equivalently an action

$$A^* \curvearrowright S$$

of the free monoid A^* on the state set S. Rather than requiring the morphisms between states to be invertible, we may instead replace S by an arbitrary category $\mathcal{S}$. The transition structure then becomes an action of A^* by endofunctors of $\mathcal{S}$.

Recall that if $(\mathcal{A},\otimes,I)$ is a monoidal category, an $\mathcal{A}$-actegory consists of a category $\mathcal{X}$ together with a functor

$$\odot : \mathcal{A}\times\mathcal{X} \longrightarrow \mathcal{X}$$

and coherent natural isomorphisms

$$(a \otimes b) \odot x \cong a \odot (b \odot x), \qquad I \odot x \cong x,$$

satisfying the usual pentagon and triangle coherence conditions. Equivalently, an $\mathcal{A}$-actegory may be regarded as a strong monoidal functor

$$\mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}],$$

where the endofunctor category is monoidal under composition. In the strict case the displayed coherence isomorphisms are identities.

Now let $\mathbf{A}^*$ denote the discrete strict monoidal category whose objects are words in A^*, with tensor product given by concatenation. Taking $\mathcal{A} = \mathbf{A}^*$, an $\mathbf{A}^*$-actegory is therefore a category $\mathcal{S}$ equipped with an action of words by endofunctors. Thus each letter $a \in A$ determines an endofunctor

$$F_a : \mathcal{S} \longrightarrow \mathcal{S},$$

and a word $w = a_1 \cdots a_n$ acts, up to the chosen coherences, by the corresponding composite F_w. Ordinary A^*-sets occur when $\mathcal{S}$ is discrete, while the groupoidal examples considered above occur when every morphism of $\mathcal{S}$ is invertible.

There is a useful monadic way to express the same point. Since $\mathbf{A}^*$ is a strict monoidal category, multiplication by it defines a 2-monad

$$T_A = \mathbf{A}^* \times - : \mathbf{Cat} \longrightarrow \mathbf{Cat}.$$

The multiplication of the 2-monad is induced by concatenation,

$$\mathbf{A}^* \times \mathbf{A}^* \times \mathcal{S} \longrightarrow \mathbf{A}^* \times \mathcal{S},$$

and its unit by the empty word. Strict algebras are strict actions, while pseudoalgebras give the usual coherent actegories. Thus the categorification is not merely the replacement

$$S \in \mathbf{Set} \qquad \rightsquigarrow \qquad \mathcal{S} \in \mathbf{Cat};$$

it is also the replacement of the ordinary action monad

$$A^* \times - : \mathbf{Set} \longrightarrow \mathbf{Set}$$

by the 2-monad

$$\mathbf{A}^* \times - : \mathbf{Cat} \longrightarrow \mathbf{Cat}.$$

For a Moore machine we should also retain the observation map. With an output category $\mathcal{O}$, the elementary categorified datum is therefore

$$\boxed{\mathbf{A}^* \curvearrowright \mathcal{S}, \qquad o : \mathcal{S} \longrightarrow \mathcal{O}.}$$

If $\mathcal{O}$ is discrete, a morphism $s \to s'$ can only exist between states having the same present output. Allowing a non-discrete output category permits morphisms of states to induce non-trivial comparison morphisms between observations. The latter is likely to be the more flexible notion in a genuinely categorical theory.

This points towards a second higher analogue of the Sweedler setting. In the set-based theory the natural ambient duoidal category is

$$\mathcal{D}_{\mathbf{Set}} = [\mathbf{Set}, \mathbf{Set}]_{\mathrm{acc}},$$

with composition as one monoidal product and the Day-type convolution induced by the cartesian structure of **Set** as the other. Monoids for composition are accessible monads. Replacing **Set** by **Cat** suggests correspondingly

$$\mathcal{D}_{\mathbf{Cat}} = [\mathbf{Cat}, \mathbf{Cat}]_{\mathrm{acc}},$$

or, more accurately, the 2-category

$$\mathbf{End}_{\mathrm{acc}}(\mathbf{Cat})$$

of accessible 2-endofunctors, transformations and modifications. Composition again supplies one monoidal structure, while the cartesian monoidal structure on **Cat** should supply the corresponding convolutional one. At the underlying 1-categorical level this is the familiar accessible-endofunctor duoidal construction; retaining the full 2-dimensional structure suggests instead a duoidal 2-category or bicategorical analogue.

