Modal homotopy type theory: The new new logic

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Philosophy and 'current' mathematics

- Plato, Aristotle – Euclidean geometry
- Descartes – analytic geometry
- Leibniz – differential calculus
- (Reaction to) Kant – non-Euclidean geometry
- Frege, Peano, Poincaré, Russell, Hilbert, Brouwer – Foundational discussions
- ...

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Colin McLarty has emphasised for decades that philosophical development in mathematics has never stopped.

Here with a particular focus not on the logicians (Gödel, Robinson, Kreisel,...), but rather on the mathematicians (Brouwer, Noether, Mac Lane, Grothendieck,...)

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Where should we turn in current mathematics for philosophically salient developments? How do we choose from the content of the 30000 articles a year on the arXiv.

There are risks in our choices, so one should look for some strong signals of their importance.

In the last few years I have been looking at *homotopy type theory* and its *modal* extensions.

I'll outline here some of its signals.

One central choice in mathematics is the basic shape of mathematical entities:

- The set as a bag of dots, completely distinct and yet indistinguishable.

Irrespective of the way one chooses to describe sets formally, ‘materially’ or ‘structurally’, it’s an astonishing idea that mathematics could rely on such a conception.

- $x, y : A$, then $(x =_A y)$ is a proposition.

We ask *whether* two elements are the same, not *how* they are the same.

However, arising from the needs of current geometry and current physics, we find that having solely such a basic shape is a restriction. Beyond sets we need

- *Homotopy types* or *n-groupoids*: points, reversible paths between points, reversible paths between paths, ...

These may seem more complicated, but they arise in systems with fewer axioms.

The internal view

For any two elements of a collection we can ask are they the same or not.

- Where we have a collection A and $x, y : A$, we form $x =_A y$.

But then we can treat the latter as a collection and iterate

- With $x =_A y$, and $p, q : x =_A y$, we form $p =_{(x =_A y)} q$.

Drop the 'Uniqueness of Identity proofs'

We need not insist that any two proofs of the sameness of entities are themselves the same.

We reject the axiom that claims this is the case, or in other words we don't insist that the following type is necessarily inhabited:

$$p =_{(x =_A y)} q \,.$$

The external view

- Gathering together all sets results in a collection which behaves nicely: a *topos*.
- Gathering together all homotopy types results in a collection which behaves *extremely* nicely: an $(\infty, 1)$-*topos*.

We need to tell a justificatory story running at least from Grothendieck to Lurie.

($(\infty, 1)$-toposes are a particularly nice environment for cohomology: https://ncatlab.org/nlab/show/cohomology)

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- *Homotopy type theory* as homotopy (type theory) is the internal language of ∞-toposes. It is a type theory in the logical sense, and may be implemented on a computer.

We see wedded together the

- Categorical logic of William Lawvere: Adjointness in foundations.
- (Constructive) intensional type theory of Per Martin-Löf.

Hierarchy of *homotopy* types

We have a hierarchy of kinds of types to be treated uniformly:

...	...
2	2-groupoid
1	groupoid
0	set
-1	mere proposition
-2	contractible type

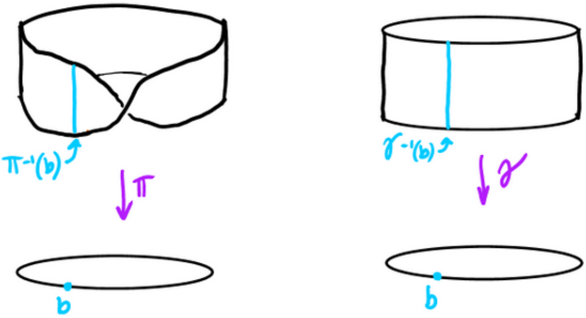
An important part of Martin-Löf type theory is the notion of a *dependent* type, denoted

$$x : A \vdash B(x) : Type.$$

Here the type $B(x)$ *depends* on an element of A, as in

- $Days(m)$ for $m : Month$
- $Players(t)$ for $t : Team$

It's helpful to have in mind the imagery of spaces fibred over other spaces:



Realising *n*-types as spaces, such spaces over other spaces are everywhere in mathematics and physics, fibre bundles and gauge fields.

