

Arity-Generated Sweedler Universes from Monoidal Sites

Representable, finitary and presentable arities

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Aim. Several parts of the tutorial changed the ambient category in which states, theories and runners live. A recurring technical difficulty was that the full endofunctor category

$$[\mathcal{C}, \mathcal{C}]_{\text{acc}} \quad \text{or} \quad \mathcal{V}\text{-}[\mathcal{C}, \mathcal{C}]_{\text{acc}}$$

need not satisfy the standard hypotheses under which Sweedler measuring comonoids are known to exist. The cost example suggested a different strategy: begin with a small monoidal category of geometric probes and use a small class generated from those probes as operational arities. The first version of this note took the representables themselves as the default arities. That is too restrictive for most algebraic purposes. The more useful picture has three levels:

$$\mathcal{J}_0 = y(\mathcal{A}) \subseteq \mathcal{J}^+ = \text{FinCoproduct}(y\mathcal{A}) \subseteq \mathcal{J}_\kappa = \mathcal{C}_\kappa.$$

The representable level is a strong, conically cocontinuous or “linear” core; finite coproducts of representables supply the strong sifted-colimit-preserving level of ordinary finite multi-input arities; a sufficiently large presentable level is the uniform technical fallback. This note develops that hierarchy and revisits the cost, cubical, smooth, Cahiers and light-condensed examples.

1 Monoidal probes in a presheaf category

Let $(\mathcal{A}, \otimes, I)$ be a small symmetric monoidal category and let

$$\mathcal{C} = \widehat{\mathcal{A}} = [\mathcal{A}^{op}, \mathbf{Set}]$$

carry Day convolution $\otimes_D$. The Yoneda embedding

$$y : \mathcal{A} \longrightarrow \mathcal{C}$$

is strong monoidal:

$$y(a) \otimes_D y(b) \cong y(a \otimes b), \quad y(I) \cong I_{\mathcal{C}}.$$

The representables therefore form a small tensor-closed family of geometric probes. In the closed Day structure one has

$$\begin{aligned} [y(a), X](b) &\cong \mathcal{C}(y(b), [y(a), X]) \\ &\cong \mathcal{C}(y(b) \otimes_D y(a), X) \\ &\cong \mathcal{C}(y(b \otimes a), X) \\ &\cong X(b \otimes a). \end{aligned}$$

Thus internal homming out of a representable is monoidal shift:

$$[y(a), -] \cong (-) \circ (- \otimes a).$$

Because presheaf colimits are pointwise, these shifts preserve conical colimits. What this does *not* by itself prove is preservation of every enriched weighted colimit required in the definition of an eleutheric system of arities. The representable case is justified instead by the density-plus-colimit-preservation argument of the next subsection.

2 Three arity levels

2.1 Level 0: representables and the conically cocontinuous core

Let

$$\mathcal{J}_0 := y(\mathcal{A})$$

denote the full $\mathcal{C}$ -enriched subcategory on the representables. The correct statement is not that the corresponding endofunctors are enriched-cocontinuous: enriched cocontinuity would force preservation of all $\mathcal{C}$ -tensors and is much too strong. Rather, $\mathcal{J}_0$ is eleutheric, and the $\mathcal{J}_0$ -ary $\mathcal{C}$ -endofunctors are precisely the strong $\mathcal{C}$ -functors whose underlying ordinary functors preserve conical colimits.

Here is the useful proof pattern. For $H : \mathcal{J}_0 \rightarrow \mathcal{C}$, enriched left Kan extension has the form

$$(\mathrm{Lan}_j H)(X) \cong \int^{a \in \mathcal{A}} [y(a), X] \otimes_D H(y(a)).$$

Each $[y(a), -]$ is a monoidal shift and preserves conical colimits; tensoring and the coend preserve them as well. Hence the underlying ordinary functor of $\mathrm{Lan}_j H$ is cocontinuous. Enriched Yoneda gives

$$(\mathrm{Lan}_j H)(y(a)) \cong H(y(a)).$$

Conversely, by ordinary density every presheaf is the canonical conical colimit of representables indexed by its category of elements. Therefore any strong functor whose underlying functor preserves conical colimits is reconstructed from its restriction to $\mathcal{J}_0$. Finally, the same density presentation proves the preservation clause in eleuthericity: for $J' \in \mathcal{J}_0$, the functor $[J', -]$ is again a shift and preserves the conical colimits occurring in $\mathrm{Lan}_j H$. Thus $\mathcal{J}_0$ is eleutheric without appealing to a free $\mathcal{C}$ -cocompletion claim.