The distinction matters. If we pass immediately to the ordinary category

$$[\mathbf{Cat}, \mathbf{Cat}]_{\mathrm{acc}},$$

then monoids for composition are ordinary monads on the underlying category **Cat**. This forgets the modifications and coherence data which distinguish strict 2-algebras from pseudoalgebras. For actegories, the more natural ambient structure is therefore genuinely 2-categorical: the input theory

$$T_A = \mathbf{A}^* \times -$$

is a 2-monad, and its pseudoalgebras are the desired categorical state spaces.

This should be regarded at present as a programme rather than as an already established extension of the Sweedler construction. The first technical problem is to establish the appropriate genuinely 2-categorical duoidal structure on a suitable 2-category of accessible endo-2-functors of **Cat**. Composition should provide one monoidal product, so that its monoids or pseudomonoids are 2-monads or pseudomonads, while the cartesian monoidal structure on **Cat** should induce the corresponding convolutional product. Crucially, these two products must be related by an appropriate coherent interchange law. Only once this structure is in place can one ask for 2-categorical analogues of the Sweedler copower, convolution hom, and measuring comonoid, and hence for categorified runners. The elementary Moore example then gives a sharp first test of what such a theory should recover.

In particular, a satisfactory 2-categorical Sweedler theory should produce, from the categorified Moore theory, coalgebras whose underlying data are

$$\mathbf{A}^* \curvearrowright \mathcal{S}, \qquad o : \mathcal{S} \longrightarrow \mathcal{O},$$

with the expected coherent morphisms between them. Restriction to groupoids should recover the groupoidal Moore machines discussed earlier.

There are therefore at least two closely related directions out of the ordinary set-based theory. One is homotopical:

$$\mathbf{Set} \longrightarrow \mathbf{Gpd} \longrightarrow \mathbf{Anima},$$

where the new higher identifications remain invertible. The other is genuinely directed:

$$\mathbf{Set} \longrightarrow \mathbf{Cat},$$

where the state category may contain non-invertible arrows expressing refinement, simulation, information loss, resource expenditure, or other asymmetric relations between states.

For the purposes of this appendix the important point is therefore not that groupoids should be replaced by categories, but that both arise naturally from the same elementary observation. The transition structure of a Moore machine is an action. Once the state space is categorified, the action becomes an actegory structure, and the monad

$$A^* \times -$$

becomes the 2-monad

$$\mathbf{A}^* \times -.$$

This suggests that the groupoidal Sweedler theory developed above is one face of a broader higher theory in which

$$[\mathbf{Set}, \mathbf{Set}]_{\mathrm{acc}}$$

is replaced not only by endofunctors of spaces, but also, in a directed 2-categorical direction, by accessible endo-2-functors of **Cat**.

We might usefully approach this topic by opting for a strict setting. Then we may use existing parts of enriched Sweedler theory to work with categories enriched in Cat, the symmetric monoidal closed category of categories with cartesian product. These are strict 2-categories. Hence we might consider

$$Cat\text{-}[\mathbf{Cat}, \mathbf{Cat}]_{acc},$$

where **Cat** is the strict 2-category of small categories, functors, and natural transformations.

11.1 Actegories in applied category theory

Actegories are not merely a formal intermediate step between ordinary and higher automata. They have acquired a substantial role in applied category theory, especially where one wishes to distinguish a category of systems from a category of agents, resources, controls, messages, or contexts acting upon those systems. Capucci and Gavranović emphasise precisely this point in their survey *Actegories for the Working Amthematician* [1]: the usefulness of actegories lies in their ability to encode forms of agency, and much of the recent applied literature treats them as a basic semantic structure rather than as an auxiliary construction.