Two central constructions we can apply to these types are **dependent sum** and **dependent product**: the total space and the sections.

In general we can think of this **dependent sum** as sitting 'fibred' above the base type A, as one might imagine the collection of league players lined up in fibres above their team name.

Likewise an element of the **dependent product** is a choice of a player from each team, such as $Captain(t)$.

Dependent sum	Dependent product
$\sum_{x:A} B(x)$ is the collection of pairs (a, b) with $a : A$ and $b : B(a)$	$\prod_{x:A} B(x)$, is the collection of functions, f, such that $f(a) : B(a)$
When A is a set and $B(x)$ is a constant set B: The product of the sets.	When A is a set and $B(x)$ is a constant set B: The set of functions from A to B.
When A is a proposition and $B(x)$ is a constant proposition, B: The conjunction of A and B.	When A is a proposition and $B(x)$ is a constant proposition, B: The implication $A \to B$.

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When A is a set and $B(x)$ is a varying proposition: Existential quantification.	When A is a set and $B(x)$ is a varying proposition: Universal quantification.

As Lawvere taught us, these are left and right adjoints.

The bottom line is that homotopy type theory for the lower levels of the hierarchy encapsulates:

- Propositional logic
- (Typed) predicate logic
- Structural set theory

Considering the full type theory, the line between logic and mathematics has blurred – homotopy groups of the spheres, group actions,...

HoTT is a structural theory *par excellence*.

Structural inference - univalence

If A and B are equivalent types, then whatever we can establish about A may be transferred to B.

(See my *Expressing 'the structure of' in homotopy type theory*, or Ahrens and North, *Univalent foundations and the equivalence principle*.)

People look to use computer assistants, Agda or Coq, to construct proofs in HoTT. (See later slide.)

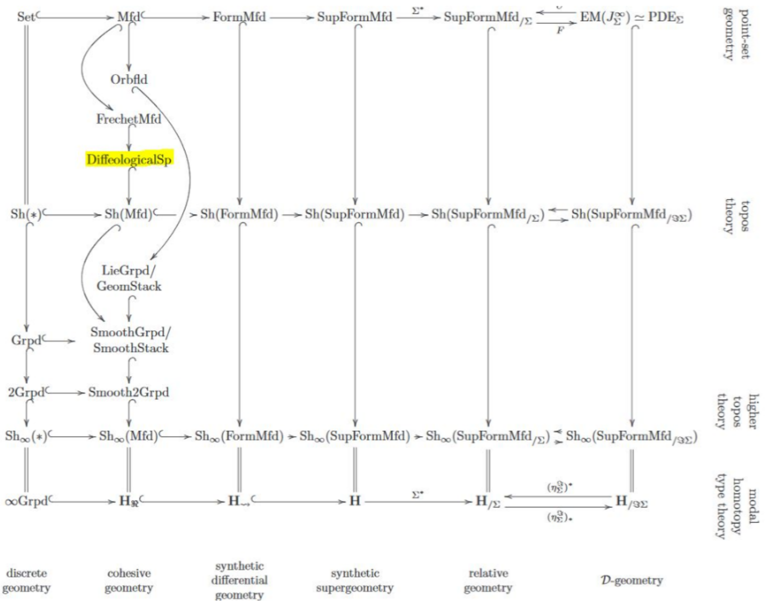
Modal variants of HoTT

For modern geometry we can add another of Lawvere's discoveries: a synthetic account of cohesion via *modalities*.

(See my *Reviving the philosophy of geometry*.)

Computer science also studies modal types for permissions, etc.

Extending geometry



Other variants on HoTT

For parameterized spectra/twisted cohomology we need *linear* homotopy type theory.

There are other varieties, such as *directed* homotopy type theory.

See *n*Lab for all your needs.

Philosophical leads

Philosophers of mathematics should already have been persuaded by the success of category theory, and by now be ready to hear about the successes of higher category theory.