This is a very small universe. In a cartesian example it generally excludes familiar non-linear constructions such as

$$X + 1, \quad X \times X, \quad \mathrm{List}(X),$$

and reader functors $X \mapsto X^S$ unless the chosen probe S is exceptionally small. What survives naturally are shift-, writer- and grading-type operations. It is therefore best regarded as the *conically cocontinuous* or *linear* core.

The extreme sanity check is $\mathcal{A} = 1$. Then $\mathcal{C} = \mathbf{Set}$ and $\mathcal{J}_0 = \{1\}$. A strong $\mathcal{J}_0$ -ary endofunctor is of the form

$$X \mapsto X \times K,$$

so the representable-only level is visibly much smaller than the world of ordinary algebraic theories.

2.2 Level 1: finite coproducts of representables

For algebraic systems the more natural default is the finite-coproduct closure

$$\mathcal{J}^+ := \text{FinCoproduct}(y\mathcal{A}) \subseteq \mathcal{C}.$$

An object of $\mathcal{J}^+$ has the form

$$J \cong y(a_1) + \cdots + y(a_n).$$

Since Day convolution preserves colimits separately,

$$\left(\coprod_i y(a_i)\right) \otimes_D \left(\coprod_j y(b_j)\right) \cong \coprod_{i,j} y(a_i \otimes b_j),$$

so $\mathcal{J}^+$ is again tensor-closed and contains the unit.

The relevant cocompletion is now sifted rather than arbitrary. If $\mathcal{A}^+$ denotes the free finite-coproduct completion of $\mathcal{A}$, then

$$[\mathcal{A}^{op}, \mathbf{Set}] \simeq \text{Sind}(\mathcal{A}^+),$$

so every presheaf is a sifted conical colimit of finite coproducts of representables. Moreover, for

$$J = \coprod_{i=1}^n y(a_i)$$

one has

$$[J, X] \cong \prod_{i=1}^n [y(a_i), X].$$

Each factor is a shift, and finite products in $\mathbf{Set}$ commute with sifted colimits. Hence the underlying ordinary functor $[J, -]$ preserves sifted colimits.

The same proof as at level 0 now applies with “conical” replaced by “sifted”. For $H : \mathcal{J}^+ \rightarrow \mathcal{C}$, $\text{Lan}_j H$ preserves sifted colimits and agrees with H on $\mathcal{J}^+$ by enriched Yoneda. Density through the sifted presentation then shows that every strong functor whose underlying ordinary functor preserves sifted colimits is recovered from its restriction to $\mathcal{J}^+$. The arity homs $[J, -]$ preserve exactly those sifted colimits, so $\mathcal{J}^+$ is eleutheric. Thus the $\mathcal{J}^+$ -ary endofunctors are precisely the strong functors whose underlying ordinary functors preserve sifted colimits.

The sanity check is exactly the desired one:

$$\mathcal{A} = 1 \implies \mathcal{J}^+ = \mathbf{FinSet} \subseteq \mathbf{Set}.$$

Thus finite coproducts of representables recover the ordinary finite arities of Lawvere theories. In cartesian cases this level includes familiar finite algebraic constructions such as

$$X + 1 \quad \text{and} \quad X \times X,$$

and more generally it is the natural strongly finitary level. This middle level is therefore the first serious candidate for a general algebraic Sweedler universe generated by geometric probes.

2.3 Level 2: a presentable fallback

Suppose $\mathcal{C}$ is locally presentable as an enriched monoidal category. Choose a regular cardinal κ large enough that the tensor unit and the geometric probes are κ -presentable and that a chosen small skeleton

$$\mathcal{J}_\kappa = \mathcal{C}_\kappa$$

is closed under the relevant tensor. Under the standard enriched local-presentability hypotheses, this gives an eleutheric arity category and hence a controlled category of κ -ary strong endofunctors.