This is particularly visible in categorical cybernetics, where an acting category describes interventions on a category of systems; in mixed optics, where actegories and the associated Tambara modules provide the mathematical backbone for equivariant profunctorial constructions; and in categorical models of concurrency, where linear actegories separate a sequential world of messages from a concurrent world of message-passing processes. Similar ideas occur in the semantics of semi-effects, where actegories mediate between values and computations, and in persistence theory, where actions of monoids of time or scale give rise to what are naturally interpreted as dynamical categories.

There is also a structural point of particular relevance to the present programme. Capucci and Gavranović study the situation in which an $\mathcal{M}$-actegory $\mathcal{C}$ is itself monoidal. Compatibility between the action and the monoidal product is then expressed by a mixed interchanger

$$\iota_{m,n,c,d} : (m \otimes n) \bullet (c \boxtimes d) \longrightarrow (m \bullet c) \boxtimes (n \bullet d).$$

This is the structure required, for example, for the **Para** construction to inherit a monoidal structure. It is not itself the duoidal interchange required for a 2-categorical Sweedler theory, but it is a nearby manifestation of the same general phenomenon: two monoidal or action-like structures cannot simply be placed alongside one another if constructions involving both are to compose coherently. Monoidal actegories therefore provide a useful applied model for the kind of interchange structure that must be understood in developing the duoidal 2-categorical setting above.

These examples suggest that a categorified Sweedler theory could have applications well beyond the categorification of Moore machines. An actegory records that some monoidal category $\mathcal{A}$ acts on a category $\mathcal{X}$,

$$\mathcal{A} \curvearrowright \mathcal{X},$$

but by itself it does not classify the possible stateful implementations, mediations, or translations between one such action theory and another. This is exactly the kind of role played by measuring objects and runners in the ordinary Sweedler setting. One may therefore hope that a higher measuring object

$$\{T, R\}_2$$

would classify suitably coherent stateful transformations between 2-monads or pseudomonads whose algebras are actegories.

The connection with mixed optics is especially suggestive. Tambara modules may be viewed as equivariant profunctors between actegories for the same acting monoidal category. Since our behavioural construction already leads from runners to a profunctorial setting, one natural question is whether higher runners induce Tambara-type structure, or conversely whether suitable Tambara modules can be represented by higher measuring objects. In such a picture, actegories would provide the actions, Tambara modules the equivariant profunctorial transport, and higher runners the stateful realisations mediating between the underlying theories.

Concurrency provides another useful test case. If a monoidal category of messages acts on a category of processes, then the actegory records how messages may be supplied to processes. A categorified runner should be able to carry additional internal state and hence describe not merely that such an action exists, but how one protocol, interface, or message discipline is implemented through another stateful process. This would make contact with the directed and higher-dimensional systems considered elsewhere in the tutorial: a categorical state space already permits non-invertible morphisms representing process development, while precubical or higher-categorical refinements can subsequently record genuinely concurrent behaviour.

The examples from semi-effects and persistence point in the same direction. In the former, one may ask for a measuring object classifying stateful equivariant translations between different effect theories. In the latter, where a monoid of time or scale acts on a category of data, higher runners could mediate between different temporal or resource actions. This is close in spirit to the resource-sensitive constructions of Part VI, but now lifted from graded sets or endofunctors to categorical actions.

Thus the passage from groupoidal Moore machines to actegories is not only a matter of increasing categorical dimension. It places the higher Sweedler programme in contact with a body of existing applied work in which actions already encode agency, control, effects, resources, and concurrency. These settings provide concrete test cases for the stronger question:

can higher Sweedler objects classify stateful mediations between categorical actions?

A positive answer would make categorified runners relevant not only to higher automata, but also to several of the main applied uses for which actegories have already proved effective.

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