Although HoTT is very young, and modal HoTT even younger, at last we have an opportunity to bring *real* mathematics to the attention of philosophy, and not just to the tiny domain of philosophy of mathematics.

Philosophical leads

- Logicism, constructivism, structuralism, formalism
- Computational trinitarianism
- Husserl, ...
- Metaphysics: Types, identity, modal types...
- Natural language
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But never forget the place of category theory here.

- HoTT and $(\infty, 1)$-toposes go hand in hand.
- Modal HoTT is about functors between $(\infty, 1)$-toposes

Physics with Modal HoTT

Schreiber

Introduction to Higher Supergeometry

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- Urs Schreiber

  Introduction to Higher Supergeometry

  lecture at *Higher Structures in M-Theory*

  Durham Symposium

  August 2018

Abstract. Due to the existence of a) gauge fields and b) fermion fields, the *geometry of physics* is higher supergeometry, i.e. super-geometric homotopy theory. This is made precise via Grothendieck's functional geometry implemented in higher topos theory. We give an introduction to the higher topos of higher superspaces and how it accomodates higher Lie theory of super L-∞ algebras. We close by indicating how geometric homotopy theory reveals that the superpoint emerges "from nothing", and that core structure of M-theory emerges out of the superpoint via a sequence of invariant universal higher central extensions. This will be discussed in more detail in other talks in the meeting.

Additional reading

Mike Shulman

- Homotopy type theory: the logic of space, arXiv:1703.03007
- Homotopy Type Theory: A synthetic approach to higher equalities, arXiv:1601.05035

Mathematical developments in HoTT

- Covering Spaces in Homotopy Type Theory, unpublished
- Higher Groups in Homotopy Type Theory, arXiv:1802.04315, Free Higher Groups in Homotopy Type Theory, arXiv:1805.02069
- Localization in Homotopy Type Theory, arXiv:1807.04155
- The James construction and $\pi_4(S^3)$ in homotopy type theory, arXiv: 1610.01134
- The Cayley-Dickson Construction in Homotopy Type Theory, arXiv:1610.01134
- Cellular Cohomology in Homotopy Type Theory, arXiv:1802.02191
- The real projective spaces in homotopy type theory, arXiv:1704.05770
- Synthetic Homology in Homotopy Type Theory, arXiv:1706.01540
- On the homotopy groups of spheres in homotopy type theory, arXiv:1606.05916
- A mechanization of the Blakers-Massey connectivity theorem in Homotopy Type Theory, arXiv:1605.03227, A Generalized Blakers-Massey Theorem, arXiv:1703.09050

Modal developments

- Brouwer's fixed-point theorem in real-cohesive homotopy type theory, arXiv:1509.07584
- Cartan Geometry in Modal Homotopy Type Theory, arXiv:1806.05966
- Sketch given for Noether's theorem

A few achievements of higher category theory

- Weil's Conjecture for Function Fields: the problem of computing Tamagawa numbers of algebraic groups over function fields.
- Moduli stacks in elliptic cohomology.
- Serre intersection formula for two varieties revised via the homotopy fiber product.
- The Cobordism hypothesis.

On teaching Lurie's Higher Algebra to graduates

One could argue this is the next logical step of a progression. Older books in homological algebra refused to use spectral sequences. Then Weibel's highly praised book does the opposite and introduces them early on, but relegates derived categories to a final chapter. Then Gelfand-Manin take it one step further and start with derived categories. They discuss dg-algebras and model categories at the very end and stop short of discussing non-abelian derived functors. Lurie's higher algebra is the next step but it's also quite big and not meant to be used for lectures...
(https://mathoverflow.net/questions/225712/teaching-higher-algebra)

Lurie and Gaitsgory explain how derived categories are deficient in 2.2.1 of *Weil's Conjecture for Function Fields I*:
http://www.math.harvard.edu/~lurie/papers/tamagawa-abridged.pdf

A graduate course on homological algebra as an aspect of homotopic algebra is here:
https://ncatlab.org/schreiber/show/Introduction+to+Homological+Algebra