This is the technically safest level, but it has two disadvantages. First, the geometry of the original test category plays much less of a role: fixing κ is essentially the standard accessibility repair. Second, the resulting measuring comonoids and even the class of admissible theories may depend on κ . The κ -level should therefore be treated as a fallback, not as the conceptual endpoint.

3 The duoidal Sweedler target

For a small tensor-closed arity category $\mathcal{J} \subseteq \mathcal{C}$, the candidate small ambient category is

$$\mathcal{D}_\mathcal{J} := [\mathcal{J}, \mathcal{C}]_\mathcal{C}.$$

There are two intended products. Composition of the associated $\mathcal{J}$ -ary endofunctors transports to a substitution product, written $\star$, while the monoidal structure on $\mathcal{J}$ gives enriched Day convolution, written $\diamond$:

$$\star = \text{substitution}, \quad \diamond = \text{Day convolution}.$$

On the Day side, for small $\mathcal{J}$ and closed cocomplete $\mathcal{C}$, standard enriched Day theory gives biclosedness. On the substitution side the relevant requirement for the measuring theorem is accessibility, not closedness. Eleuthericity is essential here: it is what identifies monoids for the transported substitution product $\star$ with the intended $\mathcal{J}$ -ary monads on $\mathcal{C}$. Without this identification the measuring theorem would enrich the wrong class of monoids.

The precise hypotheses used from López Franco–Vasilakopoulou, Theorem 20, are these. For a duoidal category

$$(\mathcal{D}, \diamond, i, \star, j),$$

they assume that $\mathcal{D}$ is a locally presentable category, that $\star$ is an accessible functor, and that the monoidal structure $(\diamond, i)$ is biclosed. Under these hypotheses there exists a tensored and cotensored $\text{Comon}_\star(\mathcal{D})$ -category whose underlying category is isomorphic to $\text{Mon}_\star(\mathcal{D})$; its enriched homs are the universal measuring comonoids [5, Thm. 20].

For the two presheaf arity levels above, the three non-duoidal hypotheses of Theorem 20 are already available. First, $[\mathcal{J}, \mathcal{C}]_\mathcal{C}$ is locally presentable because $\mathcal{J}$ is small and $\mathcal{C}$ is locally presentable. Second, enriched Day convolution is biclosed because $\mathcal{J}$ is small symmetric monoidal and $\mathcal{C}$ is closed and cocomplete. Third, substitution is accessible. Indeed,

$$(F \star G)(J) = (\text{Lan}_j F)(GJ).$$

For fixed G this is cocontinuous in F , since enriched left Kan extension is computed pointwise. For fixed F , it preserves filtered colimits in G : at level 0, $\text{Lan}_j F$ preserves all conical colimits, while

at level 1 it preserves sifted colimits, hence in particular filtered colimits. Separate filtered-colimit preservation implies preservation of filtered colimits jointly, using cofinality of the diagonal in a filtered indexing category. Thus $\star$ is in fact finitary at both presheaf levels.

The interchange map itself can also be constructed. For $F, H : \mathcal{J} \rightarrow \mathcal{C}$, consider the two bifunctors

$$(X, Y) \mapsto (\text{Lan}_{\mathcal{J}} F)(X) \otimes (\text{Lan}_{\mathcal{J}} H)(Y)$$

and

$$(X, Y) \mapsto \text{Lan}_{\mathcal{J}}(F \diamond H)(X \otimes Y).$$

At level 0 both preserve conical colimits separately in X and Y ; at level 1 both preserve sifted colimits separately. Consequently a transformation between them is uniquely determined by its restriction to $\mathcal{J} \times \mathcal{J}$, and every such transformation on arities extends uniquely. On arities we use the Day universal map

$$FJ \otimes HJ' \longrightarrow (F \diamond H)(J \otimes J').$$

Its extension is $\mathcal{C}$ -natural in X and Y because the Day universal map is enriched natural. This gives a well-defined natural transformation

$$(\text{Lan}_{\mathcal{J}} F)(X) \otimes (\text{Lan}_{\mathcal{J}} H)(Y) \longrightarrow \text{Lan}_{\mathcal{J}}(F \diamond H)(X \otimes Y).$$

Applying it to $X = GJ$ and $Y = KJ'$, and then composing with the Day universal map

$$GJ \otimes KJ' \longrightarrow (G \diamond K)(J \otimes J'),$$

gives, by the universal property of Day convolution, the required natural transformation

$$(F \star G) \diamond (H \star K) \longrightarrow (F \diamond H) \star (G \diamond K).$$

Thus the remaining presheaf-level issue is no longer construction of the interchange but verification of its duoidal coherence and unit laws. Until that verification is written down, the phrase “Sweedler universe” below should still be read as conditional on this final coherence check.

Proposition 1 (Presheaf arity principle). *Let $\mathcal{A}$ be a small symmetric monoidal category and let $\mathcal{C} = [\mathcal{A}^{op}, \mathbf{Set}]$ carry Day convolution. Then:*

1. $\mathcal{J}_0 = y\mathcal{A}$ is eleutheric, and the $\mathcal{J}_0$ -ary $\mathcal{C}$ -endofunctors are precisely the strong functors whose underlying ordinary functors preserve conical colimits;
2. $\mathcal{J}^+ = \text{FinCoproduct}(y\mathcal{A})$ is tensor-closed and eleutheric, and the $\mathcal{J}^+$ -ary $\mathcal{C}$ -endofunctors are precisely the strong functors whose underlying ordinary functors preserve sifted colimits; in particular $\mathcal{A} = 1$ gives $\mathcal{J}^+ = \mathbf{FinSet}$;
3. after fixing a suitable regular cardinal, $\mathcal{J}_{\kappa} = \mathcal{C}_{\kappa}$ supplies a larger accessibility-controlled level.

For the first two claims the proof is density plus preservation of the corresponding conical colimits by the arity homs, as described above. At these presheaf levels the resulting small functor categories are locally presentable, Day convolution is biclosed, and substitution is finitary. The canonical interchange map is constructed above. Hence, once its coherence and unit laws are verified, Theorem 20 of [5] applies: its $\star$ -monoids, equivalently by eleuthericity the corresponding $\mathcal{J}$ -ary monads, admit universal measuring comonoids and the resulting tensored and cotensored enrichment.

4 Monoidal reflective localizations and sheaves

Many ambient geometries in the tutorial are sheaf categories rather than raw presheaf categories. Suppose

$$L : [\mathcal{A}^{op}, \mathbf{Set}] \rightleftarrows \mathcal{C} : i$$

is an accessible reflective localization compatible with the monoidal structure. Day's reflection theorem gives conditions under which the reflected subcategory inherits the required closed monoidal structure; in particular a strong symmetric monoidal reflector is a standard sufficient situation [2].

Put

$$j(a) := Ly(a).$$

If the site is subcanonical these are simply the representable sheaves. Strong monoidality gives

$$j(a) \otimes j(b) \cong j(a \otimes b).$$

Thus the geometry again provides a canonical hierarchy

$$\mathcal{J}_0 = \{j(a)\}, \quad \mathcal{J}^+ = \text{FinCoproduct}\{j(a)\}, \quad \mathcal{J}_\kappa = \mathcal{C}_\kappa.$$

What changes after localization is that neither the representables nor their finite coproducts are automatically eleutheric. Colimits of sheaves are obtained by sheafification, and exponentiation/internal hom by a geometric probe need not preserve the colimits relevant to the chosen arity theory. Thus the presheaf theorem does not simply descend through sheafification.

The right sharp questions are therefore:

1. are the localized representables tiny enough to make $\mathcal{J}_0$ eleutheric?;
2. if not, do finite coproducts of them form a strongly finitary arity category $\mathcal{J}^+$?
3. failing both, what is the smallest tensor-closed presentability rank κ for which $\mathcal{J}_\kappa$ works?

This gives a genuine middle level between the representables-only and arbitrary κ -presentable constructions.

5 Cases from the tutorial

5.1 Cost-graded sets

Let C be the small commutative monoidal cost category of Part VI and

$$\mathcal{E} = C\text{-}\mathbf{Set}$$

with Day tensor. The representables satisfy

$$y_c \otimes_D y_d \cong y_{c+d}, \quad [y_c, X](p) \cong X(p+c).$$

Thus the level-0 arities give cost shifts and the strong, conically cocontinuous resource calculus. This level is useful, but the preceding correction matters: it does not by itself contain arbitrary

finite-arity algebraic operations. For a resource-sensitive analogue of ordinary Lawvere theories the more natural default is

$$\mathcal{J}_C^+ = \text{FinCoproduct}\{y_c \mid c \in C\}.$$

The hom out of such an arity is a finite product of cost shifts, hence its underlying ordinary functor preserves sifted colimits. This is the cost analogue of passing from the single Set-probe 1 to all finite sets. It contains the minimal shift/writer/grading effects while also allowing finite multi-input operations.

Accordingly, Part VI should distinguish two claims. The representable level gives a particularly transparent resource calculus. The finite-coproduct level is the better candidate for the general resource-sensitive algebraic theory used by Run. Determining exactly which monads used in Part VI already live at level 0 and which require $\mathcal{J}_C^+$ is a concrete audit still to be done.

5.2 Precubical and cubical sets

For a cube category $\square$ one again has a presheaf category

$$[\square^{op}, \mathbf{Set}]$$

and representable cubes are the evident geometric probes. If the geometric tensor satisfies

$$\square[m] \otimes \square[n] \cong \square[m + n],$$

then representables and their finite coproducts are tensor-closed exactly as above.

There is, however, a structural caveat. The geometric tensor used for (pre)cubical execution is generally monoidal but not symmetric, while the enriched system-of-arithies framework used above is formulated over a symmetric monoidal base. Thus the geometric cubical case is not yet an instance of the theorem as stated. There are two possible routes: use the cartesian monoidal structure for the arity theory, or develop the corresponding non-symmetric enriched arity theory. This is precisely the tensor distinction already encountered in Part VII and should remain explicit.

5.3 Smooth sets

Let

$$\text{SmoothSet} = \text{Sh}(\text{CartSp})$$

for a small Cartesian-space site with its usual open-cover topology. The site is cartesian monoidal and subcanonical, so the representables

$$y(\mathbb{R}^n)$$

are the canonical geometric probes. At the level of test values,

$$X^{y(\mathbb{R}^m)}(\mathbb{R}^n) \cong X(\mathbb{R}^{n+m}).$$

But exponentiation by $y(\mathbb{R}^m)$ is a right adjoint and is not automatically cocontinuous or sifted-colimit preserving. Hence neither $\mathcal{J}_0$ nor its finite-coproduct closure may simply be declared eleutheric.

The hierarchy nevertheless gives a useful programme. First test the representables; then test

$$\mathcal{J}_{\text{sm}}^+ = \text{FinCoproduct}\{y(\mathbb{R}^n)\};$$

only if those fail should one pass to a sufficiently large $\mathcal{J}_\kappa$. The necessary κ depends on the chosen site and should not be guessed from the present argument.

5.4 The Cahiers topos

Let FormalCartSp denote the usual site of formal Cartesian spaces

$$\mathbb{R}^n \times \ell W$$

with coverings in the Cartesian direction, and let

$$\mathcal{C}_{\text{Cah}} = \text{Sh}(\text{FormalCartSp}).$$

The formal Cartesian representables are dense geometric probes and are closed under finite products. Thus they again give natural candidates for $\mathcal{J}_0$ and $\mathcal{J}^+$.

The Cahiers case also shows exactly why one should distinguish density from the special behaviour of the infinitesimal direction. For a Weil infinitesimal ℓW in the Cahiers model, the external endofunctor

$$(-)^{\ell W} : \mathcal{C}_{\text{Cah}} \longrightarrow \mathcal{C}_{\text{Cah}}$$

has a right adjoint, the “amazing right adjoint” of synthetic differential geometry [7, 8, 9]. Since $(-)^{\ell W}$ is then itself a left adjoint, it preserves external colimits. This is an external categorical adjunction; it should not be described as an internalisable adjunction or as a generic property of smooth topoi. The infinitesimal factor therefore has exceptionally good external colimit behaviour in the Cahiers model, but the infinitesimal probes alone do not form the full dense family of formal Cartesian test objects. The dense probes also contain the Cartesian factor $\mathbb{R}^n$, and the amazing-right-adjoint property does not settle that direction.

So the sharp Cahiers question is not merely “are the representables eleutheric?” but more specifically:

how much of the Cartesian direction can be admitted while retaining the required arity preservation?

This makes the middle level especially attractive. Finite coproducts of formal Cartesian probes may capture the algebraic multi-input structure without immediately jumping to all κ -presentables. If that still fails, the locally presentable topos supplies the safe κ -level.

5.5 Light condensed sets

Light condensed sets were introduced in the Clausen–Scholze *Analytic Stacks* lectures as a size-controlled replacement for unrestricted condensed sets. They are sheaves on the essentially small site LightProFin with its coherent topology of finite jointly surjective covers [11]. Thus

$$\text{LightCondSet} = \text{Sh}(\text{LightProFin})$$

is a genuine Grothendieck topos. The site is cartesian monoidal, so the representable light profinite sets and their finite coproducts give the natural first two arity levels.

Here the safe level is unusually favourable. The light profinite site has finite limits and its topology is generated by finite jointly surjective covering families, so it is a coherent site. Hence the resulting

Grothendieck topos is coherent. A standard result in topos theory is that every coherent topos is locally finitely presentable; see Johnstone, §D3.3 [10]. Thus one may take

$$\mathcal{J}_\omega = (\text{LightCondSet})_\omega.$$

This arity category is also closed under the cartesian tensor. In a locally finitely presentable topos generated by the representables, finitely presentable objects are retracts of finite colimits of representables. Cartesian product preserves colimits separately, the representables are closed under product, and closure under finite colimits and retracts therefore shows that $\mathcal{J}_\omega$ is product-closed. Hence the safe level is genuinely finitary and tensor-closed in the light-condensed case. This does not by itself prove that the representables or their finite-coproduct closure are eleutheric, but it makes the gap between the geometric and safe levels much smaller than in the smooth examples.

For solid modules and other analytic categories built over light condensed sets, a separate linear/analytic step remains. Their tensor products are not simply the cartesian tensor of the underlying topos, so closedness, accessibility and compatibility of the localization with the solid tensor must be checked afresh.

6 What the hierarchy says

The corrected picture is therefore

linear probes	$\mathcal{J}_0 = y\mathcal{A},$
finite algebraic arities	$\mathcal{J}^+ = \text{FinCoproduct}(y\mathcal{A}),$
safe accessibility level	$\mathcal{J}_\kappa = \mathcal{C}_\kappa.$

For raw presheaves the first two levels have a clean cocompletion interpretation: representables control strong endofunctors whose underlying functors preserve conical colimits, while finite coproducts of representables control strong endofunctors whose underlying functors preserve sifted colimits. After sheaf localization these become questions rather than automatic facts, and the presentable level supplies the fallback.

This weakens but also clarifies the original slogan. Geometric probes do suggest operational arities, but the probes themselves may be too “linear” for the algebraic theories needed by systems. The more accurate passage is

$\text{geometric probes} \longrightarrow \text{finite probe arities} \longrightarrow \text{Sweedler test.}$

The middle arrow is what recovers ordinary Lawvere arities in **Set**, and it is likely to be the relevant one for many of the enriched extensions in the tutorial.

7 Open checks

The note now separates the established arity/cocompletion facts from the remaining Sweedler step. The main open checks are:

1. verify the coherence and unit laws for the substitution–Day interchange constructed above;

2. in the cost case, determine whether the monads actually used in Part VI lie already at $\mathcal{J}_0$ or require $\mathcal{J}_C^+$;
3. in the cubical case, choose between cartesian arities and a genuinely non-symmetric arity theory;
4. in SmoothSet and Cahiers, determine whether $\mathcal{J}^+$ is eleutheric before passing to $\mathcal{J}_\kappa$;
5. in Cahiers, isolate precisely how the externally tiny infinitesimal direction interacts with the non-tiny Cartesian direction;
6. in light condensed sets, verify accessibility of the substitution product $\star$ and construct the interchange on

$$[\mathcal{J}_\omega, \text{LightCondSet}]_{\text{LightCondSet}}.$$

Finite computations on truncated graded sets could provide useful sanity checks for the first item, but they would not replace the categorical coherence proof.

8 References

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